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The current issue features articles addressing four main themes: student learning obstacles, realistic mathematics education, technology-enriched learning, and teacher training and development. By bringing research from Croatia, Hungary, Indonesia, Malaysia, Morocco, South Africa, Türkiye, the United States, and Vietnam, the articles offer diverse perspectives that transcend one-size-fits-all solutions and enrich the global discourse in mathematics education. Specifically, they present strategies for leveraging students' thinking to inform instructional design, designing effective technology learning environments, and promoting high-quality teaching practices. Together, this issue provides valuable insights across the educational ecosystem that directly inform the work of teachers, researchers, and policymakers. We hope these papers serve as a catalyst for future research and classroom innovation, ultimately fostering a more vibrant mathematics education landscape internationally.

The editorial team extends its gratitude to all authors for their valuable contributions to the *Mathematics Teaching–Research Journal*. Their work is essential in cultivating meaningful dialogue between mathematics education research and classroom practice.

Learning Obstacles and Trajectories

Learning trajectories provide a roadmap for designing instruction that builds on students' mathematical strengths. Anticipating learning obstacles and predicting learning paths allows teachers to tailor instruction to students' evolving mathematical thinking. The first three studies explore

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students' learning obstacles from a phenomenology perspective and offer instructional strategies for addressing common learning obstacles of fractions, variables, and inequalities. **Andriatna, Sujadi, Kurniawati, Wulandari, and Nursanti** examine Indonesian junior high school students' learning obstacles of fractions and share a lesson design based on the 5Es model to address the identified ontogenic and epistemological obstacles. **Fitria, Susiswo, and Susanto** identify cognitive disconnect as a key obstacle in middle school students' understanding of variables and emphasize that instruction should deliberately link arithmetic and algebra through gradual representations and systematic scaffolding to bridge cognitive disconnection. **Siagian's** analysis of students' concept images in learning inequalities highlights the need to bridge procedural proficiency and conceptual understanding. Next, two studies integrated the RME approach to articulate hypothetical learning trajectories of learning division and developing trigonometric ratios concepts. **Duyen and Loc**'s study presents a hypothetical learning trajectory of division and a four-lesson intervention to develop first-grade students' robust understanding of partitive and quotitive division. **Andiani, Darhim, and Nurlaelah**'s study demonstrates that contextualized activities informed by a hypothetical learning trajectory enhance high school students' understanding of trigonometric ratios and facilitate their transfer to real-world contexts.

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Technology-enriched Learning

These technology-focused studies demonstrate that when integrated effectively, digital tools and instructional innovations have a transformative impact on mathematics learning by developing deeper conceptual understanding, fostering student engagement, and providing personalized learning experiences. **Sugiman, Maulyda, and Rahmawati** document the process of designing an augmented reality-based and learning tool based on the Didactical Technology Design framework and highlight its potential in creating interactive learning environments to improve primary students' numeracy skills. **Nugroho, Arliani, and Sugiman** demonstrate that integrating Realistic Mathematics Education into instructional websites (RME-Web) can enhance junior high students' computational thinking skills more effectively compared to traditional RME or direct-learning approaches. **Uyun, Mailizar, and Elizar** report that the flipped classroom model enhances Grade 11 students' mathematics learning outcomes and overall engagement, though behavioral engagement showed no significant difference. The two studies on GeoGebra by **Matić and Bošnjak** and **Torma, Kiss, and Waldhauser** highlight how the tool can enhance students' understanding through accessible visualization representations and sustained engagement. Last but not least, **Kartika and Clark** examine the argumentation structures created by generative AI and by undergraduate students using the Claim-Evidence-Reasoning (CER) framework. The findings indicate that generative AI tends to rely on pattern recognition and naïve empiricism rather than rigorous logical reasoning. Accordingly, they advocate for positioning AI as a pedagogical supporting tool in the classrooms.

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Teacher Training and Development

These four studies shed light on advancing the preparation of pre-service teachers and in-service mathematics teachers' teaching practice. **Demirci, Akkoç, and Çeziktürk** develop a Techno-pedagogical integration matrix to examine how pre-service teachers use the 5E model to plan technology-enhanced lessons. The study offers a unique case of integrating the SMAR model and the 5E model to guide effective use of technology in mathematics teaching. **Acar and Bozkurt** investigate pre-service teachers' awareness of Sustainable Development Goals and emphasize the importance of equipping future educators with knowledge and strategies to foster sustainability consciousness in students. **Gholami** reports a lesson study grounded in a collaborative, constructivist-based approach to teacher professional development. The findings highlight the value of reflective, community-based approaches in fostering effective mathematics teaching and learning. Drawing on Casa et. al's four types of mathematical writing, **Ngoako and Machaba** analyze a teacher's writing practices in a 10th-grade classroom and reveal the need to integrate various writing strategies and opportunities to enhance students' mathematics learning.

An Examination of Technology Usage Levels of Pre-Service Mathematics Teachers' Micro-teaching Lesson Plans Based on the 5E Model: A Case Survey
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Exploring Writing in Mathematics: A Case Study analyzing Grade 10 Classroom Practice through Four Types of Writing
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This issue concludes with two articles that focus on students' reasoning and emotions in mathematics learning. **Nhiry, Hanini, and Abouhanifa** present a student work analysis of inductive reasoning. They developed a set of criteria to assess Moroccan secondary school students' inductive reasoning skills and suggest that instruction should include exploratory tasks, conjecture-making opportunities, and pattern-focused discussions to support high-level inductive reasoning. **Suparman, Juandi, Turmudi, Martadiputra, and Diana** show that students' achievement emotions in doing some mathematics-related activities vary by gender, major, school type, or age. The study reveals the need for providing differentiated interventions to support stable emotional achievement in mathematics learning.

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Learning Obstacles in Fraction Concept: A Hermeneutic Phenomenology Study

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Abstract: The concept of fractions is a fundamental mathematics, particularly for junior high school students in Indonesia. Despite its importance, learning obstacles are frequently encountered in the educational process. This study investigates the learning obstacles experienced by students in the concept of fractions. This research employs a hermeneutic phenomenological approach to identify the types of learning obstacles—ontogenic, didactic, or epistemological—based on the analysis of students' test answers, which are further explored through interviews. The study involved 30 junior high school students, with eight students selected for in-depth interviews. Data analysis includes stages of explanation, naïve understanding, in-depth understanding, and appropriation. The results indicate learning obstacles primarily in the ontogenic and epistemological types. Ontogenic obstacles are categorized into three types: psychological, instrumental, and conceptual. These are attributed to students' lack of interest in mathematics, failure to apply fundamental fraction concepts, and gaps in prerequisite knowledge of fractions. Epistemological obstacles are characterized by students' limited understanding of fractions exclusively in the $\frac{a}{b}$ form without grasping alternative interpretations of fractions. Based on these results, this study also recommends instructional solutions that can be used as alternatives in anticipating the occurrence of learning obstacles in students.

Keywords: fraction concept, hermeneutic phenomenology, learning obstacle

INTRODUCTION

The concept of fractions is a crucial element in mathematics that students must master. Lamon (2020) identifies five interpretations of the fraction concept, which are further elaborated by Isnawan et al. (2022a): (1) part of a whole, tending to the proper fraction; (2) quotients; (3) ratios, namely the comparison of two numbers; (4) measure, that is, the measure of an object; and (5) operator refers to the process of scaling a quantity. In line with Lamon, Van de Walle et al. (2010) posits that the introduction to the fraction concept also includes: (1) fraction parts indicating the

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quantity needed to compose a whole; (2) the more fraction parts required, the smaller the fraction part; (3) the denominator indicating how many parts the whole is divided into; and (4) two equivalent fractions representing the same quantity using parts of different sizes.

Fractions are crucial for understanding mathematical concepts, especially at the primary and secondary education levels (Hariyani et al., 2022; Getenet & Callingham, 2021; Kullberg & Runesson, 2013; Roni et al., 2017; Unaenah et al., 2023, 2024). The concept of fractions has numerous benefits, not only related to mathematics itself but also impacting other disciplines (Ballard & Johnson, 2004; Coetzee & Mammen, 2017; Gagani & Diano JR, 2019; Lortie-Forgues et al., 2015; Tian & Siegler, 2017). It is important to note that the concept of fractions is also employed in the solving of everyday problems as an alternative representation approach when integer solutions are not adequate (Purnomo et al., 2021). Therefore, the concept of fractions plays an indispensable role in both mathematics and real-world applications.

In Indonesia, based on the Minister of Education and Culture's Decree No. 37 of 2018, the concept of fractions is studied at the primary and secondary education levels. In elementary school, the concept of fractions is introduced from grade 2 to grade 6 (Unaenah et al., 2024). At this level, students are introduced to fraction symbols, albeit not in depth, while in junior high school, fractions are taught alongside the concept of integers. According to the curriculum documents, the competencies for fractions taught in junior high school are as follows: (1) explaining and determining the order of fractions (proper, mixed, decimal, and percent); (2) solving problems related to the order of fractions; (3) explaining and performing arithmetic operations with fractions using various properties of operations; and (4) solving problems related to arithmetic operations with fractions. In addition to primary and secondary education, fractions are also studied in some university courses. This indicates that fractions are one of the fundamental concepts in mathematics.

Given the significance of fractions, it is important for students to master this concept. However, Unaenah et al. (2024) state that there are still issues in the learning process related to fractions. Several studies show that students still find it difficult to understand the concept of fractions (Isnawan et al., 2022a, 2022b; Unaenah et al., 2023, 2024). These findings indicate that students' mastery of fractions remains limited, reflecting the existence of learning obstacles as one of the urgent and unavoidable problems in the mathematics learning process (Fauzi & Suryadi, 2020; Fritz et al., 2019). Douroux states that learning obstacles are pieces of knowledge not based on students' ignorance (Brousseau, 2002). Therefore, learning obstacles do not occur because students are unaware of the mathematical concept but due to certain factors that hinder students from fully mastering the concept.

Brousseau (2002) identifies three types of learning obstacles: ontogenic, epistemological, and didactical. Hendriyanto et al. (2024) states that ontogenic obstacles arise from a mismatch between the learning process and students' mental readiness. Consequently, learning can become problem-

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atic if the demands exceed or fall short of students' mental readiness. Nurhayati et al. (2023) classifies ontogenic obstacles into three types: (1) psychological, related to a lack of interest or motivation.; (2) instrumental, which include technical aspects that are important in the learning and problem-solving process; and (3) conceptual, which arise from demands on students to think at a level that is higher or lower than their abilities.

Didactical obstacles are linked to the didactic system, emphasizing obstacles based on the sequence of material presentation that hinders students' continuity of thought (Fauzi & Suryadi, 2020). Therefore, the order of material presentation should consider both structural sequences, showing the interrelation of concepts, and functional sequence, ensuring the continuity of students' thinking processes. Regarding epistemological obstacles, Brousseau (2002) states that these obstacles occur when knowledge that was previously meaningful and effective in a certain context continues to be used, but at a certain stage produces incorrect or inadequate answers, thereby creating contradictions. This indicates that epistemological obstacles occur due to the limitations of the applicability of knowledge or only limited to specific contexts. These epistemological obstacles arise when students encounter new and different problem situations compared to those usually presented during instruction.

Research on learning obstacles in the concept of fractions has been conducted by several researchers in Indonesia (Fauzi & Suryadi, 2020; Hariyani et al., 2022; Isnawan et al., 2022b, 2022a; Unaenah et al., 2023, 2024). However, most of this research focuses on the elementary school level, with limited studies on junior high school students. As previously mentioned, the concept of fractions is taught from elementary school through university, including junior high school. According to Unaenah et al. (2024), the concept of fractions still poses difficulties for students in learning mathematics. Therefore, it is likely that learning obstacles also exist for junior high school students regarding fractions. This study aims to identify potential learning obstacles related to the concept of fractions among junior high school students. The findings are expected to serve as a reference for developing didactic designs to minimize the potential occurrence of learning obstacles in teaching fractions.

METHODS

Research Design

This study employed a hermeneutic phenomenological approach. Hermeneutic phenomenology is a research approach based on the philosophies of phenomenology and hermeneutics. According to Langdridge (2007), phenomenology focuses on revealing the meaning or understanding of an individual's experience of phenomena that occurr around them. Hermeneutics refers to the interpretation of meaning. Through hermeneutic phenomenology, one can describe and interpret an individual's experience and the meaning and interpretation related to that experience (Lindseth & Norberg, 2004; Regan, 2012). Therefore, this study focused on examining the learning obstacles

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students encountered in the concept of fractions based on their experiences after receiving instruction (phenomenology) and then investigating the causes of these learning obstacles (hermeneutics) into ontogenic, didactical, and epistemological obstacles.

Participants and Data Collection

The participants in this study were 30 seventh-grade students from a junior high school in Surakarta, Indonesia. All thirty students were given a mathematics test on the topic of fractions. The selection of grade 7 is based on the consideration of the curriculum in Indonesia, in which the fraction material is taught. This finding suggests that the subjects have gained direct learning experience. Based on the test responses, eight students (labeled as S1 to S8) were selected for further investigation through in-depth interviews. These eight students were chosen based on their test answers, considering their communication abilities as recommended by their mathematics teacher. The eight selected students represented the errors that occurred in each of the given problems. Thus, purposive sampling was used in this study.

Data collection was conducted through both test and non-test methods. The test method aimed to gather initial data on potential learning obstacles in the fractions. The instrument employed is a mathematics question concerning fractions that has been validated by experts in mathematics education from the content aspect related to suitability for measurement purposes. Additionally, one of the questions used, specifically number 1 in Table 1, was taken from the test instrument used by Isnawan et al. (2022a). Table 1 shows the test instrument blueprint used in this study. The blueprint in Table 1 refers to the Basic Competencies in the Number element, namely: (1) explaining and determining the order of integers and fractions; (2) explaining and performing arithmetic operations on integers and fractions using various operational properties; and (3) solving problems related to arithmetic operations on integers and fractions. The test was completed individually by all participants. The test was administered for a duration of 45 minutes, during which the subjects were presented with written essay as answers.

In addition to using the test instrument, data collection in this study was also conducted through in-depth interviews with students who had answered the test questions. The interviews were guided by an open-ended question sheet while referring to the students' test answers. Activities were conducted face-to-face with each selected participant. Responses from each participant were audio-recorded with the participant's verbal consent. The interviews were conducted in-depth to gather data and confirm students' responses to identify the causes of learning obstacles. This allowed the researchers to classify the learning obstacles into categories of ontogenic, didactical, or epistemological obstacles.

No	Indicators	Item Test
1	Representing fractions using images/models/diagrams	Using the given illustrations/models, visualize/draw/shade the values of $\frac{1}{6}$ and $\frac{2}{3}$ on models 1 and 2 below! Model 1

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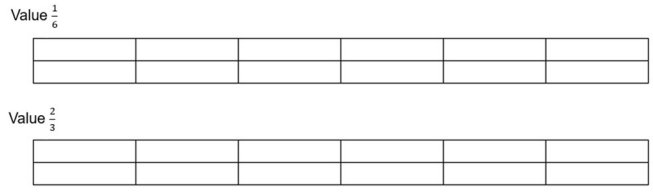
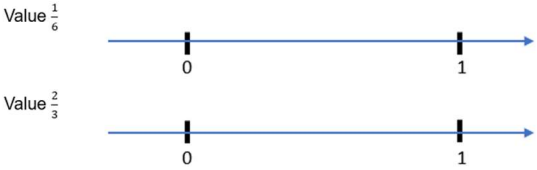
		 Model 2 
2	Calculating the value of fraction operations	Calculate the value of $1\frac{2}{3}:\frac{2}{5}\text{-}2\frac{3}{5}+\frac{1}{2}$.
3	Ordering the values of fractions	Order the following fractions from smallest to largest! Provide explanations if necessary. $\frac{2}{3};\ \frac{1}{2};\ 0{,}6;\ 20\%$
4	Applying fraction operations to solve contextual problems	A farmer has $1\frac{2}{3}$ hectares of land to plant crops. The farmer feels that the land is insufficient, so he buys another $2\frac{1}{4}$ hectares. If the land is to be planted with maize, peanuts, and cassava, calculate the area of land for each crop.

Table 1: The Mathematics Test Instrument for Identifying Learning Obstacles

Data Analysis

The data analysis in this study referred to qualitative data analysis. Specifically, the analysis was conducted using a synthesis of the phenomenological data analysis steps according to Creswell (2007) and the hermeneutic data analysis steps based on Ricouer's Interpretation Theory (Ghasemi et al., 2011; Tan et al., 2009). The steps for hermeneutic phenomenological analysis are as follows.

1. *Explanation*: In this stage, the researchers compiled the research data, including students' answers, and recorded interviews with the students. The audio recordings of the interviews were transcribed, and the test answer sheets were reviewed and analyzed.
2. *Naïve Understanding*: The researchers developed research notes to obtain significant and specific statements based on the analysis of the interview transcripts and students' answer sheets. The developed statements were then grouped into information units (meaning units) related to learning obstacles. Textural and structural descriptions were then created. The textural description involved detailing what the students experienced regarding the meaning of the concept of

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fractions, while the structural description referred to how the students acquired the concept of fractions.

3. *In-depth Understanding*: Further analysis was conducted on the relationship between the textural and structural descriptions for each respondent to derive the essence and meaning of the concept of fractions, including the learning obstacles faced by the students. Additionally, a composite description was created by combining the textural and structural descriptions based on the previous analyses.
4. *Appropriation*: In the appropriation stage, the researchers analyzed the relationship between the composite description and the significant statements from the students, including relevant theories. This process aimed to identify the learning obstacles encountered by the students.

Validity of Data

The validity of the data in this study was ensured through tests of credibility, transferability, dependability, and confirmability. Credibility was achieved through technique triangulation, comparing students' test answers with the results of the interviews conducted. Transferability was evaluated based on the research setting, participant selection, and data processing (Puspita et al., 2023), ensuring that the research findings provided sufficient information to the readers. Dependability was ensured by examining the entire research process, starting from problem identification, instrument development, data accuracy, and data analysis. Meanwhile, confirmability was achieved by examining the objectivity and transparency of the research results and discussions, ensuring the findings could be independently verified and re-examined.

RESULTS AND DISCUSSION

Based on these test responses, in-depth interviews with eight students were conducted, identifying learning obstacles related to the concept of fractions. Errors found in the test answers showed notable similarities among students. This section presents the research results and discussion, divided into three sub-chapters: (1) Learning obstacles in representation the meaning of fractions; (2) Learning obstacles in fraction operations and their application in solving contextual problems; and (3) Learning obstacles related to ordering fractions.

Learning Obstacles in Conceptual Representation of Fractions

The representation of the meaning of fractions in this section is based on question number 1 presented in Table 1. In Table 1, students were asked to represent the values of $\frac{1}{6}$ and $\frac{2}{3}$ on two models, each illustrating a specific meaning of fractions. Errors were still found in students' representations of fractions in both Model 1 and Model 2. Below is an example of a student's response to the first problem.

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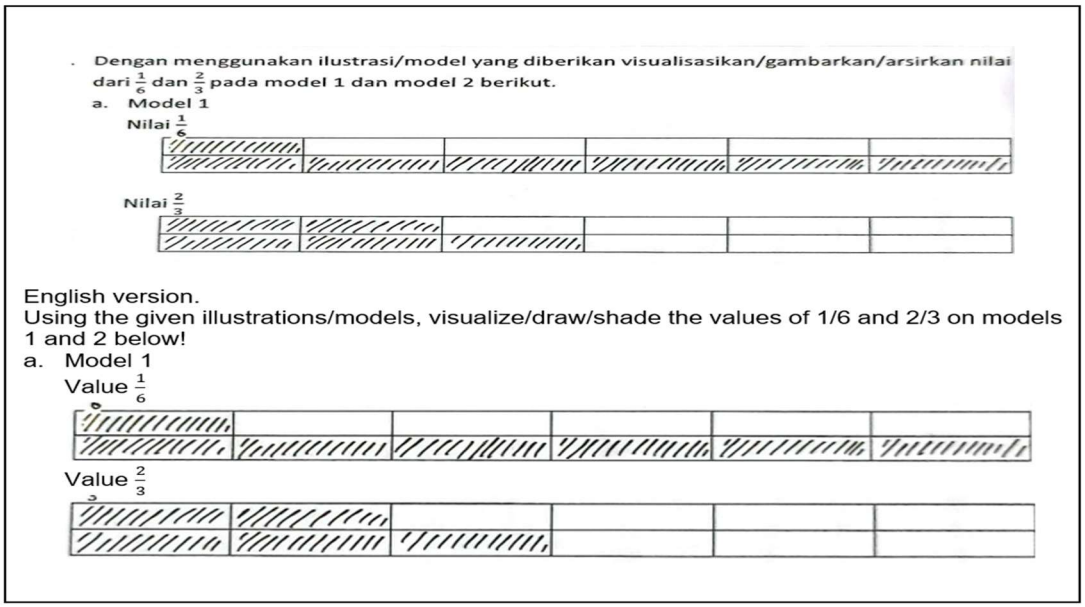


Figure 1. S1's Answer on the Representation of Fraction Meaning in Model 1

Figure 1 shows that the student represented $\frac{1}{6}$ and $\frac{2}{3}$ in Model 1 as two separate parts. The value $\frac{1}{6}$ in Figure 1 was represented with the numerator 1 on the top and the denominator 6 on the bottom. The student (S1) interpreted Model 1, consisting of 12 boxes, as representing equal parts corresponding to the numerator and denominator. A similar approach was taken by the student for the value $\frac{2}{3}$, as shown in Figure 1. This condition was explicitly stated based on an interview excerpt with S1 regarding the first problem.

Researcher	:	Okay. From your answer, in Model 1 you were asked to shade $\frac{1}{6}$ and $\frac{2}{3}$. For $\frac{1}{6}$, you shaded the part with 1 on top and 6 on the bottom. Why did you do that?
S1	:	Because the number 1 is on top, so I shaded 1, and the number 6 is on the bottom, so I shaded 6.
Researcher	:	Okay, did you understand what the boxes represent?
S1	:	Each box represents one unit.
Researcher	:	So, you shaded the numerator on top and the denominator on the bottom?
S1	:	Yes.
Researcher	:	And for $\frac{2}{3}$?
S1	:	The same.
Researcher	:	So what does it mean when you shade and don't shade parts?

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S1	:	The shaded parts are the numbers being asked for, and the unshaded parts are the rest.
Researcher	:	Has the teacher ever given a problem like this before?
S1	:	No, never.
Researcher	:	Okay, so this is the first time. Were you confused when doing this?
S1	:	Yes, I was confused.
Researcher	:	Do you think you can imagine $\frac{1}{6}$ without illustrating it?
S1	:	No, I can't.

Based on the interview excerpt, S1 understood that the values of the numerator and denominator in a fraction indicate that they can be visualized as separate parts. This condition demonstrates a misunderstanding of the meaning of fractions by the student. Specifically, the student perceived Model 1 as a representation of fractions but displayed a partial understanding of the fraction's value. The student's errors in understanding the meaning of fractions are partly attributed to their lack of familiarity with such problems. This situation reflects a potential epistemological learning obstacle, which Brousseau (2002) describes as a barrier related to limited understanding of a concept based solely on specific contexts related to one's learning experiences. The student still perceived the basic concept of a fraction as a number denoted in the $\frac{a}{b}$ form. Therefore, when expressed in a different format, such as the representation of fraction meaning, the student made errors.

According to Lamon (2020), the meaning of fractions can be understood in several contexts, one of which is that a fraction represents a part of a whole. In the context of the first problem, $\frac{1}{6}$ should be visualized as one part of six equal parts. Thus, if given 12 equally sized boxes, only two of the 12 boxes should be shaded as shown in Figure 1. The epistemological obstacle encountered by the student is further reinforced by a contradiction in their answer when asked about the meaning of fractions. When asked what a fraction is, the student responded, "a fraction is one whole divided into several equal parts." In this context, the student understood what a fraction means, but when presented with a different situation, the student became confused.

This condition indicates that the student has not yet fully applied their understanding to solve problems in different contexts. Carpenter et al. (2015) asserts that students who have a clear misunderstanding of a concept will struggle to solve or apply that concept in varied situations. This condition may arise due to reliance on rote memorization rather than understanding the underlying principles (Catrambone, 1994). Nokes et al. (2007) suggest that to address this issue, teachers should provide assistance by explaining the relationships between principles and examples, as well as offering appropriate explanations and analogies. Additionally, Quintero (1983) emphasizes that the diversity of problem types provided should be considered to clearly identify students' difficulties.

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In Model 2, Figure 2, students were asked to visualize the values $\frac{1}{6}$ and $\frac{2}{3}$ on a number line. Most students first converted $\frac{1}{6}$ and $\frac{2}{3}$ into decimal form and then placed these decimal values on the number line, with $\frac{2}{3}$ positioned to the right of $\frac{1}{6}$. This indicates that students understood how to place $\frac{1}{6}$ and $\frac{2}{3}$ on the number line. However, there are still students who are not correct in solving this problem. One of them is the answer from S2. The following is an example of a student's response and interview to the research problem.

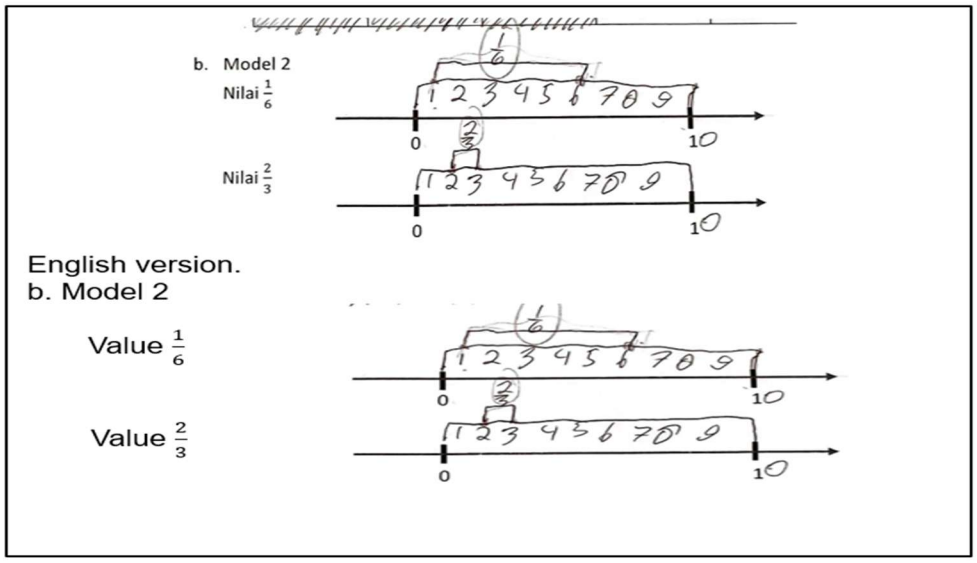


Figure 2. S2's Answer on the Representation of Fraction Meaning in Model 2

Researcher	:	Okay, let's move to Model 2. The question is the same. Can you explain where $\frac{1}{6}$ is located?
S2	:	Here it says 1-9 between 0 and 10. So, $\frac{1}{6}$ is from 1 to 6.
Researcher	:	So, is $\frac{1}{6}$ located at the far right, far left, or somewhere along 1 to 6?
S2	:	Somewhere along 1 to 6.
Researcher	:	Do you know what this line represents?
S2	:	I have seen it before but I have forgotten.
Researcher	:	What are the numbers before 0 and after 0?
S2	:	Before 0 is negative, after 0 is positive.
Researcher	:	Okay, and what about $\frac{2}{3}$?
S2	:	It's the same. Along 2 and 3.

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Based on Figure 2 and the interview excerpt, S2 incorrectly answered the problem by visualizing $\frac{1}{6}$ or $\frac{2}{3}$ as distances between the numerator and denominator values. In this case, the student converted the number line interval into integers ranging from 0 to 10. This mistake aligns with findings from Isnawan et al. (2022a), where similar errors were found, with students marking integers on the number line. A notable finding from the interview is that the student struggled to understand the real number line, leading to errors in addressing the problem. This situation indicates that the student encountered an ontogenic obstacle.

Referring to the answers in Figure 2 and the interview results, the student faced a conceptual ontogenic obstacle related to previous learning experiences, specifically the number line. The number line concept in Model 2 is crucial as a previous learning experience for understanding one meaning of fractions, such as measurement, as explained by Lamon (2020). Furthermore, Isnawan et al. (2022a) illustrate that a fraction can represent a measurement, such as a length of 50 centimeters, which can also be expressed as a fraction value $\frac{1}{2}$ meters. This suggests that fraction values can be represented on a number line. The incorrect prior experience with the number line caused an obstacle when students attempted to visualize values like $\frac{1}{6}$ or $\frac{2}{3}$, despite the number line concept having been taught at the elementary school level.

Lutfi et al. (2021) states that a conceptual ontogenic obstacle refer to the mismatch between the conceptual level targeted in teaching and the way students think, which is shaped by their previous learning experiences. In this case, the students' previous learning experiences impact the obstacles they face, indicating that students have not fully grasped their knowledge related to the number line. According to Loewenberg Ball et al. (2008), this situation occurs when the knowledge acquired has not been fully integrated with the basic concepts being studied. Ideally, at the junior high school level, the concept of fractions should build upon a broader range of concepts, requiring higher-level thinking. However, the observed situation reveals a gap between the expected cognitive demands and the actual design of the instruction.

Learning Obstacles in Fraction Operations and Application in Contextual Problem Solving

The analysis of fraction operations and their application in contextual problem solving is based on students' answers to test questions 2 and 4 from Table 1. Question 2 required students to calculate the value of arithmetic operations involving fractions, including both proper and mixed fractions, covering operations such as division, subtraction, and addition. Based on students' answers, several errors were identified in question 2: (1) Errors in converting mixed fractions to proper fractions; (2) Errors in performing operations on fractions, particularly in addition and subtraction; and (3) Errors in arithmetic operations involving integers. Examples of these errors are illustrated in Figure 3 and Figure 4.

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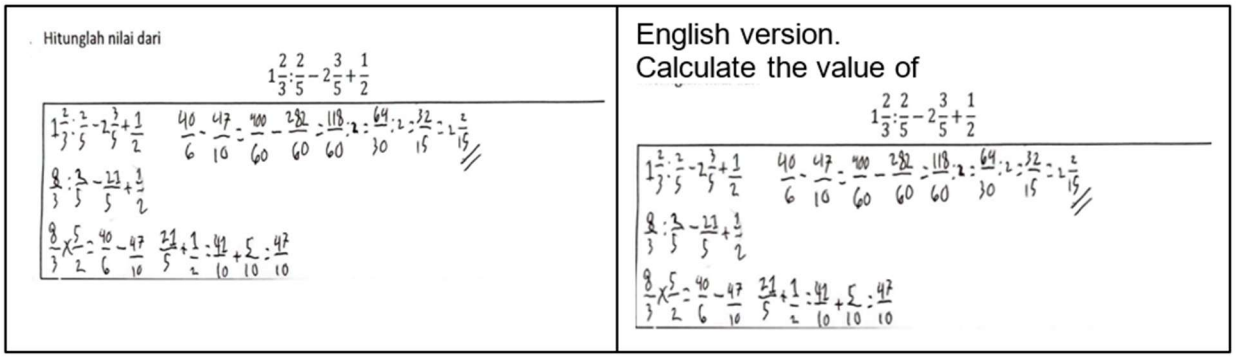


Figure 3. Errors in Converting Mixed Fractions to Proper Fractions and Performing Fraction Operations from S3

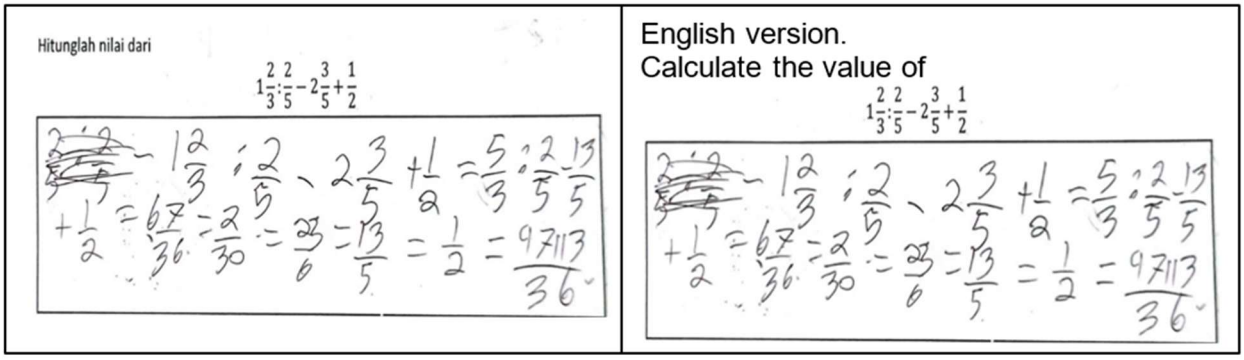


Figure 4. Errors in Fraction Operations and Integer Calculations from S4

Fraction operations differ from integer operations, particularly in the conversion of mixed fractions to proper fractions. Figure 3 shows S3 making errors in converting mixed fractions to proper fractions, such as incorrectly converting $1\frac{2}{3}$ into $\frac{8}{3}$. Similar conversion errors were noted with other mixed fractions, such as changing $2\frac{3}{5}$ into $\frac{21}{5}$, as confirmed in subsequent interviews with the researcher.

Researcher	:	Okay, let's move on to the calculations. You have just rewritten the problem. Can you explain why this conversion is done?
S3	:	So it could be easier to calculate.
Researcher	:	Okay, how is this conversion done? For instance, how does one and two third become eight-thirds.
S3	:	First, add and then multiply, add and then multiply.
Researcher	:	What is added? Can you point it out?
S3	:	Add 3 and 1.
Researcher	:	Okay, so 3 added to 1 is 4. What happens next?

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S3	:	Four times two.
Researcher	:	So, you obtained 8. If the 3 remains the same?
S3	:	Yes.

Based on Figure 3 and the interview results, S3 incorrectly converted mixed fractions $1\frac{2}{3}$ into proper fractions by first adding 1 to 3 and then multiplying by 2. The correct method should involve multiplying 1 by 3 and then adding 2, resulting in $\frac{5}{3}$. In addition to errors in converting mixed fractions to proper fractions, students also made mistakes in calculating the values for fraction operations, including integer arithmetic, as shown in Figure 3 and Figure 4. For fraction operations, specifically addition and subtraction, students need to equalize the denominators by finding the Least Common Multiple (LCM) of the denominators involved. However, S4 in Figure 4 incorrectly determined the LCM for the denominators 2, 3, and 5.

Based on the students' answers and interviews, the errors indicate the presence of ontogenic obstacles, specifically both instrumental and conceptual ontogenic obstacles. Ontogenic instrumental obstacles refer to technical errors that hinder students from understanding the given conditions. In this case, students made mistakes in fraction operations, where a crucial technical aspect is determining the LCM of the denominators in fraction addition and subtraction. Ontogenic conceptual obstacles, as defined by Nurhayati et al. (2023), refer to obstacles that arise due to a misalignment between the conceptual level of the questions designed and the students' understanding. Puspita et al. (2023) clarify that conceptual ontogenic obstacles pertain to problem conceptualization that is not aligned with the students' cognitive experiences. Hendriyanto et al. (2024) note that conceptual ontogenic obstacles may arise from a lack of foundational mathematical knowledge that students should have acquired earlier. This notion aligns with Septyawan et al. (2019), who emphasize that prior knowledge of a concept is necessary for understanding advanced topics. In this case, the conceptual obstacle is evident in the students' difficulty in converting mixed fractions to proper fractions. It is important to note that fraction operations are applicable when dealing with $\frac{a}{b}$ fractions rather than mixed fractions. Thus, mastering the conversion from mixed fractions to proper fractions is essential prior knowledge that students need before performing fraction operations.

The concept of fractional numbers is closely related to the numerical development of individuals. Siegler et al. (2011) state that one fundamental reason for students learning fractions is related to the reorganization of children's numerical knowledge, where children who have not yet learned fractions assume that the properties of integers apply to all numbers. In the concept of fractions, Lazić et al. (2017) emphasize the importance of understanding the differences and individual development in both conceptual knowledge and procedural knowledge related to fractions. Lazić et al. (2017) further explain that conceptual knowledge pertains to understanding the properties of

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fractions, including their magnitude, principles, and notation, whereas procedural knowledge involves understanding and proficiency in using the four arithmetic operations.

Subsequently, in question number 4 in Table 1, students were asked to solve a simple contextual problem by applying fraction operations. In this case, two broad categories of answers were identified, as shown in Figures 5 and 6.

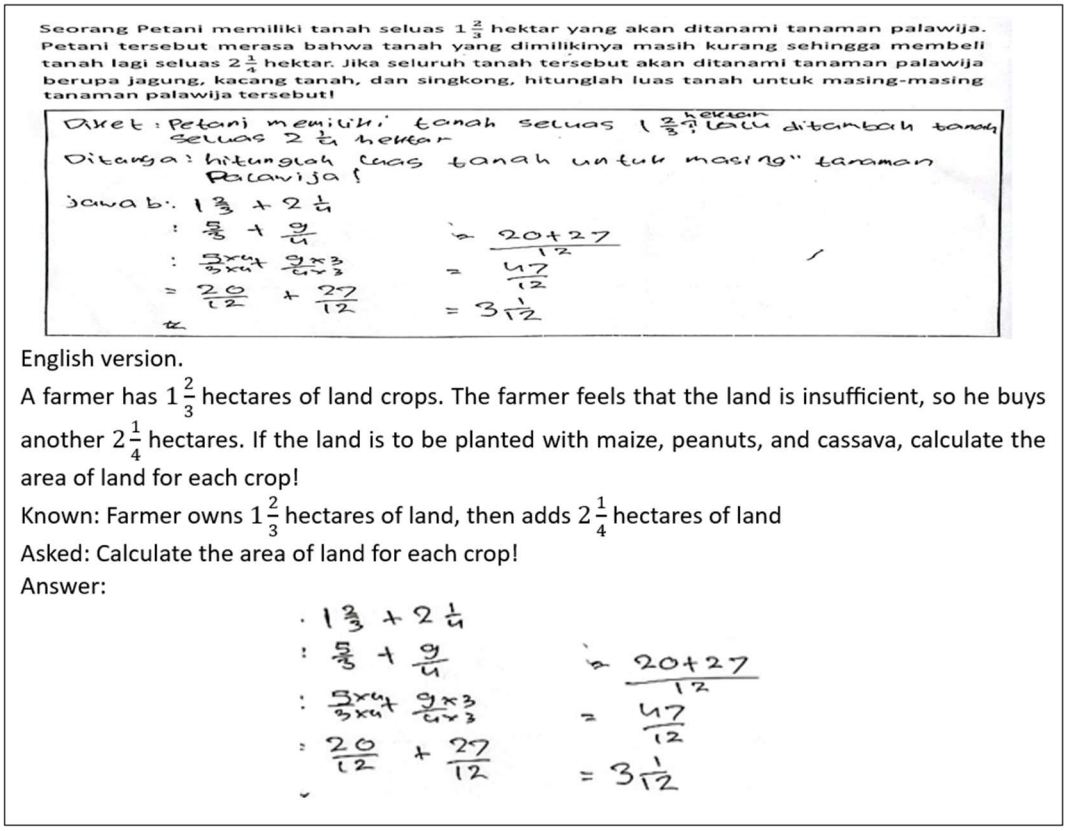


Figure 5. Example of Incorrect Answers from S5

In Figure 5, S5 answer only includes the addition of the areas of the two plots of land based on the operation of two fractions, but does not proceed to the division operation to show the area of each part of the land for intercropping. The interview with S5 revealed that the student considered the final answer to question number 4 to be the sum of the first and second land areas, which is $\frac{47}{12}$. This was confirmed by the interview when the S5 was asked, "Is this answer sufficient to address the question?" and responded, "Yes, it is sufficient." In Figure 6, S6 correctly calculated the total area of the two plots of land but mistakenly divided the area of each of the three intercropping sections by $\frac{1}{3}$ instead of dividing by 3.

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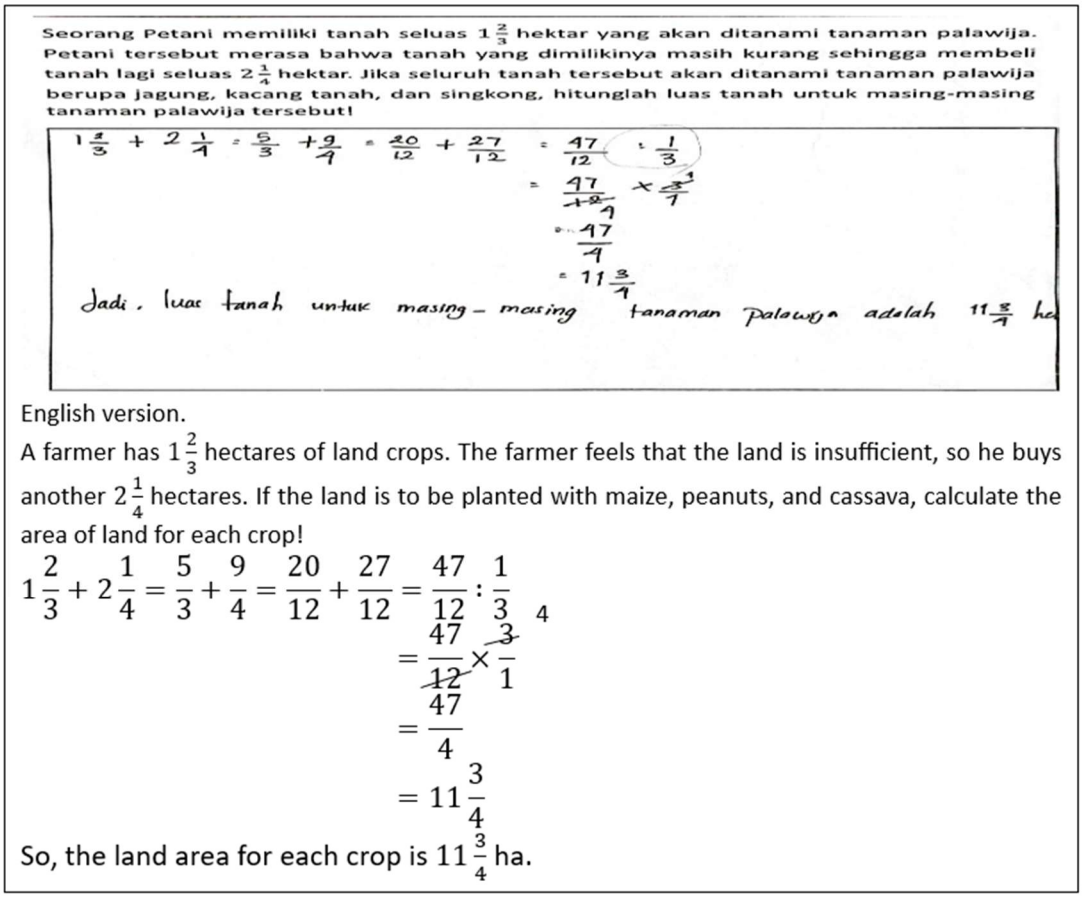


Figure 6. Example of Incorrect Answers from S6 in Divided the Area

The errors observed in both Figures 5 and 6 are not attributed to the nature of the problem, as the interviews indicated that similar problems to question number 4 had been presented to the students during fraction concept lessons. Didactically, the teacher had provided an understanding of how to solve contextual problems related to fractions, specifically fraction operations. However, the errors were due to the students' lack of attention in understanding the problem, including a tendency to forget. An interesting finding was that students appeared to perform tasks by imitation without understanding the reasoning behind the tasks. Thus, relating to the types of learning obstacles, this indicates an ontogenic obstacle in the psychological form. Hendriyanto et al. (2024) state that psychological ontogenic refers to the students' lack of readiness in aspects related to motivation or interest, which impacts errors in task completion or the learning process.

Researcher	:	Regarding the addition of fractions, do you remember the conditions?
S5	:	No, I don't understand these types of problems.
Researcher	:	Did you ask your friend about this?
S5	:	Yes, I asked a friend.

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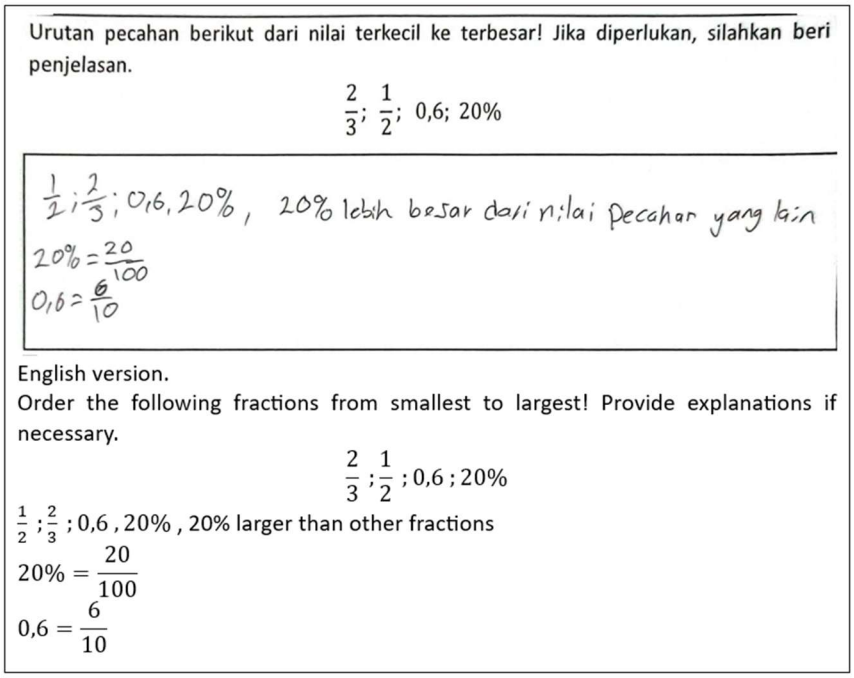
Researcher	:	Did you ask why it is done this way?
S5	:	No.

The interview results revealed a misunderstanding of the concept of fraction operations, coupled with the students' reluctance to ask questions. This lack of curiosity is likely due to insufficient motivation in the learning process. Skilling et al. (2021) suggested that low motivation to learn can lead to the emergence of learning obstacles. Furthermore, Hendriyanto et al. (2024) state that a lack of motivation and interest in mathematics can result in reduced attention in class, failure to solve problems, and a lack of enthusiasm in asking questions to understand the material.

Learning Obstacle in Ordering Fractions

The concept of ordering fractions is inherently tied to comparing two fractions. Thus, the fundamental concept of ordering fractions involves comparing two fractions to determine whether one is greater than, less than, or equal to the other. Using a number line visualization, if given two fractions A and B where $A>B$, then A is positioned to the right of B on the number line. In this study, two methods were identified for ordering fractions in question three of Table 1: one involves equalizing the denominators and performing division between the numerator and denominator, while the other method interprets fractions as quotients, as described by Lamon (2020). However, errors were found in the students' answers regarding the ordering of fractions from smallest to largest, specifically for $\frac{2}{3}, \frac{1}{2}$, 0.6, and 20%. Some students correctly ordered the fractions as 20%, $\frac{1}{2}$, 0.6, and $\frac{2}{3}$. Nonetheless, there were also incorrect answers, both in total and partially. Below are examples of incorrect student answers from S7 and S8.

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Urutan pecahan berikut dari nilai terkecil ke terbesar! Jika diperlukan, silahkan beri penjelasan.

$\frac{2}{3}$; $\frac{1}{2}$; 0,6; 20%

$\frac{1}{2}$; $\frac{2}{3}$; 0,6, 20%, 20% lebih besar dari nilai pecahan yang lain

$20\% = \frac{20}{100}$

$0,6 = \frac{6}{10}$

English version.

Order the following fractions from smallest to largest! Provide explanations if necessary.

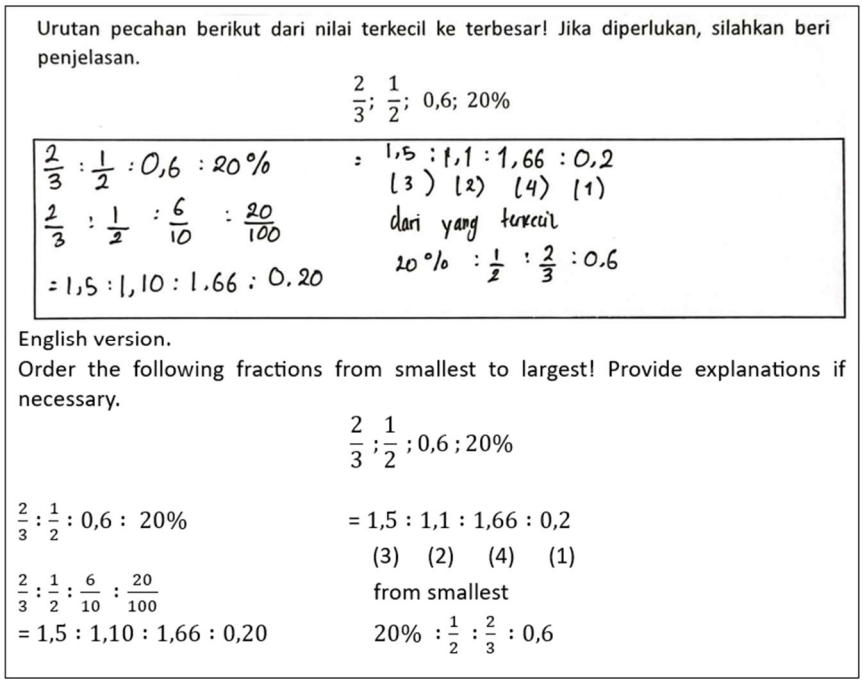
$\frac{2}{3}$; $\frac{1}{2}$; 0,6 ; 20%

$\frac{1}{2}$; $\frac{2}{3}$; 0,6 , 20% , 20% larger than other fractions

$20\% = \frac{20}{100}$

$0,6 = \frac{6}{10}$

Figure 7. Example of an Incorrect Student Response from S7 in Ordering Fractions


Urutan pecahan berikut dari nilai terkecil ke terbesar! Jika diperlukan, silahkan beri penjelasan.

$\frac{2}{3}$; $\frac{1}{2}$; 0,6; 20%

$\frac{2}{3} : \frac{1}{2} : 0,6 : 20\%$: 1,5 : 1,1 : 1,66 : 0,2

(3) (2) (4) (1)

$\frac{2}{3} : \frac{1}{2} : \frac{6}{10} : \frac{20}{100}$ dari yang terkecil

= 1,5 : 1,10 : 1.66 : 0.20 20 % : $\frac{1}{2}$: $\frac{2}{3}$: 0,6

English version.

Order the following fractions from smallest to largest! Provide explanations if necessary.

$\frac{2}{3}$; $\frac{1}{2}$; 0,6 ; 20%

$\frac{2}{3} : \frac{1}{2} : 0,6 : 20\%$ = 1,5 : 1,1 : 1,66 : 0,2

(3) (2) (4) (1)

$\frac{2}{3} : \frac{1}{2} : \frac{6}{10} : \frac{20}{100}$ from smallest

= 1,5 : 1,10 : 1,66 : 0,20 $20\% : \frac{1}{2} : \frac{2}{3} : 0,6$

Figure 8. Partial Error in Ordering Fractions from S8

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In Figure 7, S7 incorrectly ordered the fractions by stating that 20% is the largest fraction. The following is an excerpt from the interview with the student.

Researcher	:	Okay, let's continue with question 4. Could you read the question again? What does it ask?
S7	:	(Reads the question) It asks to order the fractions.
Researcher	:	Alright, how did you come to order them this way?
S7	:	Because as far as I know, $\frac{1}{2}$ is smaller than $\frac{2}{3}$.
Researcher	:	Why is that? What did you observe?
S7	:	The numbers.
Researcher	:	So 1<2 and 2<3?
S7	:	Yes. Also, I thought 0.6 is 0.60, and the largest is 20%.
Researcher	:	Why do you think 20% is the largest?
S7	:	Because 20% is $\frac{20}{100}$.
Researcher	:	So it's the largest because it's under 100?
S7	:	Yes. 100 is the largest.

S7 stated that the ordering was done by comparing the numerator of the first fraction with the numerator of the second fraction, and similarly for the denominators, resulting in the order $\frac{1}{2}$ and $\frac{2}{3}$. However, an interesting finding is that 20% was identified as the largest fraction based on the assumption that the denominator value is the largest compared to the denominators of the other fractions. This condition indicates a misunderstanding in determining the order of fractions. According to (Moyo & Machaba, 2021), this error suggests that students still perceive fractions similarly to integers. In this case, the student views the numerator and denominator in a fraction as separate integers, leading to a failure in understanding the concept of fractions (Deringöl, 2019; Moyo & Machaba, 2021). When the student identified 20% as the largest fraction based on the assumption that the denominator value of 100 is the largest, it shows that the student ordered the fractions based on the value of the denominator. This aligns with the view of Aliustaoğlu et al. (2018), who stated that in ordering fractions, students rely on comparing numerators individually and then apply this to the fractions as a whole.

The errors observed in students' ordering of fractions, as presented above, stem from their failure to understand the concept of fractions themselves. Students perceive fractions in terms of separate numerators and denominators, treating them similarly to integers. In this context, the difficulty in ordering fractions reflects an instrumental obstacle as it is one of the key technicalities in fraction concepts. In fraction ordering, a fraction is expressed as an indivisible whole rather than as separate numerators and denominators. Pavlovičová and Vargová (2020) state that comparing fractions re-

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lies on a compensatory relationship between the size and the number of equal parts within a partitioned unit. Hence, a fraction with more partitioned parts has a smaller value. This understanding aligns with the interpretation of fractions as quotients.

The Design of Alternative Instructional Strategies for Fraction Learning

The results of the learning obstacle analysis indicate the presence of ontogenic obstacles, as well as obstacles stemming from psychological, instrumental, and conceptual obstacles. Furthermore, students encounter epistemological challenges manifesting as constrained fractional context. The existence of learning obstacles can serve as a foundation for the development of learning designs, with the objective of reducing these obstacles in the future. In addressing the challenges posed by psychological barriers, Lin and Singh (2015) asserted the necessity of guided scaffolding during the learning process. This pedagogical approach has been shown to facilitate learning and enhance student motivation. Teachers should avoid delivering concepts that advance very quickly to complex concepts, but should be in accordance with the ontogenic development of students (Unaenah et al., 2023). Furthermore, the delivery of concepts through concrete manipulatives of fraction concepts has the potential to reduce the emergence of instrumental and conceptual ontogenic obstacles (Mohamed et al., 2021; Powell, 2023). The use of visual representations helps students make connections between the concept of fractions and real situations so as to encourage stable mental development, thereby reducing the occurrence of misconceptions stemming from the use of procedural that students do not understand conceptually (Pantziara & Philippou, 2012; Siegler et al., 2011). Thus, the use of concrete manipulatives or visual representations such as fraction blocks, number lines, or fraction circles, including digital learning media builds deep student understanding.

In the epistemological obstacle aspect, learning can be designed by involving multi-representation of concepts. Teachers provide a diversity of representations so that students are not limited to one meaning of fractions. The use of bar representations, pictures, number lines, or other visual models for the concept of fractions can help students in understanding the concept. Robotti et al. (2015) stated that the use of representation models such as colored number lines plays an important role in supporting the construction of fraction concepts. In addition, during the learning process teachers can develop the belief that fractions are not only limited to $\frac{a}{b}$ notation but emphasize meaning and context (Gupta & Elby, 2011). Teachers do not focus on only one meaning, but rather explore other meanings, including at the initial stage of learning (Purnomo et al., 2021). Fraction concept learning does not focus on the "how" problem but on the "why" problem. Involve a variety of real contexts to be explored such as through open-ended problems, discussions, or reflections including critical questions related to the interpretation of fractions. Thus, teachers can develop a learning design in the form of a learning trajectory based on the above considerations with the main basis of learning obstacles that occur to students. Table 2 provides a detailed description of the lesson

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plan design with instructional strategies as alternative, which aim to correct the identified student obstacles.

Phase	Instructional Activities	Strategy	Purpose
Engage	Presenting concrete-real life situations or problems related to fraction concepts, such as involving sharing food equally pizza. *Example Situations-One has 4 circular pizza pans that will each be cut into 8 pieces. 1. If the person is going to share the pizza with friends, how many pieces of pizza will each friend get? 2. If there are 10 pieces of pizza left, how many pieces have been eaten?*	Contextual Learning	Connecting fraction concepts to real life, activating prior knowledge, and over-coming psychological obstacle
Explore	Using concrete manipulations (i. e fraction strips, circles, or use virtual laboratory such as PhET) to explore fraction concepts such as part-whole use area of various shapes in paper *Example Situations-Students fold a circle piece of paper into four equal parts, then shade two of them, and then direct students to express the shaded areas as part values in the form of fractions to show how many parts are shaded.*	Concrete-Representational-Abstract	Address conceptual obstacles due to lack of intuitive understanding
Explain	Discuss in groups and present findings (can use representational forms such as number lines or symbols) *Example Situations- In groups, students are given different pieces of paper cut into equal parts (e.g. one-half, one-third, one-quarter, one-sixth) then ask students to represent each fraction on a number line and explain how* $\frac{1}{2}$ *is equal to* $\frac{2}{4}$ *and* $\frac{3}{6}$*, using visual models and symbols during group presentations. Furthermore, if it leads to arithmetic operations, namely addition, for example, students are asked to use two fraction strips*	Sociocultural Approach and Scaffolding	Encourage collaborative development of knowledge, address instrumental obstacles

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	(for example $\frac{1}{2}$ and $\frac{1}{3}$), then cut and rearrange the strips so that they have equal parts, then represent $\frac{1}{2}+\frac{1}{3}$ visually and conclude for themselves why it is necessary to equalize the denominator before adding (LCM concept).		
Elaborate	Elaborate various concepts of fractions as part-whole, measure, ratio, and operator through various problems, including contextual problems *Example Situations- Students work in groups to solve four different contextual problems, each of which reflects one meaning of fraction (part-whole with cases such as cake cutting, measure with cases such as on distance traveled, ratio with cases such as mixture or composition of food ingredients, operator with cases such as taking part of an item), then asked to classify the problem based on the meaning of the fraction used and present the explanation visually and symbolically.*	Multiple Representation	Counter epistemological obstacles by broadening conceptual understanding
Evaluate	Reflect on initial problems and present interpretations of fraction concepts in different contexts	Formative Assessment	Assess conceptual shift, metacognitive awareness

Table 2: Lesson Design with Instructional Strategies for Fraction Learning.

The results of this study showed that some of the obstacles that occurred were mainly related to the fundamentals of the concept of fractions. The series of activities designed in Table 2 are designed to construct students' understanding gradually, especially on the fundamental understanding of the concept of fractions. In the engage phase, students are introduced to problems that may occur in life. This phase emphasizes on activating prior knowledge and generating students' interest in learning so that it has an impact on anticipating students' psychological obstacles. The explore phase provides opportunities for students to gain a deeper understanding of the concept of fractions, especially as part-whole through concrete or visual manipulation. This phase can be designed as a continuation of the engage phase through problems or situations. The involvement of group discussion activities in the design is raised through the explain phase where students discuss in representing fractions. In this phase, students discuss and give each other reinforcement or correction so that the key understanding in the concept of fractions becomes correct. The next

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phase is elaborate which is designed to introduce students to various fundamental understandings of fractions, not only as part-whole but also to other meanings of fractions. Through the variety of contexts presented, students do not only recognize one meaning, but several meanings so that students' understanding of the concept of fractions is not limited to only one concept.

CONCLUSION

The research results reveal that junior high school students encounter learning obstacles in the concept of fractions, specifically in the representation of fraction meaning, fraction operations and their applications, and the ordering of fractions. The types of learning obstacles identified include ontogenic and epistemological obstacles. Ontogenic obstacles observed in students encompass psychological, instrumental, and conceptual dimensions. Psychological ontogenic obstacles refer to a lack of motivation and interest in mathematics. Instrumental ontogenic obstacles pertain to students' failure to solve fraction-related problems, particularly in determining key aspects such as the Least Common Multiple (LCM) of denominators. Conceptual ontogenic obstacles involve students' insufficient mastery of prerequisite concepts related to fractions. Epistemological obstacles in the concept of fractions are related to students' limited context, where students are confined to understanding fractions only in the form of $\frac{a}{b}$. This restriction does not extend to other contexts or interpretations of fractions, such as part of a whole and measurement. However, this study is limited to identifying learning obstacles, specifically ontogenic and epistemological, while didactical ontogenic aspects remain unexplored. Future research should delve into didactical obstacles and include the development of didactic designs based on identified learning obstacles, as well as the creation of learning trajectories for teaching fraction concepts in junior high school.

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Cognitive Disconnect in Middle School Algebra: A Phenomenological Study of Students' Understanding of Variables

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Abstract: *This study aims to explore the cognitive obstacles experienced by middle school students in understanding the concept of variables in algebraic expression learning. Using a descriptive phenomenological approach, this study introduces the term cognitive disconnect to describe a disruption of meaning caused by incompatibility between students' prior knowledge (Met-Before the Lesson/MBL) and the new concepts acquired during the lesson (Met-During the Lesson/MDL). Data were collected through classroom observations, analysis of student worksheets (LKPD), error-based interviews, and examination of textbooks and worksheets used by the teacher. The results show that students tend to interpret variables as concrete objects rather than quantitative symbols. Although algebraic expressions can be written correctly in terms of syntax, the meaning of letter symbols remains tied to prior arithmetic experiences, resulting in inaccurate interpretations of variables. The incompatibility between MBL and MDL is not explicitly bridged in the learning process and is reinforced by didactical situations, such as inconsistencies in learning resources and the dominance of procedural approaches. These findings highlight the importance of learning that connects arithmetic experiences with variables through representations and scaffolding.*

Keywords: cognitive disconnect, met-before, variable, didactical situation

INTRODUCTION

Algebra is a major branch of mathematics that forms the foundation for mastering concepts in geometry, statistics, and calculus (Bush, 2013; Edogawatte, 2011; Joanna & Jacquelynn, 2019; Nataraj & Thomas, 2017; Wettergren, 2022; Yew et al., 2020). At the middle school level, one of the essential topics is the introduction to algebraic expressions, particularly the concept of variables, which functions as a starting point for the cognitive transition from concrete arithmetic to abstract symbolization (Kieran, 2011; Öztürk, 2021; Pratiwi et al., 2019). Algebraic expressions include elements such as variables, coefficients, constants, and terms (Girit & Akyuz, 2017; Tekin-Sitrava, 2017).

However, various studies have shown that many students experience difficulties in understanding the conceptual meaning of variables (Blanton et al., 2017; Edogawatte, 2011; Ely & Adams, 2012;

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Radford, 2006; Wettergren, 2022). This phenomenon is also found in Indonesia, where students often hold misconceptions regarding the function and meaning of symbols in algebraic expressions (Das, 2020; Jupri et al., 2014; Natalia et al., 2016; Wahyuni & Herman, 2019). Fitria et al. (2023) found that although students can identify algebraic expressions symbolically, they often interpret variables as concrete objects rather than abstract quantities (Fitria et al., 2023).

The transition from arithmetic to algebra demands a cognitive restructuring in the way students think. Their prior experiences with arithmetic, which are concrete and object-oriented, often conflict with the abstract nature of algebra, leading to conceptual gaps and increasing the likelihood of misconceptions (Akkan & Baki, 2016; Humberstone & Reeve, 2017; Malisani & Spagnolo, 2009; Sudianto & Kisno, 2021). These difficulties become one of the key barriers to effective algebra learning at subsequent stages. A large proportion of students who fail to understand variables and subsequently algebraic expressions tend to struggle with mathematics at higher levels (Blanton et al., 2018; Sugiman, 2023). One of the main contributing factors to this difficulty is the activation of pre-existing thinking schemas that are misaligned with the new symbolic context.

According to Tall (2004), before receiving formal learning in the classroom, students have already been exposed to various mathematical ideas and knowledge through everyday experiences. This knowledge is informally constructed by students before they encounter the formal definition of a concept in school learning (Beyene, 2023; Levin & Walkoe, 2022; Tall, 2008). Tall refers to this prior knowledge or experience as *met-before*, which is a mental structure formed from previous interactions with mathematical concepts (De Lima & Tall, 2008; Mallet, 2013; McGowen & Tall, 2010; Tall, 2004). In learning, *met-before* can support understanding if it aligns with the concept being taught (Liaw et al., 2021; Özdemir & Yıldız, 2021). However, *met-before* can also become an obstacle when applied in a different context, namely when previous rules or ways of thinking are no longer relevant or suitable to be used in a new learning situation (De Lima & Tall, 2008). For example, experiences in arithmetic can become a hindrance when students are accustomed to the assumption that "*addition always results in a larger number*," whereas this assumption does not always apply in the context of algebra (Kieran, 2011). An obstacle can hinder the learning process and cause students to experience difficulties in understanding the concepts being taught (Beyene, 2023; Brousseau, 2006; Herscovics, 1978; Mallet, 2013; Tall, 2004).

The idea of this obstacle was first introduced by Bachelard (1938) as an epistemological obstacle, namely a disruption in thinking that hinders the understanding of new concepts (Trindade et al., 2019). This concept was later adapted by Herscovics (1978, 1997) into a cognitive obstacle in the context of learning, which arises when students' knowledge schemas conflict with new concepts, particularly during the transition from concrete to abstract understanding, such as in the shift from arithmetic to algebra (Herscovics, 1978).
Tall (2004) stated that prior experiences (*met-before*) often become a source of obstacles in learning mathematics. Students have a tendency to apply existing schemas to new contexts, so when the concept being learned conflicts with previous experiences, cognitive conflict arises and hinders the learning process. In such conditions, students prefer to rely on familiar procedures rather than

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engage in the reorganization of more complex concepts, in line with theory of equilibration by Piaget, which postulates that individuals are more likely to assimilate than to accommodate when facing knowledge discrepancies (De Lima & Tall, 2008; McGowen & Tall, 2010). Expanding on this idea, Beyene (2023) introduced the Met Before the Lesson (MBL) and Met During the Lesson (MDL) framework to explain in greater detail how cognitive obstacles occur. Cognitive obstacles arise when there is a incompatibility between MBL, which refers to students' prior experiences and knowledge schemas, and MDL, which represents the new concepts introduced during the learning process. This incompatibility may take the form of misconceptions or inaccurate interpretations of new concepts and is strongly influenced by cultural contexts and teaching approaches in each country (Artigue et al., 2014; Beyene, 2023; Jupri & Drijvers, 2016). Therefore, understanding the nature of obstacles stemming from the MBL–MDL incompatibility is a crucial step in designing more effective and conceptually grounded mathematics learning.

Since cognitive obstacles are related to the incompatibility between MBL and MDL, it is important to trace how MDL is formed during the classroom learning process, particularly in the context of introducing algebraic expressions. The Theory of Didactical Situations (TDS) is used to analyze the formation of MDL through didactical situations, which include the *milieu* and the didactical contract (Artigue et al., 2014; Brousseau, 2002; Maknun et al., 2022). In this study, the *milieu* refers to the learning environment, which includes learning resources such as textbooks, teaching modules, and student worksheets (LKPD) that are used and designed by the teacher. If the provided *milieu* is procedural and does not present challenging problematic situations, then students' understanding tends to be shallow (Artigue et al., 2014; Brousseau, 2002). Meanwhile, the didactical contract refers to the teacher's instructional strategies, including the use of analogies and the ways in which student responses are addressed during the lesson (Artigue et al., 2014; Radford, 2014). Exploration-based instructional strategies can provide opportunities for students to connect their MBL with the new concepts being learned in a more meaningful way.

Furthermore, indications of cognitive obstacles can be identified through student errors that are repeated and consistent. Such misconceptions are not random occurrences but rather reflect an underlying structure of understanding that has been formed (Beyene, 2023; Brousseau, 2002; Mallet, 2013). Persistent misconceptions indicate the presence of obstacles in the thinking process that cannot be resolved merely through additional instruction but require a deeper analysis of how students construct understanding during the lesson process.

Various studies have examined students' lesson difficulties in algebra through the perspectives of misconceptions, epistemological obstacles, and didactical design (Fitriani et al., 2023; Joanna & Jacquelynn, 2019; Jupri et al., 2014; Lenz, 2022; Noto et al., 2020; Öztürk, 2021; Rakes & Ronau, 2019; Riastuti et al., 2023; Stacey & MacGregor, 1999; Weinberg et al., 2016). However, studies that explicitly highlight cognitive obstacles in understanding the concept of variables based on the framework of incompatibility between MBL and MDL are still limited. In fact, at the early stage of introducing algebra in middle school, this incompatibility can lead to students' failure to connect

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symbolic meaning with their prior arithmetic experiences. Based on this background, this study is formulated to answer two main questions:

1. How do middle school students experience cognitive obstacles in understanding the concept of variables in algebraic expression lessons as a result of the incompatibility between MBL and MDL?
2. How do classroom instructional practices (through teacher strategies and lesson resources) contribute to the formation of these cognitive obstacles?

By using a phenomenological approach and didactical situation analysis, this article is expected to provide both theoretical and practical contributions to the design of algebra instruction that is more responsive to students' prior knowledge structures.

LITERATURE REVIEW

This section discusses the main conceptual framework used in this study, namely, cognitive obstacles, the incompatibility between MBL and MDL, and the contribution of didactical situations in shaping or obstructing students' conceptual understanding of variables in algebra.

Cognitive Obstacle in Mathematics Learning

The concept of obstacles in the lesson was first introduced by Bachelard (1938) through the idea of epistemological obstacles, namely that the development of scientific knowledge does not proceed linearly, but rather through discontinuous leaps that occur due to obstacles in thinking (Trindade et al., 2019). This idea was later adapted into the context of mathematics education by Herscovics (1978), who emphasized that difficulties in understanding mathematical concepts are often not caused by a lack of information, but by a conflict between old cognitive structures and the newly introduced concepts (Eraslan, 2005; Herscovics, 1978). In his subsequent work, Herscovics (1989) stated that cognitive obstacles occur when students fail to assimilate new concepts into the knowledge framework they have previously constructed, particularly in higher-level algebraic concepts.

This concept has become the foundation for various subsequent studies. Cornu (1991), in his study on the concept of limits, and Mallet (2013), in a lesson about line integrals, showed that students' initial mental models often remain unchanged despite the introduction of formal concepts (Cornu, 1991; Kartinah et al., 2021; Mallet, 2013). These findings emphasize that cognitive restructuring does not happen automatically following the delivery of a formal lesson, but rather requires an active process of reconstruction by the students. Other studies have also contributed to expanding the understanding of cognitive obstacles. Bishop et al. (2015) explained that cognitive obstacles may initially support the formation of understanding, but later become barriers when that knowledge is applied in a different context (Bishop et al., 2014). Therefore, gradual instructional strategies are important to facilitate students' conceptual transitions. In a fraction lesson, Yoshida and Sawano (2017) showed that students' initial understanding of parts and division often leads to cognitive conflict, which can be alleviated through the use of concrete materials and appropriate

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conceptual approaches (Yoshida & Sawano, 2002). Antonijević (2016) added that obstacles may also arise when mathematical tasks exceed students' cognitive capacity; however, this can be overcome through cooperative learning that encourages collaboration in solving complex problems.

