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Exploring the Concept of Inequality Through the Lens of Concept Image Students

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Abstract: This study explores students' concept images of inequalities, examining their procedural proficiency and conceptual gaps. Employing a qualitative phenomenological methodology, data was collected through tests and interviews with high school students to elucidate their comprehension of inequality rules, solution sets, and symbolic representations. The findings reveal that while numerous students possess the ability to execute procedures, such as manipulating inequality signs when multiplying or dividing by negative numbers, they encounter challenges in articulating their reasoning. Common misconceptions include misinterpretation of interval notation, overgeneralization of rules, and difficulties in visualizing solutions on a number line. Furthermore, students exhibit varying degrees of proficiency in translating real-world scenarios into inequality notation, with some demonstrating symbolic accuracy but lacking conceptual justification. The study also identifies limitations stemming from sample diversity and instructional influences, underscoring the necessity for broader investigations. Based on these findings, recommendations include refining instructional strategies to bridge procedural and conceptual understanding, integrating visualization techniques, and providing targeted interventions to address misconceptions. Future research should explore effective pedagogical approaches to fortify students' relational comprehension of inequalities. These insights contribute to mathematics education by informing strategies for enhancing students' comprehension and application of inequality concepts in academic and practical contexts.

Keywords: Inequalities; concept image; misconceptions; procedural proficiency

INTRODUCTION

The causes of human errors and misconceptions have been a subject of interest for numerous educators, including David Tall and Shlomo Vinner, who are among the leading figures in this field. To comprehend the mental processes involved in conceptual manipulation, Tall and Vinner (1981) proposed the idea of a "concept image" to serve as a comprehensive description of the entire cognitive structure associated with a concept. This includes all mental representations and their associated properties and processes. Concept images undergo development over time, influenced by all the experiences an individual has with a particular concept, regardless of whether it manifests in visual forms such as pictures, drawings, diagrams, representations, or metaphors.

Tall and Vinner (1981) defined "concept definition" as a verbal explanation that accurately and non-circularly describes a concept. This definition is conveyed through the words used to articulate the concept and may be acquired either through rote memorization or through meaningful engagement with tasks related to the concept, allowing for the formation of relevant associations. A concept definition can take the form of a formal definition typically found in textbooks or a verbal expression of an individual's mental representation of the concept.

The way a person verbalizes a concept often reflects their concept image—that is, the mental structure or understanding they associate with the concept. Likewise, non-verbal representations, such as graphs or diagrams, can offer further insight into an individual's concept image. Vinner (1983) noted that during problem-solving activities, individuals frequently shift between their concept image and concept definition. This back-and-forth movement, especially when accompanied by inconsistencies or disconnections between the two, can indicate the depth of a student's conceptual understanding.

Many mathematical terms have everyday meanings that can inadvertently hinder mathematical progress. In mathematical activities, mathematical concepts are not solely utilized based on their formal definitions, but also through mental representations that may vary among individuals. Pre-existing concepts, derived from everyday experiences that generate mathematical ideas, can sometimes interfere with the very definition of mathematics itself (Cornu, 1981). Students' mathematical knowledge actually consists of the concept image-concept definition that is formed. Mathematical knowledge is categorized into two distinct terms: procedural knowledge and conceptual knowledge (Hiebert & Lefevre, 1986). Procedural knowledge encompasses computational skills and knowledge of procedures for identifying mathematical components, algorithms, and definitions. Conceptual knowledge, on the other hand, refers to knowledge of the underlying structure of mathematics. It is characterized as knowledge that is rich in relationships and encompasses understanding of mathematical concepts, definitions, and facts. Both procedural and conceptual knowledge are considered essential components of mathematical understanding (Hiebert & Lefevre, 1986). Therefore, teaching mathematical knowledge should encompass instruction in both procedural and conceptual knowledge.

In the realm of mathematics, students should be encouraged to develop a formal conceptual understanding of a concept rather than solely relying on personal procedural and conceptual knowledge (Hussein, 2022). This shift in emphasis is crucial for students to grasp the formal definition of a concept. The role of definitions in mathematics is multifaceted. On the one hand, definitions serve to identify familiar concepts, ensuring their clarity and precision. Conversely, in formal mathematics, definitions establish the properties of the defined concept, defining it in the process (Pinto & Tall, 1996). However, the concept image-concept definition scheme is not solely constructed through definitions but also through experience (Vinner & Dreyfus, 1989).

By comprehending students' conceptualizations (concept images), educators can identify their learning gaps and tailor their instruction to enhance their comprehension of mathematical concepts (Siagian et al., 2021). Formal mathematical definitions are pivotal in mathematics, serving as the foundation for theorems that establish the theoretical framework of the subject. However, students frequently encounter challenges in grasping and applying formal mathematical definitions (Rasslan & Tall, 2002; Tall, 1994; Vinner & Dreyfus, 1989). Consequently, the same mathematical concept can be elucidated through various representations, and sometimes even with distinct definitions. Consequently, the concept image associated with a particular mathematical idea can differ significantly between students and their instructors (Dreyfus, 1991).

Consequently, it is imperative to conduct a research study to investigate students' conceptual understanding, particularly regarding the concept of inequality. Inequality is a widely employed term in both mathematical and practical contexts. In the realm of mathematics, inequality is explicitly utilized to denote the relationship between numbers or to specify limits for quantities. Alternatively, it may be implicitly incorporated within the domain or characteristics of a function, encompassing its behavior. In practical scenarios, inequality can be categorized as injustice, disparity, inequality, irregularity, or dissimilarity.

Inequality is a fundamental concept in mathematics education, not only serving as a prerequisite for other mathematical concepts but also finding applications in various disciplines (Bicer et al., 2014; Siagian et al., 2022; Yokoso et al., 2024). However, empirical evidence suggests that students' comprehension of the concept of inequality warrants substantial attention, as its urgency appears to be inversely proportional to its perceived importance. Previous research has consistently highlighted students' challenges in solving problems related to the concept of inequality (Palupi et al., 2022; Rofiki et al., 2017; Sitanggang & Araiku, 2022; Tamba & Saragih, 2020; Yasa et al., 2022; Zulhendri et al., 2022). Notably, these difficulties extend beyond high school students, affecting even college students (Biney et al., 2023; El-Khateeb, 2016; Siagian et al., 2022; Turner et al., 2016).

Based on the research findings presented, I have identified the significant learning challenges posed by inequality for students. The research reveals cognitive difficulties that students encounter when grasping the concept of inequality. Notably, students frequently struggle to comprehend the relational nature of inequality, particularly when transitioning from equations to inequalities, where solutions involve ranges rather than specific values. Furthermore, the reversal of the inequality sign when multiplying or dividing by a negative number is a well-documented source of confusion, as it deviates from students' prior arithmetic experiences. These challenges stem from conceptual misunderstandings and procedural errors, rendering inequality a complex subject in mathematics education.

Certainly, research related to the concept of inequality has been conducted by numerous researchers. For instance, Halmaghi (2011) conducted an analysis of students' and first-year students' comprehension of the concept of inequality. Pratiwi and Rosjanuardi (2020) conducted a study analyzing students' responses to rational and irrational inequalities. Mokh et al. (2019) conducted a study related to the concept of inequality with the objective of examining the errors made by students in utilizing logical conjunctions when simplifying algebraic equations and inequalities. Furthermore, research conducted by Jupri et al. (2022) aims to investigate the strategies employed by pre-service teachers when solving absolute value equations and inequalities from the perspective of symbol interpretation. El-Khateeb (2016) conducted a study with the intention of investigating and identifying the types of errors that occur in preparatory year students in solving inequalities. Similarly, research conducted by Bicer et al. (2014) remains pertinent to pre-service mathematics teachers' comprehension of linear and quadratic inequalities, with the aim of ascertaining whether they harbor common misconceptions and difficulties with inequalities. Additionally, research conducted by Moon (2019) proposes several alternative test solutions for comprehending two-variable inequality graphs.

Based on the urgency of the inequality concept and several relevant research findings, I conducted a study by emphasizing students' concept images in the inequality concept. This study aims to elucidate the meaning of the inequality concept as perceived by students. Students' concept images are analyzed through the examination of three distinct components: mental images, processes, and properties. Mental images encompass all mental representations employed by students to elucidate concepts or describe their mental processes in solving problems related to the concept. Processes refer to all procedures, methods, or steps utilized by students to describe concepts or approach problems related to the concept. Properties encompass all axioms, definitions, lemmas, theorems, or formulas employed by students to describe concepts or solve problems related to the concept.

In accordance with the objectives of this study, the following research questions are formulated: (1) How do students conceptualize the mathematical concept of inequality in relation to their concept images? (2) What are the prevalent misconceptions and reasoning patterns that students demonstrate when attempting to solve problems involving inequalities? and (3) To what extent do students' procedural skills correspond with their conceptual comprehension within the context of inequalities?

LITERATURE REVIEW

The Significance of Definitions in Mathematics Education

Edwards and Ward (2008) posited that definitions serve as indispensable tools for mathematicians and students to ensure effective communication. Moreover, definitions serve as the cornerstone for logical reasoning and mathematical proof construction, particularly in advanced mathematics. However, Berman and Shvartsman (2016) identified that a deficiency in comprehension of formal definitions can impede students' learning trajectory. Through empirical research, they demonstrated that students' engagement and comprehension levels surged substantially when theoretical inquiries pertaining to definitions were integrated into academic assignments. This finding aligns with the assertions of Chang et al. (2022), Dorier et al. (2000) and Carlson (1993), who underscored students' challenges in grasping mathematics, particularly due to the inadequate time allocated for understanding formal definitions at the secondary school level.

Cognitive Structures in Concept Formation

To comprehend the cognitive processes involved in mathematical concept formation, numerous cognitive models have been developed by researchers (Duval, 1998; Fischbein, 1993; Tall & Vinner, 1981). One of the most influential models is the cognitive structure proposed by Tall and Vinner (1981), which comprises two primary components: concept image and concept definition.

The concept image is defined as "all cognitive structures in an individual's mind associated with a concept" (Vinner, 2011). This component encompasses mental representations, attributes, and related processes, which can manifest as visual representations (e.g., charts, graphs, diagrams), symbolic representations (e.g., formulas), or other forms. For instance, students can interpret mathematical functions as rules, graphs, operations, tables of values, or mappings between sets (Ojo & Olanipekun, 2023; Tirosh & Tsamir, 2022).

