

Enhancing Teachers' Performance and Students' Problem-Solving Abilities through Constructivist-Based Lesson Study: Developing a Study Lesson on the Sum of Consecutive Natural Numbers

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Abstract: This qualitative research article examines the role of Lesson Study, a collaborative professional development method based on constructivist learning theory, in enhancing teachers' instructional practices and students' problem-solving abilities. Despite efforts by educational policymakers to promote modern teaching methods in mathematics, many teachers still rely on traditional approaches that prioritize rote memorization over conceptual understanding. This study focuses on a Lesson Study group consisting of four mathematics teachers and twelve seventh-grade students from an international school in Malaysia. The group designed and implemented a Study Lesson centered on the summation of consecutive natural numbers from one to n . Through data collected via observations and interviews, the study demonstrates that Lesson Study not only improves teachers' instructional methods but also enhances students' problem-solving skills within the context of the study lesson, utilizing constructivist learning theory.

Keywords: Lesson Study, Study Lesson, Mathematics problem

INTRODUCTION

Mathematics holds a prominent position in the curriculum, with students needing to acquire appropriate mathematical knowledge for practical application in their daily lives. However, the current state of mathematics education necessitates a paradigm shift due to the prevailing perception among students at various educational levels that mathematics is a challenging subject (Fritz et al., 2019; Li & Schoenfeld, 2019). In order to promote students' mathematical thinking abilities, educators must underscore the humanistic nature of mathematics education, wherein students actively engage in group discussions to explore problem-solving ideas rather than passively acquiring static knowledge and skills (Li & Schoenfeld, 2019). By shifting the focus towards mathematical inquiry and collaborative learning, teachers can foster a deeper understanding and appreciation of mathematics among students. Active engagement through independent speculation,

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exploration, and analysis of problems fosters meaningful mathematical learning, empowering students to discover alternative problem-solving methods and enhancing their understanding of mathematical concepts (Lin et al., 2017). Furthermore, promoting effective group dynamics entails ensuring that all group members possess the ability to explain and discuss their collaborative work.

In mathematics education, a problem is distinguished from an exercise by its inherent challenge, demanding higher-order thinking skills, whereas exercises entail routine practice and application of established procedures (Schoenfeld, 1985). Traditional mathematics instruction predominantly centers on solving exercises, resulting in students memorizing formulas, theorems, shortcuts, axioms, and methods without cultivating a profound comprehension of the fundamental concepts (Gholami et al., 2019; Khalid, 2017; Mon et al., 2016). In the realm of mathematics education, researchers grapple with the inquiry of how students can attain a deep understanding of mathematical concepts and appreciate their inherent beauty without actively engaging in problem-solving activities. It appears that educators' involvement in innovative professional development approaches, such as Lesson Study, can enhance their instructional effectiveness, consequently improving students' problem-solving skills and critical thinking abilities.

This study explores the impact of lesson study on teachers' ability to create educational content about the sum of natural numbers, ranging from one to any given natural number and related problems. Through collaborative group work, teachers can refine each other's teaching methods while fostering professional growth. The primary outcome of this process is the significant enhancement of students' problem-solving skills. This dual benefit underscores the value of lesson study in improving both teaching quality and student learning outcomes.

LITERATURE REVIEW

The creation and sequencing of mathematical content are crucial in lesson preparation, as they shape the structure and coherence of instruction, promoting a logical progression that enhances student understanding and engagement. Aligning content with pedagogical goals enables teachers to deliver clearer and more effective lessons, fostering meaningful learning experiences. This principle was exemplified in a study by Fernandez et al. (2003), where sixteen mathematics teachers from New Jersey collaboratively planned a sixth-grade math lesson. Initially following the textbook sequence by teaching rectangle area before triangle area, the group reconsidered their approach after a Japanese educator suggested starting with right triangles before scalene triangles, based on the conceptual link between triangles and rectangles. Although some American teachers struggled with this structured knowledge due to limited content familiarity, the cross-cultural collaboration ultimately enriched their pedagogical strategies, underscoring the value of diverse perspectives in lesson design.

The collaborative planning of lessons involves determining the topic and content order, which relies on the collective knowledge and experience of the teachers involved. Their expertise shapes

the decision-making process for effective instructional design. According to Marjanovic (2017), a group of mathematics teachers proposed an instructional approach during a discussion meeting. Their suggestion involved utilizing Figure 1 as a central component of the lesson to demonstrate the proof of the mathematical relationship $1+3+5+ \dots +(2n-1)=n^2$. Subsequently, the teachers identified the proof of the relationship $1+2+3+ \dots +n=(n^2+n) \div 2$ as a problem to be addressed within the lesson, utilizing the aforementioned relationship as a means of solving it.

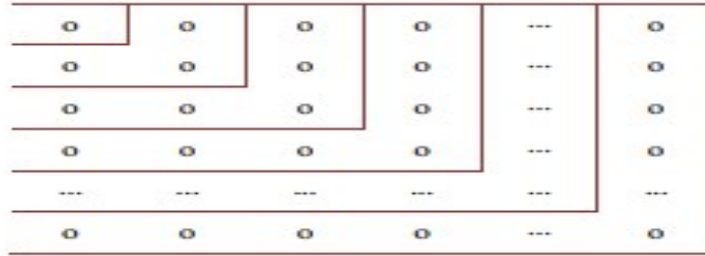


Figure 1: Summation of Odd Natural Numbers from 1 to $(2n-1)$ (Marjanovic, 2017)

Figure 2 illustrates the solution method proposed by mathematics teachers.

$$\begin{aligned}
 &1 + 3 + 5 + \dots + (2n - 1) = n^2 \\
 \Rightarrow &(2 \times 1 - 1) + (2 \times 2 - 1) + (2 \times 3 - 1) + \dots + (2n - 1) = n^2 \\
 &\Rightarrow 2(1 + 2 + 3 + \dots + n) - n = n^2 \\
 \Rightarrow &1 + 2 + 3 + \dots + n = (n^2 + n) \div 2
 \end{aligned}$$

Figure 2: Summation of Natural Numbers from 1 to n (Marjanovic, 2017)

In fact, Marjanovic (2017) initially established a relationship for the sum of odd natural numbers (see Figure 1) and subsequently derived the formula for the sum of natural numbers from 1 to n based on this relationship (see Figure 2). In contrast, the present study takes the sum of natural numbers as the foundational concept and explores numerous novel problems by applying this relation. This approach facilitates students' understanding of fundamental mathematical tools and enhances their ability to apply these tools effectively in solving related problems.