In a more recent study, Beyene (2023) developed the conceptual framework of cognitive obstacles by referring to the theoretical construct of met-before proposed by Tall (2004; 2008), which is the prior knowledge structure formed from previous lesson experiences. This *met-before* can serve as a supportive foundation, but also has the potential to become a barrier to understanding new concepts. Beyene adopted this concept by distinguishing *Met-Before the Lesso*n (MBL) as students' prior knowledge and *Met-During the Lesson* (MDL) as the new concepts introduced during the lesson. According to Beyene, cognitive obstacles happen when there is a incompatibility between MBL and MDL. This incompatibility can hinder the process of constructing and applying concepts, especially when instructional strategies or didactical transposition do not explicitly connect students' prior experiences with the new ideas being taught during the lesson.

This cognitive tension becomes increasingly complex when the students' concept image, formed from MBL, contradicts the formal concept definition provided in MDL (Tall & Vinner, 1981). This condition often causes dissonance in thinking, particularly when students are required to shift from concrete arithmetic understanding to symbolic representation in algebra. Vinner and Dreyfus (2010) refer to this phenomenon as compartmentalization, which is a condition in which students possess two systems of thinking that are not interconnected. For example, students may solve the expression $2x + 3x = 5x$ procedurally, yet still believe that x holds a specific value as in arithmetic, rather than recognizing it as a free variable in the algebraic system.

In the context of learning algebraic expressions at the middle school level, an incompatibility between MBL and MDL is highly likely to happen. This is primarily due to the transition from the concrete world of arithmetic to the more abstract forms of algebraic symbolization and manipulation. Therefore, this study directly adopts Beyene's (2023) conceptual framework to analyze how the incompatibility between MBL and MDL contributes to the emergence of cognitive obstacles in students' understanding of variables and algebraic expressions.

Met Before the Lesson (MBL), *Met During the Lesson* (MDL), and Didactical Situations as Components Shaping Cognitive Obstacles

Within the framework of this study, *Met Before the Lesson* (MBL) refers to the cognitive structures or prior knowledge that students have before receiving a formal lesson on algebraic expressions. This structure is formed through previous learning experiences, including arithmetic lessons in elementary school, textbooks previously used, and students' interactions with various mathematical situations before the algebra lesson begins (Tall, 2008; McGowen & Tall, 2010; Beyene, 2023). Meanwhile, *Met During the Lesson* (MDL) refers to the new concepts introduced during the formal classroom lesson process. MDL reflects the conceptual structures expected to be developed

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through lessons, such as the concept of variables, algebraic expressions, as well as the operations and modeling of algebraic expressions. Cognitive obstacles in this study are defined as the incompatibility between MBL and MDL. When students' prior knowledge structures (MBL) do not align with or even contradict the new concepts being taught (MDL), students experience difficulties in constructing meaningful understanding. Beyene (2023) identifies three forms of such incompatibility: (1) prior knowledge cannot be applied directly; (2) difficulty in integrating prior knowledge with new concepts; or (3) the presence of deceptive compatibility, namely the impression that prior knowledge remains relevant, while it is misleading in the new context.

Furthermore, Prediger (2008) classifies the knowledge structures within MBL into three main dimensions: (1) algorithmic knowledge, which refers to procedures or steps that students are accustomed to performing; (2) intuitive knowledge and mental models, which are informal representations and beliefs formed through previous lesson experiences; and (3) formal knowledge, which refers to symbolic and conceptual understanding. An incompatibility in one or a combination of these three dimensions with the conceptual structure of the new content in MDL may lead to cognitive obstacles in learning algebraic expressions.

To understand how the incompatibility between MBL and MDL is formed and sustained throughout the learning process, this study analyzes the accompanying didactical situations using the framework of the *Theory of Didactical Situations* (TDS) (Brousseau, 2002; Artigue et al., 2014). This analysis is not intended to conclude that didactical situations are the direct cause of cognitive obstacles, but rather to examine whether and how the elements within the didactical situation, namely the *milieu* and the didactical contract, contribute to the emergence or reinforcement of the incompatibility between students' prior knowledge (MBL) and the new concepts being taught (MDL). The *milieu* includes all learning resources used during the lesson, such as textbooks, student worksheets (LKPD), and teaching modules. LKPD (Indonesian: *Lembar Kerja Peserta Didik)* refers to teacher-designed student worksheets used as instructional materials during the lesson. If the materials in these resources are presented without considering students' prior knowledge structures, the likelihood of an incompatibility between MBL and MDL will increase. The didactical contract, on the other hand, refers to the teacher's instructional strategies and the expectations established during the learning process. When learning places greater emphasis on procedures rather than on the understanding of meaning, students are likely to memorize rules without comprehending the underlying concepts. Effects such as the Topaze effect and the Jourdain effect (Brousseau, 2002; Artigue et al., 2014) serve as indicators that students may appear to succeed procedurally but experience conceptual obstacles in developing a deeper understanding of algebraic expressions.

Thus, cognitive obstacles in learning algebraic expressions cannot be separated from the complex interaction between students' initial cognitive structures (MBL), the conceptual structures built during instruction (MDL), and how the didactical situation, including both the *milieu* and the didactical contract, either supports or weakens the connection between the two. Therefore, understanding the relationship among these three components is essential for designing an algebra lesson

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that is more conceptual and meaningful. Figure 1 presents the conceptual framework of the study, illustrating the interaction between MBL, MDL, and the didactical situation as a potential source of cognitive obstacles in learning the concept of variables.

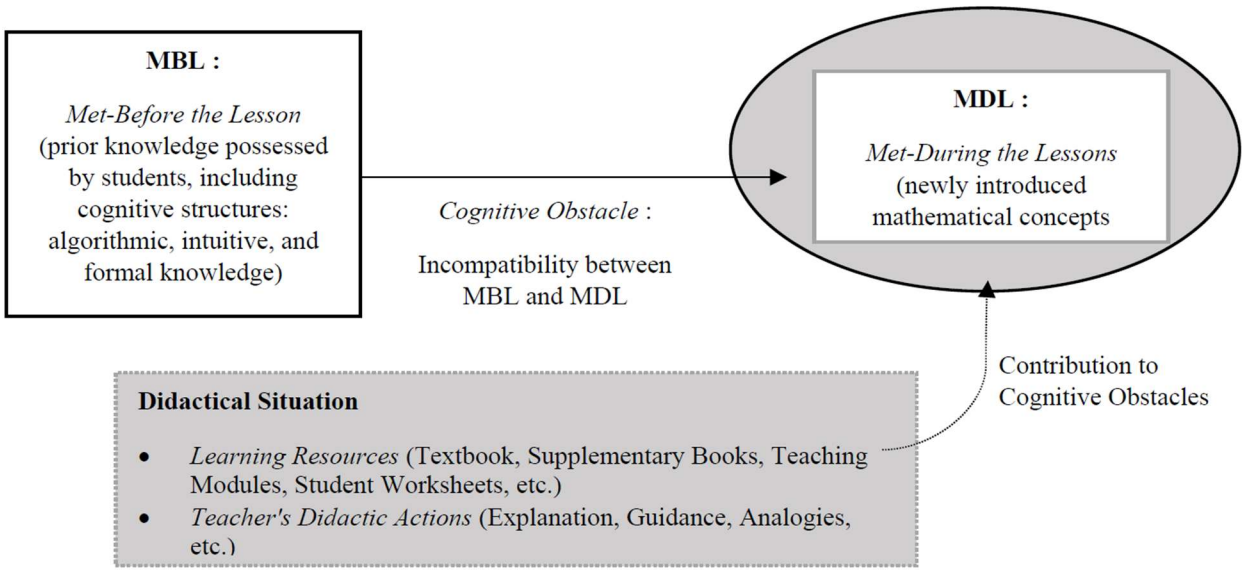


Figure 1: Conceptual Framework of the Study

METHOD

This study aims to describe the cognitive obstacles experienced by students in understanding the concept of variables in algebraic expression material, as well as to analyze the contribution of the didactical situation to the incompatibility between students' prior knowledge (Met Before the Lesson/MBL) and the new concepts acquired during lesson (Met During the Lesson/MDL). To achieve these objectives, a descriptive phenomenological approach was used, focusing on students' experiences as directly lived, without researcher intervention (Cohen et al., 2018; Polkinghorne, 1989; Qutoshi, 2018; Stolz, 2023; Zahavi, 2019). In this context, cognitive obstacles are understood not merely as procedural errors but as manifestations of the incompatibility between the MBL and MDL structures (Beyene, 2023), as observed during the classroom lesson process of algebraic expressions. To systematically implement this approach, the study adopted a descriptive phenomenological method by Giorgi (2009), which consists of three main stages: (1) Phenomenological reduction, (2) Description of experience, and (3) Essence analysis.

Research Context and Subjects

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This study was conducted in one of the seventh-grade classes at a public junior high school in Banjarmasin City, which showed an increase in numeracy scores based on the Computer-Based National Assessment (Indonesian: *Asesmen Nasional Berbasis Komputer*; ANBK) results, although students' achievement in the algebra domain was still relatively low. According to the curriculum, the topic of algebraic expressions was taught over 25 instructional periods in 10 sessions, spanning approximately 1.5 months (6–7 weeks), covering algebraic expressions and their components, operations on algebraic expressions, and modeling. The concept of variables was introduced during the first two sessions (five instructional periods), which focused specifically on the formal introduction of variable notation. In the subsequent sessions, variables continued to be applied in activities involving algebraic operations and problem modeling. The observation was conducted in class VII A, which consisted of 34 students and was taught by one mathematics teacher. The observation was carried out without intervention, with the researcher acting as a passive observer, as recommended in the phenomenological approach (Cohen et al., 2018). All learning resources, including the textbook and student worksheets (LKPD), were entirely designed by the teacher, allowing the learning context to proceed authentically. As seventh-grade students, the participants had completed six years of elementary education, during which they systematically studied arithmetic and its properties. This arithmetic background constituted their met-before knowledge, which likely influenced how they perceived and interpreted the new concept of variables introduced in algebra lessons.

Participant selection in phenomenological research is conducted purposively and based on empirical data. By the principles of phenomenology, participants are selected because they have direct experience with the phenomenon being studied (Qutoshi, 2018; Polkinghorne, 1989; Williams, 2021). Therefore, the subjects in this study were students identified as experiencing cognitive obstacles based on conceptual errors in assignments (student worksheets) and responses during lessons. These errors are understood as indications of cognitive obstacles, namely the incompatibility between students' prior knowledge (Met-Before the Lesson/MBL) and the new concepts introduced during the lesson (Met-During the Lesson/MDL) (Brousseau, 2006; Beyene, 2023). During the lesson, the teacher divided the students into six groups, each consisting of 5 to 6 members. Based on the results of observation and analysis of the student worksheets, two groups showed indications of typical conceptual errors that represent cognitive obstacles, namely Group A (5 members) and Group D (5 members). This study focused its analysis on Group A because their seating position was the most feasible for conducting observation and interviews without disrupting the flow of the lesson. Although the in-depth analysis focused on Group A, similar patterns of conceptual errors and cognitive obstacles were also observed in Group D, based on their worksheet responses and classroom discussions. While not all groups exhibited these difficulties, the recurring patterns found in Group A and Group D suggest that the focal group represents common cognitive challenges experienced by a subset of students in this classroom context. This follows the principles of the phenomenological approach, which emphasizes a natural learning context free from intervention (Cohen et al., 2018; Polkinghorne, 1989).

Data Collection

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This study has two main focuses: (1) to explore students' cognitive obstacles in understanding the concept of variables, particularly those that arise from the incompatibility between prior knowledge (MBL) and new concepts introduced during lesson (MDL); and (2) to analyze the contribution of the didactical situation to this incompatibility, both in terms of learning resources and the teacher's instructional strategies. To address these two focuses, data were collected from various sources:

1. Classroom observation, used to examine the learning situation, the teacher's instructional strategies, students' responses to tasks, and indications of cognitive obstacles.
2. Student worksheets, which were analyzed to identify errors as initial indicators of cognitive obstacles.
3. Error-based interviews, conducted in groups or individually (based on the worksheets provided by the teacher) to explore students' thinking processes and trace the forms of prior knowledge (MBL).
4. Learning resource documents, such as textbooks, student worksheets, and teaching modules, were analyzed to evaluate how the presentation of material may reinforce or reduce the incompatibility between MBL and MDL.

It is important to emphasize that all student worksheets and tasks were analyzed as they were, within the context of the regular lesson, fully designed by the teacher. The researcher did not intervene or modify the materials or tasks. This is following the phenomenological approach, which emphasizes the exploration of students' experiences in a natural context (as lived), without external manipulation.

Data Analysis

Data analysis in this study followed a descriptive phenomenological approach by Giorgi (2009) which is oriented toward the essential description of students' experiences in understanding the concept of variables in the algebraic expression lesson. The main focus is to reveal how cognitive obstacles emerge as a result of the incompatibility between students' prior knowledge (MBL) and the new concepts acquired during the lesson (MDL). The analysis process was carried out in three main stages:

1. *Phenomenological Reduction (Epoché)*: The researcher suspended theoretical assumptions or initial interpretations and focused solely on students' experiences as they are. The data were analyzed without being directly connected to a specific theoretical framework, so that the meaning that emerged truly originated from the experiences as observed in classroom activities, student work, and interviews.
2. *Description of Experience*: Data from classroom observations, responses in the student worksheets, and error-based interviews were organized into a coherent descriptive narrative. The principle of horizontalization was applied, in which each data element (students' actions,

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statements, and responses) was treated equally and then categorized based on experiential themes related to the emergence of cognitive obstacles.

3. *Essence Analysis*: this stage aims to identify the essential structure of the emerging cognitive obstacles. Using imaginative variation, the researcher explored how the incompatibility between MBL and MDL shaped recurring patterns of error. The analysis also highlighted the contribution of the didactical situation, both in terms of learning resources and teacher strategies, to the emergence of cognitive obstacles.

The final results were compiled in the form of a thematic narrative that represents the interrelation between students' thinking processes, prior knowledge structures (MBL), new concepts (MDL), and the conditions of the accompanying didactical situation. This approach enables a deep and contextual understanding of the phenomenon of cognitive obstacles, in accordance with the main principles of descriptive phenomenology.

RESULT

Initial Responses to the Teacher's Stimulating Questions

The classroom lesson on the topic of algebra began with stimulating questions delivered by the teacher to all students. The purpose was to explore students' prior knowledge (MBL) related to the concept of algebra, particularly concerning variables. Two open-ended questions were posed:

> *What do you know about algebra?*
> *Is there any algebra in daily life?*

However, the teacher did not directly call on students, resulting in only a few students responding spontaneously. The students' assumptions indicated limited and non-conceptual understanding. The majority stated that they did not know what algebra is, while a small number associated it with letter symbols, such as x, or with mathematical activities in daily life.

These observational results indicate that most students did not yet possess a strong initial conceptual structure related to algebra, even at the introductory level. The majority of students did not mention concepts such as variables, algebraic expressions, or relationships between quantities. The responses that emerged reflected recognition-based knowledge, for example, the presence of the letter x, without an understanding of the function of the symbol (Booth and McGinn, 2016). This phenomenon indicates that students' knowledge of algebra tends to be superficially symbolic and not yet rooted semantically. In the literature, however, variables are considered fundamental to the understanding of algebra and serve as an essential starting point in the cognitive transition from arithmetic to more abstract forms of symbolic representation (Knuth et al., 2005; Tekin Sitrava, 2017; Ventura et al., 2021).

The absence of a semantic foundation for mathematical symbols has the potential to create obstacles when students begin to formally study algebraic expressions. Furthermore, the teacher did not

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provide follow-up questions to explore more deeply the possible forms of MBL possessed by the students. No apperception activity connected previous lesson experiences, particularly in arithmetic, with the concept of variables. As a result, the lesson immediately shifted to the completion of student worksheets without providing the necessary conceptual transition. This situation closed the opportunity for students to realize that symbols in algebra, such as letters, have a representational function that can be linked to their previous arithmetic experiences. This condition became the starting point for the emergence of the cognitive disconnect identified in the subsequent findings.

Literal Interpretation and Pseudo-Construction in Variable Representation

After the session with stimulating questions, the teacher distributed student worksheets designed to introduce the concept of algebraic expressions through the visualization of concrete objects such as apples, mangoes, boxes, and balls. In this activity, students were asked to write algebraic expressions based on the given images. The findings in this section are focused on the analysis of Group A, which was identified as experiencing cognitive obstacles during the lesson on algebraic expressions. The selection of this group was conducted purposively based on classroom observation results and analysis of the student worksheets, as well as the consideration of their seating position, which allowed for observation and interviews without disrupting the lesson process. Data were obtained through classroom observation, student worksheet results, and error-based interviews, and were organized into thematic narratives following the principles of descriptive phenomenology. Although this section presents a qualitative analysis of Group A, a brief review of worksheets from all six groups (N = 34) showed that students in Groups A and D tended to interpret variable notation through concrete-object references. These 10 students interpreted variable symbols as physical items rather than as abstract representations of unknown or varying quantities.

The responses from students in Group A indicated that they tended to interpret the images literally as the number of concrete objects that could be counted, for example, writing 3 for a picture of three apples, or 2 + 1 for a picture of two apples and one mango. The results of Group A's work can be seen in Figure 2 below.

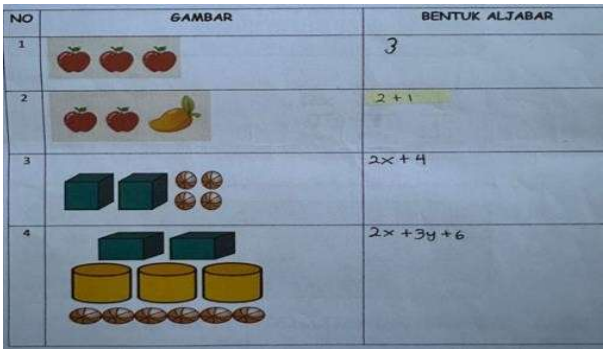


Figure 2. Student Group Responses on Worksheet (Group A)

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The worksheet results from Group A indicate a tendency among students to interpret algebraic symbols literally and concretely. In the first question, students wrote the algebraic expression as 3 for the three apple images, reasoning that the number 3 directly represented the number of apples. Meanwhile, in the second question, which displayed two apples and one mango, students wrote 2+1 as the algebraic expression, because they observed two types of fruit and recorded them according to their physical quantities. Arithmetically, the representation 2+1 reflects the total number of concrete objects. However, when simplified to 3, the expression is identical to the first problem, even though the visual context and types of objects are different. This indicates that students have not yet internalized the idea that symbolic expressions in algebra represent not only quantities, but also the identity of objects or quantities whose values may change.

The interview with Group A reinforced this finding. When asked why they did not write 3x for three apples, the students rejected the idea because, according to them, there was nothing unknown in the image, and everything could be counted. They also explained that letter symbols are only used when the objects are not directly visible or are placed inside a container, such as in the case of a box or a tube filled with balls. The following interview excerpt clarifies how the students interpreted letters in that context:

Question: In this picture, why did you answer that the algebraic expression is 3*?*
Group A: Because there are three apples in the picture.
Question: Is it acceptable to write it as 3x?
Group A: No, Miss. We also discussed it earlier, and we think there is no need to use x because all the apples are clearly visible.

In another question involving two boxes and four balls, the students wrote $2x + 4$. When asked about the meaning of x, the students explained that x represented the box because the balls inside it were not known with certainty. In contrast, the number 4 was considered not to require a letter because those balls were clearly visible. This indicates that students tended to use letters as physical labels for unseen objects rather than as mathematical symbols representing quantities. This is illustrated in the following interview quotation

Question: For this 2*x*+4*, how did you come up with it?*
Group A: From the picture, there are two boxes, so that makes 2*x, and there are* 4 *visible balls, so it remains* 4*.*
Question: What do you think x represents?
Group A: The box, Miss. Because there are two boxes, so it's 2*x. The balls do not need letters.*
Question: Why did you not use letters for the apples and mangoes earlier?
Group A: Because there were no boxes. All the fruits were visible.

These findings indicate that students understood letter symbols in a formalistic manner, but their semantic understanding remained tied to concrete objects. This phenomenon is referred to as pseudo-construction (Subanji, 2015), which occurs when students write symbolic forms that appear syntactically correct but are conceptually inaccurate. In addition, there were indications of

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deceptive compatibility, where symbolic structures such as $2x$ appeared to align with the visual context, but in fact did not represent the actual meaning of algebra. The connection with previous learning experiences in arithmetic was also strongly obvious. Students were accustomed to interpreting numbers as direct representations of concrete objects (for example, 2 rabbits means two actual rabbits).

The content of textbooks at the earlier grade levels further reinforced this association, as seen in examples that link images with quantities of objects (Figure 3). In the context of algebra, the lesson provided was insufficient to support students' transition toward understanding variables as abstract and flexible quantitative symbols. Without an explicit conceptual bridge in a lesson, the transition from arithmetic to algebra becomes hindered. When students assign object-based meanings to letter symbols rather than viewing them as variables, they are unable to comprehend the core concept of variables as quantities that are subject to change or whose values are unknown (Radford, 2011).

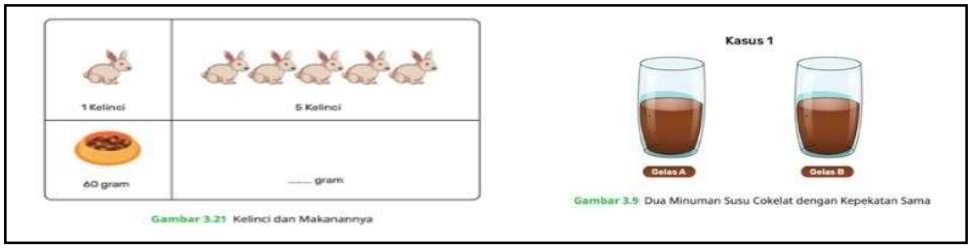


Figure 3: Textbook Material Connecting Images and Object Quantities

Figure 4 provides a schematic representation of the cognitive obstacle observed, illustrating how students' met-before experiences (MBL) in arithmetic—particularly associating numbers with visible objects—interfere with their understanding of variables as abstract symbols (MDL). This visual captures the deceptive compatibility between students' prior symbolic habits and the new algebraic representations expected in the lesson.

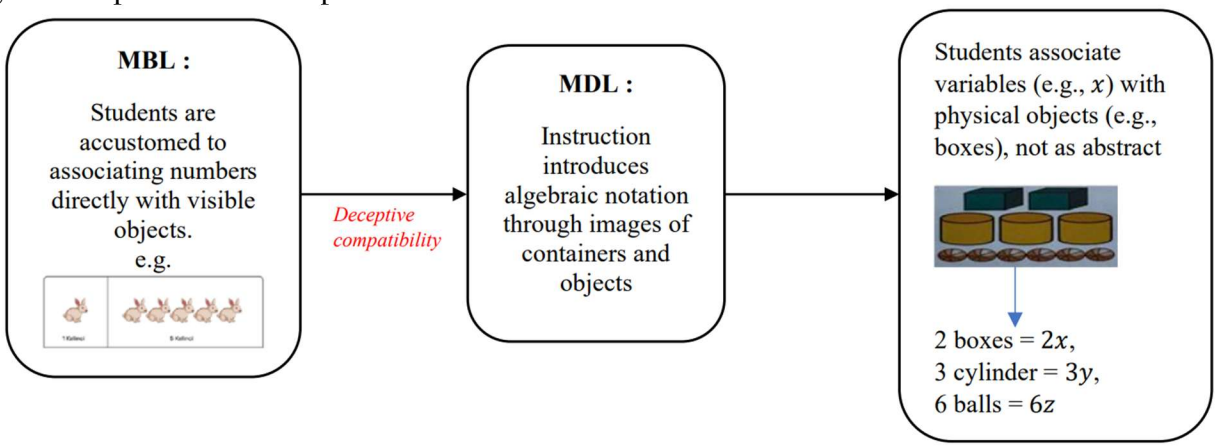


Figure 4: Deceptive Compatibility Between Students' MBL and MDL on the Concept of Variables

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***Cognitive Disconnect* in the Transition from Placeholder to Variable**

To follow up on the previous findings of pseudo-construction, the researcher administered a reflective task to all members of Group A as part of the interview preparation (Wijaya et al., 2022). The purpose was to further explore students' symbolic experiences related to the structure of MBL. The task consisted of four arithmetic statements using the symbol ⊡ (*placeholder*), such as 2+ ⊡ = 5. Students were asked to choose "Agree," "Disagree," or "Don't Know" and to provide a justification. All students in Group A responded correctly and consistently, indicating that they had prior experience using arithmetic forms involving explicitly unknown numbers. Students understood ⊡ as a *placeholder*, namely an empty space that must be filled with a specific number.

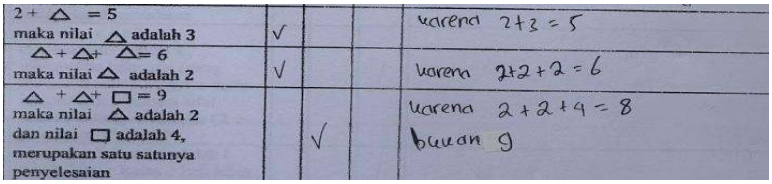


Figure 5: Student S1's Response Sheet on the Placeholder Task

To further explore the meaning structure that students constructed regarding the symbol, a follow-up interview was conducted with two group members, namely S1 and S2. The interview was carried out after the instructional activity had concluded, to ensure the process did not interfere with the learning situation, following the principles of the phenomenological approach (Cohen et al., 2018; Qutoshi, 2018). Although conducted separately, both students provided similar responses in terms of content and the reasoning behind their answers. This interview indicates that although students understood the symbol ⊡ as an unknown number, they did not automatically transfer that understanding to letter symbols. The following are excerpts from the interviews with S1 and S2 (member of Group A).

Question	***Student 1 (S1)***
Suppose it is given 2 + . . . = 5, *can you solve it?*	3
How did you get that?	*From* 5 - 2
What about this form, 2 + ⊡ = 5, *can you solve it?*	*Same as before,* 5 - 2 *is* 3
In your opinion, what does the form 2 + ⊡ = 5 *mean?*	*Two plus this (points to the box) equals five, so what is this (box)?*
What about the previous form, 2 + . . . = 5*?*	*Same, Miss, only the dots are different (box versus dots)*
What about the lesson just given earlier, do you think it is related to these forms?	*Hmm… I don't think so, earlier we were learning algebra with letters*

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Question	***Student 1 (S1)***
So that means you were learning variables?	*Hmmm… (nods)*
Can you explain what a variable is?	*A variable is a letter*
Do you know the function of the letter?	*For example, earlier it was for marbles inside a jar, also x for apples, and y for mangoes*
Earlier, you were able to fill in the box with the correct number to make 2 + ⊡ = 5, *meaning the box contains* 3. *If we replace the box with a random letter, what do you think?*	*For example, like* 2 + a = 5, *is that what you mean, Miss?*
Does it have the same meaning?	*If it's written like that, then it becomes algebra, Miss.*
What do you mean? So is it different from the earlier one?	*Yes, if it already uses a letter (a variable), then it's considered algebra.*
Can a be replaced with 3*?*	*In the previous lesson, we had to assume what the letter stands for, so I didn't really understand.*

Table 1: Excerpt Interview with S1

Interview data from S2 (a student from Group A) illustrates this interpretation

Question	***Student 2 (S2)***
For a question like this, 2 + ⊡ = 5, *can you fill it in?*	*Yes, Miss. The answer is* 3.
How do you know it's 3*?*	*Because* 5 *minus* 2, *Miss.*
If we replace the box with a letter, like 2 + a = 5, *do you think it still works?*	*It's different, Miss. If you use a letter, it becomes algebra.*
What's the difference?	*If it's a letter, we need to know first what a means. For example, a might mean apples or marbles.*
So does the letter mean an object in that case?	*Yes, Miss. Usually, a letter is used to replace the name of an object. If it's just a blank box, we just calculate it directly. But if we use a letter, we need to be told what the letter stands for.*
So, in your opinion, letters in algebra are not used for unknown numbers?	*No, Miss. If it hasn't been explained, I don't know what it's supposed to mean. Letters are like labels for objects.*

Table 2: Excerpt Interview with S2

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The statements of both students indicate that they did not interpret letters in algebraic expressions as a continuation of the placeholder concept. Letters were understood as symbols that only carry meaning when assigned to specific objects, rather than as flexible quantitative representations that can stand independently. They were unaware that letter symbols such as a or x in algebra can function in the same way as the placeholder symbol $\boxdot$, namely as a substitute for an unknown number (Knuth et al., 2005; Schoenfeld and Arcavi, 1988). Based on this analysis, the author refers to this phenomenon as *cognitive disconnect*, which is the students' failure to integrate meaning between their prior symbolic experiences (MBL) and the new concepts introduced during the lesson (MDL). This incompatibility happened due to the absence of an explicit bridge linking the function of placeholders and variables throughout the learning process.

This cognitive obstacle is not merely a procedural error, but a failure to construct a meaning structure that is conceptually connected. Although students can write symbolic forms formally, the foundational mathematical meaning is not understood. This situation indicates the presence of pseudo-construction (Subanji, 2015), which occurs when students produce semantically incorrect and syntactically correct answers because they interpret variables as object labels rather than as abstract quantities that are unknown or subject to variation. This finding highlights the importance of lessons that explicitly connect symbolic experiences in arithmetic with the concept of variables in algebra, thus, students can develop a conceptual understanding that is coherent and meaningful.

This finding is consistent with the report by Ely and Adams (2012), which shows that the transition from placeholder symbols to formal variables is often confusing for students, especially when teachers do not provide an explicit bridge connecting the two. Within the MBL and MDL framework (Beyene, 2023), this phenomenon reflects a cognitive disconnect, namely the failure of students to integrate their prior symbolic experiences (MBL) with the new concepts being taught (MDL). In summary:

- *MBL*: Students had prior experience solving expressions such as 2+□=5 with correct interpretation, where the placeholder represents an unknown number.
- *MDL*: The lesson introduced letters such as x or a as variables in algebraic expressions, but did not establish a connection between their meaning and the placeholder.
- *Cognitive Disconnect*: Students viewed *the placeholder* and the variable as two distinct entities. Letters were interpreted merely as labels for objects rather than as standalone quantitative symbols.

This finding directly addresses the research question, namely, how the incompatibility between students' MBL and the MDL, specifically the introduction of the variable concept in the classroom, contributes to the emergence of cognitive obstacles.

Contribution of Didactic Situations to the Emergence of *Cognitive Disconnect*

The *cognitive disconnect* experienced by students is closely related to the didactical situation that shapes the learning process. This situation involves various learning resources, including how materials are presented in textbooks, teaching modules, and student worksheets (LKPD), as well as

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the instructional strategies employed by the teacher. In the observed classroom, the teacher primarily referred to the Mathematics Textbook for Junior High School Grade VII (Kemendikbud, 2022), supplemented by a companion book published by PT Global Offset (2022) and custom-designed student worksheets.

1. Presentation of the Concept in the Textbook

The textbook introduces the concept of variables through the context of pattern generalization, such as constructing squares using matchsticks, and then expressing it in the form of an expression like $1+3n$. In this approach, the symbol n is positioned as an arbitrary number representing the number of pattern repetitions. However, students find it difficult to understand the meaning of this symbol because they do not see its connection to their prior experiences in arithmetic. The textbook does not connect the symbol n with the placeholder that is already familiar to students, resulting in n not being understood as a varying quantity, but rather being perceived merely as part of a form or an ordinary number (Ely and Adams, 2012). In fact, the transition from arithmetic to algebra requires students to recognize pattern structures as the foundation for constructing symbolic expressions (Apsari et al., 2020; Linchevski and Livneh, 1999; Taban and Cadorna, 2018).

Radford (2011) emphasizes that an effective algebra lesson must involve a transition from concrete representations to abstract symbols. When symbols such as n are introduced without sufficient explanation of their symbolic meaning, students will perceive them as labels for physical objects rather than as representations of quantitative relationships (Radford, 2014). This reinforces students' tendency to interpret variables in a concrete and inflexible manner.

2. Supplementary Book and Worksheet Approach

Unlike the textbook, the supplementary book and student worksheet (LKPD) frequently present variables through contextual problems, such as the unknown number of marbles in a bag. In this approach, the variable is understood as an unknown number, which is structurally closer to students' experiences in arithmetic. The following interview supports this tendency:

> *Question: Which book is easier to understand, the textbook or the worksheet (LKPD)?*
> *Student 1: The worksheet, Miss. Because you used the worksheet in class. I don't understand this one (the textbook), it's about arranging matchsticks.*
> *Question: So the worksheet is easier?*
> *Student 1: Yes. Because it already uses letters (variables) directly.*

The worksheet (LKPD) is connected to students' prior knowledge (MBL), such as the association between numbers and quantities of objects, making it cognitively more comfortable. However, this approach does not always support an abstract understanding of the variable concept.

3. Visual Representations in the Worksheet That Reinforce Misconceptions

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In the worksheet used by the teacher, students were provided with images of concrete objects such as apples, mangoes, cubes, cylinders, and marbles. Students were asked to write algebraic expressions based on these images. The question boxes contained only pictures without labels or semantic cues, while the answer boxes were left blank and filled in by students based on their own interpretations. However, due to the absence of explicit guidance, students tended to associate algebraic symbols directly with types of objects rather than with varying quantities. For example, an image such as two cubes, three cylinders, and six cakes was converted into $2x+3y+6$, indicating that students perceived each physical shape as a distinct symbol. This condition reinforces the understanding of variables as labels for objects rather than as flexible quantitative symbols. Algebraic symbols are not interpreted as representations of relationships between quantities, but merely as labels for different objects. This phenomenon illustrates a failure in the transition from concrete to symbolic representation, as described by Radford (2014). Without teacher intervention that explains that variables represent quantities that vary, students will continue to form incorrect associations based solely on visual forms.

4. Incompatibility of Approaches and Cognitive Load

The discrepancy between the approaches used in the textbook and the worksheet (LKPD) creates two inconsistent representations of variables. The textbook leads to an understanding of variables as varying quantities, while the worksheet emphasizes variables as unknowns. Without a bridge or explanation that explicitly addresses this distinction, students become confused and are unable to understand that the same symbol can carry different meanings depending on the context. According to Malisani and Spagnolo (2009), variables as unknowns are easier to understand because they are more concrete, whereas variables as varying quantities require a higher level of abstraction. When students are presented with two approaches without integration, they experience representational conflict and an increase in cognitive load (Kirschner, 2002; Prendergast and Treacy, 2018; Takir, 2011). The following interview supports this finding:

Question: Did you read the section about variables in the textbook?
Group A: Yes, Miss. But it was difficult. For example, arranging matchsticks. We did not understand what it was for.
Group A: We found the worksheet easier to understand.

5. Limited Role of the Teacher in Bridging Concepts

Observations showed that the teacher did not provide a stimulating question to connect students' prior experiences in arithmetic, such as the use of placeholders, with the concept of variables in algebra. Although the teacher designed the lesson using a Problem-Based Learning (PBL) approach, in practice, the teacher directly introduced symbolic forms such as $2x+3y+6$ through the problem, without first developing students' understanding of the meaning of each symbol. In general, the lesson activities were dominated by question-answering tasks, both in the form of visual manipulation and the construction of algebraic expressions. The main focus of the lesson appeared to be on procedural problem-solving rather than on the exploration of the underlying concepts. The teacher guided students to complete the tasks in the student worksheet (LKPD) and

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evaluated their work based on the correctness of the algebraic forms, but did not provide space for in-depth discussion about the meaning of the algebraic symbols used by the students.

When students experienced confusion, the teacher assisted in the form of technical guidance, but it remained procedural. Correct answers in terms of form were immediately accepted, without encouraging students to reflect on the symbolic meaning behind them. For example, students were given an example of writing $2x+3y+6$ as a representation of images of concrete objects, without any explanation of why the variables x and y were used or what quantitative relationships were embedded in the algebraic expression. When students gave answers that were formally correct, the teacher approved them directly without uncovering the symbolic meaning behind them. As a result of this approach, students did not undergo the abstraction process necessary to understand variables as flexible symbols and representations of mathematical relationships. They only understood variables as labels for fixed objects depicted in the images, rather than as representations of quantities that can vary. Consequently, students' understanding remained at the pre-variable stage, a stage in which letter symbols are interpreted concretely and have not yet been understood functionally within algebraic structures (Ventura et al., 2021).

A brief interview with the teacher revealed her awareness of students' difficulties in learning algebraic expressions. In response, she designed worksheets featuring contextual visual representations such as apples, boxes, and marbles. She assumed these would help students grasp the concept of algebraic expressions by linking symbols to familiar concrete objects. However, the teacher did not fully recognize that the main difficulty lay in students' limited understanding of variables—not just algebraic notation. While these visual aids supported symbol recognition, they did not facilitate students' transition toward interpreting variables as unknown or varying quantities. Instead, the representations reinforced the misconception that algebraic symbols always correspond to specific concrete objects. This suggests that the instructional strategy, although well-intentioned, did not adequately support students in developing a structural understanding of variables. Moreover, the teacher's conceptual understanding of variables may still be developing, which limits her ability to effectively support students in constructing this understanding.

Overall, the didactical situation in the learning process, including the approach used in the textbook and student worksheet (LKPD), the visual representations in the tasks, and the teacher's lesson strategies, did not succeed in bridging students' prior experiences with the symbolic concepts in algebra. The lack of integration between approaches, the absence of a conceptual bridge, and the dominance of procedural learning reinforced the cognitive disconnect. This finding addresses the second research question, indicating that an instructional design that does not explicitly connect MBL and MDL is a major factor contributing to the emergence of conceptual obstacles in understanding variables. To summarize the dynamics of the incompatibility between students' prior knowledge (MBL), the new concepts introduced during lesson (MDL), and the contribution of the didactical situation, Figure 6 presents the interaction among these three elements in shaping the *cognitive disconnect*.

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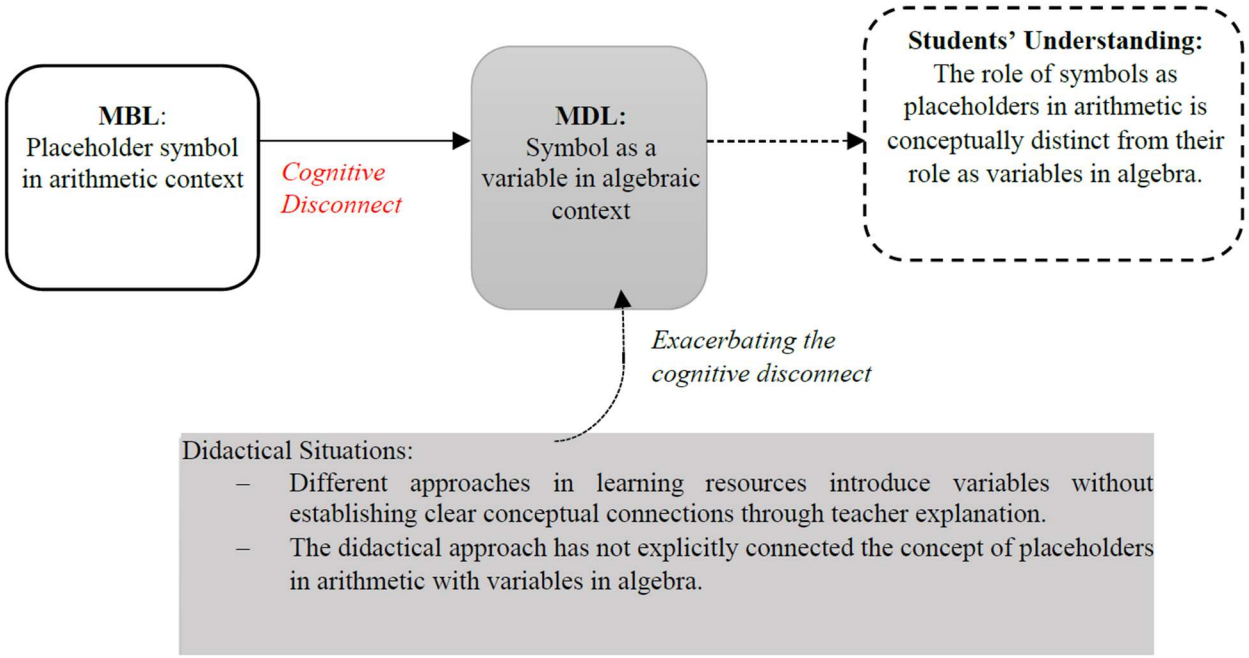


Figure 6: Cognitive Disconnect in the Concept of Variables

DISCUSSION

The findings of this study indicate that a primary conceptual obstacle in students' understanding of algebraic expressions is *cognitive disconnect*. This term, introduced in the current study, refers to students' failure to construct continuity of meaning between arithmetic placeholders (e.g., □ in 2+□=5) and letter symbols in algebraic expressions (e.g., 2+x). Rather than viewing variables as a semantic continuation of prior symbolic experiences, students perceive them as entirely new and unrelated forms. This disconnect results in a rupture between students' prior knowledge (MBL) and the mathematical concepts introduced during instruction (MDL), as framed by Beyene (2023).

Although students are able to produce syntactically correct expressions such as 2x+4, their interpretation remains literal. Variables are viewed as concrete labels—such as "apples" or "boxes"—rather than as abstract representations of varying quantities. This suggests that the semantic structure of variables has not yet been fully developed. Ely and Adams (2012) emphasize the importance of instructional approaches that explicitly guide students from using symbols as placeholders toward understanding variables as relational constructs. The inability to associate $2 + \square$ with $2 + x$ is similarly highlighted by Nathan and Koellner (2007) and Wang (2015), who stress the importance of symbolic continuity. Furthermore, Schoenfeld and Arcavi (1988), along with Nataraj and Thomas (2017), note that placeholders can serve as a foundation for understanding variables—if supported by deliberate instructional mediation. Without such scaffolding, this potential often remains unrealized (Glogger-Frey et al., 2018).

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Two cognitive phenomena further reinforce this disconnect: pseudo-construction and deceptive compatibility. Pseudo-construction occurs when students write syntactically correct expressions based on incorrect conceptual reasoning (Subanji, 2015). Deceptive compatibility refers to students' confidence in misinterpretations that appear to align with prior knowledge (Beyene, 2023). For example, interpreting $2a$ as "2 apples" may feel accurate, but this superficial alignment conceals a failure to develop the abstract concept of a variable.

From the perspective of the *Theory of Didactical Situations* (Artigue et al., 2014; Brousseau, 2002), the learning milieu often fails to provide sufficient conceptual support. The textbook introduces variables through generalized patterns (e.g., $1+3n$) without explicitly connecting them to prior symbolic experiences. Conversely, the student worksheet (LKPD) relies on concrete visuals without symbolic explanation. This inconsistency is not bridged by the teacher through reflective discussion or conceptual questioning. As a result, instruction emphasizes procedural form over semantic development. Although the textbook presents opportunities for conceptual exploration through pattern generalization, this potential is left untapped without teacher mediation. As Radford (2014) explains, the development of symbolic reasoning involves *sensuous cognition*, in which concrete and perceptual experiences gradually evolve into abstract forms. Without guided support, students interpret variables as arbitrary visual symbols rather than tools for expressing generalized relationships.

This disconnect also explains students' preference for worksheets over textbooks. The issue lies not only in the procedural nature of the worksheets, but also in the perception that textbooks are too abstract and disconnected from existing knowledge structures. As Maknun et al. (2022) argue, when conceptual and procedural aspects of instructional resources are not integrated—and when teachers fail to mediate this gap—misconceptions are likely to deepen. A similar pattern has been identified in trigonometry instruction, where procedural emphasis without conceptual clarity creates epistemological barriers (Prendergast & Treacy, 2018). In algebra, such barriers emerge when instructional resources are not aligned through a coherent representational trajectory. Finally, within Tall's (2004) framework of symbolic representation, *cognitive disconnect* reflects students' failure to transition from the concrete operations of arithmetic to the abstract reasoning of algebra. Letter symbols such as x or a are not yet perceived as generalized mathematical objects, but remain linked to concrete associations. Without reflective mediation to support this conceptual progression, students remain confined to literal interpretations. At its core, *cognitive disconnect* emerges when students fail to recognize variables as a semantic extension of placeholders—thus separating the two in meaning rather than understanding them as part of a coherent conceptual progression.

While these findings offer valuable insights into students' cognitive disconnect in understanding variables, they should be interpreted in light of the study's methodological limitations. This study focused on a single group of students, purposively selected to maintain the authenticity of the learning process given the researcher's passive role. This approach allowed for in-depth analysis of students' collective cognitive dynamics, in line with the phenomenological emphasis on lived experience. However, the findings may not be broadly generalizable, as students' meaning-making

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can vary across different groups and contexts. Future studies are recommended to involve more diverse student populations, to design instructional interventions that explicitly connect placeholders and variables through visual, symbolic, and conceptual representational transitions, and to examine how teachers mediate students' conceptual development in authentic classroom contexts.

CONCLUSION

This study concludes that one of the most fundamental cognitive obstacles in learning algebraic expressions among junior high school students is *cognitive disconnect*, defined as a break in meaning between students' prior knowledge (Met Before the Lesson or MBL)—such as placeholder symbols in arithmetic—and the new concepts introduced during instruction (Met During the Lesson or MDL), namely variables expressed through letters such as x or a. Although students are able to construct algebraic expressions that are syntactically correct, they tend to interpret the letters as labels for concrete objects (such as apples, boxes, or cylinders) rather than as abstract symbols representing unknown or varying quantities. This phenomenon indicates the presence of pseudo-construction, in which students manipulate symbols formally without understanding the mathematical meaning they represent.

These findings suggest that *cognitive disconnect* is not merely a cognitive obstacle arising internally from the student, but also a consequence of a didactical situation that fails to provide adequate conceptual support. When textbooks and worksheets (LKPD) present inconsistent instructional approaches, and teachers do not offer explicit mediation of meaning, students are unable to connect prior symbolic experiences with newly introduced concepts. This study proposes the concept of *cognitive disconnect* as a specific form of conceptual cognitive obstacle that results from the incompatibility between MBL and MDL, compounded by insufficient didactical support that fails to promote meaning integration.

Practically, the study recommends the importance of:

1. Initiating instruction from students' arithmetic experiences as a conceptual foundation for constructing the meaning of variables,
2. Designing representations progressively, beginning with concrete contexts and evolving into symbolic forms, to support gradual abstraction, and
3. Applying scaffolding strategies systematically, to assist students in transitioning from arithmetic-based thinking to symbolic understanding in algebra.

A meaningful understanding of variables can only be achieved when instruction is deliberately designed to build upon students' existing knowledge structures and anticipates the potential cognitive obstacles that may emerge during the shift toward abstract symbolic reasoning. Therefore, teachers need to develop approaches that explicitly and gradually connect arithmetic experiences with the conceptual meaning of algebraic symbols.

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Future research may focus on developing and evaluating didactical interventions that explicitly bridge students' prior experiences (MBL) with new concepts (MDL), including the use of representational strategies and scaffolding techniques that support conceptual transition. Experimental studies in various algebra learning contexts will further enrich our understanding of how to address and overcome cognitive disconnect.

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Exploring the Concept of Inequality Through the Lens of Concept Image Students

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Abstract: This study explores students' concept images of inequalities, examining their procedural proficiency and conceptual gaps. Employing a qualitative phenomenological methodology, data was collected through tests and interviews with high school students to elucidate their comprehension of inequality rules, solution sets, and symbolic representations. The findings reveal that while numerous students possess the ability to execute procedures, such as manipulating inequality signs when multiplying or dividing by negative numbers, they encounter challenges in articulating their reasoning. Common misconceptions include misinterpretation of interval notation, overgeneralization of rules, and difficulties in visualizing solutions on a number line. Furthermore, students exhibit varying degrees of proficiency in translating real-world scenarios into inequality notation, with some demonstrating symbolic accuracy but lacking conceptual justification. The study also identifies limitations stemming from sample diversity and instructional influences, underscoring the necessity for broader investigations. Based on these findings, recommendations include refining instructional strategies to bridge procedural and conceptual understanding, integrating visualization techniques, and providing targeted interventions to address misconceptions. Future research should explore effective pedagogical approaches to fortify students' relational comprehension of inequalities. These insights contribute to mathematics education by informing strategies for enhancing students' comprehension and application of inequality concepts in academic and practical contexts.

Keywords: Inequalities; concept image; misconceptions; procedural proficiency

INTRODUCTION

The causes of human errors and misconceptions have been a subject of interest for numerous educators, including David Tall and Shlomo Vinner, who are among the leading figures in this field. To comprehend the mental processes involved in conceptual manipulation, Tall and Vinner (1981)

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proposed the idea of a "concept image" to serve as a comprehensive description of the entire cognitive structure associated with a concept. This includes all mental representations and their associated properties and processes. Concept images undergo development over time, influenced by all the experiences an individual has with a particular concept, regardless of whether it manifests in visual forms such as pictures, drawings, diagrams, representations, or metaphors.

Tall and Vinner (1981) defined "concept definition" as a verbal explanation that accurately and non-circularly describes a concept. This definition is conveyed through the words used to articulate the concept and may be acquired either through rote memorization or through meaningful engagement with tasks related to the concept, allowing for the formation of relevant associations. A concept definition can take the form of a formal definition typically found in textbooks or a verbal expression of an individual's mental representation of the concept.

The way a person verbalizes a concept often reflects their concept image—that is, the mental structure or understanding they associate with the concept. Likewise, non-verbal representations, such as graphs or diagrams, can offer further insight into an individual's concept image. Vinner (1983) noted that during problem-solving activities, individuals frequently shift between their concept image and concept definition. This back-and-forth movement, especially when accompanied by inconsistencies or disconnections between the two, can indicate the depth of a student's conceptual understanding.

Many mathematical terms have everyday meanings that can inadvertently hinder mathematical progress. In mathematical activities, mathematical concepts are not solely utilized based on their formal definitions, but also through mental representations that may vary among individuals. Pre-existing concepts, derived from everyday experiences that generate mathematical ideas, can sometimes interfere with the very definition of mathematics itself (Cornu, 1981). Students' mathematical knowledge actually consists of the concept image-concept definition that is formed. Mathematical knowledge is categorized into two distinct terms: procedural knowledge and conceptual knowledge (Hiebert & Lefevre, 1986). Procedural knowledge encompasses computational skills and knowledge of procedures for identifying mathematical components, algorithms, and definitions. Conceptual knowledge, on the other hand, refers to knowledge of the underlying structure of mathematics. It is characterized as knowledge that is rich in relationships and encompasses understanding of mathematical concepts, definitions, and facts. Both procedural and conceptual knowledge are considered essential components of mathematical understanding (Hiebert & Lefevre, 1986). Therefore, teaching mathematical knowledge should encompass instruction in both procedural and conceptual knowledge.

In the realm of mathematics, students should be encouraged to develop a formal conceptual understanding of a concept rather than solely relying on personal procedural and conceptual

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knowledge (Hussein, 2022). This shift in emphasis is crucial for students to grasp the formal definition of a concept. The role of definitions in mathematics is multifaceted. On the one hand, definitions serve to identify familiar concepts, ensuring their clarity and precision. Conversely, in formal mathematics, definitions establish the properties of the defined concept, defining it in the process (Pinto & Tall, 1996). However, the concept image-concept definition scheme is not solely constructed through definitions but also through experience (Vinner & Dreyfus, 1989).

By comprehending students' conceptualizations (concept images), educators can identify their learning gaps and tailor their instruction to enhance their comprehension of mathematical concepts (Siagian et al., 2021). Formal mathematical definitions are pivotal in mathematics, serving as the foundation for theorems that establish the theoretical framework of the subject. However, students frequently encounter challenges in grasping and applying formal mathematical definitions (Rasslan & Tall, 2002; Tall, 1994; Vinner & Dreyfus, 1989). Consequently, the same mathematical concept can be elucidated through various representations, and sometimes even with distinct definitions. Consequently, the concept image associated with a particular mathematical idea can differ significantly between students and their instructors (Dreyfus, 1991).

Consequently, it is imperative to conduct a research study to investigate students' conceptual understanding, particularly regarding the concept of inequality. Inequality is a widely employed term in both mathematical and practical contexts. In the realm of mathematics, inequality is explicitly utilized to denote the relationship between numbers or to specify limits for quantities. Alternatively, it may be implicitly incorporated within the domain or characteristics of a function, encompassing its behavior. In practical scenarios, inequality can be categorized as injustice, disparity, inequality, irregularity, or dissimilarity.

Inequality is a fundamental concept in mathematics education, not only serving as a prerequisite for other mathematical concepts but also finding applications in various disciplines (Bicer et al., 2014; Siagian et al., 2022; Yokoso et al., 2024). However, empirical evidence suggests that students' comprehension of the concept of inequality warrants substantial attention, as its urgency appears to be inversely proportional to its perceived importance. Previous research has consistently highlighted students' challenges in solving problems related to the concept of inequality (Palupi et al., 2022; Rofiki et al., 2017; Sitanggang & Araiku, 2022; Tamba & Saragih, 2020; Yasa et al., 2022; Zulhendri et al., 2022). Notably, these difficulties extend beyond high school students, affecting even college students (Biney et al., 2023; El-Khateeb, 2016; Siagian et al., 2022; Turner et al., 2016).

Based on the research findings presented, I have identified the significant learning challenges posed by inequality for students. The research reveals cognitive difficulties that students encounter when grasping the concept of inequality. Notably, students frequently struggle to comprehend the

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relational nature of inequality, particularly when transitioning from equations to inequalities, where solutions involve ranges rather than specific values. Furthermore, the reversal of the inequality sign when multiplying or dividing by a negative number is a well-documented source of confusion, as it deviates from students' prior arithmetic experiences. These challenges stem from conceptual misunderstandings and procedural errors, rendering inequality a complex subject in mathematics education.

Certainly, research related to the concept of inequality has been conducted by numerous researchers. For instance, Halmaghi (2011) conducted an analysis of students' and first-year students' comprehension of the concept of inequality. Pratiwi and Rosjanuardi (2020) conducted a study analyzing students' responses to rational and irrational inequalities. Mokh et al. (2019) conducted a study related to the concept of inequality with the objective of examining the errors made by students in utilizing logical conjunctions when simplifying algebraic equations and inequalities. Furthermore, research conducted by Jupri et al. (2022) aims to investigate the strategies employed by pre-service teachers when solving absolute value equations and inequalities from the perspective of symbol interpretation. El-Khateeb (2016) conducted a study with the intention of investigating and identifying the types of errors that occur in preparatory year students in solving inequalities. Similarly, research conducted by Bicer et al. (2014) remains pertinent to pre-service mathematics teachers' comprehension of linear and quadratic inequalities, with the aim of ascertaining whether they harbor common misconceptions and difficulties with inequalities. Additionally, research conducted by Moon (2019) proposes several alternative test solutions for comprehending two-variable inequality graphs.

Based on the urgency of the inequality concept and several relevant research findings, I conducted a study by emphasizing students' concept images in the inequality concept. This study aims to elucidate the meaning of the inequality concept as perceived by students. Students' concept images are analyzed through the examination of three distinct components: mental images, processes, and properties. Mental images encompass all mental representations employed by students to elucidate concepts or describe their mental processes in solving problems related to the concept. Processes refer to all procedures, methods, or steps utilized by students to describe concepts or approach problems related to the concept. Properties encompass all axioms, definitions, lemmas, theorems, or formulas employed by students to describe concepts or solve problems related to the concept.

In accordance with the objectives of this study, the following research questions are formulated: (1) How do students conceptualize the mathematical concept of inequality in relation to their concept images? (2) What are the prevalent misconceptions and reasoning patterns that students demonstrate when attempting to solve problems involving inequalities? and (3) To what extent do students' procedural skills correspond with their conceptual comprehension within the context of inequalities?

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LITERATURE REVIEW

The Significance of Definitions in Mathematics Education

Edwards and Ward (2008) posited that definitions serve as indispensable tools for mathematicians and students to ensure effective communication. Moreover, definitions serve as the cornerstone for logical reasoning and mathematical proof construction, particularly in advanced mathematics. However, Berman and Shvartsman (2016) identified that a deficiency in comprehension of formal definitions can impede students' learning trajectory. Through empirical research, they demonstrated that students' engagement and comprehension levels surged substantially when theoretical inquiries pertaining to definitions were integrated into academic assignments. This finding aligns with the assertions of Chang et al. (2022), Dorier et al. (2000) and Carlson (1993), who underscored students' challenges in grasping mathematics, particularly due to the inadequate time allocated for understanding formal definitions at the secondary school level.

Cognitive Structures in Concept Formation

To comprehend the cognitive processes involved in mathematical concept formation, numerous cognitive models have been developed by researchers (Duval, 1998; Fischbein, 1993; Tall & Vinner, 1981). One of the most influential models is the cognitive structure proposed by Tall and Vinner (1981), which comprises two primary components: concept image and concept definition.

The concept image is defined as "all cognitive structures in an individual's mind associated with a concept" (Vinner, 2011). This component encompasses mental representations, attributes, and related processes, which can manifest as visual representations (e.g., charts, graphs, diagrams), symbolic representations (e.g., formulas), or other forms. For instance, students can interpret mathematical functions as rules, graphs, operations, tables of values, or mappings between sets (Ojo & Olanipekun, 2023; Tirosh & Tsamir, 2022).

Concept definitions, on the other hand, refer to "the form of words employed to define a concept" (Tall & Vinner, 1981). These definitions are categorized into two types: personal concept definitions, which students utilize to elucidate their comprehension, and formal concept definitions, which are recognized within the mathematical community. The dichotomy between examples and non-examples of a concept, as defined by the formal definition of the concept, plays a pivotal role in mathematics education and instruction (Ulusoy, 2019). Vinner (1983, 1991) underscored the significance of the reciprocal interaction between concept image and concept definition in comprehending a concept. He identified three potential interactions: (1) adjusting the concept image to align with a formal definition that contradicts the existing image, (2) memorizing the definition without modifying the concept image, and (3) neither internalizing the definition nor altering the concept image.

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Mathematical Knowledge: Procedural and Conceptual

Students' mathematical knowledge is characterized by a dynamic interplay between concept images and concept definitions, which can be categorized into procedural knowledge and conceptual knowledge (Hiebert & Lefevre, 1986). Sfard (1991) provided a comprehensive analysis of the development of mathematical concepts from a historical and psychological perspective, positing that mathematical concepts can be comprehended operationally (as a process) or structurally (as an object). The process of conceptual acquisition involves three distinct stages: internalization (acquiring knowledge of the concept and its operations), condensation (developing flexibility in representing the concept), and reification (gaining a comprehensive understanding of the concept as a unified entity). However, difficulties often arise in comprehending the structural aspect due to the ontological gap between the operational and structural stages.

Furthermore, Sfard (1991) distinguished between concepts and conceptions, where concepts encompass the formal aspects of mathematics, while conceptions represent individual interpretations of the concept. This distinction aligns with the theory of concept images and concept definitions, which suggest that a person's mental representation may not always precisely correspond to the formal definition. As Ulusoy (2021) elucidates, constructing a mathematical definition entail more than merely expressing a conceptual image in words.

Personal Interpretation and Misconceptions

Students not only require a comprehensive understanding of formal axiomatic systems but also the ability to construct personal interpretations of formal definitions. These interpretations can vary and may not always align with formal definitions (Tsamir & Tirosh, 2023; Pinto & Tall, 1996; 2002). Tall and Vinner (1981) elucidated that concept definitions reflect cognitive structures associated with the definition of a concept and constitute an integral component of the overall concept image. In practical terms, the application of formal definitions in reasoning necessitates personal interpretations, which can significantly impact students' comprehension.

The relationship between concept image and concept definition also provides insights into misconceptions and learning difficulties. During the initial stages of learning, students often possess limited concept images that undergo development through experience and concrete examples (Vinner, 1991). However, at the secondary and tertiary levels, this relationship is frequently perceived as a one-way process, wherein concept images are regarded as the outcome of understanding formal definitions (Vinner, 1983). This approach overlooks the significance of developing rich and adaptable concept images, which serve as the foundation for profound comprehension.

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METHODS

This research employs a qualitative research approach, which systematically organizes and interprets textual materials derived from conversations or conversations. Qualitative research is instrumental in exploring the subjective experiences and perspectives of individuals within a specific context (Grossoehme, 2014). The phenomenological approach is the research methodology employed in this study. Phenomenology can be defined as a research approach that aims to elucidate the essence of a phenomenon by examining it from the viewpoint of those who have experienced it (Teherani et al., 2015). The phenomenon investigated in this study is students' experiences in learning the concept of inequality, and how these experiences shape their concept image of the concept.

The study participants were 10th-grade high school students (aged between 15 and 16 years) who were selected through purposive sampling. To mitigate potential bias, the researcher employed purposive sampling, incorporating academic records and teacher assessments in participant selection. Specifically, students were categorized into three distinct academic ability levels: high, medium, and low achievers. Furthermore, gender diversity was considered. These criteria were meticulously selected to ensure that the exploration of students' concept images pertaining to inequality encompasses a spectrum of mathematical abilities, reasoning patterns, and diverse learning experiences. The primary objective of this study was to explore students' concept images of inequalities.

Data collection in the study was conducted through administering tests related to the concept of inequality (see Appendix) and unstructured interviews to explore the concept image formed in students based on their learning experiences. The tests designed aimed to (1) explore students' perceptions of understanding the rules of the concept of inequality and also related to how students interpret the concept of inequality; (2) explore students' understanding of the application of inequality rules, determining and utilizing the solution set of the inequality concept; and (3) explore students' understanding of using inequality notation in terms of transforming mathematical situations or problems into mathematical symbols in inequality. In accordance with the test results, the reduction process is undertaken to further elucidate students' concept images of inequality.

The validity and reliability of the test instrument were assessed prior to its implementation. The findings of these assessments were based on Cohen et al.'s (2020) evaluation. To evaluate the validity of the test instrument employed in this study and determine the extent to which contextual factors influence the concept of inequality both theoretically and practically, content validation was deemed appropriate. While the primary objective of the reliability test was to investigate the

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impact of the provided context on students' responses to questions, the researcher was also interested in assessing the level of comprehension of students' understanding of the concept of inequality.

Data analysis was conducted employing the phenomenological approach developed by Hycner (1985) and modified by Groenewald (2004), which recognizes the researcher's interpretive involvement with the data. Willig (2001) elucidated that the implications of phenomenology for data analysis are achieved by adhering to the following characteristics: focusing on the lived experiences of the subjects, maintaining openness to their perspectives, providing precise descriptions, deferring prior knowledge or biases, and seeking the essence in descriptions.

RESULTS

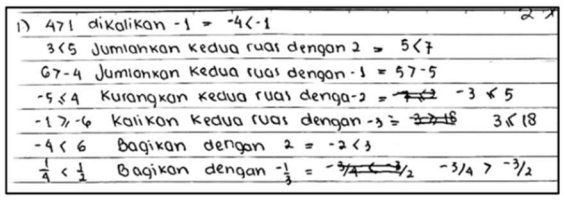
Students' Perceptions Regarding the Understanding and Interpretation of Inequality Rules

Problem 1 seeks to explore students' perceptions regarding the rules of the concept of inequality and their interpretations of the concept. Additionally, it aims to review students' understanding of the concept of numbers. Based on their perceptions and responses to Problem 1, the concept image of the rules of the concept of inequality and the meaning of the concept of inequality will be investigated. However, prior to investigating the concept image formed, the categories of students' answers will be systematically summarized based on the results of the answers and interviews, as presented in Table 1.

Response Category	Response
Correct procedure accompanied by appropriate reasons	3
Correct procedure lacking proper reasons	10
Incorrect procedure accompanied by inappropriate reasons	9

Table 1: Student response categories for problem 1

Based on the categorization of students' responses to problem 1, it can be ascertained that there are students who execute the procedure accurately. Students demonstrate proficiency in applying inequality rules, particularly when modifying inequality signs while multiplying or dividing by negative numbers. However, it is evident that many students lack a clear explanation for their results. For instance, Figure 1 provides an example of a student's solution to problem 1.



$4 > 1$ Multiply by $-1 = -4 < -1$
$3 < 5$ Add both sides together with $2 = 5 < 7$
$6 > -4$ Add both sides together with $-1 = 5 > -5$
$-5 \leq 4$ Subtract both sides by $-2 = -3 <= 5$
$-1 \geq -6$ Multiply both sides by $-3 = 3 \leq 18$
$-4 < 6$ Divide by $2 = -2 < 3$
$\frac{1}{4} < \frac{1}{2}$ Divide by $-\frac{1}{3} = -\frac{3}{4} > -\frac{3}{2}$

Figure 1: Student response with correct procedure but no justification.

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Analysis of the concept image component shows that students correctly followed the process to solve the problem. The "properties" component also aligns with the rules of inequalities. However, in the "mental image" component, students transformed the inequality notation after multiplying or dividing by a negative number. This transformation was based on their understanding of number positioning relative to zero—numbers further to the left are smaller. This mental image was derived from interview results, summarized below.

Researcher : *Based on your response to problem 1, it appears that several parts yield results that change sign. Could you please elaborate on this?*
Student : *Indeed, sir. When a number is multiplied by a negative number, its sign changes.*
Researcher : *Could you provide an example to illustrate this?*
Student : *For instance, in the first part,* 4>1 *is multiplied by* -1. *The resulting values on each side are* -4 *and* -1. *If we examine* -4, *we can see that it is smaller than* -1 *because* -4 *is farther from 0 than* -1.

The interview excerpt shows that students understand why the inequality sign changes when multiplying or dividing by negative numbers. However, their explanation lacks clarity. In problem 1, their mental image of this concept is evident. Some students also grasp the formal rules for modifying inequalities in such cases. The following interview excerpt illustrates this understanding.

Researcher : *I understand that the answer to the final part of problem 1 has undergone a change in sign.*
Student : *Indeed, sir.*
Researcher : *Could you please elaborate on the reason behind this change?*
Student : *As I recall, sir, when inequalities are multiplied or divided by a negative number, the sign is reversed.*
Researcher : *Why did you not provide this explanation yesterday?*
Student : *I apologize, sir, but I inadvertently neglected to do so due to a pressing time constraint.*

The second category of student responses, which is the ability to execute the procedure accurately and provide valid reasons for solving problem 1, is also noteworthy. An example of student responses in this category is provided in Figure 2.



Conclusion:

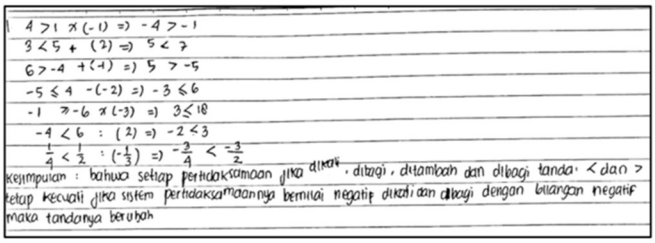
If the inequality, both sides are added or subtracted together, the inequality sign does not change. If the inequality, both sides are multiplied or divided by a positive number, the inequality sign also does not change. If the inequality, both sides are multiplied or divided by a negative number, then the inequality sign will change.

Figure 2: Student response with correct procedure and valid reasoning

Referring to Figure 2, it is evident that students have successfully executed the "process" component. Furthermore, the "properties" component has been correctly applied by students in solving problem 1. The students' mental images can be inferred from the explanations provided in their

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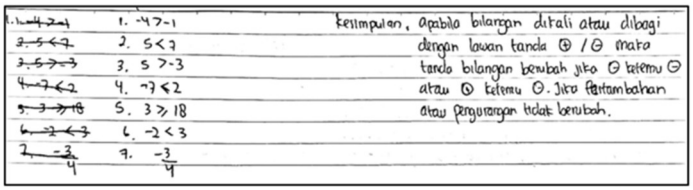
answers. However, it is noteworthy that the students' answers in the "procedure" category are incorrect, and the reasons given are also inappropriate, as illustrated in Figure 3.



Conclusion: That every inequality if multiplied, divided, added and divided by the < and > signs remain the same, except if the inequality system has negative values and is multiplied and divided by negative numbers, then the sign changes.

Figure 3: Student response with incorrect procedure and flawed reasoning

Figure 3 shows that students follow the given instructions correctly. However, they make errors in the "properties" component, focusing on procedures from Problem 1 while neglecting the rules for inequalities involving negative numbers. As a result, their answers to Problem 1 are incorrect. Students' mental images reveal that they recognize a change in the inequality sign when a negative inequality is multiplied by a negative number. Additionally, other errors related to incorrect procedures are also accompanied by flawed reasoning. Figure 4 provides a detailed view of these errors.



Conclusion, if a number is multiplied or divided by the +/− sign, the number sign changes if − meets − or + meets −. If addition or subtraction does not change.

Figure 4: Student response with incorrect procedure and reasoning

As depicted in Figure 4, students have demonstrated proficiency in performing addition, subtraction, multiplication, and division operations. However, the methodology employed to solve the students' answers is flawed. This arises from the application of incorrect properties in the rules governing inequality concepts. Specifically, the significance of altering the sign of a number from positive to negative and vice versa is not utilized as a basis for students to revise their inequality notation. Consequently, the final solution to problem 1 is erroneous.

Conversely, students' mental representations of solving problem 1 primarily focus on the numerical values within the inequality notation. This tendency to make errors, as evident in Figures 3 and 4, can be attributed to students' reliance solely on memorizing the rules applicable to a particular concept. This is evident from the interview excerpts provided in Figure 3.

Researcher : *Could you please explain why the form of inequality changes when considering the five signs of inequality?*

Student : *The signs of inequality (*-1 *and* -6*) are negative. When multiplied by a negative number, the resulting number becomes positive. Consequently, the sign of the inequality also changes.*

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This demonstrates that students' mental representations of the rule for changing inequality signs are based on the initial form of the inequality being negative and subsequently multiplied or divided by a negative number. This can be clearly illustrated in the following interview excerpt.

Researcher : *Upon reviewing the first inequality, it appears that multiplying* 4 *and* 1 *by* -1 *does not alter the number.*
Student : *Indeed, the number remains unchanged. However, the inequality sign does change.*
Researcher : *Could you please clarify why the inequality sign changes when multiplying or dividing by a negative number?*
Student : *To the best of my recollection, the sign of an inequality flips when it is divided or multiplied by a negative number because the resulting expression becomes negative, similar to the fifth part.*

The interview excerpt shows that students have inaccurate mental representations of inequality properties, as they rely on memorization rather than understanding the underlying concepts. Their inconsistent application of these rules in Problem 1 further reflects this gap in comprehension. While they handle positive and negative numbers correctly, their understanding of inequality rules remains incomplete. Figure 2, based on interview excerpts, illustrates this issue.

Researcher : *Based on your response to problem 1, part two, can you clarify if the form* $3<5$ *is equivalent to* $5<7$*?*
Student : *No, Sir. The numbers are different.*
Researcher : *Could you please elaborate on why this is the case?*

Based on the interview excerpt, it appears that students lack a comprehensive understanding of the nature and rules governing the concept of inequality. Specifically, consider the following scenario: suppose a and b are real numbers such that $a<b$, and c is another real number that is distinct from zero. In this case, the inequality $a<b$ is equivalent to the following: $a+c<b+c$; $ac<bc$, provided that $c>0$; and $ac>bc$, provided that $c<0$. Consequently, based on their misunderstanding, students perceive the form $3<5$ as equivalent to $5<7$. However, this is incorrect. In fact, the form $5<7$ is the result of adding the same quantity to both sides of the inequality $3<5$, with the quantity being 2. Therefore, $5<7$ is a new inequality that is equivalent to $3<5$.

Students' Proficiency in Applying Inequality Rules and Interpreting Solution Sets

Problem 2 examines students' understanding of linear inequalities, focusing on determining and interpreting solution sets. Their responses will be analyzed to assess how they apply these rules. Before this analysis, we will categorize their answers based on responses and interviews. The categories are: (a) solutions using visual representation (number line) and (b) solutions using interval notation. Table 2 presents these categorized responses.