Concept definitions, on the other hand, refer to "the form of words employed to define a concept" (Tall & Vinner, 1981). These definitions are categorized into two types: personal concept definitions, which students utilize to elucidate their comprehension, and formal concept definitions, which are recognized within the mathematical community. The dichotomy between examples and non-examples of a concept, as defined by the formal definition of the concept, plays a pivotal role in mathematics education and instruction (Ulusoy, 2019). Vinner (1983, 1991) underscored the significance of the reciprocal interaction between concept image and concept definition in comprehending a concept. He identified three potential interactions: (1) adjusting the concept image to align with a formal definition that contradicts the existing image, (2) memorizing the definition without modifying the concept image, and (3) neither internalizing the definition nor altering the concept image.

Mathematical Knowledge: Procedural and Conceptual

Students' mathematical knowledge is characterized by a dynamic interplay between concept images and concept definitions, which can be categorized into procedural knowledge and conceptual knowledge (Hiebert & Lefevre, 1986). Sfard (1991) provided a comprehensive analysis of the development of mathematical concepts from a historical and psychological perspective, positing that mathematical concepts can be comprehended operationally (as a process) or structurally (as an object). The process of conceptual acquisition involves three distinct stages: internalization (acquiring knowledge of the concept and its operations), condensation (developing flexibility in representing the concept), and reification (gaining a comprehensive understanding of the concept as a unified entity). However, difficulties often arise in comprehending the structural aspect due to the ontological gap between the operational and structural stages.

Furthermore, Sfard (1991) distinguished between concepts and conceptions, where concepts encompass the formal aspects of mathematics, while conceptions represent individual interpretations of the concept. This distinction aligns with the theory of concept images and concept definitions, which suggest that a person's mental representation may not always precisely correspond to the formal definition. As Ulusoy (2021) elucidates, constructing a mathematical definition entail more than merely expressing a conceptual image in words.

Personal Interpretation and Misconceptions

Students not only require a comprehensive understanding of formal axiomatic systems but also the ability to construct personal interpretations of formal definitions. These interpretations can vary and may not always align with formal definitions (Tsamir & Tirosh, 2023; Pinto & Tall, 1996; 2002). Tall and Vinner (1981) elucidated that concept definitions reflect cognitive structures associated with the definition of a concept and constitute an integral component of the overall concept image. In practical terms, the application of formal definitions in reasoning necessitates personal interpretations, which can significantly impact students' comprehension.

The relationship between concept image and concept definition also provides insights into misconceptions and learning difficulties. During the initial stages of learning, students often possess limited concept images that undergo development through experience and concrete examples (Vinner, 1991). However, at the secondary and tertiary levels, this relationship is frequently perceived as a one-way process, wherein concept images are regarded as the outcome of understanding formal definitions (Vinner, 1983). This approach overlooks the significance of developing rich and adaptable concept images, which serve as the foundation for profound comprehension.

METHODS

This research employs a qualitative research approach, which systematically organizes and interprets textual materials derived from conversations or conversations. Qualitative research is instrumental in exploring the subjective experiences and perspectives of individuals within a specific context (Grossoehme, 2014). The phenomenological approach is the research methodology employed in this study. Phenomenology can be defined as a research approach that aims to elucidate the essence of a phenomenon by examining it from the viewpoint of those who have experienced it (Teherani et al., 2015). The phenomenon investigated in this study is students' experiences in learning the concept of inequality, and how these experiences shape their concept image of the concept.

The study participants were 10th-grade high school students (aged between 15 and 16 years) who were selected through purposive sampling. To mitigate potential bias, the researcher employed purposive sampling, incorporating academic records and teacher assessments in participant selection. Specifically, students were categorized into three distinct academic ability levels: high, medium, and low achievers. Furthermore, gender diversity was considered. These criteria were meticulously selected to ensure that the exploration of students' concept images pertaining to inequality encompasses a spectrum of mathematical abilities, reasoning patterns, and diverse learning experiences. The primary objective of this study was to explore students' concept images of inequalities.

Data collection in the study was conducted through administering tests related to the concept of inequality (see Appendix) and unstructured interviews to explore the concept image formed in students based on their learning experiences. The tests designed aimed to (1) explore students' perceptions of understanding the rules of the concept of inequality and also related to how students interpret the concept of inequality; (2) explore students' understanding of the application of inequality rules, determining and utilizing the solution set of the inequality concept; and (3) explore students' understanding of using inequality notation in terms of transforming mathematical situations or problems into mathematical symbols in inequality. In accordance with the test results, the reduction process is undertaken to further elucidate students' concept images of inequality.

The validity and reliability of the test instrument were assessed prior to its implementation. The findings of these assessments were based on Cohen et al.'s (2020) evaluation. To evaluate the validity of the test instrument employed in this study and determine the extent to which contextual factors influence the concept of inequality both theoretically and practically, content validation was deemed appropriate. While the primary objective of the reliability test was to investigate the impact of the provided context on students' responses to questions, the researcher was also interested in assessing the level of comprehension of students' understanding of the concept of inequality.

Data analysis was conducted employing the phenomenological approach developed by Hycner (1985) and modified by Groenewald (2004), which recognizes the researcher's interpretive involvement with the data. Willig (2001) elucidated that the implications of phenomenology for data analysis are achieved by adhering to the following characteristics: focusing on the lived experiences of the subjects, maintaining openness to their perspectives, providing precise descriptions, deferring prior knowledge or biases, and seeking the essence in descriptions.

RESULTS

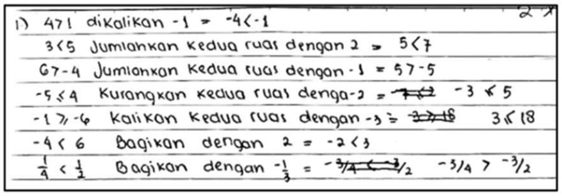
Students' Perceptions Regarding the Understanding and Interpretation of Inequality Rules

Problem 1 seeks to explore students' perceptions regarding the rules of the concept of inequality and their interpretations of the concept. Additionally, it aims to review students' understanding of the concept of numbers. Based on their perceptions and responses to Problem 1, the concept image of the rules of the concept of inequality and the meaning of the concept of inequality will be investigated. However, prior to investigating the concept image formed, the categories of students' answers will be systematically summarized based on the results of the answers and interviews, as presented in Table 1.

Response Category	Response
Correct procedure accompanied by appropriate reasons	3
Correct procedure lacking proper reasons	10
Incorrect procedure accompanied by inappropriate reasons	9

Table 1: Student response categories for problem 1

Based on the categorization of students' responses to problem 1, it can be ascertained that there are students who execute the procedure accurately. Students demonstrate proficiency in applying inequality rules, particularly when modifying inequality signs while multiplying or dividing by negative numbers. However, it is evident that many students lack a clear explanation for their results. For instance, Figure 1 provides an example of a student's solution to problem 1.



1) 4 > 1 dikalikan -1 = -4 < -1
3 < 5 Jumlahkan kedua ruas dengan 2 = 5 < 7
6 > -4 Jumlahkan kedua ruas dengan -1 = 5 > -5
-5 ≤ 4 Kurangkan kedua ruas denga-2 = ~~[illegible]~~ -3 ≤ 5
-1 ≥ -6 Kalikan kedua ruas dengan -3 = ~~3 ≥ 18~~ 3 ≤ 18
-4 < 6 Bagikan dengan 2 = -2 < 3
$\frac{1}{4} < \frac{1}{2}$ Bagikan dengan $-\frac{1}{3}$ = ~~-3/4 < -3/2~~ -3/4 > -3/2

$4 > 1$ Multiply by $-1 = -4 < -1$
$3 < 5$ Add both sides together with $2 = 5 < 7$
$6 > -4$ Add both sides together with $-1 = 5 > -5$
$-5 \leq 4$ Subtract both sides by $-2 = -3 <= 5$
$-1 \geq -6$ Multiply both sides by $-3 = 3 \leq 18$
$-4 < 6$ Divide by $2 = -2 < 3$
$\frac{1}{4} < \frac{1}{2}$ Divide by $-\frac{1}{3} = -\frac{3}{4} > -\frac{3}{2}$

Figure 1: Student response with correct procedure but no justification.

Analysis of the concept image component shows that students correctly followed the process to solve the problem. The "properties" component also aligns with the rules of inequalities. However, in the "mental image" component, students transformed the inequality notation after multiplying or dividing by a negative number. This transformation was based on their understanding of number positioning relative to zero—numbers further to the left are smaller. This mental image was derived from interview results, summarized below.

Researcher : *Based on your response to problem 1, it appears that several parts yield results that change sign. Could you please elaborate on this?*
Student : *Indeed, sir. When a number is multiplied by a negative number, its sign changes.*
Researcher : *Could you provide an example to illustrate this?*
Student : *For instance, in the first part,* $4>1$ *is multiplied by* -1*. The resulting values on each side are* -4 *and* -1*. If we examine* -4*, we can see that it is smaller than* -1 *because* -4 *is farther from 0 than* -1*.*

The interview excerpt shows that students understand why the inequality sign changes when multiplying or dividing by negative numbers. However, their explanation lacks clarity. In problem 1, their mental image of this concept is evident. Some students also grasp the formal rules for modifying inequalities in such cases. The following interview excerpt illustrates this understanding.

Researcher : *I understand that the answer to the final part of problem 1 has undergone a change in sign.*
Student : *Indeed, sir.*
Researcher : *Could you please elaborate on the reason behind this change?*
Student : *As I recall, sir, when inequalities are multiplied or divided by a negative number, the sign is reversed.*
Researcher : *Why did you not provide this explanation yesterday?*
Student : *I apologize, sir, but I inadvertently neglected to do so due to a pressing time constraint.*

The second category of student responses, which is the ability to execute the procedure accurately and provide valid reasons for solving problem 1, is also noteworthy. An example of student responses in this category is provided in Figure 2.

1] 1. $4>1 \xrightarrow{\times -1} -4<-1$
2. $3<5 \xrightarrow{+2} 5<7$
3. $6>-4 \xrightarrow{+(-1)} 5>-5$
4. $-5 \leq 4 \xrightarrow{-(-2)} -3 \leq 6$
5. $-1 \geq -6 \xrightarrow{\times(-3)} 3 \leq 18$
6. $-4<6 \xrightarrow{:2} -2<3$
7. $\frac{1}{4} < \frac{1}{2} \xrightarrow{:-\frac{1}{3}} -\frac{3}{4} > -\frac{3}{2}$

2] ~~nilai-nilai x berjarak 4 dari angka 5.~~

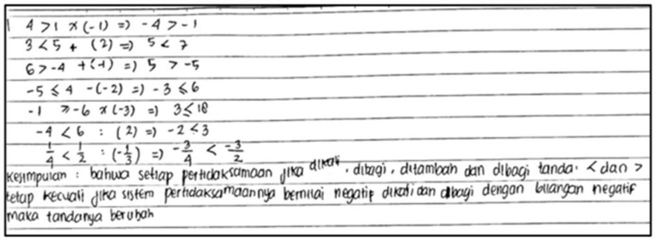
Simpulan:
- jika pertidaksamaan, kedua ruasnya sama-sama dijumlahkan atau dikurangkan, tanda pertidaksamaan tidak berubah.
- jika pertidaksamaan, kedua ruasnya sama-sama dikali atau dibagi dengan bilangan positif, tanda pertidaksamaan juga tidak berubah.
- jika pertidaksamaan, kedua ruasnya sama-sama dikali atau dibagi ~~nega~~ bilangan negatif, maka tanda pertidaksamaan akan berubah.

