GeoGebra as an Epistemic Scaffold for Reasoning and Proof: Designing Instruction for Secondary Geometry Based on the Van Hiele Framework

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Abstract: Constructing formal proofs remains a major hurdle for many prospective mathematics teachers, especially when their geometric thinking is confined to visualization and analysis levels. This study investigates how dynamic exploration with GeoGebra can support cognitive transitions across Van Hiele levels in structured reasoning and proof tasks on the parabola. Six pre-service teachers completed four GeoGebra-based tasks designed to link visual, symbolic, and deductive representations. Data from worksheets, think-aloud protocols, and interviews were analyzed thematically using the Van Hiele framework. Results show consistent development, with participants progressing from intuitive observations to formal deductive reasoning. GeoGebra features—such as sliders, measurement tools, and dynamic tracing-acted as epistemic scaffolds that enabled the formulation of conjectures and logically valid proofs. Based on these findings, a two-phase instructional model, Reasoning and Proof Learning (RPL), was developed, comprising Exploration and Representation, and Justification and Proof phases. This model was translated into a practical teaching plan for secondary classrooms, promoting geometric reasoning and proof habits. The study offers both theoretical and instructional contributions to mathematics teacher education and technology-enhanced geometry learning.

Keywords: dynamic geometry software; geometric thinking; pre-service mathematics teachers; reasoning and proof; van Hiele model

INTRODUCTION

Geometry is a fundamental domain in mathematics that supports the development of spatial reasoning, logical structuring, and the ability to construct deductive arguments. Within this field, geometric thinking and mathematical reasoning particularly in the context of proof construction are widely recognized as core competencies for mathematics learners (G. Stylianides et al., 2017); The National Council of Teachers of Mathematics, 2000). These competencies are essential not only for mastering mathematics as a discipline but also for fostering the higher-order thinking skills required in 21st-century learning. Despite longstanding curricular emphasis on proof, achieving advanced levels of geometric thinking remains a persistent challenge for both students and teachers.

Empirical evidence shows that prospective mathematics teachers often struggle to reach higher levels of geometric thinking as described by the Van Hiele model, especially in constructing formal deductive proofs. According to (Van Hiele, 1999), geometric thinking progresses hierarchically from visualization to analysis, informal deduction, formal deduction, and ultimately rigor. However, many pre-service teachers operate only at the lower levels of this hierarchy. Studies in Indonesia by by Armah, (2024); Mawarsari et al., (2023); Naufal et al., (2021) revealed that most pre-service teachers remained at the visualization or analysis levels and were unable to construct formal geometric proofs. These findings are supported by Scristia et al., (2025), who reported that only a small number of students achieved the formal deduction level, indicating a significant gap in teacher education programs.

These trends align with recent international studies. (Haj Yahya & Hershkowitz, 2024) found that many pre-service teachers struggled with constructing proofs due to reliance on prototypical representations and weak logical structuring. González et al., (2024) showed that most mathematics education students operated at the informal deduction level. (Birgin & Özkan, 2024) highlighted limited abilities in hierarchical classification and deductive reasoning in plane geometry tasks. Haj-Yahya & Olsher, (2025) emphasized the need for explicit scaffolding, while Ødegaard et al., (2024) pointed to failures in transferring empirical reasoning into formal deductive arguments. Anwar et al., (2025) further emphasized these challenges, identifying three major obstacles faced by pre-service teachers when formulating conjectures in dynamic geometry environments: (1) limited proficiency with GeoGebra, (2) difficulties in identifying robust invariants, and (3) formulating conjectures without strong visual or empirical foundations. These findings suggest that while technology can open new opportunities for reasoning, its effectiveness depends heavily on how it is pedagogically integrated to support cognitive transitions toward proof construction.

In response to these challenges, scholars have advocated for the explicit integration of reasoning and proof tasks into geometry instruction. Such tasks provide structured opportunities for learners to progress through the Van Hiele levels via conjecturing, justification, and argumentation (Buchbinder & Mccrone, 2020; Hanna & de Villiers, 2021). In this context, proof is not merely a tool for verification but a dynamic process of constructing and communicating logical relationships (Ellis et al., 2012; A. J. Stylianides, 2007). Actively constructing arguments helps learners deepen their understanding of geometric structures and the principles underlying them. Aligned with this pedagogical shift is the growing adoption of Dynamic Geometry Software (DGS), such as GeoGebra, in mathematics education. DGS enables learners to manipulate geometric objects in real time, explore invariants, and formulate conjectures through visual feedback (Laborde, 2002). Several studies have shown that DGS enhances conceptual understanding and spatial reasoning (Martinovic & Manizade, 2020; Yohannes & Chen, 2021). However, most existing literature focuses on the visual and exploratory functions of DGS, with limited research exploring how such environments can be strategically used to support transitions from informal reasoning to formal deductive proof, especially among pre-service teachers (Galili et al., 2023; Hartono et al., 2024; Komatsu & Jones, 2020).

A notable study by Mariotti & Pedemonte, (2019) demonstrated that interaction with DGS allows learners to connect spatial intuition with formal logical structures through activities such as conjecture formulation and formal proof validation. Their findings emphasize that a "need for proof" emerges as a key component in students' mathematical reasoning processes. In other words, DGS not only helps learners *see* geometry but also fosters epistemological awareness and motivation to justify observations through deductive arguments (Thapa et al., 2022).

Although the potential of Dynamic Geometry Software (DGS) for enhancing conceptual understanding and spatial reasoning is well established, few empirical studies have examined how such environments can be systematically designed to develop formal deductive reasoning, especially among pre-service mathematics teachers. Previous research often treated DGS mainly as a visualization or exploration tool, without deeply analyzing learners' progression through the Van Hiele levels or how DGS can serve as an epistemic bridge between intuition and formal proof. This study investigates how DGS-based tasks influence the reasoning pathways and engagement of prospective mathematics teachers as they construct geometric proofs, particularly during their transitions across Van Hiele levels. The research contributes to both the theoretical understanding of deductive reasoning in digital environments and the identification of pedagogical strategies that integrate visual, symbolic, and deductive representations in geometry learning. The empirical findings inform the development of an initial instructional framework grounded in GeoGebra-based activities, designed to strengthen pre-service teachers' reasoning and proof competencies. By engaging pre-service teachers as active participants in dynamic exploration, this study provides an empirical foundation for effective teaching practices that can be applied by both mathematics educators and classroom teachers. Consequently, the results are expected to have direct implications for technology-enhanced geometry instruction and to inform curriculum development in teacher education programs emphasizing reasoning and proof.

Research Problem and Questions

Although DGS is increasingly accessible in mathematics education, many pre-service mathematics teachers still struggle to construct formal geometric reasoning and proof, especially when working with abstract concepts such as conic sections. Among these, the parabola stands out as a foundational topic that integrates visual, symbolic, and deductive reasoning. Effectively guiding learners through cognitive transitions among these representations requires well-designed instructional interventions that not only support exploration but also foster conceptual understanding and formal justification. This study aims to examine how dynamic exploration using GeoGebra can support the development of geometric reasoning among pre-service mathematics teachers, particularly in reasoning and proof tasks involving the properties of the parabola. To address this overarching aim, the following research questions are posed:

1. How does dynamic exploration within a GeoGebra-based environment support the development of pre-service mathematics teachers' geometric thinking, particularly in constructing reasoning and proof related to the properties of the parabola?
2. What features of technology-integrated tasks facilitate cognitive transitions through the Van Hiele levels among pre-service teachers?

3. How do pre-service teachers utilize GeoGebra to move from visual exploration to formal justification in geometric proof construction?
4. How can instructional designs that incorporate DGS be structured to bridge visual, symbolic, and deductive representations in teaching and learning the concept of the parabola?

LITERATURE REVIEW

Geometric Thinking and Van Hiele Model

The Van Hiele model (Van Hiele, 1999) describes how learners' geometric thinking progresses through hierarchical stages: visualization, analysis, informal deduction, and formal deduction. This framework remains a cornerstone in mathematics education research and curriculum design (Clements & Battista, 1992; Usiskin, 1982). However, studies show that many pre-service teachers still operate at lower levels and struggle to construct formal proofs (Birgin & Özkan, 2024; González et al., 2024b). Recent international research emphasizes the need for instructional scaffolding to help learners transition between these levels (Haj-Yahya & Olsher, 2025; Ødegaard et al., 2024). Importantly, Van Hiele levels are not determined by age or grade (Škrbec & Čadež, 2015; Van Hiele, 1999). In this study, four levels were considered: (1) Visualization; (2) Analysis; (3) Informal Deduction; and (4) Formal Deduction. These levels collectively represent the developmental pathway through which students advance from perceptual understanding to rigorous deductive reasoning in geometry.