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Response Category	Response	
	Point a	Point b
The solution procedure and visual representation are both correct.	8	8
The solution procedure and visual representation are both incorrect.	3	4
The solution procedure is correct, but cannot be represented in visual form.	6	3
The solution procedure is incorrect, and cannot be represented in visual form.	5	7

Table 2: Categories of student responses to problem 2 based on visual representation

The concept image component shows that students who correctly identify procedures and visualize linear inequalities can effectively apply the necessary steps to solve related problems. However, in solving Problem 2, they tend to use segment displacement. As a result, their understanding of linear inequality rules appears unstable and inconsistent. The following interview excerpt illustrates this further.

Researcher : *Let us review your response* (Figure 5). *In the third step point a, why does the inequality become* $6x<9+27$*?*
Student : *The initial form is negative 27, so when it is moved, it becomes positive 27.*
Researcher : *Therefore, is the form* $6x<9+27$ *equivalent or the same as the initial form* $6x-27<9$*?*
Student : *(Pauses for clarification)*

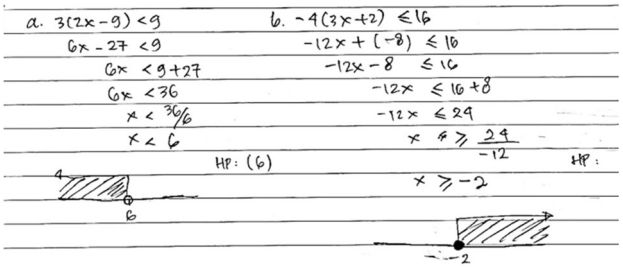


Figure 5: Student response with correct procedure and accurate visual representation

The interview excerpt shows that students rely on procedural understanding when solving Problem 2. This aligns with their grasp of the "properties" of linear inequalities, including axioms, definitions, and theorems. However, their understanding remains incomplete when it comes to determining and interpreting solution sets using precise procedures and visual representations. The following student interview results illustrate this.

Researcher : *In point a, you have* $x<6$*, correct? What does that imply?*
Student : *Indeed,* x *is less than* 6*, sir.*
Researcher : *Then why did you write the solution set* (6) *in point a and an empty set-in point b?*
Student : *I am uncertain about this, sir. The result is* 6*, but the sign is confusing me, sir.*

The interview excerpt shows that students struggle to explain, determine, and interpret solution sets for linear inequalities. While they can follow procedures, they fail to interpret each step and understand the role of inequality notation. This leads to difficulties in solving inequalities. Figure 6 illustrates this issue with categorized student responses, highlighting errors in solution procedures and visual representations.

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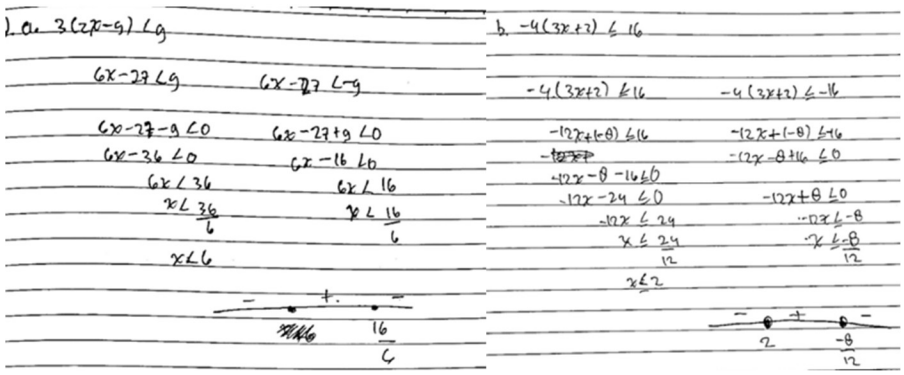


Figure 6: Student response with incorrect procedure and inaccurate visual representation

Figure 6 shows that students use an incorrect approach to solving a single-variable linear inequality. They mistakenly apply properties of absolute value inequalities to Problem 2. However, interviews reveal that they do recognize it as a single-variable inequality. Despite this, they incorrectly use properties like "if $x \in R$, $a \in R$, and $a>0$, then $x<a$ if and only if $-a<x<a$," which leads to errors. This suggests that students overgeneralize inequality rules. The following interview excerpt illustrates this misconception.

Researcher : *Could you please elaborate on the rationale behind your solution?*
Student : *To the best of my recollection, that is the formula. If I am not mistaken, we need to calculate* $x<a$ *and* $x<-a$. *In question number 2, correct? In that case,* x *is* $3(2x-9)$ *and a is* 9.
Researcher : *Yes, you are referring to the rule* $-a<x<a$.
Student : *Indeed, sir.*

The interview excerpt shows that students understand linear inequality properties procedurally, relying on memorization rather than true comprehension. As a result, they generalize these rules without fully grasping their function. For students who use correct procedures but struggle with visual representation, Figure 7 provides further insights.

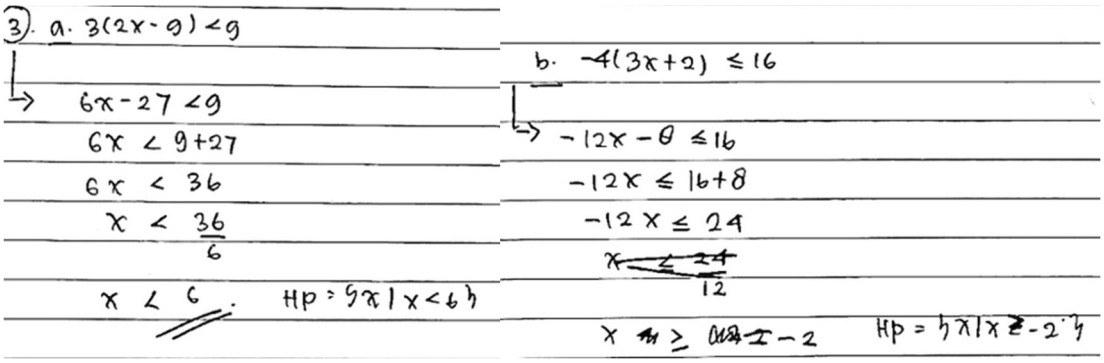


Figure 7: Student response with correct procedure but lacking visual representation

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Figure 7 demonstrates that students have successfully solved problem 2 procedurally. This indicates that their "process" and "properties" for problem 2 are guided by their procedural understanding. However, the students have not been able to represent the set of solutions on a number line, as evident from the following interview excerpt.

Researcher : Based on the results you obtained, could you please present them on a number line, adhering to the instructions provided in the question?
Student : I am somewhat confused, Sir. How would you describe it?
Researcher : Could you please clarify what the expression "$x<6$" means?
Student : It means that the value of x is less than 6, Sir.
Researcher : You have indicated that you have solved the equation and obtained the solution set "$x<6$". Could you please provide the solution set in more detail?
Student : Umm… I believe the solution set refers to the values of x that satisfy the inequality, Sir.

Based on the interview excerpt, it is evident that students' mental representations of the understanding of the solution set of a linear inequality of one variable hinder their ability to visualize it, despite their proficiency in solving the problem procedurally. The students' responses in the solution procedure category are incorrect and cannot be represented visually, as illustrated in Figure 8.

Figure 8 demonstrates the students' incorrect approach to solving the linear inequality of one variable, which involves applying the rule $-a<x<a$ (for algebraic expressions X and $k>0$). This rule states that absolute value inequalities of the form $|X|<k$ are equivalent to $-k<X<k$, $|X|>k$ is equivalent to $X<-k$ or $X>k$, and $|X| \leq k$ and $|X| \geq k$. Therefore, it can be concluded that students generalize the rules or properties of linear inequality forms involving absolute values into linear inequality forms of one variable. Consequently, their "process" for finding the solution set of the inequality form is flawed. Simultaneously, students' mental representations of the solution set in interval notation are inadequate, as they fail to represent it visually (on a number line).

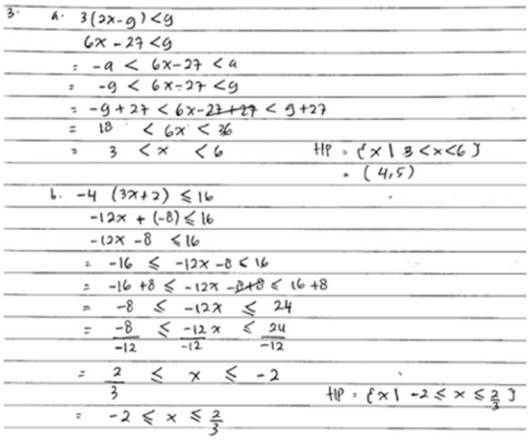


Figure 8: Student response with incorrect procedure and no visual representation

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Researcher : In Problem 2, could you please elaborate on the rule or formula you employed to determine the solution set?

Student : Certainly, Sir. I utilized the formula $-a<x<a$*, where a is set to 16 and x is* $-4(3x+2)$.

Researcher : Attached is the solution set for item a, which is $3<x<6$ *and item b, which is* $-2 \leq x \leq \frac{2}{3}$. *Could you please clarify if the solution set on the number line graph is derived from the same formula or if it is a separate representation?*

Student : I am somewhat perplexed, Sir. Could you please clarify whether the solution sets were constructed sequentially or if there is a different process involved?

Furthermore, the categories of students' responses in interpreting solution sets into interval notation are presented in Table 3.

Response Category	Response	
	Point a	Point b
The solution procedure and interpretation of the interval notation form are both correct.	1	0
The solution procedure is correct, but the interpretation of the interval notation form is incorrect.	2	2
The solution procedure and interpretation of the interval notation form are both incorrect.	1	2
The solution procedure is correct, but cannot be interpreted into the interval notation form.	11	9
The solution procedure is incorrect, and cannot be interpreted into the interval notation form.	7	9

Table 3: Categories of student responses to problem 2 based on interval notation

In addition to representing the set of solutions obtained by students using a number line, Problem 2 also requires students to express the set of solutions in interval notation. Out of the 22 students, only one student successfully represented the set of solutions obtained in interval notation for point a. Conversely, no students were able to correctly represent their set of solutions in interval notation for point b. This observation is depicted in Figure 9.

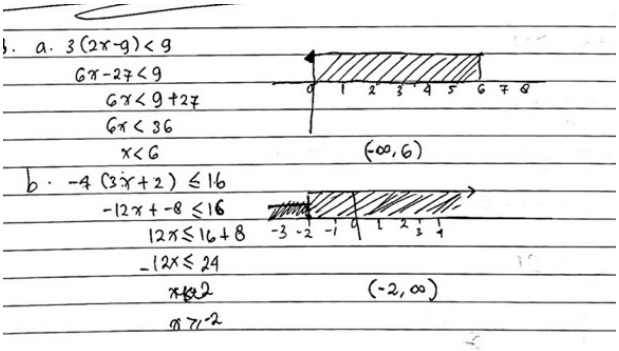


Figure 9: Student response with correct procedure and accurate interval notation

Figure 9 shows that the student correctly solved Problem 2a and represented the solution set in interval notation. However, the solution does not fully align with the correct procedure category. In point b, while the student followed the correct steps, the interval notation representation is in-

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correct. This indicates a misunderstanding of inequality properties, particularly the distinction between < and > versus ≤ and ≥ in interval notation. The interview results below further support this finding.

Researcher : In interval notation, what is the solution set?
Student : The solution set is written in brackets as I have done here, Sir.
Researcher : Could you please clarify the meaning of $(-\infty, 6)$ *and* $(-2, \infty)$*?*
Student : $(-\infty, 6)$ *indicates that the value of* x *is less than 6, while* $(-2, \infty)$ *suggests that the value of* x *is greater than* -2*.*
Researcher : Therefore, for point a, is $-\infty$ *a solution set for* $x<6$*?*
Student : No, Sir. The solution set for $x<6$ *is* $(-\infty, 6)$*.*
Researcher : Similarly, for point b?
Student : The solution set for $x>-2$ *is* $(-2, \infty)$*.*
Researcher : Based on these explanations, is there a distinction between the solution sets for "<" and "≥" signs?
Student : Yes, Sir.

The interview excerpt shows that students have an incomplete understanding of inequality properties. Their mental image of determining and interpreting the solution set for one-variable linear inequalities is also flawed. They mistakenly believe that interval notation does not differ for different inequality signs. Some errors involve correct solution procedures but incorrect interval notation representation. Figure 7 illustrates this mistake, where students incorrectly express the solution set. Similarly, Figure 8 shows students applying absolute value inequality rules to one-variable inequalities, leading to incorrect solutions. Their mental image of interval notation is also inadequate, as they struggle to express ≤ and ≥ correctly.

Researcher : Could you please clarify the meaning of $(4, 5)$*?*
Student : $(4, 5)$ *The solution set is because it was previously determined that* x *is greater than 3 and less than 6. The values between 3 and 6 are 4 and 5.*
Researcher : I understand. However, I am confused about the change in the solution set for point b in the last step. It went from $-2 \leq x \leq \frac{2}{3}$ *to* $-2 \leq x \leq \frac{2}{3}$*. Could you please explain the reason for this change?*
Student : The previous form was divided by -12*, sir. When dividing by a negative number, the sign changes to its opposite.*
Researcher : Thank you for your clarification. Could you please provide the solution set in interval notation?
Student : I apologize, but I am unable to recall the solution set in interval notation.

Figure 8 and the interview excerpt show that students' responses in the "completion procedures" and "interval notation representation" categories are incorrect. Their problem-solving process is also unsuitable, and their understanding of linear inequality properties is incomplete. As a result, they lack a clear mental picture of solution sets in interval notation.

Students who correctly follow solution procedures but struggle with interval notation lack a clear mental image of solution sets. They associate < and > with an empty circle and ≤ and ≥ with a full

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circle, as shown in Figure 5. However, some students incorrectly apply solution procedures and cannot represent them in interval notation, as seen in Figure 6. In these cases, they mistakenly generalize absolute value inequality rules to one-variable inequalities, leading to errors in determining the solution set. Their mental image of interval notation remains inadequate, as confirmed by the following interview results.

Researcher : *Regarding point a, the form is* $6x\text{-}27<\text{-}9$. *What is the result of the calculation* $\text{-}27+9$?
Student : $\text{-}18$, *Sir*
Researcher : *Therefore, the result should be* 3, *correct? Even if the result is* $\frac{16}{6}$, *which is smaller between* 6 *and* $\frac{16}{6}$?
Student : *I am unsure, Sir.*
Researcher : *Then, what about the solution set in interval notation?*
Student : *Could you please clarify, Sir?*

Based on the interview excerpt, it is evident that students lack comprehension of the solution set presented in interval notation. Consequently, they are unable to articulate their solutions effectively. Furthermore, students continue to make errors in calculating and ordering integers with fractional components. Therefore, it is concluded that the students' answers, categorized by solution procedures and representation in interval notation, are not accurate. Additionally, students' understanding of the properties applicable to linear inequalities is incomplete. Consequently, they do not possess a clear mental representation of the solution set in interval notation.

Students' Comprehension of Inequality Notation

Problem 3 examines students' ability to use inequality notation to represent mathematical situations symbolically. Their responses will be analyzed to understand their "concept image" of this process. Before exploring how they form this concept image, their answers will be categorized based on analysis (refer to Table 4) and interviews.

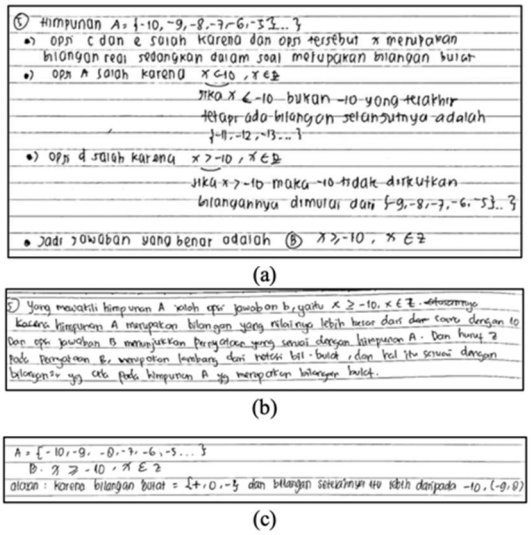
Response Category	Response
The mathematical symbol representation is correct.	9
The mathematical symbol representation is correct, but the reasoning is incorrect.	3
The mathematical symbol representation is correct, but no reasoning is provided.	2
The mathematical symbol representation is incorrect, and no reasoning is provided.	3
No response provided.	5

Table 4: Student response categories for problem 3

The assessment demonstrated a robust comprehension of numerical concepts and the accurate application of inequality notation, as evident in Problem 3. In this task, students were required to select the statement that accurately represents the given set A={-10-9, -8, -7, -6, -5, ...}. Figure 10

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illustrates student responses categorized based on their representation of mathematical symbols and logical reasoning.



(a)

(b)

(c)

Set $A = \{-10, -9, -8, -7, -6, -5, \ldots\}$

- Options c and e are wrong because from these options x is a real number while in the question it is an integer
- Option a is wrong because $x < -10, x \in \mathbb{Z}$
  If $x < -10$ is not the last -10, but there are the next numbers, namely $\{-11, -12, -13, \ldots\}$
- Option d is wrong because x>-10, $x \in \mathbb{Z}$
  If $x > -10$ then -10 is not included, the numbers start from $\{-9, -8, -7, -6, -5, \ldots\}$
- So, the correct answer is b, $x \geq -10, x \in \mathbb{Z}$

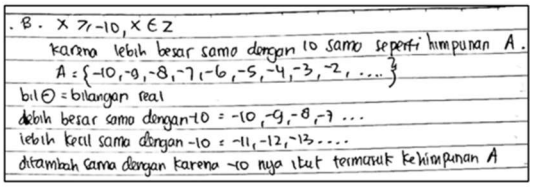
The representative of set A is answer option b, namely $x \geq -10, x \in \mathbb{Z}$. Because set A is a number whose value is greater than or equal to 10. And answer option B shows a statement that corresponds to set A. and the letter $\mathbb{Z}$ in statement B is a symbol of integer notation, and it corresponds to the numbers in set A which are integers.

Reason: Because the integers = {+, 0, −} and the number after that is more than -10, $(-9, 8)$

Figure 10: Student responses with correct symbolic representation and reasoning

Figure 10 shows that students correctly follow procedures and methods to solve Problem 3. They do more than just select answers—they also provide valid reasoning. In Figure 10(a), students offer detailed explanations and correctly apply the inequality notation ≥. A thorough review confirms that students accurately use properties such as axioms, definitions, and theorems. Additionally, their mental images of the concepts in Problem 3 are well-formed.

In the mathematical symbol representation category, the reasoning is incorrect. While students correctly justify using the inequality notation ≥, their understanding of why set A in Problem 3 is expressed as $x \in Z$ is flawed. This indicates that while their process and use of properties for inequality notation are correct, their concept image is not. Figure 11 illustrates this issue.



B. $x \geq -10, x \in \mathbb{Z}$
Because greater than or equal to 10 is the same as set A.
$A = \{-10, -9, -8, -7, -6, -5, -4, -3, -2, \ldots\}$
Number − = real number
Greater than or equal to -10 = $-10, -9, -8, -7, \ldots$
Smaller than or equal to -10 = $-11, -12, -13, \ldots$
Added equal to because -10 is also included in set A

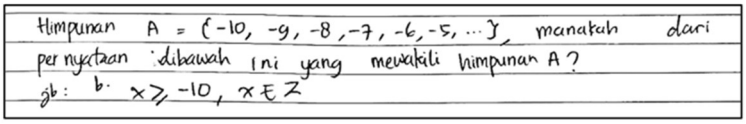
Figure 11: Student response with correct symbolic representation but incorrect reasoning

Referring to Figure 11, it becomes evident that the reasons provided by students deviate from their answer choices, which explicitly state that set A comprises elements of the set of integers ($x \in Z$). To gain insights into the students' conceptual understanding of the problem presented in problem 3, kindly refer to the following interview excerpt.

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Researcher : Could you please elaborate on why the answer is b?
Student : Sir, the answer is B because if the number is negative, the closer it is to zero, the larger its absolute value becomes. Therefore, the sign is larger because -10 *is included in the* A *set, making it equal to or greater than zero.*
Researcher : In that case, why choose Z *instead of* R*?*
Student : Sir, I need clarification on the definition of an integer.
Researcher : Could you please provide a definition of an integer?
Student : Certainly, an integer is a whole number, including negative numbers such as -10, -9*, and so on.*

The interview excerpt shows that students struggle to justify selecting set A in Problem 3 as $x \in Z$. Their problem-solving process for inequalities, especially those involving numbers, is underdeveloped due to an incomplete understanding of integer properties. While their mental image of using inequality notation ≥ is correct, their understanding of number concepts in Problem 3 is lacking. Although their mathematical symbol representation is accurate, it lacks supporting explanations, as shown in Figure 12.



Set $A = \{-10, -9, -8, -7, -6, -5, \ldots\}$, which of the statements below represents set A?

Answer: b. $x \geq -10, x \in \mathbb{Z}$

Figure 12: Student response with correct symbolic representation but no reasoning

Figure 12 shows that students can translate mathematical scenarios into symbolic representations but struggle to justify their choices. Interviews confirm they understand how to use the inequality notation ≥ but cannot explain the concept of numbers in Problem 3. They also lack an understanding of the properties and definitions of numbers, leading to weak reasoning for selecting option b. This is due to the absence of a clear mental image of number concepts.

Researcher : Could you please elaborate on your decision to represent set A with the letter "b"?
Student : Given that set A is known to contain elements such as -10, -9, -8*, and so on, I chose "*$x \geq$ -10*" as the representation. This expression ensures that the value of "*x*" is greater than or equal to* -10*, and* -10 *is included within the set.*
Researcher : In contrast, why did you select "Z*" instead of "*R*"? Since "e" and "b" represent the same value, only the symbol differs.*
Student : Indeed, you are correct.

Figure 13 shows that students' mathematical symbol representation is incorrect and lacks justification. This suggests they do not fully understand inequality notation. Interviews reveal that students use a flawed process to solve Problem 3, mistakenly believing that a negative number decreases in value as it approaches 0. As a result, they incorrectly choose the ">" notation to represent mathematical situations.

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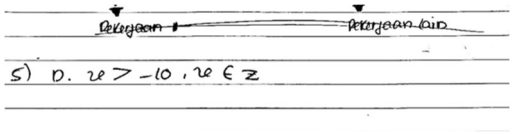


Figure 13: Student response with inaccurate symbolic representation and no justification

Researcher : Could you please elaborate on your choice of option D?
Student : In negative numbers, the closer a value is to zero from the right, the larger its value becomes. Therefore, I use the greater sign to indicate this.

Further study shows that students can distinguish between ">" and "≥" and illustrate this difference with examples. This indicates their understanding of the properties and definitions of inequality notation. However, while they have a mental image of interpreting inequalities, they struggle to understand the numerical notation in Problem 3.

Researcher : Could you please clarify the meaning of the symbol "$x \in Z$"?
Student : I am unfamiliar with this symbol, sir.
Researcher : Have you encountered any similar symbols in the past?
Student : I do not recall, sir.

The interview excerpt shows that students do not fully understand number notation, preventing them from forming a mental representation of it. As a result, they struggle to explain the correct solution process for Problem 3. Their difficulty stems from not grasping the fundamental properties and definitions of number notation. Students who do not attempt an answer face even greater challenges in solving the problem. This is further illustrated in the following interview excerpt.

Researcher : Could you please clarify why you did not respond to question number 3?
Student : I am unable to comprehend the interpretation of the question, sir.
Researcher : The question explicitly states the instruction to select one of the forms of inequality presented in the question.
Student : I acknowledge this, sir, but I remain perplexed.

The provided interview excerpt suggests that students lack a comprehensive understanding of the problem contest presented in Problem 3. This could be attributed to the absence of a "mental image" or all relevant thoughts (memory) that students can utilize to explain the concept encapsulated in Problem 3. In this instance, students are lacking a mental representation associated with the concept of the set they have acquired. The form of inequality employed in the choices presented in Problem 3 can be interpreted as a set-forming notation designed to simplify the listing of members in Set *A* within the problem. Consequently, it is unsurprising that students struggle to articulate the "process," "procedure," or even describe the "properties" that can be employed to describe the concept contained in Problem 3 and effectively solve it.

DISCUSSION

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The findings of this study provide a nuanced understanding of students' "concept images" of inequality, exposing both strengths and substantial gaps in their procedural and conceptual knowledge. These insights can be effectively analyzed through various theoretical frameworks, including Tall and Vinner's (1981) concept image and concept definition, Skemp's (2006) distinction between instrumental and relational understanding, and Vygotsky's (1978) sociocultural theory of learning. Furthermore, the results align with prior research on students' misconceptions and challenges in comprehending mathematical concepts, particularly inequalities.

Based on the data presented in the research, students' "concept image" of the concept of inequality can be categorized into several distinct groups, reflecting their comprehension, reasoning, and application of inequality rules. These categories are derived from their responses to the problems and the interview excerpts. Please refer to the attached Table 5 for a detailed categorization of students' "concept image" regarding the concept of inequality.

1	Category	Correct procedure with appropriate reasoning
	Description	Students in this category possess a robust understanding of the principles of governing inequality. They are adequate at executing correct procedures and provide sound and logical justifications for their solutions.
	Evidence	• Students accurately modify inequality notation when multiplying or dividing by negative numbers and provide lucid explanations (Figure 2). • They can elucidate the rationale behind the change in inequality sign when multiplying or dividing by negative numbers, demonstrating a profound conceptual understanding.
	Concept Image	Students possess a comprehensive and coherent "concept image" that is congruent with formal mathematical definitions. This image effectively integrates procedural and conceptual knowledge.
2	Category	Correct procedure with inadequate or missing reasoning
	Description	Students in this category may execute the correct procedure but lack sufficient justification or demonstrate no justification at all. Their comprehension is procedural rather than conceptual.
	Evidence	• Students correctly solve inequalities but lack the ability to explain the change in inequality sign when multiplying or dividing by negative numbers (Figure 1). • Interviews revealed that several students admitted to forgetting or omitting explanations due to time constraints, suggesting a reliance on memorization rather than understanding the underlying principles.
	Concept Image	Students have a procedural "concept image" but lack the conceptual depth to explain the basic principles of the inequality rules.
3	Category	Incorrect procedure with inappropriate reasoning
	Description	Students in this category demonstrate an incorrect understanding of the rules of inequality, applying incorrect procedures and providing inappropriate reasoning.
	Evidence	• Students misapplied inequality rules, including failing to change the inequality sign when multiplying or dividing by a negative number (Figure 3). • Some students provided incorrect explanations, such as assuming the inequality sign only changes if the original form is negative (Figure 4).
	Concept Image	Students often hold flawed "concept images" of inequality due to misunderstandings or incomplete comprehension of its rules. Consequently, they make errors in procedures and reasoning.
4	Category	Correct visual representation with procedural understanding

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	Description	Students in this category possess the ability to solve inequalities accurately and visually represent the solution set on a number line. However, they encounter challenges in comprehending and articulating the conceptual nature of the solution set.
	Evidence	• Students demonstrate proficiency in solving inequalities and representing solution sets on a number line. However, they lack the ability to articulate the interpretation of solution sets in interval notation (Figure 5). • During interviews, students exhibited confusion regarding the various inequality symbols, such as "<" and "≤," when representing solution sets.
	Concept Image	Students possess a procedural "concept image" that enables them to visualize solutions, but they lack the conceptual understanding to fully interpret or articulate their solutions.
5	Category	Incorrect visual representation with procedural errors
	Description	Students in this category demonstrate errors in solving inequalities and representing solution sets visually, suggesting a deficiency in comprehension of inequality rules and their practical applications.
	Evidence	• Students frequently misinterpret inequality rules, particularly the absolute value property, resulting in incorrect solution sets (Figure 6). • Additionally, they demonstrate a lack of proficiency in representing solution sets accurately on a number line or in interval notation (Figure 8).
	Concept Image	Students possess a fragmented "concept image" characterized by substantial gaps in procedural and conceptual knowledge, resulting in errors in solving and representing inequalities.
6	Category	Correct symbolic representation with inadequate reasoning
	Description	Students in this category demonstrate proficiency in translating mathematical scenarios into inequality notation. However, they lack the ability to provide valid reasons for their choices.
	Evidence	• Students accurately represented mathematical scenarios using inequality symbols (e.g., "$x \geq -10$"), but they lacked the ability to explain their choices or the underlying notation (Figure 12). • Interviews revealed a lack of conceptual understanding, as students were unable to articulate the reasons behind their decisions.
	Concept Image	Students possess a procedural "concept image" that facilitates symbolic translation, but they lack the conceptual depth to substantiate their reasoning.
7	Category	Incorrect symbolic representation with flawed reasoning
	Description	Students in this category exhibit an incorrect translation of mathematical scenarios into inequality notation, accompanied by inappropriate reasoning. This demonstrates a lack of understanding of inequality rules and numerical concepts.
	Evidence	• Students exhibited errors in the usage of inequality symbols, such as incorrectly employing ">" instead of "≥." Additionally, they provided incorrect explanations, including a misinterpretation of the value of negative numbers (Figure 13). • During interviews, students demonstrated a lack of comprehension regarding the behavior of negative numbers and the conventions of inequality notation.
	Concept Image	Students possess an erroneous "concept image" stemming from misconceptions regarding inequality rules and numerical principles. Consequently, they incur errors in symbolic representation and reasoning.
8	Category	No response or inability to solve
	Description	Students in this category demonstrate a profound lack of comprehension regarding the concept of inequality, as evidenced by their inability to solve the problem or provide any response.
	Evidence	• Students failed to attempt solving the problem or provide any answers, frequently due to confusion or a lack of comprehension (Table 4).

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	• In interviews, students indicated difficulties in comprehending the problem or the necessary steps to resolve it.
Concept Image	Students exhibit a poorly developed or absent "concept image" of inequality, indicating substantial gaps in procedural and conceptual understanding.

Table 5: Description of students' "concept image" categorization in the concept of inequality

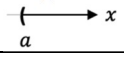
These categories categorize students' varying levels of concept image regarding the concept of inequality, ranging from a strong grasp of procedural and conceptual principles to substantial misconceptions and knowledge gaps.

Concept Image vs. Concept Definition: A Disjunction in Understanding

The distinction between concept image and concept definition, as introduced by Tall and Vinner (1981), remains a crucial framework for comprehending how individuals internalize and apply mathematical concepts. According to their theory, concept image refers to the mental representations, intuitions, and associated properties that individuals develop about a concept, while concept definition represents the formal, mathematical articulation of that concept. Tall and Vinner (1981) posited that discrepancies between these two constructs frequently lead to persistent misconceptions, particularly in mathematics education. This study corroborates their findings, demonstrating that students frequently exhibit a procedural understanding of inequality rules—such as reversing the inequality sign when multiplying or dividing by a negative number—yet struggle to articulate the underlying conceptual rationale. This reliance on procedural knowledge, devoid of deeper conceptual understanding, suggests that students' concept images are often shaped by rote memorization rather than a meaningful engagement with formal mathematical principles. Table 6 presents a comparative analysis between students' concept images and the corresponding concept definitions, categorized based on the identified concept image categories.

Category	Student Concept Image	Concept Definition
Correct procedure with appropriate reasoning	Students possess a comprehensive and coherent "concept image" that is congruent with formal mathematical definitions. This image effectively integrates procedural and conceptual knowledge. An example of evidence: Figure 2	Let $a, b, c \in R$. If $a>b$ and $c<0$, then $ac<bc$. Consider any $a, b, c \in R$, such that $a>b$ and $c<0$. According to the definition $a>b$ implies $a-b \in P$ and $c<0$ implies $-c \in P$. It will be demonstrated that $ac<bc$. Consequently, it is imperative to demonstrate that $bc-ac \in P$. Using the order properties, the product of $a-b \in P$ dan $-c \in P$, $-c(a-b)=-ac+bc=bc-ac \in P$. Thus, we obtain $bc-ac \in P$, Confirming that $ac<bc$
Correct procedure with inadequate or missing reasoning	Students have a procedural "concept image" but lack the conceptual depth to explain the basic principles of the inequality rules.	Let $a, b, c \in R$. If $a>b$ and $c<0$, then $ac<bc$. Consider any $a, b, c \in R$, such that $a>b$ and $c<0$. According to the definition $a>b$ implies $a-b \in P$ and $c<0$ implies $-c \in P$. It will be demonstrated

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Category	Student Concept Image	Concept Definition
	An example of evidence: Figure 1	that $ac<bc$. Consequently, it is imperative to demonstrate that $bc-ac \in P$. Using the order properties, the product of $a-b \in P$ dan $-c \in P$, $-c(a-b)=-ac+bc=bc-ac \in P$. Thus, we obtain $bc-ac \in P$, Confirming that $ac<bc$
Incorrect procedure with inappropriate reasoning	Students often hold flawed "concept images" of inequality due to misunderstandings or incomplete comprehension of its rules. Consequently, they make errors in procedures and reasoning. An example of evidence: Figure 3 and 4	Let $a, b, c \in R$. If $a>b$ and $b>c$, then $a>c$. If $a>b$ then $a+c>b+c$. If $a>b$ and $c>0$, then $ac>bc$. If $a>b$ and $c<0$, then $ac<bc$.
Correct visual representation with procedural understanding	Students possess a procedural "concept image" that enables them to visualize solutions, but they lack the conceptual understanding to fully interpret or articulate their solutions. An example of evidence: Figure 5	An infinite open interval is a set of the form $(a,\infty):=\{x \in R \mid x>a\}$ and $(-\infty,b):=\{x \in R \mid x<b\}$. The first set has no upper bound, while the second set has no lower bound. Interval notation: (a,∞) Form of inequality: $x>a$ Graphics: 
Incorrect visual representation with procedural errors	Students possess a fragmented "concept image" characterized by substantial gaps in procedural and conceptual knowledge, resulting in errors in solving and representing inequalities. An example of evidence: Figure 6 and 8	The properties of inequalities that facilitate transforming a given inequality into an equivalent inequality are as follows: Let a and b be real numbers such that $a<b$, and let c be another nonzero real number. Then, the inequality $a<b$ is equivalent to: $a+c<b+c$ $ac<bc$, for $c>0$ $ac>bc$, for $c<0$
Correct symbolic representation with inadequate reasoning	Students possess a procedural "concept image" that facilitates symbolic translation, but they lack the conceptual depth to substantiate their reasoning. An example of evidence: Figure 12	An interval can be classified as open, closed, half-open, or unbounded. Open interval (a,b) includes all real numbers between a and b but excludes the endpoints. Closed interval $[a,b]$ includes both endpoints. Half-open intervals $[a,b)$ and $(a,b]$ include only one endpoint. If $a=b$, the open interval is empty, while the closed interval contains a single element. Unbounded intervals use ∞ and $-\infty$ as notation for non-existent endpoints: Open infinite intervals: (a,∞) (no upper bound) and $(-\infty,b)$ (no lower bound). Closed infinite intervals: $[a,\infty)$ and $(-\infty,b]$.

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Category	Student Concept Image	Concept Definition
		The entire real number set is expressed as $(-\infty, \infty)=R$, which has no defined endpoints.
Incorrect symbolic representation with flawed reasoning	Students possess an erroneous "concept image" stemming from misconceptions regarding inequality rules and numerical principles. Consequently, they incur errors in symbolic representation and reasoning. An example of evidence: Figure 13	An interval can be classified as open, closed, half-open, or unbounded. Open interval (a, b) includes all real numbers between a and b but excludes the endpoints. Closed interval $[a, b]$ includes both endpoints. Half-open intervals $[a, b)$ and $(a, b]$ include only one endpoint. If $a=b$, the open interval is empty, while the closed interval contains a single element. Unbounded intervals use ∞ and $-\infty$ as notation for non-existent endpoints: Open infinite intervals: (a, ∞) (no upper bound) and $(-\infty, b)$ (no lower bound). Closed infinite intervals: $[a, \infty)$ and $(-\infty, b]$. The entire real number set is expressed as $(-\infty, \infty)=R$, which has no defined endpoints.
No response or inability to solve	Students exhibit a poorly developed or absent "concept image" of inequality, indicating substantial gaps in procedural and conceptual understanding. An example of evidence: Table 4	The order properties of R, the fundamental rules of inequalities, and the interpretation of inequality solution sets in interval notation and graphical representation.

Table 6: Illustration of the comparison of students' concept images and concept definitions

Based on the presented evidence regarding students' concept images of inequalities, it was observed that some students exhibit partial understanding. While they correctly apply procedural steps, they lack a clear conceptual grasp of the underlying principles. For instance, certain students are able to manipulate inequality notation but struggle to justify their reasoning or misinterpret interval notation. To bridge the gap between correct procedural execution and conceptual reasoning, instructional strategies should emphasize conceptual exploration alongside procedural practice (Veith et al., 2023; Chan et al., 2023).

The phenomenon of procedural reliance without conceptual depth has been extensively documented in prior research. For instance, Tsamir and Almog (2001) demonstrated that students frequently apply inequality rules mechanically, resulting in errors when confronted with non-routine problems. Similarly, Linchevski and Sfard (1991) underscored that students' over-reliance on procedural knowledge stems from a lack of integration between their intuitive concept images and formal definitions. These findings are further supported by more recent studies. For example, Star and Stylianides (2013) highlighted that students' procedural fluency often conceals substantial

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gaps in their conceptual understanding, particularly in algebra and calculus. They argued that instructional practices emphasizing procedural shortcuts can inadvertently reinforce superficial learning, perpetuating the disjunction between concept image and concept definition.

Furthermore, recent research has highlighted the cognitive and pedagogical challenges associated with bridging the gap between conceptual and procedural understanding in mathematics education. Jupri et al. (2014) demonstrated that students' misconceptions regarding inequalities often stem from an overemphasis on algorithmic problem-solving strategies in curricula, which neglects the development of conceptual reasoning. Similarly, Rittle-Johnson et al. (2015) showed that students who receive instruction focused solely on procedural skills are less likely to transfer their knowledge to novel contexts compared to those who engage in conceptually rich learning experiences. These findings collectively underscore the necessity of instructional strategies that explicitly address the disjunction between concept image and concept definition.

To foster a more integrated understanding of mathematical concepts, educators must adopt pedagogical approaches that prioritize conceptual depth alongside procedural fluency. For instance, the utilization of multiple representations, such as graphical, numerical, and algebraic, has been demonstrated to enhance students' ability to connect their intuitive concept images with formal definitions (Ainsworth, 2018). Additionally, inquiry-based learning and problem-solving tasks that encourage students to explore the underlying reasons behind mathematical rules can facilitate the bridge between procedural and conceptual knowledge (Hurrell, 2021; Fatqurhohman, 2016; McCormick, 1997). These strategies align with the constructivist perspective of learning, which posits that meaningful understanding arises from actively constructing connections between prior knowledge and new information (Vygotsky, 1978). In conclusion, the disjunction between concept image and concept definition remains a substantial challenge in mathematics education, particularly in the context of inequality rules. While procedural knowledge is fundamental, it must be complemented by a robust conceptual understanding to prevent persistent misconceptions.

Instrumental vs. Relational Understanding: The Crucial Gap

Skemp's (2006) seminal work introduced a critical distinction between instrumental understanding and relational understanding. Instrumental understanding encompasses the ability to execute mathematical procedures and apply rules with precision, while relational understanding entails a profound comprehension of the underlying principles, connections, and reasons that underpin these procedures. This distinction remains highly pertinent in contemporary mathematics education, as evidenced by recent research underscoring the persistent disparity between procedural proficiency and conceptual mastery among students.

The data presented in this study aligns with Skemp's framework, demonstrating that a substantial portion of students exhibit instrumental understanding rather than relational understanding. For

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instance, while numerous students demonstrated procedural competence in solving inequalities, such as correctly altering the inequality sign when multiplying by a negative number, they were unable to articulate the reasoning behind this rule. This inability to justify mathematical procedures highlights a critical deficiency in relational understanding, which is fundamental for developing a comprehensive conceptual grasp of mathematical concepts, particularly inequalities. As Rittle-Johnson and Schneider (2015) aptly state, procedural knowledge without conceptual understanding often leads to precarious learning, where students encounter difficulties in applying their knowledge flexibly or adapting it to novel contexts.

The implications of this gap are profound. Research by Skemp (2006) and Hiebert and Lefevre (1986) consistently demonstrates that students who primarily rely on instrumental understanding are more susceptible to errors, particularly when confronted with unfamiliar problems or tasks that require justification. For instance, Star and Stylianides (2013) observed that students with predominantly instrumental understanding often fail to discern the logical connections between mathematical concepts, resulting in misconceptions and difficulties in problem-solving. This underscores the limitations of rote memorization and procedural fluency in promoting long-term mathematical proficiency.

Recent studies further emphasize the paramount importance of prioritizing relational understanding in mathematics education. For instance, Boaler (2022) contends that a focus on conceptual comprehension promotes deeper engagement with mathematics, enabling students to perceive the subject as a coherent and interconnected discipline rather than a collection of isolated rules and procedures. Similarly, Tall (2013) underscores that relational understanding fosters mathematical adaptability, allowing students to adapt their knowledge to diverse problem-solving scenarios.

In conclusion, the distinction between instrumental and relational understanding remains a pivotal consideration in mathematics education. While instrumental understanding facilitates accurate procedure execution, it falls short of developing the profound conceptual knowledge necessary for advanced mathematical reasoning. By prioritizing relational understanding, educators can equip students with the means to transcend rote memorization, engage with mathematics as a dynamic and interconnected discipline, and ultimately attain a more profound and enduring mastery of the subject.

Addressing Flawed Concept Images: Unraveling Misconceptions and Errors

Misconceptions in mathematics frequently originate from students' flawed concept images, which are mental representations formed by incomplete or erroneous understandings of mathematical concepts. These misconceptions persist even after formal instruction is provided if not explicitly identified and corrected (Tall & Vinner, 1981). A prevalent example is students' errors in inequalities, such as failing to reverse the inequality sign when multiplying or dividing by a negative

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number or misapplying absolute value properties in linear inequalities. These persistent misconceptions suggest a superficial comprehension of mathematical principles, resulting in challenges in reasoning and problem-solving.

Research consistently demonstrates that students encounter difficulties comprehending the abstract nature of inequalities, frequently mistreating them as equations. Tsamir and Almog (2001) identified a significant number of students who fail to distinguish between equations and inequalities, particularly when dealing with negative coefficients. Similarly, Linchevski and Sfard (1991) observed frequent misinterpretations of inequality directionality, which adversely impacts students' ability to solve and graph inequalities accurately. More recent studies corroborate these findings, underscoring the detrimental effects of an over-reliance on procedural knowledge without a conceptual understanding of inequalities (Kieran, 2020; Bofferding & Wessman-Enzinger, 2018).

Another prevalent source of errors lies in students' misapplication of absolute value properties to inequalities. Many students assume that the rules governing absolute value in equations directly apply to inequalities, resulting in incorrect solution sets (Zazkis & Chernoff, 2008). This misinterpretation underscores the need for instructional strategies that clarify the conceptual distinctions between absolute value in different mathematical contexts.

To address these misconceptions, instructional approaches must directly target flawed concept images. Research suggests that cognitive conflict tasks, which challenge students' existing misunderstandings, are effective in promoting conceptual change (Star & Stylianides, 2013). Furthermore, visual representations, such as number lines and graphs, can reinforce students' comprehension of inequalities and their properties (Battista, 2017). Formative assessments also play a pivotal role in diagnosing misconceptions. Studies indicate that timely feedback and diagnostic questioning assist educators in uncovering and correcting students' erroneous reasoning (Wiliam, 2016). Encouraging students to articulate their thought processes when solving inequalities can further expose underlying errors and facilitate conceptual reinforcement.

In conclusion, addressing flawed concept images in mathematics necessitates a multifaceted approach that integrates conceptual instruction, visual representations, and targeted interventions. By identifying and rectifying prevalent misconceptions, particularly in inequality operations and absolute value applications, educators can facilitate students' acquisition of accurate and robust mathematical concepts. Future research should delve into the cognitive mechanisms underpinning these misconceptions and refine evidence-based instructional strategies to enhance students' mathematical reasoning.

Enhancing Conceptual Understanding in Mathematics: The Role of Visualization and Representation

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Visualization and representation are indispensable tools in the development of conceptual understanding, particularly in the context of abstract mathematical concepts such as inequalities. As Duval (2006) posits, the ability to translate between various representations—symbolic, graphical, and verbal—is crucial for students to fully grasp mathematical concepts. This study underscores a persistent challenge: students who excel in solving inequalities often encounter difficulties in visually representing solution sets, such as on a number line or in interval notation. This disparity suggests a gap in their comprehension of the relationships between different representations of inequalities, which, in turn, impedes their ability to interpret and communicate solutions effectively.

The significance of employing multiple representations in mathematics education has been extensively documented in previous research. Duval (2006) and Presmeg (2006) assert that students who possess the ability to effectively translate between diverse forms of representation are more inclined to develop a profound conceptual comprehension of mathematical concepts. For instance, Ainsworth (2018) conducted a study that revealed that students who actively engaged with multiple representations of algebraic concepts, such as inequalities, exhibited enhanced problem-solving abilities and a more comprehensive understanding of the underlying principles. This emphasizes the imperative of instructional strategies that actively encourage students to explore and establish connections between various representations of inequalities, thereby fostering a more holistic comprehension of the concept.

The findings of this study highlight the complexities of students' conceptions of inequality and underscore the necessity of instructional approaches that simultaneously address both procedural and conceptual comprehension. By integrating theoretical frameworks from Tall and Vinner (1981), Skemp (2006), and Vygotsky (1978), educators can devise interventions that facilitate the bridging of the gap between students' conceptions and formal definitions, promote relational understanding, and provide the necessary support for conceptual development. Future research should investigate the efficacy of such strategies in diverse educational settings, ensuring that all students have the opportunity to acquire a comprehensive and accurate understanding of inequality.

Furthermore, the findings of this study can be extended to college mathematics, particularly in courses that require a robust foundation in algebraic reasoning and problem-solving. The concept of inequality holds significant contributions in college mathematics and other disciplines, such as functions, calculus, and algebraic reasoning. In function analysis, inequality is employed to determine the domain, range, and limits of a function. In calculus, this concept plays a role in the definition of limits using the epsilon-delta method, derivative analysis to identify intervals of increasing and decreasing functions, and in estimating integral values. Additionally, in algebra and systems of inequalities, this concept is utilized in mathematical optimization, linear programming, and number theory. Inequalities also have practical applications in statistics and probability, such

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as in Chebyshev and Markov inequalities for estimating data distributions. Overall, a comprehensive understanding of inequalities facilitates students in grasping more intricate and practical mathematical concepts across diverse fields.

CONCLUSION

This analysis of students' concept images of inequality reveals both strengths and weaknesses in their procedural and conceptual knowledge. While many students demonstrated proficiency in solving inequalities, particularly in executing operations such as changing the inequality sign when multiplying or dividing by negative numbers, a substantial portion struggled to articulate the reasoning behind these procedures. Common errors included misapplying absolute value properties, failing to change the inequality sign appropriately, and misinterpreting solution sets, suggesting an overreliance on procedural knowledge rather than a deep conceptual understanding. Furthermore, students exhibited difficulties in translating between different representations of inequalities, such as expressing solutions on a number line or in interval notation. While some students correctly utilized inequality notation in mathematical scenarios, many were unable to justify their choices, underscoring the necessity of instructional strategies that foster relational understanding rather than mere procedural execution. These findings reinforce the imperative of teaching approaches that bridge conceptual gaps and support students in internalizing the principles governing inequalities.

Although this study provides valuable insights into students' comprehension of inequalities, certain limitations should be considered. Although the researchers have mitigated potential biases by ensuring sample diversity, they acknowledge that the findings primarily reflect the characteristics of students within the study's specific context. Further research with a larger and more diverse sample would enhance the generalizability of the results and validate these conclusions. Additionally, the study relied on a predetermined problem set and interviews to assess students' understanding, which may not have captured the full spectrum of their conceptual and procedural knowledge. Expanding assessment methods could offer a more nuanced perspective on students' difficulties. To address these gaps, educators should integrate teaching strategies that emphasize conceptual understanding, such as scaffolding, collaborative learning, and targeted interventions to correct specific misconceptions. Encouraging students to engage with multiple representations of inequality—symbolic, graphical, and verbal—can deepen their understanding and enhance their problem-solving abilities. Ultimately, this study underscores the complexity of inequality comprehension and highlights the importance of instructional approaches that balance procedural fluency with conceptual depth. Future research should continue exploring effective teaching strategies to improve students' mastery of inequalities in diverse educational settings.

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APPENDIX

Problem 1:
Thoroughly analyze and resolve each of the following inequalities. For each problem, execute the specified operation on both sides of the inequality and write the resulting inequality. Subsequently, based on your expertise, provide a response to the reflective question at the conclusion.

No	Inequality	Instruction	Resulting Inequality
1	$4 > 1$	Multiply both sides by -1	
2	$3 < 5$	Add 2 to both sides	
3	$6 > -4$	Add -1 to both sides	
4	$-5 \leq 4$	Subtract 2 from both sides	
5	$-1 \geq -6$	Multiply both sides by -3	
6	$-4 < 6$	Divide both sides by 2	
7	$¼ < ½$	Divide both sides by -⅓	

Based on your experience with the aforementioned issues, what can you infer about the impact of various operations on inequality signs? Please elaborate on your rationale.

Problem 2:
Please solve each of the following linear inequalities. Subsequently, graph the solution set on a number line.
a. $3(2x\text{-}9)<9$
b. $-4(3x+2) \leq 16$.
Could you please elaborate on the significance of the solution set you identified? How did you determine the appropriate locations for the boundaries and sketch them on the number line?

Problem 3:
Please consider the following set:
A={-10,-9,-8,-7,-6,-5, ... }
Which of the following statements correctly represents set A in symbolic mathematical notation? Please select one answer and provide a detailed explanation of your reasoning.
a. x<-10, $x \in Z$
b. $x \geq$ -10, $x \in Z$
c. $x \leq$ -10, $x \in R$
d. x>-10, $x \in Z$
e. $x \geq$ -10, $x \in R$
Provide a justification for your selection. How do inequality symbols and number set notations facilitate the accurate description of the structure of a set? Are there any alternative options that appear nearly, correct? Provide a rationale for your preference.

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Teaching Division of Two Natural Numbers to Primary School Students Independently of Multiplication Through Realistic Mathematics Education

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Abstract: This study investigates the necessity of teaching the division of natural numbers to primary school students independently of multiplication, addressing the challenges learners face. A framework grounded in Realistic Mathematics Education (RME) guided the development of a Hypothetical Learning Trajectory (HLT) that included specific learning activities, objectives, and teachers' expectations. In alignment with the designed HLT, an instructional intervention consisting of four lessons was conducted with 11 first-grade students who had completed their curriculum but had not yet learned the division and multiplication of two natural numbers. The results of this descriptive study indicate that the HLT effectively created an environment in which students could experience mathematization to grasp the concept. It enabled students to distinguish between the two types of division, write the correct division equation, and verbally describe appropriate real-world situations corresponding to division, thereby enhancing their overall comprehension.

Keywords: Realistic Mathematics Education, mathematization, division of two natural numbers, primary school mathematics.

INTRODUCTION

The division of natural numbers is a fundamental concept in primary school mathematics education, often taught in conjunction with multiplication due to their close relationship. Multiplication and division are inverse operations, and this connection is commonly leveraged in traditional teaching methods. To deepen students' understanding of division, it is widely agreed that a more robust integration of multiplication and division is necessary. Strengthening the conceptual connection between these two operations will enhance students' ability to grasp and apply division more effectively (Downton, 2008; Robinson & LeFevre, 2012).

Although the conceptual link between multiplication and division is commonly emphasized in teaching, students frequently encounter significant challenges when learning division. These dif-

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ficulties have been well-documented in studies such as that of Alghamdi et al. (2020), who highlighted several specific difficulties in students' understanding of division. For instance, students often struggle to distinguish between partitive division (dividing into equal parts) and quotitive division (determining how many times a divisor fits into a dividend), with the latter being more abstract and challenging. Additionally, they face difficulties in understanding the inverse relationship between divisor and quotient when the dividend remains constant. Lastly, interpreting remainders in real-life contexts presents another challenge, where students may apply division correctly but fail to interpret remainders correctly or to appropriate problem-solving strategies (Pacheco-Muñoz et al., 2023).

In addition, the research team conducted a survey (see Figure 1 below) to assess the ability of 273 third-grade students from three public schools in Vietnam to understand and apply division of two natural numbers. The survey, conducted in November 2022, found that despite having studied both multiplication and division, a significant number of students struggled. Specifically, 39.19% of students answered question 1 incorrectly, 65.57% struggled with question 2, and 63% answered question 3 incorrectly, indicating ongoing confusion in understanding and applying the concept of division. This result underscores the difficulties that third-grade students face in grasping the concept of division, despite prior exposure to both division and multiplication.

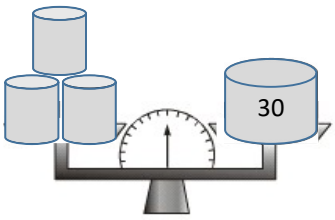
1. Có 12 người khách cần sang sông, mỗi thuyền đều chở 4 người khách (không kể người lái thuyền). Hỏi cần mấy thuyền để chở hết số khách đó? (khoanh tròn đáp án đúng) [The translation of Question 1: There are 12 passengers who need to cross the river, each boat carries 4 passengers (not including the boat driver). How many boats are needed to carry all those passengers? (circle the correct answer)]

A. 3 B. 4 C. 8 D. 48

2. Các ghế dài được dành cho trẻ em ngồi xem phim. Mỗi ghế dài ngồi được 3 em. Hỏi cần ít nhất bao nhiêu ghế dài như vậy để đủ chỗ ngồi cho 25 em? (khoanh tròn đáp án đúng) [The translation of Question 2: Benches are reserved for children to sit and watch movies. Each bench can seat 3 children. How many benches are needed to accommodate 25 children? (circle the correct answer)

A.7 B. 8 C. 9 D. 10

3. Điền số thích hợp vào chỗ trống.



Có …….. quả cân trên dĩa cân bên trái.

Các quả cân (trên dĩa cân bên trái) nặng bằng nhau và nặng …….kg.

[The translation of Question 3: Fill in the appropriate number in the blank. (Image). There are …….. weights on the scale on the left. The weights (on the scale on the left) weigh the same and weigh …….kg.]

Figure 1. Questions in the survey

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Additionally, according to a recent study conducted in Germany, about 20% of 15-year-old students failed to develop an understanding of the four basic arithmetic operations during their schooling, with division being the most challenging operation for them. The study also revealed that the development of conceptual understanding of division heavily relies on language structures that express the relationship between division and multiplication, as well as on the students' ability to verbalize the concepts of division (Götze, 2018). This suggests that foundational gaps in learning division at an early age may continue into later years, further emphasizing the importance of effective instruction in primary school.

To address these challenges, one of the most promising alternative instructional methods is Realistic Mathematics Education (RME). Developed by Dutch mathematician Hans Freudenthal, RME encourages students to engage with mathematics through situations that are meaningful and relevant to them, allowing them to construct a deeper conceptual understanding. In fact, RME positively impacts mathematics reading comprehension, helping students flexibly apply knowledge to real-life situations (Ariati et al., 2022). Moreover, RME also creates a positive learning environment, encourages the students' active participation and contributes to improved overall learning outcomes (Ardiyani et al., 2018).

In addition, this approach aligns with Vietnam's current educational trends (Ministry of Education and Training – Vietnam, 2018a, 2018b). Research shows that integrating RME enhances students' math performance and attitudes by emphasizing real-world problem-solving (Tong et al., 2022). RME supports educational reforms, particularly in curriculum and textbook development (Trung et al., 2019). A framework for RME implementation highlights key factors like national vision, curriculum, assessment, and teacher training (Trung et al., 2020). Case studies show RME improves learning outcomes and student engagement compared to traditional methods (Tong et al., 2022; Loc & Tien, 2020; Duyen & Loc, 2022; Ha et al., 2021).

Gravemeijer (1997) emphasizes the need for intermediate models, such as visual tools and manipulatives, to help students better understand complex mathematical concepts. While division is often introduced through its relationship with multiplication, the RME approach allows students to explore division through real-world problems before strongly associating it with multiplication. Furthermore, teaching division in the context of multiplication may create barriers for students who struggle with multiplication or fail to grasp the inverse relationship, and limit students' ability to apply the concept flexibly in real-life situations, where division is often encountered without explicit reference to multiplication.

The current research on division and RME focuses largely on the benefits of teaching division within the framework of multiplication. However, there is a noticeable gap in the literature regarding the potential impact of teaching division independently of multiplication. By addressing this gap, this study seeks to provide insights into how RME may foster a deeper understanding of

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division as a stand-alone concept, particularly in real-world contexts where equal sharing is essential. Thus, this study aimed to answer the following question:

Research question: *How does RME affect primary school students' understanding of division of two natural numbers, especially when the division is taught independently of multiplication through real-life equal-sharing situations?*

By exploring how RME can foster a deeper understanding of division as a stand-alone concept, this study aims to fill a critical gap in the literature. The study's findings contribute to developing a practical approach for teaching math in general and division of two natural numbers to primary school students in particular.

LITERATURE REVIEW

Contexts (contexts, realistic contexts, contextual problems) play an essential role in RME theory. Contextual problems are defined as "problems where the problem situation is experientially real to the student" (Gravemeijer & Doorman, 1999, p.111). The key point is that these contexts are situations or problems suitable for mathematization. That means these contexts could make it easier for primary students to imagine and visualize abstract mathematical concepts (Duyen & Loc, 2022). Johar et al. (2017) also asserted that problems placed in real-world contexts enable students to start from simple, informal solutions and gradually move to more complex, formal solutions. This suggests that context is a starting point and a bridge that helps students transition from intuitive to abstract thinking. In addition, they should be *relevant and essential contexts*, rather than *camouflage contexts*. For example, the problem "There are 2163 notebooks arranged evenly in 7 boxes. How many notebooks are in those 5 boxes?" is a type of *camouflage* context (Son et al., 2022). In contrast, *relevant and essential contexts* create opportunities for constructing new mathematical concepts (just new for learners). For instance, the context as "One carton of milk can fill 5 glasses. There are 42 students in the class. Each student receives 1 glass of milk. How many cartons of milk do we need? Fill in the answer in the blank box." (see Figure 2) can be considered as a suitable context and contributes to learning the knowledge of multiplication and division of two natural numbers.

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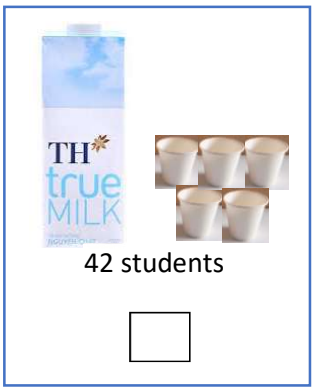


Figure 2. An example of relevant and essential contexts

In fact, students can actively explore, build and develop mathematical thinking naturally and flexibly thanks to *relevant and essential contexts.* Contexts are also tools for forming mathematical concepts and models, serving as foundations for applying knowledge, practicing skills (Treffers, 1994), and developing diverse and flexible ways to solve problems—thereby promoting the development of mathematical thinking at various levels (Widjaja et al., 2010).

In RME, Freudenthal (1991) emphasized that learning mathematics begins with students transforming real-life situations into mathematical problems – a process known as *mathematical reinvention*. Students use their existing knowledge and mathematical tools to explore and build new mathematical concepts. Students naturally form and develop mathematical ideas after being exposed to many real-life situations. In this process, the class teacher plays an essential role in creating a learning environment that helps students explore mathematical knowledge based on their existing understanding (Treffers, 1991). The process in which students take action to reinvent mathematics is called *mathematization*. It has been classified into two types: horizontal mathematization and vertical mathematization (Treffers, 1987). Horizontal mathematization is a process in which students use their solutions to describe a contextual problem by symbols. These symbols can be mathematics (formulas, algorithms, etc.) or not (figures, diagrams, etc.). Students can explicitly present them, or they even can visualize or imagine them. In the world of symbols, students continue to use their known mathematics to take action to find out new mathematics knowledge. It helps students answer the problem situated in the context problem. The process in which students take action within the mathematics system is called vertical mathematization (Duyen & Loc, 2022). Horizontal and vertical mathematization in RME can be summarized as Figure 3 below.

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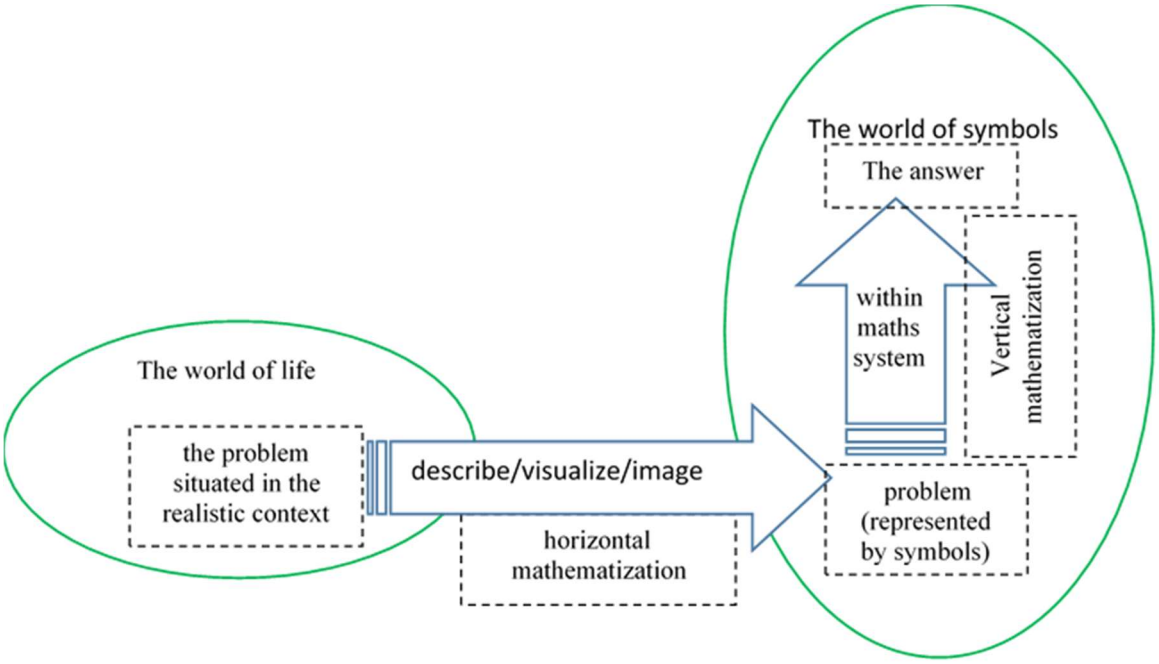


Figure 3. Horizontal mathematization and vertical mathematization in RME

It can be seen that guided reinvention via progressive mathematization is one of the major elements for designing education in RME. According to this principle, Gravemeijer (1994) argued that a Hypothetical Learning Trajectory (HLT) must be mapped out, allowing students to experience progressive mathematization. The HLT, also called a learning design, includes learning activities as well as objectives of these activities and the teacher's conjecture about how students learn and think (Simon, 1995). That means the teacher has the responsibility for providing a suitable series of contextual problems that can map out a possible HLT and in which students can experience "progressive mathematization" as a gradual change. By applying HLT in RME, teachers can create meaningful learning experiences that help students develop mathematical skills naturally and effectively. Research by Nuraida and Amam (2019) showed that HLT in RME significantly improved the mathematical communication skills of middle school students. Moreover, the HLT also helps teachers identify both the starting point and destination in the students' learning process. For example, Febriani and Sidik's (2020) study used the HLT to design activities that helped fourth graders effectively connect prior knowledge to new knowledge. In addition, Napitupulu et al.'s (2021) study also confirmed the value of the HLT in teaching complex mathematical concepts such as least common multiples.

In addition, when designing education in RME, attention should be paid to the principle of self-developed models, particularly to the role of self-developed models in students' learning processes. This principle is essential because self-developed models, as also known as emergent models, which enable students to bridge the gap between informal and formal mathematical

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knowledge. Accordingly, teachers must create opportunities for students to use and develop their own models when solving contextual problems. At first, students will develop a model that is familiar to them. After generalization and formalization, the model gradually becomes a general model to approach formal knowledge. Gravemeijer called this process the transformation from model-of to model-for (Gravemeijer, 1994, 1999). After the transformation, the model can be used as a model for mathematical reasoning (Treffers, 1991).

In this study, this principle is applied when designing activities to help students transition from concrete representations to abstract understanding. To help students practically and visually grasp the division of two natural numbers, physical objects such as math-link cubes, flowers and vases, or cakes/candies and plates are used to form a model-of. For example, in the context of partitive division, to illustrate 12 ÷ 4=3, twelve math-link cubes are arranged into 4 equal groups, each containing 3 cubes. After this, the class teacher asks questions like: "How many cubes are in each group?", "What is 12 divided by 4?". These questions serve as a model-for, guiding students toward understanding partitive division (dividing into equal parts). In Figure 4, each group of 3 cubes represents partitive division, where the total (12) is divided into equal parts (3).



Figure 4. A concrete model-of to help students engage with 12: 4=3

Similarly, for the quotitive division problem 12: 4=3, students might be asked to repeatedly form groups of 3 with 12 math-link cubes. This physical grouping gives them a concrete model-of, as shown in Figure 4. After this, the class teacher asks: "How many groups of 3 can be made from 12?". This question serves as a model-for, guiding students toward understanding quotitive division (determining how many times a divisor fits into a dividend). As students become familiar with this concrete process, they begin to see that quotitive division involves determining "how many groups of 3 can be made from 12" and they no longer need to rely on physical objects but can mentally or visually imagine the grouping process.

METHOD

This study employed a descriptive research design with an educational intervention to explore an instructional approach for teaching division to primary students. The research was grounded in RME, which informed the design of an HLT. The intervention facilitated hands-on exploration and practical application of theoretical concepts in a classroom setting, enabling the assessment of students' understanding and engagement. This approach ensures a comprehensive framework for teaching, blending theory with practice to enhance learning outcomes.

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Researchers performed the study in the following three stages:

Before Conducting the Intervention

During this phase, an HLT for the division of two natural numbers was developed, followed by a-priori analysis. The HLT comprised nine activities, structured into four lesson plans that aligned with four experimental lessons, detailing classroom organization. Both the HLT and lesson plans were thoroughly reviewed and refined through collaboration between the classroom teacher, a veteran with over 14 years of teaching experience, and the researcher, ensuring alignment with pedagogical goals.

While Implementing the Intervention

In alignment with the designed HLT, the teaching experiment consisted of four lessons conducted from July 21, 2023, to July 23, 2023. These lessons were recorded on video while classroom observations were carried out, as shown in Table 1. The study involved 11 students (7 girls and 4 boys) who had completed the first grade curriculum, but had not learned division and multiplication of two natural numbers yet. This allowed for a focused examination of teaching division independently, which is crucial since second-graders would already have learned multiplication and therefore wouldn't meet the study's criteria. All students participating in the experiment lived in Soc Trang province. They had completed grade 1 in the 2022-2023 academic year at two public primary schools located in Soc Trang province. The participants of the study were determined by using convenience sampling. They were chosen because they were willing to participate in the teaching-learning process after receiving an invitation sent by the researcher.

Lesson Title	**Dividing equally**	**Division**	**Division (continued)**	**Division (continued)**
Date	21/7/2023	21/7/2023	22/7/2023	22/7/2023
Activities in HLT	1, 2, 3	4, 5, 6	3, 5, 7, 8	9
Worksheet	Worksheet 1	Worksheet 2		Worksheet A Worksheet 3

Table 1. Four intervention lessons

Table 1 summarizes four intervention lessons on division, detailing their titles, dates, activities in the HLT, and four associated worksheets. Worksheet 1 was used in the first lesson, Worksheet 2 was used in the second lesson, and Worksheet A and Worksheet 3 were used in the last lesson.

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After the Intervention

Based on these videos, researchers continued to document the observed teaching-learning process. In addition, students' responses on the four worksheets were collected. Based on data obtained from classroom observations and worksheets, the authors aimed to conduct a qualitative analysis from the perspective of RME to evaluate the feasibility and effectiveness of the designed HLT, focusing on students' activities, attitudes towards math, and learning outcomes.

To examine the effectiveness of the intervention, criteria for evaluating students' understanding of the division of two natural numbers were established, and four worksheets were designed and used as tools for collecting students' responses, as shown in Table 2. Additionally, verbal responses and group work products, along with the HLT, were collected to assess students' level of understanding.

No	Criteria	Tools
1.	Understand: Correctly answer the question "How many [object X] does each part have? " after dividing *a* [object X] equally into *b parts.*	Worksheet 1
2.	Know: Write appropriate division equations for situations involving equal division.	Worksheet 2
3.	Know: Write the appropriate division equations based on students' manipulation of dividing objects equally.	Worksheet A
4.	Understand: Correctly write two division equations corresponding to a given image, where the image represents a context of equal groups.	Worksheet 3

Table 2. Criteria and tools for collection data

RESULTS

The HLT for the Concept of Division of Two Natural Numbers According to RME

We determined that the HLT should include the activities shown in Table 3. All activities are necessary to provide opportunities for students to experience mathematization when solving contextual problems. In the HLT, students can participate in solving contextual problems by distributing the quantities described in the problems into equal parts. The action that students perform to solve these problems involves manipulating objects to divide them equally. Consequently, the teacher will help students generalize this into new knowledge: the division of two natural numbers.

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No.	Activity	Objectives
1.	Students solve contextual problems by distributing into equal parts	By solving contextual problems, students gain an understanding of partitive division, where they determine how to divide a total equally, equal parts and answer the question: How many [object X] are in each part?
2.	The teacher guides students to express how to divide a total equally	Through the equal division process implemented in Activity 1, guided by teachers, students learn that to divide all objects (X), they distribute a certain number of X into each part and repeat the process until X is fully distributed.
3.	Students manipulate the objects to divide them into b equal parts	Continue reinforcing students' understanding of partitive division and equal parts, guiding them to answer the question: How many [object X] are in each part?
4.	The teacher introduces the concepts of the division of two natural numbers, the symbol ":", and how to read.	Students understand $a:b=q$, where a, b, q are specific non-zero natural numbers, b represents the known number of parts/groups before dividing, and q is the result found after dividing - the number of [object X] in each part. The division operations from Activity 1 serve as examples.
5.	Write appropriate division equations for dividing objects or pictures/video clips.	Continue reinforcing students' understanding of partitive division by having them write the exact division equation and verbally state the situations that are appropriate to that division.
6.	State the appropriate situation for the given division.	
7.	Students solve contextual problems: There are a objects X, if we place q objects X in each part, how many parts will there be?	Students learn how to divide by grouping; that is, they take q objects X for each group and count the number of groups. This allows them to correctly answer the question: How many groups were formed?

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8.	The teacher introduces the concept of quotitive division	Students understand that in the equation $a:b=q$, where a, b, q are specific non-zero natural numbers, dividing each part with q [object X], results in b equal parts. The division operations presented in Activity 7 serve as examples.
9.	Distinguish between partitive division and quotitive division	Students can write the correct division equation and verbally state the appropriate situations that correspond to that division.

Table 3. The HLT for the division of two natural numbers

Analysis of Nine Activities in the HLT

Activity 1: Students solve contextual problems by distributing into equal parts

Task 1. Divide the class into groups

- **Describe**: Suppose the class has a student, and the teacher announces the need to divide the class into b equal groups.
- **Analysis**: The teacher will assign the number of groups by handing out group name tags to a few representative students (for example, if there are 40 students and the class needs to be divided into 10 groups, the teacher will assign 10 students to receive group name tags numbered from 1 to 10). After that, the students will move to the corresponding groups and adjust the numbers so that the groups are equal.