Lesson Study

Lesson Study has been employed by Japanese teachers as a method for professional development in mathematics education since the 1950s (Abiko, 2011). The Japanese term "Jugyo Kenkyu" was translated into English as "Lesson Study" by Yoshida in 1999 (Doig & Groves, 2011). Lesson Study has become increasingly popular worldwide, with widespread adoption among mathematics teachers in diverse educational settings. However, Lesson Study is known by different names in other countries, such as "Learning Study" in Hong Kong (Ling et al., 2005), "Teacher Research Group" in China (Yang, 2009), and "Learning Communities" in Singapore (Chua, 2009). Muarata and Takahashi (2002) found that 91.3% of Japanese mathematics teachers believe Lesson Study is effective in enhancing problem-solving and higher-order thinking skills. The Singapore Ministry of Education's school cluster system, based on geographical location, facilitates collaborative Lesson Study among mathematics educators, allowing them to harness the contextual benefits of this approach (Chua, 2009).

In recent decades, Lesson Study has emerged as a notable educational approach, attracting attention globally. This collaborative method involves a team of educators who jointly concentrate on a specific mathematics topic, investing considerable time in planning, delivering, and discussing a lesson. Lesson Study aims to optimize students' mathematical learning by focusing on the development of problem-solving and critical thinking skills through the refinement of teaching practices (Matanluk et al., 2013). During the Lesson Study process, educators develop and implement lessons known as Research Lessons (Fujii, 2014) or Study Lessons (Yoshida, 1999), which serve as integral components of the pedagogical cycle in teaching (Lewis, 2002; McDonald, 2009). McDonald (2009) emphasizes the significance of the Study Lesson in the professional development of mathematics teachers and highlights its key role in facilitating their growth and improvement. Lesson Study is an educational methodology that empowers educators to teach mathematical concepts through a problem-solving approach, fostering deep learning and enhancing students' critical thinking skills, aligning with cognitivist learning theory and applicable across multiple subjects and educational levels (Cheng & Yee, 2011).

In Japan's educational system, Lesson Study is categorized into four main types: School-based, District-based, National-level, and Association-sponsored. Each type follows a similar cycle but differs in terms of the intended audience for the Study Lessons (Fujii, 2016; Takahashi & McDougal, 2016). School-based Lesson Study focuses on improving teaching practices within a specific school, targeting learners within that school. District-based Lesson Study targets learners within a district, often involving teachers from multiple schools in the area. National-level Lesson Study aims to benefit learners across the entire country, typically conducted in national lab schools or through subject matter organizations. Association-sponsored Lesson Study is supported by national subject matter associations, allowing teachers to focus on specific subjects or areas of interest. These types of Lesson Study work together to enhance instructional quality and curriculum development across different levels of the educational system. Among these, School-based Lesson Study is considered the most robust as it allows mathematics educators to tailor

Study Lessons with appropriate materials and open-ended problems that cater to the specific abilities of students in their school, resulting in more effective mathematics learning outcomes.

Alamri (2022) conducted a study in Saudi Arabia where mathematics teachers explored the Lesson Study approach and reported positive outcomes in teaching mathematics. Teachers gained autonomy in developing their teaching schedules and objectives, improved content knowledge and pedagogical content knowledge through collaborative work, and enhanced students' problem-solving abilities and critical thinking skills through rich Study Lessons. Lesson Study showed promising effects on teacher development and student learning outcomes. According to the results of a Lesson Study program for middle school mathematics teachers in the context of Turkey, Lesson Study not only encourages mathematics teachers to conduct reflection activities effectively but also it encourages them for self-evaluation and improving their teaching by increasing their awareness of their own teaching practices using rich Study Lessons (Bozkurt & Yetkin-Ozdemir, 2018).

The Stages of Lesson Study

Fujii (2014) proposed a framework consisting of five distinct phases for the successful implementation of the Lesson Study program, which are outlined as follows: Firstly, during the Goal Setting phase, teachers establish long-term objectives designed to create learning environments that promote learners' problem-solving abilities and higher-order thinking skills. Secondly, during the Lesson Planning phase, teachers collaborate to design a Study Lesson, using mathematics materials that are tailored to meet the specific needs and abilities of the learners. Next, during the Study Lesson phase, a designated member of the Lesson Study group teaches the prepared lesson, while other members observe the teaching process and gather relevant data to improve the quality of the Study Lesson. After that, during the Post-lesson Discussion phase, teachers engage in discussions about students' problem-solving abilities, observed misconceptions, disciplinary mathematical content, various solution methods for the given problems, and suggestions for enhancing the Study Lesson. Lastly, during the Reflection phase, teachers introduce new mathematical problems to be addressed in the next cycle of Lesson Study. They also engage in discussions to solve these problems and compile a comprehensive report that documents the findings and outcomes of the Study Lesson. Figure 3 illustrates the Lesson Study approach cycle, as informed by Fujii's work in 2014.

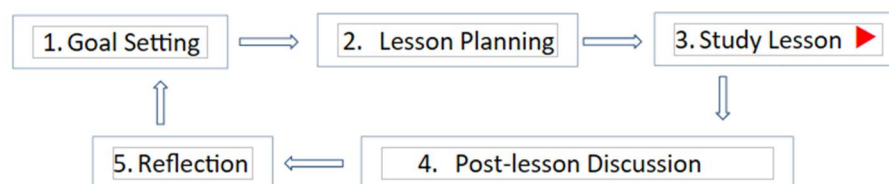


Figure 3: Phases of the Lesson Study Program (Fujii, 2014)

The theoretical framework, based on constructivist theory and Lesson Study, highlights the active role of learners in teaching the sum of natural numbers from 1 to n . This method posits that

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students actively build mathematical understanding by engaging with their educational environment (Hoover, 1996). Lesson Study offers a collaborative approach for teachers to create engaging lessons that promote meaningful mathematical discovery and effective problem-solving (Fujii, 2016). The constructivist approach highlights the crucial role of problem-solving activities, discussions, and reflection in facilitating students' active construction of knowledge (Gergen, 1995). Lesson Study enables teachers to craft a vibrant learning environment that encourages student engagement, promotes critical thinking, and cultivates a profound understanding of the concept of summing natural numbers from 1 to n and associated problems. The iterative nature of Lesson Study facilitates continuous refinement and adaptation of teaching strategies, aligning them with students' evolving responses and needs, while reinforcing the constructivist approach by emphasizing student-centered learning and active knowledge construction (Alamri, 2020; Bozkurt & Yetkin-Ozdemir, 2018). Through this cycle, educators can tailor their methods to better support students grasping complex concepts, such as summing natural numbers from 1 to n .

