Teaching Division of Two Natural Numbers to Primary School Students Independently of Multiplication Through Realistic Mathematics Education

Nguyen Thi Hong Duyen, Nguyen Phu Loc*

Can Tho University, Can Tho city, Vietnam

* nploc@ctu.edu.vn

Abstract: This study investigates the necessity of teaching the division of natural numbers to primary school students independently of multiplication, addressing the challenges learners face. A framework grounded in Realistic Mathematics Education (RME) guided the development of a Hypothetical Learning Trajectory (HLT) that included specific learning activities, objectives, and teachers' expectations. In alignment with the designed HLT, an instructional intervention consisting of four lessons was conducted with 11 first-grade students who had completed their curriculum but had not yet learned the division and multiplication of two natural numbers. The results of this descriptive study indicate that the HLT effectively created an environment in which students could experience mathematization to grasp the concept. It enabled students to distinguish between the two types of division, write the correct division equation, and verbally describe appropriate real-world situations corresponding to division, thereby enhancing their overall comprehension.

Keywords: Realistic Mathematics Education, mathematization, division of two natural numbers, primary school mathematics.

INTRODUCTION

The division of natural numbers is a fundamental concept in primary school mathematics education, often taught in conjunction with multiplication due to their close relationship. Multiplication and division are inverse operations, and this connection is commonly leveraged in traditional teaching methods. To deepen students' understanding of division, it is widely agreed that a more robust integration of multiplication and division is necessary. Strengthening the conceptual connection between these two operations will enhance students' ability to grasp and apply division more effectively (Downton, 2008; Robinson & LeFevre, 2012).

Although the conceptual link between multiplication and division is commonly emphasized in teaching, students frequently encounter significant challenges when learning division. These difficulties have been well-documented in studies such as that of Alghamdi et al. (2020), who highlighted several specific difficulties in students' understanding of division. For instance, students often struggle to distinguish between partitive division (dividing into equal parts) and quotitive division (determining how many times a divisor fits into a dividend), with the latter being more abstract and challenging. Additionally, they face difficulties in understanding the inverse relationship between divisor and quotient when the dividend remains constant. Lastly, interpreting remainders in real-life contexts presents another challenge, where students may apply division correctly but fail to interpret remainders correctly or to appropriate problem-solving strategies (Pacheco-Muñoz et al., 2023).

In addition, the research team conducted a survey (see Figure 1 below) to assess the ability of 273 third-grade students from three public schools in Vietnam to understand and apply division of two natural numbers. The survey, conducted in November 2022, found that despite having studied both multiplication and division, a significant number of students struggled. Specifically, 39.19% of students answered question 1 incorrectly, 65.57% struggled with question 2, and 63% answered question 3 incorrectly, indicating ongoing confusion in understanding and applying the concept of division. This result underscores the difficulties that third-grade students face in grasping the concept of division, despite prior exposure to both division and multiplication.

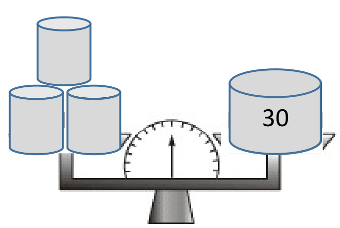
1. Có 12 người khách cần sang sông, mỗi thuyền đều chở 4 người khách (không kể người lái thuyền). Hỏi cần mấy thuyền để chở hết số khách đó? (khoanh tròn đáp án đúng) [The translation of Question 1: There are 12 passengers who need to cross the river, each boat carries 4 passengers (not including the boat driver). How many boats are needed to carry all those passengers? (circle the correct answer)]

A. 3 B. 4 C. 8 D. 48

2. Các ghế dài được dành cho trẻ em ngồi xem phim. Mỗi ghế dài ngồi được 3 em. Hỏi cần ít nhất bao nhiêu ghế dài như vậy để đủ chỗ ngồi cho 25 em? (khoanh tròn đáp án đúng) [The translation of Question 2: Benches are reserved for children to sit and watch movies. Each bench can seat 3 children. How many benches are needed to accommodate 25 children? (circle the correct answer)

A.7 B. 8 C. 9 D. 10

3. Điền số thích hợp vào chỗ trống.



Có …….. quả cân trên dĩa cân bên trái.

Các quả cân (trên dĩa cân bên trái) nặng bằng nhau và nặng …….kg.

[The translation of Question 3: Fill in the appropriate number in the blank. (Image). There are …….. weights on the scale on the left. The weights (on the scale on the left) weigh the same and weigh …….kg.]

Figure 1. Questions in the survey

Additionally, according to a recent study conducted in Germany, about 20% of 15-year-old students failed to develop an understanding of the four basic arithmetic operations during their schooling, with division being the most challenging operation for them. The study also revealed that the development of conceptual understanding of division heavily relies on language structures that express the relationship between division and multiplication, as well as on the students' ability to verbalize the concepts of division (Götze, 2018). This suggests that foundational gaps in learning division at an early age may continue into later years, further emphasizing the importance of effective instruction in primary school.

To address these challenges, one of the most promising alternative instructional methods is Realistic Mathematics Education (RME). Developed by Dutch mathematician Hans Freudenthal, RME encourages students to engage with mathematics through situations that are meaningful and relevant to them, allowing them to construct a deeper conceptual understanding. In fact, RME positively impacts mathematics reading comprehension, helping students flexibly apply knowledge to real-life situations (Ariati et al., 2022). Moreover, RME also creates a positive learning environment, encourages the students' active participation and contributes to improved overall learning outcomes (Ardiyani et al., 2018).

In addition, this approach aligns with Vietnam's current educational trends (Ministry of Education and Training – Vietnam, 2018a, 2018b). Research shows that integrating RME enhances students' math performance and attitudes by emphasizing real-world problem-solving (Tong et al., 2022). RME supports educational reforms, particularly in curriculum and textbook development (Trung et al., 2019). A framework for RME implementation highlights key factors like national vision, curriculum, assessment, and teacher training (Trung et al., 2020). Case studies show RME improves learning outcomes and student engagement compared to traditional methods (Tong et al., 2022; Loc & Tien, 2020; Duyen & Loc, 2022; Ha et al., 2021).

Gravemeijer (1997) emphasizes the need for intermediate models, such as visual tools and manipulatives, to help students better understand complex mathematical concepts. While division is often introduced through its relationship with multiplication, the RME approach allows students to explore division through real-world problems before strongly associating it with multiplication. Furthermore, teaching division in the context of multiplication may create barriers for students who struggle with multiplication or fail to grasp the inverse relationship, and limit students' ability to apply the concept flexibly in real-life situations, where division is often encountered without explicit reference to multiplication.

The current research on division and RME focuses largely on the benefits of teaching division within the framework of multiplication. However, there is a noticeable gap in the literature regarding the potential impact of teaching division independently of multiplication. By addressing this gap, this study seeks to provide insights into how RME may foster a deeper understanding of division as a stand-alone concept, particularly in real-world contexts where equal sharing is essential. Thus, this study aimed to answer the following question:

Research question: *How does RME affect primary school students*' *understanding of division of two natural numbers, especially when the division is taught independently of multiplication through real-life equal-sharing situations?*

By exploring how RME can foster a deeper understanding of division as a stand-alone concept, this study aims to fill a critical gap in the literature. The study's findings contribute to developing a practical approach for teaching math in general and division of two natural numbers to primary school students in particular.