Reasoning and Proof in Mathematics Education

Reasoning is fundamentally intertwined with proof at all levels of mathematics education, serving as the process through which learners explore patterns, form conjectures, and validate them through logical argumentation (Lesseig et al., 2019; Stylianides G, 2008). Proof provides the logical structure needed to confirm or refute conjectures, ensuring the coherence and validity of mathematical reasoning (Stylianides G, 2008; Hanna & de Villiers, 2021). As emphasized by Buchbinder & McCrone, (2022); Ellis et al., (2012); The National Council of Teachers of Mathematics, (2000), this continuum of exploration, conjecturing, and proving defines what is known as *reasoning and proof*. Stylianides G, (2008) identifies reasoning and proof as inquiry-based activities involving (1) identifying patterns, (2) forming conjectures, (3) constructing logical arguments to test validity, and (4) developing formal proofs. Similarly, Thompson et al. (2012) highlight key indicators such as generating conjectures, evaluating arguments, identifying counterexamples, and correcting flawed proofs. Ellis et al., (2012) and Jeannotte & Kieran, (2017) expand this to include exploration, generalization, justification, and critical evaluation. Synthesizing these perspectives, Utari & Hartono, (2019) propose six operational indicators: (1) generating patterns to form conjectures, (2) justifying conjectures, (3) constructing proofs, (4) evaluating argument correctness, (5) finding counterexamples, and (6) revising flawed proofs. More recently, Buchbinder & McCrone, (2022) and Alsina et al., (2024) reaffirm reasoning and proof as a cyclical process encompassing exploration, pattern observation, conjecture formation, generalization, justification, and verification ultimately fostering the development of mathematical thinking.

Dynamic Geometry Software (DGS) as Epistemic Mediator

Dynamic Geometry Software (DGS), especially GeoGebra, allows learners to manipulate objects and visualize invariant relationships in real time. Laborde, (2002) conceptualized DGS as more than a visualization tool, it mediates mathematical reasoning by encouraging students to explore and refine their ideas. Mariotti & Pedemonte, (2019) found that interaction with DGS fosters a natural progression from empirical conjecture to formal proof, provided the tasks are epistemologically well-structured. Santos-Trigo et al., (2024) confirmed this potential by showing how DGS environments promote deeper reasoning by enabling learners to construct and test their own proofs, particularly when supported by timely teacher questioning. DGS can scaffold critical reasoning behaviors, such as conjecturing, validating, and generalizing that are foundational for proof. The integration of DGS into teacher education has been shown to not only improve conceptual understanding but also shift learners' epistemological views about mathematics. (Komatsu & Jones, 2020) argue that DGS can bridge the gap between empirical observation and deductive rigor when paired with structured reflection. (Quaresma & Janičić, 2006) extend this argument by emphasizing that exploratory environments must prompt the "need for proof," turning visual patterns into formal argumentation. Studies Gulkilik, (2023); Rizos & Gkrekas, (2023); Santos-Trigo et al., (2024) underscore that student-driven constructions using DGS lead to not only better geometric intuition, but more robust proof skills.

METHOD

All procedures and instruments in this study were strategically designed to address the four research questions that had been formulated. The analysis of participants' geometric thinking development through GeoGebra-based exploratory tasks was used to answer the first and second questions (RQ1 and RQ2), which focused on reasoning construction and cognitive transitions across Van Hiele levels. Meanwhile, verbal explorations captured through think-aloud protocols and written proofs from tasks and interviews were employed to examine the transition from visual representations to deductive justification (RQ3). The findings from this process served as the foundation for designing a two-phase GeoGebra-based instructional model that integrates visual, symbolic, and deductive representations of the parabola concept (RQ4).

Research Design and Approach

This study adopted a qualitative approach within the framework of design-based research to explore and construct a reasoning and proof-based instructional design (Reasoning and Proof Learning/RPL) using GeoGebra. This approach allowed for the iterative design, implementation, and reflection of instructional activities in a real classroom context, providing both theoretical insights and practical contributions.

Participants, Instruments and Indicators

Six pre-service mathematics teachers from a mathematics education program at a public university in Indonesia were purposively selected. Selection criteria included comparable academic backgrounds and prior experience with GeoGebra. All participants engaged in the full sequence of instructional tasks and provided written informed consent. The study received ethical approval from the university's ethics committee. Table 1 summarizes the instruments, indicators, and descriptors used in the study, which were developed based on theoretical frameworks such as Van Hiele's levels of geometric thinking, reasoning and proof, and dynamic geometry-based exploration.

Instruments	Indicators
Task-Based Worksheets (Van Hiele (1999); Stylianides (2007); Laborde (2002)	Van Hiele Geometric Thinking Levels
	Reasoning Structure (Formulating conjectures and justifications through dynamic observation)
	Formal Deductive Argument (Constructing proofs using definitions and theorems)
Think-Aloud Protocols (Ellis et al., 2012; G. Stylianides et al., 2013)	Real-Time Reasoning (Revealing logical thought processes during DGS-based exploration)
	Type of Reasoning (Transitioning from intuitive ideas to formal justifications)
Interview Guidelines (Buchbinder & McCrone, 2022))	Reflective Reasoning (Connecting visual representations to formal geometric concepts)
	Argumentation Quality (Constructing and evaluating logical justifications)
Observation Field Notes (Martinovic & Manizade, 2020; Santos-Trigo et al., 2024)	Engagement in Dynamic Exploration (Actively manipulating DGS tools independently)
	Independent vs. Scaffolded Reasoning (Identifying moments requiring external prompts or scaffolding)

Table 1: Indicators and Descriptors of Research Instruments Based on Theoretical Frameworks

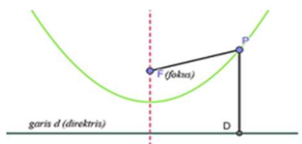
Instructional Tasks and Learning Intervention

The intervention was conducted over three weeks and consisted of four instructional tasks, each designed to facilitate specific cognitive transitions in geometric thinking. Table 2 presents the topic, cognitive focus, and targeted Van Hiele level for each task.

Task/Topic	Cognitive Focus	Targeted Level
Task 1: Definition of parabola (focus and directrix)	Foundational properties	Level 1 → Level 2
Task 2: Dynamic construction of a parabola from its focus and directrix	Invariance and symmetry	Level 2 → Level 3
Task 3: Exploring vertex form y=$a(x$-$h)^2$+k	Symbolic conjecturing	Level 3 → Level 4
Task 4: Formal proof construction	Deductive justification	Level4

Table 2: Task Descriptions and Targeted Van Hiele Level

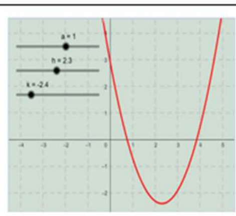
Each of these tasks was purposefully designed not only to introduce geometric content, but to provoke cognitive movement through the Van Hiele levels, from initial visualization to formal deduction. In particular, tasks 1 and 2 aim to activate spatial recognition and analytical awareness, while tasks 3 and 4 challenge students to formulate and validate conjectures using definitions and theorems, ultimately constructing structured formal proofs. This progression aligns with the research aim to trace how dynamic exploration mediates the shift from intuitive understanding to deductive argumentation.



Definition:
The locus definition of a parabola is the set of all points that are equidistant from a fixed point called the focus and a line called the directrix.

Task 1: (a) Point F is known as the focus of the parabola. What is the name of the line that passes through point F and is perpendicular to the directrix? (b) Line d is the directrix of the parabola. What is the relationship between line d and the dashed line passing through point F?
Task 2: (a) Drag point P along the shape of the parabola. How does the distance from point P to the focus compare to its distance to the directrix? What can you conclude from your observation? (b) Move the focus point (F) to different positions on the screen. Does the property you observed in Question 3 still apply? Explain your reasoning.

Figure 2: Task 1 and Task 2- Focus and Directrix



Task 3: Open the GeoGebra document titled "BENTUK PUNCAK PARABOLA", which displays the graph of a parabola in vertex form: $y = a(x - h)^2 + k$, featuring interactive sliders for the parameters a, h, and k that allow you to dynamically view the changes in the parabola's graph as the values are adjusted.
The vertex form of the parabola equation $y = a(x - h)^2 + k$ is displayed. Move the sliders to change the values of $a, h,$ and k. Explain how changes in each parameter $a, h,$ and k affect the shape and position of the graph.

Figure 3: Task 3- Exploring Vertex Form

Given the focus (m, n) the directrix $y = d$ and a point $P(x, y)$ on the parabola, use the distance formula to derive the general equation of the parabola.
Use the fact that the distance from point P to the focus is equal to its distance to the directrix. Set up an equation for the two distances and solve to find the value of y.
Using the vertex form of the parabola, $y = a(x - h)^2 + k$, and the algebraic relationship derived in Task 4, answer the following questions:

1. Explain the relationship between the focus, directrix, and the value of a.
2. What happens to the shape of the parabola as a increases? How does this relate to the distance between the focus and the directrix?"

Figure 4: Task 4 - From visual exploration to formal justification

The instructional tasks in this study were deliberately structured to support a progressive development of deductive reasoning, guiding pre-service teachers from initial visual recognition of geometric properties to the formulation of conjectures and, ultimately, to the construction of formal deductive arguments. Each task was designed to build upon the reasoning outcomes of the previous one, thereby creating a scaffolded sequence of cognitive engagement.