Conclusion:

If the inequality, both sides are added or subtracted together, the inequality sign does not change. If the inequality, both sides are multiplied or divided by a positive number, the inequality sign also does not change. If the inequality, both sides are multiplied or divided by a negative number, then the inequality sign will change.

Figure 2: Student response with correct procedure and valid reasoning

Referring to Figure 2, it is evident that students have successfully executed the "process" component. Furthermore, the "properties" component has been correctly applied by students in solving problem 1. The students' mental images can be inferred from the explanations provided in their answers. However, it is noteworthy that the students' answers in the "procedure" category are incorrect, and the reasons given are also inappropriate, as illustrated in Figure 3.

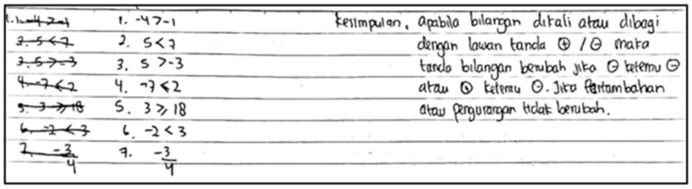


1 $4 > 1 \times (-1) \Rightarrow -4 > -1$
$3 < 5 + (2) \Rightarrow 5 < 7$
$6 > -4 + (-1) \Rightarrow 5 > -5$
$-5 \leq 4 - (-2) \Rightarrow -3 \leq 6$
$-1 \geq -6 \times (-3) \Rightarrow 3 \leq 18$
$-4 < 6 : (2) \Rightarrow -2 < 3$
$\frac{1}{4} < \frac{1}{2} : (-\frac{1}{3}) \Rightarrow -\frac{3}{4} < -\frac{3}{2}$
Kesimpulan : bahwa setiap pertidaksamaan jika dikali, dibagi, ditambah dan dibagi tanda < dan > tetap kecuali jika sistem pertidaksamaannya bernilai negatif dikali dan dibagi dengan bilangan negatif maka tandanya berubah

Conclusion: That every inequality if multiplied, divided, added and divided by the < and > signs remain the same, except if the inequality system has negative values and is multiplied and divided by negative numbers, then the sign changes.

Figure 3: Student response with incorrect procedure and flawed reasoning

Figure 3 shows that students follow the given instructions correctly. However, they make errors in the "properties" component, focusing on procedures from Problem 1 while neglecting the rules for inequalities involving negative numbers. As a result, their answers to Problem 1 are incorrect. Students' mental images reveal that they recognize a change in the inequality sign when a negative inequality is multiplied by a negative number. Additionally, other errors related to incorrect procedures are also accompanied by flawed reasoning. Figure 4 provides a detailed view of these errors.



~~1. -4 > -1~~ 1. $-4 > -1$
~~2. 5 < 7~~ 2. $5 < 7$
~~3. 5 > -3~~ 3. $5 > -3$
~~4. -7 < 2~~ 4. $-7 < 2$
~~5. 3 ≥ 18~~ 5. $3 \geq 18$
~~6. -2 < 3~~ 6. $-2 < 3$
~~7. -3/4~~ 7. $\frac{-3}{4}$

Kesimpulan, apabila bilangan dikali atau dibagi dengan lawan tanda ⊕ / ⊖ maka tanda bilangan berubah jika ⊖ ketemu ⊖ atau ⊕ ketemu ⊖. Jika pertambahan atau pengurangan tidak berubah.

Conclusion, if a number is multiplied or divided by the +/− sign, the number sign changes if − meets − or + meets −. If addition or subtraction does not change.

Figure 4: Student response with incorrect procedure and reasoning

As depicted in Figure 4, students have demonstrated proficiency in performing addition, subtraction, multiplication, and division operations. However, the methodology employed to solve the students' answers is flawed. This arises from the application of incorrect properties in the rules governing inequality concepts. Specifically, the significance of altering the sign of a number from positive to negative and vice versa is not utilized as a basis for students to revise their inequality notation. Consequently, the final solution to problem 1 is erroneous.

Conversely, students' mental representations of solving problem 1 primarily focus on the numerical values within the inequality notation. This tendency to make errors, as evident in Figures 3 and 4, can be attributed to students' reliance solely on memorizing the rules applicable to a particular concept. This is evident from the interview excerpts provided in Figure 3.

Researcher : *Could you please explain why the form of inequality changes when considering the five signs of inequality?*

Student : *The signs of inequality (*-1 *and* -6*) are negative. When multiplied by a negative number, the resulting number becomes positive. Consequently, the sign of the inequality also changes.*

This demonstrates that students' mental representations of the rule for changing inequality signs are based on the initial form of the inequality being negative and subsequently multiplied or divided by a negative number. This can be clearly illustrated in the following interview excerpt.

> *Researcher* : *Upon reviewing the first inequality, it appears that multiplying* 4 *and* 1 *by* -1 *does not alter the number.*
> *Student* : *Indeed, the number remains unchanged. However, the inequality sign does change.*
> *Researcher* : *Could you please clarify why the inequality sign changes when multiplying or dividing by a negative number?*
> *Student* : *To the best of my recollection, the sign of an inequality flips when it is divided or multiplied by a negative number because the resulting expression becomes negative, similar to the fifth part.*

The interview excerpt shows that students have inaccurate mental representations of inequality properties, as they rely on memorization rather than understanding the underlying concepts. Their inconsistent application of these rules in Problem 1 further reflects this gap in comprehension. While they handle positive and negative numbers correctly, their understanding of inequality rules remains incomplete. Figure 2, based on interview excerpts, illustrates this issue.

> *Researcher* : *Based on your response to problem 1, part two, can you clarify if the form* $3<5$ *is equivalent to* $5<7$*?*
> *Student* : *No, Sir. The numbers are different.*
> *Researcher* : *Could you please elaborate on why this is the case?*

Based on the interview excerpt, it appears that students lack a comprehensive understanding of the nature and rules governing the concept of inequality. Specifically, consider the following scenario: suppose a and b are real numbers such that $a<b$, and c is another real number that is distinct from zero. In this case, the inequality $a<b$ is equivalent to the following: $a+c<b+c$; $ac<bc$, provided that $c>0$; and $ac>bc$, provided that $c<0$. Consequently, based on their misunderstanding, students perceive the form 3<5 as equivalent to 5<7. However, this is incorrect. In fact, the form 5<7 is the result of adding the same quantity to both sides of the inequality 3<5, with the quantity being 2. Therefore, 5<7 is a new inequality that is equivalent to 3<5.

Students' Proficiency in Applying Inequality Rules and Interpreting Solution Sets

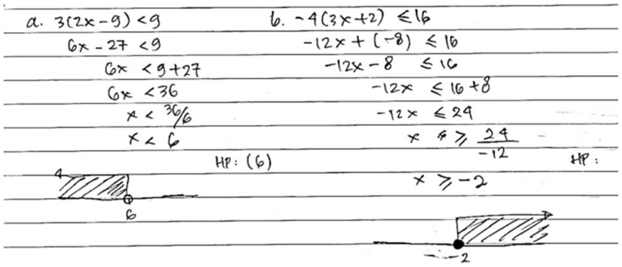
Problem 2 examines students' understanding of linear inequalities, focusing on determining and interpreting solution sets. Their responses will be analyzed to assess how they apply these rules. Before this analysis, we will categorize their answers based on responses and interviews. The categories are: (a) solutions using visual representation (number line) and (b) solutions using interval notation. Table 2 presents these categorized responses.

Response Category	Response	
	Point a	Point b
The solution procedure and visual representation are both correct.	8	8
The solution procedure and visual representation are both incorrect.	3	4
The solution procedure is correct, but cannot be represented in visual form.	6	3
The solution procedure is incorrect, and cannot be represented in visual form.	5	7

Table 2: Categories of student responses to problem 2 based on visual representation

The concept image component shows that students who correctly identify procedures and visualize linear inequalities can effectively apply the necessary steps to solve related problems. However, in solving Problem 2, they tend to use segment displacement. As a result, their understanding of linear inequality rules appears unstable and inconsistent. The following interview excerpt illustrates this further.

> *Researcher : Let us review your response* (Figure 5). *In the third step point a, why does the inequality become* $6x<9+27$?
> *Student : The initial form is negative 27, so when it is moved, it becomes positive 27.*
> *Researcher : Therefore, is the form* $6x<9+27$ *equivalent or the same as the initial form* $6x-27<9$?
> *Student : (Pauses for clarification)*



a. $3(2x-9)<9$
$6x-27<9$
$6x<9+27$
$6x<36$
$x<36/6$
$x<6$
HP: (6)

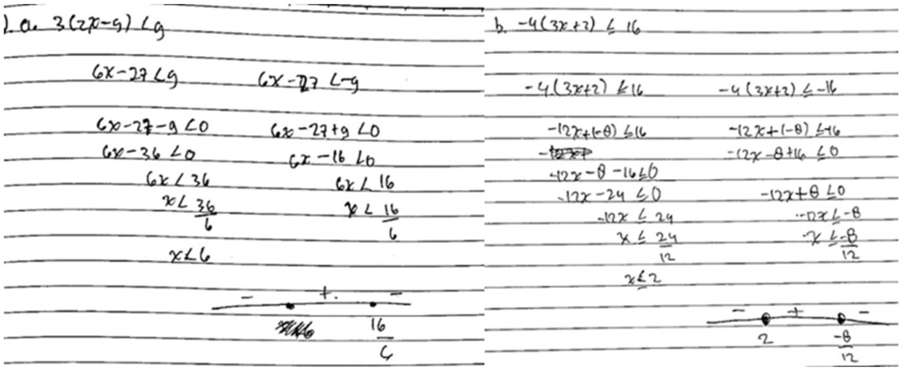
b. $-4(3x+2)\leq 16$
$-12x+(-8)\leq 16$
$-12x-8\leq 16$
$-12x\leq 16+8$
$-12x\leq 24$
$x \geq \frac{24}{-12}$
HP :
$x\geq -2$

Figure 5: Student response with correct procedure and accurate visual representation

The interview excerpt shows that students rely on procedural understanding when solving Problem 2. This aligns with their grasp of the "properties" of linear inequalities, including axioms, definitions, and theorems. However, their understanding remains incomplete when it comes to determining and interpreting solution sets using precise procedures and visual representations. The following student interview results illustrate this.