RESEARCH QUESTION

This article seeks to familiarize middle school mathematics teachers with the Lesson Study approach, aiming to refine instructional strategies and boost student achievement. In particular, it highlights the development of a Study Lesson focused on the concept of summing natural numbers from one to n . Consequently, the research question for this study is formulated as follows:

Research Question: How does the implementation of Lesson Study influence teachers' instructional skills and students' learning outcomes in the context of teaching the summation of natural numbers?

METHOD

This qualitative case study was conducted at one of the international schools of Kuala Lumpur, a prominent non-profit institution known for its learner-centered approach, providing an ideal environment for implementing innovative teaching practices like Lesson Study. The participants consisted of four mathematics teachers, namely A, B, C, and D, three of whom were male and one female. These teachers as Lesson Study group members possessed a minimum of 12 years of experience in teaching mathematics to middle school students. Additionally, twelve students from an existing Grade 7 class participated in the study. These students were divided into six groups, each consisting of two members.

Before participating, all mathematics teachers and student participants provided informed consent by signing a consent letter. The researcher convened a meeting with the teacher participants to introduce the general topic of the Study Lesson for the Lesson Study group. Following this, the teachers collaborated to develop a Study Lesson focused on the concept of summing natural numbers from one to any natural number. They refined the lesson to address various related problems. The selection of the topic and the sequence of problems within the Study Lesson were determined through collaborative decision-making, drawing on the collective expertise and experiences of the Lesson Study group members. In the textbook, this concept is presented within the

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chapter on introducing various strategies for solving mathematical problems. Consequently, this chapter provides teachers with opportunities to engage students in diverse problem-solving activities, thereby enhancing their critical thinking and problem-solving skills.

During the initial meeting, members of the Lesson Study group collaboratively planned, discussed, and designed the Study Lesson. In the subsequent meeting, one teacher delivered the lesson in a regular classroom while the remaining group members observed the instructional process. In the final meeting, the mathematics teachers conducted a post-lesson discussion to reflect on their observations and collaboratively refine the Study Lesson. Figure 4 illustrates the process of preparing the Study Lesson.

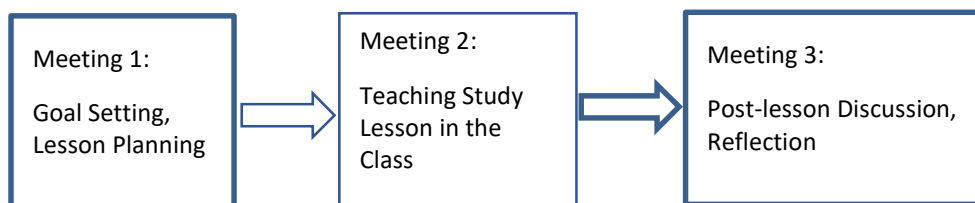


Figure 4: The Process of Preparing the Study Lesson (Author)

For example, during the initial discussion meeting, one of the teachers proposed including the following problem in the Study Lesson as a homework assignment. The group members reached a consensus that this particular problem would be suitable for enhancing students' critical thinking skills due to its level of difficulty.

Problem: Determine the value of the sum $3+7+11+ \dots +199$.

The teachers presented two potential solution methods for this problem, which are illustrated in Figure 5. In solution method 1, the teachers employed a technique where they divided each term of the series into pairs of consecutive natural numbers. By combining these pairs, they effectively converted the problem into finding the sum of the natural numbers ranging from 1 to 100. Utilizing the relevant formula for such sums, they determined that the sum of the natural numbers from 1 to 100 equals 5050, which represents the sum of the given series. In solution method 2, the Lesson Study group broke down each term of the series into pairs of consecutive natural numbers. They split the series into odd and even sequences. By applying the formulas for the sum of consecutive odd and even natural numbers, they calculated the sum of odd numbers from 1 to 99 as 2500 and the sum of even numbers from 2 to 100 as 2550. Combining these sums, they obtained the total sum of the given series as 5050. Both solution methods share a common approach of identifying the pattern within the series and transforming it into a more manageable form. These methods exhibit creativity and effectively connect to the concepts discussed in the Study Lesson.

Solution method 1:	Solution method 2:
$3+7+11+ \dots +199$ $=(1+2)+(3+4)+ \dots +(99+100)$ $=1+2+3+ \dots +100=\frac{100(100+1)}{2}=5050.$	$3+7+11+ \dots +199$ $=(1+2)+(3+4)+ \dots +(99+100)$ $=(1+3+ \dots +99)+(2+4+ \dots +100)$ $=50 \times 50+50 \times 51=5050.$

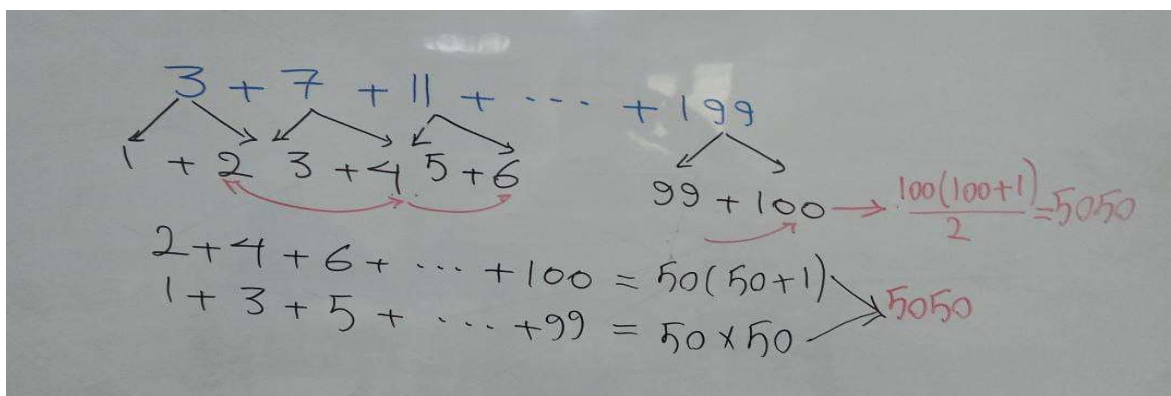


Figure 5: Two Solution Methods for the Given Problem

This study collected data from observations of the discussion meetings, the teaching process in the classroom, and interviews with teachers and students. The collected data were subjected to a descriptive analysis, which involved a detailed examination of the observations, discussions, and interview responses. This approach allowed for a comprehensive understanding of the Lesson Study process and its impact on teaching and learning outcomes.

