Learning Trajectory of Trigonometric Ratios Applying Realistic Mathematics Education

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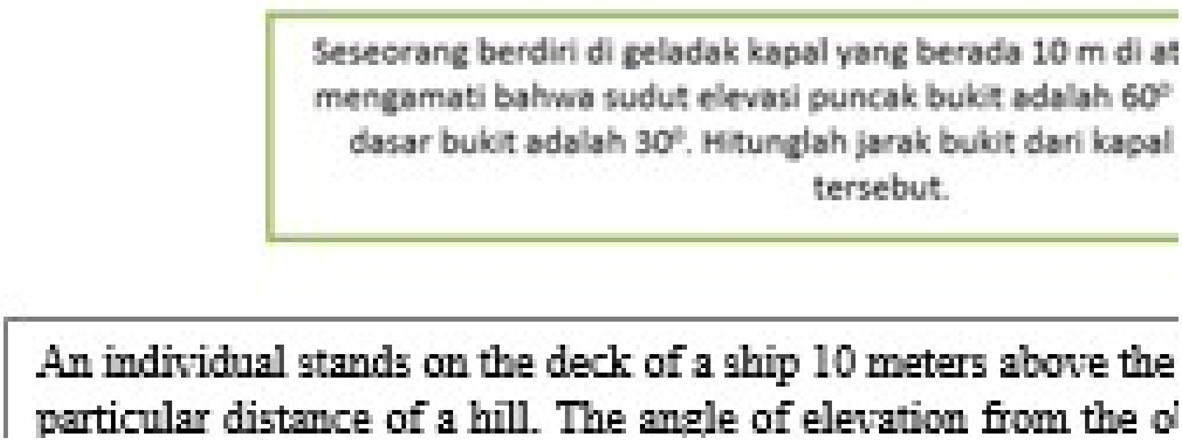
Abstract: Trigonometry materials are still considered complex materials at the secondary level. It is necessary to design an appropriate learning trajectory for *trigonometric ratios so that the students can quickly build the concept. This study aims to design a learning trajectory for trigonometric ratio material through Realistic Mathematics Education (RME). This study uses a research design involving 10th-grade high school students in Bandung City in the 2023-2024 academic year. The research data was collected through data collection using written test instruments, student worksheets, observations, interviews, and documentation. The results of the study showed that the trigonometric ratios learning trajectory consisted of raffia stretching activities from the table base to the student's head, measuring the student's height and shadow, experimenting with making three right triangles from paper, observing a picture of a calendar in the form of an equilateral triangular prism, observing a bed or bunk bed, and observing a picture of a skier going down an iceberg. Students can quickly build the concept of trigonometric ratios from several activities provided. In addition, various learning flows from various activities make it easier for students to build knowledge about trigonometric ratios and solve problems presented contextually.*

Keywords: Learning Trajectory, Trigonometric Ratios, Realistic Mathematics Education

INTRODUCTION

Trigonometry materials are vital in everyday applications such as statistics, surveying, architecture, and other engineering fields (Weber, 2005a). However, building knowledge about trigonometry is challenging (Blackett & Tall, 1991; Demir et al., 2012; Kendal & Stacey, 1997; Ngu & Phan, 2020; Sugianto et al., 2023). To understand the concept of trigonometry, students need to learn several prerequisite mathematical concepts that serve as a foundation. These prerequisite concepts include the Pythagorean Theorem and the properties of right triangles, which are essential in understanding trigonometric ratios. Kamber's Research (2017) stated that students struggle to grasp trigonometry through the unit circle (Kamber & Takaci, 2017). Additionally, they face difficulties distinguishing between trigonometry in right triangles, the unit circle, and the relationship between the two representations (Maknun et al., 2019). As a result, many students fail to understand trigonometric ratios properly, even though these ratios are fundamental in learning trigonometry (Weber, 2005; Nordlander, 2022). Several studies (Hershkowitz et al., 2016; Lessani et al., 2017; Lu'lu'il Maknun et al., 2022; Maphutha et al., 2023; Weber, 2005b) also indicate that traditional approaches lead students to often rely on memorizing formulas rather than understanding the reasoning behind trigonometric ratio concepts.

The findings of previous research provide evidence of students' ability to master trigonometric ratios (Andiani et al, 2024).The test questions given to students are presented in Figure 1.


Seseorang berdiri di geladak kapal yang berada 10 m di at
mengamati bahwa sudut elevasi puncak bukit adalah 60°
dasar bukit adalah 30°. Hitunglah jarak bukit dari kapal
tersebut.

An individual stands on the deck of a ship 10 meters above the
particular distance of a hill. The angle of elevation from the o

Figure 1. Example of Test Questions in Andiani et al.'s Research (2024)

The results of one student's answer in solving the problem are shown in Figure 2.

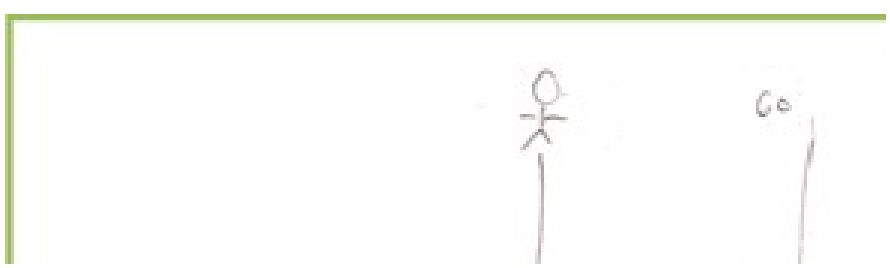


Figure 2. Example of Test Answers in Andiani et al.'s Research (2024)

The results of students' answers show that students are starting to be able to bring up their ideas or concepts in the form of mathematical models. In examining these errors, questions arise:

a. Do students misunderstand the correct mathematical symbols?
b. Do students lack understanding of trigonometry concepts?
c. Do students misunderstand the meaning of elevation angles?
d. Are the errors influenced by prior knowledge, or do they arise during the learning process?.

Student errors occur when students understand a concept in a particular context and adapt it to be applied in another context (Brousseau, 2002). The procedures that students should have written cannot be skipped because students have difficulty in the initial steps of representing the problem in a model. In this case, students have gone through the horizontal mathematization process but have not gone through the vertical mathematization process in its entirety; This is the most prevalent error among students (Meika, 2018).