LITERATURE REVIEW

Contexts (contexts, realistic contexts, contextual problems) play an essential role in RME theory. Contextual problems are defined as "problems where the problem situation is experientially real to the student" (Gravemeijer & Doorman, 1999, p.111). The key point is that these contexts are situations or problems suitable for mathematization. That means these contexts could make it easier for primary students to imagine and visualize abstract mathematical concepts (Duyen & Loc, 2022). Johar et al. (2017) also asserted that problems placed in real-world contexts enable students to start from simple, informal solutions and gradually move to more complex, formal solutions. This suggests that context is a starting point and a bridge that helps students transition from intuitive to abstract thinking. In addition, they should be *relevant and essential contexts*, rather than *camouflage contexts*. For example, the problem "There are 2163 notebooks arranged evenly in 7 boxes. How many notebooks are in those 5 boxes?" is a type of *camouflage* context (Son et al., 2022). In contrast, *relevant and essential contexts* create opportunities for constructing new mathematical concepts (just new for learners). For instance, the context as "One carton of milk can fill 5 glasses. There are 42 students in the class. Each student receives 1 glass of milk. How many cartons of milk do we need? Fill in the answer in the blank box." (see Figure 2) can be considered as a suitable context and contributes to learning the knowledge of multiplication and division of two natural numbers.

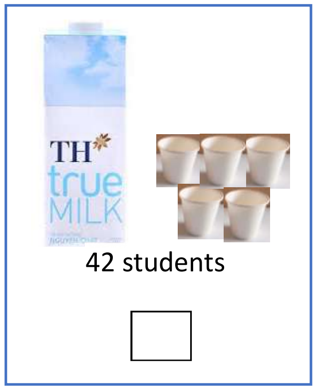


Figure 2. An example of relevant and essential contexts

In fact, students can actively explore, build and develop mathematical thinking naturally and flexibly thanks to *relevant and essential contexts.* Contexts are also tools for forming mathematical concepts and models, serving as foundations for applying knowledge, practicing skills (Treffers, 1994), and developing diverse and flexible ways to solve problems—thereby promoting the development of mathematical thinking at various levels (Widjaja et al., 2010).

In RME, Freudenthal (1991) emphasized that learning mathematics begins with students transforming real-life situations into mathematical problems – a process known as *mathematical reinvention*. Students use their existing knowledge and mathematical tools to explore and build new mathematical concepts. Students naturally form and develop mathematical ideas after being exposed to many real-life situations. In this process, the class teacher plays an essential role in creating a learning environment that helps students explore mathematical knowledge based on their existing understanding (Treffers, 1991). The process in which students take action to reinvent mathematics is called *mathematization*. It has been classified into two types: horizontal mathematization and vertical mathematization (Treffers, 1987). Horizontal mathematization is a process in which students use their solutions to describe a contextual problem by symbols. These symbols can be mathematics (formulas, algorithms, etc.) or not (figures, diagrams, etc.). Students can explicitly present them, or they even can visualize or imagine them. In the world of symbols, students continue to use their known mathematics to take action to find out new mathematics knowledge. It helps students answer the problem situated in the context problem. The process in which students take action within the mathematics system is called vertical mathematization (Duyen & Loc, 2022). Horizontal and vertical mathematization in RME can be summarized as Figure 3 below.

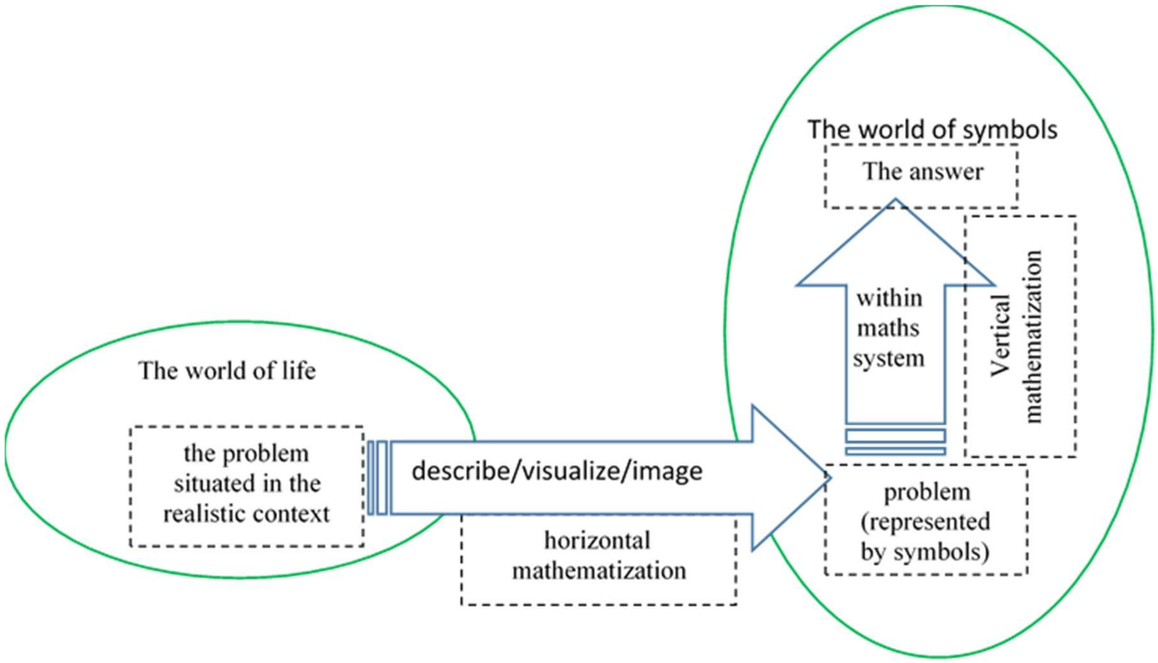


Figure 3. Horizontal mathematization and vertical mathematization in RME

It can be seen that guided reinvention via progressive mathematization is one of the major elements for designing education in RME. According to this principle, Gravemeijer (1994) argued that a Hypothetical Learning Trajectory (HLT) must be mapped out, allowing students to experience progressive mathematization. The HLT, also called a learning design, includes learning activities as well as objectives of these activities and the teacher's conjecture about how students learn and think (Simon, 1995). That means the teacher has the responsibility for providing a suitable series of contextual problems that can map out a possible HLT and in which students can experience "progressive mathematization" as a gradual change. By applying HLT in RME, teachers can create meaningful learning experiences that help students develop mathematical skills naturally and effectively. Research by Nuraida and Amam (2019) showed that HLT in RME significantly improved the mathematical communication skills of middle school students. Moreover, the HLT also helps teachers identify both the starting point and destination in the students' learning process. For example, Febriani and Sidik's (2020) study used the HLT to design activities that helped fourth graders effectively connect prior knowledge to new knowledge. In addition, Napitupulu et al.'s (2021) study also confirmed the value of the HLT in teaching complex mathematical concepts such as least common multiples.

In addition, when designing education in RME, attention should be paid to the principle of self-developed models, particularly to the role of self-developed models in students' learning processes. This principle is essential because self-developed models, as also known as emergent models, which enable students to bridge the gap between informal and formal mathematical knowledge. Accordingly, teachers must create opportunities for students to use and develop their own models when solving contextual problems. At first, students will develop a model that is familiar to them. After generalization and formalization, the model gradually becomes a general model to approach formal knowledge. Gravemeijer called this process the transformation from model-of to model-for (Gravemeijer, 1994, 1999). After the transformation, the model can be used as a model for mathematical reasoning (Treffers, 1991).