RESULTS

The primary outcome of this study entails the development of a comprehensive Study Lesson aimed at seventh-grade students, focusing on the concept of the sum of consecutive natural numbers. The Study Lesson incorporates various materials, beginning with a problem presented to the Lesson Study group, namely "find the value of $A=1+2+3+ \dots +100$." The intention behind this problem is to engage students in critical thinking, prompting them to explore the core aspects of the lesson. Initial attempts by the students to solve the problem proved challenging, leading the teacher to share the story of Carl Friedrich Gauss, a renowned German mathematician from the 18th century. Gauss's remarkable solution to determining the sum of consecutive natural numbers

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from 1 to 100 served as inspiration for the students. Gauss cleverly organized the numbers into fifty pairs, summing the first and last numbers, the second and second-to-last numbers, and so forth. Each pair exhibited a sum of 101, resulting in a total sum of the natural numbers from 1 to 100 equating to 50×101 , which simplifies to 5050.

Furthermore, the teacher introduced an additional creative strategy to compute the sum of the natural numbers from 1 to 100, aiming to foster students' creativity. This alternative method involved juxtaposing the original list of numbers with its reverse order. By adding the corresponding terms of the two series, a novel series emerged, consisting of 100 terms each equaling 101. Recognizing that the original list was duplicated, the total sum of 100 multiplied by 101 was divided by 2, yielding the sum of 5050.

The teacher provided a comprehensive explanation regarding the calculation of the sum for the series $1+2+3+ \dots +n$, considering two cases based on the parity of n . In the case where n is even, the teacher demonstrated the pairing of numbers, such that each term from 1 to n is paired with its corresponding counterpart, resulting in $\frac{n}{2}$ pairs. Each pair sums to $n+1$, thereby yielding a total sum of $\frac{n(n+1)}{2}$ for the series. Conversely, when n is odd, the teacher illustrated the pairing of numbers where each term from 1 to $n-1$ is paired with its counterpart, generating $\frac{n-1}{2}$ pairs with a sum of n for each pair. Consequently, the total sum of the series in this case is given by $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$. The classroom discussion surrounding these two cases facilitated a generalization of the concept, leading to the algebraic relation $1+2+3+ \dots +n = \frac{n(n+1)}{2}$.

The teacher emphasized the significance of geometrical shapes in enhancing students' reasoning abilities. To facilitate their understanding of the above relation, the teacher employed Figure 6 as a visual aid, encouraging students to explore and discover this relation using the shape presented. During the lesson, the students examined Figure 6 and engaged in a discussion centered around the relation $(1+2+3+ \dots +n) + (1+2+3+ \dots +n) = n(n+1)$. Through their analysis, the students made an important discovery: they realized that the sum of the natural numbers from 1 to n , denoted as $1+2+3+ \dots +n$, is equivalent to $\frac{n(n+1)}{2}$. This result was derived from their observation and reasoning based on the given figure.

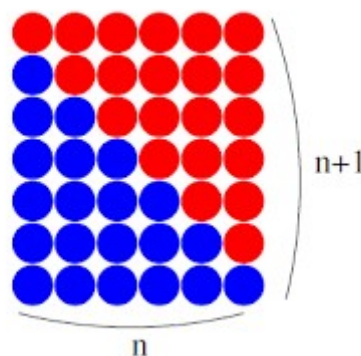


Figure 6: A proof without word for the sum of natural numbers from 1 to n (Source: <https://www.quora.com/How-do-I-find-the-sum-of-all-natural-numbers-from-1-to-n>)

Each student demonstrated their proficiency in utilizing the formula for calculating the sum of natural numbers by successfully determining the sum of $1+2+3+ \dots +200$. Through the application of the formula, $1+2+3+ \dots +200=(200 \times 201) \div 2=20100$, they efficiently computed the sum, showcasing their understanding and competence in solving such series. This exemplifies their ability to find the sum of a series of natural numbers from 1 to any given arbitrary number using the formula.

During this segment of the Study Lesson, the Lesson Study group engaged in extensive discussions and rearrangements to present a series of mathematical problems aimed at enhancing students' problem-solving abilities. These problems were strategically organized in increasing complexity, introducing novel techniques for problem resolution to the students.

Problem 1: Determine the sum of even natural numbers ranging from 2 to 300, represented as $2+4+6+ \dots +300$.

Problem 2: Find the sum of odd natural numbers from 1 to 499, represented as $1+3+5+ \dots +499$.

Problem 3: Calculate the sum of natural numbers from 201 to 600, represented as $201+202+203+ \dots +600$.

Problem 4: Determine the sum of even natural numbers from 302 to 800, represented as $302+304+ \dots +800$.

Problem 5: Find the sum of odd natural numbers from 101 to 599, represented as $101+103+105+ \dots +599$.

The Lesson Study group recognized the challenging nature of the presented problems and sought to foster students' problem-solving skills by dividing them into groups of two. This collaborative approach aimed to enhance their ability to tackle these problems effectively. The inclusion of the aforementioned problems within the Study Lesson was strategically designed to introduce new concepts and techniques to students, while also reinforcing the application of the formula for finding the sum of natural numbers from 1 to n in solving them.

By providing an appropriate framework for students to approach and solve each problem, their problem-solving capabilities and creativity were further developed. During class session, the teacher presented the problems to the students sequentially, allowing the six groups to engage in solving them. This active participation and collaborative effort among students fostered a dynamic learning environment that encouraged critical thinking and problem-solving skills.

Each student in the class successfully demonstrated their problem-solving skills by solving problem 1. Four groups of students employed solution method 1, as depicted in Figure 7, while the remaining two groups utilized solution method 2. This diversity in approaches emphasizes the

students' proficiency in logical reasoning and their ability to adapt different strategies to arrive at the correct solution. It showcases their flexibility in thinking and problem-solving, contributing to a dynamic and enriching learning environment.