These tasks functioned as the primary mechanism for addressing the study's central research question: how do pre-service mathematics teachers transition from intuitive to formal reasoning in a dynamic geometry environment? The instructional design integrated real-time dynamic manipulation via GeoGebra with structured opportunities for reflection and verbal articulation, enabling participants to externalize and evolve their reasoning processes.

Over a three-week intervention period, participants completed four sequenced GeoGebra-based tasks that required them to manipulate geometric constructions, formulate and test conjectures, and construct formal justifications. Each task was followed by a semi-structured reflective interview aimed at probing the nature and depth of participants' evolving reasoning, allowing the research team to trace their progression toward formal proof construction.

Data Collection and Analytical Focus

Data were gathered through task-based worksheets, think-aloud protocols, semi-structured interviews, classroom observations, and students' written artifacts. Each instrument was purposefully chosen to capture complementary aspects of participants' cognitive progression from intuitive reasoning to formal deductive argumentation. Think-aloud sessions allowed real-time tracing of reasoning as participants manipulated dynamic figures in DGS, revealing their transition from visual intuition to logical justification. Observation notes complemented this by recording behavioral indicators such as autonomy, engagement, and use of scaffolding. Following each task, stimulated-recall interviews explored how participants interpreted visual experiences and connected them to geometric principles, assessing the coherence and metacognitive depth of their reasoning. Written artifacts including conjectures, constructions, and proofs, provided concrete evidence of conceptual and logical development while triangulating verbal and behavioral data for analytical rigor. Assessment was integrated throughout learning. Informal assessment occurred during exploration through observation and feedback, while formal assessment within tasks required students to formulate conjectures, justify relationships, and construct deductive arguments. This structure enabled systematic evaluation of reasoning and progression across Van Hiele levels.

Data Analysis and Trustworthiness

All data were transcribed verbatim and organized using a case-based structure, treating each participant as a distinct unit of analysis. The main analytical focus was to trace individual shifts in representation, verbal justification, and deductive structure aligned with the Van Hiele levels of geometric thinking. The analysis followed Matthew Miles et al., (2014), consisting of three recursive phases: (1) data condensation: selecting, coding, and categorizing meaningful episodes from worksheets, interviews, and verbal protocols using deductive codes grounded in reasoning and proof theory; (2) data display: developing matrices and narrative timelines to visualize participants' reasoning progression; and (3) conclusion drawing and verification: identifying cognitive shifts and how dynamic exploration mediated transitions from informal reasoning to formal proof. This multi-layered analysis revealed how pre-service teachers evolved from visual intuition to logical deduction in geometry, directly addressing the research questions. To ensure rigor, methodological triangulation was applied across four instruments (worksheets, think-aloud protocols, interviews, and observations) and three data types (verbal, written, behavioral), enhancing validity and providing rich, cross-validated insights into learners' reasoning within the dynamic environment.

To trace reasoning development, an analytic rubric based on the Van Hiele model was used, delineating four hierarchical levels characterized by distinct cognitive behaviors and performance indicators. The rubric served as a deductive coding framework to map participants' reasoning shifts particularly transitions from visual to formal deductive thinking throughout the intervention. It was adapted from established interpretations of the Van Hiele model (Clements & Battista, 1992; Fuys, 1988; Usiskin, 1982; Van Hiele, 1999) to align with the indicators of each GeoGebra-based task in this study.

Level	Description of Cognitive Behavior	Task Performance Indicators
Visualization (V)	The student recognizes geometric shapes based on their global appearance without considering formal properties or definitions	Identifies a parabola solely by its shape ("Looks like an arch"), without mentioning properties or geometric relationships
Analysis (A)	The student mentions parts or properties of geometric figures (e.g., symmetry, vertex) but does not relate them logically	Mentions focus, directrix, axis of symmetry without constructing arguments or deductive connections between elements
Informal Deduction (ID)	The student formulates conjectures based on visual observations, using geometric terms intuitively but without formal proof	States hypotheses like "The distance to the focus and the line seems equal," but does not justify using definitions or theorems
Formal Deduction (FD)	The student proves conjectures using formal definitions, theorems, or deductive logic in a valid and structured manner	Constructs a proof that the distance from a point to the focus and to the line is equal based on the definition of a parabola

Table 3: Analytical Rubric for Tracing Cognitive Shifts Across Van Hiele Levels in Dynamic Geometry Tasks

Expert Validation of Instruments and Task Design

Expert validation was conducted prior to implementation to ensure theoretical alignment, clarity, and relevance of the instruments and tasks. Three experts, two mathematics education lecturers specializing in geometry and reasoning and proof, and one instructional technology expert reviewed the task-based worksheets, Van Hiele-based indicators, interview and think-aloud protocols, and observation rubrics.

The validation focused on alignment with learning objectives, conceptual accuracy, appropriateness of difficulty and language, potential to elicit reasoning and deductive processes, and instructional relevance. All four structured parabola tasks met all criteria (Table 4).

Task	Learning Objectives	Concept Alignment	Difficulty & Precision	Reasoning & Deduction Potential	Instructional Relevance
Task 1	√	√	√	√	√
Task 2	√	√	√	√	√
Task 3	√	√	√	√	√
Task 4	√	√	√	√	√

Table 4: Expert Validation Checklist of Geometry Tasks

Quantitative validation using eight indicators on a 5-point Likert scale yielded an overall mean score of 4.02, classified as *Very Valid* (Qohar et al., 2021) (Table 5). Expert feedback confirmed that the tasks effectively promoted conjecturing, justification, and deductive reasoning, with GeoGebra supporting exploratory reasoning prior to formal proof. Minor revisions were made to refine descriptors and improve instrument clarity, ensuring strong content and construct validity.

No	Indicator	Average	Category
1	Clarity of task instructions	4.00	Very Valid
2	Alignment with learning objectives on reasoning and proof	4.22	Very Valid
3	Relevance to geometric thinking levels (Van Hiele framework)	4.17	Very Valid
4	Integration with Dynamic Geometry Software (GeoGebra)	4.00	Very Valid
5	Appropriateness of difficulty level for pre-service teachers	3.78	Valid
6	Stimulation of conjecture formulation and justification	4.11	Very Valid
7	Potential to elicit deductive reasoning processes	3.89	Valid
8	Language accuracy and mathematical precision	3.94	Valid
	Overall Average Score	4.02	Very Valid

Table 5: Validation of Task-Based Worksheets Instrument

RESULT

RQ1: Supporting Geometric Thinking through Dynamic Exploration

This study found that dynamic exploration using GeoGebra significantly supported the development of geometric thinking among pre-service mathematics teachers, particularly in constructing reasoning and proof related to the properties of the parabola. As participants engaged in four structured tasks, they transitioned from intuitive and visual observations to more analytical and deductive reasoning. Their cognitive development was evaluated using the Van Hiele model, focusing on shifts in levels of reasoning.

Figure 5 illustrates that most participants progressed from lower Van Hiele levels such as visualization and analysis toward informal or formal deduction. For instance, Participant P1 advanced from the analysis level to informal deduction after using GeoGebra's measurement tool to verify the focus–directrix property by comparing distances and recognizing consistent patterns. He explained, *"I knew the definition, but seeing the distances actually equal, then measuring and trying again helped me believe it. That's when I realized I could explain it properly."* This reflects a shift from observation to structured justification—typical of movement from Van Hiele Level 2 to Level 3. Participant P2 moved from visualization to analysis while exploring symmetry using the slider parameter *h* in the vertex form $y = a(x - h)^2 + k$. Initially focused on movement, he later generalized: *"When I move h, the center line also moves. It's always at x = h. That must be the axis."* In a later interview, he reflected, *"At first, I just followed the movement, but then I realized I could generalize it and explain why it happens."* This shows growing analytical understanding.

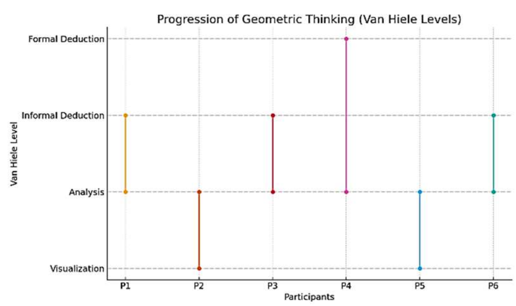


Figure 5: Van Hiele Progression of Participants

Figure 5. Each line represents a participant's progression from their initial to final Van Hiele level after the intervention.

Participant P4 exhibited the most significant improvement, reaching formal deduction by Task 4 through coordinate-based proof construction. He stated, *"Now I understand why the parabola has that shape."* Using the Euclidean distance formula, he derived the general form of the parabola and verified it algebraically: *"I set the point P(x, y), the focus F(m, n), and the directrix y = d... After simplifying, the equation matched what I had in GeoGebra."* This demonstrates full mastery at Van Hiele Level 4. Participant P6 showed meaningful growth from analysis to informal deduction. During the think-aloud protocol, he repeatedly measured distances, stating, *"Every time the point moves, the two distances stay the same. That's why it's on the parabola it has to be."* This indicates reasoning supported by pattern-based justification.