In this study, this principle is applied when designing activities to help students transition from concrete representations to abstract understanding. To help students practically and visually grasp the division of two natural numbers, physical objects such as math-link cubes, flowers and vases, or cakes/candies and plates are used to form a model-of. For example, in the context of partitive division, to illustrate 12 ÷ 4=3, twelve math-link cubes are arranged into 4 equal groups, each containing 3 cubes. After this, the class teacher asks questions like: "How many cubes are in each group?", "What is 12 divided by 4?". These questions serve as a model-for, guiding students toward understanding partitive division (dividing into equal parts). In Figure 4, each group of 3 cubes represents partitive division, where the total (12) is divided into equal parts (3).

Figure 4. A concrete model-of to help students engage with 12: 4=3

Similarly, for the quotitive division problem 12: 4=3, students might be asked to repeatedly form groups of 3 with 12 math-link cubes. This physical grouping gives them a concrete model-of, as shown in Figure 4. After this, the class teacher asks: "How many groups of 3 can be made from 12?". This question serves as a model-for, guiding students toward understanding quotitive division (determining how many times a divisor fits into a dividend). As students become familiar with this concrete process, they begin to see that quotitive division involves determining "how many groups of 3 can be made from 12" and they no longer need to rely on physical objects but can mentally or visually imagine the grouping process.

METHOD

This study employed a descriptive research design with an educational intervention to explore an instructional approach for teaching division to primary students. The research was grounded in RME, which informed the design of an HLT. The intervention facilitated hands-on exploration and practical application of theoretical concepts in a classroom setting, enabling the assessment of students' understanding and engagement. This approach ensures a comprehensive framework for teaching, blending theory with practice to enhance learning outcomes.

Researchers performed the study in the following three stages:

Before Conducting the Intervention

During this phase, an HLT for the division of two natural numbers was developed, followed by a-priori analysis. The HLT comprised nine activities, structured into four lesson plans that aligned with four experimental lessons, detailing classroom organization. Both the HLT and lesson plans were thoroughly reviewed and refined through collaboration between the classroom teacher, a veteran with over 14 years of teaching experience, and the researcher, ensuring alignment with pedagogical goals.

While Implementing the Intervention

In alignment with the designed HLT, the teaching experiment consisted of four lessons conducted from July 21, 2023, to July 23, 2023. These lessons were recorded on video while classroom observations were carried out, as shown in Table 1. The study involved 11 students (7 girls and 4 boys) who had completed the first grade curriculum, but had not learned division and multiplication of two natural numbers yet. This allowed for a focused examination of teaching division independently, which is crucial since second-graders would already have learned multiplication and therefore wouldn't meet the study's criteria. All students participating in the experiment lived in Soc Trang province. They had completed grade 1 in the 2022-2023 academic year at two public primary schools located in Soc Trang province. The participants of the study were determined by using convenience sampling. They were chosen because they were willing to participate in the teaching-learning process after receiving an invitation sent by the researcher.

Lesson Title	**Dividing equally**	**Division**	**Division (continued)**	**Division (continued)**
Date	21/7/2023	21/7/2023	22/7/2023	22/7/2023
Activities in HLT	1, 2, 3	4, 5, 6	3, 5, 7, 8	9
Worksheet	Worksheet 1	Worksheet 2		Worksheet A Worksheet 3

Table 1. Four intervention lessons

Table 1 summarizes four intervention lessons on division, detailing their titles, dates, activities in the HLT, and four associated worksheets. Worksheet 1 was used in the first lesson, Worksheet 2 was used in the second lesson, and Worksheet A and Worksheet 3 were used in the last lesson.

After the Intervention

Based on these videos, researchers continued to document the observed teaching-learning process. In addition, students' responses on the four worksheets were collected. Based on data obtained from classroom observations and worksheets, the authors aimed to conduct a qualitative analysis from the perspective of RME to evaluate the feasibility and effectiveness of the designed HLT, focusing on students' activities, attitudes towards math, and learning outcomes.

To examine the effectiveness of the intervention, criteria for evaluating students' understanding of the division of two natural numbers were established, and four worksheets were designed and used as tools for collecting students' responses, as shown in Table 2. Additionally, verbal responses and group work products, along with the HLT, were collected to assess students' level of understanding.

No	Criteria	Tools
1.	Understand: Correctly answer the question "How many [object X] does each part have? " after dividing *a* [object X] equally into *b parts.*	Worksheet 1
2.	Know: Write appropriate division equations for situations involving equal division.	Worksheet 2
3.	Know: Write the appropriate division equations based on students' manipulation of dividing objects equally.	Worksheet A
4.	Understand: Correctly write two division equations corresponding to a given image, where the image represents a context of equal groups.	Worksheet 3

Table 2. Criteria and tools for collection data

RESULTS

The HLT for the Concept of Division of Two Natural Numbers According to RME

We determined that the HLT should include the activities shown in Table 3. All activities are necessary to provide opportunities for students to experience mathematization when solving contextual problems. In the HLT, students can participate in solving contextual problems by distributing the quantities described in the problems into equal parts. The action that students perform to solve these problems involves manipulating objects to divide them equally. Consequently, the teacher will help students generalize this into new knowledge: the division of two natural numbers.

No.	Activity	Objectives
1.	Students solve contextual problems by distributing into equal parts	By solving contextual problems, students gain an understanding of partitive division, where they determine how to divide a total equally, equal parts and answer the question: How many [object X] are in each part?
2.	The teacher guides students to express how to divide a total equally	Through the equal division process implemented in Activity 1, guided by teachers, students learn that to divide all objects (X), they distribute a certain number of X into each part and repeat the process until X is fully distributed.
3.	Students manipulate the objects to divide them into b equal parts	Continue reinforcing students' understanding of partitive division and equal parts, guiding them to answer the question: How many [object X] are in each part?
4.	The teacher introduces the concepts of the division of two natural numbers, the symbol ":", and how to read.	Students understand $a:b=q$, where a, b, q are specific non-zero natural numbers, b represents the known number of parts/groups before dividing, and q is the result found after dividing - the number of [object X] in each part. The division operations from Activity 1 serve as examples.
5.	Write appropriate division equations for dividing objects or pictures/video clips.	Continue reinforcing students' understanding of partitive division by having them write the exact division equation and verbally state the situations that are appropriate to that division.
6.	State the appropriate situation for the given division.	
7.	Students solve contextual problems: There are a objects X, if we place q objects X in each part, how many parts will there be?	Students learn how to divide by grouping; that is, they take q objects X for each group and count the number of groups. This allows them to correctly answer the question: How many groups were formed?

8.	The teacher introduces the concept of quotitive division	Students understand that in the equation $a\!:\!b=q$, where a, b, q are specific non-zero natural numbers, dividing each part with q [object X], results in b equal parts. The division operations presented in Activity 7 serve as examples.
9.	Distinguish between partitive division and quotitive division	Students can write the correct division equation and verbally state the appropriate situations that correspond to that division.