Student errors indicate that the students have difficulty solving contextual problems. This finding is in line with Haryani's research that the students still find it difficult to construct an understanding of trigonometry material, especially in the context of story problems or real life (Saputri et al., 2020; Widyanti et al., 2022). In addition, a similar opinion was expressed by Insani and Kadarisma in their research that in studying trigonometry, the students face epistemological obstacles caused by the limitations of their knowledge tied to a particular context. As a result, when faced with different situations, the students struggle to apply or adapt this knowledge to the new context (Insani & Kadarisma, 2020; Nasrulloh & Zahiro, 2023). Therefore, teachers need to know the causes of the difficulties experienced by the students and what the students think in their learning process so that these difficulties occur (Pramesti et al., 2024; Rianasari & Guzon, 2024; Wilson et al., 2013, 2015).

Students' learning trajectories represent their cognitive processes in understanding concepts (Clements, 2011), with each student following a unique path (Chahine & Grinshpon, 2020; Ellis et al., 2014; Tahiri et al., 2017; Zhao et al., 2024). While previous studies have explored learning trajectories for trigonometric ratios (Hamdani et al., 2023; Hidayati & Armiati, 2024; Saputra et al., 2021). In addition, several studies have implemented the Realistic Mathematics Education (RME) approach in trigonometry instruction. However, limited research has explicitly developed a systematic learning trajectory for trigonometric ratios grounded in RME principles. Therefore, this study aims to develop a learning trajectory for trigonometric ratios using the RME approach. The main research question guiding this study is: How can a learning trajectory for trigonometric ratios be designed through the application of Realistic Mathematics Education?

This article is structured into several sections. The next section explores the theoretical framework and the methods employed in the study. Subsequently, the research findings are presented and analyzed in depth. The article concludes by summarizing the key findings and the study's implications.

THEORETICAL FRAMEWORK

Structured learning trajectories provide a clear pathway for students to develop mathematical understanding. Simon (1993) introduced the concept of Hypothetical Learning Trajectories (HLT), later expanded by Clements (2011) to describe students' cognitive processes in mathematics. This framework consists of guided learning activities that foster conceptual development, helping the students progress through different reasoning levels and achieve specific learning goals.

This study adopted the RME approach to help the students better understand trigonometric ratios. The RME approach aligns with the student's understanding of trigonometric ratios because it emphasizes contextual learning, allowing the students to build concepts based on real-world experiences (Gravemeijer, 1994). Through authentic situations, such as calculating the height of a building with a shadow or determining the elevation angle in navigation, the students can understand the concept of trigonometric ratios more meaningfully than simply memorizing formulas (M. van D. H. Panhuizen, 2003). In addition, RME encourages visual exploration using tools such as triangle diagrams, interactive math software, or physical props so that the students can directly observe the relationship between angles and side lengths in triangles (Freudenthal, 1991). This approach also emphasizes mathematical modelling, where the students learn and apply the theory to solve real problems, improving their conceptual understanding and critical thinking skills (Gravemeijer & Doorman, 1999). Thus, RME enables students to understand how to calculate trigonometric ratios, why these concepts are important, and how to use them in real-world situations.

RME emphasizes mathematization as the main principle to help students understand mathematical concepts gradually, starting from real situations to achieving formal forms. Hans Freudenthal (1991) developed the theory of mathematization in RME by distinguishing two types of mathematization, namely horizontal mathematization, which transforms real-world problems into mathematical models, and vertical mathematization, which processes mathematical models into more formal forms (Freudenthal, 1991).

In addition, the concept of mathematization in RME is also strengthened by Treffers (1987), who emphasized that mathematization is the core of the RME approach, allowing the students to explore and construct mathematical concepts through meaningful activities (Treffers, 1987). Meanwhile, Gravemeijer (1994) emphasized the importance of designing learning trajectories in RME that consider how students develop understanding progressively through the mathematization process (Gravemeijer, 1994).

In the RME approach, horizontal and vertical mathematization support modelling activities into right triangles (Freudenthal, 1991; Treffers, 1987). Horizontal mathematization occurs when the students translate real-world situations, such as measuring the length of a shadow and the height of a student and then recognizing the proportional relationship in the situation. Furthermore, through vertical mathematization, the students construct more formal mathematical models, for example, by comparing the height and shadow length ratio using the concept of triangle similarity or trigonometric ratios such as the tangent of an elevation angle.

In addition, the concept of a model as a bridge in RME helps students move from informal understanding to more abstract mathematical models (Gravemeijer, 1994). The students begin learning from real-world situation models, such as shadows formed by light in the classroom, then gradually move to formal mathematical models, such as right triangles and trigonometric ratios. These models serve as a tool to help the students understand how the relationship between elements in a physical system can be represented in a more abstract mathematical form. In this learning, a real-world context is an important starting point to facilitate the students' understanding of mathematical concepts (De Lange, 1996). Measuring the students' heights and shadows allows the students to understand the concepts of triangle similarity and trigonometric ratios more intuitively.

RME is based on the philosophy of Hans Freudenthal, who views mathematics as a human activity involving creativity and problem-solving (Freudenthal, 1973; Gravemeijer & Terwel, 2000; Treffers, 1987; Van den Heuvel-Panhuizen & Drijvers, 2020). This approach allows students to develop mathematical understanding through real-life contexts and experiences (Yuanita et al., 2018). Research by Aiyub et al. (2024) also highlights that learning designs incorporating real-world problems can improve students' algebraic thinking and overall mathematical understanding (Aiyub et al., 2024).

The students build understanding through informal processes. They observe, see, and understand the environment wherever they are (Meika, 2018). Realistic Mathematics Education encompasses three key concepts: directed reinvention and progressive mathematical processes, didactic phenomenology, and model construct development (Gravemeijer, 1994). Within the teaching and learning process, these three concepts are further developed into six distinct features of RME: 1) The principle of activity, 2) The principle of reality, 3) The principle of level, 4) The principle of intertwinement, 5) The principle of interactivity, and 6) The principle of guidance (M. V. den H. Panhuizen & Drijvers, 2014).

Based on Freudenthal's view of mathematics as a human activity, the activity principle emphasizes the student's active role in learning. The reality principle encourages learning through real-world problems that can be mathematized. The level principle describes the students' progression through different levels of understanding. The interconnection principle highlights integrating various concepts using multiple tools and methods. The interaction principle sees learning as a social and reflective process. Lastly, the guidance principle, rooted in Freudenthal's "guided reinvention," ensures learning aligns with a structured long-term trajectory. (M. V. den H. Panhuizen & Drijvers, 2014).

The students often have difficulty understanding the concept of trigonometric ratios, especially in relating the lengths of the sides of a triangle to the angles (Maknun et al., 2019). In addition, several studies have shown that they fail to understand the concept of trigonometric ratios even though they have been taught formally (Weber, 2005; Nordlander, 2022). Another difficulty often faced is distinguishing between the right triangle approach and the unit circle (Kamber & Takaci, 2017). Given the importance of understanding this concept, this study aims to develop a learning trajectory that helps students overcome these difficulties, with the RME approach as the main theoretical basis.