> *Researcher : In point a, you have* $x<6$, *correct? What does that imply?*
> *Student : Indeed,* x *is less than* 6, *sir.*
> *Researcher : Then why did you write the solution set* (6) *in point a and an empty set-in point b?*
> *Student : I am uncertain about this, sir. The result is* 6, *but the sign is confusing me, sir.*

The interview excerpt shows that students struggle to explain, determine, and interpret solution sets for linear inequalities. While they can follow procedures, they fail to interpret each step and understand the role of inequality notation. This leads to difficulties in solving inequalities. Figure 6 illustrates this issue with categorized student responses, highlighting errors in solution procedures and visual representations.



1. a. $3(2x-9)<9$

$6x-27<9$	$6x-27<-9$
$6x-27-9<0$	$6x-27+9<0$
$6x-36<0$	$6x-16<0$
$6x<36$	$6x<16$
$x<\frac{36}{6}$	$x<\frac{16}{6}$
$x<6$	

b. $-4(3x+2) \leq 16$

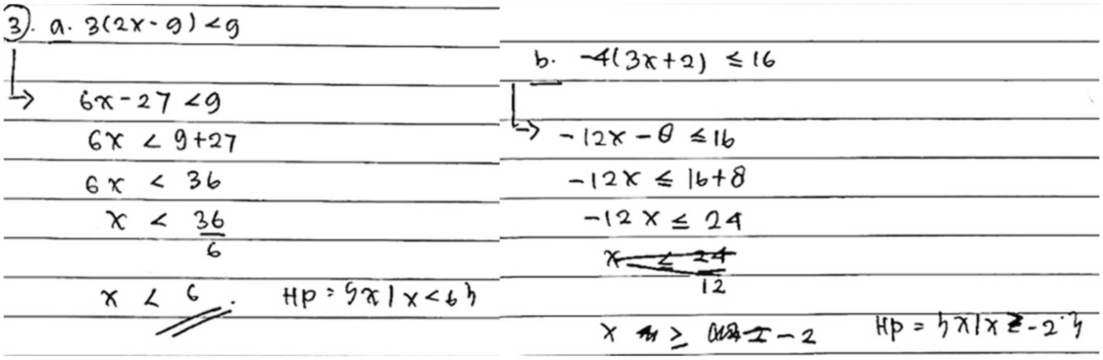
$-4(3x+2) \leq 16$	$-4(3x+2) \leq -16$
$-12x+(-8) \leq 16$	$-12x+(-8) \leq -16$
$-12x-8-16 \leq 0$	$-12x-8+16 \leq 0$
$-12x-24 \leq 0$	$-12x+8 \leq 0$
$-12x \leq 24$	$-12x \leq -8$
$x \leq \frac{24}{12}$	$x \leq \frac{-8}{12}$
$x \leq 2$	

Figure 6: Student response with incorrect procedure and inaccurate visual representation

Figure 6 shows that students use an incorrect approach to solving a single-variable linear inequality. They mistakenly apply properties of absolute value inequalities to Problem 2. However, interviews reveal that they do recognize it as a single-variable inequality. Despite this, they incorrectly use properties like "if $x \in R$, $a \in R$, and $a>0$, then $x<a$ if and only if $-a<x<a$," which leads to errors. This suggests that students overgeneralize inequality rules. The following interview excerpt illustrates this misconception.

> *Researcher* : *Could you please elaborate on the rationale behind your solution?*
> *Student* : *To the best of my recollection, that is the formula. If I am not mistaken, we need to calculate* $x<a$ *and* $x<-a$. *In question number 2, correct? In that case,* x *is* $3(2x-9)$ *and a is* 9.
> *Researcher* : *Yes, you are referring to the rule* $-a<x<a$.
> *Student* : *Indeed, sir.*

The interview excerpt shows that students understand linear inequality properties procedurally, relying on memorization rather than true comprehension. As a result, they generalize these rules without fully grasping their function. For students who use correct procedures but struggle with visual representation, Figure 7 provides further insights.



③. a. $3(2x-9)<9$

$\rightarrow 6x-27<9$

$6x<9+27$

$6x<36$

$x<\frac{36}{6}$

$x<6$ $\quad$ Hp = {x | x<6}

b. $-4(3x+2) \leq 16$

$\rightarrow -12x-8 \leq 16$

$-12x \leq 16+8$

$-12x \leq 24$

~~$x \leq \frac{24}{12}$~~

$x \geq -2$ $\quad$ Hp = {x | x ≥ -2}

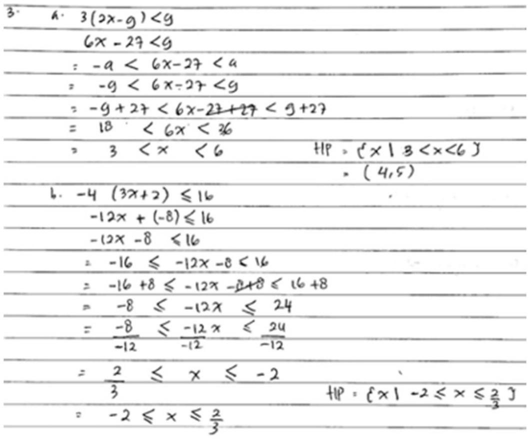
Figure 7: Student response with correct procedure but lacking visual representation

Figure 7 demonstrates that students have successfully solved problem 2 procedurally. This indicates that their "process" and "properties" for problem 2 are guided by their procedural understanding. However, the students have not been able to represent the set of solutions on a number line, as evident from the following interview excerpt.

Researcher : Based on the results you obtained, could you please present them on a number line, adhering to the instructions provided in the question?
Student : I am somewhat confused, Sir. How would you describe it?
Researcher : Could you please clarify what the expression "$x<6$" means?
Student : It means that the value of x is less than 6, Sir.
Researcher : You have indicated that you have solved the equation and obtained the solution set "$x<6$". Could you please provide the solution set in more detail?
Student : Umm… I believe the solution set refers to the values of x that satisfy the inequality, Sir.

Based on the interview excerpt, it is evident that students' mental representations of the understanding of the solution set of a linear inequality of one variable hinder their ability to visualize it, despite their proficiency in solving the problem procedurally. The students' responses in the solution procedure category are incorrect and cannot be represented visually, as illustrated in Figure 8.

Figure 8 demonstrates the students' incorrect approach to solving the linear inequality of one variable, which involves applying the rule $-a<x<a$ (for algebraic expressions X and $k>0$). This rule states that absolute value inequalities of the form $|X|<k$ are equivalent to $-k<X<k$, $|X|>k$ is equivalent to $X<-k$ or $X>k$, and $|X| \leq k$ and $|X| \geq k$. Therefore, it can be concluded that students generalize the rules or properties of linear inequality forms involving absolute values into linear inequality forms of one variable. Consequently, their "process" for finding the solution set of the inequality form is flawed. Simultaneously, students' mental representations of the solution set in interval notation are inadequate, as they fail to represent it visually (on a number line).



3. a. $3(2x-9)<9$
$6x-27<9$
$= -a < 6x-27 < a$
$= -9 < 6x-27 < 9$
$= -9+27 < 6x-\cancel{27+27} < 9+27$
$= 18 < 6x < 36$
$= 3 < x < 6$ HP $= \{x \mid 3<x<6\}$
$= (4,5)$
b. $-4(3x+2) \leq 16$
$-12x+(-8) \leq 16$
$-12x-8 \leq 16$
$= -16 \leq -12x-8 \leq 16$
$= -16+8 \leq -12x-\cancel{8+8} \leq 16+8$
$= -8 \leq -12x \leq 24$
$= \frac{-8}{-12} \leq \frac{-12x}{-12} \leq \frac{24}{-12}$
$= \frac{2}{3} \leq x \leq -2$
HP $= \{x \mid -2 \leq x \leq \frac{2}{3}\}$
$= -2 \leq x \leq \frac{2}{3}$

Figure 8: Student response with incorrect procedure and no visual representation

Researcher : In Problem 2, could you please elaborate on the rule or formula you employed to determine the solution set?

Student : Certainly, Sir. I utilized the formula $-a<x<a$*, where a is set to 16 and x is* $-4(3x+2)$*.*

Researcher : Attached is the solution set for item a, which is $3<x<6$ *and item b, which is* $-2 \leq x \leq \frac{2}{3}$*. Could you please clarify if the solution set on the number line graph is derived from the same formula or if it is a separate representation?*

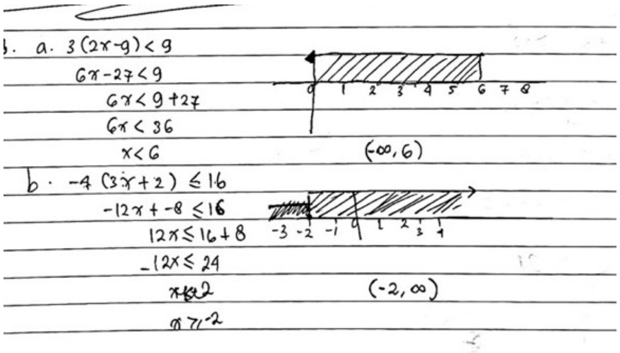
Student : I am somewhat perplexed, Sir. Could you please clarify whether the solution sets were constructed sequentially or if there is a different process involved?

Furthermore, the categories of students' responses in interpreting solution sets into interval notation are presented in Table 3.

Response Category	Response	
	Point a	Point b
The solution procedure and interpretation of the interval notation form are both correct.	1	0
The solution procedure is correct, but the interpretation of the interval notation form is incorrect.	2	2
The solution procedure and interpretation of the interval notation form are both incorrect.	1	2
The solution procedure is correct, but cannot be interpreted into the interval notation form.	11	9
The solution procedure is incorrect, and cannot be interpreted into the interval notation form.	7	9

Table 3: Categories of student responses to problem 2 based on interval notation

In addition to representing the set of solutions obtained by students using a number line, Problem 2 also requires students to express the set of solutions in interval notation. Out of the 22 students, only one student successfully represented the set of solutions obtained in interval notation for point a. Conversely, no students were able to correctly represent their set of solutions in interval notation for point b. This observation is depicted in Figure 9.



a. $3(2x-9)<9$
$6x-27<9$
$6x<9+27$
$6x<36$
$x<6$
$(-\infty,6)$

b. $-4(3x+2) \leq 16$
$-12x+-8 \leq 16$
$12x \leq 16+8$
$-12x \leq 24$
$x \cancel{\leq} 2$
$(-2,\infty)$
$x \geq -2$

Figure 9: Student response with correct procedure and accurate interval notation

Figure 9 shows that the student correctly solved Problem 2a and represented the solution set in interval notation. However, the solution does not fully align with the correct procedure category. In point b, while the student followed the correct steps, the interval notation representation is incorrect. This indicates a misunderstanding of inequality properties, particularly the distinction between < and > versus ≤ and ≥ in interval notation. The interview results below further support this finding.