Table 3. The HLT for the division of two natural numbers

Analysis of Nine Activities in the HLT

Activity 1: Students solve contextual problems by distributing into equal parts

Task 1. Divide the class into groups

- **Describe**: Suppose the class has a student, and the teacher announces the need to divide the class into b equal groups.
- **Analysis**: The teacher will assign the number of groups by handing out group name tags to a few representative students (for example, if there are 40 students and the class needs to be divided into 10 groups, the teacher will assign 10 students to receive group name tags numbered from 1 to 10). After that, the students will move to the corresponding groups and adjust the numbers so that the groups are equal.

Task 2. Divide the cakes/candies evenly among the plates

- **Describe**: The number of candies and plates given to each group is the same. For example, the task is to divide 12 cakes equally into 3 plates. The teacher gives students exactly 12 cakes and 3 plates.
- **Analysis**: In this activity, we expect students to divide the cakes in their own ways, as long as they ensure that the number of cakes on each plate is equal.

Task 3. Arrange a flowers evenly into b vases

- **Describe**: Round 1: 6 flowers into 2 vases; Round 2: 10 flowers into 2 vases; Round 3: 6 flowers into 3 vases; Round 4: 15 flowers into 3 vases.

- **Analysis:** In this activity, the number of flowers is divisible by the number of vases. Similar to Task 2, students will find their own ways to divide the flowers equally.

Activity 2: **The teacher guides students on how to divide equally**

Through the process of equal division in Activity 1, guided by teachers, students learn that to divide all objects (X), they distribute a certain nuamber of X into each part and repeat the process until X is fully distributed.

Activity 3: **Students manipulate objects (magnetic dots/math-link cubes) to divide them into b equal parts**

Task 1. Manipulate the magnetic dots to divide them into b equal parts

- **Describe**: The number of magnetic dots stuck on the board is a. The teacher will call each student in turn to divide the dots equally into b parts (b are 2, 3, 4, respectively). The whole class observes and gives feedback.
- **Analysis**: In each turn, the number of dots remains constant (e.g., always 12 dots) while the number of equal parts changes. This helps emphasize the concept of parts for the students.

Task 2. Manipulate the math link-cubes to divide them into b equal parts

- **Describe**: Students work individually or in groups on 12 math link-cubes, completing 4 turns where they divide the 12 cubes equally into 2 parts, 3 parts, 4 parts and 6 parts.
- **Analysis**: In this activity, students will employ strategies for dividing and will be successful by recognizing the equality of the parts through the visual representation of the cubes.

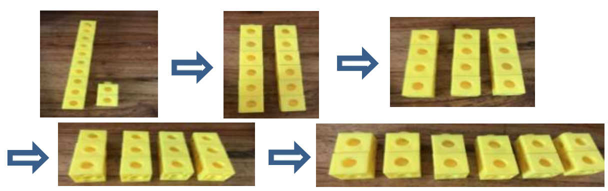


Figure 5. Divide 12 math link-cubes into b equal parts

Activity 4: The teacher introduces concepts of the division of two natural numbers, the symbol ":", and how to read

The teacher introduces to students: $a: b=q$ is a notation used to represent the act of dividing equally, where a, b and q are specific, non-zero natural numbers.

Activity 5: Write appropriate division equations for dividing objects or pictures/video clips

Task 1. Write appropriate division equations based on the manipulation of partitive division

- **Describe**: In this task, students work both individually and in groups with math-link cubes. First, students work individually to divide 12 cubes into 2 equal parts and write a division equation, divide 12 cubes into 6 equal parts and write a division equation. Students then work in groups with a block of 16 math-link cubes that are connected together. With the requirement to "divide this block of 16 math-link cubes evenly and then write the appropriate division equation", the student or group that writes the most division equations wins.
- **Analysis**: In this task, the dividend is 16, which limits students mentally from calculating the quantity in each part. Firstly, students must determine the number of parts. Then, they position each part by one math-link cube. Next, they equally distribute many math-link cubes into each part. Finally, they write the corresponding division equation.

Task 2. Write the appropriate division equation for the observed image

In this task, students work individually, observe the image and write division equations on Worksheet 2.

Activity 6: State the appropriate situation for a given division equation

Task. Identify real-life situations that can be represented by a given division equation.

- **Describe**: The teacher presents students with various division equations (e.g., $12: 3=4$, $15: 5=3$). Students work individually or in groups to create appropriate word problems or scenarios that correspond to each division equation. For example, for the equation $12: 3=4$, a possible scenario could be "If there are 12 apples and we want to put them into 3 baskets, how many apples will be in each basket?"
- **Analysis:** This activity encourages students to think critically about how division can be applied to real-life situations. By creating their own scenarios, students

deepen their understanding of the concept of division and its practical applications.

Activity 7: Solve contextual problems like "There are $a$objects X. Divide them so that each part contains q objects X. How many parts do we have?"

Task 1. How many gift bags?

Each group receives a cakes và b transparent bags with the requirement that each gift bag must contain q cakes.

Task 2. How many flower vases can be arranged?

- **Describe**: Each group receives a flowers và b vases with the requirement that each vase must contain q flowers.
- **Analysis**: Activity 7 is designed to achieve the objectives that students know how to take a group of q objects X for each part, then count the number of parts. In Task 1 and Task 2, students may initially divide incorrectly if they do not pay attention to the requirement. For example, in Task 1, if a group has 6 cakes and 6 bags but does not notice that each bag must have 2 cakes, students may mistakenly divide 6 cakes into 6 bags equally. This activity emphasizes correct grouping to ensure accurate division.

Task 3. How many numbers n can be separated? (n is the number made up of n math-link cubes)

- **Describe**: Each student receives 12 math-link cubes. Ask students to manipulate the cubes and answer the following questions: "From this number 12, how many numbers 3 can be separated?", "From this number 12, how many numbers 4 can be separated?", "From this number 12, how many numbers 2 can be separated?", "From this number 12, how many numbers 6 can be separated?", "From this number 12, how many numbers 1 can be separated?".
- **Analysis**: This task helps students mathematize one cube as number 1, two cubes as number 2, three cubes as number 3, and twelve cubes as number 12. This task also helps students to be familiar with seeing a number as the sum of equal numbers.

Activity 8: Introduces the division in the context of quotitive division

The teacher places a dots on the board (e.g., a=8) for the whole class to observe. Students are asked to verbally state the situation and write appropriate division equations based on the observed image. Once students provide the correct answers, the teacher emphasizes: In the division $8:2$=4, we know the number of parts is 2, so we divide the dots equally into 2 parts. Next, the teacher introduces the concept of quotitive division and presents the corresponding division equation as $8:4$=2. The teacher highlights: In this division, we know that each part must have 4 dots. With 8 dots, we can divide them into 2 parts.

Activity 9: **Distinguish between partitive division and quotitive division**

Task 1. Two divisions with flowers and vases

- **Describe**: In this task, vases are created by gluing colored craft paper onto an A4 sheet, forming vase shapes, so that hand-made flowers can be placed in them. Each A4 sheet has three vases. Each group receives six vases (on two A4 sheets) and fifteen flowers. Students discuss how to evenly arrange the flowers into the vases based on the teacher's instructions. Afterward, they write the appropriate division equation and verbally describe the corresponding situations. For example, when students are asked to "evenly place fifteen flowers into three vases" or "place five flowers into each vase", their arrangement will match what is illustrated in Figure 6.