<p>Solution 1: $2+4+6+ \dots +300$ $=2(1+2+3+ \dots +150)$ $=2\left(\frac{150 \times 151}{2}\right)=150 \times 151$ $=22650.$</p>	<p>Solution 2: $2+4+6+ \dots +300$ $=(1+1)+(2+2)+ \dots +(150+150)$ $=(1+2+ \dots +150)+(1+2+ \dots +150)$ $=\frac{150 \times 151}{2} + \frac{150 \times 151}{2}=22650.$</p>
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Figure 7: Comparative Analysis of Solution Methods for Problem 1

Solution 1 demonstrates a methodical approach by recognizing patterns, simplifying the series, and leveraging existing mathematical formula ($1+2+3+ \dots +n=\frac{n(n+1)}{2}$). In fact, this method showcases creativity by employing techniques that reduce complexity and facilitate more straightforward calculations. Also, Solution 2 showcases an alternative approach to the problem by pairing each term with itself and simplifying the expression into identical sums. It cleverly utilizes the relation $1+2+3+ \dots +n=\frac{n(n+1)}{2}$ to calculate the sum of all given natural numbers. This approach demonstrates creative thinking and adaptability in finding different perspectives to solve the same problem.

During the discussion, the teacher introduced the relation $2+4+6+ \dots +2n=n(n+1)$ with the help of Figure 8 to efficiently find the sum of even natural numbers from 2 to $2n$. This pedagogical approach incorporates visual aid to enhance understanding, demonstrating the application of mathematical concepts and logical reasoning in deriving a concise formula for the sum. Such teaching methods encourage critical thinking, exploration of different techniques, and foster a deeper understanding of mathematical principles among students.

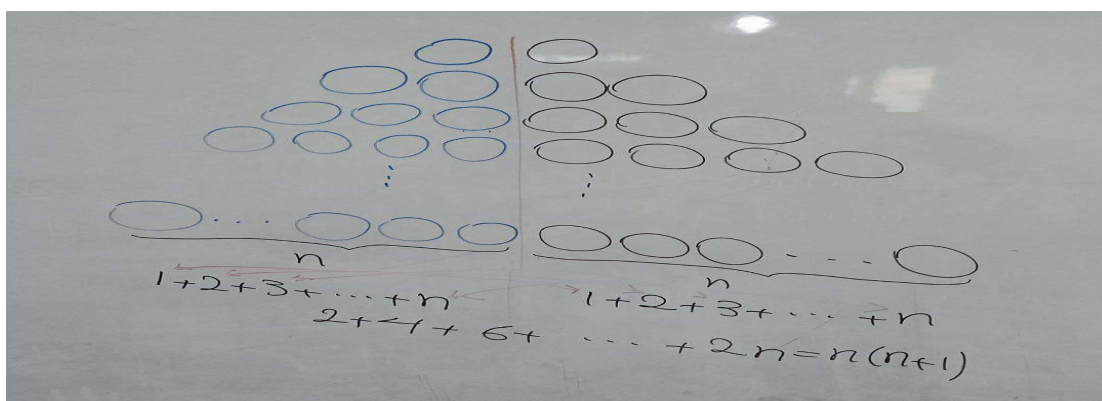


Figure 8: An Approach for Summing Even Natural Numbers from 2 to $2n$

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Problem 2 served as an effective task to enhance critical thinking skills among the students. Since the task involved finding the sum of odd numbers from 1 to 499, it required students to employ logical reasoning and devise appropriate methods to arrive at a solution. Notably, only three groups managed to successfully solve the problem, as depicted in Figure 9. Two groups proposed Solution 1, while the remaining group presented Solution 2 as their approach to solving the problem. These findings highlight the students' ability to think critically and creatively when confronted with mathematical challenges.

<p>Solution 1:</p> $1+3+5+ \dots +49$ $=(1+2+ \dots +500)-(2+4+ \dots +500)$ $=(1+2+ \dots +500)-2(1+2+ \dots +250)$ $= \frac{500 \times 501}{2} - 2 \times \frac{250 \times 251}{2}$ $=250 \times 501 - 250 \times 251 = 62500.$	<p>Solution 2:</p> $1+3=4=2 \times 2$ $1+3+5=9=3 \times 3$ $1+3+5+7=16=4 \times 4$ \vdots \vdots \vdots $1+3+5+ \dots +499=250 \times 250=62500.$
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Figure 9: Comparative Analysis of Solution Methods for Problem 2

Solution 1 demonstrates the students' creative approach in finding the sum of odd natural numbers from 1 to 499 indirectly. Rather than directly calculating the sum of the odd numbers, they devised a method by calculating the sum of all natural numbers from 1 to 500 and the sum of all even natural numbers from 2 to 500. By subtracting the sum of even numbers from the sum of all numbers, they were able to determine the sum of odd natural numbers from 1 to 499. This inventive solution showcases the students' ability to think critically and devise alternative methods to solve complex problems. Solution 2 displays the students' ingenuity by recognizing the pattern of consecutive odd numbers and their relationship to perfect squares. They creatively observed that the sum of the first n odd natural numbers is equal to n squared, enabling them to determine the sum of odd numbers from 1 to 499. This method demonstrates their analytical thinking and ability to identify mathematical patterns to solve problems efficiently.

The teacher facilitated the calculation of the sum of odd natural numbers from 1 to an arbitrary odd number by utilizing Figure 10. This geometrical shape provided students with a straightforward method for determining the sum. By examining specific examples, such as $1+3+5+7=4 \times 4$ and $1+3+5+7+9=5 \times 5$, students were able to discern a general pattern. Consequently, they derived a relation for calculating the sum of all odd natural numbers from 1 to $2n-1$ as $n \times n$. This approach showcases the teacher's effective use of visual aids and the students' ability to identify patterns and generalize them into a mathematical relation.



Figure 10: The sum of odd natural numbers

(Source: <https://www.studysmarter.co.uk/explanations/math/pure-maths/sum-of-natural-numbers/>)

Problem 3 solved by four groups of students, who demonstrated their creativity through the development of two distinct approaches. Figure 11 showcases these solutions, with Solution 1 being successfully implemented by three groups, while Solution 2 was devised and executed by a single group.