Researcher : In interval notation, what is the solution set?
Student : The solution set is written in brackets as I have done here, Sir.
Researcher : Could you please clarify the meaning of $(-\infty, 6)$ *and* $(-2, \infty)$*?*
Student : $(-\infty, 6)$ *indicates that the value of* x *is less than 6, while* $(-2, \infty)$ *suggests that the value of* x *is greater than* -2.
Researcher : Therefore, for point a, is $-\infty$ *a solution set for* $x<6$*?*
Student : No, Sir. The solution set for $x<6$ *is* $(-\infty, 6)$.
Researcher : Similarly, for point b?
Student : The solution set for $x>-2$ *is* $(-2, \infty)$.
Researcher : Based on these explanations, is there a distinction between the solution sets for "$<$*" and "*$\geq$*" signs?*
Student : Yes, Sir.

The interview excerpt shows that students have an incomplete understanding of inequality properties. Their mental image of determining and interpreting the solution set for one-variable linear inequalities is also flawed. They mistakenly believe that interval notation does not differ for different inequality signs. Some errors involve correct solution procedures but incorrect interval notation representation. Figure 7 illustrates this mistake, where students incorrectly express the solution set. Similarly, Figure 8 shows students applying absolute value inequality rules to one-variable inequalities, leading to incorrect solutions. Their mental image of interval notation is also inadequate, as they struggle to express ≤ and ≥ correctly.

Researcher : Could you please clarify the meaning of $(4, 5)$*?*
Student : The solution set is $(4, 5)$ *because it was previously determined that* x *is greater than 3 and less than 6. The values between 3 and 6 are 4 and 5.*
Researcher : I understand. However, I am confused about the change in the solution set for point b in the last step. It went from $-2 \leq x \leq \frac{2}{3}$ *to* $-2 \leq x \leq \frac{2}{3}$*. Could you please explain the reason for this change?*
Student : The previous form was divided by -12*, sir. When dividing by a negative number, the sign changes to its opposite.*
Researcher : Thank you for your clarification. Could you please provide the solution set in interval notation?
Student : I apologize, but I am unable to recall the solution set in interval notation.

Figure 8 and the interview excerpt show that students' responses in the "completion procedures" and "interval notation representation" categories are incorrect. Their problem-solving process is also unsuitable, and their understanding of linear inequality properties is incomplete. As a result, they lack a clear mental picture of solution sets in interval notation.

Students who correctly follow solution procedures but struggle with interval notation lack a clear mental image of solution sets. They associate < and > with an empty circle and ≤ and ≥ with a full circle, as shown in Figure 5. However, some students incorrectly apply solution procedures and cannot represent them in interval notation, as seen in Figure 6. In these cases, they mistakenly generalize absolute value inequality rules to one-variable inequalities, leading to errors in determining the solution set. Their mental image of interval notation remains inadequate, as confirmed by the following interview results.

Researcher : Regarding point a, the form is $6x-27<-9$. *What is the result of the calculation* $-27+9$*?*
Student : -18, Sir
Researcher : Therefore, the result should be 3*, correct? Even if the result is* $\frac{16}{6}$*, which is smaller between* 6 *and* $\frac{16}{6}$*?*
Student : I am unsure, Sir.
Researcher : Then, what about the solution set in interval notation?
Student : Could you please clarify, Sir?

Based on the interview excerpt, it is evident that students lack comprehension of the solution set presented in interval notation. Consequently, they are unable to articulate their solutions effectively. Furthermore, students continue to make errors in calculating and ordering integers with fractional components. Therefore, it is concluded that the students' answers, categorized by solution procedures and representation in interval notation, are not accurate. Additionally, students' understanding of the properties applicable to linear inequalities is incomplete. Consequently, they do not possess a clear mental representation of the solution set in interval notation.

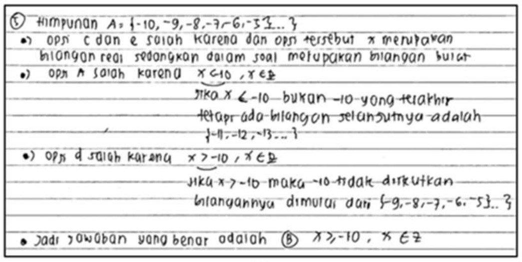
Students' Comprehension of Inequality Notation

Problem 3 examines students' ability to use inequality notation to represent mathematical situations symbolically. Their responses will be analyzed to understand their "concept image" of this process. Before exploring how they form this concept image, their answers will be categorized based on analysis (refer to Table 4) and interviews.

Response Category	Response
The mathematical symbol representation is correct.	9
The mathematical symbol representation is correct, but the reasoning is incorrect.	3
The mathematical symbol representation is correct, but no reasoning is provided.	2
The mathematical symbol representation is incorrect, and no reasoning is provided.	3
No response provided.	5

Table 4: Student response categories for problem 3

The assessment demonstrated a robust comprehension of numerical concepts and the accurate application of inequality notation, as evident in Problem 3. In this task, students were required to select the statement that accurately represents the given set $A=\{-10\text{-}9, -8, -7, -6, -5, \dots\}$. Figure 10 illustrates student responses categorized based on their representation of mathematical symbols and logical reasoning.

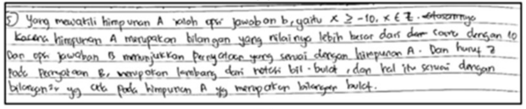


⑤ Himpunan A = {-10, -9, -8, -7, -6, -5, ...}
•) opsi c dan e salah karena dan opsi tersebut x merupakan bilangan real sedangkan dalam soal merupakan bilangan bulat
•) opsi A salah karena x < -10, x ∈ Z
jika x < -10 bukan -10 yang terakhir tetapi ada bilangan selanjutnya adalah {-11, -12, -13, ...}
•) opsi d salah karena x > -10, x ∈ Z
jika x > -10 maka -10 tidak diikutkan bilangannya dimulai dari {-9, -8, -7, -6, -5, ...}
• Jadi jawaban yang benar adalah (B) x ≥ -10, x ∈ Z

(a)

Set $A = \{-10, -9, -8, -7, -6, -5, \dots\}$

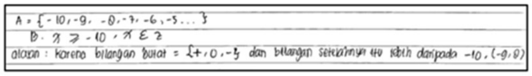
- Options c and e are wrong because from these options x is a real number while in the question it is an integer
- Option a is wrong because $x < -10, x \in \mathbb{Z}$
  If $x < -10$ is not the last -10, but there are the next numbers, namely $\{-11, -12, -13, \dots\}$
- Option d is wrong because x>-10, $x \in \mathbb{Z}$
  If $x > -10$ then -10 is not included, the numbers start from $\{-9, -8, -7, -6, -5, \dots\}$
- So, the correct answer is b, $x \geq -10, x \in \mathbb{Z}$



⑤ Yang mewakili himpunan A ialah opsi jawaban b, yaitu x ≥ -10, x ∈ Z. ~~alasannya~~ karena himpunan A merupakan bilangan yang nilainya lebih besar dari ~~dan~~ sama dengan 10 Dan opsi jawaban B menunjukkan pernyataan yang sesuai dengan himpunan A. Dan huruf Z pada pernyataan B, merupakan lambang dari notasi bil. bulat, dan hal itu sesuai dengan bilangan2 yg ada pada himpunan A yg merupakan bilangan bulat.

(b)

The representative of set A is answer option b, namely $x \geq -10, x \in \mathbb{Z}$. Because set A is a number whose value is greater than or equal to 10. And answer option B shows a statement that corresponds to set A. and the letter $\mathbb{Z}$ in statement B is a symbol of integer notation, and it corresponds to the numbers in set A which are integers.



A = {-10, -9, -8, -7, -6, -5 ...}
B. x ≥ -10, x ∈ Z
alasan : karena bilangan bulat = {+, 0, -} dan bilangan setelahnya itu lebih daripada -10, (-9, 8)

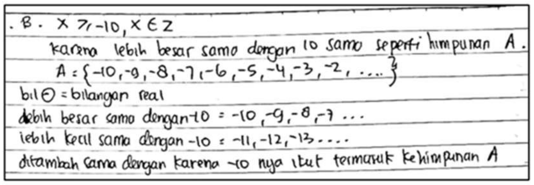
(c)

Reason: Because the integers $= \{+, 0, -\}$ and the number after that is more than -10, $(-9, 8)$

Figure 10: Student responses with correct symbolic representation and reasoning

Figure 10 shows that students correctly follow procedures and methods to solve Problem 3. They do more than just select answers—they also provide valid reasoning. In Figure 10(a), students offer detailed explanations and correctly apply the inequality notation ≥. A thorough review confirms that students accurately use properties such as axioms, definitions, and theorems. Additionally, their mental images of the concepts in Problem 3 are well-formed.

In the mathematical symbol representation category, the reasoning is incorrect. While students correctly justify using the inequality notation ≥, their understanding of why set A in Problem 3 is expressed as $x \in Z$ is flawed. This indicates that while their process and use of properties for inequality notation are correct, their concept image is not. Figure 11 illustrates this issue.



. B. x ≥ -10, x ∈ Z
karena lebih besar sama dengan 10 sama seperti himpunan A.
A = {-10, -9, -8, -7, -6, -5, -4, -3, -2, ...}
bil ⊖ = bilangan real
lebih besar sama dengan -10 = -10, -9, -8, -7 ...
lebih kecil sama dengan -10 = -11, -12, -13 ....
ditambah sama dengan karena -10 nya itu termasuk ke himpunan A

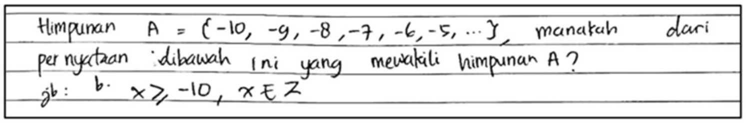
B. $x \geq -10, x \in \mathbb{Z}$
Because greater than or equal to 10 is the same as set A.
$A = \{-10, -9, -8, -7, -6, -5, -4, -3, -2, \dots\}$
Number $-$ = real number
Greater than or equal to $-10 = -10, -9, -8, -7, \dots$
Smaller than or equal to $-10 = -11, -12, -13, \dots$
Added equal to because -10 is also included in set A

Figure 11: Student response with correct symbolic representation but incorrect reasoning

Referring to Figure 11, it becomes evident that the reasons provided by students deviate from their answer choices, which explicitly state that set A comprises elements of the set of integers ($x \in Z$). To gain insights into the students' conceptual understanding of the problem presented in problem 3, kindly refer to the following interview excerpt.