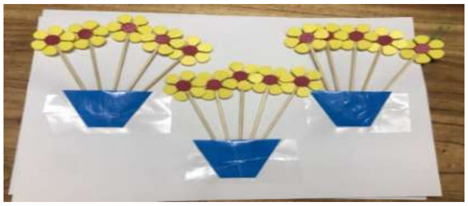
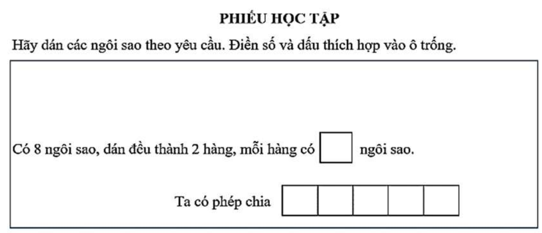


Figure 6. 15 flowers into three vases

- **Analysis**: This task is designed to help students understand the concept of division by both equally distributing objects and identifying groups. By physically arranging flowers into vases, students can see how division works in practice. The first scenario: evenly placing fifteen flowers into three vases demonstrates division when the total number of objects is known and the goal is to divide them into a specific number of parts (vases), which represents partitive division. The second scenario: placing five flowers into each vase helps students conceptualize division where the number of objects per part is fixed, and the task is to determine how many parts (vases) can be formed, which illustrates quotitive division.

Task 2. Two divisions with stars (stickers) and rows

- **Describe**: Each student receives two rows of stickers, with eight stars in each row, and Worksheet A as shown in Figure 7.


PHIẾU HỌC TẬP

Hãy dán các ngôi sao theo yêu cầu. Điền số và dấu thích hợp vào ô trống.

Có 8 ngôi sao, dán đều thành 2 hàng, mỗi hàng có ☐ ngôi sao.

Ta có phép chia ☐☐☐☐☐

Figure 7. A part of the worksheet A
[The translation of Figure 7: Worksheet A. Paste the star stickers as instructed. Fill in the blanks with the correct numbers and symbols. [Space for pasting stickers] There are 8 stars, evenly distributed into 2 rows, each row having [blank] stars. We have the division: [5 consecutive blank spaces]].

- **Analysis**: In this task, students explore division through the physical manipulation of stars (stickers) on worksheet A. Similar to Task 1, this task reinforces two types of division: partitive division and quotitive division.

Task 3. Observe the picture of b groups of a – state two division situations

The teacher attaches each picture of b groups of a on the blackboard. Students are asked to observe the picture and state two different division situations that could apply. For each situation, students must write the corresponding division equation. The teacher only moves on to a new picture once the class has correctly identified two situations and stated the matching division equations.

Task 4. Write two division equations with a given image of b groups of a

- **Describe**: Students work individually on Worksheet 3, which contains images of b groups of a objects (e.g., 4 bouquets of 3 roses). Students are required to observe the image and write two corresponding division equations.
- **Analysis**: This task serves as an assessment tool to gauge students' grasp of division after completing the HLT. As presented in Tables 1 and 2, Worksheet 3 will be used to evaluate whether students can independently identify both partitive and quotitive division situations.

Results of Implementing Activities in the HLT and Students' Learning Outcomes

Based on video recordings of each group's activities throughout the four intervention lessons, as well as data collected from the students' work on worksheets, we summarize the results of implementing the activities in the HLT and students' learning outcomes for each lesson.

During the first lesson, students transitioned from concrete representations to abstract understanding through carefully designed activities involving real-life contextual problems and manipulatives. In the first task of Activity 1, the whole class participated in dividing into groups. After the teacher assigned five students to hold group name cards for Group 1 through 5, the remaining students moved to their respective groups. In total, ten students participated in this task, as one student arrived late. In the second task, students worked in groups to divide 12 cakes or candies evenly among 3 plates (three groups received 12 cakes, while two groups received 12 candies). The students collaborated well, employing different methods for equal distribution (see Table 4), and successfully arranged 4 cakes or candies per plate. This hands-on experience served as a tangible representation of partitive division, allowing students to visually and physically explore the concept. After manipulating the items, they articulated their understanding by answering questions like "Each plate has four cakes". Moving on to Activity 2, the teacher guided students in expressing how to divide a total equally, encouraging them to articulate their thought processes and strategies for equitable distribution.

Way	Describe	Students
1)	Students took each cake and placed it on each plate, repeating the action 4 times. After verifying that no cakes remained, they counted the number of cakes on each plate.	B2 & B11 B5 & B6
2)	One student poured a bag of 12 candies onto the table. Each student took one candy, equally dividing the candies between 2 students. Then, each student divided their 6 candies into 3 plates. Each plate ended up with 4 candies.	B4 & B8
3)	One student placed 3 cakes on one plate, then repeated the process for the second and third plates. The remaining 3 cakes were added to each plate by two students, one cake at a time.	B9 & B10
4)	Three students each took a plate. Student B7 placed 2 cakes on each plate, paused, and then continued placing 2 more cakes on each plate. Afterward, students B1, B7, and B3 counted the cakes on all three plates.	B1, B3 & B7

Table 4. Ways students used to divide cakes or candies equally

In Activity 3, students further manipulated objects like magnetic dots and math-link cubes to divide them into equal parts. By dividing 18 magnetic dots into 3 equal parts and various colored cubes into 2, 3, 4, and 6 parts, students gained additional concrete experiences (model-of). Different groups employed various methods for equal distribution, as detailed in Table 5. The progression of these tasks helped solidify their understanding, eventually allowing students to internalize and abstract the process (model-for). This was evident in their ability to complete Worksheet 1 independently, with 9 out of 11 students correctly filling in all blanks based on how the objects were divided. Overall, the results illustrate how physical manipulation, followed by guided reflection, supported students in moving from concrete actions to abstract reasoning about division.

Way	Describe	Group
1.1)	Students broke and placed a cube to represent a part. After repeating this 3 times, they set up 3 parts. They then continued breaking each cube and distributed one cube evenly into each part until all the cubes were gone.	B9 & B10 B5 & B6
1.2)	The teacher drew 3 circles on the table to support the group. Students placed each cube into each circle, repeating this 4 times. They then linked 4 cubes to form a block of 4 cubes at each circle, resulting in 3 blocks of 4.	B1& B3
2)	Students split 4 cubes at a time, breaking them 3 times until finished. They then counted again to confirm there were 3 equal parts.	B2 & B11 B4 & B8; B7

Table 5. The ways students used in activity "divide equally 12 math-link cubes into 3 parts"

By the end of the first lesson, the concepts of equal sharing and equal parts were emphasized to lay the foundation for introducing the concept of division in the second lesson (Activity 4). After forming the concept, students continued with Activities 5 and 6 to further solidify their grasp of partitive division through both concrete manipulation and abstract reasoning. Classroom observations in Activity 5 show that, in the first task, students initially manipulated 12 cubes, dividing them into 2 equal parts and then 6 equal parts. Through this concrete manipulation (model-of), all students correctly wrote the corresponding division equations, $12:2=6$and $12:6=2$, demonstrating their ability to connect physical division to abstract numerical equations (model-for). This hands-on activity reinforced their understanding of partitive division. In a subsequent task, students worked in groups to divide a block of 16 math-link cubes evenly and wrote the appropriate division equation. With the teacher's support, Figure 8 shows that nearly all groups correctly wrote the equations, showing a successful transition from concrete manipulation to abstract reasoning. However, one group (B1 & B3) made some errors, indicating a need for further reinforcement. In the second task, students were asked to write division equations based on an image in Worksheet 2. Ten out of eleven students correctly wrote all five equations, while one student (B3) wrote four out of five. This result suggests that most students had successfully applied their understanding of division from earlier manipulations to abstract representations, though some may need additional support.