<p>Solution 1: $201+202+203+ \dots +600$ $= (1+2+ \dots +600) - (1+2+ \dots +200)$ $= \frac{600 \times 601}{2} - \frac{200 \times 201}{2}$ $= 300 \times 601 - 100 \times 201$ $= 180300 - 20100 = 160200.$</p>	<p>Solution 2: $201+202+203+ \dots +600$ $= (200+1) + (200+2) + \dots + (200+400)$ $= 400 \times 200 + (1+2+3+ \dots +400)$ $= 80000 + \frac{400 \times 401}{2}$ $= 80000 + 80200 = 160200.$</p>
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Figure 11: Comparative Analysis of Solution Methods for Problem 3

In Solution 1, the students cleverly approached the problem of finding the sum of numbers from 201 to 600. They utilized the concept of subtracting the sum of numbers from 1 to 200 from the sum of numbers from 1 to 600. By applying the formula for the sum of consecutive natural numbers from 1 to n , they simplified the expression and obtained the final result of 160200. This solution demonstrates the students' ability to decompose complex problems and apply mathematical principles to efficiently find the solution.

In Solution 2, the students subtracted 200 from each number in the given series and applied the formula for the sum of natural numbers from 1 to n to find the solution. By grouping the terms as $(200+1) + (200+2) + \dots + (200+400)$, they simplified the expression and calculated the sum of the modified series. Utilizing the relation for the sum of natural numbers, they efficiently obtained the result of 160200. This approach demonstrates their strategic thinking in manipulating the series and utilizing mathematical concepts to arrive at the solution.

Four groups of students demonstrated their success in solving Problem 4 by employing the solu-

tion method depicted in Figure 12. In this solution method, the students employed the sum formula for even natural numbers from 2 to $2n$. By subtracting the sum of even numbers from 2 to 300 from the sum of even numbers from 2 to 800, they were able to find the sum of the given series. This approach showcases the students' ability to utilize mathematical formulas and techniques to efficiently solve the problem.

Solution method:

$$302+304+306+ \dots +800=(2+4+6+ \dots +800)-(2+4+6+ \dots +300)$$

$$=400(400+1)-150(150+1)=160400-22650=137750.$$

Figure 12: A solution method for problem 4

Two groups of students exhibited their proficiency by successfully presenting a comprehensive solution to Problem 5 within the designated time-frame, as illustrated in Figure 13. By subtracting the sum of odd numbers from 1 to 99 from the sum of odd numbers from 1 to 599, they effectively eliminated the terms up to 99 and obtained the desired sum. Utilizing the property of the sum of consecutive odd numbers, the students calculated the result as 87500. This approach demonstrates their ability to decompose the problem, apply mathematical properties, and devise an efficient solution strategy.

Solution method:

$$101+103+105+ \dots +599=(1+3+5+ \dots +599)-(1+3+5+ \dots +99)$$

$$=300 \times 300-50 \times 50=87500.$$

Figure 13: A solution method for problem 5

The Lesson Study group members recognized the significance of engaging and captivating students in their learning process. Consequently, teachers incorporated the following enjoyable puzzles related to the topic in the Study Lesson to foster student participation and enthusiasm in problem-solving activities.

Puzzle 1: Given a set of 50 bags, each containing 100 coins, all but one bag contains genuine gold coins, while a single bag holds counterfeit coins. The counterfeit coins weigh 3 grams each, whereas the genuine gold coins weigh 4 grams each. Your task is to identify the bag containing the counterfeit coins, with the constraint that you can use a digital scale for measurement only once.

The teacher commenced by presenting the puzzle to the students and elucidating its solution through the utilization of concepts introduced during the session. The teacher proceeded to resolve the puzzle in the following manner: Firstly, the teacher instructed the students, "I will assign a label to each bag from 1 to 50 and extract a number of coins from each bag corresponding to its label." In other words, one coin was extracted from bag number 1, two coins from bag number 2, three coins from bag number 3, and so forth, until a total of 50 coins were obtained from bag number 50. The cumulative count of coins extracted from these 50 bags amounted to $1+2+3+ \dots +50=\frac{50 \times 51}{2}=1275$. If all the coins were genuine, their combined weight would be 5100 grams. However, since it was known that one bag contained counterfeit coins, the disparity be-

tween 5100 grams and the actual weight of the coins determined the bag harboring the counterfeits. For instance, if the actual weight of the coins measured 5090 grams, then the fake coins resided in bag number 10 ($5100-5090=10$).

The teacher presented the students with Puzzle 2, challenging them to employ the concepts covered in the current session to find the solution. The puzzle is as follows:

Puzzle 2: The natural numbers are grouped into categories as follows:

$$(1), (2,3), (4, 5, 6), \dots$$

In the first category, there is only one number. In the second and third categories, there are two and three numbers respectively, and so on. The task is to determine the sum of all the numbers in the twentieth category.

The teacher provided an explanation to the students on how to approach and solve this puzzle. It was emphasized that finding the first number of the twentieth category would suffice. Consequently, the number of natural numbers classified in categories one to nineteen was determined to be $1+2+3+ \dots +19 = \frac{19 \times 20}{2} = 190$. Hence, the first number in the twentieth category was found to be 191. To calculate the sum of all numbers in this category, the approach involved subtracting the sum of numbers from 1 to 190 from the sum of numbers from 1 to 210. Thus, the sum was calculated as $(1+2+3+ \dots +210) - (1+2+3+ \dots +190) = 4010$.

The students displayed great enthusiasm in response to these logical explanations. Through engaging in discussions surrounding these puzzles, they had the opportunity to witness the real-world application of the properties taught in the Study Lesson multiple times. As a result, they were able to solidify their conceptual understanding of the topic, benefiting from the incorporation of enjoyable materials and activities throughout the learning process.

Research studies have indicated that students who engage in homework assignments containing suitable mathematical problems exhibit enhanced conceptual understanding compared to those who solely rely on conventional textbook tasks (Jacobse & Harskamp, 2009; Ozcan & Erkin, 2015). In fact, the inclusion of challenging mathematical problems in homework assignments contributes to the holistic process of mathematics learning among students. Thus, the Lesson Study group members deemed it essential to incorporate the following problems as part of the homework for this particular Study Lesson.

Problem 1: Determine the value of the sum $1001+1003+1005+ \dots +2023$.

Problem 2: Calculate the value of the sum $3+7+11+ \dots +199$.

Problem 3: Find the value of the sum $215+219+223+ \dots +611$.

Problem 4: Consider a lively gathering consisting of 50 attendees, where each guest is assigned a card bearing a randomly selected natural number from 1 to 50. The number written on the card corresponds to the quantity of chocolates received

by the cardholder, with each individual receiving three times the value indicated on their card. What is the total number of chocolates distributed among the guests at this event?