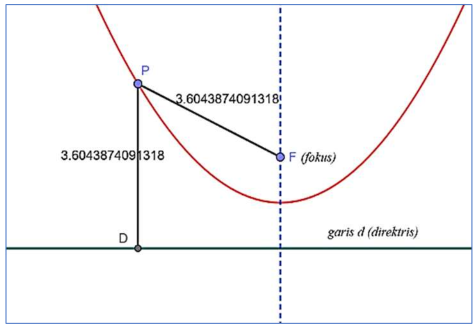


Figure 6: Participant P4's GeoGebra construction exploring the parabola's equidistance property

Finally, Participant P5, although remaining at the analysis level, showed conceptual improvement. Initially struggling with parameter interpretation, by Task 3 he explained, *"When a gets bigger, the curve is tighter because the focus and directrix get closer."* This reflects increased symbolic awareness despite not reaching formal proof.

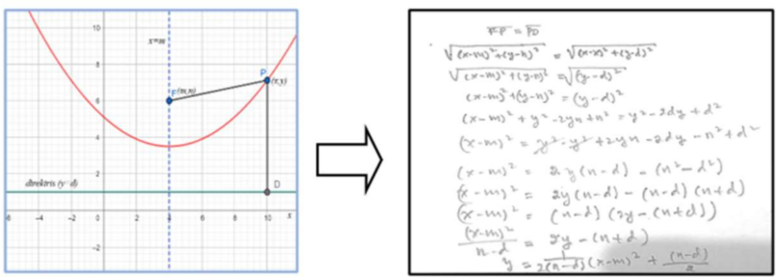


Figure 7: P4 used GeoGebra measurements to demonstrate that $\overline{FP}=\overline{PD}$

Overall, five of six participants advanced in their Van Hiele levels, from visualization or analysis to informal deduction, and one (P4) reaching formal deduction showing that GeoGebra-based dynamic exploration effectively supported their cognitive progression in geometric reasoning and proof.

RQ2: Technology-Based Task Features that Facilitate Van Hiele Transitions

The analysis revealed that certain GeoGebra features interactive visual manipulation, measurement tools, sliders, and dual algebraic graphical displays were instrumental in supporting participants' transitions across Van Hiele levels. These tools helped learners connect geometric objects, dynamic patterns, and symbolic representations, fostering movement from intuition to structured reasoning.

Task 1 introduced the parabola's geometric form through point plotting and tracing, which promoted initial visual awareness. Participant P5 reflected, *"At first, I just followed the curve without thinking, but when the path started showing a pattern, I wondered why it behaved that way."*

Task 2 emphasized symmetry exploration through the vertex form $y = a(x - h)^2 + k$. Using sliders, Participant P2 observed, *"When I move the slider, the center line also moves. It's always at x = h.*

That must be the axis." Later he added, *"At first, I was just watching the movement, but then I realized I could explain it with the equation."* This progression indicates a shift from visualization to analysis.

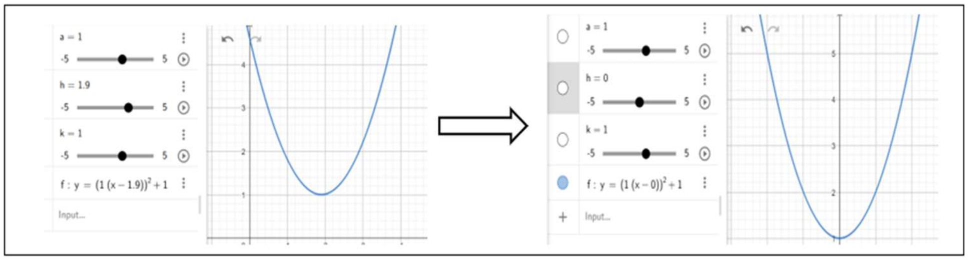


Figure 8: Dynamic Exploration of Symmetry in Vertex Form of a Parabola

Figure 8 showcases the dynamic manipulation of the vertex form of a parabola using sliders in GeoGebra. As seen, participants adjusted the value of h and observed consistent alignment of the axis of symmetry at $x = h$, fostering a connection between symbolic representation and geometric structure. Such features particularly real-time algebraic-visual interaction supported participants like P2 and P5 in recognizing mathematical invariants, even before full deductive proof was constructed.

Tasks 3 and 4 focused on the parabola's equidistance property using measurement and trace tools. Participant P1 stated, *"After measuring again and again, the distances are always equal. That's when I believed it wasn't just coincidence,"* showing the development of informal deduction. Similarly, Participant P6 shared, *"When the trace appeared, I knew that every point followed the same rule,"* reflecting a transition from intuition to rule-based reasoning. Participants responded differently to features. Trace and measurement tools encouraged pattern generalization, while sliders required reflection to deepen understanding. Participant P3, for example, initially changed parameters without clear meaning but later realized, *"Now I see that a doesn't just stretch the curve-it changes the focus and the curve at once."*

Table 7 summarizes how different GeoGebra features, linked to specific tasks, supported reasoning development across participants. Five out of six participants progressed through at least one Van Hiele level, with varied pathways depending on their interaction with technological features and reflective opportunities.

Parti-cipant	Ini-tialLe vel	Final Level	Related Task	Observed Reasoning Development
P1	A	ID	Task 3	Moved from observation to structured justification
P2	V	A	Task 2	Began linking visual changes to symbolic structure
P3	A	ID	Task 2–3	Developed language of informal justification
P4	A	FD	Task 4	Produced formal argument using definition/theorem
P5	V	A	Task 3	Improved understanding of function-graph relationships
P6	A	ID	Task 3	Transitioned from intuitive to rule-based explanation

Table 7: Van Hiele Progressions, GeoGebra Features, and Reasoning Outcomes

These findings highlight that effective technology-based tasks must align with learners' cognitive stages. Dynamic features should be sequenced intentionally to scaffold reasoning transitions—from visualization, to identifying relationships, to constructing logical arguments. Furthermore, guided reflection and teacher scaffolding are essential for helping learners transform empirical observations into formal mathematical justifications.

RQ3: Transitioning from Visual Exploration to Formal Justification through DGS

The analysis showed a clear shift from intuitive visual exploration to structured deductive reasoning facilitated by Dynamic Geometry Software (DGS). GeoGebra enabled participants to dynamically interact with geometric objects, bridging empirical observation and formal proof.

Initially, participants relied on visual cues such as symmetry and trace patterns. Participant P1 moved from observation to informal deduction, explaining: *"At first, I just knew the definition, but after I saw the distances were always the same, then I measured and tried again, I started to believe it. That's when I felt I could explain why it works."* Participant P4 showed the most significant progress, achieving formal deduction in Task 4. He algebraically proved the equidistance property of the parabola using coordinate-based reasoning: *"Now I understand why the parabola has that shape."* He derived the vertex form from the distance formula, connecting visual intuition to symbolic proof.

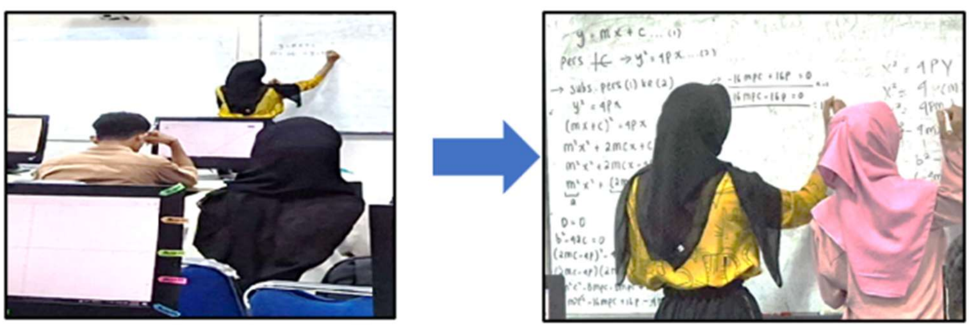


Figure 9: Transition from Dynamic Visualization to Formal Deductive Proof (*P4 and a peer moving from computer-based exploration to collaborative proof construction*)

A follow-up interview with P4 further demonstrated his conceptual integration of visual and algebraic understanding:

"After trying several times in GeoGebra, I saw that when 'a' increases, the parabola becomes narrower. Measuring the vertex-to-focus and vertex-to-directrix distances showed they get shorter because a = 1/4p". "Using the distance formula, I matched the vertex form to the geometric definition. It made the proof feel logical, not just memorized."