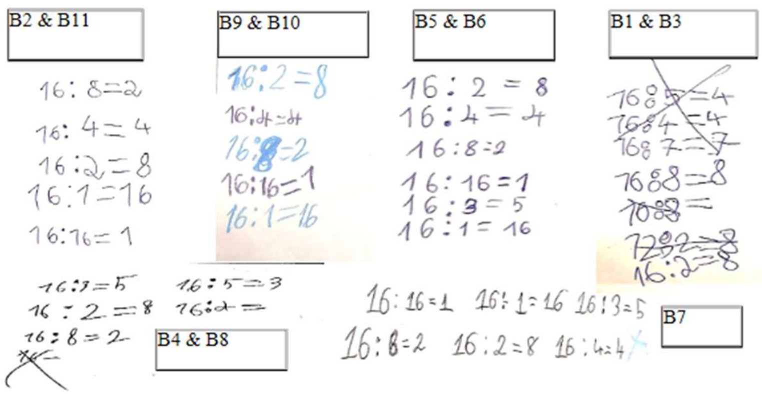


Figure 8. Results of the groups in Activity "evenly divide a block of 16 math-link cubes"

Classroom observations in Activity 6 show that all five called-upon students correctly stated the situation of a given division equation. For example, when given 10: 5=2, student B9 stated, "There are 10 teddy bears, divided among 5 people, each person gets 2 teddy bears", while student B4 stated, "There are ten cars divided among 5 people; each person gets 2 cars". In interviews conducted at the end of the lesson, students expressed enjoyment of the class and demonstrated their understanding of equal division. Seven out of eight students correctly answered questions such as: 6: 2=?, 6: 1=?, 6: 6=?, 10: 2=?, 24: 2=?,30: 3=?.

In the third lesson, students continued with tasks of Activities 3 and 5, then progressed to Activity 7 before being introduced to quotitive division in Activity 8. Each student manipulated 10 cubes, dividing them into 2 and 5 equal parts, and 9 cubes into 3 parts. Classroom observations showed that students correctly answered with something like, "Each part has ... cubes", and wrote the corresponding division equations (task of Activity 3). They then divided 9 cubes into 3 equal parts, 8 cubes into 4 equal parts, and 8 cubes into 2 equal parts, successfully writing the equations 9: 3=3; 8: 4=2; 8: 2=4 (task of Activity 5). In tasks 1, 2 of Activity 7, students demonstrated their understanding by dividing objects equally and correctly identifying the number of parts or people. In task 3 of Activity 7, students experienced mathematization, recognizing numbers as sums of equal numbers (Table 6). Based on images received, students correctly answered that 12 can be divided into four numbers 3, three numbers 4, two numbers 6, six numbers 2, and twelve numbers 1.

Turn	Images received after manipulating	Mathematics
1)		$12=3+3+3+3$ $12:3=4$
2)		$12=4+4+4$ $12:4=3$
3)		$12=2+2+2+2+2+2$ $12:2=6$
4)		$12=6+6$ $12:6=2$
5)		$12=1+1+1+1+1+1+1+1+1+1+1+1$ $12:1=12$

Table 6. Mathematization in task 3 of Activity 7

In the fourth lesson – the last lesson in the HLT – students perform four tasks in Activity 9 to clearly distinguish between partitive division and quotitive division. In Task 1, students worked with two division scenarios involving flowers and vases: 6 flowers and 3 vases, 9 flowers and 3 vases, 12 flowers and 2 vases, and 15 flowers and 5 vases. All groups followed the instructions and correctly stated the corresponding division equations, demonstrating a clear distinction between partitive and quotitive division. Classroom observations, students' work in Task 2, and the results of 11 students on Worksheet A show that students demonstrated correct understanding. After evenly pasting 8 stars (stickers) into 2 rows, students correctly filled in the number 4 and wrote $8:2=4$. Likewise, after pasting 4 stars in each row, students filled in the number 2 and wrote $8:4=2$. In Task 3, students were asked to observe a picture representing "b groups of a" and state two division situations. Eight out of nine called-upon students correctly stated the division situations, while student B3 was unable to respond. Finally, in Task 4, students were asked to write two division equations with a given image of group a. Results from Worksheet 3 show that 8 out of 11 students correctly wrote all 6 division equations, 2 students correctly wrote only 2 equations, and student B3 did not write any equations correctly. The following Figure 9 shows student B8's work in the sticker activity and student B6's work on the worksheet 3. Overall, the findings indicate that most students successfully distinguished between the two types of division, though a few may need further support.