DISCUSSIONS

During the development of this Study Lesson, teachers introduced captivating topics related to the summation of natural numbers from 1 to n , resulting in enhanced classroom engagement and performance. They prioritized conceptual learning through a problem-solving approach, rooted in constructivist learning theory. When designing the problems for this lesson, the teachers intentionally structured them to facilitate valuable insights and discussions for the students with each new problem. While there was initially disagreement among group members regarding the order of the problems in the Study Lesson, they reached a consensus after thorough deliberation, ultimately deciding to present the problems to the students in the prescribed order outlined in this Study Lesson.

Furthermore, the implementation of the prepared Study Lesson in an actual classroom setting contributed to the refinement of teaching methodologies among the Lesson Study group members. Post-discussion meetings allowed the group to identify areas for improvement in the Study Lesson's quality. For instance, during the class, it was observed that some students encountered difficulties in grasping the concept of summing odd natural numbers. To address this, the teacher recommended the utilization of Figure 10 as a visual aid to enhance the understanding of the concept. The class was divided into six groups of two students each, and the teacher dedicated approximately ten minutes to introducing the main topic of the Study Lesson: the method for finding the sum of consecutive natural numbers from 1 to an arbitrary value. Subsequently, the class adopted a student-centered approach, wherein the teacher assigned problems to the students and monitored the progress of the different groups. At times, the teacher provided gentle guidance to assist the students in devising suitable problem-solving strategies. The performance of the students during this session indicated that the prepared Study Lesson was well-suited to their needs, although certain problems may require modification based on the students' proficiency levels in various contexts. Figure 14 illustrates the performance of each group in solving five assigned problems. Specifically, it highlights that only Groups 1 and 5 successfully solved Problem 5 in its entirety. In the Results section, the detailed performance of each group on all five problems is described, indicating which problems were fully solved and which were not. The pie charts in Figure 14 provide a visual summary of these group outcomes, showing the proportion of groups that completely solved each problem versus those that did not. This representation helps to quickly convey the overall success rates across the different problems.

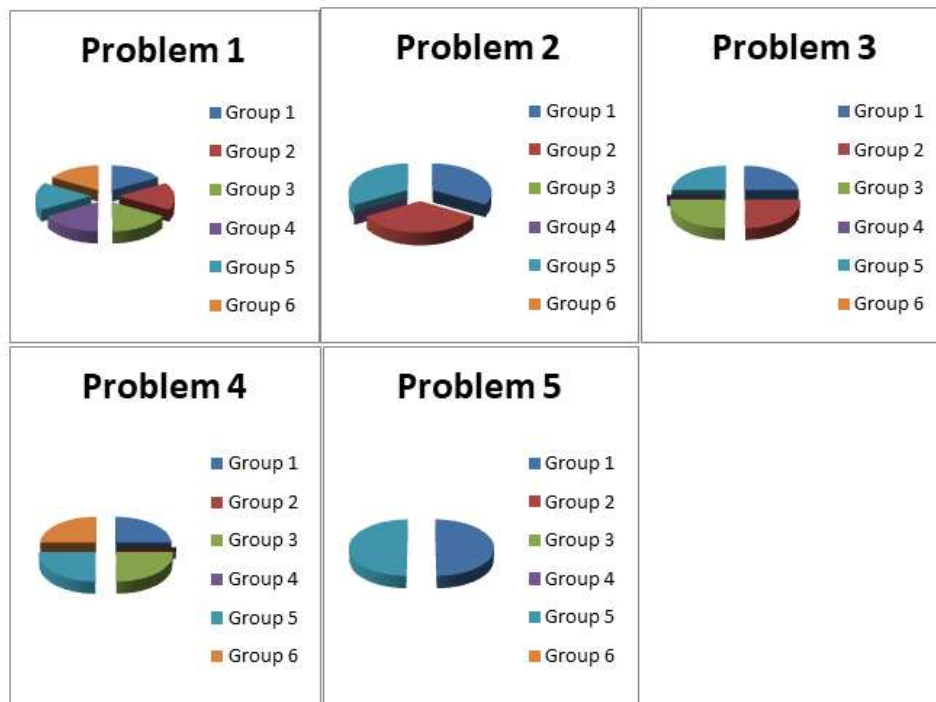


Figure 14: Comparative Analysis of Student Performance in Problem Solving

The Lesson Study program played a significant role in enhancing students' proficiency in mathematics problem-solving. Through the implementation of the problem-solving approach based on the model of Polya (1945), students were able to engage in a systematic process encompassing problem understanding, strategizing, solution implementation, and solution verification. The study lesson adopted a student-centered teaching model, fostering the development of students' problem-solving skills and critical thinking abilities in accordance with the principles of constructivist learning theory. As a result, students experienced considerable growth in their problem-solving aptitude and critical thinking capabilities. A student provided an explanation, stating:

During this session, I engaged in contemplation on various mathematical problems, all centered on the summation of series of natural numbers. Each problem necessitated the exploration of innovative solution methods, leading to the acquisition of new and creative techniques through subsequent discussions. Additionally, I found great enjoyment in participating in classroom discussions centered on two intriguing mathematical puzzles.

In traditional teaching approaches, the classroom environment tends to be teacher-centered, with students dedicating substantial class time to copying and transcribing materials presented by the teacher (Gholami, Yunus, et al., 2021). However, the members of the study group adopted an alternative method that eliminated the need for students to invest time in note-taking. Instead, class

time was effectively utilized for engaging students in optimal problem-solving activities. A student expressed their perspective, stating:

During this session, the teaching method employed eliminated the need for me to engage in the task of copying mathematical notes. Instead, I actively engaged with a diverse range of problems, making optimal use of the class time. This approach proved to be both captivating and enjoyable, allowing me to fully immerse myself in the problem-solving process.

All members of the Lesson Study group reached a consensus that this instructional approach not only enhanced their pedagogical expertise but also actively engaged students through carefully selected mathematics problems. This engagement fostered the development of students' problem-solving abilities and supported a constructivist learning environment. Within the framework of the Lesson Study program, teachers collaboratively identified appropriate problems and deliberated on the most effective strategies for conveying mathematical concepts. By emphasizing the importance of problem sequencing, they examined a variety of solution methods to stimulate students' creativity. The Study Lesson functioned as a repository of diverse solution strategies and pedagogical techniques, thereby serving as a valuable resource to inform and improve future teaching practices. As an example, Figure 15 showcases three additional solution methods incorporated by the teachers for problem 4 in the Study Lesson.