Task 2. Divide the cakes/candies evenly among the plates

- **Describe**: The number of candies and plates given to each group is the same. For example, the task is to divide 12 cakes equally into 3 plates. The teacher gives students exactly 12 cakes and 3 plates.
- **Analysis**: In this activity, we expect students to divide the cakes in their own ways, as long as they ensure that the number of cakes on each plate is equal.

Task 3. Arrange a flowers evenly into b vases

- **Describe**: Round 1: 6 flowers into 2 vases; Round 2: 10 flowers into 2 vases; Round 3: 6 flowers into 3 vases; Round 4: 15 flowers into 3 vases.

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- **Analysis:** In this activity, the number of flowers is divisible by the number of vases. Similar to Task 2, students will find their own ways to divide the flowers equally.

Activity 2: **The teacher guides students on how to divide equally**

Through the process of equal division in Activity 1, guided by teachers, students learn that to divide all objects (X), they distribute a certain nuamber of X into each part and repeat the process until X is fully distributed.

Activity 3: **Students manipulate objects (magnetic dots/math-link cubes) to divide them into b equal parts**

Task 1. Manipulate the magnetic dots to divide them into b equal parts

- **Describe**: The number of magnetic dots stuck on the board is a. The teacher will call each student in turn to divide the dots equally into b parts (b are 2, 3, 4, respectively). The whole class observes and gives feedback.
- **Analysis**: In each turn, the number of dots remains constant (e.g., always 12 dots) while the number of equal parts changes. This helps emphasize the concept of parts for the students.

Task 2. Manipulate the math link-cubes to divide them into b equal parts

- **Describe**: Students work individually or in groups on 12 math link-cubes, completing 4 turns where they divide the 12 cubes equally into 2 parts, 3 parts, 4 parts and 6 parts.
- **Analysis**: In this activity, students will employ strategies for dividing and will be successful by recognizing the equality of the parts through the visual representation of the cubes.

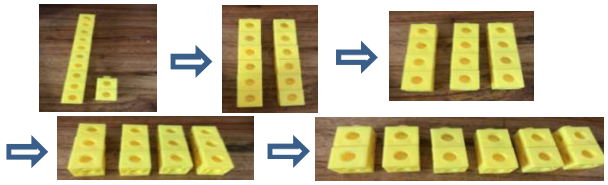


Figure 5. Divide 12 math link-cubes into b equal parts

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Activity 4: The teacher introduces concepts of the division of two natural numbers, the symbol ":", and how to read

The teacher introduces to students: a: b=q is a notation used to represent the act of dividing equally, where a, b and q are specific, non-zero natural numbers.

Activity 5: Write appropriate division equations for dividing objects or pictures/video clips

Task 1. Write appropriate division equations based on the manipulation of partitive division

- **Describe**: In this task, students work both individually and in groups with math-link cubes. First, students work individually to divide 12 cubes into 2 equal parts and write a division equation, divide 12 cubes into 6 equal parts and write a division equation. Students then work in groups with a block of 16 math-link cubes that are connected together. With the requirement to "divide this block of 16 math-link cubes evenly and then write the appropriate division equation", the student or group that writes the most division equations wins.
- **Analysis**: In this task, the dividend is 16, which limits students mentally from calculating the quantity in each part. Firstly, students must determine the number of parts. Then, they position each part by one math-link cube. Next, they equally distribute many math-link cubes into each part. Finally, they write the corresponding division equation.

Task 2. Write the appropriate division equation for the observed image

In this task, students work individually, observe the image and write division equations on Worksheet 2.

Activity 6: State the appropriate situation for a given division equation

Task. Identify real-life situations that can be represented by a given division equation.

- **Describe**: The teacher presents students with various division equations (e.g., 12: 3=4, 15: 5=3). Students work individually or in groups to create appropriate word problems or scenarios that correspond to each division equation. For example, for the equation 12: 3=4, a possible scenario could be "If there are 12 apples and we want to put them into 3 baskets, how many apples will be in each basket?"
- **Analysis:** This activity encourages students to think critically about how division can be applied to real-life situations. By creating their own scenarios, students

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deepen their understanding of the concept of division and its practical applications.

Activity 7: Solve contextual problems like "There are $a$objects X. Divide them so that each part contains q objects X. How many parts do we have?"

Task 1. How many gift bags?

Each group receives a cakes và b transparent bags with the requirement that each gift bag must contain q cakes.

Task 2. How many flower vases can be arranged?

- **Describe**: Each group receives a flowers và b vases with the requirement that each vase must contain q flowers.
- **Analysis**: Activity 7 is designed to achieve the objectives that students know how to take a group of q objects X for each part, then count the number of parts. In Task 1 and Task 2, students may initially divide incorrectly if they do not pay attention to the requirement. For example, in Task 1, if a group has 6 cakes and 6 bags but does not notice that each bag must have 2 cakes, students may mistakenly divide 6 cakes into 6 bags equally. This activity emphasizes correct grouping to ensure accurate division.

Task 3. How many numbers n can be separated? (n is the number made up of n math-link cubes)

- **Describe**: Each student receives 12 math-link cubes. Ask students to manipulate the cubes and answer the following questions: "From this number 12, how many numbers 3 can be separated?", "From this number 12, how many numbers 4 can be separated?", "From this number 12, how many numbers 2 can be separated?", "From this number 12, how many numbers 6 can be separated?", "From this number 12, how many numbers 1 can be separated?".
- **Analysis**: This task helps students mathematize one cube as number 1, two cubes as number 2, three cubes as number 3, and twelve cubes as number 12. This task also helps students to be familiar with seeing a number as the sum of equal numbers.

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Activity 8: Introduces the division in the context of quotitive division

The teacher places a dots on the board (e.g., a=8) for the whole class to observe. Students are asked to verbally state the situation and write appropriate division equations based on the observed image. Once students provide the correct answers, the teacher emphasizes: In the division 8: 2=4, we know the number of parts is 2, so we divide the dots equally into 2 parts. Next, the teacher introduces the concept of quotitive division and presents the corresponding division equation as 8: 4=2. The teacher highlights: In this division, we know that each part must have 4 dots. With 8 dots, we can divide them into 2 parts.

Activity 9: **Distinguish between partitive division and quotitive division**

Task 1. Two divisions with flowers and vases

- **Describe**: In this task, vases are created by gluing colored craft paper onto an A4 sheet, forming vase shapes, so that hand-made flowers can be placed in them. Each A4 sheet has three vases. Each group receives six vases (on two A4 sheets) and fifteen flowers. Students discuss how to evenly arrange the flowers into the vases based on the teacher's instructions. Afterward, they write the appropriate division equation and verbally describe the corresponding situations. For example, when students are asked to "evenly place fifteen flowers into three vases" or "place five flowers into each vase", their arrangement will match what is illustrated in Figure 6.

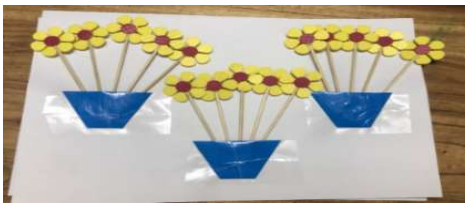


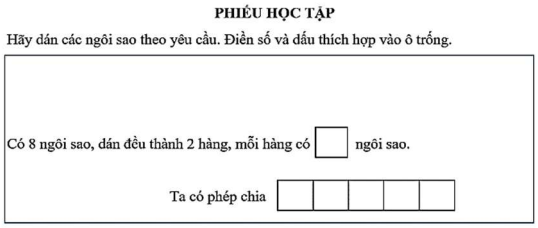
Figure 6. 15 flowers into three vases

- **Analysis**: This task is designed to help students understand the concept of division by both equally distributing objects and identifying groups. By physically arranging flowers into vases, students can see how division works in practice. The first scenario: evenly placing fifteen flowers into three vases demonstrates division when the total number of objects is known and the goal is to divide them into a specific number of parts (vases), which represents partitive division. The second scenario: placing five flowers into each vase helps students conceptualize division where the number of objects per part is fixed, and the task is to determine how many parts (vases) can be formed, which illustrates quotitive division.

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Task 2. Two divisions with stars (stickers) and rows

- **Describe**: Each student receives two rows of stickers, with eight stars in each row, and Worksheet A as shown in Figure 7.


PHIẾU HỌC TẬP

Hãy dán các ngôi sao theo yêu cầu. Điền số và dấu thích hợp vào ô trống.

Có 8 ngôi sao, dán đều thành 2 hàng, mỗi hàng có ☐ ngôi sao.

Ta có phép chia ☐☐☐☐☐

Figure 7. A part of the worksheet A

[The translation of Figure 7: Worksheet A. Paste the star stickers as instructed. Fill in the blanks with the correct numbers and symbols. [Space for pasting stickers] There are 8 stars, evenly distributed into 2 rows, each row having [blank] stars. We have the division: [5 consecutive blank spaces]].

- **Analysis**: In this task, students explore division through the physical manipulation of stars (stickers) on worksheet A. Similar to Task 1, this task reinforces two types of division: partitive division and quotitive division.

Task 3. Observe the picture of b groups of a – state two division situations

The teacher attaches each picture of b groups of a on the blackboard. Students are asked to observe the picture and state two different division situations that could apply. For each situation, students must write the corresponding division equation. The teacher only moves on to a new picture once the class has correctly identified two situations and stated the matching division equations.

Task 4. Write two division equations with a given image of b groups of a

- **Describe**: Students work individually on Worksheet 3, which contains images of b groups of a objects (e.g., 4 bouquets of 3 roses). Students are required to observe the image and write two corresponding division equations.
- **Analysis**: This task serves as an assessment tool to gauge students' grasp of division after completing the HLT. As presented in Tables 1 and 2, Worksheet 3 will be used to evaluate whether students can independently identify both partitive and quotitive division situations.

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Results of Implementing Activities in the HLT and Students' Learning Outcomes

Based on video recordings of each group's activities throughout the four intervention lessons, as well as data collected from the students' work on worksheets, we summarize the results of implementing the activities in the HLT and students' learning outcomes for each lesson.

During the first lesson, students transitioned from concrete representations to abstract understanding through carefully designed activities involving real-life contextual problems and manipulatives. In the first task of Activity 1, the whole class participated in dividing into groups. After the teacher assigned five students to hold group name cards for Group 1 through 5, the remaining students moved to their respective groups. In total, ten students participated in this task, as one student arrived late. In the second task, students worked in groups to divide 12 cakes or candies evenly among 3 plates (three groups received 12 cakes, while two groups received 12 candies). The students collaborated well, employing different methods for equal distribution (see Table 4), and successfully arranged 4 cakes or candies per plate. This hands-on experience served as a tangible representation of partitive division, allowing students to visually and physically explore the concept. After manipulating the items, they articulated their understanding by answering questions like "Each plate has four cakes". Moving on to Activity 2, the teacher guided students in expressing how to divide a total equally, encouraging them to articulate their thought processes and strategies for equitable distribution.

Way	Describe	Students
1)	Students took each cake and placed it on each plate, repeating the action 4 times. After verifying that no cakes remained, they counted the number of cakes on each plate.	B2 & B11 B5 & B6
2)	One student poured a bag of 12 candies onto the table. Each student took one candy, equally dividing the candies between 2 students. Then, each student divided their 6 candies into 3 plates. Each plate ended up with 4 candies.	B4 & B8
3)	One student placed 3 cakes on one plate, then repeated the process for the second and third plates. The remaining 3 cakes were added to each plate by two students, one cake at a time.	B9 & B10
4)	Three students each took a plate. Student B7 placed 2 cakes on each plate, paused, and then continued placing 2 more cakes on each plate. Afterward, students B1, B7, and B3 counted the cakes on all three plates.	B1, B3 & B7

Table 4. Ways students used to divide cakes or candies equally

In Activity 3, students further manipulated objects like magnetic dots and math-link cubes to divide them into equal parts. By dividing 18 magnetic dots into 3 equal parts and various colored cubes into 2, 3, 4, and 6 parts, students gained additional concrete experiences (model-of). Different groups employed various methods for equal distribution, as detailed in Table 5. The progression

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of these tasks helped solidify their understanding, eventually allowing students to internalize and abstract the process (model-for). This was evident in their ability to complete Worksheet 1 independently, with 9 out of 11 students correctly filling in all blanks based on how the objects were divided. Overall, the results illustrate how physical manipulation, followed by guided reflection, supported students in moving from concrete actions to abstract reasoning about division.

Way	Describe	Group
1.1)	Students broke and placed a cube to represent a part. After repeating this 3 times, they set up 3 parts. They then continued breaking each cube and distributed one cube evenly into each part until all the cubes were gone.	B9 & B10 B5 & B6
1.2)	The teacher drew 3 circles on the table to support the group. Students placed each cube into each circle, repeating this 4 times. They then linked 4 cubes to form a block of 4 cubes at each circle, resulting in 3 blocks of 4.	B1& B3
2)	Students split 4 cubes at a time, breaking them 3 times until finished. They then counted again to confirm there were 3 equal parts.	B2 & B11 B4 & B8; B7

Table 5. The ways students used in activity "divide equally 12 math-link cubes into 3 parts"

By the end of the first lesson, the concepts of equal sharing and equal parts were emphasized to lay the foundation for introducing the concept of division in the second lesson (Activity 4). After forming the concept, students continued with Activities 5 and 6 to further solidify their grasp of partitive division through both concrete manipulation and abstract reasoning. Classroom observations in Activity 5 show that, in the first task, students initially manipulated 12 cubes, dividing them into 2 equal parts and then 6 equal parts. Through this concrete manipulation (model-of), all students correctly wrote the corresponding division equations, $12:2=6$and $12:6=2$, demonstrating their ability to connect physical division to abstract numerical equations (model-for). This hands-on activity reinforced their understanding of partitive division. In a subsequent task, students worked in groups to divide a block of 16 math-link cubes evenly and wrote the appropriate division equation. With the teacher's support, Figure 8 shows that nearly all groups correctly wrote the equations, showing a successful transition from concrete manipulation to abstract reasoning. However, one group (B1 & B3) made some errors, indicating a need for further reinforcement. In the second task, students were asked to write division equations based on an image in Worksheet 2. Ten out of eleven students correctly wrote all five equations, while one student (B3) wrote four

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out of five. This result suggests that most students had successfully applied their understanding of division from earlier manipulations to abstract representations, though some may need additional support.

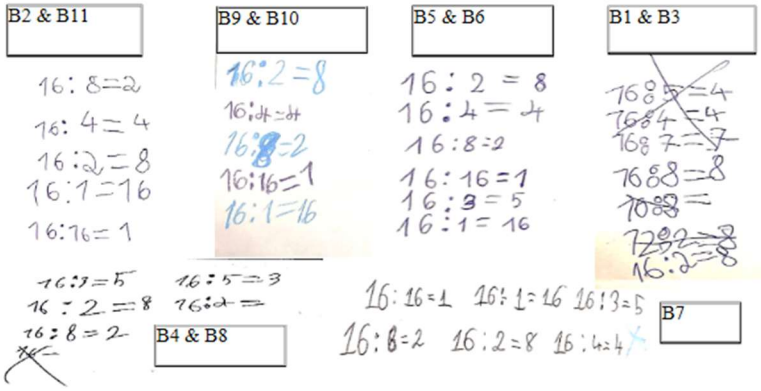


Figure 8. Results of the groups in Activity "evenly divide a block of 16 math-link cubes"

Classroom observations in Activity 6 show that all five called-upon students correctly stated the situation of a given division equation. For example, when given 10: 5=2, student B9 stated, "There are 10 teddy bears, divided among 5 people, each person gets 2 teddy bears", while student B4 stated, "There are ten cars divided among 5 people; each person gets 2 cars". In interviews conducted at the end of the lesson, students expressed enjoyment of the class and demonstrated their understanding of equal division. Seven out of eight students correctly answered questions such as: 6: 2=?, 6: 1=?, 6: 6=?, 10: 2=?, 24: 2=?,30: 3=?.

In the third lesson, students continued with tasks of Activities 3 and 5, then progressed to Activity 7 before being introduced to quotitive division in Activity 8. Each student manipulated 10 cubes, dividing them into 2 and 5 equal parts, and 9 cubes into 3 parts. Classroom observations showed that students correctly answered with something like, "Each part has ... cubes", and wrote the corresponding division equations (task of Activity 3). They then divided 9 cubes into 3 equal parts, 8 cubes into 4 equal parts, and 8 cubes into 2 equal parts, successfully writing the equations 9: 3=3; 8: 4=2; 8: 2=4 (task of Activity 5). In tasks 1, 2 of Activity 7, students demonstrated their understanding by dividing objects equally and correctly identifying the number of parts or people. In task 3 of Activity 7, students experienced mathematization, recognizing numbers as sums of equal numbers (Table 6). Based on images received, students correctly answered that 12 can be divided into four numbers 3, three numbers 4, two numbers 6, six numbers 2, and twelve numbers 1.

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Turn	Images received after manipulating	Mathematics
1)		12=3+3+3+3 12: 3=4
2)		12=4+4+4 12: 4=3
3)		12=2+2+2+2+2+2 12: 2=6
4)		12=6+6 12: 6=2
5)		12=1+1+1+1+1+1+1+1+1+1+1+1 12: 1=12

Table 6. Mathematization in task 3 of Activity 7

In the fourth lesson – the last lesson in the HLT – students perform four tasks in Activity 9 to clearly distinguish between partitive division and quotitive division. In Task 1, students worked with two division scenarios involving flowers and vases: 6 flowers and 3 vases, 9 flowers and 3 vases, 12 flowers and 2 vases, and 15 flowers and 5 vases. All groups followed the instructions and correctly stated the corresponding division equations, demonstrating a clear distinction between partitive and quotitive division. Classroom observations, students' work in Task 2, and the results of 11 students on Worksheet A show that students demonstrated correct understanding. After evenly pasting 8 stars (stickers) into 2 rows, students correctly filled in the number 4 and wrote 8: 2=4. Likewise, after pasting 4 stars in each row, students filled in the number 2 and wrote 8: 4=2. In Task 3, students were asked to observe a picture representing "b groups of a" and state two division situations. Eight out of nine called-upon students correctly stated the division situations, while student B3 was unable to respond. Finally, in Task 4, students were asked to write two division equations with a given image of group a. Results from Worksheet 3 show that 8 out of 11 students correctly wrote all 6 division equations, 2 students correctly wrote only 2 equations, and student B3 did not write any equations correctly. The following Figure 9 shows student B8's work in the sticker activity and student B6's work on the worksheet 3. Overall, the findings indicate that most students successfully distinguished between the two types of division, though a few may need further support.

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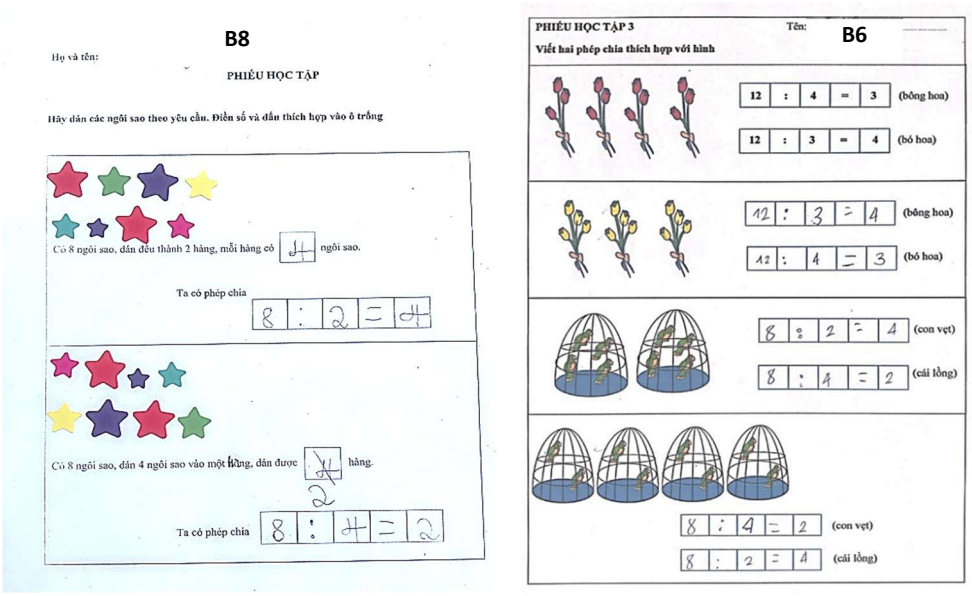


Figure 9. Student B8's work on Worksheet A and student B6's work on Worksheet 3

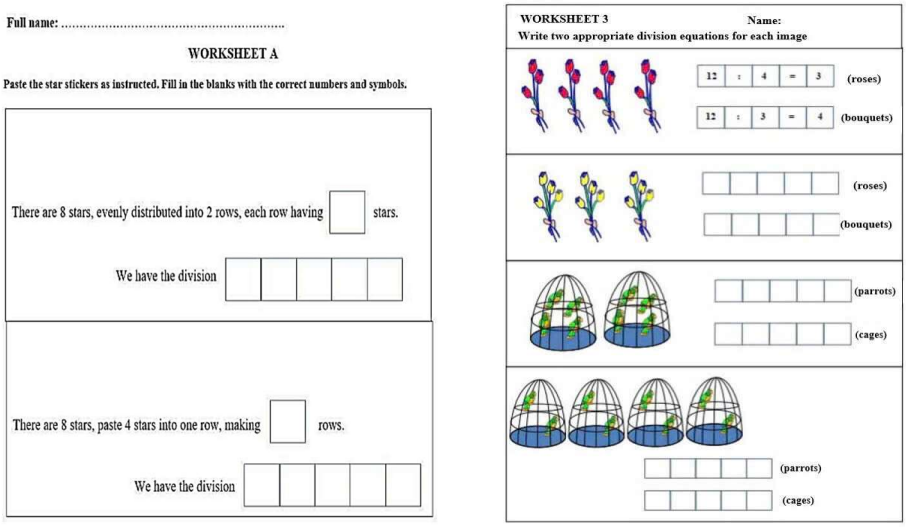


Figure 10. The translation of Figure 9 into English

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DISCUSSION

The results of this study show that the designed HLT for teaching division independently of multiplication offers a promising approach to helping primary students build a conceptual understanding of division. By incorporating RME principles—especially hands-on manipulatives and realistic contexts—students engaged more deeply with division, focusing on grasping core concepts instead of memorizing procedures. This aligns with studies demonstrating the effectiveness of RME in improving students' understanding, problem-solving skills, and test performance (Leng et al., 2020; Drijvers et al., 2019; Syafriafdi et al., 2019; Laurens et al., 2018; Papadakis et al., 2017; Lestari & Surya, 2017). Moreover, RME enhances the meaningfulness of math learning and strengthens reasoning (Fauzan et al., 2020).

The research also highlights the pivotal role of classroom teachers in the RME approach: organizing activities align with the goals of the HLT (Duyen & Loc, 2022) and supporting learners (Wahyudi et al., 2017). Throughout the instructional process, teachers actively observe students' learning activities and provide timely guidance, ensuring that learners can independently reinvent mathematical concepts.

Challenges emerge particularly in helping low-ability students transition from concrete to abstract understanding. While manipulatives supported an initial grasp of division, these students struggled with abstract representation, highlighting the need for additional scaffolding. This aligns with research on students' problem-solving abilities, which shows that high-ability students tend to exhibit the greatest fluency and flexibility when solving problems. In contrast, low-ability students often struggle to comprehend the problems and make frequent errors during the problem-solving process (Arifin et al., 2021).

Overall, the study highlights both the strengths and potential limitations of using the RME approach. Future studies could explore how additional scaffolding or technology-based tools might further assist low-ability students in mastering the abstract aspects of division.

CONCLUSIONS

The findings of this study underscore the significance of teaching the division of two natural numbers to primary school students independently of multiplication through the principles of RME. The designed HLT facilitated a conducive learning environment, enabling students to engage deeply with the concept of division. By experiencing mathematization, students were able to distinguish between partitive and quotitive division, articulate their understanding through appropriate real-world contexts, and write accurate division equations.

The positive outcomes suggest that RME not only supports the understanding of division as a standalone concept but also enhances students' ability to apply mathematical reasoning in practical

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situations. This approach addresses the challenges learners face when learning division, offering a framework that encourages exploration and critical thinking.

Future research could expand on these findings by investigating the long-term effects of teaching division independently of multiplication, exploring additional real-world contexts, and examining the impact on diverse learning styles. Overall, this study contributes valuable insights into effective mathematics education practices for primary school students, emphasizing the need for instructional strategies that prioritize conceptual understanding over rote learning.

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Learning Trajectory of Trigonometric Ratios Applying Realistic Mathematics Education

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Abstract: Trigonometry materials are still considered complex materials at the secondary level. It is necessary to design an appropriate learning trajectory for *trigonometric ratios so that the students can quickly build the concept. This study aims to design a learning trajectory for trigonometric ratio material through Realistic Mathematics Education (RME). This study uses a research design involving 10th-grade high school students in Bandung City in the 2023-2024 academic year. The research data was collected through data collection using written test instruments, student worksheets, observations, interviews, and documentation. The results of the study showed that the trigonometric ratios learning trajectory consisted of raffia stretching activities from the table base to the student's head, measuring the student's height and shadow, experimenting with making three right triangles from paper, observing a picture of a calendar in the form of an equilateral triangular prism, observing a bed or bunk bed, and observing a picture of a skier going down an iceberg. Students can quickly build the concept of trigonometric ratios from several activities provided. In addition, various learning flows from various activities make it easier for students to build knowledge about trigonometric ratios and solve problems presented contextually.*

Keywords: Learning Trajectory, Trigonometric Ratios, Realistic Mathematics Education

INTRODUCTION

Trigonometry materials are vital in everyday applications such as statistics, surveying, architecture, and other engineering fields (Weber, 2005a). However, building knowledge about trigonometry is challenging (Blackett & Tall, 1991; Demir et al., 2012; Kendal & Stacey, 1997; Ngu & Phan, 2020; Sugianto et al., 2023). To understand the concept of trigonometry, students need to learn several prerequisite mathematical concepts that serve as a foundation. These

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prerequisite concepts include the Pythagorean Theorem and the properties of right triangles, which are essential in understanding trigonometric ratios. Kamber's Research (2017) stated that students struggle to grasp trigonometry through the unit circle (Kamber & Takaci, 2017). Additionally, they face difficulties distinguishing between trigonometry in right triangles, the unit circle, and the relationship between the two representations (Maknun et al., 2019). As a result, many students fail to understand trigonometric ratios properly, even though these ratios are fundamental in learning trigonometry (Weber, 2005; Nordlander, 2022). Several studies (Hershkowitz et al., 2016; Lessani et al., 2017; Lu'lu'il Maknun et al., 2022; Maphutha et al., 2023; Weber, 2005b) also indicate that traditional approaches lead students to often rely on memorizing formulas rather than understanding the reasoning behind trigonometric ratio concepts.

The findings of previous research provide evidence of students' ability to master trigonometric ratios (Andiani et al, 2024).The test questions given to students are presented in Figure 1.

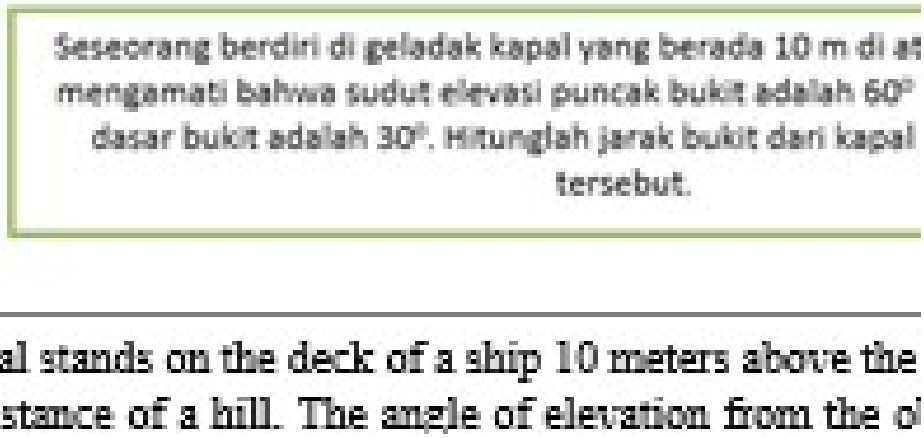


Figure 1. Example of Test Questions in Andiani et al.'s Research (2024)

The results of one student's answer in solving the problem are shown in Figure 2.

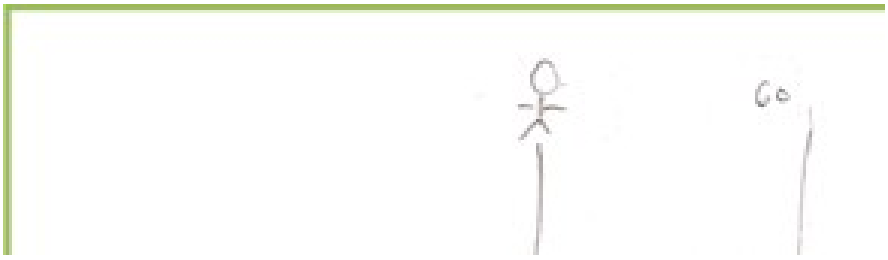


Figure 2. Example of Test Answers in Andiani et al.'s Research (2024)

The results of students' answers show that students are starting to be able to bring up their ideas or concepts in the form of mathematical models. In examining these errors, questions arise:

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a. Do students misunderstand the correct mathematical symbols?
b. Do students lack understanding of trigonometry concepts?
c. Do students misunderstand the meaning of elevation angles?
d. Are the errors influenced by prior knowledge, or do they arise during the learning process?.

Student errors occur when students understand a concept in a particular context and adapt it to be applied in another context (Brousseau, 2002). The procedures that students should have written cannot be skipped because students have difficulty in the initial steps of representing the problem in a model. In this case, students have gone through the horizontal mathematization process but have not gone through the vertical mathematization process in its entirety; This is the most prevalent error among students (Meika, 2018).

Student errors indicate that the students have difficulty solving contextual problems. This finding is in line with Haryani's research that the students still find it difficult to construct an understanding of trigonometry material, especially in the context of story problems or real life (Saputri et al., 2020; Widyanti et al., 2022). In addition, a similar opinion was expressed by Insani and Kadarisma in their research that in studying trigonometry, the students face epistemological obstacles caused by the limitations of their knowledge tied to a particular context. As a result, when faced with different situations, the students struggle to apply or adapt this knowledge to the new context (Insani & Kadarisma, 2020; Nasrulloh & Zahiro, 2023). Therefore, teachers need to know the causes of the difficulties experienced by the students and what the students think in their learning process so that these difficulties occur (Pramesti et al., 2024; Rianasari & Guzon, 2024; Wilson et al., 2013, 2015).

Students' learning trajectories represent their cognitive processes in understanding concepts (Clements, 2011), with each student following a unique path (Chahine & Grinshpon, 2020; Ellis et al., 2014; Tahiri et al., 2017; Zhao et al., 2024). While previous studies have explored learning trajectories for trigonometric ratios (Hamdani et al., 2023; Hidayati & Armiati, 2024; Saputra et al., 2021). In addition, several studies have implemented the Realistic Mathematics Education (RME) approach in trigonometry instruction. However, limited research has explicitly developed a systematic learning trajectory for trigonometric ratios grounded in RME principles. Therefore, this study aims to develop a learning trajectory for trigonometric ratios using the RME approach. The main research question guiding this study is: How can a learning trajectory for trigonometric ratios be designed through the application of Realistic Mathematics Education?

This article is structured into several sections. The next section explores the theoretical framework and the methods employed in the study. Subsequently, the research findings are presented and analyzed in depth. The article concludes by summarizing the key findings and the study's implications.

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THEORETICAL FRAMEWORK

Structured learning trajectories provide a clear pathway for students to develop mathematical understanding. Simon (1993) introduced the concept of Hypothetical Learning Trajectories (HLT), later expanded by Clements (2011) to describe students' cognitive processes in mathematics. This framework consists of guided learning activities that foster conceptual development, helping the students progress through different reasoning levels and achieve specific learning goals.

This study adopted the RME approach to help the students better understand trigonometric ratios. The RME approach aligns with the student's understanding of trigonometric ratios because it emphasizes contextual learning, allowing the students to build concepts based on real-world experiences (Gravemeijer, 1994). Through authentic situations, such as calculating the height of a building with a shadow or determining the elevation angle in navigation, the students can understand the concept of trigonometric ratios more meaningfully than simply memorizing formulas (M. van D. H. Panhuizen, 2003). In addition, RME encourages visual exploration using tools such as triangle diagrams, interactive math software, or physical props so that the students can directly observe the relationship between angles and side lengths in triangles (Freudenthal, 1991). This approach also emphasizes mathematical modelling, where the students learn and apply the theory to solve real problems, improving their conceptual understanding and critical thinking skills (Gravemeijer & Doorman, 1999). Thus, RME enables students to understand how to calculate trigonometric ratios, why these concepts are important, and how to use them in real-world situations.

RME emphasizes mathematization as the main principle to help students understand mathematical concepts gradually, starting from real situations to achieving formal forms. Hans Freudenthal (1991) developed the theory of mathematization in RME by distinguishing two types of mathematization, namely horizontal mathematization, which transforms real-world problems into mathematical models, and vertical mathematization, which processes mathematical models into more formal forms (Freudenthal, 1991).

In addition, the concept of mathematization in RME is also strengthened by Treffers (1987), who emphasized that mathematization is the core of the RME approach, allowing the students to explore and construct mathematical concepts through meaningful activities (Treffers, 1987). Meanwhile, Gravemeijer (1994) emphasized the importance of designing learning trajectories in RME that consider how students develop understanding progressively through the mathematization process (Gravemeijer, 1994).

In the RME approach, horizontal and vertical mathematization support modelling activities into right triangles (Freudenthal, 1991; Treffers, 1987). Horizontal mathematization occurs when the

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students translate real-world situations, such as measuring the length of a shadow and the height of a student and then recognizing the proportional relationship in the situation. Furthermore, through vertical mathematization, the students construct more formal mathematical models, for example, by comparing the height and shadow length ratio using the concept of triangle similarity or trigonometric ratios such as the tangent of an elevation angle.

In addition, the concept of a model as a bridge in RME helps students move from informal understanding to more abstract mathematical models (Gravemeijer, 1994). The students begin learning from real-world situation models, such as shadows formed by light in the classroom, then gradually move to formal mathematical models, such as right triangles and trigonometric ratios. These models serve as a tool to help the students understand how the relationship between elements in a physical system can be represented in a more abstract mathematical form. In this learning, a real-world context is an important starting point to facilitate the students' understanding of mathematical concepts (De Lange, 1996). Measuring the students' heights and shadows allows the students to understand the concepts of triangle similarity and trigonometric ratios more intuitively.

RME is based on the philosophy of Hans Freudenthal, who views mathematics as a human activity involving creativity and problem-solving (Freudenthal, 1973; Gravemeijer & Terwel, 2000; Treffers, 1987; Van den Heuvel-Panhuizen & Drijvers, 2020). This approach allows students to develop mathematical understanding through real-life contexts and experiences (Yuanita et al., 2018). Research by Aiyub et al. (2024) also highlights that learning designs incorporating real-world problems can improve students' algebraic thinking and overall mathematical understanding (Aiyub et al., 2024).

The students build understanding through informal processes. They observe, see, and understand the environment wherever they are (Meika, 2018). Realistic Mathematics Education encompasses three key concepts: directed reinvention and progressive mathematical processes, didactic phenomenology, and model construct development (Gravemeijer, 1994). Within the teaching and learning process, these three concepts are further developed into six distinct features of RME: 1) The principle of activity, 2) The principle of reality, 3) The principle of level, 4) The principle of intertwinement, 5) The principle of interactivity, and 6) The principle of guidance (M. V. den H. Panhuizen & Drijvers, 2014).

Based on Freudenthal's view of mathematics as a human activity, the activity principle emphasizes the student’s active role in learning. The reality principle encourages learning through real-world problems that can be mathematized. The level principle describes the students' progression through different levels of understanding. The interconnection principle highlights integrating various concepts using multiple tools and methods. The interaction principle sees learning as a social and reflective process. Lastly, the guidance principle, rooted in Freudenthal's "guided reinvention," ensures learning aligns with a structured long-term trajectory. (M. V. den H. Panhuizen & Drijvers, 2014).

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The students often have difficulty understanding the concept of trigonometric ratios, especially in relating the lengths of the sides of a triangle to the angles (Maknun et al., 2019). In addition, several studies have shown that they fail to understand the concept of trigonometric ratios even though they have been taught formally (Weber, 2005; Nordlander, 2022). Another difficulty often faced is distinguishing between the right triangle approach and the unit circle (Kamber & Takaci, 2017). Given the importance of understanding this concept, this study aims to develop a learning trajectory that helps students overcome these difficulties, with the RME approach as the main theoretical basis.

METHOD

The methodology employed is design research as described by Gravemeijer and Van Eerde in 2009 (Gravemeijer & Van Eerde, 2009). This research methodology is focused on learning trajectory to enhance the quality of learning by fostering effective collaboration between researchers and teachers (Gravemeijer & Van Eerde, 2009). Design research will enable the attainment of the goals of the research paradigm in the development of sequential activities and the enhancement of learning quality. The primary objective of this work is to develop a comparative trigonometry learning trajectory by utilizing the RME approach. The study was carried out in two cyclical phases, each following the design research stages outlined by Gravemeijer (2004): 1) preliminary design, 2) teaching experiment, and 3) retrospective analysis. The outcomes of the initial phase activities that have been executed and the subsequent analytical review have contributed valuable insights (Gravemeijer, 2004).

In the initial design stage, the researcher observed the school and interviewed teachers and students. Additionally, the researcher collected and analyzed various sources related to the challenges the students and teachers faced in understanding trigonometric ratios and reviewed materials on learning activities based on the RME approach. These analyses served as a reference for developing the Hypothetical Learning Trajectory (dHLT), which includes learning objectives, learning activities, and hypotheses on students' cognitive development (Simon, 1993). These learning hypotheses form the foundation for designing the learning activities developed by the researcher.

Learning objectives	Learning Activities	Student Thinking Conjecture
The students can identify the position of the right angle, opposite side, and hypotenuse in a right triangle.	Stretch the raffia from the table's base to the top of the head to form a right triangle.	A group of students can model the activity into a right triangle. One of the students was able to sketch the formed right triangle.

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Learning objectives	Learning Activities	Student Thinking Conjecture
		The students can identify the position of right angles, front sides, hypotenuses, and side sides in sketches formed with various answers without knowing the reasons. The students need to be corrected in identifying the position of right angles.
The students can apply the definition of trigonometric ratios sin θ, cos θ, and tan θ to contextual problems using the concept of ratios in right triangles.	Measuring students' height and shadow	The students can draw sketches formed from the models presented. The students can use the definitions of the trigonometric ratios tan θ, sin θ, and cos θ to measure the heights of both students and their shadows. The students can determine the position θ of different angles to obtain different trigonometric ratio values.
The students can understand the relationship between similarity and trigonometric ratios.	Experiment with making three right triangles from paper	The students can make three similar right triangles from paper. The students can make sketches from paper formed into three similar right triangles. The students can identify the shortest and longest sides of each triangle formed. The students can determine the ratio between the shortest side and the longest side of each right triangle. The students can relate the concept of similarity to trigonometric ratios.
The students can determine the value of trigonometric ratios at particular angles (0^0, 30^0, 45	Observations on the calendar image in the form of an equilateral triangular prism.	The students can sketch a calendar model as an equilateral triangular prism. The students can construct a right triangle in which one of the angles corresponds to a specifically

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Learning objectives	Learning Activities	Student Thinking Conjecture
^{0}, 60 ^{0}, and 90 ^{0}).		defined angle. The students can determine the value of trigonometric ratios at particular angles (0 ^{0}, 30 ^{0}, 45 ^{0}, 60 ^{0}, and 90 ^{0}).
Applying trigonometric ratios enables students to ascertain a right triangle's unknown length of sides	Bed or bunk bed observation	The students can observe bed or bunk beds by sketching or drawing bed models. The students can identify the position of right angles, oblique angles, side angles, and front side positions in sketches. The students can apply trigonometric ratios to determine the length of unknown sides in problems presented related to beds.
The students can solve contextual problems related to right triangles using trigonometric ratios.	Observing an image of a skier descending an iceberg	The students can identify the position of the angles, opposite sides, and hypotenuses in a right triangle, which is formed as a sketch of a picture of a skier descending an iceberg. The students can apply the definition of trigonometric ratios to skier problems. The students can solve the problems given in the case of skiers.

Table 1: Hypothetical Learning Trajectory in Teaching Experiment 1

Within phase 1, the research participants comprised 25 10th-grade students attending a Bandung City high school. In phase 2, the research participants comprised 25 10th-grade students from a different high school within the same municipal area. The initial implementation of the HLT took place in the first phase of the teaching experiment to assess the efficacy of the prepared learning activity design in investigating students' cognitive patterns.

Moreover, the results of teaching experiment 1 indicated the need to refine the Hypothetical Learning Trajectory (HLT). The enhancement consisted of restructuring the sequence of learning activities, providing more explicit scaffolding in constructing right-angled triangles, and incorporating intermediate modeling tasks to support students' transition from contextual situations to formal trigonometric representations. Additional guiding questions were also

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integrated to facilitate students' vertical mathematization. Furthermore, contextual problems were redesigned to better align with students' prior knowledge and to strengthen the progression from model-of to model-for reasoning. The revised HLT was thus grounded more explicitly in the principles of RME to address the identified difficulties.

First, the instructional design was improved by embedding more meaningful real-world contexts to help the students relate trigonometric concepts to their daily experiences, reinforcing the idea of mathematics as a human activity. Second, scaffolding strategies were strengthened by structuring learning activities progressively, guiding students from informal visual representations (e.g., real-life diagrams and sketches) to more formal mathematical models. Third, interactive learning moments were emphasized by incorporating collaborative discussions and teacher questioning techniques to encourage student reflection and active participation. These refinements maintain the core activities of the previous HLT while improving students' engagement with mathematical representations through a more realistic and interactive learning approach.

The revised Hypothetical Learning Trajectory rHLT was implemented in Phase 2 of the teaching experiment to examine its effectiveness in a different school setting and diverse learning environment. The data-collection protocols included classroom observations, video recordings of learning activities, students' workbooks, and interview recordings with students.

Before implementing the dHLT, a diagnostic test was conducted. The results showed that 47% of students had difficulty constructing right triangles, while 86% experienced difficulties in applying trigonometric ratios in contextual problems. After the intervention, the results of the final test indicated a significant improvement: 83% of students successfully constructed right triangles correctly, and 50% were able to apply trigonometric ratios appropriately. Classroom observations also revealed increased student engagement and improved accuracy in representing mathematical concepts. These results indicate that the revised Hypothetical Learning Trajectory (rHLT), which integrates real-world contexts, scaffolding, and interactive discussions, effectively helps students bridge their understanding from informal reasoning to formal mathematical representations, in line with the principles of RME.

Phase 1 of the retrospective analysis involved collecting and analyzing all research data. The analysis evaluated the outcomes of implementing the developed Hypothetical Learning Trajectory (dHLT) during Teaching Experiment 1. Furthermore, the researcher examined the learning activities to determine the extent of students' understanding of trigonometric ratio concepts. The analysis also evaluated the outcomes of the improvements made to the trajectory and their implementation during Teaching Experiment 2. This analysis compared the dHLT implemented in the first teaching experiment with the revised Hypothetical Learning Trajectory (rHLT) applied in the second teaching experiment.

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The researchers selected six student participants for interviews. These participants represented different levels of mathematical ability: two students with high ability, two with medium ability, and two with low ability. In addition, interviews were conducted with the mathematics teacher who taught trigonometric ratios at the school where the study was conducted.

RESULTS

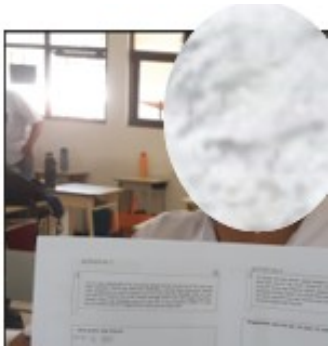
The learning trajectory design discussed in this paper consists of a series of student learning activities intended to support students' understanding of trigonometric ratios through the Realistic Mathematics Education (RME) approach. The developed Hypothetical Learning Trajectory (dHLT) in this study was constructed based on empirical data obtained from classroom observations, student interviews, and worksheet analyses during the learning process. The findings indicate that students experienced gradual progress in understanding trigonometric ratios through a sequence of exploration-based learning activities.

Based on observation data, the students initially experienced difficulty connecting the concept of trigonometric ratios with the right triangles encountered in everyday contexts. However, through a sequence of structured learning activities, their understanding gradually developed from concrete exploration toward a more formal understanding of trigonometric ratios.

Activity 1: Observation of "Stretching the Raffia from the Base of the Table to the Top of the Student's Head"

Activity 1 aims to help the students accurately determine the right angle, opposite side, and hypotenuse in a right triangle. The students, divided into groups of 4–5, use contextual equipment like raffia, a table, and a peer. One of the students holds one end of the raffia while the other is attached to the table base, forming a right triangle with the student's height and the floor. Observations are illustrated in Figure 3. Each group sketches their demonstration (Figure 4), identifying key triangle components. Students explain their reasoning for positioning angle θ and triangle sides. Those struggling receive guided reinvention through teacher-led discussions.





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Figure 3: Activity 1 "Stretching a Raffia from the Base of the Table to the Top of a Student's Head"

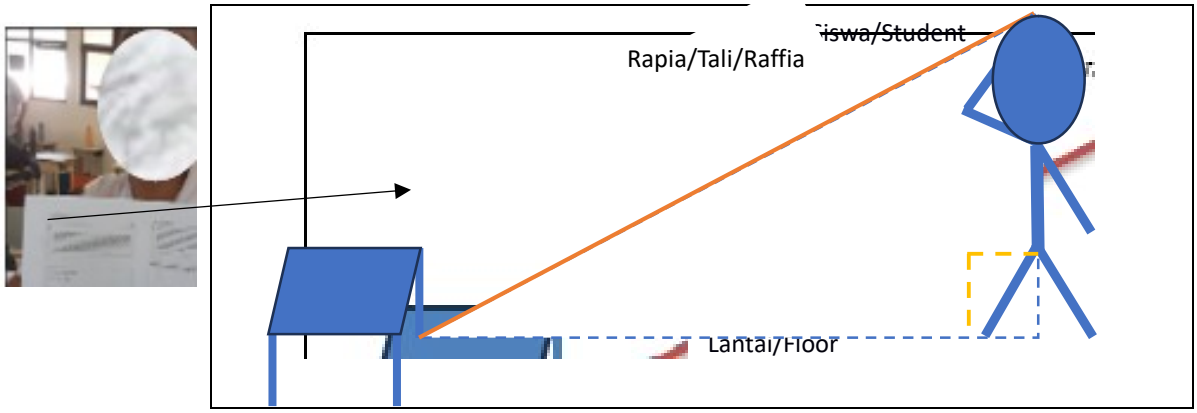


Figure 4: Example of the Student Sketches in Activity 1

Teacher : *"Okay, S-1, please explain what you drew."*

S-1 : *"Okay, ma'am, if you look at the instructions in the worksheet, we are asked to practice what is written in activity 1, and then we share roles; S-2 ties the raffia to the base of the bottom table, S-3 holds the end of the raffia that results from extending S-2's raffia, S-4 reads what is on the worksheet, and I make the sketch."*

Teacher : *"Okay, have you considered whether your sketch will be a right triangle?"*

S-1 : *"At first, I did not, Ma'am, but when you said, 'Look if you draw it, is a right angle formed?" I remembered."*

Teacher : *"Okay, how do you determine the corner points?"*

S-1 : *"The benchmarks are at the end of the table legs, Leg S-3, and head S-3, Ma'am."*

Teacher : *"Why is that, S-1?"*

S-1 : *"Yes, that is what it looks like when drawn, ma'am."*

Teacher : *"Okay, the benchmarks that S-1 mentioned earlier make a move from the rapid raffia, right?"*

S-1 : *"What do you mean, ma'am?"*

Teacher : *"Mom, look, you represent the rapier raffia, the floor, and the height of S-2 with a line, right?"*

S-1 : *"That is right, ma'am."*

Teacher : *"Well, when you make that line, there is a phase where you turn; that is called a*

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r		*benchmark."*
S-1	:	*"Oh yes, that is right, ma'am, that is right."*
Teache r	:	*"S-1, try to determine the right angle, slanted side, front side, and side in the sketch you have made!"*
S-1	:	*"This one is a right angle, this one is the hypotenuse, this one is the front side, and this one is the side side." (pointing to the picture).*
Teache r	:	*"How do you determine a right angle, S-1?"*
S-1	:	*"There is a line like this and like this, ma'am." (pointing at the picture).*
Teache r	:	*"Now, try to observe what degree angle the two lines form, or what angle is it called then?"*
S-1	:	*"Oh yes, ma'am, right angles."*
Teache r	:	*"How do you determine the hypotenuse?"*
S-1	:	*"The side in front of the right angle, ma'am."*
Teache r	:	*"How do you determine the front side?"*
S-1	:	*"This one, ma'am? Yes, the one in front of this corner (pointing to one of the corners of the picture)".*
Teache r	:	*"Okay, how do you determine the side?"*
S-1	:	*"The one on the front side, ma'am."*

In this activity, the students identify the expected position, which depends on the angle θ. This process helps them determine angles and sides in a right triangle. After completing the task, the teacher and the students reflect on positioning the right angle, hypotenuse, and θ, which influence side identification. The raffia stretching activity from the table to a student's head helps visualize the relationship between height and shadow length. Observations indicate that the students begin recognizing this pattern, even without fully grasping the ratio concept, providing a concrete experience before introducing the formal definition.

Activity 2: Observation "Measurement of The Students' Height and Shadow"

Activity 2 aims to help the students apply trigonometric ratios (sin θ, cos θ, and tan θ) in real-life contexts using right triangle comparisons. The students work in groups to transform comparison problems into hands-on tasks. One student stands with their back to a lamp to create a shadow, which the group then sketches (Figure 5). Each member takes turns measuring their height and shadow length, recording the data in a table (Figure 6). The teacher guides the students in determining angle θ and calculating the height-to-shadow length ratio. This activity reinforces the concept of trigonometric ratios through visual and practical learning.

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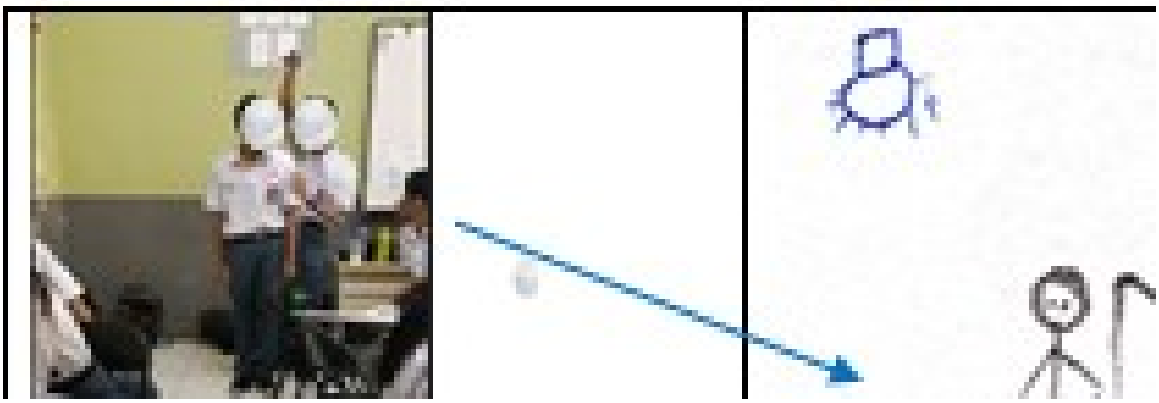


Figure 5: Activity 2 and Example of The Student Sketches in Activity 2

	Kamu	Nama temanmu	Nama temanmu
	Nabila	Bunga	Naufal M
Tinggi badan	155	153	173

	Your name Nabila	Your Classmate's name Bunga	Your Classmate's name Naufal M	Your Classmate's name Jahsyi	Your Classmate's name Nalvin
Height	155	153	173	169	161
Shadow Height	165	163	183	179	171

Figure 6: Example of a table of results measuring height and shadow in Activity 2.

This project aims to help the students accurately calculate sin θ, cos θ, and tan θ in their triangle sketches (Figure 7). The teacher monitors each group, guiding the students in problem-solving while allowing them to use formal or informal methods. Scaffolding is provided to help students recall and apply their knowledge of trigonometric ratios.

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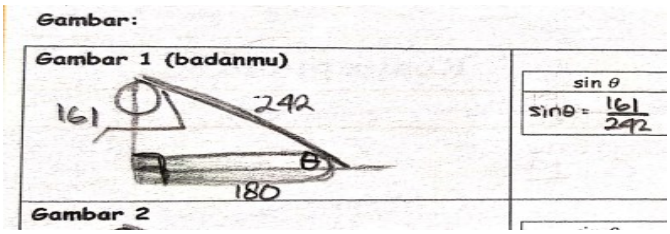


Figure 7: Example of Sketch and Application of Trigonometric Ratios Values in Activity 2

One of the students (S-5), as a representative of one of the group members, participated in presenting activity two and explained how the values of sin θ, cos θ, and tan θ were obtained. After sketching the experiment, this group identified the right angle to determine the other positions. The following is a dialogue explaining the concept of trigonometric ratios described by the students.

Teacher	:	*"Okay, S-5. Explain how your group determines each trigonometric ratio value requested in the table."*
S-5	:	*"Okay, Ma'am, first we sketch according to our height in the group, then, from the sketch formed, we determine the right angle, draw the symbol on the sketch, then determine the angle θ....."*
Teacher	:	*"Okay, do you mean identifying the position of a right angle?*
S-5	:	*"Yes, ma'am."*
Teacher	:	*"What do you mean by determining the angle?"*
S-5	:	*"So we can determine the front and side, ma'am."*
Teacher	:	*"Okay, what is the next process?"*
S-5	:	*"Earlier, after determining the angle position θ, we then wrote down each of our heights according to what was measured earlier and wrote down the height of the shadow."*
Teacher	:	*"Okay, how do you determine the hypotenuse?"*
S-5	:	*"Using the Pythagorean theorem, ma'am, with the help of a calculator."*
Teacher	:	*"Okay, how do you calculate sin θ, cos θ, and tan θ?*
S-5	:	*"Well, because the value of sin θ is the front per hypotenuse, look at the picture (while pointing to picture 1 in UBA 2 Activity 2); the front of the angle $\theta$is the front side, which is 161, and the hypotenuse is 242 so the sin θ is 161 per 242.*

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Teacher : *"How to calculate cos θ and tan θ?*

S-5 : *"Same here, ma'am, just a different position according to what was learned previously: cos θ means side per hypotenuse, and tan θ means front per side."*

The conversation above helped all the students understand the concept of trigonometric ratios correctly. At the end of the activity, one of the students reflected on the lesson and concluded how to apply trigonometric ratios correctly. Measuring the height and shadow length helped the students understand that the ratio between an object's height and its image can be quantified. After multiple measurements, most students recognized a consistent pattern in this relationship, reinforcing the concept of constant ratios as the foundation for sine, cosine, and tangent.

Activity 3: Experiment with "Making Three Right Triangles from Paper"

Exercise 3 aims to help the students understand the relationship between similarity and trigonometric ratios. In groups of 4–5, the students use scissors, paper, rulers, and worksheets to cut rectangular paper into two equal right triangles and identify similarities in three right triangles. They explore similarity using diagrams and concrete teaching aids, as illustrated in Figures 8 and 9, where some groups present findings through diagrams while others use models. The teacher guides the students in recognizing corresponding sides and discovering formal patterns of similarity through directed questions, leading them to apply Geometry theorems by comparing three similar right triangles.



Figure 8. Activities and Examples of the Student Responses using Teaching Aids in Activity 3.

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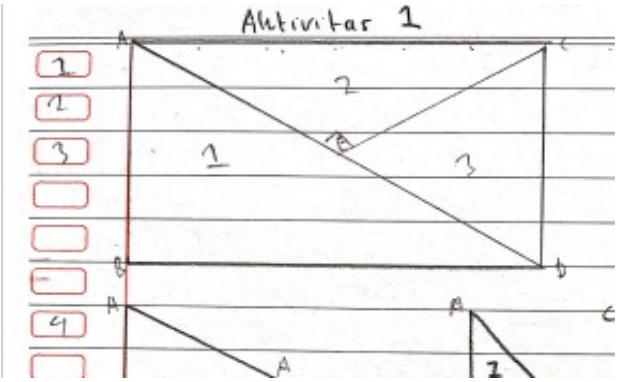


Figure 9. Example of the Student Responses using Diagrams in Activity 3

The teacher guides the students in identifying the shortest and longest sides of each triangle, helping them develop comparative thinking patterns (Figure 10). To deepen understanding, the teacher poses questions that draw analogies between similarity and trigonometric ratios (Figure 9). One example involves a logo on a garbage can placed on an equilateral triangle with a 6 cm side length. The students are asked to calculate the logo's height, sketch the triangle, and explain the concepts used in their solution.

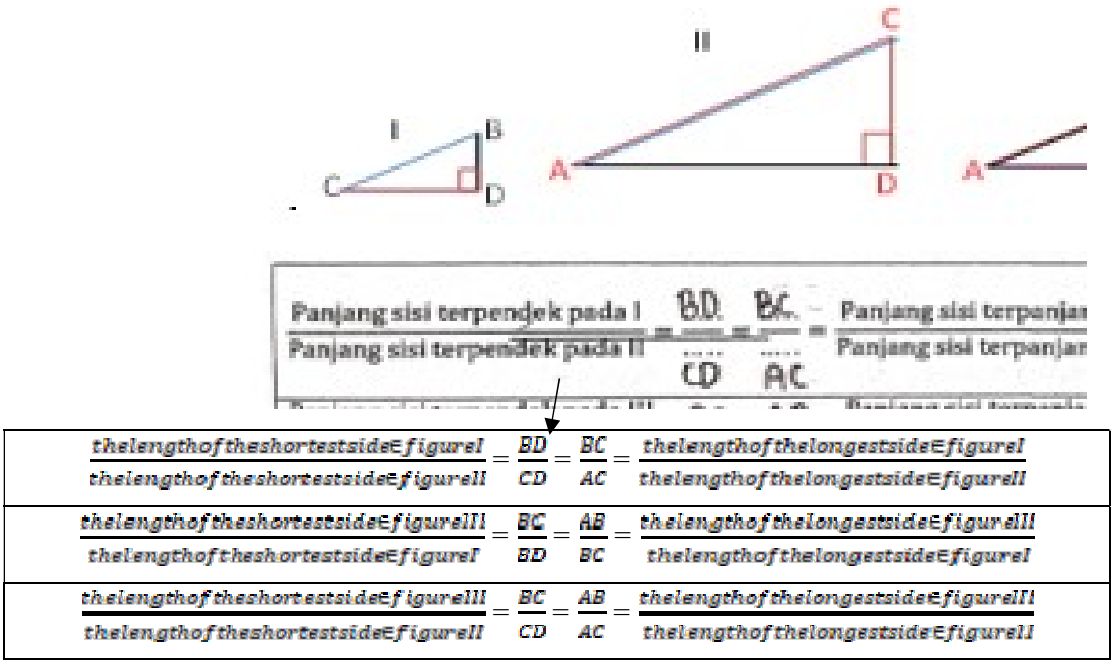


Figure 10. Example of the Student Responses in Using the Concept of Comparison in Activity 3.

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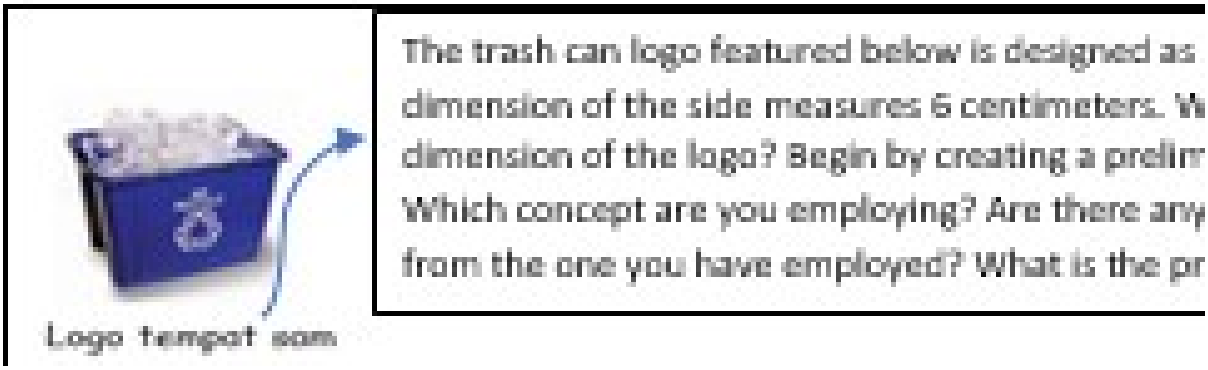


Figure 11. Example of Questions on the Relationship between the Concept of Similarity and Trigonometric Ratios in Activity 3.

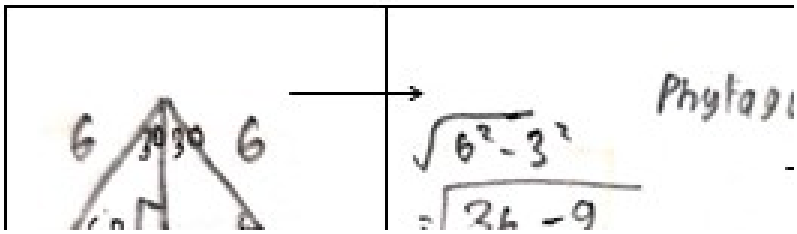


Figure 12. Example of the Student Responses to Contextual Questions in Activity 3.

In Figure 12, a group sketches the shape and calculates the logo's height using the Pythagorean Theorem, then verifies their results through similarity. The teacher and the students discuss the relationship between similarity and trigonometric ratios. By creating three right triangles of different sizes, the students observe that side length ratios remain constant for equal angles. Some believed the ratios would change initially, but experimentation confirmed their invariance, reinforcing a key principle of trigonometric functions.

Activity 4: Observation of "Calendar Image in the form of an Equilateral Triangular Prism"

Activity 4 helps the students determine trigonometric ratios for specific angles (0°, 30°, 45°, 60°, and 90°). Working in groups of 4–5, they analyze a calendar image as an equilateral triangular prism (Figure 13), extract an equilateral triangle, construct a symmetrical one with a 2-unit side, and divide it into two right triangles. Using the Pythagorean Theorem, they calculate unknown side lengths and determine trigonometric ratios for 30° and 60°, compiling their results in a table (Figure 14). For a 45° angle, the students draw an isosceles right triangle with 1-unit sides, calculate the hypotenuse, and determine sin θ, cos θ, and tan θ (Figure 15). By the end of the

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activity, the students summarize their findings in a table, reinforcing their understanding of trigonometric ratios.

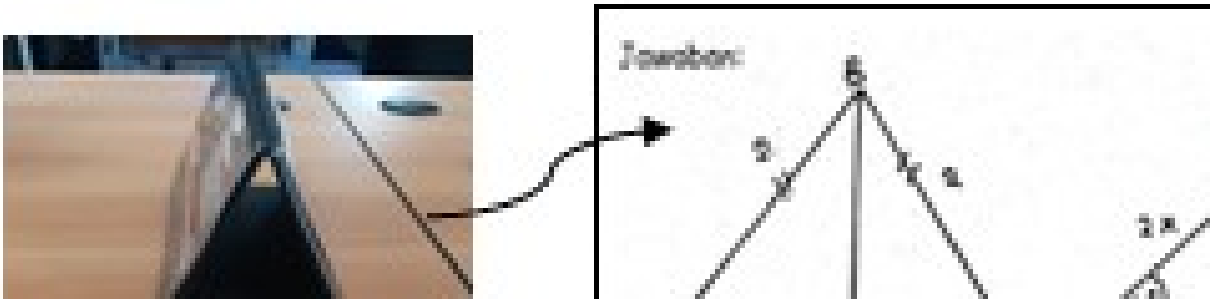


Figure 13. Calendar Image and Example of the Student Responses in Making Sketches in Activity 4

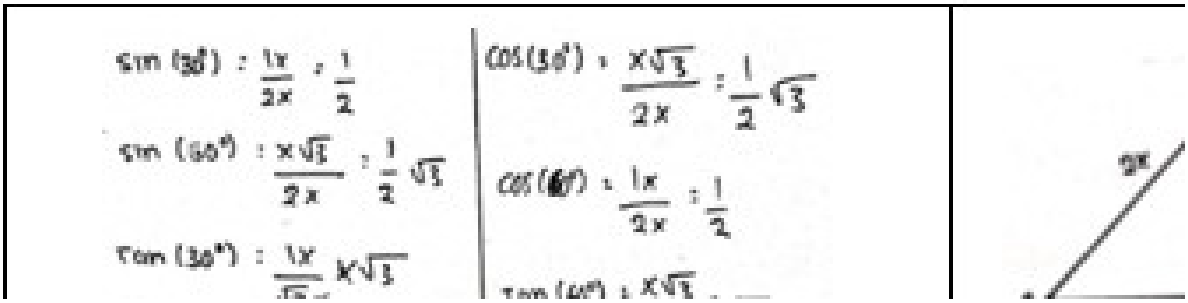


Figure 14. Example of the Student Responses regarding memorable and unique 30^0angles 60^0 in Activity 4

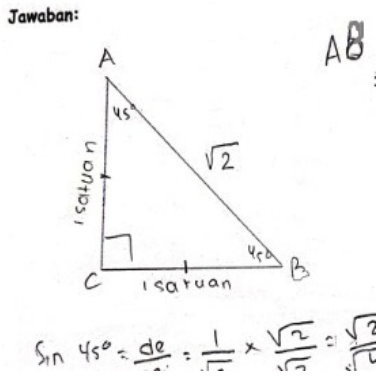
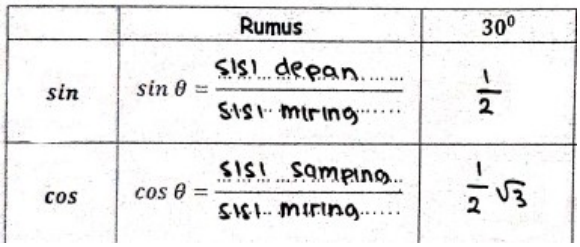


Figure 15. Example of the Student Responses Regarding Particular Angles 45^0 In Activity 4

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	Rumus	30^0
sin	$\sin\theta = \frac{\text{sisi depan}}{\text{sisi miring}}$	$\frac{1}{2}$
cos	$\cos\theta = \frac{\text{sisi samping}}{\text{sisi miring}}$	$\frac{1}{2}\sqrt{3}$

Figure 16. Example of Student Responses Represented in a Table in Activity 4

By the end of the exercise, the teacher and the students had reached a general conclusion on determining the value of trigonometric ratios at particular angles.

Observing the calendar image as an equilateral triangle helps the students recognize right triangles in various visual contexts. This activity increases their ability to identify right triangles in everyday objects, making it easier for them to connect trigonometric ratios to real-world applications.

Activity 5: Observation of "Bunk Bed"

Exercise 5 helps the students use trigonometric ratios to determine unknown side lengths in a right triangle. The students work in groups of 4-5, analyzing an image (Figure 17) and a triangular figure formed by stairs, a bunk bed, and the floor (Figure 18). They then apply horizontal mathematization by creating a diagram representing the triangle. The students identify whether the shape is a right triangle and describe the given data in a sketch. Using the tangent ratio, they calculate the displacement between the floor and the end of the stairs.



Figure 17. Activities and Questions in Activity 5

Each group's findings from activity 5 are illustrated in Figure 18.

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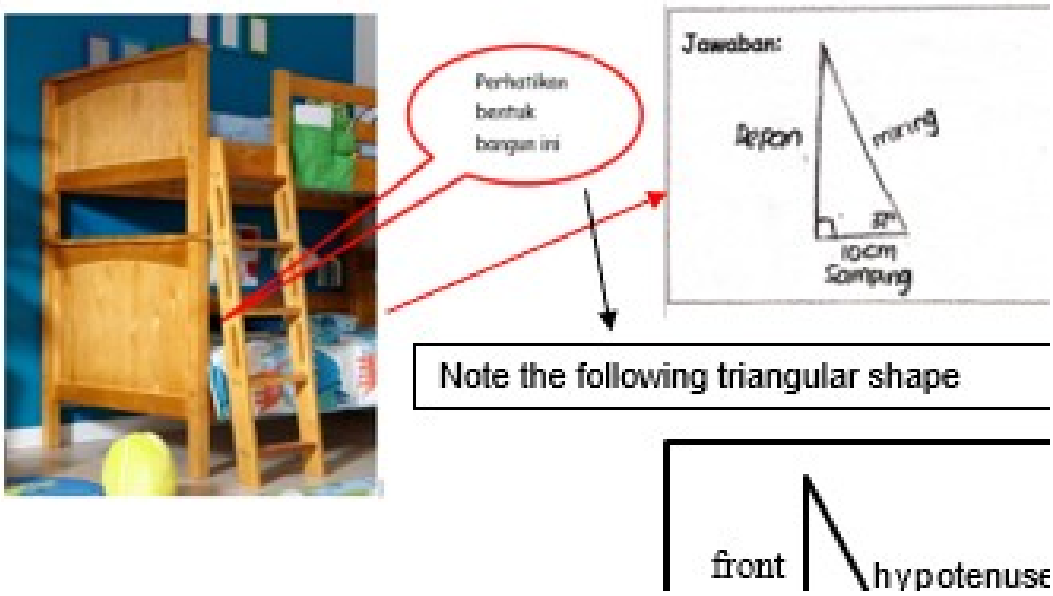


Figure 18. Students' sketches illustrating the results of Activity 5.

With teacher guidance, the students sketch a right triangle from the bunk bed, floor, and stairs, identifying the right angle and using the given values (37° and 10 cm) to calculate the height with the tangent ratio. After solving the problems, one group presents their results, explaining their understanding of trigonometric ratios in determining unknown side lengths. The teacher then asks questions to assess the student's comprehension, as shown in the dialogue below.

Teacher	:	*"Okay, how do you find the length of the unknown side in a right triangle?"*
S-6	:	*"Begin by sketching the problem, identifying given information such as side lengths (opposite, hypotenuse, or adjacent) and known angles. Then, analyze the available data to determine the appropriate trigonometric ratio to apply."*
Teacher	:	*"Can you give an example?"*
S-6	:	*"In the bunk bed problem, after sketching a right triangle, we label the given values: a 37° angle between the stairs and floor and a 10 cm distance from the bunk bed to the stairs. Using the tangent ratio, we calculate the unknown side length, which relates the opposite and adjacent sides."*

Observing the bed frame helps the students see how trigonometric ratios apply in construction. Those who initially struggled with trigonometry begin to connect triangle-side ratios to real-world structures. This activity reinforces the practical significance of trigonometric ratios in design and construction.

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Activity 6: Observation of "Picture of a Skier Descending an Iceberg"

Task 6 helps the students apply trigonometric ratios to real-world problems by analyzing worksheet images (Figure 19) and solving open-ended questions in groups. Referring to Larson et al. (2007), they calculate a skier's track length on an iceberg (Larson et al., 2007), and one group presents their findings to conclude the learning outcomes (Figure 20).