Participant P6 also transitioned from intuition to generalization: *"Every time the point moves, the two distances stay the same. That's why it's on the parabola."* Participant P3 stated, *"I didn't just try; I wanted to prove it works."*

Participant P2 reflected, *"The diagrams helped me connect the movement to the formula"*. Meanwhile, Participant P5, though remaining at the analysis level, reported deeper symbolic understanding: *"The real-time changes helped me think deeper."*

Collectively, these findings confirm that DGS not only promotes pattern recognition but also transforms empirical exploration into formal justification. Participants evolved from passive observers to active justifiers of geometric truths, demonstrating how dynamic visual interaction can foster symbolic and logical reasoning.

RQ4: Designing an Instructional Framework to Bridge Visual, Symbolic, and Deductive Representations in the Concept of Parabola

Building on the findings from RQ1–RQ3, an instructional framework was designed to integrate dynamic GeoGebra tasks with reasoning and proof activities. The framework aimed to help preservice mathematics teachers transition smoothly between visual, symbolic, and deductive representations of the parabola while addressing common learning challenges such as limited GeoGebra proficiency, difficulty distinguishing invariants, and reliance on memorized theory (Anwar et al., 2025).

Phase 1: Dynamic Exploration and Symbolic Representation

This phase focused on developing geometric understanding by dynamically manipulating the focus, directrix, and vertex using GeoGebra. The activities aimed to (1) build a visual understanding of the parabola as a locus of points equidistant from the focus and directrix, and (2) connect this concept to its symbolic representation, particularly the vertex form. Participants used sliders to adjust parameters and observed how graph changes reflected algebraic relationships. Participant P3 noted, *"When I moved the focus closer to the vertex, the graph became steeper, and the a value increased. That's when I realized that the distance actually affects the equation."* This phase strengthened the coordination between visual and symbolic reasoning.

Phase 2: From Symbolic Patterns to Deductive Reasoning

In the second phase, participants engaged in structured proof tasks-deriving the parabola's general equation from its geometric definition using the distance formula and constructing algebraic justifications. Participant P4 shared, *"I compared the lengths FP and PD, then simplified it to get the vertex form. That's when I really understood-it wasn't just told to me, I discovered it myself."*

As shown in Figure 9, learners transitioned from digital exploration to collaborative proof construction, linking symbolic reasoning with formal deduction.

Transition	Learning Activity	Van Hiele Level Transition	Instructional Support	Evidence of Transition
Visual → Symbolic	Manipulating focus, directrix, and vertex using GeoGebra sliders	Level 1 → Level 2	Real-time algebraic feedback, guided questions	Students linked visual changes to the vertex form $y = a(x–h)^2 + k$, e.g., "As I moved the focus closer to the vertex, the curve became sharper."
Symbolic → Deductive	Constructing general parabola equation using distance formula	Level 3 → Level 4	Structured proof task, visual-symbolic scaffolding	Students justified the vertex form algebraically: "I proved it myself, step by step using the distance formula."
Full Integration	Project-based modeling (e.g., water fountain, projectile motion)	Across Levels 2-4	Open-ended tasks, collaborative discussion	Students integrated visual models, symbolic equations, and logical arguments coherently to explain real phenomena.

Table 8: Representational Transitions Supported by the Instructional Design in Parabola

Collectively, the instructional design promoted representational flexibility and addressed key challenges in learning reasoning and proof. The integration of GeoGebra, structured guidance, and open-ended projects empowered pre-service teachers to construct and validate their understanding of parabolic concepts through connected visual, symbolic, and deductive reasoning.

Instructional Design Model for Reasoning and Proof Learning

Findings from classroom observations, task analyses, and student interviews revealed a consistent pattern of cognitive growth supported by a systematically structured instructional approach. As pre-service mathematics teachers engaged with GeoGebra-based tasks, their reasoning evolved progressively across visual, symbolic, and deductive representations in a clear and traceable sequence.

From this evidence, a two-phase data-driven model Reasoning and Proof Learning (RPL) model was developed to represent how participants navigated the reasoning cycle during instruction. *Phase 1: Exploration and Representation*, this phase involved dynamic interaction with visual aspects of the parabola using GeoGebra and the gradual articulation of symbolic relationships. Manipulation of parameters, algebraic interpretation, and reflective dialogue were key indicators of reasoning growth. *Phase 2: Justification and Proof*, in this phase, participants constructed algebraic arguments to validate independently formed conjectures. Data from worksheets, whiteboard collaborations, and group discussions showed clear transitions from symbolic reasoning to formal deduction, grounded in definitions and structured proof strategies.

These two phases were identified inductively from empirical classroom data rather than imposed by prior theory. Hence, the RPL model is not a theoretical abstraction but a grounded, evidence-based framework reflecting how learners authentically experienced reasoning and proof within a technology-enhanced environment.

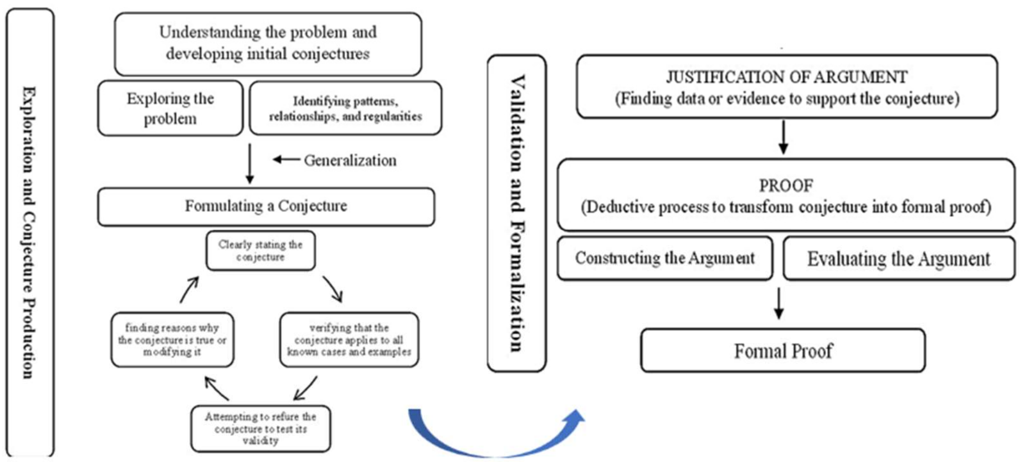


Figure 10: Conceptual Framework of the Reasoning and Proof Process

This visualization highlights how the instructional design effectively supported both the formulation of conjectures and the validation of mathematical arguments, capturing authentic learning trajectories observed throughout the study.

Implementation of the Instructional Design: A Lesson Plan for Secondary School Teachers

Drawing from the instructional design research, a practical lesson plan was developed based on the Reasoning and Proof Learning (RPL) model to help secondary school mathematics teachers bridge visual, symbolic, and deductive representations in teaching the parabola. The plan maintains the pedagogical integrity of the RPL model while ensuring classroom applicability.

The lesson plan was structured across two 120-minute sessions, each corresponding to one of the two main phases of the model: (1) *Exploration and Representation*, and (2) *Justification and Proof*. The first phase focused on dynamic visual exploration using GeoGebra, where students manipulated the focus, directrix, and vertex of a parabola to construct its shape and observe real-time algebraic feedback. The second phase emphasized the construction of formal justifications and proofs derived from the geometric definition of a parabola, using the distance formula as the basis for algebraic reasoning. Collaborative group work, discussion, and whiteboard presentations were incorporated to support deductive thinking.

Developed through synthesis of classroom observations, task analyses, and student interviews, this plan is both flexible for teaching and grounded in cognitive transitions identified during the study.

Phase	Learning Objectives	Core Activities	Tools and Media	Expected Representational Transition
1	Construct a parabola as a locus of points- Connect visual and symbolic forms (vertex form)	GeoGebra exploration with parameter sliders- Guided questions and group discussion- Individual reflection	GeoGebra, worksheets, projector	Visual to Symbolic
2	Derive the vertex form from the geometric definition- Construct a formal proof using algebraic reasoning	Derivation using the distance formula- Collaborative proof construction- Whole-class discussion	Proof worksheets, whiteboard	Symbolic to Deductive

Table 9: Lesson Plan Based on the Two-Phase RPL Model

This lesson plan serves as a practical application of the RPL model and forms a foundation for the ongoing dissertation on implementing reasoning and proof-oriented instruction in dynamic, technology-enhanced learning environments. By aligning visual exploration, symbolic manipulation, and deductive reasoning, it offers a structured, research-based pathway for improving geometry teaching and fostering deeper mathematical understanding.

DISCUSSION

This study set out to examine how dynamic exploration through GeoGebra could support preservice mathematics teachers in developing geometric reasoning, particularly concerning the properties and proofs of the parabola. The findings provide compelling evidence that a well-designed instructional sequence can facilitate meaningful cognitive transitions across visual, symbolic, and deductive representations. These outcomes confirm the relevance of Van Hiele's geometric thinking model Van Hiele, (1999) and align with prior studies on technology-supported reasoning and proof development (Stylianides & Stylianides, 2008; Baccaglini-Frank & Mariotti, 2010).