METHOD

The methodology employed is design research as described by Gravemeijer and Van Eerde in 2009 (Gravemeijer & Van Eerde, 2009). This research methodology is focused on learning trajectory to enhance the quality of learning by fostering effective collaboration between researchers and teachers (Gravemeijer & Van Eerde, 2009). Design research will enable the attainment of the goals of the research paradigm in the development of sequential activities and the enhancement of learning quality. The primary objective of this work is to develop a comparative trigonometry learning trajectory by utilizing the RME approach. The study was carried out in two cyclical phases, each following the design research stages outlined by Gravemeijer (2004): 1) preliminary design, 2) teaching experiment, and 3) retrospective analysis. The outcomes of the initial phase activities that have been executed and the subsequent analytical review have contributed valuable insights (Gravemeijer, 2004).

In the initial design stage, the researcher observed the school and interviewed teachers and students. Additionally, the researcher collected and analyzed various sources related to the challenges the students and teachers faced in understanding trigonometric ratios and reviewed materials on learning activities based on the RME approach. These analyses served as a reference for developing the Hypothetical Learning Trajectory (dHLT), which includes learning objectives, learning activities, and hypotheses on students' cognitive development (Simon, 1993). These learning hypotheses form the foundation for designing the learning activities developed by the researcher.

Learning objectives	Learning Activities	Student Thinking Conjecture
The students can identify the position of the right angle, opposite side, and hypotenuse in a right triangle.	Stretch the raffia from the table's base to the top of the head to form a right triangle.	A group of students can model the activity into a right triangle. One of the students was able to sketch the formed right triangle.

Learning objectives	Learning Activities	Student Thinking Conjecture
		The students can identify the position of right angles, front sides, hypotenuses, and side sides in sketches formed with various answers without knowing the reasons. The students need to be corrected in identifying the position of right angles.
The students can apply the definition of trigonometric ratios sin θ, cos θ, and tan θ to contextual problems using the concept of ratios in right triangles.	Measuring students' height and shadow	The students can draw sketches formed from the models presented. The students can use the definitions of the trigonometric ratios tan θ, sin θ, and cos θ to measure the heights of both students and their shadows. The students can determine the position θ of different angles to obtain different trigonometric ratio values.
The students can understand the relationship between similarity and trigonometric ratios.	Experiment with making three right triangles from paper	The students can make three similar right triangles from paper. The students can make sketches from paper formed into three similar right triangles. The students can identify the shortest and longest sides of each triangle formed. The students can determine the ratio between the shortest side and the longest side of each right triangle. The students can relate the concept of similarity to trigonometric ratios.
The students can determine the value of trigonometric ratios at particular angles (0^0, 30^0, 45	Observations on the calendar image in the form of an equilateral triangular prism.	The students can sketch a calendar model as an equilateral triangular prism. The students can construct a right triangle in which one of the angles corresponds to a specifically

Learning objectives	Learning Activities	Student Thinking Conjecture
^{0}, 60 ^{0}, and 90 ^{0}).		defined angle. The students can determine the value of trigonometric ratios at particular angles (0 ^{0}, 30 ^{0}, 45 ^{0}, 60 ^{0}, and 90 ^{0}).
Applying trigonometric ratios enables students to ascertain a right triangle's unknown length of sides	Bed or bunk bed observation	The students can observe bed or bunk beds by sketching or drawing bed models. The students can identify the position of right angles, oblique angles, side angles, and front side positions in sketches. The students can apply trigonometric ratios to determine the length of unknown sides in problems presented related to beds.
The students can solve contextual problems related to right triangles using trigonometric ratios.	Observing an image of a skier descending an iceberg	The students can identify the position of the angles, opposite sides, and hypotenuses in a right triangle, which is formed as a sketch of a picture of a skier descending an iceberg. The students can apply the definition of trigonometric ratios to skier problems. The students can solve the problems given in the case of skiers.

Table 1: Hypothetical Learning Trajectory in Teaching Experiment 1

Within phase 1, the research participants comprised 25 10th-grade students attending a Bandung City high school. In phase 2, the research participants comprised 25 10th-grade students from a different high school within the same municipal area. The initial implementation of the HLT took place in the first phase of the teaching experiment to assess the efficacy of the prepared learning activity design in investigating students' cognitive patterns.

Moreover, the results of teaching experiment 1 indicated the need to refine the Hypothetical Learning Trajectory (HLT). The enhancement consisted of restructuring the sequence of learning activities, providing more explicit scaffolding in constructing right-angled triangles, and incorporating intermediate modeling tasks to support students' transition from contextual situations to formal trigonometric representations. Additional guiding questions were also integrated to facilitate students' vertical mathematization. Furthermore, contextual problems were redesigned to better align with students' prior knowledge and to strengthen the progression from model-of to model-for reasoning. The revised HLT was thus grounded more explicitly in the principles of RME to address the identified difficulties.

First, the instructional design was improved by embedding more meaningful real-world contexts to help the students relate trigonometric concepts to their daily experiences, reinforcing the idea of mathematics as a human activity. Second, scaffolding strategies were strengthened by structuring learning activities progressively, guiding students from informal visual representations (e.g., real-life diagrams and sketches) to more formal mathematical models. Third, interactive learning moments were emphasized by incorporating collaborative discussions and teacher questioning techniques to encourage student reflection and active participation. These refinements maintain the core activities of the previous HLT while improving students' engagement with mathematical representations through a more realistic and interactive learning approach.

The revised Hypothetical Learning Trajectory rHLT was implemented in Phase 2 of the teaching experiment to examine its effectiveness in a different school setting and diverse learning environment. The data-collection protocols included classroom observations, video recordings of learning activities, students' workbooks, and interview recordings with students.

Before implementing the dHLT, a diagnostic test was conducted. The results showed that 47% of students had difficulty constructing right triangles, while 86% experienced difficulties in applying trigonometric ratios in contextual problems. After the intervention, the results of the final test indicated a significant improvement: 83% of students successfully constructed right triangles correctly, and 50% were able to apply trigonometric ratios appropriately. Classroom observations also revealed increased student engagement and improved accuracy in representing mathematical concepts. These results indicate that the revised Hypothetical Learning Trajectory (rHLT), which integrates real-world contexts, scaffolding, and interactive discussions, effectively helps students bridge their understanding from informal reasoning to formal mathematical representations, in line with the principles of RME.