Figure 19. Questions in Activity 6

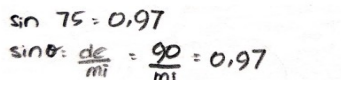


Figure 20. Example of the Student Responses to Activity 6

Observing an image of a skier descending an iceberg introduces the concept of slope about terrain and movement. The students connect slope angles to right triangle ratios, enhancing their understanding of the tangent function. This activity reinforces that trigonometry applies not only to static shapes but also to dynamic phenomena.

DISCUSSION

Learning activities using the Realistic Mathematics Education approach regarding trigonometric ratios followed by the students consist of six activities, namely raffia stretching activities, the student height measurement activities, activities to make three right triangles from paper, activities to observe calendar images, activities to observe bunk bed images, and activities to observe skier images. Each activity has its purpose and is interrelated and continuous. The goals of each action are outlined below: Firstly, stretching the raffia allows the students to determine the location of the angles and sides of a right triangle. Secondly, measuring the height of the students and their shadows allows the students to apply the concept of trigonometric ratios. 3) Creating a teaching tool including three right triangles is intended to facilitate the student's comprehension of the correlation between resemblance and trigonometric ratios. 4) Observing the

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calendar image permits the students to ascertain the numerical value of trigonometric ratios at specific angles. The objective of viewing the bunk bed image is to help the students apply trigonometric ratios to ascertain the length of the desired side in a right triangle. Similarly, observing the skier image seeks to empower the students in solving contextual difficulties by employing the concept of trigonometric ratios. Every worksheet provided incorporates the specific features of the Realistic Mathematics Education pedagogy about trigonometric ratios. These findings corroborate Subekti's assertion that student worksheets will enhance learning (Subekti & Prahmana, 2021).

Applying a realistic approach to teaching mathematics in various activities in the learning process should facilitate the student's understanding of mathematical skills and allow teachers to help the students effectively achieve learning goals (Ekowati & Nenohai, 2017). Students' understanding improves because the process of learning mathematics that follows Realistic Mathematics Education utilizes the environment around the world to encourage students to produce ideas and develop their knowledge (Nirawati et al., 2021).

According to research done by Sukendra (2020), a series of learning process activities with the RME approach can significantly facilitate students' understanding of the trigonometric comparison concept (Sukendra, 2020). Therefore, deliberate learning paths encourage students to build their knowledge, improve their current skills, and effectively overcome challenges in many subjects (Prahmana et al., 2017; Nursyahidah et al., 2020; Wijaya et al., 2021; Kuncoro et al., 2023).

CONCLUSIONS

This study aims to develop a learning trajectory for trigonometric ratios using the Realistic Mathematics Education (RME) approach. Prior to the intervention, the results of the diagnostic test indicated that many students experienced difficulties in understanding trigonometric ratio concepts, particularly in constructing right triangles and applying trigonometric ratios in contextual problems. The findings of this study indicate that the implementation of the revised Hypothetical Learning Trajectory (rHLT) based on the RME approach can improve students' mathematical proficiency in understanding trigonometric ratio concepts.

The learning trajectory designed in this study consists of six learning activities. Through a sequence of continuous learning activities, students gradually develop their conceptual understanding of trigonometric ratios. Their understanding develops from identifying the positions of angles and sides, applying the definition of trigonometric ratios in contextual situations, understanding the relationship between similarity and trigonometric ratios, determining the values of trigonometric ratios at specific angles, applying trigonometric ratios, and solving contextual problems involving trigonometric ratios.

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Empowering Primary School Student's Numeracy Skills Through Augmented Reality: The Didactical Technology Design

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Abstract: This study aims to enhance primary school students' numeracy skills through the Didactical Technology Design (DTD) framework integrated with Augmented Reality (AR) technology. It focuses on identifying learning obstacles, developing and validating AR-based instructional media, and evaluating their effectiveness through teaching experiments. A total of 301 participants, including teachers and students, were involved across three phases: Preliminary, Development, and Teaching Experiment. Data collection utilized a numeracy test, analyzed through descriptive and parametric tests. Results showed significant improvement in numeracy skills, with a mean score of 69.68, highlighting the potential of DTD with AR in fostering interactive, effective learning environments.

Keywords: early years; learning trajectory; mathematics; visualization

INTRODUCTION

The rapid development of technology in education has transformed the way knowledge is delivered, particularly in mathematics education (McKnight, 2020). One promising innovation is Augmented Reality (AR), which enhances learning by overlaying digital information onto the real world (Barteit, 2021; Elmqaddem, 2019). AR offers an interactive and immersive experience, bridging the gap between abstract mathematical concepts and their real-world applications, ultimately improving educational outcomes in primary mathematics (Zhengtao & Hidayat, 2025)(Zhengtao & Hidayat, 2025). For primary school students, who often struggle with numeracy, AR can provide visual and hands-on experiences that facilitate understanding (Kobayashi, 2018). In Indonesia, however, the integration of AR in primary mathematics education remains limited, despite its potential to address persistent learning challenges (Maulyda et al., 2024). This context underscores the need for a didactic approach that leverages AR to enhance numeracy skills.

Despite the growing recognition of AR's potential, many primary school students in Indonesia face significant barriers to mastering numeracy (Perso, 2006; Tout, 2020). These in-

clude limited exposure to engaging and contextual learning media, as well as instructional methods that fail to cater to diverse learning needs. Traditional teaching methods, often focused on rote memorization, struggle to foster deep conceptual understanding (Purpura et al., 2011). As a result, students frequently encounter difficulties in applying mathematical knowledge to real-life problems. This issue is compounded by learning obstacles such as low motivation and limited spatial reasoning abilities (Lusardi, 2012; Tariq, 2014). Addressing these challenges requires innovative approaches that make learning mathematics more engaging and effective, particularly for younger learners.

A review of the existing literature reveals a gap in the integration of AR technology within a structured didactical framework (Kobayashi, 2018; Lai, 2019; Rau, 2018). Most studies on AR in education focus on its technical implementation or its general impact on student engagement and motivation (Kiv, 2020; Molnár, 2018; Rossano, 2020). However, few studies have examined its potential within a systematic model that combines technology with didactic principles tailored to numeracy development (Klimova, 2018; Sorko, 2019). Additionally, while some research explores AR's role in secondary education, its application in primary school settings remains underexplored. This gap highlights the need for a model that not only incorporates AR but also aligns with the specific pedagogical and cognitive needs of primary school students.

This study introduces the Didactical Technology Design (DTD), a novel model that integrates AR into mathematics education through a structured, three-phase approach: Preliminary Phase, Development Phase, and Teaching Experiment Phase. The DTD model combines elements of Design-Based Research (DBR) (Nieveen et al., 2013), Technology Integration Planning (TIP) (Roblyer & Doering, 2014), and ASSURE (Smaldino et al., 2013) to create an iterative, evidence-based framework. This combination ensures that AR is not only effectively integrated but also adapted to address specific learning obstacles and enhance students' numeracy skills. The novelty of this research lies in its didactical focus, offering a new lens for leveraging AR to foster mathematical understanding among primary school students.

The primary aim of this study is to empower primary school students' numeracy skills through the implementation of the DTD model with AR technology. Specifically, this research seeks to: (1) identify learning obstacles and product needs in numeracy education, (2) develop and validate AR-based instructional media within the DTD framework, and (3) evaluate the effectiveness of these media in improving numeracy skills through teaching experiments. By addressing the identified gaps, this study aims to contribute to the growing body of knowledge on the didactical use of AR in education, providing practical insights for educators and policymakers.

LITERATURE REVIEW

Integer Operation Knowledge

Understanding integer operations (Integer Operation Knowledge) is an essential aspect of mathematics education in primary schools. According to de Walle et al. (2016), learning integers requires an approach that simultaneously involves both conceptual and procedural

understanding. Students must be able to comprehend the meaning of addition, subtraction, multiplication, and division of integers in real-world contexts before mastering formal algorithms. This approach aims to build mathematical reasoning skills, which involves linking real-world experiences with formal mathematical concepts.

Creating mathematical models actually originates from real-life situations and contexts (Mulbar et al., 2023). In the context of Realistic Mathematics Education (RME), the process of mathematical modeling plays a crucial role in helping students understand integer operations. According to (Heuvel-Panhuizen, 1996; Oonk et al., 2019), RME emphasizes the use of contextual problems that can be bridged through two stages of mathematization: horizontal and vertical mathematization. Horizontal mathematization involves transforming real-life situations into mathematical models, while vertical mathematization focuses on refining these models into formal procedures (Hidayati et al., 2020). For example, to understand the concept of adding negative integers, teachers may use contexts such as temperature or depth below sea level.

This process can be applied, for instance, by presenting a problem like: "The initial temperature in city A is -3°C. Afterward, the temperature increases by 5°C. What is the current temperature?" Students first translate the situation into the mathematical equation $(-3) + 5$ through horizontal mathematization. Then, they solve the equation using informal strategies before eventually mastering the formal algorithm, as part of vertical mathematization. Students' understanding of negative integers is very important as a basis for understanding broader mathematical concepts (Zuhriawan et al., 2024). Through this approach, students not only gain an intuitive understanding of integers but also develop the ability to apply these concepts in various contexts, strengthening their mathematical thinking skills. Thus it is expected that students are able to solve problems because, problem solving is one of the important activities in learning mathematics (Kamariah et al., 2023).

Augmented Reality Facilitation

Augmented Reality (AR) has emerged as a promising technological innovation in education, particularly at the primary school level. AR allows for the integration of digital elements into the real-world environment, creating an interactive and immersive learning experience. According McKnight (2020), AR combines the real and virtual worlds in real-time, enabling students to interact with abstract concepts through three-dimensional visualizations. The use of augmented reality-based interactive media is one of the answers to the implementation of a non-discriminatory information and communication technology-based learning system (Buliali et al., 2022). In the context of primary education, AR provides opportunities to develop experiential learning, where students can gain a deeper understanding of mathematical concepts through concrete and interactive visual representations (Jesionkowska, 2020; Molnár, 2018; Nechypurenko, 2018).

One of the key contributions of AR in mathematics education at the primary level is its role in the process of mathematization. According to Rossano (2020), mathematization consists of two types: horizontal and vertical mathematization. AR can facilitate both of these

through interactive learning scenarios. For instance, in the context of measuring area and volume, students can use AR applications to visualize how various three-dimensional geometric shapes can be filled with specific measurement units (Rashevska, 2020). In horizontal mathematization, students solve real-world problems, such as calculating the volume of water in a virtual tank. Then, through vertical mathematization, they are guided to formulate mathematical equations based on their observations, such as calculating volume using the formula $V=l \times w \times h$.

The implementation of AR also aligns with constructivist theory, which emphasizes active learning. Piaget stated that students build knowledge through direct experience, and AR provides a platform that supports this process (Slavin, 2018). By actively engaging students in exploring and manipulating virtual objects, AR not only enhances student engagement but also helps reinforce their conceptual understanding. Thus, AR becomes an effective facilitator in supporting meaningful mathematical learning at the primary school level.

Didactical Technology Design

Didactical Technology Design (DTD) is a development model designed to integrate technology into didactic learning. This model aims to create an interactive and effective learning environment (milieu) through the use of technology, particularly in mathematics education. DTD is based on a combination of three key frameworks: Design-Based Research (DBR) from Nieveen et al. (2013), Technology Integration Planning (TIP) from Roblyer & Doering (2014), and ASSURE from Smaldino et al. (2013). By combining the principles from these three models, DTD provides a structured approach to designing and implementing learning solutions that are oriented toward the didactic needs of students.

Several recent studies have begun to adopt and refine the DTD model to address various educational contexts. For instance, Kerr (2020) applied DTD to develop augmented reality (AR) media for elementary mathematics, showing improvements in students' conceptual understanding and engagement. Similarly, Ahsani et al. (2022) implemented DTD in developing digital flipbooks for numeracy learning, which enhanced both teachers' instructional quality and students' learning motivation. In another study, Firdaus et al. (2024) utilized DTD to design culturally responsive digital modules for multicultural classrooms, highlighting its adaptability in diverse learning environments. These findings support the practicality and effectiveness of DTD, reinforcing its potential to meet varied didactic needs.

DTD consists of three main phases. The Preliminary Phase is the initial stage where learning obstacles and product development needs are identified. This process refers to DBR, which emphasizes in-depth exploration of learning issues in real-world contexts (Nieveen et al., 2013). Additionally, TIP is used to plan the integration of technology that aligns with learning needs (Roblyer & Doering, 2013). The results of this analysis become the foundation for developing products that are pedagogically and didactically relevant. Past research

(Bunari et al., 2024; Firdaus et al., 2024) demonstrates how thorough needs analysis and contextual understanding at this stage contribute to creating targeted learning solutions.

The next stage, the Development Phase, involves the development of learning media using an iterative approach based on the ASSURE framework (Smaldino et al., 2013). Studies by Fahmi et al. (2019) and Kerr (2020) highlight the importance of continuous feedback loops and iterative improvements, which lead to more refined and user-centered educational products.

The final stage, the Teaching Experiment Phase, aims to evaluate the effectiveness of the developed product in real learning contexts. This phase also serves as an empirical validation process, ensuring that the learning media is not only effective but also adaptable to various learning situations. Previous studies utilizing DTD (Abdullah et al., 2022; Bunari et al., 2024) have shown positive outcomes in both student achievement and teacher satisfaction, underscoring the model's value in practical educational settings.

By integrating DBR, TIP, and ASSURE, DTD offers a comprehensive and didactic solution for developing technology-based learning media. Its use in recent studies highlights its growing relevance and ability to address contemporary educational challenges. This model is expected to contribute to improving the quality of education through an evidence-based approach.

Conjecture of Study

This study hypothesizes that integrating Augmented Reality (AR) into mathematics learning through Didactical Technology Design (DTD) can significantly enhance primary school students' numeracy skills. AR, with its interactive and visual elements, enriches learning experiences by facilitating students' understanding of abstract mathematical concepts and boosting active engagement. AR enables direct virtual interaction with mathematical objects such as 3D shapes and arithmetic operations, thus supporting the development of numeracy skills through horizontal and vertical mathematization.

The DTD framework ensures a systematic design and evaluation process, incorporating Preliminary, Development, and Teaching Experiment Phases. This approach guarantees that AR-based learning tools are both pedagogically sound and effective. The hypothesis suggests that AR, when designed within the DTD model, enhances numeracy by promoting interaction, visualization, and conceptual understanding, ultimately improving students' mathematical competencies in a structured and impactful manner.

METHODS

Research Design

The research employs a mixed-method approach using the Didactical Technology Design (DTD) framework to enhance primary students' numeracy skills through Augmented Reality (AR). As explained earlier, this research took place in three stages: Introduction, Development, and Teaching Experiment. This design ensures AR media aligns with didactic

goals, fostering interactive and effective learning environments while empirically validating its impact on student numeracy. The details of DTD framework are in figure 1.

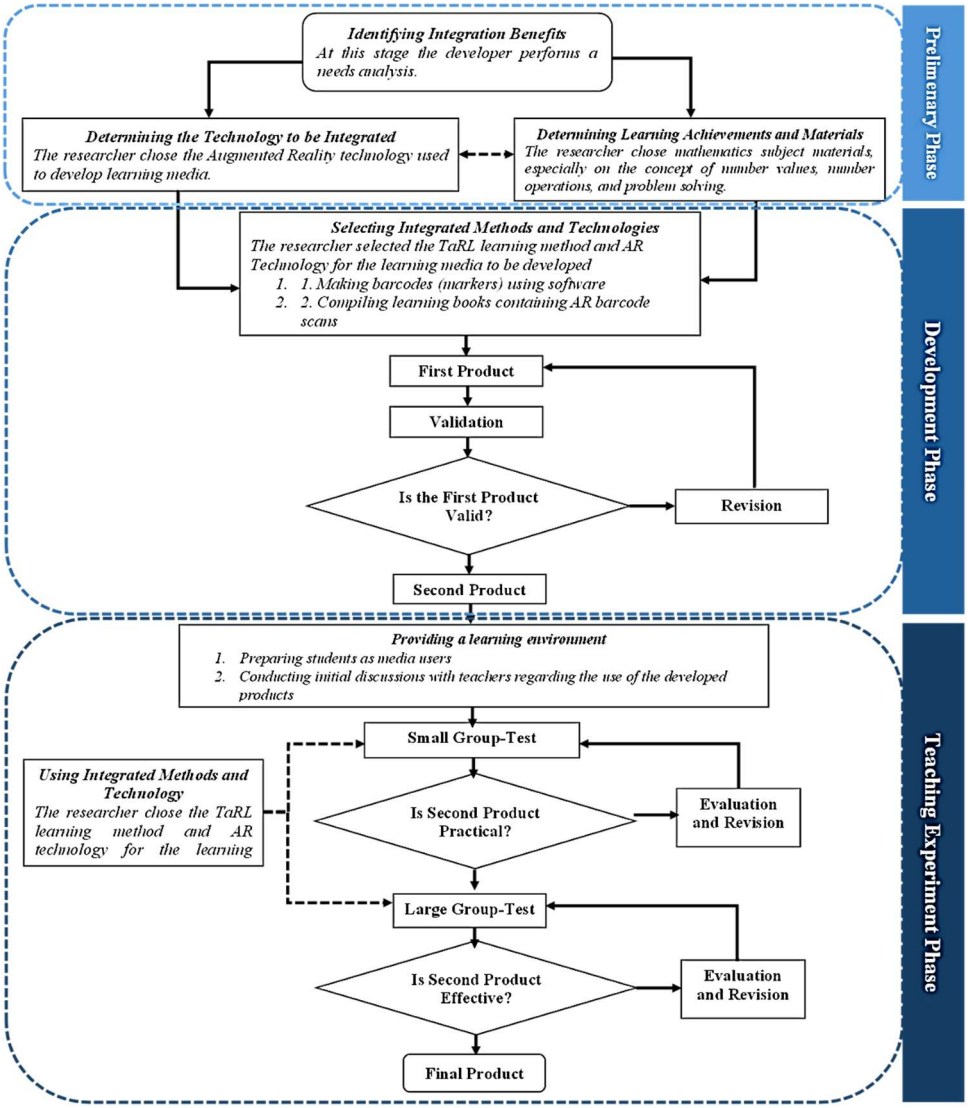


Figure 1: The Dicactical Technology Design Frameworks

Participant

This study involved a total of 301 participants, comprising both teachers and students, across different phases of the research. During the Preliminary Phase, 26 students from SDN Kraton were interviewed to analyze the learning obstacles they encountered. Concurrently, 213 teachers were surveyed using questionnaires to gather data on the perceived needs for educational products that could address these learning challenges. Following the analysis and development stages in the Development Phase, the study proceeded to the Teaching Experiment Phase, where the developed product was tested on 62 students. These

participants were drawn from two different schools: 28 students from Kleco Public Elementary School (SDN) and 34 students from Baluwarti Public Elementary School (SDN). The demographic data of the students involved in the Teaching Experiment Phase is detailed in Table 1. This structured participant recruitment ensured comprehensive data collection, aligning with the study's aim of addressing learning obstacles through the Didactical Technology Design framework.

Aspect	Count (%)
Gender	
Male	25 (40.3)
Female	37 (59.7)
Age	
11 years old	42 (67.7)
12 years old	20 (32.3)
Geographical Location	
Rural	34 (54.8)
Urban	28 (45.2)

Table 1: Demographic data of participant in teaching experiment

The demographic distribution of the 62 students involved in the Teaching Experiment Phase is summarized as follows. In terms of gender, the sample comprised 25 male students (40.3%) and 37 female students (59.7%). Regarding age, the majority of participants were 11 years old (42 students, 67.7%), while the remaining 20 students (32.3%) were 12 years old. Geographically, 34 students (54.8%) resided in rural areas, whereas 28 students (45.2%) were from urban areas. This diverse demographic composition provided a balanced representation of students across different backgrounds, ensuring the generalizability of the study's findings.

Measurement and Data Collection

Learning Obstacle Interviews

The data collection process began with sequential face-to-face interviews conducted separately with teachers and students. These interviews aimed to gather in-depth and accurate information directly from the research subjects. The primary focus of the student interviews was to identify specific learning obstacles they encountered, particularly in understanding mathematical concepts. The interview instrument was adapted from Carvalho et al. (2004). It consists of 12 in-depth questions designed to explore the learning obstacles faced by students. This instrument has a reliability of 0.78, indicating its effectiveness in gathering data. The details of the instrument are presented in Table 2 below.

No	Dimensions	Items	Example item wording
1	Student	1-7	Explain if there are any topics you haven't understood so far.
2	Teacher	8-10	What teaching style do you like from your teacher?
3	Milieu	11-12	Have you used a gadget? If so, have you used it for learning mathematics?

Table 2: Learning Obstacle Interviews

Need Assesment Survey

This study utilized a survey questionnaire to assess elementary school teachers' digital competencies, covering the 27 item in three dimensions: Pedagogical Knowledge, Didactic Skills, and Teacher Attitudes. The instrument, based on indicators by Torres-Hernández & Gallego-Arrufat (2022), was validated by education and technology experts. Reliability tests were conducted using Pearson Product Moment and Cronbach's Alpha via IBM SPSS 25. The reliability test yielded a Cronbach's Alpha of 0.893, indicating high reliability. The reliable items were subsequently used with the primary study respondents, ensuring robust data collection to evaluate teachers' digital competencies. The details of the instrument can be found in Table 3 below.

No	Dimensions	Items	Example item wording
1	Pedagogical Knowledge	1-9	Can you display learning videos using an LCD?
2	Didactic Skills	10-17	Have you ever accessed the Quizizz website or other learning evaluation platforms?
3	Teacher's Attitude	18-27	Have you ever heard the term digital security?

Table 3: Need Assessment Survey

Numeracy Test

The numeracy test developed for this study consists of several key dimensions, aligned with the standards from PISA and AKM, as well as mathematical knowledge for teaching (Ball et al., 2008). The test includes a total of 20 items, distributed across different content and context areas. The content areas include Geometry, Measurement, Numbers and Algebra, and Data and Uncertainty. The context areas are Socio-Cultural, Personal, and Scientific, along with a Teaching Work dimension. The instrument has been tested for reliability, yielding a result of 0.83, indicating very high reliability. The details of the instrument can be found in Table 4 below.

No	Dimensions	Items	Example item wording
1	Socio-Cultural	1, 2, 4, 10, 13, 15	The Sekatenan celebration is a traditional festival regularly held at the Yogyakarta town square. This year, the head of Baleharjo Village intends to participate in the festival. One of the required items is 500 traditional cakes. The price of one box containing 72 traditional cakes is Rp. 199,500. How many boxes of traditional cakes need to be purchased, and what is the total cost?
2	Personal	3, 5, 8, 9, 11, 12, 14	Andi lives on the coast of Depok Beach, Yogyakarta. During his work holiday, he plans to travel to Mount Merapi. He departs from Depok Village to Pakem Village, located at the foot of Mount Merapi, using a private car. The following map shows the distance, route, and time required to reach Pakem Village. Andi departs from Depok Village at 8:30 AM. After driving for 3 hours and 10 minutes, he stops to rest at Ambarukmo Mall. After resting for 45 minutes, Andi continues his journey to Pakem Village. What time does Andi arrive at Pakem Village?
3	Scientific	6, 7, 20	The World Health Organization (WHO) recommends limiting sodium intake to 2,400 mg, or about one teaspoon of salt per day. The likelihood of sodium deficiency for those living in Indonesia is very low because most foods in the country are high in sodium content. Even diet menus are high in sodium, such as the South Beach Diet, which includes 2,300 to 6,700 mg of sodium per day. Foods high in sodium, such as chips, canned soups, processed meats, salted fish, soy sauce, ketchup, chicken bouillon cubes, and cheese, should be avoided or minimized as they can cause high blood pressure (hypertension), which can trigger heart disease, stroke, and kidney problems.
4	Teaching Work	16, 17, 18, 19	An elementary school teacher is teaching multiplication. The students provide various answers. As a teacher, it is essential to under-

stand the students' thinking. Below are some of their work results.

Siswa A	Siswa B	Siswa C
45 x 15 75 +60 675	45 x 15 225 +450 675	45 x 15 25 200 50 +400 675

Based on the students' answers above, which student correctly applied the method for multiplying two numbers? Provide an explanation!
Mark an (x) on the correct answers for each method.

Table 4: Numeracy Test

Development Phase

Workshop on Didactic Design Development

The development of Augmented Reality (AR) media to address students' learning obstacles began with a workshop on refining the didactic design for numeracy. This workshop aimed to improve learning tools that focus on reducing gaps in students' numeracy understanding. The activity involved elementary school teachers as active partners in discussions and Q&A sessions. The development team, consisting of the researchers (authors) themselves and the teachers as practitioners, facilitated the workshop. The researchers played a key role in presenting the initial AR-based didactic design, moderating discussions, and documenting feedback. Teachers, as participants, provided constructive feedback and raised relevant questions, contributing valuable insights to the refinement of the instructional materials. This collaborative approach ensured that both theoretical and practical perspectives were integrated into the development process. The outcomes of these discussions were summarized to refine the didactic design and to assign teachers the task of studying the Student Worksheet (LKPD, which stands for Lembar Kerja Peserta Didik in Indonesian) and AR media in preparation for the next session. This participatory workshop ensured that the resulting learning tools were practical and aligned with classroom needs.

In the second phase, a simulation workshop was conducted to test the implementation of the AR-assisted didactic design in a classroom-like setting. Teachers were given the opportunity to practice the design, while the development team conducted simulations and provided guidance. This simulation aimed not only to validate the learning tools but also to identify potential improvements through hands-on experience. This phase reinforced the synergy between didactic theory and classroom practice, ensuring the tools' applicability and effectiveness in real teaching scenarios.

FGD with Experts and Practitioners

Following the workshops, the development process continued with a Focus Group Discussion (FGD) involving education experts and practitioners, specifically elementary school teachers. This stage aimed to obtain a comprehensive validation of the AR-assisted numeracy didactic design. The FGD began with a presentation of the workshop and simulation results, followed by an in-depth discussion on the effectiveness of AR media in addressing learning obstacles. Experts provided feedback based on didactic and pedagogical theories, while teachers shared their empirical experiences and potential challenges in classroom implementation.

The FGD yielded a multidimensional evaluation of the AR media, covering aspects such as design, numeracy content, and interactivity. Input from experts ensured the tools aligned with constructivist learning principles, while feedback from teachers enhanced the tools' applicability in real classroom contexts. Discussions also included developing adaptive learning scenarios to meet the needs of students with varying numeracy skills. The development team recorded all feedback to refine the LKPD and AR media, integrating it into the final product.

The combination of workshops and FGD ensured that the development process was collaborative, and evidence based. By involving diverse stakeholders, the resulting AR media effectively addressed learning obstacles, enhanced student engagement, and supported more meaningful numeracy learning.

Data Analysis

The data analysis process begins with the use of descriptive analysis techniques to identify and summarize data related to learning obstacles and needs assessment. Descriptive analysis aims to describe the general conditions of the observed variables, such as frequency, percentage, mean, and data distribution. In this context, data on learning obstacles and students' needs are analyzed to identify common patterns or issues faced by students during the learning process. The results of this descriptive analysis provide an initial overview of the challenges faced by students and areas that require further attention in designing more effective learning interventions.

Following the descriptive analysis, normality testing is conducted to assess whether the data follows a normal distribution, which is a prerequisite for conducting parametric tests. This step ensures that the data meets the assumptions necessary for valid statistical analysis. If the data is found to be normally distributed, parametric tests such as the One-Sample T-Test can be applied. This test compares the meaning of the teaching experiment data with the theoretical or expected average, allowing the identification of whether there is a significant difference between the experimental condition and the standard or expected condition. In addition, covariate analysis is performed to control for potential confounding variables that may affect the results, ensuring that the effects observed are attributed to the intervention itself.

These steps, normality testing and covariate analysis are crucial in ensuring the robustness and validity of the statistical findings, and they strengthen the interpretation of the results. By integrating these analyses, researchers can confidently draw conclusions about whether the teaching method applied in the experiment has had a significant impact on improving learning outcomes and whether the changes observed can be considered statistically strong enough.

RESULTS

Product Validation Phase

To validate the content of the AR-based instructional materials, the product was assessed by three experts in education and numeracy. The experts reviewed the relevance, accuracy, and clarity of the AR media and accompanying materials. The validation process was quantified using Aiken's V coefficient, a statistical measure to assess content validity based on expert ratings. Aiken's V ranges from 0 to 1, where a value closer to 1 indicates higher validity. The experts provided their ratings for each item on a scale from 1 (not valid) to 4 (highly valid), and Aiken's V was calculated for each item of the AR media and worksheets. The final content validity for the entire product was determined by calculating the average Aiken's V score across all items, which yielded a score of 0.87, indicating that the content was highly valid for educational purposes.

For construct validity, the AR-based instructional materials were analyzed to ensure they effectively measure the intended construct numeracy skills and engagement with AR tools. A factor analysis was conducted to assess the underlying structure of the materials and to determine if the items loaded onto the expected factors. The results of the factor analysis indicated that all items of the AR media and worksheets were strongly associated with the construction of numeracy learning, with factor loadings ranging from 0.65 to 0.85. This shows that the items contributed significantly to the construction being measured. Additionally, model fit indices were assessed to evaluate the overall construct validity. The indices indicated a good fit for the model, with a Comparative Fit Index (CFI) of 0.91, a Tucker-Lewis Index (TLI) of 0.90, and a Root Mean Square Error of Approximation (RMSEA) of 0.048, all of which suggest that the materials effectively measure the intended constructs and are suitable for use in classroom settings.

During the development of the AR-based instructional materials, several iterations were conducted to refine the content, design, and functionality. Feedback from expert panels and FGD sessions played a critical role in shaping the final materials. Based on expert input, the language and examples used in the materials were refined for clarity and relevance, resulting in a higher Aiken's V score. Additionally, based on feedback regarding the AR design, interactivity was enhanced, and visual elements were added to ensure better engagement and usability for students. The final materials, therefore, reflect both the content and design improvements made in response to the feedback, ensuring that the AR-based materials are pedagogically sound, engaging, and aligned with the intended educational goals.

Prelimenary Phase

In this study, the interviews conducted with 26 students were part of a broader data collection process aimed at understanding learning obstacles. However, the responses from only three students (S1, S2, and S3) are presented in this section due to their representative nature in highlighting the key challenges encountered by students in mathematics. These students were selected because their individual struggles with multiplication exemplified distinct levels of difficulty, ranging from memorization issues to fundamental misconceptions, thus providing a comprehensive view of the variety of obstacles faced. The focus on these three students allows for a detailed exploration of their specific challenges and the types of interventions needed, which directly informs the design of educational products to address similar issues among other students in the sample.

The interviews with three students, identified as S1, S2, and S3, revealed various learning obstacles in mathematics, particularly in multiplication. S1, despite understanding the concept of multiplication, struggled with memorizing multiplication tables from 1 to 9 and often made errors due to inaccuracy in calculations. She noted that practice exercises were insufficient, although she felt comfortable with story problems and preferred when teachers provided initial explanations before problem-solving. Conversely, S2, who had memorized multiplication tables up to 10, encountered difficulties in multi-digit multiplication, especially involving hundreds and thousands. S2 often made errors in carrying numbers due to limited practice in this type of problem. Although familiar with story problems, the limited exposure to them affected her ability to master this skill.

Meanwhile, S3 exhibited more fundamental learning obstacles, particularly in understanding basic mathematical operations. She often confused addition and multiplication and experienced misconceptions in basic operations. Her scores were generally low, except in topics that were repeatedly practiced. Moreover, S3 was unfamiliar with the concept of story problems and struggled to understand and solve them unless the teacher read them aloud. This highlights her limited exposure to and understanding of different types of mathematical problems.

The findings indicate that these students require more intensive and varied instructional approaches. S1 and S2 need additional practice, particularly in story problems and multi-digit multiplication. S3, on the other hand, requires foundational support in understanding basic mathematical operations and a gradual introduction to story problems. These insights emphasize the crucial role of teachers in providing consistent practice, leveraging engaging instructional media, and fostering a supportive learning environment, particularly for topics perceived as difficult or uninteresting.

Teaching Experiment Phases

Descriptive statistics aim to provide an initial overview of the numeracy data. Based on the analysis, from a total of 62 samples, the minimum value is 60, and the maximum value is 85, with a mean of 69.68. The standard deviation of 8.818 indicates a degree of variation in the numeracy scores. This interpretation suggests that, on average, numeracy scores fall

within the medium range, with a relatively varied data distribution. The results of the descriptive statistics analysis are presented in Table 5.

Descriptive Statistics					
	N	**Minimum**	**Maximum**	**Mean**	**Std. Deviation**
Numeracy	62	60	85	69.68	8.818
Valid N (listwise)	62				

Table 5: Descriptive Test

Subsequently, the researcher conducted a normality test to examine the distribution of the data. This step is crucial to determine the appropriate hypothesis test to be employed in addressing the research hypothesis. The results of the normality test are presented in Table 6 below.

Tests of Normality						
	Kolmogorov-Smirnov[a]			Shapiro-Wilk		
	Statistic	df	Sig.	Statistic	df	Sig.
Numeracy	.251	62	.248	.848	62	.098

a. Lilliefors Significance Correction

Table 6: Normality Test

The normality test using Kolmogorov-Smirnov and Shapiro-Wilk aimed to determine whether the numeracy data followed a normal distribution. Based on the Kolmogorov-Smirnov test, the significance value was 0.248, while the Shapiro-Wilk test showed a significance value of 0.098. Since the significance values of both tests exceed 0.05, it can be concluded that the numeracy data is normally distributed. Given the normal distribution, the researcher proceeded with parametric testing. Accordingly, a One-Sample Test was conducted. The results of this test are presented in Table 7 below.

One-Sample Test						
	Test Value = 0					
	t	df	Sig. (2-tailed)	Mean Difference	95% Confidence Interval of the Difference	
					Lower	Upper

Numeracy	62.216	61	.000	69.677	67.44	71.92

Table 7: One-Sample Test

The hypothesis test was conducted to determine whether the mean numeracy score differs significantly from zero. The analysis results show a t-value of 62.216 with 61 degrees of freedom (df) and a significance level of 0.000. Since the significance value is less than 0.05, the null hypothesis is rejected, indicating that the mean numeracy score (69.68) is statistically significantly different from zero. The 95% confidence interval for the mean numeracy score ranges from 67.44 to 71.92. To further explore these findings, the researcher examined other variables that might influence students' numeracy performance. The results of the covariate analysis are presented in Table 8 below.

Tests of Between-Subjects Effects

Dependent Variable: Numeracy

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	3400.701[a]	3	1133.567	48.961	.000
Intercept	10407.440	1	10407.440	449.516	.000
GND	2251.307	1	2251.307	97.238	.000
AGE	65.089	1	65.089	2.811	.099
GEO	242.667	1	242.667	10.481	.002
Error	1342.847	58	23.153		
Total	305750.000	62			
Corrected Total	4743.548	61			

a. R Squared = .717 (Adjusted R Squared = .702)

Table 8: Covariate Test

The covariate analysis aimed to assess the influence of the independent variables (GND, AGE, and GEO) on numeracy, while accounting for the covariate effect. The analysis results indicate a significant corrected model (F = 48.961, Sig. = 0.000) with an Adjusted R Squared of 0.702, suggesting that 70.2% of the variance in numeracy can be explained by this model. Both GND (F = 97.238, Sig. = 0.000) and GEO (F = 10.481, Sig. = 0.002) have a significant effect on numeracy. However, AGE (F = 2.811, Sig. = 0.099) does not show a significant effect. This interpreta-

tion indicates that gender and geographical location significantly influence numeracy, whereas age does not have a significant impact.

DISCUSSION

Product Validation

The results of the content and construct validity assessments for the AR-based instructional. The results of the content and construct validity assessments for the AR-based instructional materials demonstrate that the developed tools are highly valid and effectively aligned with their intended educational purpose. The content validity, as indicated by an Aiken's V score of 0.87, suggests that the materials were considered highly relevant, accurate, and appropriate by the expert panel. The strong content and construct validity of the AR-based instructional materials developed in this study are consistent with findings from previous research that underscores the importance of expert validation and rigorous assessment in educational material development. According to Ariffin et al. (2010), expert validation plays a crucial role in ensuring that instructional materials are not only accurate but also contextually appropriate for the intended educational settings. The high Aiken's V score of 0.87, which improved to 0.90 after refinements based on feedback, aligns with this notion, emphasizing that expert feedback is pivotal in enhancing the relevance and clarity of instructional content. This expert validation process ensures that the materials align with pedagogical goals and can meet the specific needs of students. Furthermore, the expert panel's positive assessment of the AR design further supports the findings of Nieveen et al. (2013), who argued that the integration of engaging technology, such as AR, can significantly enhance numeracy instruction by fostering interactive and hands-on learning experiences.

Moreover, to content validity, the construct validity results reinforce the effectiveness of the AR-based materials in measuring and improving numeracy skills, as well as fostering student engagement. The high factor loadings and strong model fit indices (CFI = 0.91, TLI = 0.90, RMSEA = 0.048) support the notion presented by Hair et al. (2019) that technology-enhanced learning tools can play a significant role in supporting constructivist approaches to education. The refined AR materials, which emphasize gamification and interactive elements, align with Slavin's argument that such tools can promote deeper engagement and improve cognitive outcomes. The details of validation and FGD are shown in table 9.

Validated Aspect	Validity Score	Improvements Made	FGD Feedback	Final Evaluation Score
Content	Aiken's V = 0.87	Refinement of language and example problems	The materials were found to be highly relevant and easy to understand	Aiken's V = 0.90
AR Design	-	Added interactivity and visualizations	FGD appreciated the use of AR in numeracy materials but suggested adjustments to the interface controls	Score 4.5/5

Student Engagement	-	Strengthened gamification elements to increase motivation	FGD showed that students were more engaged with more interactive elements	Score 4.8/5

Table 9: Validation and FGD Summary

These findings suggest that the AR-based instructional materials developed in this study show considerable potential in addressing common numeracy learning challenges faced by students. Previous research has demonstrated that technology-enhanced learning tools, such as augmented reality (AR), can offer innovative solutions to improving numeracy skills by engaging students in interactive learning experiences (Nieveen et al., 2013). The materials developed in this study are designed with pedagogical best practices in mind, ensuring that they not only enhance numeracy understanding but also foster deeper cognitive engagement. This approach aligns with the constructivist learning theory, which emphasizes active learning through interaction with technology to build a more meaningful understanding of complex concepts (Purpura et al., 2011). By incorporating interactive features and adaptive learning elements, the AR materials provide an immersive environment that supports students in mastering numeracy concepts at their own pace.

Moreover, the alignment of both content and construct validity with established educational frameworks further strengthens the potential impact of the AR materials on student learning. The expert validation process confirmed that the content is relevant and accurate, while the construct validity analysis demonstrated that the AR-based materials effectively measure numeracy skills and student engagement. These results are consistent with studies by Ariffin et al. (2010) and Sari et al. (2024), which emphasize the importance of expert feedback and validation in ensuring that instructional materials are not only pedagogically sound but also effective in achieving desired educational outcomes. As the use of AR technology in education continues to grow, these findings suggest that the developed materials could offer valuable contributions to improving numeracy education and student engagement in the classroom.

Thinking Trajectories

Thinking trajectories describe the cognitive steps students take as they develop their numeracy skills. In the context of Augmented Reality (AR), these trajectories are dynamic and interactive, providing students with opportunities to explore, manipulate, and reflect on mathematical concepts through direct engagement with virtual environments (Szilágyi et al., 2013). This approach aligns with Sztajn et al. (2016) theory of teaching and learning trajectories, which helps educators understand the typical developmental stages of students' mathematical reasoning.

For instance, in our study, when students engaged with Augmented Reality (AR) tools to learn number operations such as addition and subtraction, they first interacted with visual representations that allowed them to manipulate objects directly on their devices. The results indicated that students in the AR group demonstrated a significant improvement in their ability to understand

and perform these operations compared to those in the traditional learning group. The concrete level of interaction provided by AR allowed students to visualize the mathematical concepts in a dynamic way, enhancing their comprehension and retention of the material. Initially, students might use virtual manipulatives, such as blocks or counters, to represent numbers and perform basic operations. This hands-on approach mirrors the concrete operational stage described by Piaget, where children understand mathematical concepts through tangible experiences (Slavin, 2018).

Moreover, the thinking trajectory has several stages (Panhuizen, 2019). At the first stage, students use AR tools to visualize and manipulate objects. For example, in an addition task, the AR app might display 5 virtual cubes and ask students to add 3 more. The student interacts with the AR by virtually adding 3 cubes to the existing 5, seeing the total number increase to 8. This operation is grounded in the real-world experience of counting physical objects, which aids students in forming the concept of addition.

In mathematical notation, this stage can be represented as:

$$5+3=8$$

This stage is foundational as students connect the abstract symbol "+" with the physical act of combining quantities. Once students are comfortable with the concrete representation, the AR environment may prompt them to focus on the abstract aspects of number operations. The AR system might hide the virtual cubes and present the equation 5+3=? on the screen. Here, students must rely on their previous experiences with physical manipulatives to understand and solve the problem. In this stage, students begin to make connections between the physical action of counting and the symbolic representation of numbers and operations. This stage represents the transition from concrete manipulatives to abstract mathematical reasoning. The equation becomes a symbol-based problem rather than a physical one, leading to a deeper understanding of numerical relationships.

Next, students begin to fully understand and apply mathematical symbols and operations independently of the physical objects. In AR, students may interact with virtual number lines or calculators that display abstract problems like 8-3=?. The use of symbols such as "+" and "=" is now internalized, and students can perform operations without needing the visual aid of objects. For example, in a subtraction task, students may see the equation:

$$8-3=5$$

Here, the student no longer needs to manipulate virtual cubes but instead relies on their mental model of the operation. This shift represents the formalization of mathematical thinking, where the student can now use symbolic operations fluidly and with confidence.

At the final stage, students begin to reflect on their own thinking and justify their solutions. Using AR, students might be asked to explain why 8-3=5 or to explore different methods of solving the problem, such as counting backward on a number line or using the inverse relationship between addition and subtraction. This stage promotes higher-order thinking, where students not only perform operations but also understand the reasoning behind them. For example, students might explain: "To find 8-3, I know that 3 is the same as taking away 3 from 8, which leaves 5.

This is the opposite of adding 3 to 5 to make 8." The ability to articulate mathematical reasoning indicates a mature understanding of both the symbols and the underlying concepts.

Through AR, students' thinking trajectories evolve from concrete manipulation of objects to abstract mathematical reasoning as they internalize the symbols and operations that define numeracy. These stages demonstrate how AR not only supports cognitive development but also aligns with Piaget's stages of cognitive development and Vygotsky's socio-constructivist theory, where social interactions and tools such as AR mediate learning and help students move from concrete experiences to abstract concepts (Glassman, 2001; Liu & Matthews, 2005). These findings have significant implications for real-world educational practices, particularly in designing numeracy instruction that is contextual and responsive to students' needs.

For educators seeking to implement AR-based numeracy interventions, it is recommended to integrate features such as adaptive problem-solving prompts tailored to students' cognitive development levels. Teachers can also leverage AR to create learning environments that foster collaboration and discussion, thereby strengthening the internalization of mathematical concepts. Future research could further explore how different AR features influence students' progression through these thinking stages and how their implementation can be adapted to diverse classroom needs.

AR for Numeracy

Augmented Reality (AR) has emerged as a transformative tool in mathematics education, particularly in enhancing numeracy skills. AR integrates virtual content with the real world, offering an immersive learning environment that promotes engagement and motivation (Elmqaddem, 2019; McKnight, 2020). In this study, AR was utilized to teach number operations, providing interactive simulations where students could explore and practice addition, subtraction, multiplication, and division.

The hypothesis test confirmed that students' numeracy scores were significantly above the baseline, with a 95% confidence interval between 67.44 and 71.92. These results are consistent with prior research (Bacca, 2019; Hwang et al., 2023), which found that AR fosters active learning by enabling students to visualize abstract concepts in tangible ways. These findings have practical implications for real-world educational settings, particularly in classrooms where engaging students in numeracy learning can be challenging. By incorporating AR, teachers can create a more interactive and student-centered learning environment, which may lead to improved long-term retention and understanding of mathematical concepts.

AR environments encourage active exploration, aligning with the experiential learning model proposed by Kolb (2014). For example, when solving arithmetic problems, students can manipulate virtual counters or number lines, supporting their procedural fluency and conceptual understanding simultaneously. The immediate feedback and gamified elements in AR also cater to intrinsic motivation, as highlighted by self-determination theory (Abramovich et al., 2019; Kobus et al., 2008). Integrating AR within a Desain Teknologi Didaktik (DTD) framework ensures that technology use is aligned with instructional goals. The DTD approach, which combines the TIP and ASSURE models, emphasizes planning, implementation, and evaluation stages that help

teachers guide students to focus on learning objectives rather than merely interacting with technology.

In classroom practice, educators can apply the DTD framework by first identifying learning goals related to numeracy, selecting AR tools that align with these objectives, and designing activities that encourage meaningful student interaction (Rahmatih et al., 2021). For instance, during lessons on number operations, teachers can incorporate AR-based number lines to help students visualize numerical relationships, while structured guidance ensures that technological engagement remains purposeful (Meeks et al., 2014). Future research could explore the longitudinal impacts of AR on numeracy, particularly in varying instructional settings, to provide further insights into its sustained educational benefits.

Age and Geographical Location

The covariate analysis revealed significant effects of gender (GND) and geographical location (GEO) on numeracy scores, while age (AGE) did not emerge as a significant predictor. Specifically, the model's Adjusted R Squared value of 0.702 indicates that 70.2% of the variance in numeracy outcomes can be attributed to the model, underscoring the critical importance of gender and geographical location in explaining differences in numeracy achievement. The significance of geographical location aligns with findings from Schleicher (2019), which highlight disparities in educational resources and opportunities that exist across different regions.

These disparities are often linked to access to high-quality teaching, technological resources, and infrastructure. Students from urban areas, for instance, typically benefit from more advanced educational technologies, such as digital tools and online learning platforms, which can facilitate enhanced numeracy learning experiences. In contrast, students from rural or remote areas may face challenges related to limited access to such resources, which can hinder their learning progress, particularly in the context of AR-based education. The importance of infrastructure and technological literacy in the successful implementation of AR learning is well-documented by Ennis & Chen (1995), Rakhmawati et al. (2024), and Sumardi (2019), who emphasize the need for robust technological infrastructure to ensure that students can fully engage with and benefit from AR-based learning environments.

Conversely, the lack of a significant effect of age on numeracy scores suggests that developmental differences among the sampled students were relatively minimal within the age range represented in this study. This finding is particularly interesting because it contrasts with Piaget's theory of cognitive development, which suggests that cognitive growth and mathematical reasoning are strongly influenced by age-related stages (Slavin, 2018). According to Piaget, students at different ages may exhibit distinct cognitive abilities, which in turn affect their capacity to understand and apply mathematical concepts. However, the current study's results align more closely with research by Allcoat (2021), who found that numeracy skills are more closely linked to the quality of the learning environment and instructional methods than to age itself. This suggests that factors such as teaching strategies, classroom support, and access to educational technologies may play a more significant role in shaping numeracy outcomes than the biological age or developmental stage of students.

The findings further underscore the need for an inclusive and effective learning environment that transcends age-related assumptions about students' capabilities. More importantly, these results point to the crucial role of equitable educational policies that address geographical disparities in access to resources. For instance, the implementation of AR-based learning programs in rural schools could help bridge the gap in numeracy achievement between urban and rural students. By providing innovative and interactive learning experiences, AR technology has the potential to offer rural students the same high-quality educational opportunities that their urban peers enjoy, overcoming traditional limitations such as lack of access to specialized teaching resources. Moving forward, future studies should focus on examining how regional policies and improvements in infrastructure can positively influence the efficacy and sustainability of AR-based numeracy programs. Investigating the relationship between these factors would provide valuable insights into how technology can be leveraged to create more equitable educational systems that benefit all students, regardless of their geographical location.

CONCLUSIONS

Based on the results of this study, it can be concluded that the use of Augmented Reality (AR) in numeracy learning has a significant impact on improving students' numeracy skills. Descriptive analysis shows that the average numeracy score of students is 69.68 with a standard deviation of 8.818, indicating a reasonable variation in the data distribution, but generally falling within the medium range. The normality test results demonstrate that the numeracy data follows a normal distribution, which allows for the use of parametric testing. The hypothesis test shows that the mean numeracy score is significantly different from zero, with a t-value of 62.216 and a significance level of 0.000, indicating that AR-based numeracy learning has a positive effect. Additionally, the covariate analysis reveals that gender and geographical location significantly influence students' numeracy abilities, while age does not have a significant effect.

The impact of using AR in numeracy learning is highly positive, as this technology offers a more interactive and visual approach for students to understand abstract concepts like arithmetic operations. Through AR, students can see and manipulate virtual objects that help them conceptualize mathematical operations in a way that is easier to understand. This is consistent with previous findings that suggest technology-based learning can enhance students' comprehension of mathematical content and strengthen their numeracy skills. AR also supports the development of students' mathematical thinking through more structured thinking stages, from manipulating concrete objects to understanding more abstract mathematical symbols.

However, while AR has proven effective in improving numeracy skills, this study also indicates that factors such as gender and geographical location can affect student learning outcomes. Therefore, it is recommended that educators consider these factors when designing AR-based learning experiences to ensure inclusivity and accommodate the diverse needs of students. Furthermore, further research is needed to explore the influence of age and how AR implementation can be tailored to students' age groups to maximize their learning outcomes.

Overall, the use of AR in numeracy education holds great potential to enrich students' learning experiences and improve their learning outcomes. Therefore, it is recommended that education

be further integrated with AR technology, particularly in mathematics, to create a more engaging and effective learning environment.

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Integrating Realistic Mathematics Education Approach into Instructional Websites to Improve Students' Computational Thinking Skills in Mathematics

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Abstract: Many studies have tried to improve students' computational thinking (CT) skills in mathematics learning. However, there are still gaps in improving students' CT skills, especially in integrating the realistic mathematics education (RME) approach into instructional websites. To overcome this gap, this study aims to analyze the effectiveness of applying RME-integrated instructional websites (RME-Web) in improving students' CT skills in mathematics learning. To address the study's purpose, the researchers compared the RME-Web approach with the RME and direct-learning (DL) approaches. The research method is a quantitative quasi-experimental design with a non-equivalent control group, involving up to 90 junior high school students. The data analysis techniques used are the One-Way ANOVA, Tukey HSD, and N-Gain score. The research results show that the RME-Web approach improves students' CT skills more effectively than the other two approaches. The research findings are expected to be useful for practitioners, stakeholders, and academics in designing and implementing innovative learning approaches that integrate technology and contextual pedagogy to improve students' computational thinking skills in mathematics.

Keywords: realistic mathematics education, instructional websites, computational thinking, mathematics, ANOVA

INTRODUCTION

In the 21st century, education is increasingly focused on developing students' higher-order thinking skills, including computational thinking (CT). The term CT first appeared in the context of *LOGO* programming developed by Seymour Papert in the 1980s (Papert, 1980). This idea was later developed and expanded by Jeannette Wing in 2006, who emphasized that CT is not only a technical skill limited to computer science but also a way of thinking that can be applied in various disciplines, including mathematics (Wing, 2008, 2011). These skills include data collection, data

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analysis, data representation, decomposition, abstraction, algorithms, automation, parallelization, and simulation. However, many studies show that students' CT skills are still low (Angraini et al., 2024; Ersozlu et al., 2023; Valovičová et al., 2020). Based on previous research, students' low CT skills are caused by several primary factors, such as a lack of exploratory and problem-solving-based learning, a lack of use of media or technology, and limitations in designing learning contextual and related to students' real lives. Therefore, one approach that can be used to overcome this problem is realistic mathematics education (RME). In RME, mathematical concepts are constructed by exploring real-world situations that are meaningful to students. This approach emphasizes the process of horizontal and vertical mathematization, where students first translate contextual problems into mathematical models before developing a more abstract understanding.

Furthermore, to support technological developments in the 21st century, RME can be integrated through various mathematics learning media to increase student engagement, clarify concepts, and support the mathematization process (Lisnani et al., 2023; Nurlisna et al., 2020). Various studies show that integrating RME into learning technology can help students improve their CT skills (Abdul Hanid et al., 2022; Abdullah et al., 2019; Andriyani, 2023; Payadnya et al., 2023; Triswidrananta et al., 2024; Zafrullah et al., 2023). However, does the same apply to RME integration into instructional websites? To address this question, the present study compares three groups: (1) students learning through instructional websites integrated with RME (RME-Web group), (2) students learning through face-to-face RME instruction (RME group), and (3) students learning through direct instruction without contextual exploration or technology integration (DL group). Therefore, this study aims to analyze the effectiveness of these approaches in improving students' CT skills in mathematics learning.

LITERATURE REVIEW

Computational Thinking (CT) in Mathematics

CT is a cognitive process that employs strategies to define a problem and devise its solution in a way that enables the solution to be executed effectively by information-processing agents (Wing, 2011). In addition, Yadav et al. (2014) define CT as cognitive thinking that focuses on abstracting problems and developing solutions that can be automated and computationally executed to address problems effectively. Based on these two definitions, a common thread can be drawn that CT is an analytical and systematic thinking ability that involves formulating solutions that can be implemented effectively to solve problems. According to Weintrop et al. (2016), Angeli et al. (2016), Barr & Stephenson (2011), Lee et al. (2011), and Selby (2013) state that CT includes four main indicators, as presented in Table 1. When students can understand and apply these four components of CT, they can improve their skills in solving mathematical problems effectively and

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efficiently (Cansu & Cansu, 2019; Rodríguez del Rey et al., 2021). In the context of mathematics education, CT is very important because it helps students to see mathematics not only as a series of procedures to find the right answer but as a problem-solving process that involves various steps of critical thinking (Büscher, 2024; Li et al., 2020; Ye et al., 2023). In this sense, problem-solving methodologies aim to develop students' thinking skills to gain a deeper understanding of mathematics and realize its relevance in everyday life.

CT Aspects	Description
Decomposition	Breaking a problem into parts
Pattern Recognition	Analyze data and look for patterns to gain an understanding of the data
Abstraction	Remove unnecessary details and focus on what is important
Algorithm	Create a series of sequential steps to take to solve the problem

Table 1: Computational Thinking Skill Indicators

Realistic Mathematics Education (RME)

RME is an approach that views mathematics as a human activity and must be associated with reality (Gravemeijer, 1994). On the other hand, Wijaya (2012) defines RME as a mathematical learning approach that emphasizes contextual problems as a starting point for learning, where students are invited to discover and build mathematical concepts through exploration and interaction. Based on these two definitions, a common thread can be drawn that RME is a mathematical learning approach oriented to the connection between mathematical concepts and real situations. However, it is essential to understand that the word "realistic" in RME does not necessarily mean the real world but rather a situation that students can imagine and relate to in their experiences (van den Heuvel-Panhuizen, 2001; van den Heuvel-Panhuizen & Drijvers, 2014). According to Wijaya (2012), the meaning of mathematical concepts is the main concept of RME. The concept's meaning will occur when mathematics is presented based on the student's experience (Van Oers, 1998). Therefore, mathematical concepts should be taught gradually, starting from an informal understanding and moving on to the formal aspect. RME provides a framework that helps students understand and apply mathematics more contextually. Based on this, Treffers (2012) formulated five characteristics of RME, namely: (1) The use of context, (2) The use of emergence models, (3) Student contribution, (4) Interactivity, and (5) Intertwinement. Based on these five characteristics, according to Wijaya (2012) realistic contexts or problems are used as a basis for developing mathematical concepts or as a source for learning.

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Instructional Websites in Mathematics Education

An instructional website is a digital platform designed to support learning by providing materials, interactive activities, and evaluations in a structured environment (Harsanto, 2017). In the context of mathematics learning, instructional websites have great potential to accommodate various student needs, including facilitating conceptual understanding through visual media, simulations, and contextual problems (Borba et al., 2016; Nyoto Kurniawan, 2013). The platform allows for flexible learning, where students can access the material at any time and from anywhere and engage in a more personalized and immersive learning experience (Cacheiro-Gonzalez et al., 2019). In addition, the platform can be designed to integrate learning approaches such as RME, which emphasizes the use of contextual problems to build student understanding. Various studies have shown that instructional websites have a positive impact on students' mathematical thinking skills (Buchori et al., 2023; Istiqomarie & Mulyono, 2023; Lisnani et al., 2023; Sosa et al., 2023).

Research Gaps and Novelty

Several previous studies have attempted to improve students' CT skills through the integration of RME into instructional technology, such as augmented-reality (Abdul Hanid et al., 2022), mobile applications (Abdullah et al., 2019), android media (Zafrullah et al., 2023), virtual-reality (Payadnya et al., 2023), and educational games (Andriyani, 2023; Triswidrananta et al., 2024). However, research gaps exist in integrating RME into instructional websites to improve students' CT skills in mathematics learning. Therefore, this study offers novelty in analyzing the effectiveness of RME-integrated instructional websites (RME-Web) in improving students' CT skills. Practitioners, stakeholders, and academics can utilize the significance of this research. Practitioners can utilize real-life context-based learning approaches and technology to improve students' CT skills. At the same time, stakeholders and academics can use these results as a reference in teacher training, curriculum development, and literature on innovative mathematics learning approaches.

METHOD

Research Design

The research method used is a quantitative quasi-experimental type of non-equivalent control group design, comparing three sample groups representing three learning approaches: (1) RME-integrated instructional websites (RME-Web) approach as the experimental group 1; (2) the RME

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approach as the experimental group 2, and (3) the direct-learning (DL) approach as the control group. The research design can be seen in Table 2.

Samples Groups	Pretest	Treatment	Posttest
Experimental 1	O_1	T_1	O_2
Experimental 2	O_3	T_2	O_4
Control	O_5	X	O_6

Table 2: Research Design

Table 2 shows the research design where O_1, O_3, and O_5 represent the pretest for the experiment group 1, experimental 2, and control. Meanwhile, O_2, O_4, and O_6 represent the posttest in the same group. Based on the treatment column, T_1 represents the treatment for experimental group 1 using the RME-Web approach, T_2 represents the treatment for experimental 2 using the RME approach, and X represents the control group using the DL approach.

Participants

This research was carried out in one of the private junior high schools in Yogyakarta, Indonesia, in September-October 2024. The population is all grade VIII students (ages 13-15 years), with as many as eight classes and 209 students. Three classes were randomly selected, and a sample of class VIII G was obtained, with 30 students as experimental group 1, class VIII H with 30 students as experimental group 2, and class VIII F with 30 students as the control group. The selection of schools is based on the availability of internet access and the overall readiness of students to engage in web-based learning media. The students have expressed their willingness to participate voluntarily. All treatment of students is carried out ethically following the guidelines set by the American Psychological Association (American Psychological Association, 1992)

Instruments

The researchers used a data collection technique in the form of a cognitive test on the topic of number patterns. The type of test is a description consisting of five questions. Each question item represents every CT skill. The grid of question items can be seen in Table 3. Pretest and posttest questions are identical to accurately compare students' skills before and after the learning intervention. The test duration of 120 minutes provides enough time for students to read, understand, and solve problems carefully. The total score in the assessment is 100 points, giving

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equal weight to each question, which is 20 points, which are scored based on completeness, accuracy, and clarity of the answers.

Computational Thinking Indicators	Description	Question Items
Decomposition	Students should break down the beats by instrument and analyze where the beats occur and overlap.	(1)
Pattern recognition	▪ Students identify consistent patterns in the packing process (number of pieces per bundle and bundles per basket).	(2)
	▪ Students identify recurring geometric patterns in Batik motifs and how they develop at each level of enlargement.	(5)
Abstraction	Students generalize the ticket price increase pattern and formulate it as a mathematical model	(3)
Algorithm	Students design a step-by-step algorithm to calculate the number of visitors on day 10 based on a growth pattern.	(4)

Table 3: Question Instrument Grid

Data Analysis Techniques

After the test instrument is completed, the next stage is the internal validity test. Internal validity involves the validation of material by three validators using the Likert Scale of four categories: "1" (strongly disagree), "2" (disagree), "3" (agree), and "4" (strongly agree). The validation instrument was analyzed using the content validity ratio (CVR) formula, namely Cohen's Kappa inter-raters (McHugh, 2012), with a score of "0.87", which indicates the level of reliability between strong validators (strong agreement). The validity of the content of each question was tested using the content validity index (CVI) formula, namely Aiken's Coefficient Value (Aiken, 1985), which resulted in a score of "0.88" (question number 1), "0.90" (question number 2), "0.92" (question number 3), "0.94" (question number 4), and "0.96" (question number 5), indicating high validity (very valid) for each question item. Furthermore, a pilot study was conducted to assess the instrument's readability, involving six students with low, medium, and high abilities. The results of the pilot study showed a number error in question 2, which prevented the answer from being searched, as well as difficulty understanding question 3, especially for students with low and medium ability. Based on these findings, revisions were made to improve the instrument's clarity and accuracy. This process ensured that the instrument was both theoretically valid and practically applicable, reducing potential measurement bias.

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After the instrument was declared valid, large-scale data was collected, and the results of the pretest-posttest were tested for normality and homogeneity using RStudio software version 4.3.2 (RStudio Team, 2023) with packages "dplyr" as the assumption of normality (Wickham et al., 2023), "car" as the assumption of homogeneity (Fox & Weisberg, 2019), and "lsr" to determine the eta-squared (Navarro, 2015) with $\alpha = 5\%$. The normality assumption hypothesis is H_0: Normally distributed data and H_1: Data is not distributed normally. Assumption of normality using statistics of the Shapiro-Wilk test where H_0 was rejected if p-value < α. Furthermore, the hypothesis of the homogeneity assumption of variance is H_0: The homogeneity of the variance is the same, and H_1: There is at least a pair of heterogeneous variances that are not the same. Homogeneity assumptions using Levene test statistics where H_0 was rejected if p-value < α. This is in line with Zimmerman & Zumbo (1992) where the criteria for normally distributed data meet the level of homogeneity of variance if the p-value > 0.05.

Furthermore, after confirming that the data is normally distributed and the variance is homogeneous, the data can be analyzed parametrically using the One-Way ANOVA test to see whether there are significant differences in students' average CT skill scores from the three sample groups. If the results of the One-Way ANOVA test are significant, a post hoc test is carried out using the Tukey HSD test introduced by Tukey (1953) to see which approach is better for improving students' CT skills during mathematics learning. Furthermore, to determine the magnitude of the influence of the intervention of the three approaches on the average CT skill score, the effect size formula of the eta-squared was used (η^2). It should be noted that the effect size criteria are based on Abbott (2014) with categories of "0.01" (small), "0.06" (medium), and "0.15" (large). In addition, the N-Gain score is used to measure the effectiveness of implementing each intervention approach. The N-Gain model used is an average of the N-Gain score because the researchers have student response data that can be analyzed directly. The interpretation of the N-Gain score is based on Hake (1999) with the categories: "<40%" (ineffective), "40-55%" (less effective), "56-75%" (quite effective), and ">76%" (effective).

Research Procedure

This study used three sample groups: experimental group 1 with the RME-Web approach, experimental group 2 with the RME approach without learning technology, and the control group with the DL approach. The topic used in this study is number patterns. The RME approach used by the researchers has several syntaxes, namely: (1) Contextual Problem, (2) Contextual Exploration, (3) Concept Exploration, (4) Mathematization, (5) Generalization, and (6) Reflection and Evaluation. The learning syntax of RME-Web can be seen in Figure 5, while RME is in Figure 6. In the RME-Web group, the principles of RME were embedded into the instructional website,

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where contextual problems, exploration, mathematization, and reflection were facilitated through interactive simulations, automated feedback, and online discussions. In the RME group, the same syntaxes were applied in a face-to-face classroom setting, where students engaged directly in contextual exploration and discussions guided by the teacher. In contrast, the DL group followed a conventional teacher-centered approach, focusing on the direct delivery of mathematical concepts and procedures without contextual exploration or technology integration.

To implement the learning activities, both experimental groups engaged in projects based on real-life and cultural contexts related to number patterns. In the RME-Web group, projects such as analyzing the rhythmic structure of *Gending Lancaran*, exploring the folding patterns of *Wiru Jarik*, and identifying Batik motif repetitions were transformed into interactive website content. In the RME group, the same projects were carried out through hands-on activities such as arranging classroom chairs, folding paper, or discussing Batik motifs directly with peers under the teacher's guidance.

Contextual Problem Stage

Experimental group 1 implemented the RME-Web approach, incorporating instructional websites featuring interactive simulations, visualizations, and step-by-step guidance. This group used local cultural contexts from Yogyakarta, Indonesia, such as *Gending Lancaran* (a traditional Javanese song scale) and the *Wiru Jarik* tradition (folding Batik cloth). The research procedure begins with the "contextual problem" stage, where students are given contextual problems related to the number patterns. Experimental group 1 with the RME-Web approach uses the local cultural context of Yogyakarta, where students were presented with digital problems that contextualized number patterns using visual depictions of *Gending Lancaran* scales or animated representations of the *Wiru Jarik* folding process. These problems were carefully designed to align with the RME principles of connecting mathematics to real-world and cultural phenomena. This stage is categorized as an informal or experience-based phase, where students explore problems related to their surroundings, specifically their cultural environment. By engaging with cultural practices such as the *Gending Lancaran* and *Wiru Jarik* traditions, students are encouraged to discover and understand number patterns through their own cultural experiences. Meanwhile, experimental group 2 with the RME approach without learning technology used the context of chairs in the classroom and folding paper, where students were given contextual problems related to number patterns through physical and manipulative activities. In this context, students are asked to arrange chairs in the classroom following a particular pattern or fold the paper in different shapes to understand the number of patterns formed. The control group applied the DL approach, where the

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teacher provided direct explanations without incorporating cultural context or learning technology. This approach introduced students to number patterns through traditional textbook problems.

Contextual Exploration Stage

In the "contextual exploration" stage, experimental group 1 explored the problem using available media or tools. By interacting with simulations depicting the *Gending Lancaran* scale and visualizing the folding process of Batik cloth, students could see real-time changes and patterns, providing immediate feedback on their exploration. For example, see Figure 1, in the notation of the song "*Gugur Gunung*", it can be observed that each time signature (*gatra*) consists of four notes. Thus, in each stanza, there are 16 notes. Generally, a song has a minimum of five stanzas, so the total tone in this song reaches 80 notes. Experimental group 2 explored patterns by manually arranging class seats or folding paper to understand the patterns (see Figure 2). For instance, students would arrange chairs in rows to form a visible pattern and explore how the number of chairs changed based on a particular rule. Similarly, by folding paper, students could see physical transformations mirrored the number patterns concept. This exploration allowed for an experiential learning process where students could apply their observations to formulate hypotheses and test their ideas. The control group only received explanations of the material topic from the teacher without in-depth exploration or special aids. Students worked through practice problems provided by the teacher, focusing on basic problem-solving techniques without any contextual or interactive support.

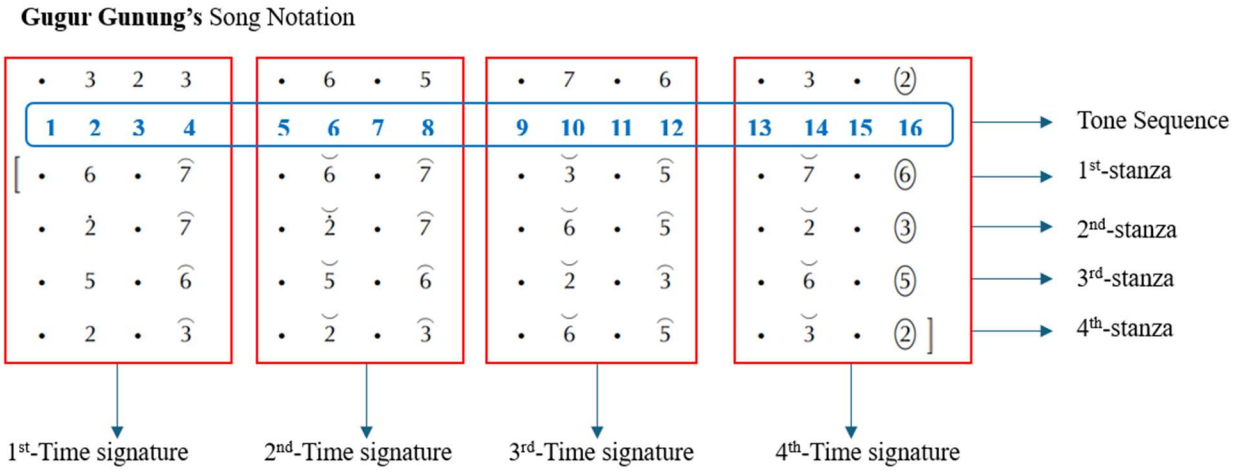


Figure 1: Contextual Exploration in *Gugur Gunung's* Song Notation

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Figure 2: Contextual Exploration Process in Paper Folding

Concept Formulation Stage

The "concept formulation" stage involves students formulating the concept of number patterns based on their exploration results. In experimental group 1, students identified the mathematical structure underlying the cultural activities through guided prompts on the website. The website provided feedback mechanisms that allowed students to refine their understanding based on their responses, helping them link the cultural context with mathematical concepts. In addition, students engaged in online discussions through the platform, where they could share responses, comment on peers' ideas, and collaboratively refine their reasoning under the teacher's moderation. To illustrate, the notes in Figure 1 are played on the *Kethuk*, which serves as a rhythm guard in the musical composition. The notation generated by this tool, as shown in Figure 3, forms a pattern of odd numbers in constant increments. In experimental group 2, students used physical aids or engaged in group discussions, with the teacher guiding them to analyze their explorations and formulate mathematical concepts collaboratively. For example, in Figure 4, the process of folding paper produces a certain number pattern. Each paper is folded, the number of parts formed doubles from before. In contrast, the control group was directly instructed by the teacher, who explained the concept of number patterns without involving any contextual exploration or interactive feedback.

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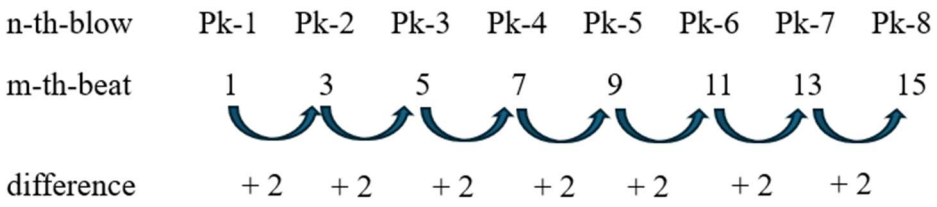


Figure 3: Concept Formulation in *Kethuk's* Beat Notation

Number of paper folding	1 time	2 times	3 times	4 times
The number of equal parts formed	2 parts	4 parts	8 parts	16 parts

Figure 4: Concept Formulation in Paper Folding

Mathematization Stage

The "mathematization" stage, students are tasked with transforming the identified patterns into mathematical models. In experimental group 1, students used the instructional website to convert cultural contexts into mathematical representations. For example, when studying the *Gending Lancaran* scale, students identified the numerical intervals between notes as part of a pattern and expressed these intervals as a series of numbers. In the case of the *Wiru Jarik* folding process, students recognized a step-by-step sequence and mapped it to a numerical pattern, such as counting the number of folds or measuring the length of fabric after each fold. The website's tools helped students visualize and input these patterns into mathematical formulas, such as identifying arithmetic progressions or simple number patterns. In experimental group 2, students manually arranged chairs or folded paper to create patterns. For instance, when arranging chairs, students noted the sequence of chair numbers in rows, identifying a progression, such as increasing by one or two chairs per row. They then translated these arrangements into mathematical notation, such as writing a formula to represent the relationship between the number of chairs and the row number. Similarly, while folding paper, they recognized repeated patterns in the number of folds and expressed this as a number pattern. In the control group, students worked with textbook problems where they were asked to identify patterns in a list of numbers or shapes. The teacher provided instructions on recognizing number patterns, such as identifying whether a pattern was increasing or decreasing. However, students did not engage in any contextual or hands-on learning.