Researcher : Could you please elaborate on why the answer is b?
Student : Sir, the answer is B because if the number is negative, the closer it is to zero, the larger its absolute value becomes. Therefore, the sign is larger because -10 *is included in the* A *set, making it equal to or greater than zero.*
Researcher : In that case, why choose Z *instead of* R*?*
Student : Sir, I need clarification on the definition of an integer.
Researcher : Could you please provide a definition of an integer?
Student : Certainly, an integer is a whole number, including negative numbers such as -10, -9*, and so on.*

The interview excerpt shows that students struggle to justify selecting set A in Problem 3 as $x \in Z$. Their problem-solving process for inequalities, especially those involving numbers, is underdeveloped due to an incomplete understanding of integer properties. While their mental image of using inequality notation ≥ is correct, their understanding of number concepts in Problem 3 is lacking. Although their mathematical symbol representation is accurate, it lacks supporting explanations, as shown in Figure 12.



Himpunan A = {-10, -9, -8, -7, -6, -5, ...}, manakah dari pernyataan dibawah ini yang mewakili himpunan A?
jb: b. $x \geq -10$, $x \in Z$

Set $A = \{-10, -9, -8, -7, -6, -5, \ldots\}$, which of the statements below represents set A?

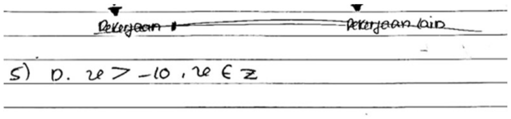
Answer: b. $x \geq -10, x \in \mathbb{Z}$

Figure 12: Student response with correct symbolic representation but no reasoning

Figure 12 shows that students can translate mathematical scenarios into symbolic representations but struggle to justify their choices. Interviews confirm they understand how to use the inequality notation ≥ but cannot explain the concept of numbers in Problem 3. They also lack an understanding of the properties and definitions of numbers, leading to weak reasoning for selecting option b. This is due to the absence of a clear mental image of number concepts.

Researcher : Could you please elaborate on your decision to represent set A with the letter "b"?
Student : Given that set A is known to contain elements such as -10, -9, -8*, and so on, I chose "*$x \geq -10$*" as the representation. This expression ensures that the value of "*x*" is greater than or equal to* -10*, and* -10 *is included within the set.*
Researcher : In contrast, why did you select "Z*" instead of "*R*"? Since "e" and "b" represent the same value, only the symbol differs.*
Student : Indeed, you are correct.

Figure 13 shows that students' mathematical symbol representation is incorrect and lacks justification. This suggests they do not fully understand inequality notation. Interviews reveal that students use a flawed process to solve Problem 3, mistakenly believing that a negative number decreases in value as it approaches 0. As a result, they incorrectly choose the ">" notation to represent mathematical situations.



5) D. $x > -10, x \in Z$

Figure 13: Student response with inaccurate symbolic representation and no justification

Researcher : Could you please elaborate on your choice of option D?
Student : In negative numbers, the closer a value is to zero from the right, the larger its value becomes. Therefore, I use the greater sign to indicate this.

Further study shows that students can distinguish between ">" and "≥" and illustrate this difference with examples. This indicates their understanding of the properties and definitions of inequality notation. However, while they have a mental image of interpreting inequalities, they struggle to understand the numerical notation in Problem 3.

Researcher : Could you please clarify the meaning of the symbol "$x \in Z$"?
Student : I am unfamiliar with this symbol, sir.
Researcher : Have you encountered any similar symbols in the past?
Student : I do not recall, sir.

The interview excerpt shows that students do not fully understand number notation, preventing them from forming a mental representation of it. As a result, they struggle to explain the correct solution process for Problem 3. Their difficulty stems from not grasping the fundamental properties and definitions of number notation. Students who do not attempt an answer face even greater challenges in solving the problem. This is further illustrated in the following interview excerpt.

Researcher : Could you please clarify why you did not respond to question number 3?
Student : I am unable to comprehend the interpretation of the question, sir.
Researcher : The question explicitly states the instruction to select one of the forms of inequality presented in the question.
Student : I acknowledge this, sir, but I remain perplexed.

The provided interview excerpt suggests that students lack a comprehensive understanding of the problem contest presented in Problem 3. This could be attributed to the absence of a "mental image" or all relevant thoughts (memory) that students can utilize to explain the concept encapsulated in Problem 3. In this instance, students are lacking a mental representation associated with the concept of the set they have acquired. The form of inequality employed in the choices presented in Problem 3 can be interpreted as a set-forming notation designed to simplify the listing of members in Set *A* within the problem. Consequently, it is unsurprising that students struggle to articulate the "process," "procedure," or even describe the "properties" that can be employed to describe the concept contained in Problem 3 and effectively solve it.

DISCUSSION

The findings of this study provide a nuanced understanding of students' "concept images" of inequality, exposing both strengths and substantial gaps in their procedural and conceptual knowledge. These insights can be effectively analyzed through various theoretical frameworks, including Tall and Vinner's (1981) concept image and concept definition, Skemp's (2006) distinction between instrumental and relational understanding, and Vygotsky's (1978) sociocultural theory of learning. Furthermore, the results align with prior research on students' misconceptions and challenges in comprehending mathematical concepts, particularly inequalities.

Based on the data presented in the research, students' "concept image" of the concept of inequality can be categorized into several distinct groups, reflecting their comprehension, reasoning, and application of inequality rules. These categories are derived from their responses to the problems and the interview excerpts. Please refer to the attached Table 5 for a detailed categorization of students' "concept image" regarding the concept of inequality.

1	Category	Correct procedure with appropriate reasoning
	Description	Students in this category possess a robust understanding of the principles of governing inequality. They are adequate at executing correct procedures and provide sound and logical justifications for their solutions.
	Evidence	• Students accurately modify inequality notation when multiplying or dividing by negative numbers and provide lucid explanations (Figure 2). • They can elucidate the rationale behind the change in inequality sign when multiplying or dividing by negative numbers, demonstrating a profound conceptual understanding.
	Concept Image	Students possess a comprehensive and coherent "concept image" that is congruent with formal mathematical definitions. This image effectively integrates procedural and conceptual knowledge.
2	Category	Correct procedure with inadequate or missing reasoning
	Description	Students in this category may execute the correct procedure but lack sufficient justification or demonstrate no justification at all. Their comprehension is procedural rather than conceptual.
	Evidence	• Students correctly solve inequalities but lack the ability to explain the change in inequality sign when multiplying or dividing by negative numbers (Figure 1). • Interviews revealed that several students admitted to forgetting or omitting explanations due to time constraints, suggesting a reliance on memorization rather than understanding the underlying principles.
	Concept Image	Students have a procedural "concept image" but lack the conceptual depth to explain the basic principles of the inequality rules.
3	Category	Incorrect procedure with inappropriate reasoning
	Description	Students in this category demonstrate an incorrect understanding of the rules of inequality, applying incorrect procedures and providing inappropriate reasoning.
	Evidence	• Students misapplied inequality rules, including failing to change the inequality sign when multiplying or dividing by a negative number (Figure 3). • Some students provided incorrect explanations, such as assuming the inequality sign only changes if the original form is negative (Figure 4).
	Concept Image	Students often hold flawed "concept images" of inequality due to misunderstandings or incomplete comprehension of its rules. Consequently, they make errors in procedures and reasoning.
4	Category	Correct visual representation with procedural understanding

	Description	Students in this category possess the ability to solve inequalities accurately and visually represent the solution set on a number line. However, they encounter challenges in comprehending and articulating the conceptual nature of the solution set.
	Evidence	• Students demonstrate proficiency in solving inequalities and representing solution sets on a number line. However, they lack the ability to articulate the interpretation of solution sets in interval notation (Figure 5). • During interviews, students exhibited confusion regarding the various inequality symbols, such as “<” and “≤,” when representing solution sets.
	Concept Image	Students possess a procedural “concept image” that enables them to visualize solutions, but they lack the conceptual understanding to fully interpret or articulate their solutions.
5	Category	Incorrect visual representation with procedural errors
	Description	Students in this category demonstrate errors in solving inequalities and representing solution sets visually, suggesting a deficiency in comprehension of inequality rules and their practical applications.
	Evidence	• Students frequently misinterpret inequality rules, particularly the absolute value property, resulting in incorrect solution sets (Figure 6). • Additionally, they demonstrate a lack of proficiency in representing solution sets accurately on a number line or in interval notation (Figure 8).
	Concept Image	Students possess a fragmented “concept image” characterized by substantial gaps in procedural and conceptual knowledge, resulting in errors in solving and representing inequalities.
6	Category	Correct symbolic representation with inadequate reasoning
	Description	Students in this category demonstrate proficiency in translating mathematical scenarios into inequality notation. However, they lack the ability to provide valid reasons for their choices.
	Evidence	• Students accurately represented mathematical scenarios using inequality symbols (e.g., “$x \geq -10$”), but they lacked the ability to explain their choices or the underlying notation (Figure 12). • Interviews revealed a lack of conceptual understanding, as students were unable to articulate the reasons behind their decisions.
	Concept Image	Students possess a procedural “concept image” that facilitates symbolic translation, but they lack the conceptual depth to substantiate their reasoning.
7	Category	Incorrect symbolic representation with flawed reasoning
	Description	Students in this category exhibit an incorrect translation of mathematical scenarios into inequality notation, accompanied by inappropriate reasoning. This demonstrates a lack of understanding of inequality rules and numerical concepts.
	Evidence	• Students exhibited errors in the usage of inequality symbols, such as incorrectly employing “>” instead of “≥.” Additionally, they provided incorrect explanations, including a misinterpretation of the value of negative numbers (Figure 13). • During interviews, students demonstrated a lack of comprehension regarding the behavior of negative numbers and the conventions of inequality notation.
	Concept Image	Students possess an erroneous “concept image” stemming from misconceptions regarding inequality rules and numerical principles. Consequently, they incur errors in symbolic representation and reasoning.
8	Category	No response or inability to solve
	Description	Students in this category demonstrate a profound lack of comprehension regarding the concept of inequality, as evidenced by their inability to solve the problem or provide any response.
	Evidence	• Students failed to attempt solving the problem or provide any answers, frequently due to confusion or a lack of comprehension (Table 4).