From Dynamic Exploration to Proof Construction: A Teaching Pathway

This study demonstrates that dynamic environments, when supported by structured tasks, can move learners from visual exploration toward mathematical justification. Through parabola-based explorations, participants formed conjectures grounded in measurement and progressively articulated justifications using distance formulas and symbolic reasoning. Participant P4 exhibited Van Hiele Level 4 reasoning through an algebraic derivation of the parabola's definition, while others (e.g., P1, P2, P6) produced informal or semi-formal justifications during the transition from empirical observation.

These findings support prior research showing that reasoning embedded in dynamic environments promotes both conceptual understanding and logical structure (Hanna & de Villiers, 2012; Ellis et al., 2012) and highlight the epistemological role of proof in developing logical reasoning (Hanna & de Villiers, 2012; Buchbinder et al., 2020; Kokeb et al., 2025). However, consistent with Anwar et al. (2025), dynamic tools alone were insufficient; without structured reflection and teacher prompts, students tended to rely on visual intuition, faced difficulties identifying invariants, and drew on memorized theory. Participants with stronger manipulation and observation skills (e.g., P4) engaged more deeply with invariants and were more likely to construct deductive arguments.

Overall, the results provide empirical support for positioning Dynamic Geometry Software as an epistemic mediator rather than a visualization tool (Laborde, 2002; Sinclair, 2003). The observed shift from memorization and passive observation to experimentation, justification, and proof aligns with prior studies emphasizing DGS in supporting the transition from intuitive reasoning to formal thinking (Gulkilik, 2023; Komatsu & Jones, 2020). The use of traces, sliders, and interactive constructions further enabled immediate feedback in exploring symmetry and equidistance properties of the parabola, strengthening both conceptual understanding and learner confidence.

Strengthening Learning Through Representational Transition

The structured sequence of tasks in this study proved effective in guiding participants through Van Hiele levels, from visualization, to analysis, to informal and formal deduction. This aligns with (Gravemeijer, 1999) *guided reinvention* model and confirms the role of task design in reasoning development. GeoGebra, when used purposefully, enabled participants to detect invariants such as symmetry and equidistance, supporting the transition from perceptual reasoning to symbolic generalization. Nonetheless, as Anwar et al., (2025) noted, the inability to differentiate between construction-based premises and emergent conclusions (e.g., confusing given constraints with discovered properties) poses a cognitive risk in DGS environments. Teachers must therefore emphasize the logical structure of geometric reasoning, including how dynamic constructions provide premises from which conclusions must be derived, not assumed. This insight has powerful pedagogical implications. Secondary teachers can adopt the Reasoning and Proof Learning (RPL) model to structure their lessons into two phases: (1) *Exploration and Representation*, where students interact with parameters and detect patterns visually and symbolically, and (2) *Justification and Proof*, where they transition toward deductive argumentation. Without this progression, the learning risks becoming observationally rich but logically shallow.

Teaching Implications: Managing Technology-Related Challenges

The findings of Anwar et al., (2025) further caution that students' lack of familiarity with GeoGebra tools can lead to construction errors that undermine conceptual understanding. As seen in both studies, poor tool usage results in inaccurate figures, misinterpretation of measurements, and flawed reasoning. This highlights the importance of deliberate training in DGS before embedding it within reasoning and proof tasks. Strategies such as collaborative task design, tutorial walkthroughs, and feedback loops are essential for optimizing learning. Moreover, Anwar et al., (2025) emphasized that conjectures formed without observation often reflect passive, recall-based learning an issue also noted in our study when participants initially hesitated to test and generalize visual phenomena. As Borba et al., (2016); Mariotti, (2014) suggest, mathematics teachers must explicitly cultivate habits of active observation, pattern recognition, and generalization, especially when using dynamic technology.

Bridging Intuition and Formalism Through the RPL Model

Our proposed RPL model emerging inductively from student learning trajectories offers a practical framework for teachers to bridge empirical observation and formal reasoning. By integrating GeoGebra into classroom instruction, educators can create learning experiences where students construct understanding through dynamic manipulation and consolidate it through proof. However, consistent with Anwar's findings, our data show that students still struggle with formalizing their ideas into structured proofs. Many arguments remained intuitive or lacked logical rigor. This underscores the need to incorporate proof scaffolds such as two-column formats or Toulmin diagrams in classroom use of DGS, to support the full transition to deductive reasoning (Stylianides, 2007; Weber, 2001).

Implications for Teaching Practice

The findings of this study provide several key implications for integrating reasoning and proof into geometry instruction.

First, the results highlight the importance of a sequenced instructional design that begins with dynamic exploration and gradually guides students toward formal justification. Teachers are encouraged to adopt the two-phase Reasoning and Proof Learning (RPL) model-*Exploration and Representation* followed by *Justification and Proof*-which helps students move from intuitive visualization to deductive reasoning by experimenting, observing invariants, verbalizing patterns, and justifying their claims logically or algebraically. Second, while tools like GeoGebra and other Dynamic Geometry Software (DGS) can powerfully support cognition, their effectiveness depends on structured scaffolding. Teachers should design investigative and reflective tasks, not just demonstrations, to promote active exploration, generalization, and justification rather than passive observation. Third, students often face challenges such as limited DGS skills, difficulty identifying invariants, and overreliance on memorized theorems. Teachers should provide progressive DGS training, guided modeling, and collaborative opportunities, emphasizing the difference between premises (conditions) and conclusions (results) in constructing valid conjectures and proofs. Fourth, many students can make insightful observations but struggle to formalize them. Teachers can address this by incorporating explicit proof structures such as two-column proofs, guided worksheets, and flow diagrams to help students organize logical reasoning. Finally, the RPL model offers a practical, classroom-ready approach for teaching geometric topics such as parabolas, loci, or triangle centers-through dynamic, reasoning-oriented instruction. By embedding reasoning within everyday classroom practice and leveraging technology, teachers can foster deeper engagement and help students view proof not merely as an endpoint but as an integral process of mathematical thinking.

Limitations and Future Research

Although promising, this study was conducted with a small sample and focused on a single geometric topic. Moreover, the proof strategies used were emergent rather than pre-taught. Future research should explore how combining RPL with explicit proof instruction enhances student outcomes. Longitudinal studies could examine the lasting effects of such instructional models across different topics, such as triangle centers, loci, or transformations. Additionally, the complexity of navigating DGS especially for those with limited prior experience should inform professional development for both pre-service and in-service teachers. Future studies may consider how RPL can be scaled for classroom use while addressing barriers in GeoGebra fluency and proof literacy.

CONCLUSION

This study underscores the importance of targeted instructional design to help pre-service mathematics teachers develop reasoning and proof skills, particularly in geometric topics like the parabola, which are central to secondary education. Achieving Van Hiele's formal deduction level is vital not only for deepening teachers' own understanding but also for nurturing structured mathematical thinking in their future students.

The findings confirm that Dynamic Geometry Software (DGS) can effectively support the transition from intuitive visualization to formal proof when embedded within a well-sequenced instructional framework. Through dynamic exploration, participants identified patterns, formulated conjectures, and validated them using algebraic and logical reasoning. In this context, DGS served as more than a visualization tool it acted as a conceptual bridge that promoted deductive reasoning through interactive feedback and representational fluency.

The resulting instructional design provides teachers with practical, classroom-ready activities that guide learners through representational transitions from visual to symbolic to deductive reasoning enhancing students' geometric understanding of familiar concepts such as the parabola. The pedagogical implications are clear: by adopting the Reasoning and Proof Learning (RPL) model, teachers can foster meaningful cognitive transitions and cultivate students' engagement with proof as a central element of mathematical learning, rather than mere observation.

REFERENCES

[1] Alsina, Á., Pincheira, N., & Delgado-Rebolledo, R. (2024). The professional practice of designing tasks: how do pre-service early childhood teachers promote mathematical

processes in early algebra? ZDM - Mathematics Education. https://doi.org/10.1007/s11858-024-01636-1

[2] Anwar, L., Sa'dijah, C., Listiawan, T., Dewi Utami, A., & Zulnaidi, H. (2025). Challenges of Prospective Mathematics Teachers in Formulating Geometrical Conjecture through Interaction with GeoGebra. In *MATHEMATICS TEACHING RESEARCH JOURNAL Early Spring* (Vol. 17, Issue 7).