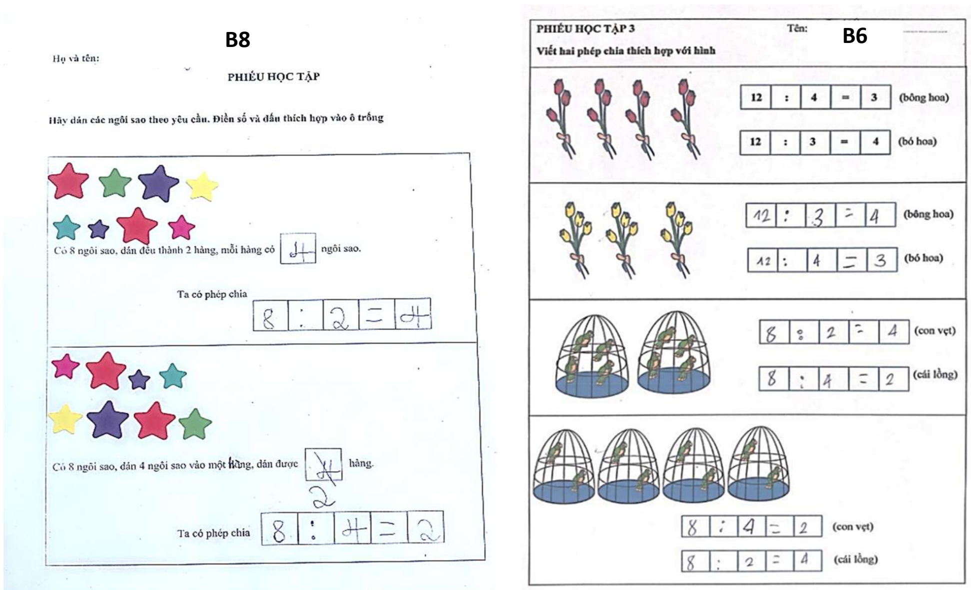


Figure 9. Student B8's work on Worksheet A and student B6's work on Worksheet 3

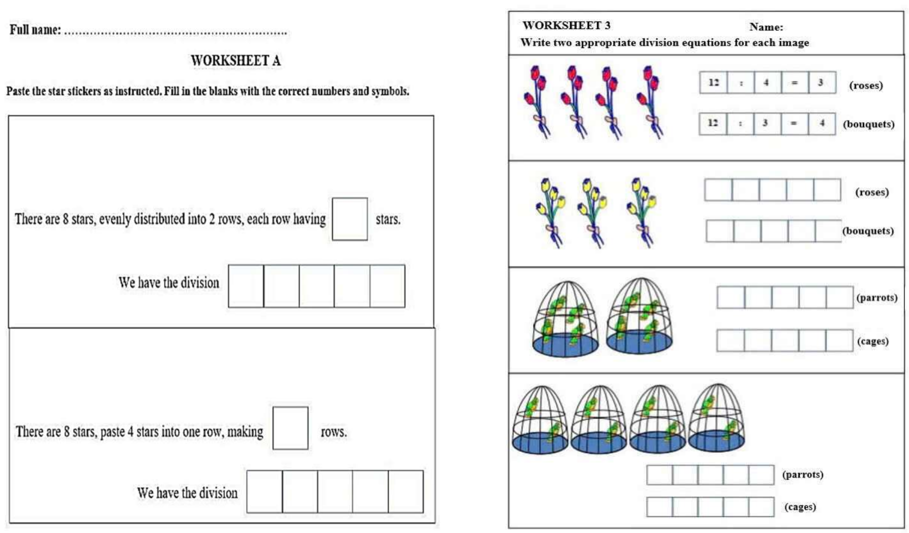


Figure 10. The translation of Figure 9 into English

DISCUSSION

The results of this study show that the designed HLT for teaching division independently of multiplication offers a promising approach to helping primary students build a conceptual understanding of division. By incorporating RME principles—especially hands-on manipulatives and realistic contexts—students engaged more deeply with division, focusing on grasping core concepts instead of memorizing procedures. This aligns with studies demonstrating the effectiveness of RME in improving students' understanding, problem-solving skills, and test performance (Leng et al., 2020; Drijvers et al., 2019; Syafriafdi et al., 2019; Laurens et al., 2018; Papadakis et al., 2017; Lestari & Surya, 2017). Moreover, RME enhances the meaningfulness of math learning and strengthens reasoning (Fauzan et al., 2020).

The research also highlights the pivotal role of classroom teachers in the RME approach: organizing activities align with the goals of the HLT (Duyen & Loc, 2022) and supporting learners (Wahyudi et al., 2017). Throughout the instructional process, teachers actively observe students' learning activities and provide timely guidance, ensuring that learners can independently reinvent mathematical concepts.

Challenges emerge particularly in helping low-ability students transition from concrete to abstract understanding. While manipulatives supported an initial grasp of division, these students struggled with abstract representation, highlighting the need for additional scaffolding. This aligns with research on students' problem-solving abilities, which shows that high-ability students tend to exhibit the greatest fluency and flexibility when solving problems. In contrast, low-ability students often struggle to comprehend the problems and make frequent errors during the problem-solving process (Arifin et al., 2021).

Overall, the study highlights both the strengths and potential limitations of using the RME approach. Future studies could explore how additional scaffolding or technology-based tools might further assist low-ability students in mastering the abstract aspects of division.

CONCLUSIONS

The findings of this study underscore the significance of teaching the division of two natural numbers to primary school students independently of multiplication through the principles of RME. The designed HLT facilitated a conducive learning environment, enabling students to engage deeply with the concept of division. By experiencing mathematization, students were able to distinguish between partitive and quotitive division, articulate their understanding through appropriate real-world contexts, and write accurate division equations.

The positive outcomes suggest that RME not only supports the understanding of division as a standalone concept but also enhances students' ability to apply mathematical reasoning in practical situations. This approach addresses the challenges learners face when learning division, offering a framework that encourages exploration and critical thinking.

Future research could expand on these findings by investigating the long-term effects of teaching division independently of multiplication, exploring additional real-world contexts, and examining the impact on diverse learning styles. Overall, this study contributes valuable insights into effective mathematics education practices for primary school students, emphasizing the need for instructional strategies that prioritize conceptual understanding over rote learning.

REFERENCES

[1] Alghamdi, A., Jitendra, A.K. & Lein, A.E. (2020). Teaching students with mathematics disabilities to solve multiplication and division word problems: the role of schema-based instruction. *ZDM Mathematics Education 52*(5), 125–137. https://doi.org/10.1007/s11858-019-01078-0

[2] Ardiyani, S., Gunarhadi, G., & Riyadi, R. (2018). Realistic mathematics education in cooperative learning viewed from learning activity. *Journal on Mathematics Education, 9*(2), 301-310.

[3] Ariati, C., Anzani, V., Juandi, D., & Hasanah, A. (2022). Meta-analysis study: Effect of realistic mathematics education approach on student's mathematical literacy ability. *AKSIOMA: Jurnal Program Studi Pendidikan Matematika.* https://doi.org/10.24127/ajpm.v11i4.6182

[4] Arifin, S., Zulkardi, Putri, R. I. I., & Hartono, Y. (2021). On creativity through mathematization in solving non-routine problems. *Journal on Mathematics Education*, *12*(2), 313-330.

[5] Götze, Daniela (2018). Fostering a conceptual understanding of division: A language and mathematics integrated project in primary school. *Proceedings of the IV ERME Topic Conference' Classroom-based research on mathematics and language',* (pp. 73-80). Dresde, Germany.

[6] de Lange, J. (1995). Assessment: No change without problems. In T. A. Romberg (Ed.), *Reform in school mathematics* (pp. 87-172). Albany, NY: SUNY Press.

[7] Downton, A. (2008). Links between children's understanding of multiplication and solution strategies for division. In M. Goos, R. Brown, & K. Makar (Eds.), *Navigating currents and charting directions* (pp. 171-178). Brisbane, Australia: MERGA.

[8] Drijvers, P., Kodde-Buitenhuis, H., & Doorman, M. (2019). Assessing mathematical thinking as part of curriculum reform in the Netherlands. *Educational Studies in Mathematics, 102*(3), 435–456. https://doi.org/10.1007/s10649-019-09905-7

[9] Duyen, N. T. H., & Loc, N. P. (2022). Developing primary students' understanding of mathematics through mathematization: A case of teaching the multiplication of two natural numbers. *European Journal of Educational Research, 11*(1), 1-16. https://doi.org/10.12973/eu-jer.11.1.1

[10] Pacheco-Muñoz, Ever., Nava-Lobato, Stefany., Juárez-López, José Antonio., Ponce de León-Palacios, Manuel. (2023). Division problems with remainder: A study on strategies and interpretations with fourth grade Mexican students. *Mathematics Teaching Research Journal*, *15*(2), 120-141.

[11] Fauzan, A., Yerizon, Y., & Yulianti, D. (2020). The RME-based local instructional theory for teaching LCM and GCF in primary school. *Journal of Physics: Conference Series, 1554*, 1-7. https://doi.org/10.1088/1742-6596/1554/1/012078

[12] Febriani, W., & Sidik, G. (2020). The effect of realistic mathematics education (RME) on the understand mathematical concepts skills of elementary students using hypothetical learning trajectory (HLT). *Proceedings of the International Conference on Education, 4*(1), 89-99. https://doi.org/10.22460/pej.v4i1.1509

[13] Freudenthal, H. (1973). *Mathematics as an educational task.* Dordrecht: Reidel.

[14] Freudenthal, H. (1991). *Revisiting mathematics education: China lectures.* Netherlands: Kluwer Academic Publishers.

[15] Gravemeijer, K. (1994). *Developing realistic mathematics education.* Utrecht, The Netherlands: Freudenthal Institute.

[16] Gravemeijer, K. (1997). Instructional design for reform in mathematics education. In M. Beishuizen, K. P. E. Gravemeijer, & E. C. D. M. van Lieshout (Eds.), *The role of contexts and models in the development of mathematical strategies and procedures* (pp. 13-34). Utrecht, The Netherlands: Freudenthal Institute.