	- In interviews, students indicated difficulties in comprehending the problem or the necessary steps to resolve it.
Concept Image	Students exhibit a poorly developed or absent "concept image" of inequality, indicating substantial gaps in procedural and conceptual understanding.

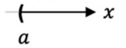
Table 5: Description of students' "concept image" categorization in the concept of inequality

These categories categorize students' varying levels of concept image regarding the concept of inequality, ranging from a strong grasp of procedural and conceptual principles to substantial misconceptions and knowledge gaps.

Concept Image vs. Concept Definition: A Disjunction in Understanding

The distinction between concept image and concept definition, as introduced by Tall and Vinner (1981), remains a crucial framework for comprehending how individuals internalize and apply mathematical concepts. According to their theory, concept image refers to the mental representations, intuitions, and associated properties that individuals develop about a concept, while concept definition represents the formal, mathematical articulation of that concept. Tall and Vinner (1981) posited that discrepancies between these two constructs frequently lead to persistent misconceptions, particularly in mathematics education. This study corroborates their findings, demonstrating that students frequently exhibit a procedural understanding of inequality rules—such as reversing the inequality sign when multiplying or dividing by a negative number—yet struggle to articulate the underlying conceptual rationale. This reliance on procedural knowledge, devoid of deeper conceptual understanding, suggests that students' concept images are often shaped by rote memorization rather than a meaningful engagement with formal mathematical principles. Table 6 presents a comparative analysis between students' concept images and the corresponding concept definitions, categorized based on the identified concept image categories.

Category	Student Concept Image	Concept Definition
Correct procedure with appropriate reasoning	Students possess a comprehensive and coherent "concept image" that is congruent with formal mathematical definitions. This image effectively integrates procedural and conceptual knowledge. An example of evidence: Figure 2	Let a, b, $c \in R$. If $a>b$ and $c<0$, then $ac<bc$. Consider any a, b, $c \in R$, such that $a>b$ and $c<0$. According to the definition $a>b$ implies $a\text{-}b \in P$ and $c<0$ implies $\text{-}c \in P$. It will be demonstrated that $ac<bc$. Consequently, it is imperative to demonstrate that $bc\text{-}ac \in P$. Using the order properties, the product of $a\text{-}b \in P$ dan $\text{-}c \in P$, $\text{-}c(a\text{-}b)\text{=-}ac\text{+}bc\text{=}bc\text{-}ac \in P$. Thus, we obtain $bc\text{-}ac \in P$, Confirming that $ac<bc$
Correct procedure with inadequate or missing reasoning	Students have a procedural "concept image" but lack the conceptual depth to explain the basic principles of the inequality rules.	Let a, b, $c \in R$. If $a>b$ and $c<0$, then $ac<bc$. Consider any a, b, $c \in R$, such that $a>b$ and $c<0$. According to the definition $a>b$ implies $a\text{-}b \in P$ and $c<0$ implies $\text{-}c \in P$. It will be demonstrated

Category	Student Concept Image	Concept Definition
	An example of evidence: Figure 1	that $ac<bc$. Consequently, it is imperative to demonstrate that $bc-ac \in P$. Using the order properties, the product of $a-b \in P$ dan $-c \in P$, $-c(a-b)=-ac+bc=bc-ac \in P$. Thus, we obtain $bc-ac \in P$, Confirming that $ac<bc$
Incorrect procedure with inappropriate reasoning	Students often hold flawed "concept images" of inequality due to misunderstandings or incomplete comprehension of its rules. Consequently, they make errors in procedures and reasoning. An example of evidence: Figure 3 and 4	Let $a, b, c \in R$. If $a>b$ and $b>c$, then $a>c$. If $a>b$ then $a+c>b+c$. If $a>b$ and $c>0$, then $ac>bc$. If $a>b$ and $c<0$, then $ac<bc$.
Correct visual representation with procedural understanding	Students possess a procedural "concept image" that enables them to visualize solutions, but they lack the conceptual understanding to fully interpret or articulate their solutions. An example of evidence: Figure 5	An infinite open interval is a set of the form $(a,\infty):=\{x \in R \mid x>a\}$ and $(-\infty,b):=\{x \in R \mid x<b\}$. The first set has no upper bound, while the second set has no lower bound. Interval notation: (a,∞) Form of inequality: $x>a$ Graphics: 
Incorrect visual representation with procedural errors	Students possess a fragmented "concept image" characterized by substantial gaps in procedural and conceptual knowledge, resulting in errors in solving and representing inequalities. An example of evidence: Figure 6 and 8	The properties of inequalities that facilitate transforming a given inequality into an equivalent inequality are as follows: Let a and b be real numbers such that $a<b$, and let c be another nonzero real number. Then, the inequality $a<b$ is equivalent to: $a+c<b+c$ $ac<bc$, for $c>0$ $ac>bc$, for $c<0$
Correct symbolic representation with inadequate reasoning	Students possess a procedural "concept image" that facilitates symbolic translation, but they lack the conceptual depth to substantiate their reasoning. An example of evidence: Figure 12	An interval can be classified as open, closed, half-open, or unbounded. Open interval (a,b) includes all real numbers between a and b but excludes the endpoints. Closed interval $[a,b]$ includes both endpoints. Half-open intervals $[a,b)$ and $(a,b]$ include only one endpoint. If $a=b$, the open interval is empty, while the closed interval contains a single element. Unbounded intervals use ∞ and $-\infty$ as notation for non-existent endpoints: Open infinite intervals: (a,∞) (no upper bound) and $(-\infty,b)$ (no lower bound). Closed infinite intervals: $[a,\infty)$ and $(-\infty,b]$.

Category	Student Concept Image	Concept Definition
		The entire real number set is expressed as $(-\infty, \infty)=R$, which has no defined endpoints.
Incorrect symbolic representation with flawed reasoning	Students possess an erroneous "concept image" stemming from misconceptions regarding inequality rules and numerical principles. Consequently, they incur errors in symbolic representation and reasoning. An example of evidence: Figure 13	An interval can be classified as open, closed, half-open, or unbounded. Open interval (a, b) includes all real numbers between a and b but excludes the endpoints. Closed interval $[a, b]$ includes both endpoints. Half-open intervals $[a, b)$ and $(a, b]$ include only one endpoint. If $a=b$, the open interval is empty, while the closed interval contains a single element. Unbounded intervals use ∞ and $-\infty$ as notation for non-existent endpoints: Open infinite intervals: (a, ∞) (no upper bound) and $(-\infty, b)$ (no lower bound). Closed infinite intervals: $[a, \infty)$ and $(-\infty, b]$. The entire real number set is expressed as $(-\infty, \infty)=R$, which has no defined endpoints.
No response or inability to solve	Students exhibit a poorly developed or absent "concept image" of inequality, indicating substantial gaps in procedural and conceptual understanding. An example of evidence: Table 4	The order properties of R, the fundamental rules of inequalities, and the interpretation of inequality solution sets in interval notation and graphical representation.

Table 6: Illustration of the comparison of students' concept images and concept definitions

Based on the presented evidence regarding students' concept images of inequalities, it was observed that some students exhibit partial understanding. While they correctly apply procedural steps, they lack a clear conceptual grasp of the underlying principles. For instance, certain students are able to manipulate inequality notation but struggle to justify their reasoning or misinterpret interval notation. To bridge the gap between correct procedural execution and conceptual reasoning, instructional strategies should emphasize conceptual exploration alongside procedural practice (Veith et al., 2023; Chan et al., 2023).

The phenomenon of procedural reliance without conceptual depth has been extensively documented in prior research. For instance, Tsamir and Almog (2001) demonstrated that students frequently apply inequality rules mechanically, resulting in errors when confronted with non-routine problems. Similarly, Linchevski and Sfard (1991) underscored that students' over-reliance on procedural knowledge stems from a lack of integration between their intuitive concept images and formal definitions. These findings are further supported by more recent studies. For example, Star and Stylianides (2013) highlighted that students' procedural fluency often conceals substantial gaps in their conceptual understanding, particularly in algebra and calculus. They argued that instructional practices emphasizing procedural shortcuts can inadvertently reinforce superficial learning, perpetuating the disjunction between concept image and concept definition.

Furthermore, recent research has highlighted the cognitive and pedagogical challenges associated with bridging the gap between conceptual and procedural understanding in mathematics education. Jupri et al. (2014) demonstrated that students' misconceptions regarding inequalities often stem from an overemphasis on algorithmic problem-solving strategies in curricula, which neglects the development of conceptual reasoning. Similarly, Rittle-Johnson et al. (2015) showed that students who receive instruction focused solely on procedural skills are less likely to transfer their knowledge to novel contexts compared to those who engage in conceptually rich learning experiences. These findings collectively underscore the necessity of instructional strategies that explicitly address the disjunction between concept image and concept definition.

To foster a more integrated understanding of mathematical concepts, educators must adopt pedagogical approaches that prioritize conceptual depth alongside procedural fluency. For instance, the utilization of multiple representations, such as graphical, numerical, and algebraic, has been demonstrated to enhance students' ability to connect their intuitive concept images with formal definitions (Ainsworth, 2018). Additionally, inquiry-based learning and problem-solving tasks that encourage students to explore the underlying reasons behind mathematical rules can facilitate the bridge between procedural and conceptual knowledge (Hurrell, 2021; Fatqurhohman, 2016; McCormick, 1997). These strategies align with the constructivist perspective of learning, which posits that meaningful understanding arises from actively constructing connections between prior knowledge and new information (Vygotsky, 1978). In conclusion, the disjunction between concept image and concept definition remains a substantial challenge in mathematics education, particularly in the context of inequality rules. While procedural knowledge is fundamental, it must be complemented by a robust conceptual understanding to prevent persistent misconceptions.

Instrumental vs. Relational Understanding: The Crucial Gap

Skemp's (2006) seminal work introduced a critical distinction between instrumental understanding and relational understanding. Instrumental understanding encompasses the ability to execute mathematical procedures and apply rules with precision, while relational understanding entails a profound comprehension of the underlying principles, connections, and reasons that underpin these procedures. This distinction remains highly pertinent in contemporary mathematics education, as evidenced by recent research underscoring the persistent disparity between procedural proficiency and conceptual mastery among students.

The data presented in this study aligns with Skemp's framework, demonstrating that a substantial portion of students exhibit instrumental understanding rather than relational understanding. For instance, while numerous students demonstrated procedural competence in solving inequalities, such as correctly altering the inequality sign when multiplying by a negative number, they were unable to articulate the reasoning behind this rule. This inability to justify mathematical procedures highlights a critical deficiency in relational understanding, which is fundamental for developing a comprehensive conceptual grasp of mathematical concepts, particularly inequalities. As Rittle-Johnson and Schneider (2015) aptly state, procedural knowledge without conceptual understanding often leads to precarious learning, where students encounter difficulties in applying their knowledge flexibly or adapting it to novel contexts.