Phase 1 of the retrospective analysis involved collecting and analyzing all research data. The analysis evaluated the outcomes of implementing the developed Hypothetical Learning Trajectory (dHLT) during Teaching Experiment 1. Furthermore, the researcher examined the learning activities to determine the extent of students' understanding of trigonometric ratio concepts. The analysis also evaluated the outcomes of the improvements made to the trajectory and their implementation during Teaching Experiment 2. This analysis compared the dHLT implemented in the first teaching experiment with the revised Hypothetical Learning Trajectory (rHLT) applied in the second teaching experiment.

The researchers selected six student participants for interviews. These participants represented different levels of mathematical ability: two students with high ability, two with medium ability, and two with low ability. In addition, interviews were conducted with the mathematics teacher who taught trigonometric ratios at the school where the study was conducted.

RESULTS

The learning trajectory design discussed in this paper consists of a series of student learning activities intended to support students' understanding of trigonometric ratios through the Realistic Mathematics Education (RME) approach. The developed Hypothetical Learning Trajectory (dHLT) in this study was constructed based on empirical data obtained from classroom observations, student interviews, and worksheet analyses during the learning process. The findings indicate that students experienced gradual progress in understanding trigonometric ratios through a sequence of exploration-based learning activities.

Based on observation data, the students initially experienced difficulty connecting the concept of trigonometric ratios with the right triangles encountered in everyday contexts. However, through a sequence of structured learning activities, their understanding gradually developed from concrete exploration toward a more formal understanding of trigonometric ratios.

Activity 1: Observation of "Stretching the Raffia from the Base of the Table to the Top of the Student's Head"

Activity 1 aims to help the students accurately determine the right angle, opposite side, and hypotenuse in a right triangle. The students, divided into groups of 4–5, use contextual equipment like raffia, a table, and a peer. One of the students holds one end of the raffia while the other is attached to the table base, forming a right triangle with the student's height and the floor. Observations are illustrated in Figure 3. Each group sketches their demonstration (Figure 4), identifying key triangle components. Students explain their reasoning for positioning angle θ and triangle sides. Those struggling receive guided reinvention through teacher-led discussions.

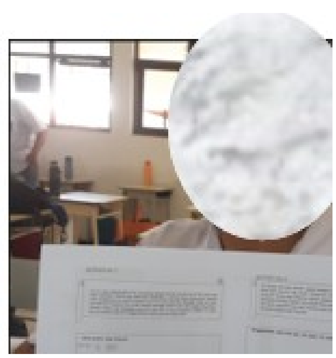


Figure 3: Activity 1 "Stretching a Raffia from the Base of the Table to the Top of a Student's Head"

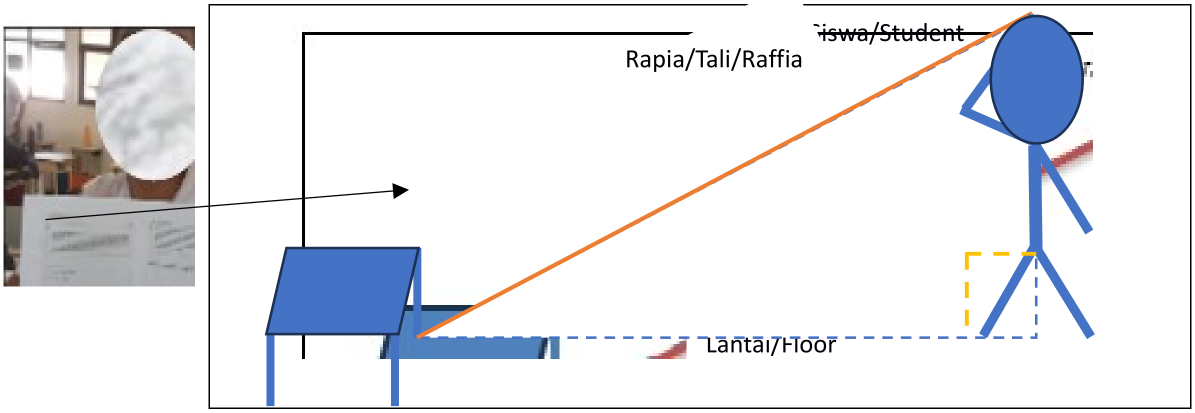


Figure 4: Example of the Student Sketches in Activity 1

Teacher	:	*"Okay, S-1, please explain what you drew."*
S-1	:	*"Okay, ma'am, if you look at the instructions in the worksheet, we are asked to practice what is written in activity 1, and then we share roles; S-2 ties the raffia to the base of the bottom table, S-3 holds the end of the raffia that results from extending S-2's raffia, S-4 reads what is on the worksheet, and I make the sketch."*
Teacher	:	*"Okay, have you considered whether your sketch will be a right triangle?"*
S-1	:	*"At first, I did not, Ma'am, but when you said, 'Look if you draw it, is a right angle formed?" I remembered."*
Teacher	:	*"Okay, how do you determine the corner points?"*
S-1	:	*"The benchmarks are at the end of the table legs, Leg S-3, and head S-3, Ma'am."*
Teacher	:	*"Why is that, S-1?"*
S-1	:	*"Yes, that is what it looks like when drawn, ma'am."*
Teacher	:	*"Okay, the benchmarks that S-1 mentioned earlier make a move from the rapid raffia, right?"*
S-1	:	*"What do you mean, ma'am?"*
Teacher	:	*"Mom, look, you represent the rapier raffia, the floor, and the height of S-2 with a line, right?"*
S-1	:	*"That is right, ma'am."*
Teacher	:	*"Well, when you make that line, there is a phase where you turn; that is called a*

r		*benchmark."*
S-1	:	*"Oh yes, that is right, ma'am, that is right."*
Teacher	:	*"S-1, try to determine the right angle, slanted side, front side, and side in the sketch you have made!"*
S-1	:	*"This one is a right angle, this one is the hypotenuse, this one is the front side, and this one is the side side." (pointing to the picture).*
Teacher	:	*"How do you determine a right angle, S-1?"*
S-1	:	*"There is a line like this and like this, ma'am." (pointing at the picture).*
Teacher	:	*"Now, try to observe what degree angle the two lines form, or what angle is it called then?"*
S-1	:	*"Oh yes, ma'am, right angles."*
Teacher	:	*"How do you determine the hypotenuse?"*
S-1	:	*"The side in front of the right angle, ma'am."*
Teacher	:	*"How do you determine the front side?"*
S-1	:	*"This one, ma'am? Yes, the one in front of this corner (pointing to one of the corners of the picture)".*
Teacher	:	*"Okay, how do you determine the side?"*
S-1	:	*"The one on the front side, ma'am."*

In this activity, the students identify the expected position, which depends on the angle θ. This process helps them determine angles and sides in a right triangle. After completing the task, the teacher and the students reflect on positioning the right angle, hypotenuse, and θ, which influence side identification. The raffia stretching activity from the table to a student's head helps visualize the relationship between height and shadow length. Observations indicate that the students begin recognizing this pattern, even without fully grasping the ratio concept, providing a concrete experience before introducing the formal definition.