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Generalization Stage

In the "generalization" stage, students applied their understanding of number patterns to new, more complex problems. In experimental group 1, students used additional simulations and mathematical problems provided on the instructional websites. They extended the mathematical concepts they had learned, such as creating new sequences and predicting future terms based on the number of patterns identified in cultural contexts. In experimental group 2, students used manual tools like classroom objects and paper folding to explore more complex problems, applying their previously identified patterns. The control group worked with teacher-provided textbook problems, focusing on mathematical tasks without the contextual or interactive elements.

Reflection and Evaluation Stage

All groups reflected on their learning in the "reflection and evaluation" stage. Experimental group 1 received instant feedback through the website's automatic evaluation feature, allowing students to quickly assess their understanding and refine their approaches. The teacher also provided feedback both online, through discussion forums on the platform, and offline, during follow-up sessions, to guide students in evaluating their reasoning and conceptual understanding. Experimental group 2 and the control group were reflected through class discussions, during which the teacher provided feedback. However, the feedback in the control group was limited to general comments, while in experimental group 2, the feedback was more focused on students' manual problem-solving processes.

This study used pretest and posttest before and after treatment to measure students' CT skills. The test includes CT skill indicators, namely decomposition, pattern recognition, abstraction, and algorithms, which are tailored to the topic of the number patterns. The pretest was conducted to measure the students' initial ability, while the posttest aimed to evaluate the improvement of CT skills after treatment in the form of a learning approach.

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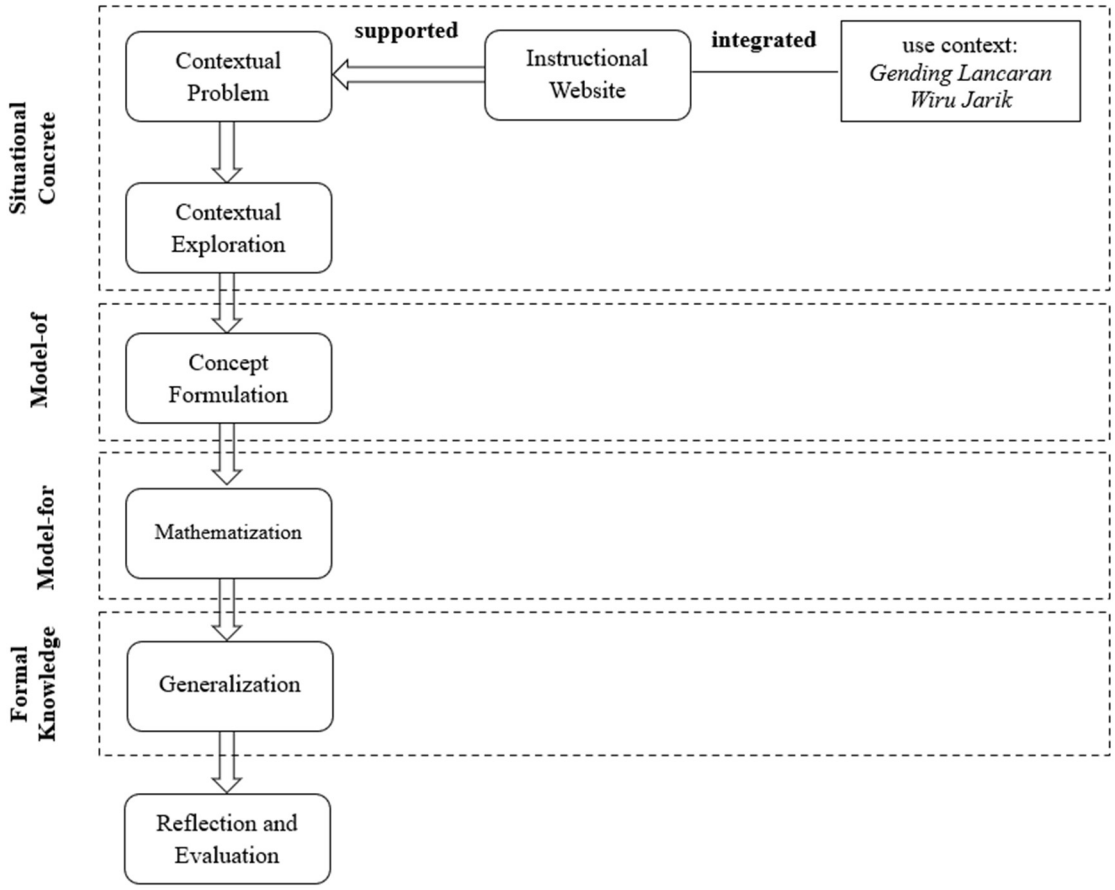


Figure 5: RME-Web syntax

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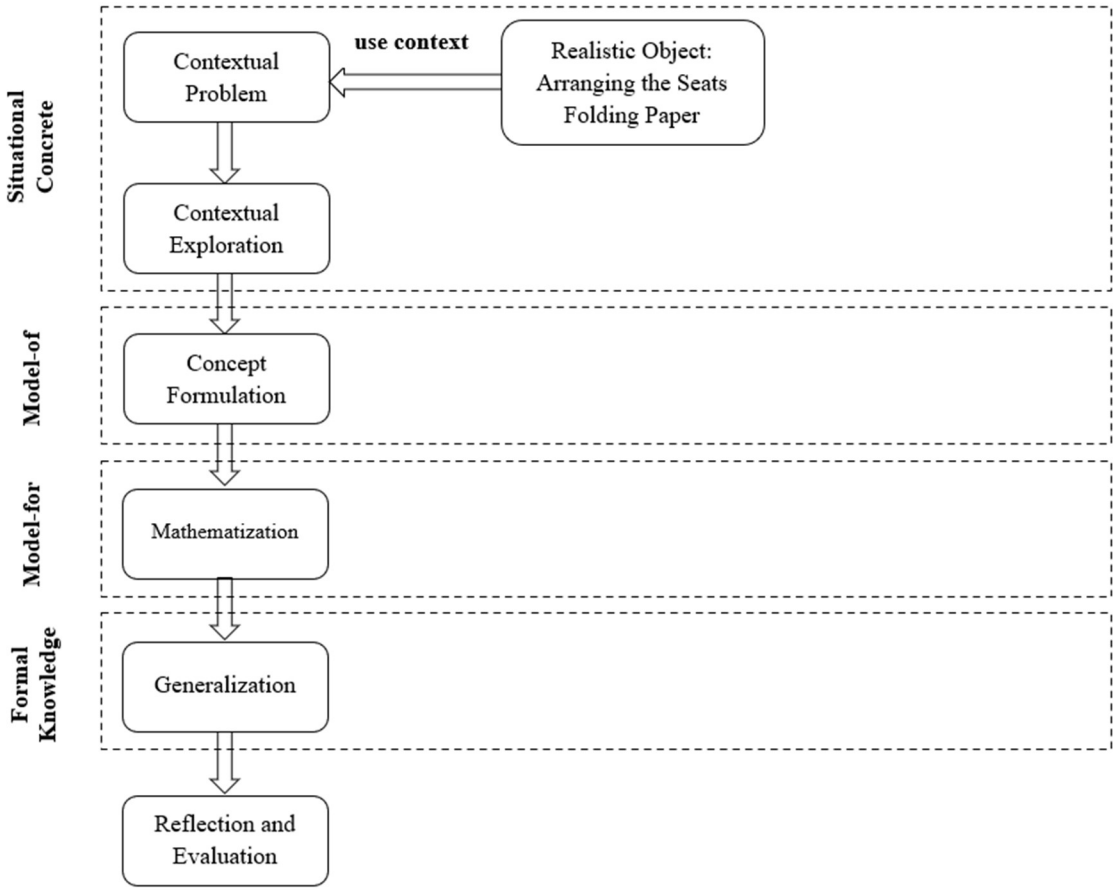


Figure 6: RME Syntax

Research Hypothesis

Several previous studies have tried to integrate RME into instructional media to improve critical thinking skills (Wijayanti et al., 2022) and problem-solving (Sukmaningthias et al., 2023). In addition, several previous studies have also used instructional websites to improve students' CT skills (Bers, 2018; Latifah et al., 2023). Based on this, researchers can hypothesize that integrating RME into instructional websites can improve students' CT skills. The hypothesis can be formulated as follows:

H_1: There was a significant difference in the average CT skill scores of the three sample groups

H_2: Experimental group 1 had a higher average CT skill score than experimental 2

H_3: Experimental group 1 had a higher average CT skill score than the control

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RESULT

This section describes the results of the analysis of two data groups, namely the analysis of initial ability and final ability after treatment. The data collection process is offline, starting from pretest data, treatment provision, and posttest data.

Results of Pretest Data Analysis

Before providing treatment, the researchers conducted a balance test on the initial ability data to find out whether the student population who received the RME-Web treatment approach (experimental 1), RME (experimental 2), and DL (control) had the same initial ability. Before the balance test, a prerequisite test is carried out, namely a normality assumption test and variance homogeneity. The normality assumption test was carried out on each sample group. The normality assumption test in the three groups used the Shapiro-Wilk formula, as presented in Table 4.

Sample Groups	Variable Statistic	p-value
RME-Web	Pretest	0.121
RME	Pretest	0.118
DL	Pretest	0.486

Table 4: Results of Pretest Data Normality Test Analysis

After the normality test assumption analysis, a group of data can be said to be normally distributed if the p-value > 0.05. Table 4 shows that the p-value of each group > 0.05 (H_0 was not rejected). This means the data in the groups with the RME-Web, RME, and DL approaches is normally distributed.

Levene Statistic	df1	df2	p-value
0.389	2	87	0.679

Table 5: Results of Pretest Data Variance Homogeneity Test Analysis

After a variance homogeneity assumption test was carried out using the Levene formula, the three sample groups were said to have homogeneous data variance if the p-value > 0.05. Table 5 shows that the p-value is at 0.679 > 0.05. These results indicate that the three groups have homogeneous variances (H_0 is not rejected). Thus, the process of analyzing pretest data can be continued to the next stage, namely the One-Way ANOVA test, to see if the average score of the student's CT skills from the three groups before treatment has a significant difference. Meanwhile, according to Kim (2017), a significant difference can be identified if the p-value of the One-Way ANOVA test is below 0.05. Table 6 shows that the p-value is at 0.718 or above 0.05. This means H_0 was not

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rejected. There was no significant difference in students' average CT skills scores from the three sample groups.

This is strengthened by the descriptive analysis test in Table 7, where the average scores of the three sample groups do not show a sharp and significant difference. This shows that the three sample groups have a balanced initial understanding to continue to provide treatment, as described in Table 2.

Condition	df	Sum of Squares	Mean Square	F	p-value
Model	2	107	53.39	0.332	0.718
Residuals	91	14630	160.77		

Table 6: Results of Pretest Data of One-Way ANOVA Test Analysis

Sample Groups	Mean	Minimum	Maximum	Std. Deviation
RME-Web	55.74	30.00	76.00	12.78
RME	57.19	36.00	92.00	13.34
DL	56.59	32.00	80.00	11.90

Table 7: Results of Pretest Data of Descriptive Test Analysis

Results of Posttest Data Analysis

After the three sample groups were given learning treatment as stated in the research design (see Table 2), the researchers gave a posttest to the students. This posttest aims to determine whether there is a significant difference in the average student's CT skill score between the RME-Web, RME, or DL groups. Therefore, the students' work results were first tested for statistical prerequisites, namely the normality assumption test using the Shapiro-Wilk formula and the homogeneity of variance with the Levene formula. The analysis results can be seen in Table 8 and Table 9.

Sample Groups	Variable Statistic	p-value
RME-Web	Posttest	0.141
RME	Posttest	0.081
DL	Posttest	0.139

Table 8: Results of Posttest Data Normality Test Analysis

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Table 8 shows that the p-values of the three groups are above 0.05 (H_0 was not rejected). This means that the data in all three groups are normally distributed. Furthermore, Table 9 shows the p-value at 0.755 (H_0 was not rejected). This means that all three groups are categorized as homogeneous.

Levene Statistic	df1	df2	p-value
0.281	2	87	0.755

Table 9: Results of Posttest Data Variance Homogeneity Test Analysis

After confirming that the posttest data was normally distributed and homogeneous, the One-Way ANOVA test was continued to determine if there was a significant difference in the average score of students' CT skills from the three sample groups. Table 10 shows the analysis results of the One-Way ANOVA test with a significance value of p-value = 0.000 < 0.05 (H_0 was rejected), meaning a significant difference in the average students' CT skills score from the three sample groups.

Condition	df	Sum of Squares	Mean Square	F	p-value
Model	2	2697	1348.3	54.63	0.000
Residuals	87	2147	24.7		

Table 10: Results of Posttest Data of One-Way ANOVA Test Analysis

Furthermore, to find out how much the difference in learning approach interventions affects students' CT skills, the effect size can be used, namely eta-squared (η^2). The result was η^2= 0.5567, meaning that 55.67% of the variance in students' CT skills was obtained from the difference in the three interventions of the learning approach (RME-Web, RME, and DL). This effect size value is relatively large.

Furthermore, a post hoc test analysis was carried out using the Tukey HSD formula to determine which approach is better for improving students' CT skills. Based on the 95% confidence interval containing 0 and the p-value < 0.05 contained in Table 11, it shows that the application of the RME-Web approach significantly influences students' CT skills compared to the RME approach. Furthermore, the RME-Web approach also has significant differences compared to the application of the DL approach. This result is confirmed by the analysis of the descriptive test in Table 12, where the group with the RME-Web approach has a higher average score than the other two approaches.

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Sample Groups		Mean Dif.	Lower	Upper	p-value	Meaning
RME-Web	RME	3.16	0.11	6.22	0.041	Significant
	DL	12.86	9.81	15.93	0.000	Significant
RME	RME-Web	-3.16	-6.22	-0.11	0.041	Significant
	DL	9.70	6.64	12.76	0.000	Significant
DL	RME-Web	-12.86	-15.93	-9.81	0.000	Significant
	RME	-9.70	-12.76	-6.64	0.000	Significant

Table 11: Results of Posttest Data of Tukey HSD Test Analysis

Sample Groups	Mean	Minimum	Maximum	Std. Deviation
RME-Web	83.53	74.00	92.00	4.86
RME	80.37	72.00	88.00	6.66
DL	70.67	58.00	80.00	5.36

Table 12: Results of Posttest Data of Descriptive Test Analysis

Finally, the average type of the N-Gain score can be used to determine the effectiveness of each intervention approach. Based on the results of the analysis in Table 13, the RME-Web approach is included in the category of quite effective, the RME approach is in the category of less effective, and the DL approach is in the category of ineffective. Therefore, it can be concluded that the RME-Web approach is more effective in improving students' CT skills than the RME and DL approaches.

Learning Approach	Average of N-Gain Score
RME-Web	63.61%
RME	54.19%
DL	32.43%

Table 13: Results of N-Gain Score

DISCUSSION

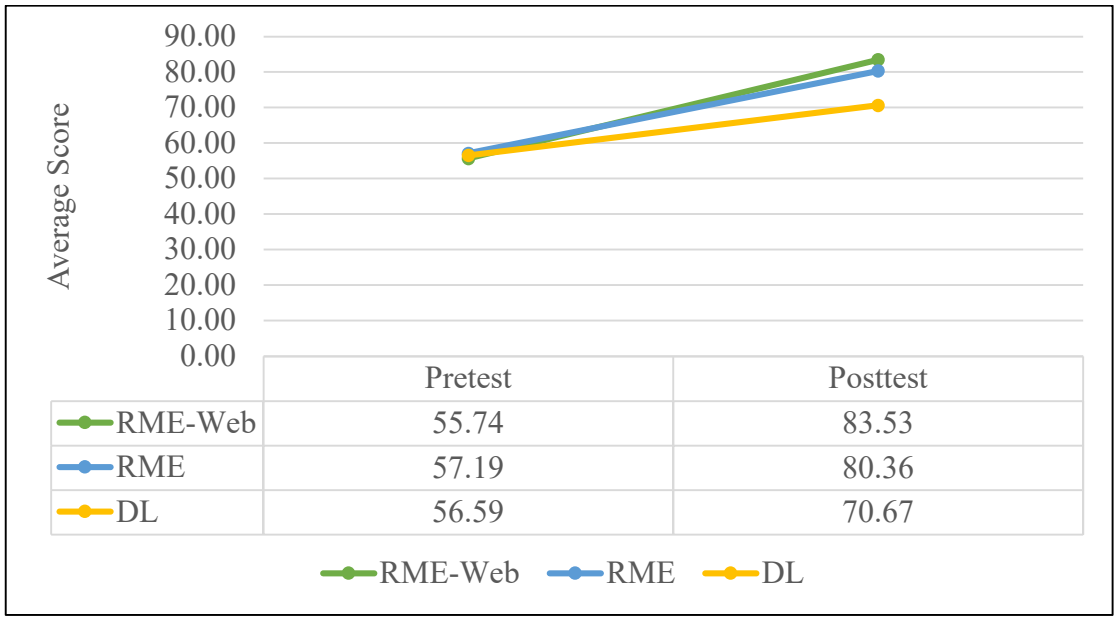
The big idea of this study is to analyze the effectiveness of integrating the RME approach into instructional websites (RME-Web) in improving students' CT skills in mathematics learning. To see the level of effectiveness, the researchers compared the application of the RME-Web approach with the RME and DL approaches. As previously explained, this section discusses the three hypotheses that have been proposed. Hypothesis 1 (H_1) was to see whether there was a significant

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difference in the average students' CT skills score from the three sample groups tested. Based on Table 10, it can be seen that the p-value < 0.05, which is at 0.000. So, the decision that can be taken is to reject H_0 or accept H_1. This means that there is a significant difference in the average CT skill scores of students in the three sample groups.

Furthermore, Hypothesis 2 (H_2) was used to test whether experimental group 1 had a higher average CT skill score than experimental 2. Based on Table 11, comparing the average CT skill score of students in the RME-Web approach and RME in the Mean Difference column of the first row obtained an average score of 3.16, which is positive. Therefore, based on the 95% confidence interval containing 0 and p-value < 0.05, it can be concluded that there is a significant difference in the average score of the two learning approaches, with the average score of the RME-Web approach higher than that of the RME (H_0 was rejected). In other words, the RME-Web approach improves students' CT skills better than RME.

Hypothesis 3 (H_3) was used to test whether experimental group 1 had a higher average CT skill score than the control. Based on Table 11, comparing the average CT skill scores of students in the RME-Web and DL approaches in the second row of the Mean Difference column obtained an average score of 12.86, which is positive. Therefore, based on the 95% confidence interval containing 0 and a p-value < 0.05, it can be concluded that there is a significant difference in the average score of the two learning approaches, with the average score of the RME-Web approach higher than that of DL (H_0 was rejected). In other words, the RME-Web approach improves students' CT skills better than DL.



	Pretest	Posttest
RME-Web	55.74	83.53
RME	57.19	80.36
DL	56.59	70.67

Figure 7: Graph of Average Increase in Student's CT Skill Score

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Based on the results of the three hypothesis tests, it can be concluded that the average CT skill score of students in experimental group 1 is higher than that of the other two sample groups. This was also clarified by the increase in the average pretest-to-posttest scores from the three sample groups. Figure 7 shows an escalation graph of students' average CT skill scores from the three sample groups. It can be seen that the average score of the group with the RME-Web approach (red line) has the most significant improvement compared to the group with RME (purple line) and DL (light blue line). The results of the N-Gain score analysis further reinforce these findings, with the RME-Web approach being included in the "quite effective" category. This means that this approach is not only effective in improving students' CT skills but also contributes significantly to the achievement of better learning outcomes compared to the RME approach, which is only based on a contextual approach without technology integration, as well as the DL approach, which is more teacher-centered.

The findings of this study are important because they confirm the success of using websites as a learning medium that can enrich students' learning experiences in learning mathematics in the 21st century. This is also confirmed by previous research, which shows that instructional websites can improve the quality of mathematics learning (Buchori et al., 2023; Lisnani et al., 2023; Suripah & Susanti, 2022). The RME-Web approach combines the power of RME based on contextual problem-solving with the benefits of technology, allowing students to engage in more authentic, contextual, and meaningful learning. RME, as an approach emphasizing real-world context-based mathematics learning, has proven effective in improving students' understanding of mathematics. This is reinforced by research conducted by Dhayanti et al. (2018); Nuraida & Amam (2019); Risdiyanti & Prahmana (2021); Umbara & Nuraeni (2019); and Payadnya et al. (2023) where the main principle of RME is to utilize relevant everyday contexts to help students understand mathematical concepts through real-life experience and problem-solving.

According to Gravemeijer (1994), RME has been recognized as an effective approach to developing students' understanding of mathematics. Furthermore, as conducted in this study, integrating technology such as instructional websites in the learning process can significantly improve the quality and outcomes of learning. In this study, the RME-Web approach expands the RME approach by integrating technology as an instructional website that provides various exercises, simulations, and exploration of context-based mathematical problems. This strengthens the principle of RME in giving context to math learning and allows students to develop indispensable CT skills in the 21st century.

The RME-Web approach significantly impacts all four aspects of CT skills. In *Gending Lancaran*, students must break down the structure of the music into smaller components, such as the sequence of beats in a specific unit of time and the relationships between notes in a single *gending* (song).

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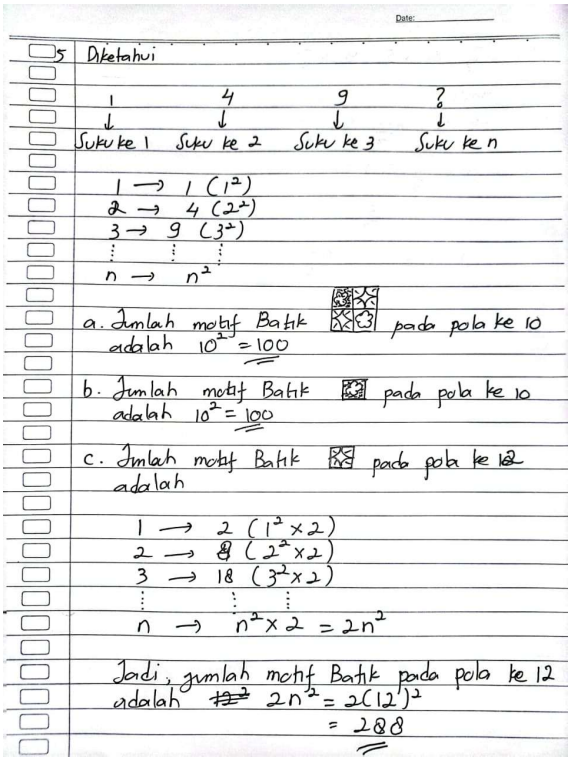
This process trains the "decomposition" aspect because students learn to identify complex rhythmic patterns before generalizing regularity in number patterns. Meanwhile, in *Wiru Jarik*, the repetitive and consistent pattern of fabric folds reflects the mathematical principle in the number pattern. Students analyze the regularity of these folds, practicing the "pattern recognition" aspect by recognizing geometric patterns and regularity in the structure of *Wiru*. Students generalize their findings into more abstract mathematical concepts after understanding the patterns in *Gending Lancaran* and *Wiru Jarik*. In *Gending Lancaran*, they convert the beat pattern into a series of numbers that represent the duration or frequency of the beat, while in *Wiru Jarik*, they model the fold pattern as a number pattern. This process develops the skills of the "abstraction" aspect. In addition, in both *Gending Lancaran* and *Wiru Jarik*, there are specific rules in the pattern that must be followed. Students devise systematic steps to understand, reproduce, or modify patterns. For example, arrange the order of the beats in *Gending Lancaran* or determine the steps for folding the fabric in *Wiru Jarik*. This skill trains the “algorithmic thinking” aspect as students design procedures that can be tested and reapplied in various mathematical contexts. Figure 8 shows the students' answers to question number 5, which measures the pattern recognition aspect in the group that received the RME-Web approach treatment. Based on the image, it can be seen how students recognize Batik motif patterns, allowing him to determine the number of Batik motifs in a particular pattern. This process demonstrates students' skill to identify similarities and regularities in a complex pattern.

At this point, it is clear that the RME-Web approach fosters the development of CT skills by integrating cultural contexts into mathematics learning. Next, based on observations in experimental group 1, the RME-Web approach helps students reinvention number patterns through interactive simulations, digital visualizations, and automated feedback. This supports the “guided reinvention” process, where students gradually mathematize their experiences from contextual problems to abstractions. The “progressive mathematization” principle is also well facilitated because students can take advantage of technological features to learn mathematical patterns, relationships, and structures gradually. This confirms the research by Lisnani et al. (2023), and Payadnya et al. (2023), which shows that the technology in the RME approach improves guided reinvention and progressive mathematization through more independent interactions and visualizations. In contrast, the RME approach without technology in experimental group 2 still applies RME principles. However, learning relies more on group discussions and manual representation, so the scaffolding provided is limited to direct intervention from the teacher. In previous research findings, as described by Nugroho et al. (2025) and Laurens et al. (2018), group discussions and manual representations can help students build mathematical understanding collaboratively. This collaborative process, supported by the teacher’s role as facilitator, enabled students to engage in contextual exploration, share ideas, and collectively formulate mathematical

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concepts. As a result, the RME group showed better outcomes than the DL group, indicating that contextual, discussion-based learning can significantly enhance students' CT skills.

In addition, the principle of "didactical phenomenology" is seen in both experiments, where contextual problems relevant to students' lives (Yogyakarta culture) are selected to motivate them to understand mathematical concepts. However, experimental 1 showed a better increase in interactivity than experimental 2 because technology allows students to be more independent and active in the exploration process and provides a more dynamic learning experience. This is in line with research conducted by Payadnya et al. (2023) and Shahidayanti et al. (2024), where the principles of didactical phenomenology are strengthened by the use of technology that supports students' more intensive experience and reflection on mathematical phenomena. In learning in the control group, teachers directly convey mathematical concepts through lectures, demonstrations, and practice questions without much involvement in exploring contextual problems. Nugroho & Septianisha (2025) and Nugroho et al. (2024) note that a more didactic and structured teaching approach can hinder students' creative exploration in the context of mathematics learning.



Known

1st Term 2nd Term 3rd Term nth Term

a. The number of Batik motifs  in the 10th pattern is $10^2 = 100$

b. The number of Batik motifs  in the 10th pattern is $10^2 = 100$

c. The number of Batik motifs  in the 12th pattern is ….

So, the number of Batik motifs in the 12th pattern is $2n^2 = 2(12)^2 = 288$

Figure 8: Sample Answers of Experimental Group 1 Students for Question Number 5

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In addition, the integration of local cultures in mathematics learning can increase students' engagement and understanding of the material because students feel closer to the content being taught, as revealed by previous research (Lidinillah et al., 2022; Nurmala Sari Agustina et al., 2021; Risdiyanti & Prahmana, 2021; Utari et al., 2024). This aligns with the principle of didactical phenomenology, which brings cultural phenomena closer to students as a starting point for learning mathematics (Prahmana, 2022). By connecting mathematical concepts to students' cultural contexts, learning becomes more meaningful and relevant, allowing students to see mathematics not as an abstract discipline but as a tool for understanding and interpreting the world around them. In this study, using Yogyakarta's local culture, such as *Gending Lancaran* and the *Wiru Jarik* tradition, provides a rich context of patterns and structures, making it very relevant to help students understand the concept of number patterns more meaningfully.

Meanwhile, the RME approach based on the mathematical context of everyday life without the integration of technology (such as the one used in experimental group 2) still yielded good results but was not comparable to the success achieved with RME-Web. This can be seen in the results of the Tukey HSD test in Table 11, where the p-value between RME-Web and RME is 0.041, which shows a value close to the significance limit of 0.05. This means that although the difference in students' average CT skill scores between these two groups is statistically significant, the difference in average is not very large. This is in line with research results that show that although contextual approaches such as RME can deepen student understanding, technology integration provides an additional dimension that enriches the student learning experience (Hilda & Siswanto, 2021; Lestari et al., 2023; Sukmaningthias et al., 2023; Yuliasari et al., 2021). Meanwhile, the control group that used the DL approach showed lower results in CT skills. This approach focuses more on direct and procedural instruction without much contextual exploration, so students are less involved in the critical and analytical thinking process. As a result, aspects of CT do not develop optimally. This result is in line with previous research, which stated that context-based learning is more effective in improving conceptual understanding compared to direct learning, which tends to be mechanistic (Wijaya et al., 2015, 2018). Overall, all three learning approaches significantly impacted students' CT skills. This can be seen from the relatively large effect-size value (η^2). In mathematics education, this value indicates that changes in learning approaches, such as the use of RME-Web, RME, and DL, strongly influence improving students' CT skills.

These findings open up opportunities for further research, especially in expanding the application of RME-Web to various levels of education other than junior high school, such as elementary, high school, or college. Advanced research can also explore integrating advanced technologies, such as augmented-reality (AR), virtual-reality (VR), gamification, and mobile apps to create more immersive and enjoyable learning experiences. In addition, the study focuses solely on computational thinking skills so that future research may expand the scope to other 21st-century

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skills, such as collaboration, critical thinking, problem-solving, communication, and creativity, while considering students' factors, such as motivation, interests, and learning styles, that may affect the effectiveness of learning approaches.

CONCLUSIONS

This study aims to analyze the effectiveness of the RME-integrated instructional websites (RME-Web) approach in improving students' CT skills in mathematics learning. Based on the One-Way ANOVA and Tukey HSD test analysis, the RME-Web approach has a better average score for students' CT skills than the RME and DL. These findings are supported by the results of the N-Gain score analysis, where applying the RME-Web approach is quite effective in improving students' CT skills. Therefore, it can be concluded that the RME-Web approach is more effective in improving students' CT skills than the other two approaches. The effectiveness of the RME-Web approach was supported by the integration of contextual problems, interactive simulations, cultural-based tasks, and online discussions that enabled students to explore, collaborate, and reflect more independently. These features strengthened the principles of guided reinvention and progressive mathematization, allowing students to gradually move from contextual experiences to abstract mathematical concepts. Although the difference between the RME-Web and RME groups was statistically significant, it was relatively small, suggesting that the RME group also achieved notable success. This can be attributed to the use of contextual exploration, group collaboration, and teacher facilitation, which enabled students to actively engage in problem-solving and concept-building. These elements made the RME approach more effective than the DL group, which relied primarily on direct, teacher-centered instruction.

Based on these findings, the researchers recommend that educators and curriculum designers integrate the RME-Web approach into mathematics education. For educators, this study offers an innovative technology-enhanced, culture-based learning strategy that fosters independent learning and helps students develop CT skills in local cultural contexts. Educators can implement this approach by integrating instructional websites with structured RME-based activities, fostering active engagement and deeper mathematical understanding. For curriculum designers, the RME-Web approach can serve as a framework for developing curricula, and it underscores the need to integrate digital learning modules that incorporate local traditions to ensure instructional materials are conceptually rich and relevant to students' cultural backgrounds. Additionally, teacher training programs should emphasize culturally contextualized technology integration to optimize the effectiveness of RME-Web in various educational settings. By bridging technology, culture, and contextual mathematics learning, this study provides a scalable and effective model for developing

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students' CT skills while preserving and promoting local heritage. This is an important step in modernizing education without losing cultural identity.

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APPENDIX 1

Pretest and Posttest Instrument

1. *Gending Lancaran* has a complex but regular rhythmic pattern in the Javanese music tradition. You are asked to analyze the beat patterns of the main instruments: *gong*, drum, and *saron*, in a single cycle of 16 beats. The rules of the sound are as follows:
   - Gong: Sounds every 8 beats
   - Drums: Sound every 4 beats
   - Saron: Sounds every 2 beats.

   However, a musician wanted to add a new instrument, *kenong*, with the rule that it sounds in multiples of prime numbers in a cycle of 16 beats.

   a. Create a structured beat table based on the sound rules for each instrument in a 16-beat cycle.
   b. Analyze which beat patterns produce sounds simultaneously
   c. Which beats sound simultaneously to two instruments? List those beats
   d. Which beats sound simultaneously to three instruments? List those beats

2. In the Javanese culinary tradition, Mr. Dalang's family makes *jenang* (traditional Indonesian sweets) for traditional events. After the *jenang* is finished cooking, they pack it to be distributed to the guests. They pack according to the following rules:
   - *Jenang* is put in a banana leaf. Each package of banana leaves contains 8 pieces of *jenang*. If the remaining *jenang* is less than 8 pieces, *jenang* is left unwrapped.
   - The banana leaf wrapper will be put into a bamboo basket. Each basket contains 8 packs. If the number of banana leaf wrappers left is less than 8, the remaining wrappers are not put in the basket.

   Today, Mr. Dalang's family makes 275 pieces of *jenang*. How many pieces of *jenang* are not wrapped in banana leaves?

3. An amusement park offers tickets at promotional prices that change monthly following a specific pattern of increases. In the first month, the ticket price is IDR 80,000; in the second month, it is IDR 85,000; and in the third month, it is IDR 90,000.
   a. Abstract the pattern of ticket price increases into a general formula to calculate ticket prices in the nth month. Describe your steps in compiling this formula.

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b. Use this formula to determine the ticket price in the 15th month without manually calculating it month by month.
c. If the theme park has a target to sell tickets for IDR 200,000, in what month will this target be achieved? Describe your calculation process.

4. An art festival is held for 10 consecutive days. On the first day, 200 visitors attended the festival. The number of visitors increases daily in a specific pattern, namely 20% from the previous day.
   a. Design a step-by-step algorithm that can be used to count the number of visitors on day 10.
   b. Explain how the algorithm can be modified if the pattern of increasing the number of visitors changes to a fixed increase of 50 people per day

5. A Batik artist is designing a Batik *Cepoko Kotak Besar* motif by using repeating geometric patterns. At first, he made the motif in small sizes, as seen in the first image. Then, he zoomed in on the pattern by adding new elements to each level, as seen in the second and third images.



   a. Without drawing, how many motif patterns  are there in the 10th pattern?
   b. In the 10th pattern, how many motif patterns  are there?
   c. In the 12th pattern, how many motifs patterns  are there?

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APPENDIX 2

Lesson Plan of Number Pattern Based on RME

General Information	
Education Level	Junior High School
Mathematics Topic	Number Pattern
Class	VIII
Time Allotment	12 x 40 minutes (6 meetings)
Instructional Outcomes	**Instructional Objective**
At the end of the lesson, students can recognize, predict, and generalize patterns in the form of the arrangement of objects and numbers	• Students can generalize patterns and rows of numbers using tables correctly • Students can generalize the pattern of an object configuration correctly
Facilities and Infrastructure	
LCD projector, computer, markers, eraser, whiteboard, tables, chairs, and colorful folding paper	
Instructional Model	**Instructional Media**
Realistic Mathematics Education Lecture and Discussion	Chairs, and colorful folding paper
Instructional Activities (First Meeting)	
Pre-Activity	
• The teacher opens the learning with greetings and prayers together • Teachers check the attendance of students as a form of discipline • The teacher conveyed the learning objectives at the meeting • The teacher informed the assessment technique, namely, the pretest work	
In-Activity	
Teacher's Activity	**Pupil's Activity**
• The teacher provides a pretest to find out the initial ability of the students	• Students work on a pretest given by the teacher
Post-Activity	
• Teachers and students reflect on meeting learning 1 • The teacher informed about the learning that would be carried out at the next meeting, namely the number pattern	
Instructional Activities (Second Meeting)	

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Pre-Activity		
• The teacher opens the learning with greetings and prayers together. • Teachers check the attendance of students as a form of discipline. • The teacher gave an overview of the benefits of studying number patterns daily. • The teacher conveyed the learning objectives at the meeting. • The teacher introduces the realistic context, namely, the arrangement of class seats. • The teacher explained that learning uses the RME approach.		
In-Activity		
Emergent Modeling Activity	**Teacher's Activity**	**Pupil's Activity**
Contextual Problem	• Instruct students to form 5 groups • Convey a real scenario: "*We will arrange a chair in the classroom for a performance or event.*" • Asking the lighter question: "*If the first row contains 3 seats, what about the second row? Third? The 10th row*?" • Invite students to pay direct attention to the chairs in the classroom as learning objects	• Observe examples of real seating arrangements in the classroom • Predict the number of seats in each row based on real observations • Write down initial ideas or conjectures about the pattern of chair arrangement
Contextual Exploration	• Instruct students to arrange classroom seats according to instructions (e.g., first row 3 seats, second row 5 seats, etc.). • Directing students to record the results of observations in a table systematically. • Ask a question: *"What changes from one line to the next?"* • *"Is there a fixed pattern in the arrangement of the chairs?"*	• Lift, move, and arrange the original chairs in the classroom in groups. • Record the number of seats per row and fill in the table. • Discuss in groups: the increase in the number of seats, the shape of the arrangement, and the patterns that appear.

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Concept Formulation	• Facilitate discussions between groups to compare results. • Directing students concludes that the number of seats increases regularly (e.g., +2 per row). • Conveying mathematical terms such as "arithmetic rows".	• Presenting the group's findings based on the arrangement of the chairs they made. • Explain the pattern of changes in the number of seats and try to write in verbal form (words). • Draw initial conclusions collectively.
Post-Activity		
• The teacher closes the first meeting by greeting the students and leading a prayer. • The teacher informs the students that the learning will continue in the third meeting		
Instructional Activities (Third Meeting)		
Pre-Activity		
• The teacher opens the learning with greetings and prayers together • Teachers check the attendance of students as a form of discipline • The teacher informed that today's learning continued the activities from the previous meeting • The teacher informed that the assessment was carried out through the work on the student's activity sheet		
In-Activity		
Emergent Modeling Activity	**Teacher's Activity**	**Pupil's Activity**

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Mathematization	• Guiding students to convert verbal patterns into mathematical notation. • Ask questions like: *"Can you write down the general formula for the number of seats in the nth row?"* • Provide feedback on mathematical formulas or expressions that students have compiled. • Provide additional examples to test the accuracy of the student's formula	• Trying to formulate a general formula from the observed pattern • Use the formula to calculate the seats in the 10th, 15th, etc. rows. • Revise the formula, if necessary, based on the calculation results. • Explain the math process to a group or class friend
Generalization	• Provide a variation of real context, for example, the seating arrangement in the school hall. • Encourage students to compile their contextual problems using the formulas provided.	• Solve advanced questions based on the formula they made. • Creating new similar problems based on the experience of arranging chairs. • Finding relationships between context and number patterns.
Reflection and Evaluation	• Invite students to reflect on today's learning process: *"What did you guys learn about the chair pattern today?"* *"Which part is the most challenging?"* • Provide feedback directly to each group or student. • Close the lesson by summarizing the process from start to finish.	• Write down personal reflections as notes or exit tickets. • Deliver a memorable or confusing learning experience. • Give feedback to your groupmates. • Answer the teacher's questions honestly and openly.

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Instructional Activity (Fourth Meeting)		
Pre-Activity		
• The teacher opens the learning with greetings and prayers together. • Teachers check the attendance of students as a form of discipline. • The teacher conveyed the learning objectives at the meeting. • The teacher reminds students of the previous material and relates it to the material to be studied • The teacher introduces the realistic context, namely, folding the paper.		
Activity	**Teacher's Activity**	**Pupil's Activity**
Contextual Problem	• Conveying the real situation: *"Imagine you fold the paper in half, then fold it again, and keep folding."* • How many parts are formed after several folds?" • Show examples of colorful folding paper and demonstrate the first and second folds. • Asking open-ended questions: *"What happens to the number of pieces of paper each time it is folded?"*	• Pay attention to the paper that the teacher folds. • Gives an initial guess of the number of parts formed after a few folds. • Record initial answers individually as a pattern prediction
Contextual Exploration	• Distribute colorful folding paper to each student or group • Giving instructions: *"Fold the paper into two parts, then four, and so on, up to a maximum of 5 times."* • Have students record the number of parts each time the fold is performed.	• Fold the paper repeatedly according to the instructions. • Count and record the number of paper parts after each fold. • Discussion in small groups to recognize the growth pattern of the number of parts

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	• Guiding students to fill out tables of n-fold and the number of sections	
Concept Formulation	• Facilitate discussions between groups regarding the pattern of paper parts. • Ask: *"How many parts if folded 6 times? 7 times?"* *"What changed? What's fixed?"* • Introduce the concept of exponential growth if students haven't already been introduced.	• Discuss the results of the table and compare them between groups. • Realize that the number of parts is always double what it was before • Conveying that the patterns that appear are: 2, 4, 8, 16, 32... • Try to write a pattern in the form of a sentence and draw an initial conclusion.
Post-Activity		
• The teacher closes the first meeting by greeting the students and leading a prayer. • The teacher informs the students that the learning will continue in the fifth meeting.		
Instructional Activities (Fifth Meeting)		
Pre-Activity		
• The teacher opens the learning with greetings and prayers together. • Teachers check the attendance of students as a form of discipline. • The teacher informed that today's learning continued the activities from the previous meeting • The teacher informed that the assessment was carried out through the work of the student's activity sheet		
Activity	**Teacher's Activity**	**Pupil's Activity**
Mathematization	• Guiding students to formulate a general formula of the number of parts as a function of the nth fold: *"Try to write this pattern as a formula. For example, B(n) = ...?"* • Provide additional training: *"How many parts if folded 10 times?"*	• Compiling the general formula: $B(n) = 2^n$ • Use the formula to calculate the number of paper parts after 6, 8, and 10 folds. • Testing the accuracy of formulas with table data.

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	• Associate the formula found with the exponential	• Convey the process of discovering formulas orally and in writing.
Generalization	• Asking follow-up questions based on predictions: *"How many parts are formed if it can be folded 20 times?"* • Providing new context: *"What if the folding is done vertically and horizontally alternately?"* • Encourage students to make their variations based on patterns.	• Apply formulas to solve predictive problems. • Develop a variety of folds by creating your problems. • Test and compare the results of vertical and horizontal folding. • Demonstrate an understanding of generalizations outside of the initial context.
Reflection and Evaluation	• Invite students to reflect on the learning process through questions: *"What did you learn from folding paper?"* *" What are the advantages of using this activity to understand number patterns?"* • Provide individual or group feedback.	• Conveying reflections: what's easy, what's challenging. • Fill out a reflection journal or answer open-ended questions. • Explain how they can recognize and formulate real activities.

Instructional Activity (Sixth Meeting)

Pre-Activity

- The teacher opens with an opening greeting and prays to start learning
- The teacher checks the presence of students as a disciplined attitude
- Convey the objectives of the learning at the ongoing meeting
- Inform the assessment technique, namely, posttest work

In-Activity

Teacher's Activity	Pupil's Activity
• The teacher gives a posttest to find out the final ability of the students	• Students work on the posttest given by the teacher

Post-Activity

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- The teacher closes the first meeting by greeting the students and leading a prayer
- Teachers and students reflect on the whole learning process

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APPENDIX 3

Student Worksheet Based on RME

Member Name:
1.
2.
3.
4.
5.

Class:

Activity 1: Contextual Problem

Imagine that you are preparing for an important event in class, such as an art performance, a class meeting, or a small performance. For this reason, you need to arrange the seats so that all participants can sit neatly and comfortably. Your teacher gives instructions on how to set up the chair.

The following are the rules for setting up the seats:

- In the first row, the chairs are arranged as many as 3 pieces.
- In each next row, the number of seats increased by 2 seats compared to the previous row.

- **Activity Instructions:**
  1. Pay close attention to the seat arrangement instructions above.
  2. Try to imagine or organize it in real life in your class if possible.
  3. Predict and record how many seats there are in the second, third, fourth, and so on.
  4. Write down your observations in the table below.

- **Initial Questions:**
  - If there are 3 seats in the first row, how many seats are in the second row?
  - What about the third row?
  - Then, how many seats will there be in the 10th row?

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Write down your initial idea (can be in the form of a sentence):

Activity 2: Contextual Exploration

Now, let's experiment with it live!

- **Instruction**

In groups, lift, move, and arrange your classroom chairs according to the instructions:

- The first row contains 3 seats.
- The second row increased by 2 seats from the previous row.
- The third row adds another 2 seats, and so on.
- Arrange it up to the 6th row

Activity 3: Concept Formulation

After you have arranged the chairs in the classroom with the teacher, now continue by filling in the following Table 1 based on your observations. Pay close attention to how many chairs are in each row, and what the pattern of increase is!

Table 14. Observation of Chair Arrangement

Row	Number of Seats	Difference from Previous Row
1		
2		
3		
4		
5		
6		

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- **Discussion**
  - What changes from one row to the next?
  - Is there a pattern in the change in the number of seats?
  - Try to describe the pattern of change in your own words.

Activity 4: Mathematization

- **Introducing the Concept of the n-th term**

Before creating a formula, let's first understand the "term" in a number pattern!

- The first term (n = 1) is the number of seats in the first row or can be symbolized by U_1
- The second term (n = 2) is the number of seats in the second row, or it can be symbolized by .....
- The third term (n = 3) is the number of seats in the third row, or can be symbolized by .....
- The fourth term (n = 4) is the number of seats in the fourth row, or can be symbolized by .....
- The fifth term (n = 5) is the number of seats in the fifth row, or can be symbolized by .....
- The sixth term (n = 6) is the number of seats in the sixth row, or can be symbolized by .....

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Table of Terms Number of Seats:

n (row number)	Number of seats
U_1	
U_2	
U_3	
U_4	
U_5	
U_6	

Identifying the Relationship Between Terms

Observe how the terms are related:

- From the first term to the second term, the number of seats increases by _____.
- From the second term to the third term, the number of seats increases by _____.
- From the third term to the fourth term, the number of seats increases by _____.
- From the fourth term to the fifth term, the number of seats increases by _____.
- From the fifth term to the sixth term, the number of seats increases by _____.

- **Question:**
  - **Is there an increase in the number of seats each time it stays?**

    Answer:
  - **How much does it increase?**

    Answer:

- **Developing a General Pattern**

Pay attention to the results of the table. If the increment is always fixed, then you can write:

- Second term = first term + _____
- Third term = second term + _____

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- Fourth term = third term + _____
- Fifth quarter = fourth quarter + _____
- Sixth quarter = fifth quarter + _____

From that pattern, try writing:

General pattern:

"The number of seats in the n row is obtained by adding ____ seats to the number of seats in the previous row."

Create an Explicit Formula

Now, compile the general formula for the number of seats on the 6th row:

- Lots of starting seats = 3
- Each increases 2 seats for each row increase

So:

U_n = ... + (n - 1) × .....

U_n = ……..

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Activity 5: Generalization

- Formula Trials

Use the n-th formula to calculate:

Number of 10th rows

Number of 25th rows

Number of 55th rows

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EXERCISE



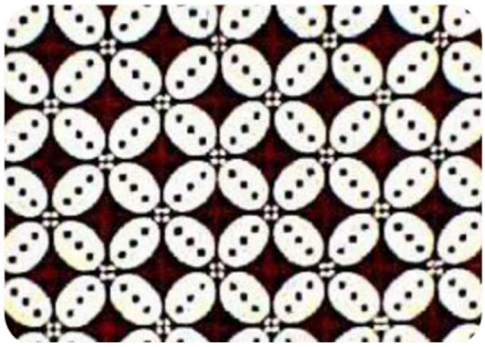
The picture above is one of the Batik motifs from the Yogyakarta Palace, Indonesia, namely the Batik *Grompol* motif. The Batik *Grompol* motif has a characteristic in the form of a geometric pattern that is neatly arranged and uniform, similar to a grid or grid pattern. Notice the arrangement of motifs  below that make up the square number pattern!



Mr. Andi, a Batik *Grompol* maker, made a motif in which the number of ornaments on each row and column followed a square number pattern. In the first row, there are 1 ornament, in the second row, there are 4 ornaments, in the third row, there are 9 ornaments, and so on.

a. How many ornaments does Mr. Andi use if he completes the Batik *Grompol* motif up to the 20th row?
b. What is the total ornament from the 1st to 10th row?
c. Create a nth formula based on the case

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The picture above is one of the Batik motifs from Yogyakarta, namely the Batik *Kawung* motif. The Batik *Kawung* motif with its geometric pattern consisting of circles that cross each other, can be represented as a form of a rectangular number pattern. Notice the arrangement of motifs  below that make up the rectangular number pattern!



Ani wanted to make a handkerchief from a cloth with a Batik *Kawung* pattern. He wanted to make the handkerchief with a precise size, so that each motif was perfectly formed and not cut.

a. If one motif on a fabric is 2x2 cm in size, how much area is the fabric formed if the largest number pattern that can be loaded is the 6th pattern of the rectangular number pattern?
b. Create a nth formula based on the case

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Member Name:
1.
2.
3.
4.
5.

Class:

Activity 1: Contextual Problem

- **Scenario:**
  Imagine you have a standard sheet of paper. Each time you fold it in half, the number of layers doubles.

- **Instructions:**
  1. Take a sheet of paper.
  2. Fold it in half once.
  3. Unfold and observe the number of layers.
  4. Repeat the folding process up to four times, folding the paper in half each time.
  5. After each fold, record the total number of layers.

- **Initial Questions:**
  - How many layers are there after the first fold?
  - How many layers are there after the second fold?
  - What about the third fold?
  - Can you predict the number of layers after the 10th fold?

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Write down your initial ideas:

Activity 2: Contextual Exploration

- **Objective:**
  Engage students in hands-on exploration to observe the pattern of exponential growth.

- **Instructions:**
  1. In groups, perform the paper folding activity as described.
  2. After each fold, count and record the number of layers.
  3. Discuss within your group the pattern you observe.

Activity 3: Concept Formulation

Objective:
Identify and articulate the pattern observed in the paper-folding activity.

Table 1: Observation of Paper Folding

Fold Number (n)	Number of Layers	Difference from Previous Fold

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Discussion Questions:

- **What changes from one fold to the next?**
- **Is there a pattern in the change in the number of layers?**
- **Try to describe the pattern of change in your own words.**

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Activity 4: Mathematization

Objective:
Develop a mathematical formula to represent the observed pattern.

Understanding the Terms:

- First term (n = 1): $U_1 = 2$
- Second term (n = 2): $U_2 = 4$
- Third term (n = 3): $U_3 = 8$
- Fourth term (n = 4): $U_4 = 16$

Notice the ratio between two adjacent terms!



$$\frac{U_2}{U_1} = \frac{\dots}{\dots} = \dots$$

$$\frac{U_3}{U_2} = \frac{\dots}{\dots} = \dots$$

$$\frac{U_4}{U_3} = \frac{\dots}{\dots} = \dots$$

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Questions:

Is the ratio between two adjacent terms always the same?

Answer:

Suppose the first term = U_1 and ratio = r, so

n-th term	Value	Multiplication	Common Forms
1	2	-	U_1
2	4	2×2	$U_2 \times r$
3	8	$2 \times$	$U_3 \times r^{\cdots}$
4	16	$2 \times$	$U_4 \times r^{\cdots}$

From that pattern, try writing:

General pattern:

"The number of paper folds in row n is found by multiplying the number of folds in the previous row by the ratio ____ "

- **Create an Explicit Formula**

Now, compile the general formula for the number of paper layers after the nth fold:

- First term: a = 2 (after the first fold, there are 2 layers)
- Common ratio: r = 2 (each fold doubles the number of layers)

So, the general formula is:

U_n = ...× .....

U_n = ...× .....

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Activity 5: Generalization

- Formula Trials

Use the n-th formula to calculate:

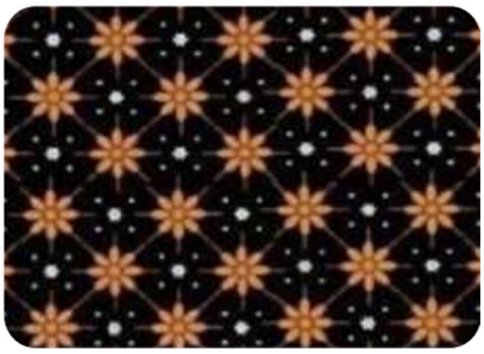
Number of 5th rows

Number of 7th rows

Number of 8th rows

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EXERCISE



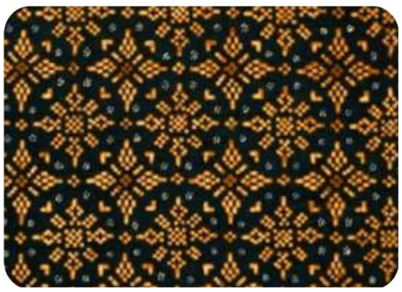
The picture above is one of the Batik motifs from Yogyakarta. Batik *Truntum*, with an irregular flower pattern, can be interpreted as a form of a triangular number pattern. In this case, we can pay attention to the arrangement of the flowers and identify the growth patterns associated with the number of triangles. Take a look at the arrangement of motifs below that make up the triangle number pattern!



Mr. Joko, a Batik craftsman in Yogyakarta, is making Batik *Truntum* motifs with ornamental arrangements that form triangles. In the first row there are 1 ornaments, in the second row there are 3 ornaments, and in the third row there are 6 ornaments, following the pattern of triangle numbers.

a. How many ornaments are there in total if Mr. Joko completes the Batik *Truntum* motif up to the 17th row?
b. Make an nth formula based on the case!

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The picture above is one of the Batik motifs from Yogyakarta, namely *Nitik* Batik motifs. The Batik *Nitik* motif depicts an odd number pattern through an arrangement of dots that form an orderly geometric pattern. Notice the arrangement of motifs below that form an odd number pattern!



A Batik *Nitik* artisan in Yogyakarta is designing a new motif that follows an odd number pattern to arrange the dots on the Batik fabric. In such a design, each row has many points that follow a sequence of odd numbers.

a. If each point takes 2 minutes to draw, how long does it take to complete the Batik *Nitik* motif to the 10th row?
b. Is this time realistic to complete in a day? Explain your reasons

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Mathematics Learning Outcomes and Student Engagement Through the Implementation of a Flipped Classroom Learning Model

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Abstract: This study compares the improvement in mathematics learning outcomes and student engagement between students participating in the flipped classroom model and those participating in conventional learning. This quantitative research employed a pretest-posttest control group design. The research sample consisted of 127 Grade 11 students from one of the high schools in Banda Aceh. We collected data using two instruments: a mathematics learning outcome test and a student engagement questionnaire. The results of the data analysis indicate that the improvement in mathematics learning outcomes and student engagement for students who participated in flipped classroom learning was better than that of students who participated in conventional learning. However, there was no significant difference in behavioral engagement between the experimental and control groups. The findings demonstrate that the flipped classroom model can improve student engagement and mathematics learning outcomes.

Keywords: flipped classroom learning model, mathematics learning outcomes, and student engagement

INTRODUCTION

Technological developments have stimulated the education system and teachers to integrate technology into learning, thus providing more experiences for students to interact with technology as a powerful learning tool (Clark-Wilson et al., 2015). This development allows teachers and students to communicate through the Internet and increases opportunities for better learning (Martinovic et al., 2013). These findings highlight how technological advancement supports improvements in students' mathematics outcomes by offering interactive, flexible, and engaging tools for learning.

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However, in the 21st century generation of learners, technology-enabled mathematics learning in the traditional classroom has made little progress in improving mathematics learning (Limniou et al., 2018). This is also seen in one of the Senior High Schools in Banda Aceh, Indonesia. Based on these observations, students had insufficient preparation before entering the class. Students learn when the teacher presents the material using PowerPoint, discusses concepts, and solves problems at home. These materials are collected before the next session without further presentation or discussion related to the assigned tasks.

Another concern is the use of class time. Pre-discussion activities before group work in class consume significant time (Chang & Brickman, 2018), whereas the typical duration of mathematics classes in schools is only 90 minutes per session. This limited time-frame allows teachers enough time to thoroughly and accurately explain each mathematical concept, leaving fewer opportunities for students to actively engage in learning. Consequently, this contributes to students' low understanding of the material and poor mathematics learning outcomes.

To address these challenges, a learning model is required to actively engage students in learning and enable them to solve diverse mathematical problems while effectively utilizing information and communication technology. One such model is the flipped classroom learning model. The flipped classroom represents a strategic reversal of the traditional classroom structure (Bergmann & Sams, 2012). In this model, students engage in initial exposure or exploration of learning content outside the classroom, while in-class time is dedicated to activities such as discussion, reflection, problem-based learning, and evaluation (Abah et al., 2017; Bergmann & Sams, 2012; Lo et al., 2017; Singh et al., 2018; Yang et al., 2019).

Students watch the video on the Canvas LMS platform prior to in-class learning. Canvas LMS enables teachers to create various content, including assignments, quizzes, discussions, and learning videos (Pahliansyah, 2021). After studying the basic material, students must make individual notes and compile a list of questions related to the content on Canvas LMS for in-class discussion (Ramadhani et al., 2019; Wei et al., 2020). This model fosters increased teacher-student interactions during classroom learning sessions (Singh et al., 2018; Utami, 2017).

The flipped classroom model has garnered significant attention in education (Goodwin & Miller, 2013; Hamdan et al., 2013). Its benefits have been widely documented over the past decade (Bishop & Verleger, 2013; Chua & Lateef, 2014; Love et al., 2014; Missildine et al., 2013; Yarbro et al., 2014). Studies have shown that students engaged in flipped classroom learning often achieve better mathematics learning outcomes than those in traditional classes (Nurfadilla, 2022; Pratiwi, 2021; Tirani et al., 2022), unlike previous research, which primarily compared learning outcomes between two groups, this study examines differences in the improvement of learning outcomes for students engaged in flipped classroom learning versus those in conventional learning.

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Recent studies have also explored the impact of the flipped classroom model on student engagement (Cevikbas & Kaiser, 2022; Cronhjort et al., 2018; Lo & Hew, 2020; Hodgson et al., 2017). However, few have specifically addressed student engagement within the context of high school mathematics education, nor have they compared the enhancement of engagement between flipped classrooms and conventional classes. Cronhjort et al. (2018) holistically examined student engagement in flipped classrooms without focusing on individual engagement components, such as behavioral, cognitive, and emotional dimensions. Conversely, some researchers have examined only specific dimensions, such as cognitive engagement (Lo & Hew, 2020) or behavioral engagement (Hodgson et al., 2017). Therefore, in contrast to previous studies, this research delves deeper into student engagement by examining all three dimensions: behavioral, cognitive, and emotional. In addition, this study will specifically measure students' mathematics learning outcomes, focusing on their mathematical understanding and representation ability related to functions and their modeling. To support out-of-class instruction and learning activities, the Canvas LMS platform will be utilized, allowing for structured access to instructional content and interaction beyond the classroom. Based on the aforementioned considerations, this study proposes the following research questions.

a. Is the improvement in mathematics learning outcomes of students participating in flipped classroom learning greater than that of students participating in conventional learning?

b. Is the improvement in engagement of students participating in flipped classroom learning greater than that of students participating in conventional learning?

c. Are the behavioral, emotional, and cognitive engagement levels of students participating in flipped classroom learning greater than those participating in conventional learning?

LITERATURE REVIEW

Student Engagement

This research focuses on the three dimensions of student engagement described by Fredricks et al. (2004). This three-dimensional framework is based on a comprehensive synthesis of educational research, encompassing all aspects of student engagement in the school environment. These dimensions have been widely adopted in student engagement research (Bond, 2020).

First, the core concept of behavioral engagement is participation, which includes both positive behaviors (e.g., following rules) and the absence of disruptive behaviors (e.g., skipping class) (Fredricks et al., 2004). Behavioral engagement also encompasses completing pre-class learning tasks in flipped classroom models (Cevikbas & Kaiser, 2022). Second, emotional engagement pertains to students' affective reactions, such as interest, satisfaction, and anxiety. Third, cogni-

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tive engagement refers to the degree of investment in learning (Fredricks et al., 2004). This dimension is characterized by students' willingness to exceed learning objectives and their preference for challenges, such as a desire to comprehend complex material (Mursalina et al., 2019).

Flipped Classroom Learning Model

The literature offers various definitions of the flipped classroom. According to Kellinger (2012), a flipped classroom is a learning model in which students utilize technology to master content at their own pace at home and engage in group discussions to prepare for subsequent lessons. Kordyban and Kinash (2013) described the flipped classroom as a model in which students access traditional content outside of class through mediums such as videos, podcasts, interactive games, blogs, or traditional reading materials. Their homework may involve watching videos, taking notes, and preparing questions or discussion points for the next class session. Based on these definitions, the flipped classroom can be characterized as a learning model with two components: activities conducted outside the classroom, such as watching videos, listening to podcasts, or reading articles, and interactive learning during class sessions, often facilitated through group discussions.

The flipped classroom offers several significant advantages. It fosters teamwork and class discussion, enables students to watch videos anytime and anywhere, allows learners to set their own pace according to their needs, and encourages critical thinking inside and outside the classroom (Kellinger, 2012). Additionally, this model allows teachers to employ various learning strategies, providing more time for teacher-student interactions and enabling educators to better understand students' emotional needs (Susanti et al., 2019). Parents can also play a supervisory role, offering support to their children. Furthermore, the flipped classroom model gives students more creative exploration time (Goodwin & Miller, 2013).

Despite its advantages, implementing the flipped classroom model poses several challenges. The most significant challenge is the time required to prepare teaching materials, such as videos, images, and written content, particularly during the first year of implementation in schools or institutions (Susanti et al., 2019). Another challenge is ensuring that students fulfill their responsibilities outside the classroom (Bristol, 2014). Additionally, some students face obstacles in adopting the flipped classroom model due to a lack of access to essential equipment, such as smartphones, tablets, or computers, and issues with internet connectivity (Kordyban & Kinash, 2013). Thus, the flipped classroom model's benefits and challenges may arise from the model's design, the learning environment, or the way educators implement it.

Mathematics Learning Outcomes

Learning outcomes are a central indicator of the effectiveness of the educational process. According to Aditya (2016), learning outcomes refer to the abilities acquired by students due to their learning experiences in school. These relatively stable outcomes represent potential behavior changes resulting from knowledge and training. Similarly, Pandey (2017) defines learning

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outcomes as the acquisition of knowledge or information demonstrated through students' understanding, skills, and techniques developed in a particular subject at a given stage. In this context, mathematics learning outcomes reflect students' mathematical abilities gained through the teaching and learning process over time, typically measured through test scores.

Multiple factors influence students' mathematics learning outcomes. Gunaseelan and Pazhanivelu (2016) categorize these factors into demographic, instructional, and individual domains. Expanding on this, Maamin et al. (2021) highlight that in ASEAN countries, student-related factors (such as gender, demographics, skills, knowledge, attitudes, and engagement) and family-related factors (such as socioeconomic status) significantly affect mathematics achievement. Additionally, teacher-related aspects—including pedagogical content knowledge, teaching strategies, and attitudes—and school-level factors, such as institutional policies and engagement, are also influential. These interconnected factors must be considered early in the learning design to enhance student outcomes effectively.

Given these influencing factors, the flipped classroom model has emerged as a promising way to improve mathematics learning outcomes. In this model, direct instruction occurs outside of class, freeing classroom time for collaborative problem-solving, discussions, and deeper learning activities. Research consistently shows that the flipped classroom can positively influence students' performance in mathematics. Research consistently shows that this model can positively impact students' mathematics learning outcomes. For instance, Baybayon and Lapinid (2024) reported a significant difference between pre-assessment and summative assessment scores, with a substantial effect size, indicating improved student performance in solving quadratic equations. Similarly, O'Flaherty and Phillips (2019) found that incorporating quizzes and maintaining face-to-face class time further strengthened the effectiveness of flipped classrooms. Other studies, such as those by Sopamena et al. (2023), have shown moderate but consistent positive effects on student achievement in university-level mathematics.

While benefits are evident, challenges such as technological barriers and varied student readiness remain, as noted in Indonesian-based studies (e.g., IOP Science, 2020). Addressing these issues through thoughtful instructional design and support is critical for optimizing the impact of flipped learning environments. In conclusion, individual, instructional, familial, and institutional factors shape students' mathematics learning outcomes. When effectively implemented and supported, the flipped classroom model provides a conducive framework for addressing these variables and enhancing student achievement in mathematics.

METHODS

Research Design

This study employed a quantitative research approach with a pre-post-test control group design. Both pretests and post-tests were administered to the control and experimental groups. The experimental group utilized the flipped classroom learning model, while the control group adhered to the conventional learning model. The study was conducted over a total of 12 hours

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spanning four weeks. During this period, the experimental group was required to bring a cell phone or mobile wireless device to school.

Sample

The research population comprised 175 Grade 11 students from a high school in Banda Aceh, Indonesia, distributed across five classes with 35 students per class. The sampling technique used was cluster random sampling, resulting in a sample of 127 students, with 66 students assigned to the experimental group and 61 to the control group.

Instrument

The primary instrument used in this study was a mathematics learning outcomes test. This test consisted of nine long-answer questions incorporating various problems related to functions and their modeling. The test was designed based on indicators of mathematical understanding and representation skills. The indicators of mathematical understanding were adapted from Kilpatrick and Swafford (2020) and included 1) Representing learned concepts, 2) Classifying objects based on whether they meet the criteria defining a concept, 3) Applying concepts algorithmically, 4) Providing examples and non-examples of learned concepts, 5) Presenting concepts in various mathematical representations, and 6) Establishing connections among different concepts, both internal and external to mathematics.

The indicators of mathematical representation refer to three types proposed by Villegas (2009) (see Table 1).

Representation	Indicator
Pictorial Representation	Create tables, diagrams, or graphs to solve the given problem.
Symbolic Representation	Mathematical modeling is used to solve the problem based on the given information.
Verbal Representation	Solve problems using written or spoken words.

Table 1: Indicator of Mathematical Representation

This study also utilized a modified student engagement questionnaire based on Norazmi et al. (2017). The questionnaire employed a Likert scale of 24 positive and negative statements, each with five response options. Each response was assigned a score ranging from 1 to 5. In a study on student engagement by Fredricks et al. (2004), student engagement encompasses multiple dimensions. Therefore, the questionnaire was adapted from their meta-construct to measure student engagement across three dimensions: behavioral, cognitive, and emotional, in alignment with the objectives of this study.

The following modifications were made to ensure the questionnaire's relevance and clarity for this research: The original English version was translated into Indonesian to ensure students

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could easily comprehend each statement. All "flipped classroom" instances were replaced with "math learning" to avoid potential confusion among students.

The questionnaire items were tailored to reflect the mathematics material taught. Specifically, the term "Writing" from English learning was replaced with "Functions and their Modeling" to align with the mathematics content of this study. The instruments were validated by two experts: one specializing in Mathematics Education and the other in Measurement and Evaluation. Both instruments were subsequently trial-tested. The reliability coefficient for the math learning outcomes test, calculated using the Kuder-Richardson Formula 20 (KR-20), was 0.718. Similarly, the internal consistency of the student engagement questionnaire was determined using Cronbach's Alpha, yielding a reliability coefficient of 0.837.

Data Analysis

We examined the normality of the data on mathematics learning outcomes and student engagement to determine the appropriate statistical tests. The results indicated that the N-Gain data for mathematics learning outcomes were normally distributed ($p > 0.05$). Consequently, the data were analyzed using the Independent Sample t-test to address the first research question. In contrast, the N-Gain data for student engagement were not normally distributed ($p < 0.05$). Therefore, non-parametric statistical tests, specifically the Mann-Whitney U Test, were employed to address the second research question. For the third research question, post-test data were analyzed using the Independent Sample *t*-test. All statistical analyses were conducted using SPSS version 25.

RESULTS

After the pretest, students were assisted in creating an account and instructed on using the Canvas application. They watched a video outside the classroom and posted their questions in the comment section. Other students responded to these questions, while the teacher corrected any inaccurate answers (see Figure 1). Following the discussion activity outside the classroom, the students completed a quiz to assess their understanding after watching the video.

In class, the teacher organized students into groups to solve the problems provided in the worksheet. Students were encouraged to present the outcomes of their discussions and engage in debates on the given problems. Additionally, this process was continued until the end of the learning session.

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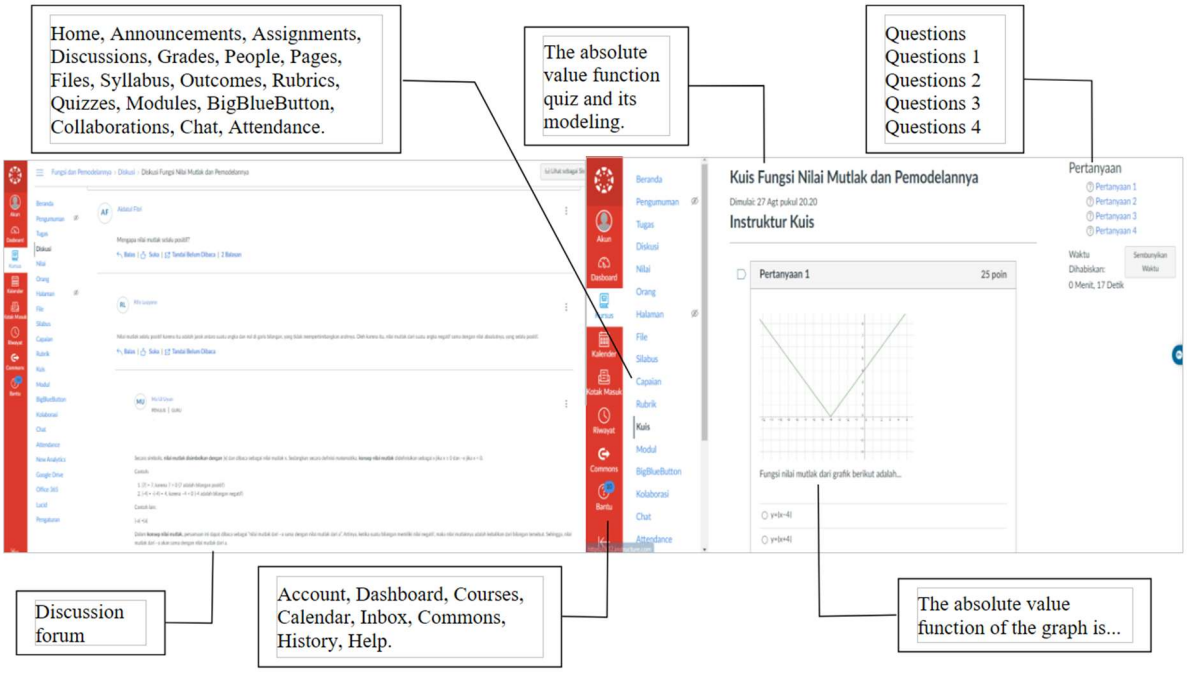


Figure 1: Students' Activities Outside the Classroom Through Canvas Application

Findings Related to Question 1

The descriptive statistics and the t-test results of the N-Gain for the mathematics learning outcomes data are presented in Table 2.

Group	Test	*N*	Mean	*SD*	Average N-Gain	*t*	*p*
Experiment	Pretest	66	38.33	11.57	0.57	4.754	<0.001*
	Posttest	66	73.98	9.73			
Control	Pretest	61	40.04	9.72	0.42		
	Posttest	61	66.35	7.30			

Table 2: The t-test results of the N-Gain for the mathematics learning outcomes data
*$p < 0.001$ is reported as 0.000 in the statistical output.