The implications of this gap are profound. Research by Skemp (2006) and Hiebert and Lefevre (1986) consistently demonstrates that students who primarily rely on instrumental understanding are more susceptible to errors, particularly when confronted with unfamiliar problems or tasks that require justification. For instance, Star and Stylianides (2013) observed that students with predominantly instrumental understanding often fail to discern the logical connections between mathematical concepts, resulting in misconceptions and difficulties in problem-solving. This underscores the limitations of rote memorization and procedural fluency in promoting long-term mathematical proficiency.

Recent studies further emphasize the paramount importance of prioritizing relational understanding in mathematics education. For instance, Boaler (2022) contends that a focus on conceptual comprehension promotes deeper engagement with mathematics, enabling students to perceive the subject as a coherent and interconnected discipline rather than a collection of isolated rules and procedures. Similarly, Tall (2013) underscores that relational understanding fosters mathematical adaptability, allowing students to adapt their knowledge to diverse problem-solving scenarios.

In conclusion, the distinction between instrumental and relational understanding remains a pivotal consideration in mathematics education. While instrumental understanding facilitates accurate procedure execution, it falls short of developing the profound conceptual knowledge necessary for advanced mathematical reasoning. By prioritizing relational understanding, educators can equip students with the means to transcend rote memorization, engage with mathematics as a dynamic and interconnected discipline, and ultimately attain a more profound and enduring mastery of the subject.

Addressing Flawed Concept Images: Unraveling Misconceptions and Errors

Misconceptions in mathematics frequently originate from students’ flawed concept images, which are mental representations formed by incomplete or erroneous understandings of mathematical concepts. These misconceptions persist even after formal instruction is provided if not explicitly identified and corrected (Tall & Vinner, 1981). A prevalent example is students’ errors in inequalities, such as failing to reverse the inequality sign when multiplying or dividing by a negative number or misapplying absolute value properties in linear inequalities. These persistent misconceptions suggest a superficial comprehension of mathematical principles, resulting in challenges in reasoning and problem-solving.

Research consistently demonstrates that students encounter difficulties comprehending the abstract nature of inequalities, frequently mistreating them as equations. Tsamir and Almog (2001) identified a significant number of students who fail to distinguish between equations and inequalities, particularly when dealing with negative coefficients. Similarly, Linchevski and Sfard (1991) observed frequent misinterpretations of inequality directionality, which adversely impacts students' ability to solve and graph inequalities accurately. More recent studies corroborate these findings, underscoring the detrimental effects of an over-reliance on procedural knowledge without a conceptual understanding of inequalities (Kieran, 2020; Bofferding & Wessman-Enzinger, 2018).

Another prevalent source of errors lies in students' misapplication of absolute value properties to inequalities. Many students assume that the rules governing absolute value in equations directly apply to inequalities, resulting in incorrect solution sets (Zazkis & Chernoff, 2008). This misinterpretation underscores the need for instructional strategies that clarify the conceptual distinctions between absolute value in different mathematical contexts.

To address these misconceptions, instructional approaches must directly target flawed concept images. Research suggests that cognitive conflict tasks, which challenge students' existing misunderstandings, are effective in promoting conceptual change (Star & Stylianides, 2013). Furthermore, visual representations, such as number lines and graphs, can reinforce students' comprehension of inequalities and their properties (Battista, 2017). Formative assessments also play a pivotal role in diagnosing misconceptions. Studies indicate that timely feedback and diagnostic questioning assist educators in uncovering and correcting students' erroneous reasoning (Wiliam, 2016). Encouraging students to articulate their thought processes when solving inequalities can further expose underlying errors and facilitate conceptual reinforcement.

In conclusion, addressing flawed concept images in mathematics necessitates a multifaceted approach that integrates conceptual instruction, visual representations, and targeted interventions. By identifying and rectifying prevalent misconceptions, particularly in inequality operations and absolute value applications, educators can facilitate students' acquisition of accurate and robust mathematical concepts. Future research should delve into the cognitive mechanisms underpinning these misconceptions and refine evidence-based instructional strategies to enhance students' mathematical reasoning.

Enhancing Conceptual Understanding in Mathematics: The Role of Visualization and Representation

Visualization and representation are indispensable tools in the development of conceptual understanding, particularly in the context of abstract mathematical concepts such as inequalities. As Duval (2006) posits, the ability to translate between various representations—symbolic, graphical, and verbal—is crucial for students to fully grasp mathematical concepts. This study underscores a persistent challenge: students who excel in solving inequalities often encounter difficulties in visually representing solution sets, such as on a number line or in interval notation. This disparity suggests a gap in their comprehension of the relationships between different representations of inequalities, which, in turn, impedes their ability to interpret and communicate solutions effectively.

The significance of employing multiple representations in mathematics education has been extensively documented in previous research. Duval (2006) and Presmeg (2006) assert that students who possess the ability to effectively translate between diverse forms of representation are more inclined to develop a profound conceptual comprehension of mathematical concepts. For instance, Ainsworth (2018) conducted a study that revealed that students who actively engaged with multiple representations of algebraic concepts, such as inequalities, exhibited enhanced problem-solving abilities and a more comprehensive understanding of the underlying principles. This emphasizes the imperative of instructional strategies that actively encourage students to explore and establish connections between various representations of inequalities, thereby fostering a more holistic comprehension of the concept.

The findings of this study highlight the complexities of students' conceptions of inequality and underscore the necessity of instructional approaches that simultaneously address both procedural and conceptual comprehension. By integrating theoretical frameworks from Tall and Vinner (1981), Skemp (2006), and Vygotsky (1978), educators can devise interventions that facilitate the bridging of the gap between students' conceptions and formal definitions, promote relational understanding, and provide the necessary support for conceptual development. Future research should investigate the efficacy of such strategies in diverse educational settings, ensuring that all students have the opportunity to acquire a comprehensive and accurate understanding of inequality.

Furthermore, the findings of this study can be extended to college mathematics, particularly in courses that require a robust foundation in algebraic reasoning and problem-solving. The concept of inequality holds significant contributions in college mathematics and other disciplines, such as functions, calculus, and algebraic reasoning. In function analysis, inequality is employed to determine the domain, range, and limits of a function. In calculus, this concept plays a role in the definition of limits using the epsilon-delta method, derivative analysis to identify intervals of increasing and decreasing functions, and in estimating integral values. Additionally, in algebra and systems of inequalities, this concept is utilized in mathematical optimization, linear programming, and number theory. Inequalities also have practical applications in statistics and probability, such as in Chebyshev and Markov inequalities for estimating data distributions. Overall, a comprehensive understanding of inequalities facilitates students in grasping more intricate and practical mathematical concepts across diverse fields.

CONCLUSION

This analysis of students' concept images of inequality reveals both strengths and weaknesses in their procedural and conceptual knowledge. While many students demonstrated proficiency in solving inequalities, particularly in executing operations such as changing the inequality sign when multiplying or dividing by negative numbers, a substantial portion struggled to articulate the reasoning behind these procedures. Common errors included misapplying absolute value properties, failing to change the inequality sign appropriately, and misinterpreting solution sets, suggesting an overreliance on procedural knowledge rather than a deep conceptual understanding. Furthermore, students exhibited difficulties in translating between different representations of inequalities, such as expressing solutions on a number line or in interval notation. While some students correctly utilized inequality notation in mathematical scenarios, many were unable to justify their choices, underscoring the necessity of instructional strategies that foster relational understanding rather than mere procedural execution. These findings reinforce the imperative of teaching approaches that bridge conceptual gaps and support students in internalizing the principles governing inequalities.

Although this study provides valuable insights into students' comprehension of inequalities, certain limitations should be considered. Although the researchers have mitigated potential biases by ensuring sample diversity, they acknowledge that the findings primarily reflect the characteristics of students within the study's specific context. Further research with a larger and more diverse sample would enhance the generalizability of the results and validate these conclusions. Additionally, the study relied on a predetermined problem set and interviews to assess students' understanding, which may not have captured the full spectrum of their conceptual and procedural knowledge. Expanding assessment methods could offer a more nuanced perspective on students' difficulties. To address these gaps, educators should integrate teaching strategies that emphasize conceptual understanding, such as scaffolding, collaborative learning, and targeted interventions to correct specific misconceptions. Encouraging students to engage with multiple representations of inequality—symbolic, graphical, and verbal—can deepen their understanding and enhance their problem-solving abilities. Ultimately, this study underscores the complexity of inequality comprehension and highlights the importance of instructional approaches that balance procedural fluency with conceptual depth. Future research should continue exploring effective teaching strategies to improve students' mastery of inequalities in diverse educational settings.

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APPENDIX

Problem 1:

Thoroughly analyze and resolve each of the following inequalities. For each problem, execute the specified operation on both sides of the inequality and write the resulting inequality. Subsequently, based on your expertise, provide a response to the reflective question at the conclusion.

No	Inequality	Instruction	Resulting Inequality
1	$4 > 1$	Multiply both sides by -1	
2	$3 < 5$	Add 2 to both sides	
3	$6 > -4$	Add -1 to both sides	
4	$-5 \leq 4$	Subtract 2 from both sides	
5	$-1 \geq -6$	Multiply both sides by -3	
6	$-4 < 6$	Divide both sides by 2	
7	$\frac{1}{4} < \frac{1}{2}$	Divide both sides by $-\frac{1}{3}$	

Based on your experience with the aforementioned issues, what can you infer about the impact of various operations on inequality signs? Please elaborate on your rationale.

Problem 2:

Please solve each of the following linear inequalities. Subsequently, graph the solution set on a number line.

a. $3(2x\text{-}9)<9$
b. $-4(3x+2) \leq 16$.

Could you please elaborate on the significance of the solution set you identified? How did you determine the appropriate locations for the boundaries and sketch them on the number line?

Problem 3:

Please consider the following set:

$A=\{-10,-9,-8,-7,-6,-5, \dots\}$

Which of the following statements correctly represents set A in symbolic mathematical notation? Please select one answer and provide a detailed explanation of your reasoning.

a. $x<-10,\ x \in Z$
b. $x \geq -10,\ x \in Z$
c. $x \leq -10,\ x \in R$
d. $x>-10,\ x \in Z$
e. $x \geq -10,\ x \in R$

Provide a justification for your selection. How do inequality symbols and number set notations facilitate the accurate description of the structure of a set? Are there any alternative options that appear nearly, correct? Provide a rationale for your preference.