Activity 2: Observation "Measurement of The Students' Height and Shadow"

Activity 2 aims to help the students apply trigonometric ratios (sin θ, cos θ, and tan θ) in real-life contexts using right triangle comparisons. The students work in groups to transform comparison problems into hands-on tasks. One student stands with their back to a lamp to create a shadow, which the group then sketches (Figure 5). Each member takes turns measuring their height and shadow length, recording the data in a table (Figure 6). The teacher guides the students in determining angle θ and calculating the height-to-shadow length ratio. This activity reinforces the concept of trigonometric ratios through visual and practical learning.

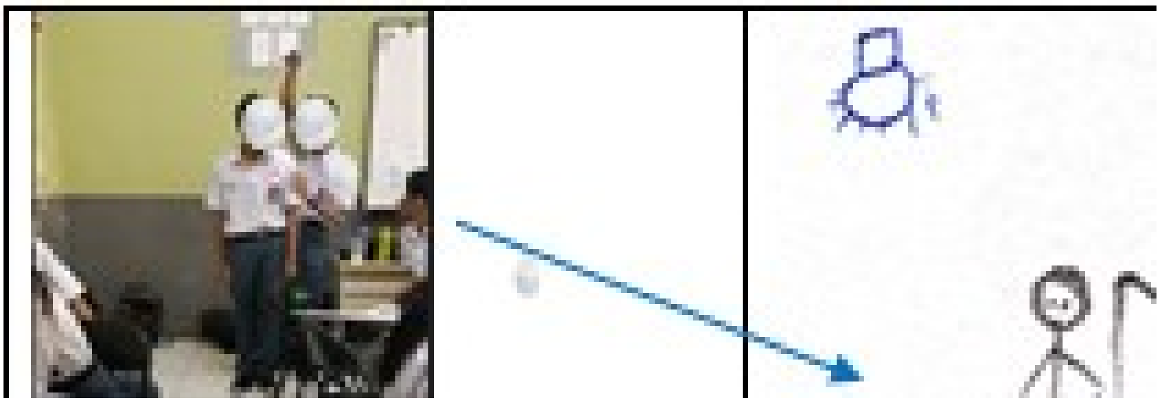
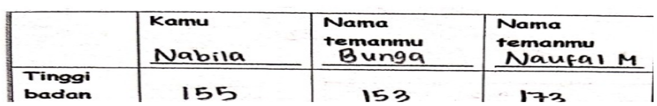


Figure 5: Activity 2 and Example of The Student Sketches in Activity 2



	Kamu Nabila	Nama temanmu Bunga	Nama temanmu Naufal M
Tinggi badan	155	153	173

	Your name Nabila	Your Classmate's name Bunga	Your Classmate's name Naufal M	Your Classmate's name Jahsyi	Your Classmate's name Nalvin
Height	155	153	173	169	161
Shadow Height	165	163	183	179	171

Figure 6: Example of a table of results measuring height and shadow in Activity 2.

This project aims to help the students accurately calculate sin θ, cos θ, and tan θ in their triangle sketches (Figure 7). The teacher monitors each group, guiding the students in problem-solving while allowing them to use formal or informal methods. Scaffolding is provided to help students recall and apply their knowledge of trigonometric ratios.

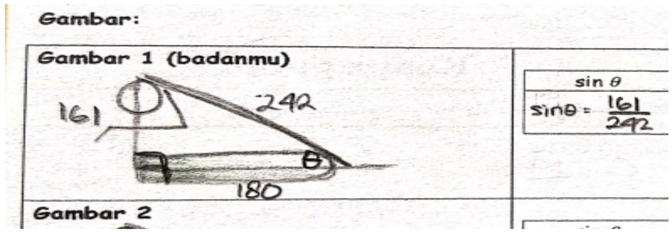


Figure 7: Example of Sketch and Application of Trigonometric Ratios Values in Activity 2

One of the students (S-5), as a representative of one of the group members, participated in presenting activity two and explained how the values of sin θ, cos θ, and tan θ were obtained. After sketching the experiment, this group identified the right angle to determine the other positions. The following is a dialogue explaining the concept of trigonometric ratios described by the students.

Teacher : *"Okay, S-5. Explain how your group determines each trigonometric ratio value requested in the table."*

S-5 : *"Okay, Ma'am, first we sketch according to our height in the group, then, from the sketch formed, we determine the right angle, draw the symbol on the sketch, then determine the angle θ....."*

Teacher : *"Okay, do you mean identifying the position of a right angle?*

S-5 : *"Yes, ma'am."*

Teacher : *"What do you mean by determining the angle?"*

S-5 : *"So we can determine the front and side, ma'am."*

Teacher : *"Okay, what is the next process?"*

S-5 : *"Earlier, after determining the angle position θ, we then wrote down each of our heights according to what was measured earlier and wrote down the height of the shadow."*

Teacher : *"Okay, how do you determine the hypotenuse?"*

S-5 : *"Using the Pythagorean theorem, ma'am, with the help of a calculator."*

Teacher : *"Okay, how do you calculate sin θ, cos θ, and tan θ?*

S-5 : *"Well, because the value of sin θ is the front per hypotenuse, look at the picture (while pointing to picture 1 in UBA 2 Activity 2); the front of the angle $\theta$is the front side, which is 161, and the hypotenuse is 242 so the sin θ is 161 per 242.*

Teacher : *"How to calculate cos θ and tan θ?*

S-5 : *"Same here, ma'am, just a different position according to what was learned previously: cos θmeans side per hypotenuse, and tan θmeans front per side."*

The conversation above helped all the students understand the concept of trigonometric ratios correctly. At the end of the activity, one of the students reflected on the lesson and concluded how to apply trigonometric ratios correctly. Measuring the height and shadow length helped the students understand that the ratio between an object's height and its image can be quantified. After multiple measurements, most students recognized a consistent pattern in this relationship, reinforcing the concept of constant ratios as the foundation for sine, cosine, and tangent.

Activity 3: Experiment with "Making Three Right Triangles from Paper"

Exercise 3 aims to help the students understand the relationship between similarity and trigonometric ratios. In groups of 4–5, the students use scissors, paper, rulers, and worksheets to cut rectangular paper into two equal right triangles and identify similarities in three right triangles. They explore similarity using diagrams and concrete teaching aids, as illustrated in Figures 8 and 9, where some groups present findings through diagrams while others use models. The teacher guides the students in recognizing corresponding sides and discovering formal patterns of similarity through directed questions, leading them to apply Geometry theorems by comparing three similar right triangles.