[3] Armah, R. (2024). Geometric Thinking of Prospective Mathematics Teachers: Assessing the Foundation Built by University Undergraduate Education in Ghana. Teacher Education and Curriculum Studies, 9(2), 40–51. https://doi.org/10.11648/j.tecs.20240902.12

[4] Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. Zentralblatt Für Didaktik Der Mathematik, 34(3), 66–72. https://doi.org/10.1007/BF02655708

[5] Baccaglini-Frank, A., & Mariotti, M. A. (2010). Generating Conjectures in Dynamic Geometry: The Maintaining Dragging Model. *International Journal of Computers for Mathematical Learning*, *15*(3), 225–253. https://doi.org/10.1007/s10758-010-9169-3

[6] Borba, M. C., Askar, P., Engelbrecht, J., Gadanidis, G., Llinares, S., & Aguilar, M. S. (2016). Blended learning, e-learning and mobile learning in mathematics education. *ZDM*, *48*(5), 589–610. https://doi.org/10.1007/s11858-016-0798-4

[7] Birgin, O., & Özkan, K. (2024). Comparing the concept images and hierarchical classification skills of students at different educational levels regarding parallelograms: a cross-sectional study. International Journal of Mathematical Education in Science and Technology, 55(4), 850–882. https://doi.org/10.1080/0020739X.2022.2052196

[8] Buchbinder, O., & Mccrone, S. (2020). Preservice teachers learning to teach proof through classroom implementation: Successes and challenges. The Journal of Mathematical Behavior, 58, 100779. https://doi.org/10.1016/j.jmathb.2020.100779

[9] Buchbinder, O., & McCrone, S. (2022). Guiding principles for teaching mathematics via reasoning and proving. https://hal.archives-ouvertes.fr/hal-03746878v2

[10] Cihan, F., & Akkoç, H. (2023). An Intervention Study for Improving Pre-service Mathematics Teachers' Proof Schemes. MATHEMATICS TEACHING RESEARCH JOURNAL (Vol. 15, Issue 2).

[11] Clements, D. H., & Battista, M. (1992). Geometry and spatial reasoning. https://www.researchgate.net/publication/258932007

[12] Ellis, A. B., Bieda, K., & Knuth, E. (2012). Developing essential understanding of proof and proving for teaching mathematics in grades 9-12. NCTM.

[13] Fuys, D. J. (1988). The van Hiele Model of Thinking in Geometry among Adolescents. Journal for Research in Mathematics Education Monograph Number 3. https://api.semanticscholar.org/CorpusID:118373716

[14] Galili, R., Hartono, Y., & Hiltiramartin, C. (2023). Desain Pembelajaran Identitas Trigonometri Dengan Proof Based Learning di SMA. Jurnal Cendekia : Jurnal Pendidikan Matematika, 7(3), 2287–2298. https://doi.org/10.31004/cendekia.v7i3.2677

[15] González, A., Manero, V., Arnal-Bailera, A., & Puertas, M. L. (2024a). Proof levels of graph theory students under the lens of the Van Hiele model. International Journal of

Mathematical Education in Science and Technology, 55(8), 1938–1956. https://doi.org/10.1080/0020739X.2022.2113467

[16] Gravemeijer, K. (1999). How Emergent Models May Foster the Constitution of Formal Mathematics. *Mathematical Thinking and Learning*, *1*(2), 155–177. https://doi.org/10.1207/s15327833mtl0102_4

[17] Gulkilik, H. (2023). Analyzing preservice secondary mathematics teachers' prompts in dynamic geometry environment tasks. Interactive Learning Environments, 31(1), 22–37. https://doi.org/10.1080/10494820.2020.1758729

[18] Haj Yahya, A., & Hershkowitz, R. (2024). Interference of prototypical geometry representations in students' construction of concepts and development of proofs. Mathematical Thinking and Learning, 1–22. https://doi.org/10.1080/10986065.2024.2386619

[19] Haj-Yahya, A., & Olsher, S. (2025). Exploring proof building blocks as scaffolds for proving high school geometry theorems. International Journal of Mathematical Education in Science and Technology, 1–30. https://doi.org/10.1080/0020739X.2025.2475933

[20] Hanna, G., & de Villiers, M. (2021). Correction to: Proof and Proving in Mathematics Education (pp. C1–C1). https://doi.org/10.1007/978-94-007-2129-6_20

[21] Hartono, S., Yuli, T., Siswono, E., & Ekawati, R. (2024). From Informal to Formal Proof in Geometry: a Preliminary Study of Scaffolding-based Interventions for Improving Pre-service Teachers' Level of Proof. In MATHEMATICS TEACHING RESEARCH JOURNAL (Vol. 48, Issue 2).

[22] Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. Educational Studies in Mathematics, 96(1), 1–16. https://doi.org/10.1007/s10649-017-9761-8

[23] Kokeb, A., Atnafu, M., & Mohammed, A. (2025). The effects of visually assisted guided discovery learning strategy on tenth grade students' problem solving skill of plane geometry. In MATHEMATICS TEACHING RESEARCH JOURNAL. (Vol. 17, Issue 2).

[24] Komatsu, K., & Jones, K. (2020). Interplay between Paper-and-Pencil Activity and Dynamic-Geometry-Environment Use during Generalisation and Proving. Digital Experiences in Mathematics Education, 6(2), 123–143. https://doi.org/10.1007/s40751-020-00067-3

[25] Laborde, C. (2002). Integration of Technology in the Design of Geometry Tasks with Cabri-Geometry. International Journal of Computers for Mathematical Learning, 6(3), 283–317. https://doi.org/10.1023/A:1013309728825

[26] Lesseig, K., Hine, G., Na, G. S., & Boardman, K. (2019). Perceptions on proof and the teaching of proof: a comparison across preservice secondary teachers in Australia, USA and Korea. Mathematics Education Research Journal, 31(4), 393–418. https://doi.org/10.1007/s13394-019-00260-7

[27] Leung, F. K. S., Graf, K.-D., & Lopez-Real, F. (2006). Mathematics Education in Different Cultural Traditions-A Comparative Study of East Asia and the West. https://api.semanticscholar.org/CorpusID:53505585

[28] Manero, V., & Arnal-Bailera. (2021). Understanding Proof Practices of pre-Service Mathematics Teachers in Geometry. In MATHEMATICS TEACHING RESEARCH JOURNAL (Vol. 99, Issue 3).

[29] Mariotti, M. A. (2014). Transforming Images in a DGS: The Semiotic Potential of the Dragging Tool for Introducing the Notion of Conditional Statement. In *Transformation - A Fundamental Idea of Mathematics Education* (pp. 155–172). Springer New York. https://doi.org/10.1007/978-1-4614-3489-4_8

[30] Mariotti, M. A., & Pedemonte, B. (2019). Intuition and proof in the solution of conjecturing problems'. ZDM, 51(5), 759–777. https://doi.org/10.1007/s11858-019-01059-3

[31] Martinovic, D., & Manizade, A. G. (2020). Teachers Using GeoGebra to Visualize and Verify Conjectures About Trapezoids. Canadian Journal of Science, Mathematics and Technology Education, 20(3), 485–503. https://doi.org/10.1007/s42330-020-00103-9

[32] Matthew Miles, Michael Hberman, & JohnnySdana. (2014). Qualitative Data Analysis A Methods (3rd ed.).

[33] Mawarsari, V. D., Waluya, St. B., & Dewi, N. R. (2023). Profile of Students' Geometric Thinking Ability in Terms of Van Hiele Level (pp. 109–117). https://doi.org/10.2991/978-2-38476-078-7_13

[34] Naufal, M. A., Abdullah, A. H., Osman, S., Abu, M. S., Ihsan, H., & Rondiyah. (2021). Reviewing the Van Hiele model and the application of metacognition on geometric thinking. International Journal of Evaluation and Research in Education, 10(2), 597–605. https://doi.org/10.11591/ijere.v10i2.21185

[35] Ødegaard, R. P., Arnesen, K. K., & Langfeldt, M. B. (2024). Tools to support learning to plan discussions that promote students' mathematical reasoning. Journal of Mathematics Teacher Education. https://doi.org/10.1007/s10857-024-09668-w

[36] Qohar, Abd., Susiswo, S., Nasution, S. H., & Wahyuningsih, S. (2021). Development of Android-Based Mathematics Learning Game on the Topic of Congruence and Similarity. International Journal of Interactive Mobile Technologies (IJIM), 15(09), 52. https://doi.org/10.3991/ijim.v15i09.20723

[37] Quaresma, P., & Janičić, P. (2006). Integrating Dynamic Geometry Software, Deduction Systems, and Theorem Repositories (pp. 280–294). https://doi.org/10.1007/11812289_22

[38] Rizos, I., & Gkrekas, N. (2023). Is there room for conjectures in mathematics? The role of dynamic geometry environments. European Journal of Science and Mathematics Education, 11(4), 589–598. https://doi.org/10.30935/scimath/13204

[39] Santos-Trigo, M., Camacho-Machín, M., & Barrera-Mora, F. (2024). Focusing on foundational Calculus ideas to understand the derivative concept via problem-solving tasks that involve the use of a Dynamic Geometry System. ZDM - Mathematics Education, 56(6), 1287–1301. https://doi.org/10.1007/s11858-024-01607-6

[40] Scristia, Herman, T., & Septy Sari Yukans. (2025). Redesigning geometry assessments to promote advanced geometric thinking: A case study on formal deduction and rigor. Jurnal Elemen, 11(1), 186–205. https://doi.org/10.29408/jel.v11i1.27723

[41] Sinclair, M. P. (2003). Some implications of the results of a case study for the design of preconstructed, dynamic geometry sketches and accompanying materials. Educational Studies in Mathematics, 52(3), 289–317. https://doi.org/10.1023/A:1024305603330

[42] Škrbec, M., & Čadež, T. H. (2015). Identifying and fostering higher levels of geometric thinking. Eurasia Journal of Mathematics, Science and Technology Education, 11(3), 601–617. https://doi.org/10.12973/eurasia.2015.1339a

[43] Stylianides, A. J. (2007). Proof and proving in school mathematics. Journal for Research in Mathematics Education, 38(3), 289–321. https://www.scopus.com/inward/record.uri?eid=2-s2.0-34248588003&partnerID=40&md5=653faded3b3ed2b0012da31415e13b6d

[44] Stylianides G. (2008). Reasoning and proof, framework Stylianedes. FLM Publishing Association, Edmonton, Alberta, Canada.

