

# Robust construction of the incipient infinite cluster in high-dimensional percolation

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## Percolation: a model of random subgraphs of the lattice

- On the integer lattice  $\mathbb{Z}^d$ , place edges independently with probability  $p$ .



$p = 0.3$



$p = 0.5$



$p = 0.7$

- Phase transition** in  $\mathbb{P}_p(0 \leftrightarrow \infty)$ . Zero for  $p < p_c$ , positive for  $p > p_c$ .
- $\mathbb{P}_{p_c}(0 \leftrightarrow \infty)$  known to be zero in two dimensions and high dimensions ( $d \geq 11$ ).
- However, large critical clusters are a natural “labyrinth”: return time of random walk decays at rate  $n^{-2/3}$ .

## Incipient infinite cluster: an infinite random labyrinth

- Setting: critical cluster “conditioned to be infinite”.
- Kesten 1986: two dimensions, using RSW crossing estimates, by conditioning on the **one-arm event**:

$$\nu(\cdot) = \lim_{n \rightarrow \infty} \mathbb{P}_{p_c}(\cdot \mid 0 \leftrightarrow \partial B(n)), \quad (1)$$

and **from above**:

$$\nu(\cdot) = \lim_{p \downarrow p_c} \mathbb{P}_p(\cdot \mid 0 \leftrightarrow \infty). \quad (2)$$

- van der Hofstad and Jarai 2003: high dimensions, using the lace expansion, by conditioning on the **two-point event**:

$$\nu(\cdot) = \lim_{|x| \rightarrow \infty} \mathbb{P}_{p_c}(\cdot \mid 0 \leftrightarrow x). \quad (3)$$

## Robust construction: theorems

Consider nearest-neighbor bond percolation on  $\mathbb{Z}^d$  with  $d \geq 11$ .

### Theorem (CCHS '24+)

For any sequence of vertex sets  $V_n, D_n$  escaping to infinity, with  $\mathbb{P}_{p_c}(0 \overset{\mathbb{Z}^d \setminus D_n}{\longleftrightarrow} V_n) \neq 0$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}_{p_c}(E \mid 0 \overset{\mathbb{Z}^d \setminus D_n}{\longleftrightarrow} V_n)$$

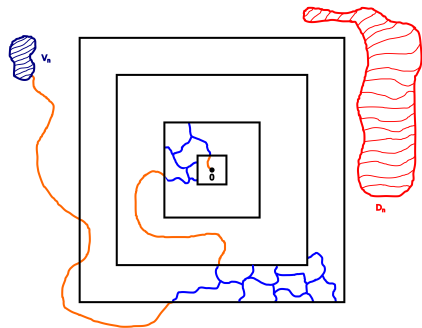
exists for all cylinder events  $E$ .

### Theorem (CCHS '24+)

$$\lim_{p \downarrow p_c} \mathbb{P}_p(E \mid 0 \leftrightarrow \infty)$$

exists and equals the previous limit for all cylinder events  $E$ .

## Gluing.



- Select a sequence of scales so with high probability, they have unique spanning clusters (u.s.c).
- Decompose  $\mathbb{P}(E, 0 \overset{\mathbb{Z}^d \setminus D_n}{\leftrightarrow} V_n)$ , and exploit independence.

$$\sum_{\mathcal{C}} \mathbb{P}(E, 0 \leftrightarrow \mathcal{C}, \mathcal{C} \text{ u.s.c.}, \mathcal{C} \leftrightarrow V_n).$$

- This turns establishing

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}(E, 0 \leftrightarrow V_n)}{\mathbb{P}(0 \leftrightarrow V_n)} \quad (4)$$

into establishing

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}(\mathcal{C} \leftrightarrow V_n \mid \mathcal{C} \text{ s.c.})}{\mathbb{P}(\mathcal{C}' \leftrightarrow V_n \mid \mathcal{C}' \text{ s.c.})}. \quad (5)$$

### Mixing.

- By iteration, (5) follows from

$$\frac{\mathbb{P}(\mathcal{C} \leftrightarrow \mathcal{D})\mathbb{P}(\mathcal{C}' \leftrightarrow \mathcal{D}')}{\mathbb{P}(\mathcal{C}' \leftrightarrow \mathcal{D})\mathbb{P}(\mathcal{C} \leftrightarrow \mathcal{D}')} \leq \kappa, \quad (6)$$

uniformly in all clusters  $\mathcal{C}, \mathcal{C}', \mathcal{D}, \mathcal{D}'$ .

- (6) is shown to be true for “good” spanning clusters using diagrammatic estimates.
- Kozma and Nachmias’ regularity method is used to show spanning clusters are “good” with high probability.