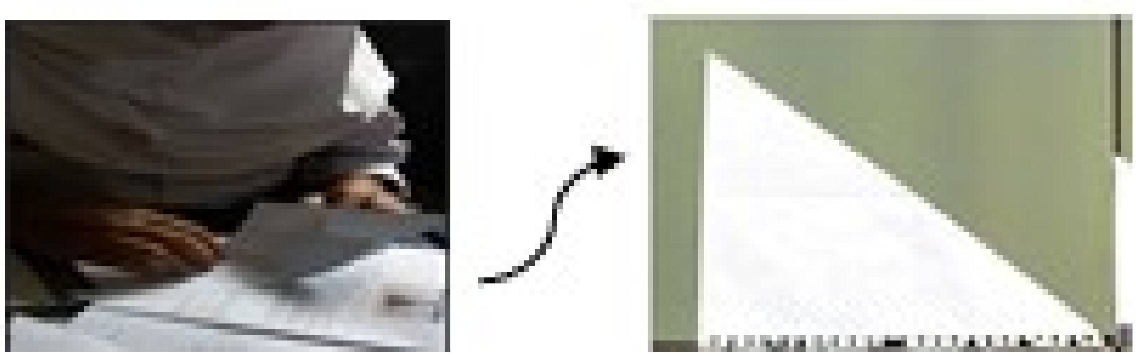


Figure 8. Activities and Examples of the Student Responses using Teaching Aids in Activity 3.

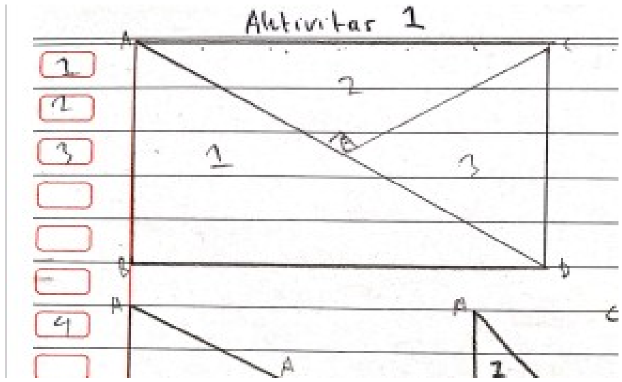
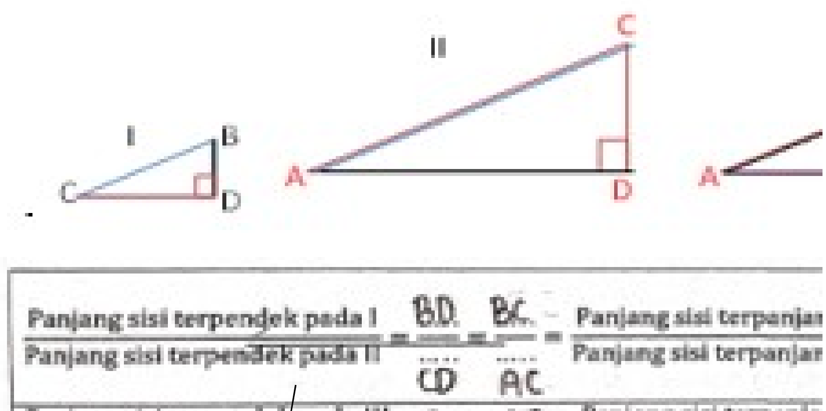


Figure 9. Example of the Student Responses using Diagrams in Activity 3

The teacher guides the students in identifying the shortest and longest sides of each triangle, helping them develop comparative thinking patterns (Figure 10). To deepen understanding, the teacher poses questions that draw analogies between similarity and trigonometric ratios (Figure 9). One example involves a logo on a garbage can placed on an equilateral triangle with a 6 cm side length. The students are asked to calculate the logo's height, sketch the triangle, and explain the concepts used in their solution.



$\frac{thelengthoftheshortestside\in figureI}{thelengthoftheshortestside\in figureII}=\frac{BD}{CD}=\frac{BC}{AC}=\frac{thelengthofthelongestside\in figureI}{thelengthofthelongestside\in figureII}$
$\frac{thelengthoftheshortestside\in figureIII}{thelengthoftheshortestside\in figureI}=\frac{BC}{BD}=\frac{AB}{BC}=\frac{thelengthofthelongestside\in figureIII}{thelengthofthelongestside\in figureI}$
$\frac{thelengthoftheshortestside\in figureIII}{thelengthoftheshortestside\in figureII}=\frac{BC}{CD}=\frac{AB}{AC}=\frac{thelengthofthelongestside\in figureIII}{thelengthofthelongestside\in figureII}$

Figure 10. Example of the Student Responses in Using the Concept of Comparison in Activity 3.

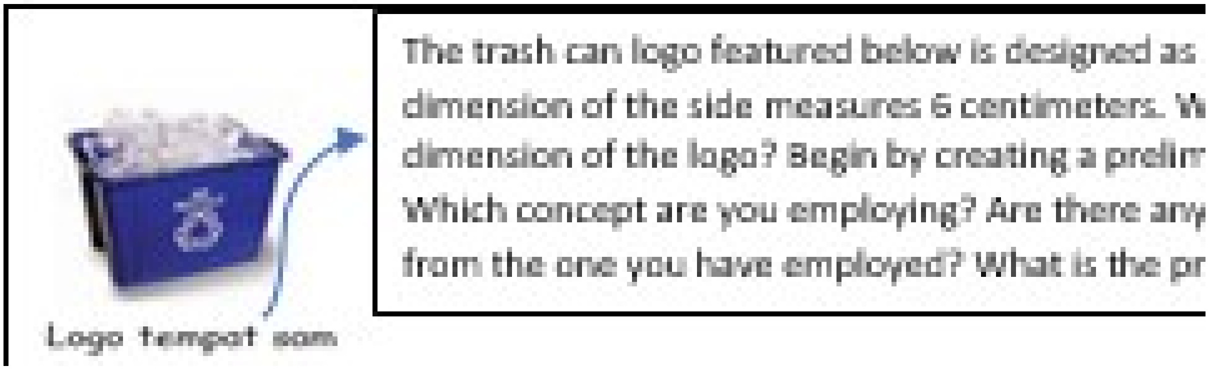


Figure 11. Example of Questions on the Relationship between the Concept of Similarity and Trigonometric Ratios in Activity 3.