[45] Stylianides, G., Stylianides, A., & Shilling-Stouffer, L. (2013). PROSPECTIVE TEACHERS' CHALLENGES IN TEACHING REASONING-AND-PROVING. International Journal of Science and Mathematics Education, 11. https://doi.org/10.1007/s10763-013-9409-9

[46] Stylianides, G., Stylianides, A., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward (pp. 237–266).

[47] Thapa, R., Dahal, N., & Pant, B. P. (2022). GeoGebra Integration in High School Mathematics: An Experiential Exploration on Concepts of Circle. In MATHEMATICS TEACHING RESEARCH JOURNAL (Vol. 16, Issue 5).

[48] The National Council of Teachers of Mathematics. (2000). Principles and Standards for School Mathematics. The National Council of Teachers of Mathematics, Inc.

[49] Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to Learn Reasoning and Proof in High School Mathematics Textbooks. Journal for Research in Mathematics Education, 43(3), 253–295. https://doi.org/10.5951/jresematheduc.43.3.0253

[50] Usiskin, Z. P. (1982). Van Hiele Levels and Achievement in Secondary School Geometry. CDASSG Project. https://api.semanticscholar.org/CorpusID:115884211

[51] Utari, T., & Hartono, H. (2019). Muatan penalaran dan pembuktian matematis pada buku teks matematika SMA kelas X Kurikulum 2013. Jurnal Riset Pendidikan Matematika, 6(1), 1–13. https://doi.org/10.21831/jrpm.v6i1.17002

[52] Van Hiele, P. M. (1999). Developing Geometric Thinking through Activities that Begin With Play. Teaching Children Mathematics. . In (pp 310-316). Reston VA: National Council of Teachers of Mathematics. (pp. 310–316). Reston VA: National Council of Teachers of Mathematics.

[53] Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. Educational Studies in Mathematics, 48(1). https://doi.org/10.1023/A:1015535614355

[54] Yohannes, A., & Chen, H.-L. (2021). GeoGebra in mathematics education: a systematic review of journal articles published from 2010 to 2020. Interactive Learning Environments, 1–16. https://doi.org/10.1080/10494820.2021.2016861

APPENDIX

Dynamic Tasks on the Parabola

Parabola

Name: ______________________

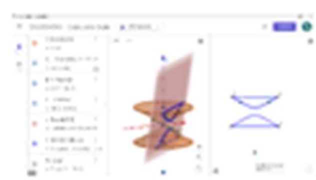
Group: ______________________

Course Learning Outcomes (CLO)

- Students will identify how each conic section is generated through the slicing of a cone.
- Students will understand the definition of the locus of a parabola.
- Students will describe how the values of a, h, and k in the vertex form of a parabola equation affect its graph.
- Students will use the locus definition of a parabola to derive its equation and explain the relationship between the focus, directrix, and the parameters in the vertex form.
- Students will strategically use appropriate tools and make use of mathematical structure

Open the document: Introduction to Conics – GeoGebra

Is there a connection between the definition of a locus and the vertex form of a parabola? In this activity, you will explore conic sections and the concept of the parabola.



Use the slider at point D to view the different types of conic sections one by one.
Based on your observations:

1. How does the position of the cutting plane relative to the cone determine the shape such as a circle, ellipse, parabola, or hyperbola? Briefly explain by referring to the position of the plane in relation to the cone's axis or side. Complete the table below.

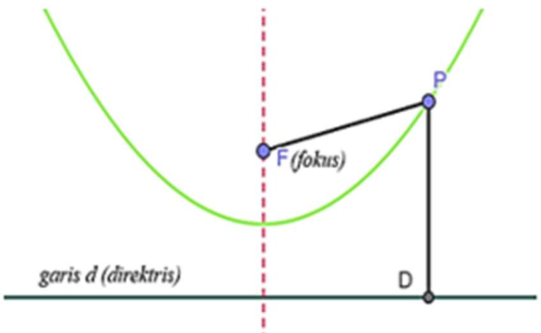
Conic Section	Description
Circle	
Ellipse	
Parabola	
Hyperbola	

Parabola

Name: ______________________

Group: ______________________

Definition:
The locus definition of a parabola is the set of all points that are equidistant from a fixed point called the focus and a line called the directrix.
Buka Dokumen PARABOLA – GeoGebra



Task 1:

a. Point F is known as the focus of the parabola. What is the name of the line that passes through point F and is perpendicular to the directrix?
b. Line d is the directrix of the parabola. What is the relationship between line d and the dashed line passing through point F?

Task 2:

a. Drag point P along the shape of the parabola. How does the distance from point P to the focus compare to its distance to the directrix? What can you conclude from your observation?
b. Move the focus point (F) to different positions on the screen. Does the property you observed in Question 3 still apply? Explain your reasoning.

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Parabola

Name: ______________________

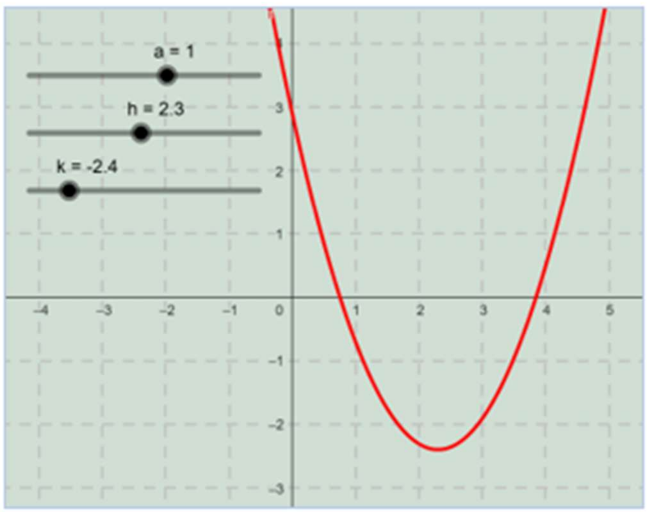
Group: ______________________

Task 3:

Open the GeoGebra document titled "BENTUK PUNCAK PARABOLA – GeoGebra, which displays the graph of a parabola in vertex form:

$$y = a(x - h)^2 + k$$

featuring interactive sliders for the parameters a, h, and k that allow you to dynamically view the changes in the parabola's graph as the values are adjusted.



The vertex form of the parabola equation $y = a(x - h)^2 + k$ is displayed. Move the sliders to change the values of $a, h,$ and k.
Explain how changes in each parameter $a, h,$ and k affect the shape and position of the graph.

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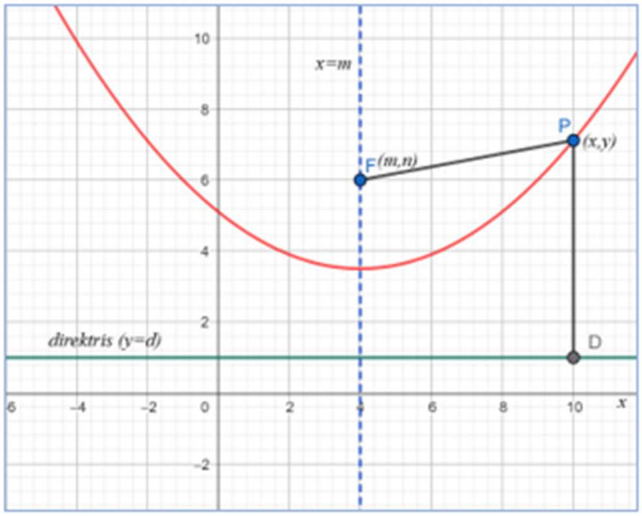
Parabola

Name: ____________________

Group: ____________________

Given the focus (m, n) the directrix $y = d$ and a point $P(x, y)$ on the parabola, use the distance formula to derive the general equation of the parabola.



a. Find the distance between the focus and point P.

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b. Find the distance between point P and directrix

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a. Use the fact that the distance from point P to the focus is equal to its distance to the directrix. Set up an equation for the two distances and solve to find the value of y.

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Parabola

Name: ______________________

Group: ______________________

Using the vertex form of the parabola,

$$\mathbf{y = a(x - h)^2 + k}$$

and the algebraic relationship derived in Task 4, answer the following questions:

a. Explain the relationship between the focus, directrix, and the value of a

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b. What happens to the shape of the parabola as a increases? How does this relate to the distance between the focus and the directrix?"

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