Based on Table 2, the t and p values demonstrate a significant difference between the experimental and control classes' N-Gain ($t = 4.754$, $p < 0.001$). This finding indicates that the average improvement in mathematics learning outcomes for students who participated in flipped classroom learning was significantly higher than that of students who participated in conventional learning.

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Findings Related to Question 2

Table 3 presents the descriptive statistics of student learning engagement and the results of the Mann-Whitney U Test on the N-Gain scores for the experimental and control groups.

Group	Test	*N*	Mean	*SD*	Average N-Gain	*U*	*p*
Experiment	Pretest	66	71.89	13.27	0.09	1417.000	0.004
	Posttest	66	77.47	12.72			
Control	Pretest	61	73.11	13.78	-0.05		
	Posttest	61	71.02	16.51			

Table 3: The t-test results of the N-Gain of Student Learning Engagement

Table 3 shows a significant difference between the N-gains of the experimental and control classes (U = 1417.000, p = 0.004). It can be concluded that the average increase in learning engagement of students who participated in flipped classroom learning was better than that of students who participated in conventional learning.

Findings Related to Question 3

To determine the level of students' behavioral, cognitive, and emotional involvement in each item of the involvement questionnaire statement, we first calculated using the Independent Sample t-test on the post-test data of the two sample groups.

	Group	N	Mean	SD	*t*	*p*
Behavior	Experiment	66	3.25	0.52	1.674	0.097
	Control	61	3.10	0.43		
Emotional	Experiment	66	3.31	0.52	4.008	<0.001*
	Control	61	2.90	0.62		
Cognitive	Experiment	66	3.39	0.60	4.187	<0.001*
	Control	61	2.88	0.73		

Table 4: T-test Results of Engagement Data Based on Behavioral, Emotional, and Cognitive Dimensions
*$p < 0.001$ is reported as 0.000 in the statistical output.

Based on Table 4, the t and p values indicate no significant difference in behavioral engagement between the experimental and control classes (t = 1.674, p = 0.097). This suggests that the average behavioral engagement of students who participated in flipped classroom learning was not higher than that of students who engaged in conventional learning. However, for the emotional dimension, there was a significant difference between the two classes (t = 4.008, $p < 0.001$), indicating that the average emotional engagement of students who participated in flipped classroom learning was higher than that of students in conventional learning. Similarly, the cognitive

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dimension also showed a significant difference between the two classes ($t = 4.187$, $p < 0.001$), demonstrating that the average cognitive engagement of students in the flipped classroom model outperformed those in conventional learning.

Statement	Experiment	Control	Sig.
B1: I can focus well during math lessons.	3.09	2.40	<0.001*
B2: I actively participate in math education.	3.52	2.95	<0.001*
B3: I can complete the assigned tasks effectively.	3.26	3.47	0.179
B4: I can complete assigned tasks using the learning materials provided.	3.52	3.07	0.006
B5: I strive to complete activities and tasks during math learning.	3.09	3.08	0.897

Table 5: Average Difference in Behavioral Engagement Items
*$p < 0.001$ is reported as 0.000 in the statistical output.

Based on Table 5, item B2: "I actively participate in math education" (mean = 3.52), and item B4: "I can complete assigned tasks using the learning materials provided" (mean = 3.52) received the highest scores among all items in the experimental class. In contrast, item B1: "I can focus well during math lessons" (mean = 3.09), and item B5: "I strive to complete activities and tasks during math learning" (mean = 3.09), received the lowest scores among all items in the control class. According to the significance values (Sig.) in Table 5, there is a significant difference ($p < 0.001$ for B1 and B2, $p = 0.006$ for B4) between the two groups on most items, except items B3 and B5.

Statement	Experiment	Control	Sig.
E1: I am happy with the math I have learned in this class.	3.52	3.06	0.007
E2: I think the math lessons are interesting.	3.52	2.46	<0.001*
E3: I feel confident doing the tasks given to me during the math lesson.	3.10	2.62	0.004
E4: I liked the math lesson because it allowed me to learn at my own pace.	3.09	2.46	<0.001*
E5: I like how mathematics is conducted because it allows me to learn anytime and anywhere.	3.52	3.15	0.039
E6: The math lesson was boring.	3.32	2.96	0.038
E7: The math lesson motivated me to learn about functions and their modeling.	3.52	3.07	0.006
E8: I feel anxious (worried, anxious, afraid, nervous) learning about functions and their modeling without a teacher.	3.52	3.33	0.089
E9: I enjoy receiving feedback on the tasks I have completed.	3.10	3.07	0.843
E10: I am worried about my ability to solve function problems and their modeling after participating in	3.26	2.36	<0.001*

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	math learning.			
E11:	I feel ready to learn about functions and their modeling with the implemented math learning.	3.52	3.07	0.007

Table 6: Average Difference in Emotional Engagement Items
*$p < 0.001$ is reported as 0.000 in the statistical output.

Based on Table 6, the items E1: "I am happy with the math I have learned in this class" (mean = 3.52), E2: "I think the math lessons are interesting" (mean = 3.52), E5: "I like the way mathematics is conducted because it allows me to learn anytime and anywhere" (mean = 3.52), E7: "The math lesson motivated me to learn about functions and their modeling" (mean = 3.52), E8: "I feel anxious (worried, anxious, afraid, nervous) learning about functions and their modeling without a teacher" (mean = 3.52), and E11: "I feel ready to learn about functions and their modeling with the implemented math learning" (mean = 3.52) received the highest scores among all items in the experimental class. According to the significance values (Sig.) in Table 6, most items showed significant differences between the experimental and control groups. Items E2, E4, and E10 were highly significant ($p < 0.001$), items E1, E3, E5, E6, E7, and E11 were significant ($p = 0.006–0.039$), except items E8 and E9.

Statement	Experiment	Control	Sig.
C1: I experience confusion when learning mathematics.	3.52	3.07	0.008
C2: I find learning functions and modeling materials easy with the implemented math learning.	3.52	2.62	0.003
C3: I often review learning materials to complete my assignments.	3.26	2.76	0.003
C4: I can clearly express my ideas to complete the task with the implemented math learning.	3.52	2.36	0.006
C5: The math lesson helped me review the lesson.	3.57	2.62	<0.001*
C6: The math lessons helped me remember what I had learned and what I was learning in class.	3.55	2.62	<0.001*
C7: I believe that I can solve problems with functions and their modeling well.	2.98	2.69	0.072
C8: Math learning encouraged me to explore more about functions and their modeling online to complete assignments.	3.26	2.76	0.003

Table 7: Average Difference in Cognitive Engagement Items
*$p < 0.001$ is reported as 0.000 in the statistical output.

Item number C5: The math lesson helped me review the lesson (mean = 3.57) in Table 7, which received the highest score among all items in the experimental class. Item C7: I believe I can solve problems or problems of functions, and their modeling well (mean = 2.98) got the lowest score among all items in the control class. Based on Table 7, most items showed significant differences between the experimental and control groups. Items C5 and C6 were highly significant

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($p < 0.001$), items C1, C2, C3, C4, and C8 were significant ($p = 0.003$–0.008), whereas item C7 did not show a significant difference.

DISCUSSION

Improvement of Students' Mathematics Learning Outcomes

The data analysis revealed that students who participated in flipped classroom learning demonstrated a greater average improvement in mathematics learning outcomes than in conventional learning. This result can be attributed to several advantages of the flipped classroom model. One of its key strengths is the incorporation of various interactive activities into the classroom, such as problem-solving, discussions, question-and-answer sessions, and group work (Bergmann & Sams, 2012; Bishop & Verleger, 2013; Tekin & Emmioğlu-Sarikaya, 2020).

Another notable advantage of the flipped classroom model is that students can revisit video content as needed, allowing repeated and personalized engagement with the material (Bergmann & Sams, 2012). For instance, students in the study by Clark-Wilson et al. (2015) reported that watching videos enhanced their comprehension of lesson content. Abeysekera and Dawson (2015) further highlighted that video-based learning can help students manage cognitive load and support those underprepared during the initial stages of learning. Similarly, it is noted that this model benefits students by offering additional support early in the learning process (Lo & Hew, 2020).

These findings align with those by Uy (2022), which demonstrated that students in the experimental group who learned mathematics through the flipped classroom model significantly outperformed their peers in the control group. Harmini et al. (2022) also reported that students studying calculus with the flipped classroom model achieved higher scores than those in conventional classrooms. Wei et al. (2020) found that students utilizing the flipped classroom model outperformed those using traditional instructional methods.

Increased Student Learning Engagement

The results of the data analysis showed that the average learning engagement of students who participated in flipped classroom learning was better than that of students who participated in conventional learning. Cassum et al. (2017) suggested that when teachers provide active learning opportunities in the classroom, such activities should encourage communication, student engagement, creativity, independence, and critical thinking. Students' experiences in both groups showed that the flipped classroom model positively impacted student engagement. Therefore, active learning, peer instruction, and discussion in the flipped classroom were critical for the success of this model.

The results of this study are in line with several other studies. Lo and Hew (2020) showed that students in gamification-based flipped classrooms significantly outperformed those with a non-

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flipped classroom model (lecture-based and online teaching). The findings indicate that students in flipped classrooms have higher levels of cognitive engagement. Similarly, Muir and Geiger (2016) and Muir (2017) demonstrated that the flipped classroom increased student engagement in mathematics at the school level. Clark (2015) also found that the flipped classroom positively affects engagement, especially during classroom activities. In addition, Hodgson et al. (2017) attempted to assess behavioral engagement in three pre-college settings using an observation instrument. However, they found that the flipped classroom intervention did not increase behavioral engagement more than traditional learning, contrary to commonly reported results.

Behavioral, Cognitive, and Emotional Engagement of Students

The study results show that the behavioral engagement of students who participated in flipped classroom learning was not better than that of students who participated in conventional learning. The questionnaire results revealed that students in the flipped classroom could not effectively complete the learning activities. It is possible that this finding of student behavioral engagement is an impact of students' lack of ability to use technological media. This causes students to not perform all activities that should be performed outside the classroom. Students who do not conduct activities outside the classroom watch videos and take quizzes at the beginning of learning in class, while students who have done quizzes still ask questions about things they do not understand.

Akçayır & Akçayır (2018) found that inadequate student preparation before class is one of the main challenges in flipped classrooms. Other studies have also reported decreased student attendance (Giannakos et al., 2014). This is supported by Larsen (2015), who observed that secondary students tend to withdraw from class discussions and even deliberately skip class. In his case, the flipped classroom had a negative effect on students' behavioral engagements.

One of the main challenges in the flipped classroom is students' readiness or preparedness to participate in in-class activities, e.g., watching the assigned recorded videos before class. Failure to do this will affect in-class learning because students who have not completed their homework will not have the basic concepts or knowledge of the topic and cannot fully participate in the in-class activities (Skamp et al., 2019). According to Akçayır & Akçayır (2018), students who have not worked on out-of-class activities affect their preparation and learning when in class. Therefore, adequate preparation and support should be considered when planning lessons implementing the flipped classroom model (Leo & Puzio, 2016; Ngo & Eichelberger, 2021).

The results also revealed that the emotional and cognitive engagement of students who participated in flipped classroom learning was better than that of students who participated in conventional learning. Several concurrent studies (e.g., Dove & Dove, 2015) have reported that positive flipped classroom learning experiences helped students reduce their anxiety when learning mathematics. Lo and Hew (2020) found that using game design elements (e.g., digital points and badges) in the flipped classroom can motivate students to complete more difficult tasks with higher quality than students in a traditional learning environment. Meanwhile, Van Sickle (2016)

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found that students in an algebra class that implemented a flipped classroom were less satisfied with the class than those in a traditional classroom. Research findings related to emotional engagement in flipped classrooms appear to be inconsistent.

CONCLUSIONS

Based on the research analysis results, it can be concluded that learning with the flipped classroom model has a significant positive impact on students' mathematics learning outcomes, which are better than conventional learning. In addition, the flipped classroom model also increases student engagement, emotionally and cognitively, which is higher than the conventional model. However, although students' emotional and cognitive engagement was more optimal, students' behavioral engagement in flipped classroom learning did not show a significant difference compared to conventional learning. This finding shows that the flipped classroom can potentially improve the quality of learning. However, some aspects must be considered to improve students' behavioral engagement.

LIMITATIONS AND RECOMMENDATIONS

This study has several limitations. First, the experiment was conducted over a relatively short period—four weeks, encompassing four meetings and a pretest and posttest. While this timeframe was sufficient to cover the target material, extending the duration of the study in future research is recommended. A longer study period would enhance the reliability and generalizability of the findings.

The second limitation concerns students' readiness to actively engage in flipped classroom activities. Some students faced challenges downloading teaching materials, watching instructional videos, and completing quizzes on the Canvas application. As a result, they could not fully participate in all the activities intended for completion outside the classroom. To address this, teachers and future researchers should allocate additional time beyond a single day to train students on navigating the LMS platform, watching educational videos, extracting key ideas, and completing quizzes before implementing the flipped classroom model.

In addition, future studies may consider incorporating student perspectives to better understand the challenges they encounter in a flipped classroom environment. Including interviews or open-ended questions in the questionnaire could allow students to elaborate on their experiences and the reasons behind their level of engagement or preparedness, particularly regarding out-of-class learning tasks. Gaining qualitative insights into students' perceptions of non-readiness or lack of preparedness during synchronous sessions can provide valuable context for identifying specific barriers to effective learning and inform more responsive instructional strategies.

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The findings of this study, conducted in Aceh, Indonesia, contribute to the growing body of international research on the flipped classroom model and its effectiveness in enhancing student learning outcomes and engagement in mathematics. While this study demonstrates the model's success in an Indonesian context, its implications extend beyond local settings. Similar trends have been observed globally, with research in various educational systems, such as the United States engagement (Lo & Hew, 2017) and Australia (Muir & Geiger, 2016), reporting improvements in students' cognitive, behavioral, and emotional engagement through flipped learning. These parallels suggest that the flipped classroom model is effective in specific cultural and educational contexts and can be adapted to different learning environments worldwide. However, the challenges identified in this study, such as students' readiness to engage in out-of-class activities, highlight the need for context-specific strategies to optimize the model's effectiveness. Future research should explore how socio-cultural factors, technological access, and institutional support influence the success of flipped learning in diverse global settings, ensuring that educators can tailor this pedagogical approach to meet the unique needs of students across various regions.

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APPENDIX

Appendix 1. Student Engagement Questionnaire

Name : .....
Grade : .....
Date of Completion : .....

Filling Instructions :

1. This questionnaire is solely for academic purposes. Please provide an honest answer.
2. Read and answer all questions carefully without missing any information.
3. Select the answer that is considered most appropriate to the conditions and circumstances that occur by placing a check mark (√) in the column provided with the following information:
   SD : Strongly Disagree
   D : Disagree
   MA : Moderately Agree
   A : Agree
   SA : Strongly Agree

No.	Statement	SD	D	MA	A	SA
1.	I can focus well during math lessons.					
2.	I actively participate in math education.					
3.	I can complete the assigned tasks effectively.					
4.	I can complete assigned tasks using the learning materials provided.					
5.	I strive to complete activities and tasks during math learning.					
6.	I am happy with the math I have learned in this class.					
7.	I think the math lessons are interesting.					
8.	I feel confident doing the tasks given to me during the math lesson.					
9.	I liked the math lesson because it allowed me to learn at my own pace.					
10.	I like how mathematics is conducted because it allows me to learn anytime and anywhere.					
11.	The math lesson was boring.					

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No.	Statement	SD	D	MA	A	SA
12.	The math lesson motivated me to learn about functions and their modeling.					
13.	I feel anxious (worried, anxious, afraid, nervous) learning about functions and their modeling without a teacher.					
14.	I enjoy receiving feedback on the tasks I have completed.					
15.	I am worried about my ability to solve function problems and their modeling after participating in math learning.					
16.	I feel ready to learn about functions and their modeling with the implemented math learning.					
17.	I experience confusion when learning mathematics.					
18.	I find learning functions and modeling materials easy with the implemented math learning.					
19.	I often review learning materials to complete my assign-ments.					
20.	I can clearly express my ideas to complete the task with the implemented math learning.					
21.	The math lesson helped me review the lesson.					
22.	The math lessons helped me remember what I had learned and what I was learning in class.					
23.	I believe that I can solve problems with functions and their modeling well.					
24.	Math learning encouraged me to explore more about functions and their modeling online to complete assign-ments.					

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Appendix 2. Math Learning Outcomes Pretest

Math Learning Outcomes Pretest

Subject : Math
Grade : XI
Problem Form : Essay
Material : Functions and Modeling
Number of Questions : 9 Items
Time : 60 min

Problem-solving Instructions:

1. Write your name and grade in the upper right corner of your answer sheet.
2. Look at each question provided. If there are questions that are not clear, ask the supervisor.
3. Do the questions honestly and independently.
4. Problems may be solved out of order.
5. Answer the given problem to the best of your ability and include the steps for solving the problem.

Questions:

1. Explain the definition of a s function.
2. Explain the difference between the shape and properties of a *piecewise* function graph and an absolute value function graph.
3. Name one example of each absolute value function and ladder function.
4. Tari is a 200-m runner. She participated in a race to commemorate the Independence Day of the Republic of Indonesia. She ran on a 200-m track in her village. For documentation purposes, a photographer stands next to the exact center of the track. Write down the function that expresses Tari's distance from the photographer in terms of Tari's distance from the track.
5. Determine the number of bacteria at the 6th minute present on a surface that started out as 50 if their proliferation is determined by an exponential function$f(t) = J_0(2)^t$ with t in minutes.
6. An employee in a company receives a salary increase because he or she has performed well. The company applies the rule that the deviation in the salary of employees with the same rank (position) is Rp 350,000.00. If the employee's salary was originally Rp2,000,000.00, determine the lowest and highest salary of employees with the same rank as the employee who received a salary increase.
7. *Playstation* rental place has the following rules. For the first hour, a fee of 20,000 is charged. If it exceeds 1 hour, up to 2 hours, the renter pays 4000. If it exceeds 2 hours until the three-hour limit, the renter must pay 60,000. The rule applies for the following hours as well. Draw a graph explaining the relationship between the length of *playstation* rental and the fee charged.
8. He walked from his house next to the highway. He cycled along the sidewalk beside the highway to the playground. The distance from his house to his playground was 10 km. Halfway between the shopping center and Made's house, there is a Singkawang Restaurant. Find the function that expresses Made's distance to Singkawang Restaurant in Made's distance to home.
9. Consider the following function:

$$f(x) = \begin{cases} 20x, 0 \le x < 2 \\ 30x - 20, 2 < x \le 3 \\ 10x + 20, 4 < x \le 6 \end{cases}$$

The function above represents the total distance traveled divided by the travel time of the three competitions of kayaking, cycling, and running in which Budi participated consecutively. Write down the information that can be obtained from the function.

Appendix 3. Math Learning Outcomes Posttest

Math Learning Outcomes Posttest

Subject : Math
Grade : XI
Problem Form : Essay
Material : Functions and Modeling
Number of Questions : 9 Items
Time : 60 min

Problem-solving Instructions:

1. Write your name and grade in the upper right corner of your answer sheet.
2. Look at each question provided. If there are questions that are not clear, ask the supervisor.
3. Do the questions honestly and independently.
4. Problems may be solved out of order.
5. Answer the given problem to the best of your ability and include the steps for solving the problem.

Questions:

1. What is an exponential function? Write down its general form.
2. Here, we explain the difference between the shape and properties of an exponential function graph and an absolute value function graph.
3. Name at least one example of a *piecewise* function.
4. Anang is a 100-m runner. He participated in a race to commemorate the Independence Day of the Republic of Indonesia. He ran on a 100-m track in his village. For documentation purposes, a photographer stands next to the exact center of the track. Write a function that expresses Anang's distance from the photographer in terms of Anang's distance from the track.
5. Determine the number of bacteria at the 4th minute present on a surface that started out as 100 if their proliferation is determined by an exponential function$f(t) = J_0(3)^t$ with t in minutes.
6. An employee in a company receives a salary increase because he or she has performed well. The company applies the rule that the deviation in the salary of employees with the same rank (position) is Rp500,000.00. If the employee's salary was originally Rp3,000,000.00, determine the lowest and highest salary of employees of the same rank as the employee who received a salary increase.
7. In an internet cafe, the internet rental payment is calculated in the following way. For the first 10 minutes, a fee of Rp2,000 is charged. This means that if an internet cafe user accesses his/her computer and internet for less than or equal to 10 minutes, he/she must pay two thousand rupiah. Likewise, for every 10 minutes thereafter, Rp2,000 will be charged. Draw a graph explaining the relationship between the length of time using an internet cafe facility and the cost of using it.
8. Kevin participated in a competition involving four activities: kayaking, cycling, swimming, and running. He traveled 30 km in a kayak for 3 hours. Next, he spent 2 hours covering 40 km on a bike course. Then, he swam 1 km for 20 minutes. Finally, Kevin ran for 2.5 hours 25 km. Let us write the function of total distance traveled divided by time traveled.
9. Consider the following function:

$$f(x) = \begin{cases} 10x, 0 \le x < 2 \\ 25x - 30, 2 < x \le 4 \\ 10x + 30, 4 < x \le 5 \end{cases}$$

The function above represents the total distance traveled divided by the travel time of the three competitions of kayaking, cycling, and running in which Amin participated. Write down the information that can be obtained from the function.

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Improving Understanding of Trigonometric Functions Using GeoGebra

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Abstract: This study aimed to investigate the effect of the digital tool GeoGebra on vocational school students' understanding of trigonometric functions. The research used a quasi-experimental approach with a single group of students (N = 20), gathering both quantitative and qualitative data via pre-tests, post-tests, and structured interviews. Students initially engaged in regular instruction, followed by a number of interactive activities in GeoGebra. The findings indicated that the difference between pre-test and post-test outcomes was not statistically significant; nonetheless, the effect size (r = 0.46; g = 0.4742) suggests a moderate practical importance of the intervention. On the other hand, the interview findings indicated that students regard GeoGebra as an effective instrument for visualizing, accelerating problem-solving, and minimizing errors. They highlighted that the integration of regular lessons and interactive digital activities enhanced their understanding of the topic and increased motivation to learn. The findings suggest that incorporating GeoGebra into teaching trigonometry, even for short time as was the case in this study, can enhance the comprehension of abstract mathematical ideas and improve student interest and understanding.

Keywords: trigonometric functions, GeoGebra, quasi-experiment, vocational school students

INTRODUCTION

Due to its broad range of applications in science, engineering, and technology, trigonometry has become an important component of mathematics education. However, it also presents substantial challenge for both students and teachers. As Delice and Roper (2006) noted, the effectiveness of instruction in trigonometry often depends heavily on the teaching approach: some methods focus mainly on the mechanical application of trigonometric formulas, often at the expense of conceptual understanding and proofs, while others emphasize formal algebraic reasoning, which can reduce student engagement. In Croatia, where this research was conducted, trigonometry is introduced in the first year of upper secondary school (students age 15-16) and continues to be taught in the second and third grades, with varying degrees of emphasis (MZO, 2019). Despite its early inclusion in the curriculum, traditional teaching methods often fall short in addressing students' conceptual difficulties, especially in relation to functions and their graphical representations, which many students perceive as abstract and challenging.

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In recent years, the integration of digital tools in mathematics instruction has gained an increasing attention, both as a means of modernizing teaching practices and as a way to enhance students' understanding and motivation. Among these tools, GeoGebra stands out as a widely used, free, interactive software that combines dynamic geometry with algebraic, graphical, and numerical representations (Hedi et al., 2023; Ziatdinov & Valles, 2022). Its usefulness in the teaching of functions has been particularly highlighted: GeoGebra enables students to manipulate parameters and instantly observe corresponding changes in the graph, thereby promoting exploration, conceptual insight, and intuitive understanding (Bedada & Machaba, 2022). This potential is particularly important in vocational education, where students frequently report low motivation to learn mathematics, often perceiving it as disconnected from their future professions (Schmid et al., 2021). This paper investigates the impact of using GeoGebra in teaching trigonometric functions to vocational school students enrolled in a program for electrical technicians whose digital skills and career orientation made them especially receptive to applied and technology-enhanced instruction. The study adopts a teaching-research approach, where a teacher-researcher designs, implements, and evaluates instructional interventions based on reflective practice. Specifically, the study aims to examine how the use of GeoGebra influences students' cognitive and affective characteristics. Therefore, we posed the following research question: *How does the use of GeoGebra affect vocational school students' understanding and engagement with learning trigonometric functions?*

LITERATURE REVIEW

Difficulties and Misconceptions Related to the Concept of Functions

Mathematics curricula around the world attempt to help students build competency in understanding and applying functions. This learning process begins in the early grades, when students investigate patterns; continues through middle school with activities involving covarying quantities; and concludes in secondary school with the formal study of functions as mappings between sets (Carlson & Oehrtman, 2005; Cooney et al., 2010; Hatisaru & Erbas, 2017). Difficulties in understanding the concept of functions emerge as early as students' first exposure to the topic (Panaoura et al., 2015; Walde, 2017; Widada et al., 2020). Numerous studies confirm that most secondary school students are unable to define a function clearly and precisely. Typically, they find it easier to provide concrete examples than to formulate coherent and accurate definitions (Dogan-Dunlap, 2007). However, a further issue is that the examples they offer are often mathematically incorrect, reflecting intuitive yet flawed conceptions of functions (Panaoura et al., 2015; Dogan-Dunlap, 2007). These challenges are exacerbated by students' limited use of precise mathematical terminology. Without mastering mathematical language, they find it difficult to express the formal meaning of a function (Panaoura et al., 2015). The abstract nature of the concept further hinders understanding, and textbooks often contribute to misconceptions by presenting different definitions of functions. Although these definitions generally convey the same mathematical meaning, students frequently perceive them as unrelated or even contradictory (Widada et al., 2020).

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A particularly common obstacle is misunderstanding of mathematical notation, especially expressions such as $f(x)=x$, where students often struggle to interpret the role and meaning of the symbol "x" (O'Shea et al., 2016; Sajka, 2003; Walde, 2017). Related to this issue is the difficulty to transition between different representations of functions, such as moving from algebraic expressions to graphical forms and vice versa (Panaoura et al., 2015; Walde, 2017). Because the fundamental concept of the function often remains unclear, students frequently fail to connect algebraic notation with the representation of ordered pairs in a coordinate system (Eames et al., 2021). Furthermore, they often misinterpret function graphs as literal depictions of real-world situations rather than as visualizations of the relationships between input values (on the x-axis) and corresponding output values (on the y-axis) (Carlson & Oehrtman, 2005). They commonly overlook the fact that the values on the y-axis represent the codomain of the function, and that the notation $f(x)=y$ expresses the mathematical relationship between two sets (Parhizgar et al., 2021).

These difficulties become especially pronounced when learning trigonometric functions. According Hamzah et al. (2021), students exhibit various misconceptions, ranging from the incorrect application of trigonometric formulas and computational errors when determining angles to a weak grasp of the function concept in the context of trigonometry. They are often confused by expressions such as $sinx$, as they struggle to distinguish between the symbolic representation of a function and its specific value at a given angle. The study in this paper focuses specifically on trigonometric functions.

Use of Technology in Mathematics Education

The integration of technology into mathematics education has transformed how students engage with mathematical concepts. Among the most influential tools are dynamic mathematics software (DMS) applications, formerly referred to as dynamic geometry software (DGS), such as GeoGebra, Cabri, Sketchpad etc. (He et al., 2025). A key strength of DMS lies in its ability to make abstract concepts more accessible. These programs allow students to interact with mathematical objects in real time by dragging points, modifying shapes, and dynamically measuring properties (Baccaglini-Frank, 2019). By enabling students to manipulate objects and test mathematical properties, for example, transforming a triangle and observing which characteristics remain unchanged, learners move beyond surface-level observation toward deeper theoretical insight. This hands-on approach supports intuitive and reflective mathematical thinking (Segal et al., 2018). Furthermore, DMS has proven especially valuable in promoting inquiry-based learning. Guan et al. (2024) found that, compared to physical manipulatives, digital tools like DMS more effectively support both immediate learning gains and long-term retention.

Research consistently shows that such tools can enhance a wide range of mathematical skills. A recent meta-analysis by He et al. (2025), which synthesized 68 effect sizes from as many studies, found a strong overall effect on K–12 students' mathematics learning (Hedges' $g = 0.820$, $p < .01$). While the overall gains were substantial, the effects were somewhat smaller in areas related to spatial reasoning. This finding suggests that the effectiveness of DMS may depend on the specific

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mathematical domain and underscores the importance of aligning digital tools with targeted instructional goals. When thoughtfully implemented, tools like DMS tools not only improve outcomes but also foster more engaging and meaningful learning experiences.

GeoGebra in Mathematics Instruction

GeoGebra is an open and free DMS tool for math education. Although it is most frequently linked with geometry instruction, it can be used for various mathematical areas. GeoGebra's interactive interface and visual representation of mathematical expressions allow students to investigate relationships between variables, modify function parameters, and immediately observe how these changes affect graphical representations. This type of visual, experiential learning promotes a greater comprehension of mathematical topics and connects abstract theory to actual applications (Breda & Santos, 2021; Zulnaidi & Zamri, 2017). Research indicates that utilizing GeoGebra can improve students' grasp of various functions, including quadratic, exponential, and trigonometric (Birgin & Acar, 2020; Bedada & Machaba, 2022; Hidayat et al., 2024; Pospos & Piñero, 2024; Tuda & Rexhepi, 2023). GeoGebra can be also used in the context of complex topics, such as multivariable functions, where it proves to be especially useful because students frequently struggle to visualize and grasp them (Hedi et al., 2023; Lepellere, 2021). Studies have demonstrated that GeoGebra not only enhances conceptual knowledge but also increases student motivation and engagement (Hidayat et al. 2024; Pospos & Piñero 2024; Zakaria et al., 2024). Lessons become more dynamic and captivating, and students are more engaged in the learning process. Students also report that mathematics feels more approachable when visuals and interactive tools are part of instruction.

Teachers also benefit from GeoGebra in their lessons. Its use improves explanations, saves time during education, and encourages classroom participation and collaboration (Bedada & Machaba, 2022; Triet et al., 2024; Zulnaidi et al., 2020). A recent meta-analysis by Zhang et al. (2025) synthesized 20 years of research on the use of GeoGebra in mathematics education. Their findings indicate a statistically significant medium-to-large effect (Hedges's $g = 0.653$) of GeoGebra-supported instruction on students' mathematics achievement. Notably, the study found that smaller sample sizes (fewer than 50 students) and shorter treatment durations (less than 4 weeks) were associated with stronger educational effects.

METHODOLOGY

This study used a quasi-experimental design, specifically a one-group pretest–posttest model. In this approach, no control group was used; instead, participants in a single experimental group were assessed both before and after the educational intervention. The sample consisted of 20 students aged 17 and 18, enrolled in the program for electrical technicians at the vocational secondary school Electrotechnics and Traffic School Osijek. These students possess digital skills, as their

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curriculum includes subjects focused on programming and application development. All participants completed both the pre-test and post-test assessments, and four of them also took part in follow-up interviews. In previous school years, they had studied trigonometric ratios involving the sides of right-angled triangles. However, during the current academic year, they encountered for the first time the formal definitions and graphical representations of trigonometric functions. Prior to the study, all participants were informed about the research objectives, content, and duration. They were explicitly informed that participation was voluntary, that they could withdraw at any time without negative consequences, and that all collected data would remain confidential and be used exclusively for scientific purposes. The study was conducted in accordance with ethical guidelines governing educational research.

Role of Teacher

Motivated by a desire to enhance her students' understanding of trigonometric functions and to explore the potential of digital tools in mathematics education, the second author, a mathematics teacher at the above-mentioned secondary school, created a set of GeoGebra applets. These applets were designed to support visual exploration of the relationships between parameters in trigonometric functions. In addition to developing the digital materials, the second author also designed the accompanying mathematical activities. The first author, a mathematics educator, reviewed these activities and provided advisory support throughout the implementation phase.

During the intervention, the second author assumed a dual role as both teacher and researcher. As a teacher, she guided students through the content and activities, offering explanations and support. As a researcher, she created and collected data on student learning using formal instruments, including a pre-test, a post-test, and interviews. This integration of teaching and research roles enabled her to gain a deeper understanding of students' learning experiences and knowledge acquisition.

Procedure

The instructional process began with regular classroom teaching, followed by the administration of the pre-test. Students then participated in the intervention involving GeoGebra applets, and finally completed the post-test. Each phase of instruction and assessment is described in detail below.

Classroom Instruction

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The students spent three class periods, each 45 minutes long, studying trigonometric functions. The topic was introduced as follows. Students began by hand-drawing the basic graphs of $f(x)=sinx, f(x)=cosx, f(x)=tanx$ and $f(x)=cotx$ on paper. This initial activity familiarized them with the fundamental properties of these functions, such as periodicity, symmetry, and key points on the graphs. After mastering the basic graphs, the students, guided by the teacher explored how the parameters a, b, c, and d affect the graphs of the of functions such as $f(x)=asinx$, $f(x)=sin(bx)$, $f(x)=sin(x+c)$, and $f(x)=sinx+d$. For each type of transformation, they plotted three examples on the same coordinate plane to observe how changing the parameter values influenced the shape and position of the graphs. This approach enabled them to systematically analyze the effect of each parameter.

When investigating the effect of parameter a, students plotted the functions $f(x)=sinx, f(x)=2sinx$, and $f(x)=-3sinx$, and concluding that a affects the amplitude of the graph. They then examined the parameter b by plotting $f(x)=sin(x), f(x)=sin(2x)$, and $f(x)=sin(0.5x)$, and observing that the period depends on b according to $P=\frac{2\pi}{|b|}$. To analyze the horizontal shift caused by parameter c, they graphed $f(x)=sinx$, $f(x)=sin\left(x+\frac{\pi}{6}\right)$, and $f(x)=sin\left(x-\frac{\pi}{3}\right)$, noting that the graph shifts left or right depending on the sign and magnitude of c. Finally, they studied the effect of parameter d by plotting $f(x)=sinx+d, f(x)=sinx+2$, and $f(x)=sinx-3$, and concluded that the graph moves up or down based on the value of d.

A similar procedure was applied to the functions $f(x)=acosx, f(x)=cos(bx)$, $f(x)=cos(x+c)$, and $f(x)=cosx+d$. Working with more complex functions such as $f(x)=asin(bx+c)+d$ and $f(x)=acos(bx+c)+d$, students learned to identify key characteristics including amplitude, period, and phase shift. They also practiced sketching graphs from algebraic expressions and deriving expressions from given graphs.

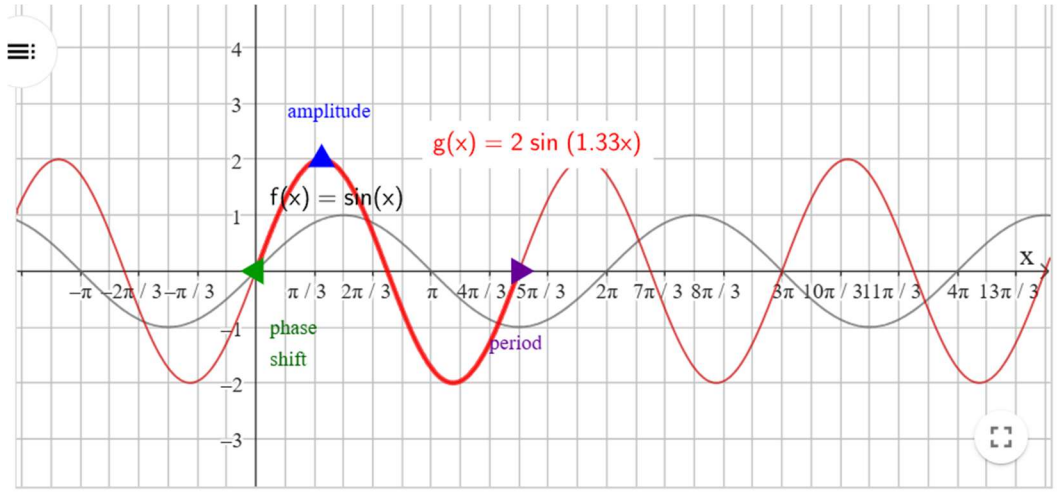
Intervention

The intervention involved the use of specially designed GeoGebra applets, which were assigned as formative homework. A GeoGebra Classroom was set up by the teacher, and students were provided with access credentials, requiring them to log in using their real names. This setup enabled the teacher to monitor student engagement on a daily basis, specifically, whether students accessed the platform and how many points they earned while completing the assigned tasks. It should be noted that the system did not record the duration of student interactions, but only logged task access, completion status, and point accumulation. Students had access to the applets for one week. Throughout this time, the teacher used engagement data from the GeoGebra Classroom to identify students who had not yet begun the tasks and issued reminders during regular lessons to encourage their participation. The formative nature of the assignment allowed students to work at their own pace while still being held accountable through visible progress tracking. While students were working on the applets at home, they were encouraged to ask questions in class. The teacher

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did not provide general demonstrations but responded to student-initiated inquiries, offering clarifications only when students expressed difficulties. This ensured that teacher support was focused and directly responsive to students' individual learning needs.

During the intervention, students engaged in two types of activities within GeoGebra Classroom. The first type consisted of eight interactive applets, allowing students to manipulate the coefficients of trigonometric functions in the forms $f(x)=asin(bx+c)+d$ and $f(x)=acos(bx+c)+d$. After exploring these applets, students responded to questions addressing how changes in parameters affected amplitude, period, and horizontal and vertical shifts of the graphs. These questions were designed to deepen students' conceptual understanding of how each parameter influences the function's shape (see Figure 1). All questions were presented in a multiple-choice format, with a total of 72 questions associated with this set of applets. (Examples of given activities can be found here: https://www.geogebra.org/m/xa4hfbds)



1. Changing the coefficient b, 0<b<1 in the function f(x) = a sin(bx + c) will affect:

Uključite sve točne odgovore

A ☐ the x-coordinates of the extrema

B ☐ the fundamental period

C ☐ the amplitude

D ☐ the phase shift

Figure 1: Example of the first type of GeoGebra activities

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The second type of activity, also consisting of eight applets, focused on assessing students' understanding of trigonometric function graphs through interactive problem-solving tasks. In these activities, students were asked to determine the parameters a, b, c and d based on graphical representations of $f(x)=asin(bx+c)+d$ and $f(x)=acos(bx+c)+d$. They were also required to plot graphs from algebraic expressions after identifying key features such as amplitude, period, and horizontal and vertical shifts. (This set of activities may be found here: https://www.geogebra.org/m/rzbjxpbr) Within these applets, points were awarded for correct answers and deducted for incorrect ones (see Figure 2 for an example). This scoring system encouraged careful reasoning, as each error affected the final score. When students made mistakes, the applets provided immediate feedback and contextual hints, guiding them toward the correct solution and highlighting key aspects to consider. This feature fostered self-regulated learning and allowed students to reflect on their reasoning in real time. The teacher informed students that earning ten points in each assigned activity indicated full mastery of the content.

At the end of the intervention period, the teacher reviewed students' interactions with the applets to identify common misunderstandings. She focused on activities in which students had earned the fewest points. Based on this analysis, a class discussion was conducted to address the most frequent errors and misconceptions revealed in students' digital work. This review session provided an opportunity for immediate feedback, clarification of misconceptions, and reinforcement of key concepts. Following the discussion, students completed a post-test.

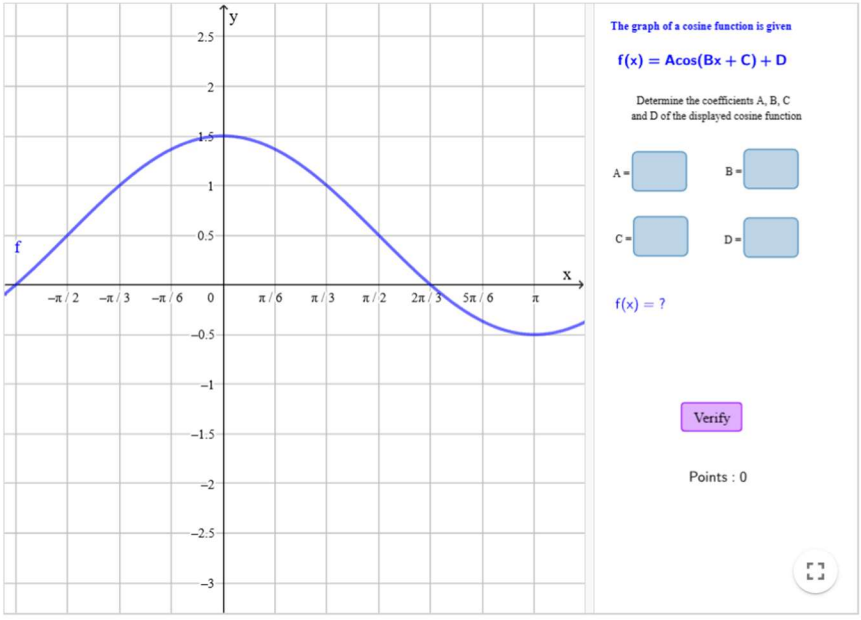


Figure 2: Example of the second type of GeoGebra activities

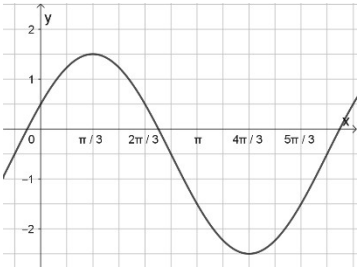
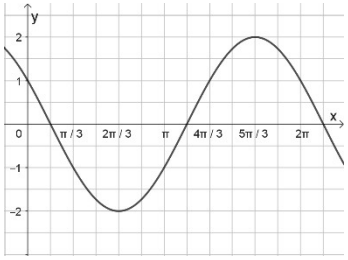
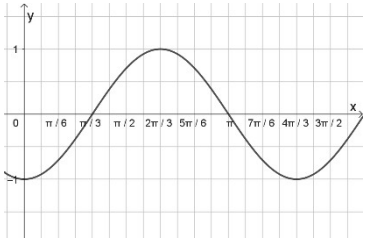
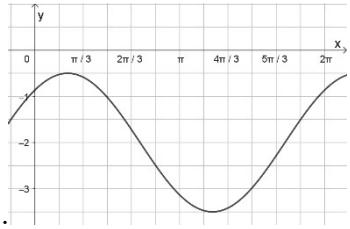
Pre-Test Assessment

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Following the classroom instruction, students completed a pre-test aimed at assessing their initial knowledge. Each question/sub-question was assessed as either 1 (correct) or 0 (incorrect). Students could achieve 15 points in total. The pre-test tasks were presented below:

A. Tasks involving data interpretation from graphical representations

1. Read and identify the required coefficients from the graphs of trigonometric functions.

a) Determine the parameter a of the graph of the function $f(x)=acos(bx+c)+d$ shown in the figure. 	b) Determine the parameter b of the graph of the function $f(x)=acos(bx+c)+d$ shown in the figure. 
c) Determine the parameter c of the graph of the function $f(x)=acos(bx+c)+d$ shown in the figure. 	d) Determine the parameter d of the graph of the function $f(x)=acos(bx+c)+d$ shown in the figure. 

B. Multiple choice

2. Determine the amplitude and the fundamental period of the function

$$f(x)=2cos(3x-4)+5$$

Amplitude	a) 2	b) 3	c) 4	d) 5
Fundamental period	a) 2π	b) $\frac{\pi}{3}$	c) $\frac{\pi}{2}$	d) π

3. If the function is given as $f(x)=-3sinx-2$, by how much is the original function $g(x)=-3sinx$ translated along the y-axis?

a) 3 units up

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b) 2 units down
c) 2 units up
d) 3 units down

4. If the function is given as $f(x)=2cos\left(x+\frac{\pi}{3}\right)$, by how much is the original function $g(x)=2cosx$ translated along the x-axis?

a) $\frac{\pi}{3}$ to the right
b) $\frac{\pi}{3}$ to the left
c) π to the right
d) $\frac{\pi}{2}$ to the left

C. Open tasks.

5. Write a cosine function with an amplitude of 4, a fundamental period of $\frac{\pi}{2}$, and a vertical shift of 2 units upwards.

6. Determine the amplitude, fundamental period, and vertical shift for the function $f(x)=-3sin2x-4$.

7. Given the function $f(x)=4cos\left(3x-\frac{\pi}{2}\right)$ find its fundamental period and horizontal shift.

D. Drawing graphs

8. Draw the graph of $f(x)=sinx$ over the interval $[0, 4\pi]$.

9. Draw the graph of $f(x)=cosx$ over the interval $[0, 4\pi]$.

10. Draw the graph of $f(x)=-2cos0.5x-1$ over its fundamental period.

11. Draw the graph of $f(x)=3sin\left(2x-\frac{2\pi}{3}\right)$ over its fundamental period.

Post-Test Administration

The post-test was conducted to evaluate students' progress following the interactive activities in GeoGebra. The test targeted the same learning outcomes as the pre-test but featured modified numerical values in the tasks, enabling an objective assessment of students' understanding of trigonometric functions. The post-test results were used to analyze student progress, identify areas requiring further attention, and provide feedback to help students recognize their strengths and weaknesses. This approach also offered the teacher the opportunity to adjust instructional methods according to individual student needs and to foster active engagement in the learning process.

Interviews

The student interviews focused on their experiences and perceptions regarding the use of GeoGebra as a tool for learning about graphs of trigonometric functions. During the interviews, students shared their views on how working with GeoGebra enhanced their understanding of function

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graphs, influenced their motivation to learn mathematics, and how this method compared to classroom teaching. The following questions were posed:

- Do you believe that working in GeoGebra helped you better understand the graphs of trigonometric functions? Please provide an example.
- Did using GeoGebra increase your motivation to learn mathematics compared to traditional methods? Explain why.
- What advantages did you notice when changing the coefficients of trigonometric functions in GeoGebra compared to manually drawing graphs?
- Do you think that combining theoretical instruction with GeoGebra was more effective than traditional learning alone? Why?
- Can you explain how GeoGebra helped you avoid certain mistakes you made at the beginning?

Data Analysis

Descriptive statistics was used to analyze pre-test and post-test results as well as statistical methods appropriate for small sample sizes. Differences between the two sets of results were examined using the Wilcoxon signed-rank test for paired samples, which is suitable given the small sample size (N = 20) and the assumption of non-normal data distribution. Additionally, effect sizes were calculated using the *r* statistic and Hedges' *g*, both of which are recommended for small samples.

The qualitative part of the study was analyzed using thematic analysis (Saldaña, 2015). Although the interview questions were predefined, the analysis was not limited to direct responses. Instead, the answers were coded and interpreted inductively, and themes were organized to capture recurring patterns that went beyond literal responses. As a result, thematic categories were developed to reflect deeper aspects of students' perceptions rather than just the explicit content of the questions. This approach allowed for the identification of the meanings students' attributed to using GeoGebra, including their insights into understanding processes, motivation, effectiveness, feedback, and the integration of different teaching methods.

RESULTS

Assessment of Intervention Effectiveness

To evaluate the effectiveness of the implemented intervention, basic descriptive statistics were calculated for the pre-test and post-test results (Table 1). The mean score on the pretest was 11.45 ($SD = 4.01$), while the post-test showed a higher mean of 12.15 ($SD = 3.79$). This increase suggests a slight improvement in student performance following the intervention. Additionally, the median increased from 12 to 13, further indicating that most students achieved better results on the post-test. A decrease in standard deviation was also observed, suggesting reduced variability in scores after the intervention.

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Measure	Mean (*SD*)	Median	Min	Max
Pre-test	11.45 (4.01)	12	3	16
Post-test	12.15 (3.79)	13	6	18

Table 1: Results of pre-test and post-test

The Wilcoxon test yielded a value of $W = 36.5$ with a corresponding p-value of 0.056. Since $p > 0.05$, the null hypothesis of no statistically significant difference between the two measurements was not rejected, meaning that the observed change was not significant at the conventional level of significance. Nevertheless, the calculated effect size using Wilcoxon's r measure was $r = 0.46$, which is interpreted as a moderate to large effect. In addition, the effect size calculated using Hedges' $g = 0.4742$, further supported the conclusion of moderate practical importance. These indicators suggest that, although the difference was not statistically significant, there was meaningful progress in student knowledge that merits attention from an educational practice perspective. It is important to note that the Wilcoxon test is based on ranks rather than raw scores, which enhances the reliability of the analysis when assessing changes.

Although the post-test results were generally higher, progress was not universal; some students showed little or no improvement. This indicates individual differences in response to the intervention. Analysis of students' engagement with the GeoGebra Classroom revealed those students did not log into GeoGebra Classroom. With regard to their prior performance, they had varying levels of knowledge on the pre-test, indicating that lack of progress was not strictly associated with low initial achievement. Other students' engagement with GeoGebra varied also in both participation and task success. While most students engaged with the platform, the depth and completeness of engagement varied. In the first type of activities (manipulation of parameters in functions having form $f(x)=asin(bx+c)+d$ and $f(x)=acos(bx+c)+d$), 8 students completed all 72 questions, 4 students completed approximately 62 questions, and the remaining completed fewer than 25. One student did not access this activity type at all. The second type of activities had lower engagement. Participation per applet ranged from 7 to 15 students, with the highest engagement in applets 1 and 2, and the lowest in applets 7 and 8. Only 8 students completed all eight applets. In terms of points earned, this instance showed great variability; for instance, 11 of 12 students achieved more than 10 points in applet 4, while only 4 of 13 students did so in applet 3. Overall, a decline in both participation and performance across later applets was observed, suggesting either increased difficulty or decreased motivation over time.

Students' Attitudes Toward GeoGebra Activities

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To gain deeper insight into students' experiences and attitudes toward using GeoGebra for learning trigonometric functions, the interview results are presented below, organized into thematic categories.

a) Visualization and Understanding

Students emphasized that GeoGebra helped them better understand the concepts of amplitude, phase shift, and period due to visual and interactive features. One student noted that progress in understanding depended on the amount of effort invested in working with GeoGebra, highlighting the importance of consistent engagement with the tool.

- "It really helped me because I could visualize the information. I didn't have to imagine it."
- "When I started using GeoGebra, I could picture amplitude, shifts, and everything else."
- "It helped me understand trigonometric functions when the coefficients changed. But I didn't practice enough in GeoGebra. If I had used it more, it would have helped me even more."

b) Increased Motivation and Engagement

Students' motivation to learn mathematics was enhanced by making the content more accessible, engaging, and less abstract. They expressed being more engaged, largely due to the dynamic nature of the digital tool, which supported autonomy and faster comprehension. They also noted that GeoGebra made mathematics feel "more approachable," which is particularly important given the subject's inherent challenges.

- "I think learning with GeoGebra is better than just doing textbook exercises."
- "Math is not many students' favorite subject, so technology makes the material more appealing."

c) Learning Efficiently and Faster Task Completion

Students almost unanimously agreed that GeoGebra enabled them to complete tasks faster and more accurate compared to manual graphing. One student specifically pointed out the advantage of being able to compare multiple graphs simultaneously, while another emphasized that hand-drawn graphs can introduce confusion due to inaccuracies.

- "Using GeoGebra, we can complete tasks much faster compared to drawing by hand."
- "I don't need to draw separate graphs to notice differences and understand the rules."
- "When drawing graphs on paper, mistakes can happen—it's not as precise as in GeoGebra."

d) Feedback

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GeoGebra provided immediate visual feedback, allowing students to quickly identify errors. This feature enhanced their ability to self-correct and facilitated a deeper understanding of their mistakes, which traditional instruction does not offer to the same extent or speed.

- "It's easier when the program corrects you. You know you made a mistake, and it's easier to find the problem."
- "When working traditionally, we don't even know when we make mistakes."

e) Synergy Between Theory and Practice

Overall, students advocated for a combination of classroom teaching and using technology, though with varying emphasis. While some appreciated how GeoGebra made the content more concrete, others pointed out that the tool cannot replace the teacher's role in developing mathematical reasoning.

- "In traditional lessons, you learn more theory, but this way, you apply what you've learned."
- "GeoGebra helps, but it cannot provide as much as a teacher can to strengthen our thinking."
- "I think both should be combined."

DISCUSSION AND CONCLUSION

This study examined the impact of GeoGebra on vocational school students' understanding and engagement in learning trigonometric functions. Although the Wilcoxon test did not reveal a statistically significant difference between pre- and post-intervention results ($p = 0.056$), the moderate effect size ($r = 0.46$; Hedges' $g = 0.4742$) suggests that the observed change has practical relevance. In small-sample studies, effect size often provides a more informative measure of an intervention's educational impact than statistical significance alone (Sullivan & Feinn, 2012). The results align with previous research confirming the positive influence of GeoGebra on students' ability to connect algebraic expressions with their corresponding graphical representations (Bedada & Machaba, 2022). In the specific context of trigonometric functions, students demonstrated an improved understanding of the relationships between function parameters and their graphs and were better able to correct errors that occurred during manual graphing. Qualitative findings from student interviews enriched the quantitative results, revealing that most participants perceived GeoGebra as a helpful tool that enabled them to complete tasks more quickly and accurately. They emphasized that visualizing amplitude, period, and phase shift facilitated their conceptual understanding and that immediate digital feedback supported error detection and correction. Students particularly highlighted the value of combining traditional instruction with GeoGebra-based activities to achieve a balance between theoretical explanation and practical application.

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Although overall results improved, progress was not uniform across students. Engagement within the GeoGebra Classroom platform varied considerably: while some students completed all assigned tasks, others did not access certain activities at all. The noticeable decline in participation in the later applets may indicate reduced motivation or increased task difficulty. These differences highlight the crucial role of the teacher in monitoring progress and encouraging participation. Previous research suggests that the effectiveness of digital tools such as GeoGebra depends heavily on clear teacher guidance (Bedada & Machaba, 2022). Merely providing access to digital technology is not sufficient; its full educational potential can be realized only when teachers are adequately prepared to integrate such tools into their instruction and when students actively engage with the material. In this study, the GeoGebra activities were assigned as homework, placing the responsibility for engagement on students and limiting the teacher's ability to provide real-time feedback or guide exploration. Ideally, GeoGebra activities should be embedded into regular classroom instruction, allowing the teacher to scaffold students' learning and ensure consistent participation. However, such integration is frequently constrained by limited instructional time and dense curricula, particularly in vocational schools, where priority is often given to technical and vocational subjects (Rodrigues & Pimenta, 2025; Rosvall et al., 2016).

Despite these limitations, the results suggest that even out-of-class engagement with GeoGebra can enhance students' understanding, provided that they actively and meaningfully participate in the activities. It is important to note that GeoGebra Classroom did not track the duration of student activity but only recorded task access, completion status, and points earned. Consequently, it was not possible to accurately estimate the amount of time students spent working on the tasks, which represents a limitation when interpreting student engagement. Although students who showed little or no progress were identified as those who had not accessed the digital activities, no clear relationship was observed between lack of engagement and prior knowledge. Some of these students had achieved relatively strong pre-test scores. This finding suggests that the lack of progress cannot necessarily be attributed to weaker prior knowledge but is more likely related to lower levels of engagement, motivation, or self-regulation. Future research should include measures of students' digital skills, learning strategies, and time-on-task data to gain a more comprehensive understanding of individual differences in performance.

Overall, the results of this study can be linked to the findings of Held and Mejeh (2024), who studied the effect of self-regulated learning in vocational schools. In their study, students were given greater autonomy in planning and organizing their own learning, with minimal but targeted support from teachers. Although the approach in our study was technically different (students worked on digital GeoGebra applets at home), the role of the teacher as a facilitator through progress monitoring and occasional interventions was similar. In both studies, students who actively participated showed increased understanding and engagement, while inactive students remained without visible progress, regardless of prior knowledge. This parallel confirms an important implication for practice: independent digital activities can support deeper understanding of mathematical concepts only if they are clearly structured and accompanied by systematic support and supervision. Autonomy, while valuable, does not guarantee success in itself. As Held and Mejeh

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(2024) show, autonomy can lead to lower engagement, procrastination, or stress for some students. Therefore, we argue that when introducing digital tools and methods of independent learning, teachers should ensure a balance between freedom and structured support, in order to maximize their educational impact.

LIMITATIONS AND FUTURE RESEARCH

Several limitations of this study should be acknowledged, as they affect the generalizability of the findings. First, the research was conducted with a limited sample, consisting of a single class from one school, and no control group was included. Second, the small number of participants reduced the statistical power of the analysis, making it difficult to draw robust conclusions. Although a moderate effect size was observed, statistically significant differences in student performance before and after the intervention could not be established.

In light of these limitations, replicating this study with students from general secondary schools would be particularly valuable. In Croatia, these students are required to take a compulsory mathematics graduation exam that includes trigonometric functions, which may lead to higher engagement and potentially different learning outcomes. A comparative study across different school types could provide deeper insight into how GeoGebra supports conceptual understanding and motivation in diverse student populations. Future research should also include larger and more diverse samples, involve multiple schools, and incorporate control groups to enhance the external validity of the results.

RECOMMENDATIONS FOR USING GEOGEBRA

For teachers who are just starting to use GeoGebra, it is recommended to start with simple applets that allow manipulation of function parameters. Teachers can use existing resources available on the platform (www.geogebra.org). GeoGebra Classroom allows for tracking student progress, and Classroom integration can begin with short demonstrations, followed by independent exploration by students with teacher support. The key is to provide clear instructions, gradually introduce functionalities, and encourage students to reflect on the changes they observe.

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Using GeoGebra in Teaching Advanced Geometry: A Case Study

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Abstract: Developing digital skills in schools is essential in the 21st century. GeoGebra is suitable for the development of various digital and mathematical skills. The program provides an opportunity to create visual representations that can facilitate the learning process and motivate students. This article reports on a case study that aims to test an online workbook. The workbook covers the topics of similarity and circle-related concepts at an advanced level and was tested among 10th graders. The workbook contains definitions, theorems, and proofs, as well as related tasks and homework. GeoGebra files are linked to each section to ease the work of both students and teachers. Students learned from the workbook during the experimental classes. Their opinions were collected through two short questionnaires. Based on the students' answers, the visual representations and dynamic figures in GeoGebra helped them understand and acquire concepts related to circles and similarity. Most students could easily use the program and enjoyed the lessons. This case study is part of a larger research, and the workbook will be further developed based on students' opinions and classroom experiences.

Keywords: circle, GeoGebra, geometry, mathematics education, similarity

INTRODUCTION

Digital tools are a part of everyday life. Today's children are digital natives, growing up in an environment where technology-based opportunities are endless. Avoiding these tools and the potential benefits they offer during learning would be unnatural. However, being a digital native does not necessarily mean that they can use these tools effectively. Digital skills must be learned, just like any other skill, and, according to Prievara (2015), they are essential in the 21st century.

The 2020 Hungarian National Core Curriculum (NCC) suggests that digital tools offer numerous methodological opportunities in the teaching and learning process. Teachers must be familiar with and use these tools, as they are indispensable. Technology can foster interaction between teachers and students and transform traditional teaching methods.

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Developing digital skills is essential in teaching mathematics. The NCC aims to introduce the use of digital tools at the beginning of primary education. Younger students should be introduced to digital tools through games that enhance problem-solving skills and provide opportunities to practice mathematical operations. In the upper grades of primary school, students are exposed to mathematical software. By the end of secondary school, students should be able to use digital tools to facilitate their mathematical studies and create visual representations (NCC, 2020).

Researchers suggest that several non-digital skills are also developed through the use of digital tools in math classes. Students can become more independent, and collaboration is also possible. They can be innovative and flexible. The development of these skills can prepare them for future challenges (Tomaschko & Hohenwarter, 2017). Technology enables students to perform multiple tasks and receive immediate feedback, which can contribute to a deeper understanding of mathematical concepts (Hollebrands, 2007). Ayub et al. (2010) point out that classroom instruction has improved with the use of digital tools.

Many digital tools can be used in mathematics education. One of them is GeoGebra, which enables users to create visual representations in the field of mathematics. The program allows teachers to reduce the time spent drawing figures on the board and gives students the opportunity to explore mathematical concepts themselves (Haciomeroglu et al., 2009). The creators aimed to provide materials for students everywhere; therefore, the program is free and can be downloaded from the internet. It runs on a wide variety of platforms (Ancsin, 2011) and has been translated into 82 languages to date.

Researchers have found many advantages to using GeoGebra in a classroom setting. It offers new ways of presenting the material (Zöchbauer et al., 2022). By visualizing mathematical concepts, GeoGebra promotes students' understanding and facilitates the learning process (Hohenwarter et al., 2009; Arbain & Shukor, 2015; Joshi & Singh 2020; Adelabu, F. M. et al., 2022; Dahal et al., 2022; Thapa et al., 2022; Zakaria et al., 2024). Saha et al. (2010) examined students with high and low spatial visualization abilities. According to their results, both groups performed better after using GeoGebra compared to the control group. However, simply looking at pictures and figures will not foster conceptual learning. Students must actively engage with GeoGebra to gain a deeper understanding (Haciomeroglu et al., 2009).

Students find the GeoGebra platform user-friendly, which makes it easy for them to use the program independently (Dahal et al., 2022; Thapa et al., 2022). Active participation (Dahal et al., 2022; Thapa et al., 2022; Zöchbauer et al., 2022) and increased motivation were also observed (Arbain & Shukor, 2015; Dahal et al., 2022; Zakaria et al., 2024). Moreover, students became more confident in their mathematics studies (Arbain & Shukor, 2015). GeoGebra also aids in developing skills such as reasoning, problem-solving, and decision-making (Saha et al., 2010; Joshi & Singh 2020; Zakaria et al., 2024). It provides an opportunity for a student-centered environment (Thapa et al., 2022), promotes discovery learning (Dahal et al., 2022; Hamidah et al., 2024), and gives immediate feedback on students' work (Saha et al., 2010). Teachers can also

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benefit from the program, as it makes sharing resources easier and faster (Zöchbauer et al., 2022). However, Anwar et al. (2025) emphasize that teachers should not only be familiar with the tool itself, but also with appropriate pedagogical methods that can help make lessons more effective.

As research suggests, the use of GeoGebra can aid in developing mathematical skills, digital skills, and skills that are useful outside the field of mathematics as well. Based on students' opinions, using GeoGebra in mathematics classes is better than traditional teaching methods (Hohenwarter et al., 2009; Saha et al., 2010; Juandi et al., 2021; Nguyen et al., 2023). However, it has been shown to be more effective when used in groups of up to 30 students. Individual work can also be a better option if there are enough devices for each student (Juandi et al., 2021).

Building on these insights, this research aims to explore the implementation of GeoGebra in Hungarian mathematics classrooms through the testing of an online interactive workbook developed as part of a larger educational project. The workbook focuses on the topics of similarity and circle-related concepts. The objectives of this study are:

- To examine how GeoGebra-based materials influence student engagement and motivation.
- To assess the clarity and effectiveness of the developed workbook through classroom application.
- To gather student feedback for improving the workbook's design and pedagogical content.
- To identify any technical or conceptual challenges during implementation.

This case study contributes to understanding how digital tools like GeoGebra can enhance mathematics education, in line with national curriculum goals. Furthermore, by testing a structured, curriculum-aligned digital resource in a real classroom environment, the study extends previous research by offering practical insights into the usability, reception, and impact of GeoGebra-based materials. These findings may inform future material development and support the broader integration of digital tools in mathematics instruction.

While many earlier studies have focused on the general benefits of GeoGebra in fostering conceptual understanding and student engagement, our research specifically investigates its application through a newly developed online workbook focused on similarity and circle-related concepts. Unlike broader interventions, this case study integrates GeoGebra into a structured, topic-specific learning material. Therefore, the study offers insights into both the pedagogical effectiveness and the practical usability of GeoGebra-based resources in real classroom settings.

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CASE STUDY

Workbook

The workbook contains definitions, theorems, and their proofs related to the topic, tasks for students, and a project for potential group work. For several exercises, a link to the relevant GeoGebra illustrations is provided, allowing the students to visualize the problem and experiment with it in an interactive way. Additionally, there are tasks that require students to create their own GeoGebra files based on classwork. The following subtopics are presented in the workbook:

- Theorem of Inscribed and Central Angles
- Angle of View
- Cyclic Quadrilaterals
- Tangential Quadrilaterals
- Basic Proportionality Theorems
- Angle Bisector Theorem
- Central Dilations and Contractions
- Similarity Transformations
- Similarity of Shapes
- Some Applications of Similarity (midlines, centroid of a triangle, diagonals of trapezoids, common tangents of two circles)
- Geometric Means in Right Triangles
- Tangent-Secant Theorem
- Ratio of the Areas of Similar Plane Figures

The material can be covered in approximately 21 lessons. The unit concludes with a review lesson and an end-of-unit test. Definitions and theorems are presented, followed by tasks and homework assignments. In some cases, there are more challenging problems that are not compulsory. Solutions to each homework assignment are provided at the end of the workbook. Links to GeoGebra files are included for many of the tasks and theorems.

In teaching this topic, GeoGebra can play a central role in both the discovery process and the development of conceptual understanding. Lessons can begin with a warm-up task. After solving the problem analytically, GeoGebra can be used to visually represent the situation. This allows students to verify their calculations and consider how the problem might change with different parameters. When a theorem is formulated, its consequences can be further explored using interactive models. The interactive construction makes it easier to test hypotheses and visualize

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hidden symmetries. Additionally, GeoGebra enhances the continuity of learning. Students can access prepared files at home, revisit steps they may not have fully understood during the lesson, and even modify the constructions to explore alternative cases. These files serve not just as visual aids but as interactive spaces for mathematical inquiry. Below, we present some examples from the workbook.

Figure 1 illustrates a theorem related to the angle subtended by an arc. Using the dynamic diagram, we can explore the locations of points P on the plane, from which the segment AB can be seen at the given angle. During the lesson, one student can find the solution for an acute angle and another for an obtuse angle. Students will discover that by moving point P, in such a way that the angle APB equals the given angle β, the path of P forms two circular arcs.

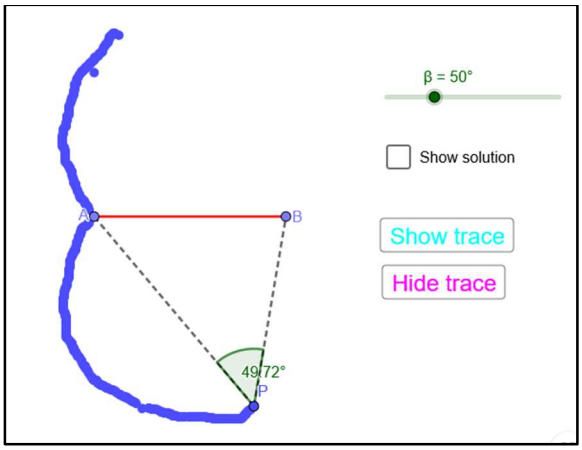


Figure 1: Subtended angle. GeoGebra link:
https://www.geogebra.org/m/ex6vez6c#material/gxrgnsd2

The construction of these two arcs is shown in Figure 2. GeoGebra allows students to follow the construction process step by step for any angle. By pressing the back and forward buttons, the steps and corresponding parts of the figure are displayed one by one.

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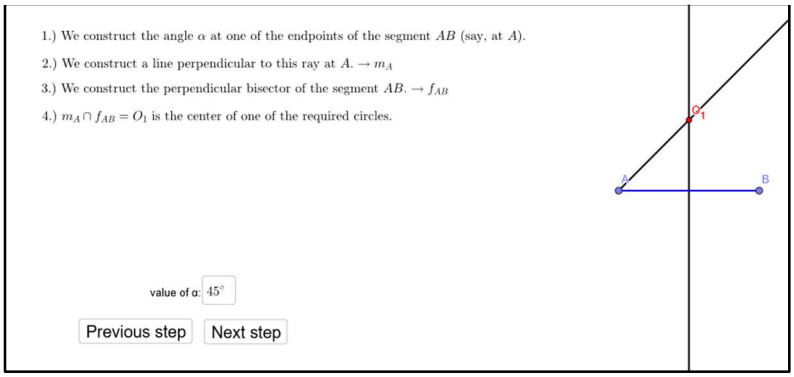


Figure 2: Constructing the arcs from where a segment is seen under a given angle. GeoGebra link: https://www.geogebra.org/m/ex6vez6c#material/z9ueupct

Figure 3 provides a visual representation of the following problem: *"The figure shows the auditorium of a theater from a top view. There are boxes on the longer sides of the auditorium on the ground floor. Find the side box (the point on the longer side of the auditorium) from which the stage can be seen at the greatest angle."* As a first step, students can experiment by moving point P in the figure. Then, by checking the "show solution" box, they can observe the arc, from which the stage can be seen from a given angle α. Using the slider, students can find the largest value of α, where the arc has a common point with the longer side of the auditorium. In this case the arc is tangential to the side, and the two points of tangency (J and K in Figure 3b) give the locations offering the best view.

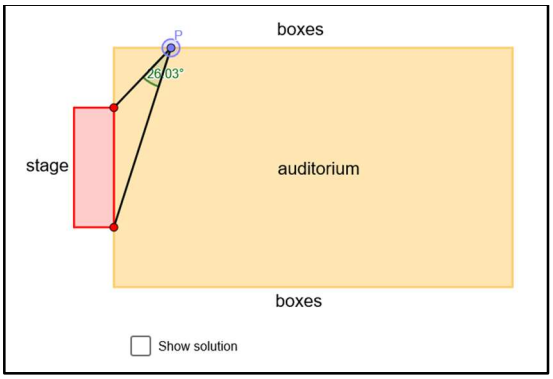


Figure 3a: Where should we sit for the best view? GeoGebra link: https://www.geogebra.org/m/ex6vez6c#material/h598qujw

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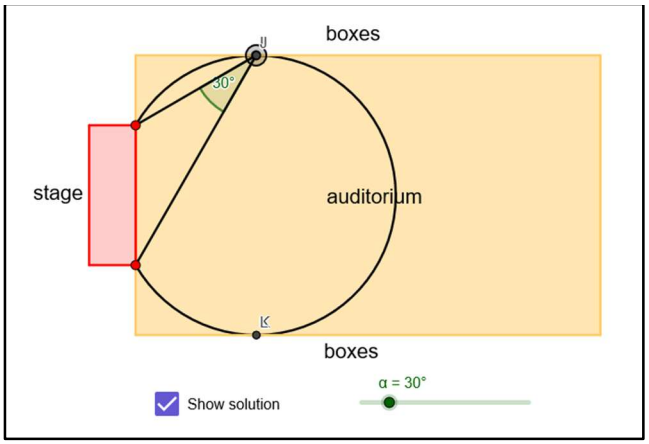


Figure 3b: Where should we sit for the best view? - solution.

GeoGebra facilitates the visual exploration of tasks. A good example is the task illustrated in Figure 4: *"Construct a triangle, given the length of one side and its corresponding median, as well as the size of the angle opposite the given side."* The construction process can be followed step by step. The location of the points C providing a given median is a circle centered at the midpoint F of AB (see the red circle in Figure 4). The prescribed angle at C constrains the location of C to two circular arcs (see the blue arcs in Figure 4). The intersection points of the red circle and the blue arcs give the possible points C. By changing the length of the side, the median, or the angle, students can see how many solutions exist for the problem: there are either 0, 2 or 4 solutions (we have 2 solutions when the blue arcs are tangential to the red circle).