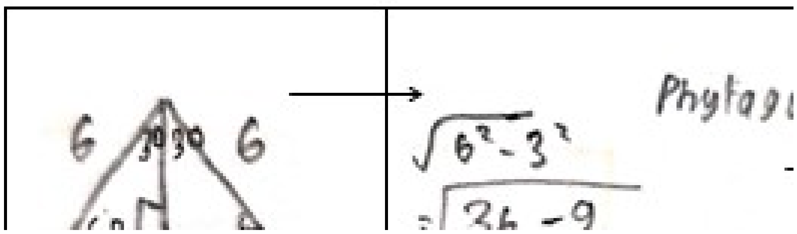


Figure 12. Example of the Student Responses to Contextual Questions in Activity 3.

In Figure 12, a group sketches the shape and calculates the logo's height using the Pythagorean Theorem, then verifies their results through similarity. The teacher and the students discuss the relationship between similarity and trigonometric ratios. By creating three right triangles of different sizes, the students observe that side length ratios remain constant for equal angles. Some believed the ratios would change initially, but experimentation confirmed their invariance, reinforcing a key principle of trigonometric functions.

Activity 4: Observation of "Calendar Image in the form of an Equilateral Triangular Prism"

Activity 4 helps the students determine trigonometric ratios for specific angles (0°, 30°, 45°, 60°, and 90°). Working in groups of 4–5, they analyze a calendar image as an equilateral triangular prism (Figure 13), extract an equilateral triangle, construct a symmetrical one with a 2-unit side, and divide it into two right triangles. Using the Pythagorean Theorem, they calculate unknown side lengths and determine trigonometric ratios for 30° and 60°, compiling their results in a table (Figure 14). For a 45° angle, the students draw an isosceles right triangle with 1-unit sides, calculate the hypotenuse, and determine sin θ, cos θ, and tan θ (Figure 15). By the end of the activity, the students summarize their findings in a table, reinforcing their understanding of trigonometric ratios.

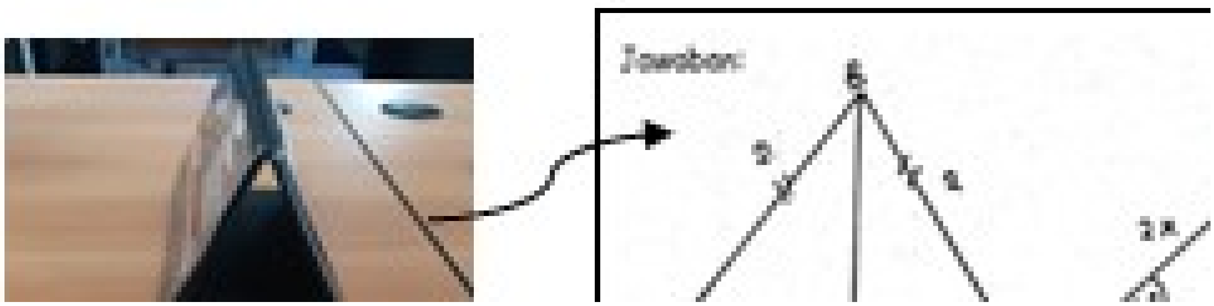


Figure 13. Calendar Image and Example of the Student Responses in Making Sketches in Activity 4

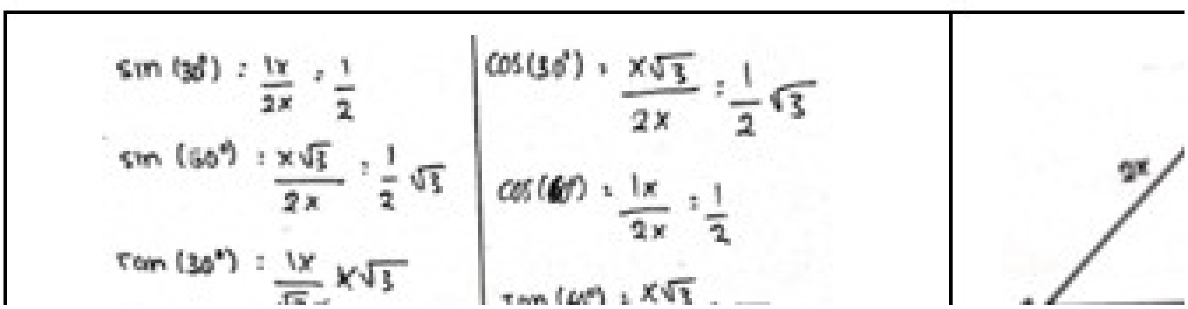


Figure 14. Example of the Student Responses regarding memorable and unique 30^0angles 60^0 in Activity 4

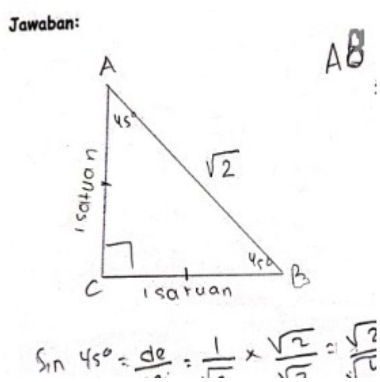


Figure 15. Example of the Student Responses Regarding Particular Angles 45^0 In Activity 4

	Rumus	30^0
sin	$\sin\theta = \frac{\text{sisi depan}}{\text{sisi miring}}$	$\frac{1}{2}$
cos	$\cos\theta = \frac{\text{sisi samping}}{\text{sisi miring}}$	$\frac{1}{2}\sqrt{3}$

Figure 16. Example of Student Responses Represented in a Table in Activity 4

By the end of the exercise, the teacher and the students had reached a general conclusion on determining the value of trigonometric ratios at particular angles.

Observing the calendar image as an equilateral triangle helps the students recognize right triangles in various visual contexts. This activity increases their ability to identify right triangles in everyday objects, making it easier for them to connect trigonometric ratios to real-world applications.

Activity 5: Observation of "Bunk Bed"

Exercise 5 helps the students use trigonometric ratios to determine unknown side lengths in a right triangle. The students work in groups of 4-5, analyzing an image (Figure 17) and a triangular figure formed by stairs, a bunk bed, and the floor (Figure 18). They then apply horizontal mathematization by creating a diagram representing the triangle. The students identify whether the shape is a right triangle and describe the given data in a sketch. Using the tangent ratio, they calculate the displacement between the floor and the end of the stairs.

Figure 17. Activities and Questions in Activity 5

Each group's findings from activity 5 are illustrated in Figure 18.