[17] Gravemeijer, K. (1997). Mediating between Concrete and Abstract. In T. Nunes & P. Bryant (Eds.), *Learning and Teaching Mathematics: An International Perspective* (pp. 315-343). Hove: Lawrence Erlbaum Associates.

[18] Gravemeijer, K. P. E. (1999). Developmental research: Fostering a dialectic relation between theory and practice. In J. Anghileri (Ed.), *Principles and practice in arithmetic teaching* (pp. 147-161). London: Open University Press.

[19] Gravemeijer, K. P. E., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics*, *39*, 111-129. https://doi.org/10.1023/A:1003749919816

[20] Gravemeijer, K., & Terwel, J. (2000). Hans Freudenthal: A mathematician on didactics and curriculum theory. *Journal of Curriculum Studies, 32*(6), 777-796. https://doi.org/10.1080/00220270050167170

[21] Ha, T. T., & Cuong, T. (2021). Potential of Applying Realistic Mathematics Education to the Teaching of Probability - Statistics at the Universities of Medicine and Pharmacy. *Journal of Science Educational Science*, *66*(5), 209-216.

[22] Johar, R., Patahuddin, S., & Widjaja, W. (2017). Linking pre-service teachers' questioning and students' strategies in solving contextual problems: A case study in Indonesia and the Netherlands. *The Mathematics Enthusiast.* https://doi.org/10.54870/1551-3440.1390

[23] Laurens, T., Batlolona, F. A., Batlolona, J. R., & Leasa, M. (2018). How does realistic mathematics education (RME) improve students' mathematics cognitive achievement? *Eurasia Journal of Mathematics, Science and Technology Education, 14*(2), 569-578. https://doi.org/10.12973/ejmste/76959

[24] Lestari, L., & Surya, E. (2017). The effectiveness of realistic mathematics education approach on ability of students' mathematical concept understanding. *International Journal of Sciences: Basic and Applied Research, 34*, 91-100.

[25] Leng, K., Razali, F., & Ayub, A. (2020). The effectiveness of realistic mathematics education approach toward students learning: A systematic literature review of empirical evidence. *Journal of Critical Reviews, 7*(13), 99-104. https://doi.org/10.31838/jcr.07.13.99

[26] Loc, N.P., & Tien, N.T. (2020). Approach To Realistic Mathematics Education In Teaching Mathematics: A Case Of Cosine Theorem – Geometry 10. *International Journal of Scientific & Technology Research, 9*, 1173-1178.

[27] Ministry of Education and Training - Vietnam. (2018a). *Chương trình giáo dục phổ thông - Chương trình môn Toán* (General education program - Mathematics program). Hanoi, Vietnam.

[28] Ministry of Education and Training - Vietnam. (2018b). *Chương trình giáo dục phổ thông - Chương trình tổng thể* (General education program - Comprehensive program). Hanoi, Vietnam.

[29] Napitupulu, R., Alim, J., Hermita, N., & Ibrahim, B. (2021). Development of an RME-based hypothetical learning trajectory of least common multiple for elementary school students. *Journal of Teaching and Learning in Elementary Education, 4*(2), 247-258.

[30] Nuraida, I., & Amam, A. (2019). Hypothetical learning trajectory in realistic mathematics education to improve the mathematical communication of junior high school students. *Infinity Journal, 8*(2), 247-258. https://doi.org/10.22460/infinity.v8i2.p247-258

[31] Papadakis, S., Kalogiannakis, M., & Zaranis, N. (2017). Improving mathematics teaching in kindergarten with realistic mathematical education. *Early Childhood Education Journal, 45*(3), 369-378. https://doi.org/10.1007/s10643-015-0768-4

[32] Robinson, K. M., & LeFevre, J. A. (2012). The inverse relation between multiplication and division: Concepts, procedures, and a cognitive framework. *Educational Studies in Mathematics*, 79, 409-428.

[33] Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145. https://doi.org/10.2307/749205

[34] Sơn, T.C., Duyen, N.T.H., & Chau, P. T.H. (2022). Một số hướng khai thác yếu tố thực tiễn trong dạy học tích hợp môn Toán ở tiểu học [Some directions to exploit practical factors in teaching mathematics in primary schools]. *Tạp chí Giáo dục, 22*(5), 20-25.

[35] Syafriafdi, N., Fauzan, A., Arnawa, I. M., Anwar, S., & Widada, W. (2019). The tools of mathematics learning based on realistic mathematics education approach in elementary school to improve math abilities. *Universal Journal of Educational Research, 7*(7), 1532-1536. https://doi.org/10.13189/ujer.2019.070707

[36] Tong, D. H., Nguyen, T. T., Uyen, B. P., Ngan, L. K., Khanh, L. T., & Tinh, P. T. (2022). Realistic Mathematics Education's Effect on Students' Performance and Attitudes: a Case of Ellipse Topics Learning. *European Journal of Educational Research*, 11(1), 403-421. https://doi.org/10.12973/eu-jer.11.1.403

[37] Treffers, A. (1987). *Three Dimensions. A Model of Goal and Theory Description in Mathematics Instruction* – The Wiskobas Project, Reidel Publishing Company, Dordrecht, The Netherlands.

[38] Treffers, A. (1991). Didactical background of a mathematics program for primary education. In L. Streefland (Ed.), *Realistic mathematics education in primary schools* (pp. 21-56). Utrecht: Freudenthal Institute, Utrecht University.

[39] Trung, N.T, Thao, T.T.P, Trung, T. (2019). *J. Phys.: Conf. Ser.* 1340 012032 https://doi.org/10.1088/1742-6596/1340/1/012032

[40] Trung, N.T., Thao, T.P., Hang, N.T.T, Anh, H.N., N.T., Trung, T., Hung, P., H, H., & Nghi, B. V. (2020). Realistic Mathematics Education in Vietnam: Recent Policies and Practices. *International Journal of Education and Practice*. Trung, N.T., Thao, T.P., Hang, N.T.T, Anh, H.N., N.T., Trung, T., Hung, P., H, H., & Nghi, B. V. (2020). Realistic Mathematics Education in Vietnam: Recent Policies and Practices. International Journal of Education and Practice. https://doi.org/10.18488/journal.61.2020.81.57.71

[41] Van den Heuvel-Panhuizen, M. (1996). *Assessment and realistic mathematics education* (Doctoral dissertation, University of Utrecht, The Netherlands).

[42] Van den Heuvel-Panhuizen, M. (2001). Mathematics education in the Netherlands: A guided tour. Utrecht: Utrecht University.

[43] Van den Heuvel-Panhuizen, M. (2002). Realistic mathematics education as work in progress. In F. L. Lin (Ed.), *Proceedings of 2001 The Netherlands and Taiwan Conference*

on Mathematics Education: Common sense in mathematics education (pp. 1-43). National Taiwan Normal University.

[44] Wahyudi, Joharman, & Ngatman. (2017). The development of realistic mathematics education (RME) for primary schools' prospective teachers. Advances in Social Science, Education and Humanities Research, 158, 814-826. https://doi.org/10.2991/ictte-17.2017.83

[45] Widjaja, W., Dolk, M., & Fauzan, A. (2010). The role of contexts and teacher's questioning to enhance students' thinking. *Journal of Science and Mathematics Education in Southeast Asia, 33*(2), 168-186.