<p>Solution 1:</p> $ \begin{aligned} 302+304+306+ \dots +800 &= 2(151+152+153+ \dots +400) \\ &= 2[(1+2+3+ \dots +400)-(1+2+3+ \dots +150)] \\ &= 2\left(\frac{400 \times 401}{2} - \frac{150 \times 151}{2}\right) \\ &= 2(80200-11325)=137750. \end{aligned} $
<p>Solution 2:</p> $ \begin{aligned} 302+304+306+ \dots +800 &= (300+2)+(300+4)+(300+6)+ \dots +(300+500) \\ &= 250 \times 300+(2+4+6+ \dots +500) \\ &= 75000+2(1+2+3+ \dots +250) \\ &= 75000+2\left(\frac{250 \times 251}{2}\right) \\ &= 75000+62750=137750. \end{aligned} $
<p>Solution 3:</p> $ \begin{aligned} 302+304+306+ \dots +800 &= (2+4+6+ \dots +800)-(2+4+6+ \dots +300) \\ &= 2(1+2+3+ \dots +400)-2(1+2+3+ \dots +150) \\ &= 2\left(\frac{400 \times 401}{2}\right)-2\left(\frac{150 \times 151}{2}\right) \\ &= 160400-22650=137750. \end{aligned} $

Figure 15: Three Added Solution Methods for Problem 4 by Teachers in the Study Lesson

Teacher B described that:

In the teaching session, the teacher actively interacted with various groups, carefully evaluating their work and providing guidance when necessary. This instructional approach underscores the significance of adopting a problem-solving approach in mathematics education. In one particular case, a group of students presented an interesting strategy to solve Problem 2. The group classified the numbers in the series $1+3+5+ \dots +499$ into pairs such as $(1,499)$, $(3, 497)$, $(5, 495)$, and so on, up to $(249, 251)$. They correctly identified that the sum of the two numbers in each pair is always 500. Utilizing this insight, they calculated the sum of all the numbers in the series as $250 \times 500 = 125000$. While acknowledging the group's commendable strategy, the teacher pointed out an error in their counting of the number of pairs. The group struggled to identify the correct count. Recognizing this, the teacher provided additional guidance to help them rectify their mistake and complete the solution.

Despite having taught this concept to students for several years, the participating teachers expressed that they acquired a new understanding of mathematics teaching through the utilization of a problem-solving approach. Teacher D reflected on the teaching approach, explaining that:

Within the Lesson Study program, the participating teachers directed their attention towards enhancing problem-solving and critical thinking skills in the Study Lesson, thereby refining their ability to design effective lesson plans for their students. While the classroom discussions primarily revolved around the following mathematical relationships:

$$1+2+3+ \dots +n = \frac{n(n+1)}{2}$$

$$2+4+6+ \dots +2n = n(n+1)$$

$$1+3+5+ \dots +2(n-1) = n \times n.$$

Both teachers and students actively engaged with numerous new problems that could be solved by utilizing these relationships. Thus, the Lesson Study program presented a valuable opportunity for me to enhance my teaching methodology through the application of a problem-solving approach.

CONCLUSION

Based on the stated curriculum aims, mathematics educators commonly engage in formal professional development programs as a means to meet institutional requirements (Friedman & Phillips, 2010) and enhance students' mathematical learning outcomes (Bautista & Ortega-Ruiz,

2015; Jacobs et al., 2017). However, educational studies in the field of mathematics teaching have shown that educators may only incorporate newly acquired methods, skills, and knowledge from these programs into their teaching on a limited basis, reverting back to their previous approaches, or not implementing the new knowledge and methods at all (Supriatna, 2011). Soebari (2012) explained that this phenomenon occurs among mathematics teachers due to the inapplicability of the materials, skills, methods, and concepts obtained from professional development programs to their specific classes, which can be attributed to variations in class size, differences in students' problem-solving abilities in mathematics, and inadequate resources. Thus, continuous professional development programs for mathematics educators are imperative and should be carried out alongside their regular teaching responsibilities (Jalal et al., 2009). It appears that Lesson Study holds promise in assisting mathematics educators in obtaining reliable evaluations of their teaching practices, thereby aiding their future instructional endeavors (Bozkurt & Yetkin-Ozdemir, 2018).

Creating and designing innovative problems is one of the essential skills for mathematics educators, which play an important role in improving the quality of their teaching (Burgos et al., 2024). Hence, Lesson Study emerges as a valuable approach for supporting educators in optimizing their classroom performance and fostering learners' problem-solving abilities. By engaging in Lesson Study, educators actively work towards improving the design and implementation of study lessons tailored to the specific context of their classrooms, thereby continuously enhancing their content knowledge and pedagogical content knowledge (Fujii, 2016). Indeed, a well-designed study lesson with a strong mathematical content equips teachers with a deeper understanding of students' conceptions and misconceptions, enabling them to identify effective mathematical problems and models that promote enhanced mathematical thinking (Kuennen & Beam, 2020). Numerous studies have underscored the pivotal role of study lessons within the Lesson Study program (Fujii, 2016; Lewis, 2002). Through the preparation of appropriate materials and activities, such as math problems and engaging puzzles, mathematics educators strive to cultivate learners' problem-solving skills and critical thinking abilities. Furthermore, at the conclusion of the Lesson Study program, mathematics teachers compile reports on their study lessons, facilitating the sharing of insights and experiences among their colleagues. Consequently, each Lesson Study group can iteratively enhance the quality of their study lessons based on their students' problem-solving proficiency.

Study lessons in mathematics teaching offer a fundamental advantage by fostering deep conceptual learning among students, in alignment with the principles of constructivist learning theory. This approach emphasizes the active construction of knowledge and understanding through students' meaningful engagement with mathematical concepts and problem-solving tasks. Therefore, considering suitable mathematical problems in the study lessons enhance students' interest in problem solving activities (Gholami, Ayub, et al., 2021). By actively participating in study lessons, students are encouraged to explore mathematical ideas, construct their own mental models, and make connections between different mathematical concepts and procedures. In line with constructivist principles, study lessons provide opportunities for students to engage in authentic mathematical tasks, encouraging them to think critically, collaborate with peers, and reflect on

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their own learning processes. Through this process, students develop a deeper understanding of the underlying principles and relationships within mathematics, rather than simply relying on rote memorization or formulaic approaches.

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