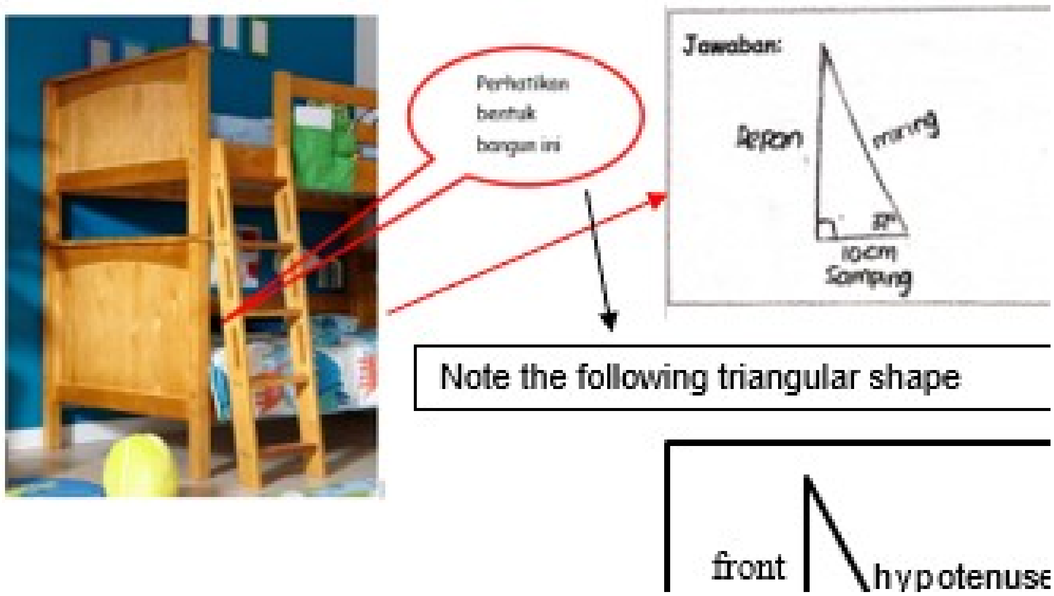


Figure 18. Students' sketches illustrating the results of Activity 5.

With teacher guidance, the students sketch a right triangle from the bunk bed, floor, and stairs, identifying the right angle and using the given values (37° and 10 cm) to calculate the height with the tangent ratio. After solving the problems, one group presents their results, explaining their understanding of trigonometric ratios in determining unknown side lengths. The teacher then asks questions to assess the student's comprehension, as shown in the dialogue below.

Teacher : *"Okay, how do you find the length of the unknown side in a right triangle?"*

S-6 : *"Begin by sketching the problem, identifying given information such as side lengths (opposite, hypotenuse, or adjacent) and known angles. Then, analyze the available data to determine the appropriate trigonometric ratio to apply."*

Teacher : *"Can you give an example?"*

S-6 : *"In the bunk bed problem, after sketching a right triangle, we label the given values: a 37° angle between the stairs and floor and a 10 cm distance from the bunk bed to the stairs. Using the tangent ratio, we calculate the unknown side length, which relates the opposite and adjacent sides."*

Observing the bed frame helps the students see how trigonometric ratios apply in construction. Those who initially struggled with trigonometry begin to connect triangle-side ratios to real-world structures. This activity reinforces the practical significance of trigonometric ratios in design and construction.

Activity 6: Observation of "Picture of a Skier Descending an Iceberg"

Task 6 helps the students apply trigonometric ratios to real-world problems by analyzing worksheet images (Figure 19) and solving open-ended questions in groups. Referring to Larson et al. (2007), they calculate a skier's track length on an iceberg (Larson et al., 2007), and one group presents their findings to conclude the learning outcomes (Figure 20).

Figure 19. Questions in Activity 6

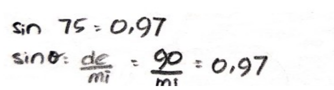


Figure 20. Example of the Student Responses to Activity 6

Observing an image of a skier descending an iceberg introduces the concept of slope about terrain and movement. The students connect slope angles to right triangle ratios, enhancing their understanding of the tangent function. This activity reinforces that trigonometry applies not only to static shapes but also to dynamic phenomena.

DISCUSSION

Learning activities using the Realistic Mathematics Education approach regarding trigonometric ratios followed by the students consist of six activities, namely raffia stretching activities, the student height measurement activities, activities to make three right triangles from paper, activities to observe calendar images, activities to observe bunk bed images, and activities to observe skier images. Each activity has its purpose and is interrelated and continuous. The goals of each action are outlined below: Firstly, stretching the raffia allows the students to determine the location of the angles and sides of a right triangle. Secondly, measuring the height of the students and their shadows allows the students to apply the concept of trigonometric ratios. 3) Creating a teaching tool including three right triangles is intended to facilitate the student's comprehension of the correlation between resemblance and trigonometric ratios. 4) Observing the calendar image permits the students to ascertain the numerical value of trigonometric ratios at specific angles. The objective of viewing the bunk bed image is to help the students apply trigonometric ratios to ascertain the length of the desired side in a right triangle. Similarly, observing the skier image seeks to empower the students in solving contextual difficulties by employing the concept of trigonometric ratios. Every worksheet provided incorporates the specific features of the Realistic Mathematics Education pedagogy about trigonometric ratios. These findings corroborate Subekti's assertion that student worksheets will enhance learning (Subekti & Prahmana, 2021).

Applying a realistic approach to teaching mathematics in various activities in the learning process should facilitate the student’s understanding of mathematical skills and allow teachers to help the students effectively achieve learning goals (Ekowati & Nenohai, 2017). Students' understanding improves because the process of learning mathematics that follows Realistic Mathematics Education utilizes the environment around the world to encourage students to produce ideas and develop their knowledge (Nirawati et al., 2021).

According to research done by Sukendra (2020), a series of learning process activities with the RME approach can significantly facilitate students' understanding of the trigonometric comparison concept (Sukendra, 2020). Therefore, deliberate learning paths encourage students to build their knowledge, improve their current skills, and effectively overcome challenges in many subjects (Prahmana et al., 2017; Nursyahidah et al., 2020; Wijaya et al., 2021; Kuncoro et al., 2023).

CONCLUSIONS

This study aims to develop a learning trajectory for trigonometric ratios using the Realistic Mathematics Education (RME) approach. Prior to the intervention, the results of the diagnostic test indicated that many students experienced difficulties in understanding trigonometric ratio concepts, particularly in constructing right triangles and applying trigonometric ratios in contextual problems. The findings of this study indicate that the implementation of the revised Hypothetical Learning Trajectory (rHLT) based on the RME approach can improve students’ mathematical proficiency in understanding trigonometric ratio concepts.

The learning trajectory designed in this study consists of six learning activities. Through a sequence of continuous learning activities, students gradually develop their conceptual understanding of trigonometric ratios. Their understanding develops from identifying the positions of angles and sides, applying the definition of trigonometric ratios in contextual situations, understanding the relationship between similarity and trigonometric ratios, determining the values of trigonometric ratios at specific angles, applying trigonometric ratios, and solving contextual problems involving trigonometric ratios.

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