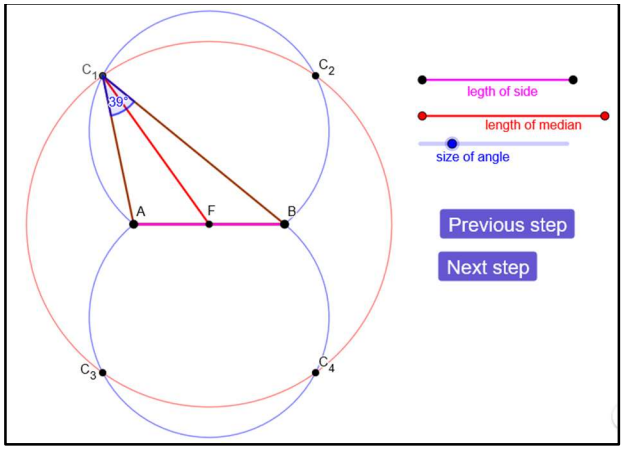


Figure 4: Constructing a triangle from a side, a median and an angle. GeoGebra link: https://www.geogebra.org/m/ex6vez6c#material/m77husfm

Figure 5 presents a task involving the division of a line segment in a prescribed proportion. Students can see, through a dynamic diagram, how a segment can be divided into any given rational

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ratio. This allows for more visual practice in class and enables students to easily check their homework assignments.

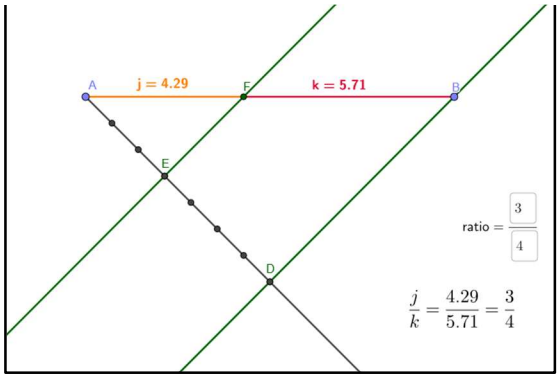


Figure 5: Dividing a segment. GeoGebra link: https://www.geogebra.org/m/ex6vez6c#material/ey9qhqwh

The inequality between the arithmetic and geometric means is illustrated in Figure 6. It proves the inequality with the help of the altitude theorem of right triangles. In the figure, the altitude of triangle ABE is shown in pink, while the radius of the circle is shown in orange. By moving point C, students can observe that segment CE will always be shorter than segment OD, except when they coincide – this happens when the length a of the red segment AC is the same length b of the blue segment CB .

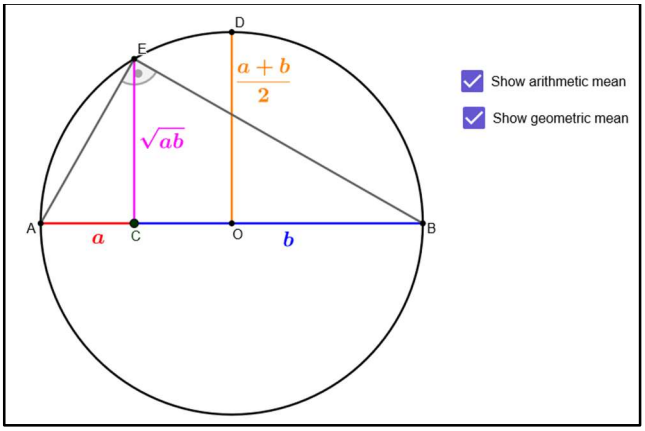


Figure 6: Inequality of arithmetic and geometric means. GeoGebra link: https://www.geogebra.org/m/ex6vez6c#material/xk9cjbkt

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Methods and Participants

During the case study, 21 lessons were conducted using the workbook created for this purpose. Students received the workbook via the school's social platform, allowing them to access it on their mobile phones during lessons. Unfortunately, computers were not available in class, so individual tasks requiring the creation of GeoGebra files were assigned as homework. In a few instances, students were able to complete the tasks during class using the GeoGebra app on their phones. In other cases, the teacher demonstrated the process of creating the files during the lessons.

The workbook was tested with 10th-grade students studying mathematics at an advanced level. They had three math classes per week. The group consisted of 14 students – 6 boys and 8 girls. Students completed a short online questionnaire twice during the experimental period. The first questionnaire included 6 statements with 5-point Likert scales, 3 open-ended questions, 1 preference question, and a box for additional suggestions. The second questionnaire included 4 statements with 5-point Likert scales, 2 open-ended questions, and a suggestion box. The statements were formed based on the questionnaire that Arbain and Shukor (2015) used in their study.

At the end of the observation period, students completed a final test covering the main topics. The test assessed their conceptual understanding of similarity and circle-related mathematical concepts, as well as their ability to apply mathematical reasoning in problem-solving. In addition to the test, homework assignments were used to evaluate students' independent use of the software. During classroom activities, attention was paid to the clarity and accuracy of student responses, particularly their use of precise mathematical language and the correct application of definitions. The questionnaires measured student motivation and to gather feedback on their experience with GeoGebra — including how much they enjoyed using it, what their general opinion of the software was, and what difficulties or positive experiences they encountered during the activities.

Results

Based on students' responses to the open-ended questions, we can draw the following conclusions about the benefits of using GeoGebra in our case study. The GeoGebra figures offer students with weaker motor skills the opportunity to analyze accurate diagrams of any size during home study. Additionally, it can be challenging for some teachers to draw precise figures on the board, so providing students with well-drawn examples is beneficial. Visual representations of the concepts make it easier for students to grasp the material.

> *"Since the figure is accurate, it's easier to understand and more transparent."*

> *"It helps us think not only based on theory but also with the help of figures."*

With dynamic diagrams, students can quickly and easily explore multiple examples. By manipulating the diagram, special cases can be generated effortlessly, something that would

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require drawing separate figures with the traditional paper-pencil method. This provides an opportunity to address any doubts students may have by seeing proofs or solutions in various forms.

> *"It illustrates the theorems—for example, when we move a point, the theorem still holds. It's much harder to imagine this on paper."*

> *"By creating the diagram together, I understand the material more easily, and it encourages me to think rather than just accept everything."*

Despite teachers' best efforts to actively plan lessons, there are times when students are tired and fall behind during class. In such cases, as well as during home preparation, the figures can be reviewed step by step.

> *"It's much more transparent, and it can be broken down into steps that can be revisited. If someone unintentionally zones out in class, it's harder to follow the steps of a task [without GeoGebra]."*

GeoGebra helps students become more motivated and confident when experimenting with geometry.

> *"I like to start my homework with it, and even in my free time I often feel like doing geometry. I open it up to create simple things, although the functional part is still somewhat unfamiliar."*

While some students were quite confident, others needed more support during individual work.

> *"I get completely lost in GeoGebra on my own and just end up confused. It would be enough for me if it were just there as a tool, nothing more."*

Students filled out the first questionnaire after the topics related to the circle. The second questionnaire was at the end of the research period. Table 1 shows three items that appeared in both of the questionnaires. The means of the responses to these items slightly decreased. It seems that students enjoy using GeoGebra and believe it is helpful. However, the second half of the workbook did not contain as many GeoGebra exercises as the first one. This could be a possible reason for the decrease in the scores. A second reason could be that using the same software for a longer period can become boring for the students. Thirdly, GeoGebra allows more interactivity in exercises related to circles than in the topic of similarity, hence its advantages compared to traditional teaching methods are less prominent in the latter topic. During further developments of the workbook, we will create more exercises and better examples for the second part. In our future research, we will concentrate on the long-term use of GeoGebra.

A fourth possible explanation for the decline could be the increasing complexity of the tasks in the second part of the workbook. As the mathematical content becomes more abstract, students might find it harder to explore concepts through GeoGebra, especially without sufficient guidance or

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scaffolding. This suggests that not only the quantity but also the variety and structure of GeoGebra activities play a crucial role in maintaining students' interest and engagement. In future revisions of the workbook, we plan to diversify the types of interactive tasks, for example by including more exploratory challenges, and real-life applications, to better align with students' varying levels of motivation and skill.

Item	First questionnaire	Second questionnaire
GeoGebra helps to learn mathematics concepts.	4.25	4.00
GeoGebra can help to increase my achievement in mathematics.	3.92	3.36
I am happy if the teacher uses the GeoGebra in teaching mathematics.	4.47	4.46

Table 1: Means of the same questions in the two questionnaires.

Based on the responses to the questionnaires, the majority of students felt that GeoGebra helped them understand the concepts better. They were more motivated to learn the material using digital tools in addition to traditional methods. Students appreciated the teacher's demonstrations of how to use the platform during class and how to create different figures, which not only helped them understand the topic but also allowed them to learn how to use GeoGebra. However, some students found the pace too fast or would have preferred to always have ready-made figures available.

According to the teacher, GeoGebra led most students to participate in lessons with much more enthusiasm, even though many of them had disliked geometry in primary school. The group enjoyed working together to create files, and some students even began exploring the program further, either as a computer or phone application. These topics are mostly found in the advanced-level curriculum of the Hungarian two-level matura exam. Although the topics are challenging, these visual representations make it easier to understand them.

LIMITATIONS

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One of the limitations of this study is the small number of participants, which limits the generalizability of the findings. No control group was used in this pilot phase, as the primary aim was to test the functionality of the workbook and gather initial feedback on its structure, content, and the integration of GeoGebra. As such, the focus was not on comparing digital and non-digital teaching methods, but rather on evaluating the digital material itself.

We acknowledge that the limited sample size and the lack of a comparison group may restrict the broader applicability of our findings to different topics, educational levels, or international contexts. In future phases of the project, we aim to test the workbook in more classrooms and with larger, more diverse groups of students.

The workbook is still in progress. More GeoGebra files will be added to the workbook, and students' feedback will be taken into account during further development of the teaching material. The case study presented here is part of a larger project in which we aim to create an online workbook that can be useful for students. Since it is time-consuming to collect suitable tasks for a topic and create visual representations in GeoGebra, we hope that teachers can profit from this workbook as well.

CONCLUSION

This study highlights the importance of using digital tools in mathematics education. Based on the literature review, GeoGebra is a suitable platform for use with students. It can be used for creating visual representations in different fields of mathematics, which can ease the learning process for students and motivate them during the lessons. Besides digital and mathematical skills, GeoGebra can also be useful in developing skills that are relevant for the 21st century.

A case study related to an online workbook was performed. The workbook was created for advanced-level 10th graders to study similarity and circle-related concepts. It consists of definitions, theorems, proofs, tasks, and homework assignments related to the topic. Several links can be found in the workbook that lead to GeoGebra files with visual representations. A teacher tested the material with a small group of students and asked for their opinion about it with two short questionnaires.

Based on students' answers, this workbook seems to be on a good track. Students enjoyed the lessons with GeoGebra and became more motivated to study geometry after successfully creating different figures. The visual representations and the dynamic figures provided more opportunities than traditional paper-pencil methods. This helped students with conceptual learning and gaining a deeper understanding of the topic. Most students could easily use the program individually; however, some of them needed more help. GeoGebra promoted student-centered teaching, as creating files together during classes encouraged students to participate actively. Individual work

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was mostly done at home; however, a project for group work is also provided in the workbook. It would be useful to create groups based on students' levels of GeoGebra knowledge, allowing those who are better at it to help others.

During the research process and the development of the workbook, we gathered several practical insights that can be useful for mathematics teachers who wish to integrate GeoGebra or other digital tools into their lessons. When introducing GeoGebra in the classroom, it is advisable to start with simple tasks that help students become familiar with the interface and basic functions of the software, such as plotting points or using sliders. These initial steps support students in navigating the digital environment with greater confidence and gradually prepare them for more complex constructions.

It is important to recognize that technical difficulties are a natural part of the learning process. Some students may initially struggle with certain GeoGebra tools, especially when working with transformations or custom tools. To maintain motivation, it is useful to incorporate a variety of task types. In addition to discovery-based activities such as mini-projects. Alternating task formats helps sustain students' interest and engagement throughout the lessons. If students encountered difficulties while using GeoGebra, they could post their questions on a forum hosted on the school's online platform. This allowed all students to view both the questions and the teacher's responses, creating an opportunity for shared learning and collaborative problem-solving.

Finally, it is beneficial to provide a brief opportunity for reflection at the end of each lesson. Ask students to write down one thing they learned during the lesson and one part they found difficult or confusing. These reflections offer valuable insight for planning future lessons and help identify students who may need additional support.

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Comparative Analysis of Mathematical Arguments in Algebraic Proofs: Generative AI vs. University Students Approaches

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Abstract: The integration of artificial intelligence (AI) into mathematics education has attracted growing attention, particularly with the emergence of generative AI models capable of constructing mathematical arguments and proofs. This study compares the argumentation structures used by generative AI and undergraduate mathematics students when proving statements in number theory. Using a qualitative comparative approach, we analyzed responses generated by ChatGPT 3.5, YouChat, and 94 fifth-year undergraduate students at a state university in Indonesia. Participants were asked to justify a mathematical statement, and their responses were examined using the Claim–Evidence–Reasoning (CER) framework and argument mapping techniques. The results show that AI and most students reached similar conclusions. However, 9.6% of students produced different and more detailed responses, suggesting a higher level of sophistication in human mathematical reasoning. While AI systematically categorized cases based on integer values, some students extended their reasoning to non-integer domains, demonstrating greater interpretative flexibility. Moreover, AI predominantly relied on naïve empiricism, whereas students displayed varying levels of deductive reasoning. These findings underscore the need for further research to enhance AI's reasoning capabilities and to better integrate AI into pedagogical strategies that support deeper mathematical understanding beyond the current capacities of AI tools.

Keywords: Algebra, Argument Quality, Mathematical Argumentation

INTRODUCTION

The comparison between human and artificial intelligence reveals fundamental and consequential distinctions (Cope et al., 2020; Korteling et al., 2021a). Empirical evidence consistently demonstrates that humans outperform AI systems across a broad spectrum of cognitive and social tasks, particularly in novel and unpredictable contexts (Korteling et al., 2021b). Human superiority

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is especially evident in social and psychosocial domains, where AI continues to encounter significant limitations. Specifically, interpreting human language and symbolism requires a nuanced and extensive contextual framework—an area in which current AI technologies remain markedly deficient (Korteling et al., 2021a). Nevertheless, AI systems exhibit a clear advantage in executing routine cognitive operations with high efficiency, particularly in domains where human performance is constrained (Markauskaite et al., 2022). These disparities underscore the necessity of a rigorous comparative analysis between AI-generated and student-generated mathematical arguments to clarify core differences in problem-solving strategies, logical structures, and conceptual depth. Whereas AI is driven by data-centric algorithms and pattern recognition, human cognition is characterized by the integration of intuition, creativity, and experiential knowledge.

Artificial intelligence (AI) has increasingly permeated various sectors of society, including the field of mathematics and its instructional methodologies (Engelbrecht & Borba, 2023). Mathematics is frequently identified as a particularly suitable domain for evaluating AI's reasoning capabilities, given its structured logic and definable problem sets (Glazer et al., 2024). Contemporary AI systems possess the capacity to analyze data, extract meaningful patterns, and generate informed inferences or predictions based on learned information (Celik et al., 2022). Recent advancements in generative AI have further demonstrated its proficiency in solving mathematical problems across a range of topics, including algebra, calculus, and geometry, thereby prompting intensified scholarly interest in its applicability to diverse mathematical disciplines (Oh, 2024). Moreover, empirical studies indicate that AI's functionality extends beyond mere computational efficiency; it also encompasses advanced competencies such as pattern recognition and logical reasoning, reinforcing its potential to engage with and solve complex mathematical problems (Lu et al., 2022; Xu et al., 2024).

In mathematics education, a growing body of research has explored the integration of artificial intelligence (AI) to support and enhance student learning. Graham (2023) identifies a range of innovative applications, including AI-generated visualizations and interactive simulations designed to facilitate the comprehension of complex mathematical concepts. AI also contributes to increased student engagement through the use of gamified learning experiences and personalized mathematical tasks.

Furthermore, AI systems offer targeted remediation for learners who encounter difficulties, thereby supporting differentiated instruction. Recent studies have focused specifically on the role of generative AI in mathematical reasoning. Park and Manley (2024) examine the use of ChatGPT in assisting students with the construction of mathematical arguments that qualify as formal proofs. Similarly, Dilling and Herrmann (2024) explore the utility of generative AI in supporting pre-service primary mathematics teachers in developing geometric proofs. Yoon et al. (2024) investigate how undergraduate students engage with generative AI when formulating mathematical proofs. Despite these promising developments, the integration of generative AI into

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mathematics education presents significant challenges (Dwivedi et al., 2023; Wardat et al., 2023), particularly concerning the accuracy and reliability of AI-generated content. Moreover, the extent to which generative AI can meaningfully support the development, evaluation, and comparison of mathematical proofs alongside human-generated arguments remains unresolved. As such, a critical next step is to systematically assess the quality of AI-generated mathematical argumentation in relation to that produced by human learners. This evaluation is essential for advancing a deeper understanding of both the capabilities and limitations of AI within the context of mathematical reasoning and instruction.

Literature Review

Mathematical argumentation refers to the structured process through which reasoning is developed, articulated, and substantiated to analyze, explain, and solve mathematical problems (Kartika et al., 2024; Stylianides et al., 2013). It serves as a foundational precursor to formal proof, a core pillar of mathematical thinking and practice (Marks Krpan & Sahmbi, 2024). A mathematical argument consists of a coherent set of statements supported by evidence—such as data, warrants, and justifications—intended to establish the validity or refute the falsity of a specific claim or conclusion (Cardetti & LeMay, 2018; Zambak & Magiera, 2020). Within this framework, data supports the claim, and the logical connection between them must be explicitly validated through established mathematical principles, including axioms, definitions, theorems, or rules (Urhan & Zengin, 2023). Thus, a mathematical argument emerges as a reasoned sequence of propositions derived through rigorous argumentation (Kartika et al., 2023).

While extensive research has investigated students' abilities to construct mathematical arguments, comparatively less attention has been paid to how these arguments align or differ from those generated by artificial intelligence. This research gap is particularly salient given the growing role of AI tools in mathematical problem-solving, necessitating a deeper examination of how mathematical reasoning manifests in both human and machine-generated contexts.

The integration of generative AI technologies—such as ChatGPT and adaptive learning platforms—has introduced significant shifts within mathematics education. These technologies not only influence instructional practices but also provoke critical theoretical discussions about the nature of mathematical knowledge and the evolving roles of agents within human-technology collectives (Engelbrecht & Borba, 2023). Although generative AI models exhibit impressive capabilities in generating coherent, human-like text (Rane, 2023), they remain susceptible to producing contextually irrelevant, inaccurate, or misleading outputs (Wardat et al., 2023; Biton & Segal, 2025). Their operation as opaque "black box" systems further complicate this issue, as they can yield plausible yet erroneous responses that may even mislead subject-matter experts (Mishra et al., 2023; Fütterer et al., 2023). Instances of fabricated content highlight the risk of undermining

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user confidence and distorting perceptions of one's own mathematical competence (Emsley, 2023).

It is therefore imperative to view generative AI as a supplementary educational tool rather than a replacement for human educators (Tenhundfeld, 2023). AI-generated content must be subject to critical scrutiny by individuals with the requisite disciplinary expertise to assess its validity (Sabzalieva & Valentini, 2023). The ability to detect and correct mathematical errors in AI outputs is essential for their responsible use, particularly in educational contexts where such tools may be employed to construct examples or instructional materials (Noster et al., 2024; Schorcht et al., 2024). To harness the full potential of generative AI in mathematics education, it is essential to integrate its capabilities with human critical thinking and disciplinary knowledge.

Beyond verifying accuracy, a deeper and more pressing concern lies in evaluating the nature and quality of AI-generated mathematical reasoning. In particular, comparative analyses between AI- and human-generated arguments are essential to understanding the degree to which AI systems can replicate—or diverge from—the logical coherence, conceptual depth, and justification that characterize rigorous mathematical argumentation.

METHOD

This study adopted a qualitative research design, utilizing document analysis as the primary methodological approach. Document analysis entails the systematic examination and interpretation of written texts—both handwritten and digitally generated—to extract meaningful patterns and insights (Bowen, 2009). In this context, the analysis focused on identifying similarities and differences in the argumentative strategies employed by generative AI and student participants. The objective was to develop a nuanced understanding of the distinctive reasoning methods and epistemic contributions each source brings to the domain of mathematical argumentation.

Participants

The generative AI models utilized in this study were ChatGPT 3.5 and YouChat, both of which are publicly accessible and widely recognized for their capabilities in natural language processing and mathematical problem-solving. The human participants comprised 94 fifth-year undergraduate mathematics education students, aged 19 to 21, from Singaperbangsa Karawang University in West Java, Indonesia. All participants had previously completed a course in Number Theory during their first semester, thereby ensuring a foundational competence relevant to the mathematical statement under investigation. Participation was voluntary, and students were informed that all data would be anonymized and used solely for research purposes, with no implications for their academic standing.

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The selection of both advanced undergraduate students and specific generative AI models was deliberate. Fifth-year students were chosen due to their extensive mathematical training, including exposure to proof-based reasoning and formal argumentation, rendering them well-positioned to critically assess and construct mathematical statements. Their academic maturity enabled a meaningful exploration of human mathematical reasoning at a level characterized by abstract and deductive thought.

ChatGPT 3.5 and YouChat were selected as AI comparators based on their accessibility and their documented ability to generate coherent mathematical explanations and structured arguments. Both models, trained on large-scale datasets, are capable of producing algorithmically organized responses that simulate human-like reasoning. By comparing the outputs of these AI systems with those of advanced mathematics students, the study aimed to evaluate the reasoning capabilities of generative AI in relation to human cognition. This comparative analysis also illuminated key distinctions in argument construction, treatment of implicit assumptions, and the use of counterexamples—critical components of rigorous mathematical argumentation.

Data Collection Process

In this study, participants were given 30–40 minutes to justify a mathematical statement in number theory, adapted from Lee (2016). They were required to construct their arguments independently, without assistance from online sources. To address any potential misunderstandings related to terminology or definitions, participants were permitted to seek clarification from the researcher.

The task was intentionally designed to align with the participants' prior coursework, integrating both fundamental and advanced elements of number theory to assess a broad spectrum of mathematical reasoning competencies. Central to the task was the construction of a deductive proof, requiring students to engage in formal logical reasoning and structured argumentation. While some participants responded with generalized explanations—often relying on mathematical language rather than specific examples (e.g., “The sum is even because odd plus odd is even”)—a rigorous refutation of the given statement required the identification of a specific counterexample and a logical demonstration that the implication was false (Lee, 2016).

To ensure the task's appropriateness and efficacy, a pilot study was conducted involving 31 fifth-year mathematics education students from the same institution. The final version of the task was selected based on criteria including clarity, the potential to generate meaningful engagement, and the capacity to elicit nontrivial and analytically rich responses. This validation process confirmed the task’s effectiveness in prompting essential components of mathematical reasoning, such as the construction of formal justifications, identification of logical inconsistencies, and formulation of counterexamples.

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Handwritten student responses were collected and subsequently categorized by thematic content. The same task prompt was also submitted to ChatGPT 3.5 and YouChat, with their outputs systematically recorded for comparative analysis. As illustrated in Figure 1, the task did not explicitly define the variable x as belonging to a specific numerical set. This omission proved to be a critical factor in evaluating both student and AI-generated arguments, as it directly impacted the interpretation and validity of the logical structure underlying each response.

Benar atau salah pernyataan ini: "Jika $3x$ dikalikan dengan $(1 + x^2)$, maka hasilnya adalah bilangan genap."
(Translation) Is this statement true or false? "If $3x$ is multiplied by $(1 + x^2)$, then the product is an even number."

Figure 1: Argumentative Tasks Frame

Before analyzing the data, the responses obtained by the students were compared to determine similarities and differences. Students who had given similar answers were grouped and selected one data from each group for in-depth analysis. In addition, we compared the responses obtained from ChatGPT and YouChat to assess consistency between the two sources. For instance, ChatGPT and YouChat evaluate the statement under two conditions, namely x is even and x is odd. Then ChatGPT and YouChat conclude that this statement is true. This indicates that the responses from both sources are consistent and, although mathematically flawed, this establishes the credibility of this data for further analysis.

Data Analysis Process

All arguments were qualitatively analyzed through close reading based on the dimensions of structure, content, and recipient orientation (Meyer & Schnell, 2020). To develop coding classifications for analyzing participants' argument structures in Deductive-Proof Construction, this research utilized the CER (Claim, Evidence, Reasoning) model (McNeill & Krajcik, 2008), which is a three-step process: making statements, providing supporting evidence, and explaining the logic behind their statement. Using the CER model, the researchers focused on identifying the conclusions or claims made by each participant. In particular, a claim refers to the participant's response to the question, *"Is this statement true or false?"*

In addition to claims, the researchers examined the evidence and reasoning provided by the participants (Kartika et al., 2023). Evidence refers to the data or information gathered to back up the claim, while reasoning involves the mathematical rule or principle that connects the evidence to the claim. Meyer and Schnell (2020) suggested several guiding questions to identify the structure of an argument. For example: Does the claim answer the question? Is there sufficient

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evidence to support the claim? Is the reasoning adequate to explain how the evidence supports the claim? These types of questions assist in evaluating the strength and validity of participants' arguments.

Furthermore, the researchers constructed argument maps by synthesizing all arguments formulated by the generative AI models and student participants. Each map illustrated the relationships among the claims, evidence, and reasoning presented in the respective solution narratives. Following the construction of these maps, the researchers analyzed the data to identify recurring patterns and draw conclusions regarding the quality of the arguments. Argument quality was evaluated using the framework developed by McNeill and Krajcik (2008), as adapted by Kartika et al. (2023) and presented in Table 1.

Component of Argument	Quality	Description	Code
Claim	Low	Not making any claims, or making inaccurate or false claims.	C_L
	Moderate	Making claims that are accurate but incomplete	C_M
	High	Making a claim that is both accurate and complete.	C_H
Evidence	Low	Not providing evidence, or only providing inappropriate evidence (evidence that does not support the claim).	E_L
	Moderate	Providing precise but insufficient evidence to support a claim. This may include some incongruous pieces of evidence.	E_M
	High	Providing precise and sufficient evidence to support a claim.	E_H
Reasoning	Low	Failure to provide reasoning, or only providing reasoning that does not connect evidence to claims.	R_L
	Moderate	Providing reasoning that connects claims and evidence, reiterating evidence and/or incorporating some, but insufficient scientific principles.	R_M
	High	Providing reasoning that connects evidence with claims, encompassing appropriate and adequate scientific principles.	R_H

Table 1: Guidelines for Assessing Argument Quality

We also identified the levels of proof formulated by Generative AI and the students. According to Balacheff (1988), there are four levels of proof.

1. Naïve Empiricism: This level involves testing a mathematical statement by verifying a few specific cases.

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2. Crucial Experiment: At this level, a special or extreme case is used to prove the truth of a statement. Students at this stage recognize the limitations of generalizing from a few examples and begin to address the problem of insufficient verification.
3. Generic Example: This level involves demonstrating the truth of a mathematical statement through a representative operation, which serves as a basis for generalization.
4. Thought Experiment: The highest level, this involves investigating the properties of operations to justify the truth of mathematical statements. A thought experiment eliminates the reliance on specific examples by building on the concept of a generic example.

Stylianides (2019) categorized Naïve Empiricism and Crucial Experiments as weak forms of argumentation, whereas Generic Examples and Thought Experiments represent strong and more rigorous approaches to proof.

The results of the analysis are presented clearly and concisely, supported by tables, graphics, generative AI outputs, and student artifacts to highlight the main findings. The coding process, which involved iterative discussions and collaboration, ensured the reliability and validity of the findings. Specifically, the first author conducted individual coding, followed by discussions to review and refine the proposed categorizations. A final agreement on the categorization was reached through this collaborative approach.

RESULTS AND DISCUSSIONS

In this study, participants were tasked with evaluating a mathematical statement in number theory: *"If 3x is multiplied by* $(1+x^2)$*, then the product is an even number."* The findings related to the question *"Is this statement true or false?"* are summarized in Table 2. This table presents a comparative overview of responses from two participant groups: generative AI models and undergraduate students. Each response was classified as either "True" or "False," with the corresponding frequency and percentage reported for each category, thereby facilitating a clear comparison of the interpretative tendencies across human and AI-generated reasoning.

Responses	Participants	
	Generative AI (n=2)	Students (n=94)
True	2 (100%)	85 (90.4%)
False	0 (0.0%)	9 (9.6%)

Table 2: Frequency of Responses between Generative AI and Students

Table 2 reveals that the generative AI systems uniformly classified the statement as "True," with both responses (100%) indicating full agreement. Likewise, the majority of student participants (n = 94) concurred with this assessment, with 85 students (90.4%) selecting "True." However, a

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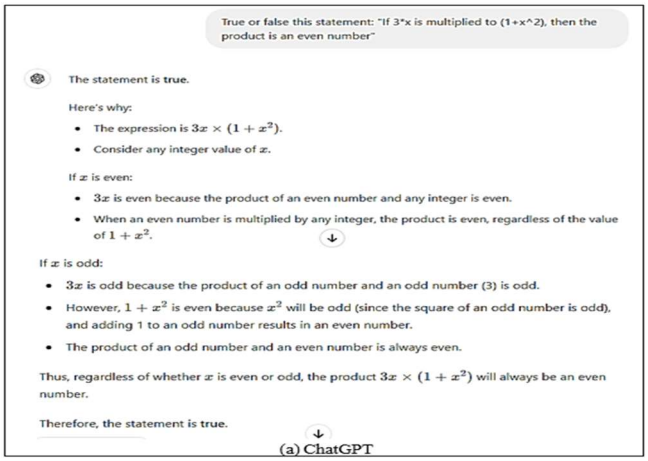
minority of students—9 individuals (9.6%)—responded with "False," indicating a degree of divergence in reasoning between subsets of the human participants and the AI models.

The comparative analysis of responses from the two groups reveals noteworthy patterns. The high proportion of "True" responses from both generative AI (100%) and student participants (90.4%) suggests a significant convergence between AI-generated reasoning and prevailing human interpretations. This alignment may be interpreted as an indicator of the apparent validity of the mathematical statement as perceived by the majority of respondents. However, the presence of a 9.6% deviation among students warrants closer examination, as it highlights the potential influence of alternative reasoning pathways in human cognition.

This divergence may stem from various underlying factors. Differences in conceptual versus procedural understanding likely play a role in how individuals evaluate mathematical statements. Students with greater exposure to formal proof techniques or training in identifying counterexamples may be more predisposed to scrutinize assertions and challenge superficially plausible conclusions. Moreover, cultural and pedagogical contexts—including norms that emphasize critical thinking, questioning of assumptions, and diverse problem-solving strategies—may further contribute to these differing interpretations. These findings suggest that while generative AI can replicate dominant reasoning patterns, it remains limited in capturing the nuanced, contextually informed, and often experience-driven dimensions of human mathematical thought.

Argumentative Structure Generated by Artificial Intelligence

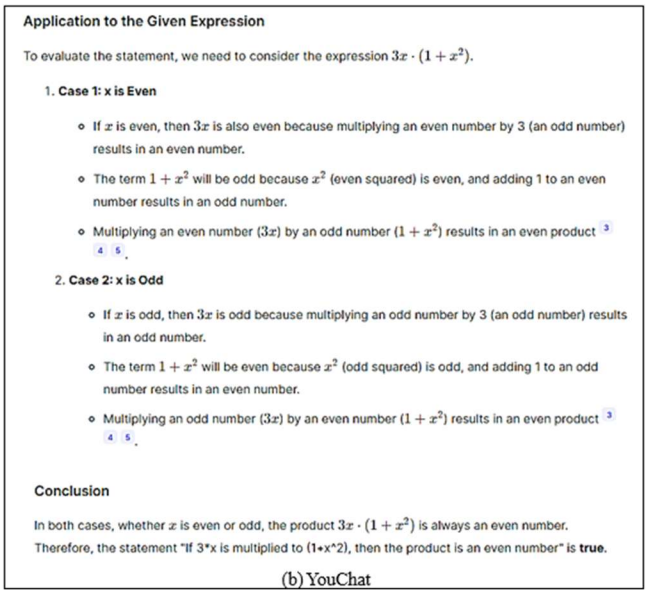
Figure 2 and 3 presents the response generated by the artificial intelligence model in relation to the given mathematical task. This figure illustrates the AI's approach to constructing a mathematical argument, highlighting its reasoning process, logical structure, and the methods employed to assess the validity of the statement.



(a) ChatGPT

Figure 2: ChatGPT Responses to the Mathematical Task

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Application to the Given Expression

To evaluate the statement, we need to consider the expression $3x \cdot (1 + x^2)$.

1. **Case 1: x is Even**
   - If x is even, then $3x$ is also even because multiplying an even number by 3 (an odd number) results in an even number.
   - The term $1 + x^2$ will be odd because x^2 (even squared) is even, and adding 1 to an even number results in an odd number.
   - Multiplying an even number ($3x$) by an odd number ($1 + x^2$) results in an even product [3] [4] [5].
2. **Case 2: x is Odd**
   - If x is odd, then $3x$ is odd because multiplying an odd number by 3 (an odd number) results in an odd number.
   - The term $1 + x^2$ will be even because x^2 (odd squared) is odd, and adding 1 to an odd number results in an even number.
   - Multiplying an odd number ($3x$) by an even number ($1 + x^2$) results in an even product [3] [4] [5].

Conclusion

In both cases, whether x is even or odd, the product $3x \cdot (1 + x^2)$ is always an even number.
Therefore, the statement "If 3*x is multiplied to (1+x^2), then the product is an even number" is **true**.

(b) YouChat

Figure 3: YouChat Responses to the Mathematical Task

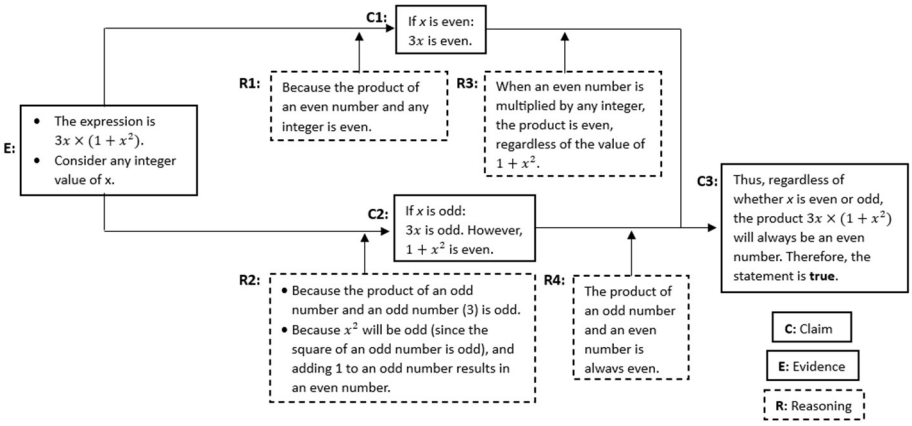


Figure 4: Argument Map Representing the Reasoning of the Generative AI

As illustrated in Figure 4, the generative AI tools—specifically ChatGPT and YouChat—began their analysis by restating the algebraic expression $3x \times (1+x^2)$ and proceeded under the assumption that the variable x is an integer. The AI then employed a case-based approach, identifying two exhaustive categories for x: namely, that x is either an even or an odd integer.

In the first case, where x is even, the AI reasoned that $3x$ must also be even, based on the mathematical principle that the product of an even number and any integer is always even. Consequently, multiplying this even result by any value—including the expression $1+x^2$—will yield an even product, in accordance with established properties of integer multiplication.

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In the second case, where x is odd, the AI asserted that $3x$ is odd, as the product of two odd numbers remains odd. It further noted that x^2 would also be odd (given that the square of an odd number is odd), and therefore, $1+x^2$ must be even. Since the product of an odd number and an even number is always even, the entire expression $3x \times (1+x^2)$ would again yield an even result.

Based on these two scenarios, both of which result in an even product, the AI concluded that the expression produces an even number for all integer values of x. Accordingly, the AI affirmed the truth of the given mathematical statement, asserting its universal validity within the set of integers.

Argumentative Structure Demonstrated by a Student

The mathematical reasoning constructed by a student participant is presented in Figure 5. The figure illustrates the student's reasoning process, including the formulation of claims, use of mathematical principles, and the logical progression employed to justify or refute the validity of the statement.

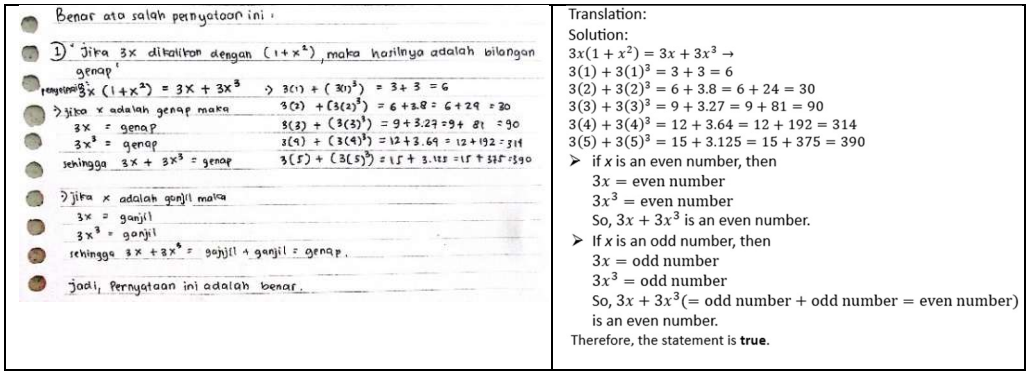


Figure 5: Responses of Students to the Mathematical Task

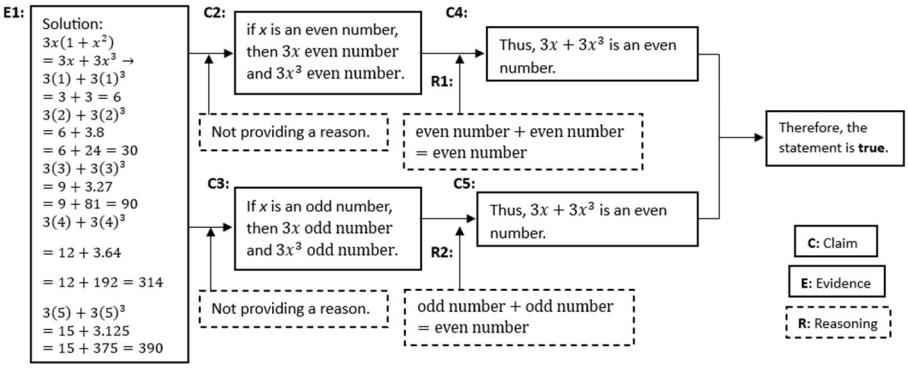


Figure 6: Structure of the Student's Argument

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Figure 6 illustrates the structure of a student's argument when evaluating the given mathematical statement. The student initially approached the problem by rewriting the expression $3x \times (1+x^2)$ and expanding it to $3x+3x^2$. Following this, the student substituted several integer values into the expanded equation. For example, when x=1, the calculation $3x+3x^2=3(1)+3(1)^2=3+3=6$, and for x=2, the equation becomes $3(2)+3(2)^3=6+3.8=6+24=30$. Based on these computations, the student identified two key conditions:

1. If x is an even number, then both $3x$ and $3x^2$ are even, and therefore, $3x+3x^3$ is even.
2. If x is an odd number, then both $3x$ and $3x^2$ are odd, and since the sum of two odd numbers is even, $3x+3x^3$ is even.

   In conclusion, the student determined that the mathematical statement was true.

The Claim-Evidence-Reasoning (CER) framework was employed to analyze the structure, content, and recipient orientation of the mathematical arguments produced by both Generative AI and university students when evaluating number theory statements. The analysis revealed that both Generative AI and the majority of students reached the same conclusion, affirming the truth of the statement under consideration. However, a subset of students diverged, concluding that the statement was false, as demonstrated in Figure 7.

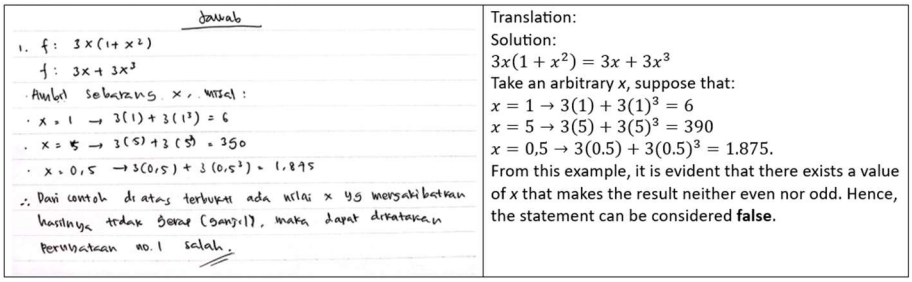


Figure 7: Responses to the Mathematical Task from an Alternative Student Group

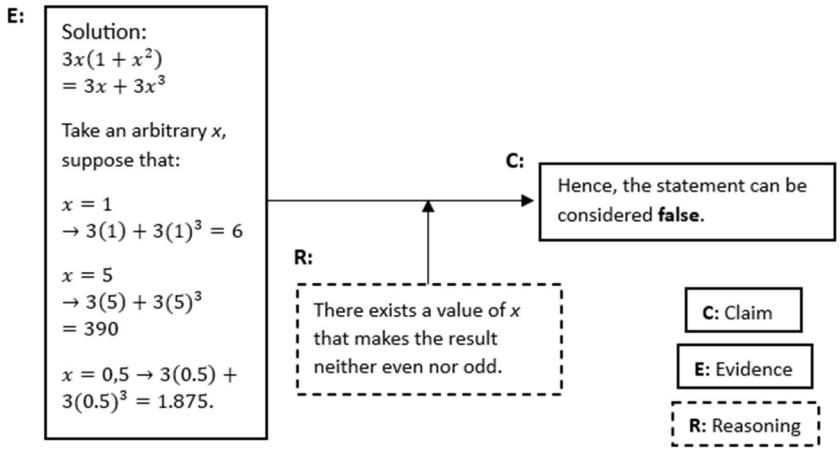


Figure 8: Structure of an Argument Involving a Counterexample

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As shown in Figure 8, this alternative group of students employed counterexamples in their reasoning. Rather than presuming that x was restricted to integer values, they systematically explored non-integer inputs. For example, substituting $x = 0.5$ into the equation $3(0.5)+3(0.5)^3=1.875$ yielded a result that contradicted earlier conclusions. This divergence arose from the fact that the problem did not explicitly define the domain of x. While the Generative AI consistently assumed x to be an integer, the students extended their reasoning to include non-integer values, thereby revealing a fundamental conceptual difference in interpretative strategy. This discrepancy highlights the critical role of counterexamples in mathematical reasoning, as they challenge implicit assumptions and contribute to the refinement and validation of mathematical arguments.

Generative AI models, such as ChatGPT and YouChat, frequently assume that x is an integer without explicitly defining this constraint within the problem context—an approach that aligns with what may be termed naïve empiricism (Stylianides, 2019). These systems predominantly rely on pattern recognition rather than formal deductive reasoning, often extrapolating from prototypical cases without systematically accounting for the complete domain of possible values for x. This methodological shortcoming compromises the validity of their reasoning, as it neglects edge cases and alternative numerical domains, thereby exposing a fundamental limitation in AI-generated mathematical argumentation.

In contrast, several students demonstrated a more analytical approach—characterized by the use of thought experiments (Stylianides, 2019)—through the systematic interrogation of implicit assumptions and the rigorous testing of a range of values for x, thereby engaging in fundamental practices of mathematical reasoning. Unlike generative AI systems, which tend to draw conclusions based on unexamined assumptions, these students employed formal justifications grounded in the use of counterexamples and logical constraints—elements that the AI models failed to incorporate. This distinction aligns with the findings of Pielsticker et al. (2024), who reported that ChatGPT exhibits difficulty maintaining logical consistency and rigor in the construction of mathematical proofs. Similarly, Frieder et al. (2023) concluded that although ChatGPT is occasionally capable of generating insightful arguments, its overall proficiency in mathematical proof remains limited due to its reliance on heuristic-based reasoning rather than structured formal logic.

The comparison between AI-generated and student-generated mathematical reasoning reveals a significant pedagogical opportunity: the strategic utilization of AI not merely as a computational tool, but as an instructional catalyst to enhance students' engagement with mathematical argumentation. These findings underscore the imperative for educators to transition from a passive reception of AI outputs to a critical, inquiry-driven approach—one that actively interrogates the logical structure, assumptions, and limitations inherent in AI-generated reasoning. This shift can be operationalized through the deliberate implementation of targeted instructional strategies.

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The findings also point to the necessity of explicitly addressing the epistemological limitations of AI-based tools within the mathematics classroom. It is essential that students are not led to accept AI-generated arguments uncritically or regard them as definitive sources of mathematical truth. Rather, educators should frame AI as a heuristic device—useful for exploration and conjecture, but ultimately subordinate to disciplined reasoning and formal justification.

CONCLUSION

The study found that Generative AI, exemplified by ChatGPT and YouChat, relies on algorithmic reasoning based on specific assumptions, such as treating x as an integer. This methodology, classified as naïve empiricism, simplifies the argumentation process but limits its adaptability to broader mathematical contexts. In contrast, students demonstrated greater flexibility by employing algebraic manipulation and thought experimentation, enabling them to consider possibilities beyond integer values. These findings underscore both the potential and limitations of Generative AI in mathematical reasoning. While AI tools can generate structured and accurate responses in well-defined scenarios, their dependence on predefined assumptions and occasional struggles with contextual nuances highlight the necessity of human oversight in mathematical problem-solving. The study reinforces the role of AI as a complementary tool in education, enhancing learning and fostering critical thinking rather than replacing human educators. By integrating Generative AI with the reasoning skills of educators and students, mathematics education can adopt a more dynamic and robust framework that encourages analytical thinking and deeper conceptual understanding.

While the present study offers meaningful insights into the comparative argumentation patterns of generative AI and undergraduate students, several limitations merit critical consideration. First, the sample of generative AI systems was limited to two models—ChatGPT and YouChat—which, although representative of current state-of-the-art tools, do not encompass the full range of available AI technologies. Consequently, the findings may not generalize to other AI systems with different training data, architectures, or reasoning mechanisms. Second, the student sample was drawn from a single population of undergraduate participants, whose prior exposure to mathematical proof techniques, familiarity with number theory, or educational background may not reflect broader or more diverse student cohorts. Such homogeneity in the student group introduces the potential for selection bias, limiting the external validity of the results.

Moreover, the study's focus on a single mathematical statement, while analytically productive, constrains the scope of inference regarding AI and human reasoning across other mathematical domains or problem types. Finally, although the Claim-Evidence-Reasoning (CER) framework facilitated a structured analysis of argumentation, its application may have introduced interpretive subjectivity, particularly in categorizing the quality and nature of students' reasoning. Future research should consider expanding the range of mathematical tasks, incorporating a more diverse

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set of AI models and student populations, and employing complementary analytical frameworks to enhance generalizability and interpretive rigor.

Future research should examine how generative AI systems handle mathematical problems characterized by undefined variables or ambiguous conditions—scenarios that require inference beyond explicitly stated assumptions. Investigating AI reasoning across a broader range of mathematical domains, including non-integer, irrational, and complex numbers, may further illuminate the strengths and limitations of AI-generated argumentation. Additionally, studies exploring the capacity of AI to construct advanced mathematical proofs and provide step-by-step justifications in higher-level mathematics could yield critical insights into areas for algorithmic refinement. Research might also focus on how AI engages with students in open-ended mathematical discussions and adapts to diverse pedagogical strategies, thereby offering a more nuanced understanding of its potential role in educational settings. Furthermore, longitudinal investigations into the sustained impact of AI integration on students' mathematical reasoning could reveal how such tools influence the development of critical thinking over time.

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An Examination of Technology Usage Levels of Pre-Service Mathematics Teachers' Microteaching Lesson Plans Based on the 5E Model: A Case Survey

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Abstract: The knowledge and skills that pre-service mathematics teachers (PMTs) should have for effective technology-enhanced mathematics teaching are essential for efficient classroom practices. This study aims to examine the PMTs' planning processes of technology-enhanced microteaching lessons based on the 5E (engage, explore, explain, elaborate, evaluate) model and determine their technology usage levels during the 5E phases. We designed a case survey model. The participants were 24 PMTs enrolled in a mathematics teacher preparation program at a state university in Istanbul. Data consisted of 24 PMTs' high school-level lesson plans and teaching notes. We analyzed the data using the Techno-Pedagogical Integration Matrix (TPIM), whose x-axis consists of the phases of the 5E instructional model, and the y-axis consists of the technology usage levels of the SAMR (substitution, augmentation, modification, redefinition) model. While the 5E phases during which participants used technology most were the engagement and exploration phases, the highest level of technology use, according to the SAMR model, was at the modification level in the exploration phase of the 5E model. This study serves as a valuable resource for researchers and educators seeking to incorporate the SAMR model into mathematics teaching, offering a clearer, more context-specific interpretation of the model.

Keywords: Technology-enhanced mathematics teaching, 5E model, SAMR, microteaching, pre-service mathematics teachers

INTRODUCTION

Digital technological tools in mathematics teaching have become widespread, especially in the last 40 years. Research on technology-enhanced mathematics teaching emphasizes that technology contributes to teaching by visualizing concepts (Zhang et al., 2023), using multiple representations and supporting conceptual understanding (Kaput, 2018; Goldin, 2020). Teachers should have adequate pedagogical and content knowledge to use technology effectively in the classroom. It is essential to support PMTs in this respect. In this context, in Türkiye, there are many courses in the

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undergraduate programs of faculties of education for PMTs to support them in successfully integrating technology into their lessons. Understanding how PMTs integrate technology into their microteaching lessons is crucial for improving teacher preparation programs. One approach to investigating this is analyzing PMTs' microteaching practices. In undergraduate teacher preparation programs in Türkiye, these practices are incorporated into courses such as Teaching Numbers and Operations, Teaching Algebra, and Teaching Geometry, among others. Still, the evaluation of microteaching lessons within the scope of technology integration is left to the initiative of the instructor giving the course. As such, it is essential to examine how PMTs use various teaching methods and strategies regarding technology integration in the planning and implementation processes of microteaching to train effective teachers. In this context, this study aims to examine PMTs' planning processes of the 5E model in technology-enhanced microteaching lessons and determine their technology usage levels during the 5E phases. In this context, this study attempts to answer the following research questions:

- How do PMTs plan each phase of the 5E model in technology-enhanced microteaching lessons?
- At which level do PMTs plan to use technology in each phase of the 5E model in technology-enhanced microteaching lessons?

LITERATURE REVIEW AND THE CONCEPTUAL FRAMEWORK

The Literature on the 5E Model

Karplus and Thier (1967) proposed the 5E model as a three-phase structure of a lesson: explore, explain, and elaborate. Later, the evaluation phase was introduced, transforming it into the 4E model. Finally, with the infusion of the constructivist approach, the engagement phase was integrated, completing the 5E model. The sequential phases are *engage*, *explore*, *explain*, *elaborate*, and *evaluate* (Bybee, 2014).

Engagement Phase: This phase aims to capture students' attention and interest. The teacher directs students' focus toward a situation, event, demonstration, or problem related to the content. Strategies such as questioning, problem-posing, and presenting unique events engage students and activate their prior knowledge.

Exploration Phase: At this stage, students engage in activities to resolve the disequilibrium introduced earlier. These activities are designed to introduce and clarify the concepts, practices, and skills associated with the instructional sequence. Students need experiences that help them explain, explore, observe, and develop cognitive and practical skills. In this phase, the teacher initiates the activity, explains the necessary background, provides adequate materials and equipment, and addresses misconceptions.

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Explanation Phase: During this phase, emphasis is placed on the scientific comprehension of phenomena. Concepts and practices that students previously encountered and explored are now elucidated. The teacher concentrates on the essential elements from the preceding phases and seeks students' explanations. Subsequently, leveraging these explanations and experiences, the teacher introduces scientific or technological concepts. While verbal explanations are prevalent in this phase, videos, web resources, or software can also be effective instructional aids.

Elaboration Phase: Students participate in learning experiences to enrich the concepts established in the preceding phases. The objective is to facilitate transferring what has been learned to novel situations. A crucial aspect of this phase involves employing challenging activities that students can effectively accomplish. The teacher introduces a new scenario and fosters student interactions with resources like written materials, simulations, and web-based searches.

Evaluation Phase: Students receive ongoing informal feedback on their explanations throughout instruction, while the evaluation phase focuses on formally assessing and reporting learning outcomes. In this phase, the teacher engages students in activities that are comprehensible, consistent with previous phases, and aligned with their explanations. The teacher must discern the evidence of student learning and the methods for acquiring this evidence as part of the evaluation phase. This phase encourages students to evaluate their understanding and abilities while enabling teachers to assess their progress toward learning objectives.

The literature on the 5E model demonstrates its effectiveness in enhancing students' academic achievement across various levels and contexts (Cakir, 2017; Cakir & Güven, 2019; Guzel, 2016; Parveen, 2017; Dahal, 2023). Studies indicate that the 5E model positively influences students' attitudes toward specific subjects (Cakir, 2017; Guzel, 2016; Sotáková & Ganajová, 2023). It promotes conceptual learning (Grau et al., 2021) and enhances 21st-century skills such as critical and creative thinking (Asrizal et al., 2022). Consequently, this instructional model facilitates knowledge construction and aligns well with student-centered approaches (Triet et al., 2024; Yaman & Karaşah, 2018). However, incorporating the 5E model into education can pose challenges for teachers (Nawastheen, 2014) and pre-service teachers (Turan, 2021), underscoring the necessity for professional development in this domain (Nawastheen, 2014).

In addition to the previously mentioned studies, literature also examines the incorporation of the 5E model into mathematics lessons. These studies indicate that the 5E model enhances students' mathematics achievement (Akinyemi & Okwatojin, 2017; Tezer & Cumhur, 2017), their levels of conceptual and procedural knowledge (Bakri, 2021), and consequently, their conceptual understanding (Grau et al., 2021). Furthermore, it improves their performance in non-routine problems (Adu & Folson, 2023), increases their engagement in lessons (Turan & Matteson, 2020), and ignites their interest in the subject (Nguyen, 2021). Similarly, the literature underscores the positive impact of integrating the 5E model with technology in teaching. This fusion significantly boosts

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students' higher-order thinking skills (Shivam & Mohalik, 2022). Integrating educational technology, such as e-modules, into the 5E model enhances engagement, interactivity, and effectiveness in attaining learning objectives (Penelitian et al., 2023).

Literature on the SAMR Model

The SAMR model, introduced by Dr. Ruben Puentedura, serves as a framework for comprehending technology integration in education (Puentedura, 2010; 2014; 2020). It encourages teachers to contemplate how and why they use technology and how its integration can enhance their pedagogical practices as they gain proficiency (Puentedura, 2016). This model facilitates the assessment of the level and purposes of technology integration in instruction, organized into four sub-levels across two primary levels (see Table 1). The *enhancement* level encompasses the *substitution* and *augmentation* levels. Technology enhances and improves the learning experience at the *enhancement* level (Caukin & Trail, 2019). The *transformation* level encompasses the *modification* and *redefinition* levels. This level aids in transforming the learning experience (Caukin & Trail, 2019). The SAMR model can be conceptualized as a ladder, with *substitution* at the lowest rung and *redefinition* at the highest (Caukin & Trail, 2019). *Substitution* occurs when technology directly replaces a task without altering its function (Puentedura, 2010). At the *augmentation* level, technology is a direct tool for enhancing functionality (Puentedura, 2010). At the *modification* level, the task undergoes significant redesign through technology. Finally, at the *redefinition* level, technology enables the creation of entirely new tasks that are not feasible without its use (Caukin & Trail, 2019). Additionally, although not explicitly part of the SAMR model, the absence of technology, as emerged from the data of this study, has also been incorporated into Table 1.

Main levels	Levels	Brief descriptions
Transformation	Redefinition	Creation of entirely new tasks that are not possible without technology integration.
	Modification	Tasks are rearranged.
Enhancement	Augmentation	Technology is implemented for a functional reason.
	Substitution	Technology directly replaces a task without altering its function
	Not using technology	Technology is not utilized.

Table 1. Revised SAMR Model

Reviewing the literature on the SAMR model reveals that its integration into instruction has resulted in positive changes in student attitudes and achievements (Adulyasas et al., 2021; Romrell

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et al., 2014). Additionally, students have demonstrated increased reflection in their classes (Hilton, 2015). These findings underscore the SAMR model's contribution to enhancing instruction.

Conceptual Framework

This study examines how PMTs plan for technology-enhanced microteaching lessons and assesses the extent of technology integration across the phases of the 5E model. We developed a conceptual framework called the TPIM to accomplish this goal. The TPIM has two dimensions: the *x*-axis corresponds to the phases of the 5E instructional model, and the *y*-axis indicates the levels of technology utilization according to the SAMR model (see Figure 1). Şahin (2020) and Şahin & Akkoç (2020; 2021) integrated the Teaching Quality framework (Klieme et al., 2009) and the SAMR model (Puentedura, 2010) to investigate the pedagogical strategies employed by PMTs in addressing misconceptions regarding the concept of derivative. They termed this framework TPIM. In the framework applied in these studies, the *x*-axis represented the Teaching Quality framework, while the *y*-axis indicated SAMR levels. For this study, the *x*-axis has been adjusted as the phases of the 5E model to explore how technology is integrated into lessons at these stages.

METHODOLOGY

Research model

We designed this study as a case survey model. The survey model aims to describe a particular phenomenon. The case survey model, a specific type of this model, provides a more detailed examination of the phenomenon described (Yin, 2018). This study prefers this model since it allows the exploration of the lesson plans of all PMTs in a specific course, combining the breadth of a survey with the depth of case analysis.

Context of the study and participants

The study participants consisted of 24 PMTs enrolled in a mathematics teacher preparation program of an education faculty at a state university in Istanbul. The study was conducted within the scope of the Mathematics Teaching II course in the spring semester of the program's third year. All of the PMTs registered on the Mathematics Teaching II course participated in the study. The course instructor was the second author of this study. Due to the pandemic, the classes were taught using a hybrid method. We divided the class into two groups: one attended weekly face-to-face sessions, while the other participated simultaneously via the university's distance education platform. During the fall semester, the *Mathematics Teaching I* course covered various topics, includ-

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ing procedural and conceptual understanding, misconceptions, and student difficulties. Additionally, it addressed van Hiele's geometric thinking levels, principles of task design, and evaluation of mathematical tasks (Dede et al., 2020; Yeşildere-İmre, 2020).

Within the scope of the final assignment of this course, PMTs evaluated tasks based on the design principles of mathematical tasks (Yeşildere-İmre, 2020). First, they identified a learning outcome from the upper secondary mathematics curriculum (MoNE, 2018) and selected a task from various sources that aimed at this learning outcome. Then, they evaluated the tasks by considering task design principles (Yeşildere-İmre, 2020). They prepared and presented their evaluations as a written report. The Mathematics Teaching II course focused on teaching strategies and methods. The expository teaching approach (Ausubel, 1963), teaching through discovery, and inquiry-based teaching (Bruner, 1960) were discussed with examples. Afterward, the course instructor explained the 5E model using exemplary lesson plans and analyzed them according to the five phases of the model. The sample lesson plans covered topics such as the Pythagorean theorem, interpreting function graphs, right prisms, definite integrals, etc., and were divided into subsections according to each stage of the 5E model. A whole-class discussion focused on identifying the characteristics of each stage. The training on effective technology integration was conducted by the first author as part of the *Instructional Technologies and Material Design* course held during the same term. This course focused on the use of software such as Geogebra and Desmos and their integration into lesson planning. For the final assignment of the *Mathematics Teaching II* course, PMTs selected a learning outcome from the curriculum (MoNE, 2018) and designed a lesson plan according to the 5E model. The assignment instructions included selecting learning outcomes that were suitable for the 5E model, ensuring that the lesson plan contained subsections for each phase of the model, and writing teacher and student activities for each phase (See Table 2).

5E phases	Teacher activity	Student activity
Engagement	The teacher asks students about the perimeter, area, and circle. The teacher asks for their opinions to assess their prior knowledge.	Students engage in peer discussions to answer the questions posed by the teacher and articulate their perspectives.
Exploration	Teacher asked students to review the GeoGebra activity called "Icosahedron Planet."	Students are expected to derive the formula for the circumference of a circle through the GeoGebra activity *"Icosahedron Planet."* Throughout this process, they actively exchange information and engage in discussions with their group members.

Table 2. A section from the lesson plan template

The course instructor (the second author) gave feedback on participants' microteaching lessons based on the 5E model and technology integration.

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Data Analysis

The data sources for this study consist of only lesson plans and teaching notes (e.g., worksheets, homework, slides) rather than microteaching lessons. Since the framework for data analysis and the lesson plans were well-structured, we analyzed the data manually. We analyzed the lesson plans to address the first research question by segmenting them according to the phases of the 5E instructional model. We identified the distinct types of teacher and student activities present in the plans and then calculated their frequency values. Then, each section underwent content analysis to examine how PMTs structured their lessons. Based on the findings, categories were then developed for each phase. In this context, teaching notes served as a supplementary data source. To address the second research question, we analyzed the level of technology integration in each phase of the 5E model using the SAMR framework. This methodological approach provided a comprehensive perspective on how PMTs incorporated technology into their lesson planning.

Validity and Reliability of the Study

Lincoln and Guba (1985) proposed four key criteria for ensuring trustworthiness in qualitative research: credibility, transferability, dependability, and confirmability. Credibility relates to how accurately findings reflect participants' experiences, transferability concerns their applicability to other contexts, dependability addresses consistency, and confirmability ensures findings are shaped by participants rather than researcher bias.

To enhance credibility, a case survey approach and data triangulation—using lesson plans and teaching notes—were employed. The second author observed and engaged with PMTs throughout the semester, contributing to data authenticity. The first author led the analysis, with consistency checked via Miles and Huberman's (1994) method. Two additional researchers—the course instructor and a qualitative expert—independently reviewed the lesson plans, adding diverse perspectives. The TPIM matrix supported systematic data analysis.

Transferability was addressed through detailed descriptions of the context and participant selection. An audit trail ensured dependability by documenting the research process. Confirmability was strengthened through triangulation and rigorous analysis. After the first author coded the data, inter-coder agreement was examined with the second author to ensure coding reliability. Coding reliability was confirmed with 100% inter-coder agreement between the first and second authors. Table 3 presents the results of this analysis.

RESULTS

Findings about the Phases of the 5E Model

This subsection will present findings regarding the first research question: "How do PMTs plan each phase of the 5E model in technology-enhanced microteaching lessons?" To answer this question, we examined the teacher and student activities outlined in the lesson plan template (See Table 2). The analysis focused on identifying the activities planned by PMTs for each phase of the 5E

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model in technology-enhanced microteaching sessions. Table 3 summarizes the lesson plans' teacher and student activities and their corresponding frequency values.

5E Phases	Teacher Activities	N	Student Activities	N
Engagement	• Examining students' readiness levels	20	• Participating in class discussions	20
	• Establishing a connection between the concept and daily life	15	• Finding examples related to the concept from daily life	5
	• Drawing attention to the lesson	11	• Watching videos	5
	• Pre-informing students about the lesson	4	• Being attentive to the lesson	4
	• Grouping students	1	• Sharing ideas through related visuals	2
			• Relating concepts/ideas	2
			• Being a passive listener	1
			• Making predictions about the topic	1
	TOTAL	**51**	**TOTAL**	**40**
Exploration	• Initiating the process of discovery with the help of activities and questions to students	24	• Participation in the activities and class discussions	24
			• Expressing the conclusions reached	3
	TOTAL	**24**	**TOTAL**	**27**
Explanation	• Explaining the ideas discovered by students in more detail	12	• Making sense of the definition provided by the teacher	7
	• Giving the definition after student explorations	12	• Expressing the topic/concept/situation correctly	6
	• Informing students about the use of the concept in other subjects	1	• Comparing/relating the concept with other concepts	2
	• Asking students questions about the concept	1	• Concentrating on conceptual understanding	1
	• Associating the concept with other concepts	1	• Interpreting peers' hypotheses	1
			• Presenting examples from daily life	1
			• Making connections between what emerges in classroom interaction and their conclusions	1
			• Learning the key points of the concept	1
			• Answering the teacher's questions	1
			• Producing different methods for solutions	1
	TOTAL	**27**	**TOTAL**	**22**
Elaboration	• Reinforcing the concept with new questions involving connections with other concepts	15	• Answering new questions using previously discovered facts	13
	• Relating the concept to other concepts	3	• Gaining more detailed information about the concept	3
	• Relating the concept to daily life	2	• Connecting the concept with other concepts	2
	• Mentioning situations that may create misconceptions about the subject	1	• Recognizing misconceptions	1
	• Providing information about the historical development of the concept	1	• Making the most comprehensive definition of the concept	1
	• Asking questions about the concept in more than one context	1	• Presenting examples from daily life and different disciplines	1
	• Building a model related to the concept	1	• Clarify parts that are mislearned or not fully understood	1
			• Model-building	1
	TOTAL	**24**	**TOTAL**	**23**
Evaluation	• Evaluating students through various platforms (Desmos, websites, GeoGebra, etc.), tests from textbooks, or worksheets	16	• Answering questions	16

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• Asking students to self-assess	4	• Conducting self-assessment	4
• Asking students to evaluate the lesson	2	• Asking the teacher about unclear points	3
• Asking students to make presentations about their evaluations	1	• Creating a concept map	1
		• Presentation	1
		• Lesson evaluation	1
TOTAL	**23**	**TOTAL**	**26**

Table 3. Teacher and student activities identified in the lesson plan

In the *engagement* phase, *examining students' readiness levels (20 occurrences*), *establishing a connection between the concept and daily life (15)*, and *drawing attention to the lesson (11)* were the most favored teacher activities. Other teacher activities included *pre-informing students about the lesson (4)* and *grouping students (1)*. Regarding student activities, notable actions included *participating in class discussions (20)*, *finding examples related to the concept from daily life (5)*, *watching videos (4)*, and *being attentive to the lesson (4)*. Thus, while *determining students' readiness levels* emerged as the most emphasized teacher activity in this phase, *participating in class discussions* was the prominent student activity.

All 24 PMTs unanimously stated that teachers could initiate the exploration phase, by posing questions to students and guiding their discovery process through tasks. Regarding the student activities, they emphasized *student participation in the activity and class discussion* (24) and students' statements about *the conclusions reached* (3). The unanimity among PMTs regarding the teacher activity in the *exploration* phase is notable. In this respect, PMTs think teachers should realize this phase by asking questions or implementing tasks.

Teacher activities such as *explaining the ideas discovered by the students in more detail* (12) and *providing definitions after the student explorations* (12) are noteworthy activities in the *explanation* phase. Moreover, the *explanation* phase exhibited the highest variety of student activities. The most common ones were *making sense of the definition provided by the teacher* (7) and *correctly expressing the topic/concept/situation* (6). The findings underscore the importance of teachers providing the ideas discovered by students in the previous phase in a detailed and formal manner. Conversely, although student activities varied, the focus was on understanding the concept, expressing it, interpreting it, and establishing connections between concepts.

The *elaboration* phase exhibited the highest variety of teacher activities among the 5E phases. Notably, *reinforcing the concept with new questions involving connections with other concepts* (15) stood out. Additionally, *relating the concept to other concepts* (3) and *relating the concept to daily life* (2) were other teacher activities. Regarding student activities, *answering new questions using previously discovered facts* (15) were the most frequently cited. In this phase, both teachers and students aimed to deepen their knowledge by applying it to different situations and contexts.

The most noteworthy teacher activity in the *evaluation* phase was *evaluating student learning through various platforms (Desmos, websites, GeoGebra, etc.), tests from textbooks, or worksheets* (16). Other assessment approaches were *asking students to self-assess* (4) and *asking students to*

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evaluate the lesson (2). *Having students answer questions* (16) was the most mentioned student activity in the *evaluation* phase.

Table 4 shows a PMT's lesson plan excerpt, coded as "examining students' readiness levels" for teacher activity and "participating in class discussions" for student activity.

5E phases	Teacher activity	Student activity
Engagement (7 minutes)	Using the activity "Are We Driving Carefully?", students are asked questions related to perimeters, areas, and circles. Their ideas are elicited to assess their prior knowledge. Students are asked questions based on bicycle images to assess the level of their prior knowledge regarding the concept of perimeter.	Students engage in discussions with their peers and express their ideas.

Table 4. An excerpt from a lesson plan (x-axis: engagement; y-axis: not using technology)

While there was no use of technology as can be seen in Table 4, another PMT used technology at the *substitution* level during the *engagement* phase as seen in Table 5.

5E phases	Teacher activity	Student activity
Engagement (5 minutes)	The teacher opens the quiz "Circumference and Area Questions in a Circle" in Desmos and gives students some time to complete the questions while closely monitoring their progress. The teacher reminds students that both their work process and outcomes will be assessed and will contribute to their grades.	The student works individually to solve the questions in Desmos and seeks assistance from the teacher when encountering difficulties.

Table 5. An excerpt from a lesson plan (*x*-axis: engagement and *y*-axis: substitution)

This excerpt was coded as "evaluating students through various platforms (Desmos, websites, GeoGebra, etc.), tests from textbooks, or worksheets" under teacher activity and "answering questions" under student activity.

While some PMTs accurately structured the phases of the 5E model in their lesson plans, others struggled to organize these phases effectively. Table 6 displays the errors identified in the lesson plans of the PMTs regarding identifying the phases of the 5E model.

PMTs	The 5E phase specified	Correct 5E phase
PMT1	Exploration	Explanation
PMT1	Elaboration	Evaluation
PMT3	Exploration	Engagement
PMT4	Exploration	Engagement

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PMT4	Explanation	Exploration
PMT4	Explanation	Elaboration
PMT5	Exploration	Engagement
PMT13	Evaluation	Elaboration
PMT16	Elaboration	Exploration
PMT16	Elaboration	Explanation

Table 6. Errors detected in lesson plans (wrong positioning of the phases)

According to the table, we found ten errors in the lesson plans of six PMTs. For instance, an activity intended for the *exploration* phase was sometimes presented in other phases. In these cases, the study repositioned the activity to the phase where it should be and conducted the analysis accordingly. As seen in Table 6, PMT4 included a teacher activity in the *exploration* phase. He stated, "*The teacher relates the topic of probability to daily life and asks students to find examples.*" This activity, however, should be part of the *engagement* phase of the lesson rather than the *exploration* phase. Similarly, PMT1 mentioned that the *teacher gives the definition of the concept* during the *exploration* phase. However, the definition of the relevant concept should be presented during the *explanation* phase. Upon reviewing the lesson plan of PMT13, *comprehensive understanding utilized to underscore the relationship between the concept and other disciplines and everyday life* should ideally be situated within the *elaboration* phase rather than the *evaluation* phase. Consequently, this statement was re-coded as the *elaboration* phase. We detected analogous misplacements in the lesson plans of other PMTs, as outlined in Table 6 and rectified accordingly for the analysis of this study.

The Levels of Technology Use Planned in the Stages of the 5e Model

This sub-heading presents the findings concerning the second research question regarding the level of technology use in each phase of the 5E model in lesson plans. The levels of technology use in each phase of the 5E model in their technology-enhanced microteaching lessons are presented in Figure 1 within the framework of TPIM.

As illustrated in Figure 1, PMTs did not incorporate technology in 62 activities outlined in their lesson plans. Furthermore, activities at this level are uniformly dispersed across all phases of the 5E model, with 12 activities allocated to *engagement*, 11 to *exploration*, 13 to *explanation*, 11 to *elaboration*, and 15 to *evaluation*. In activities categorized as *not utilizing technology*, PMTs intended to use materials such as whiteboards, activity sheets, textbooks, etc., instead of technological resources.

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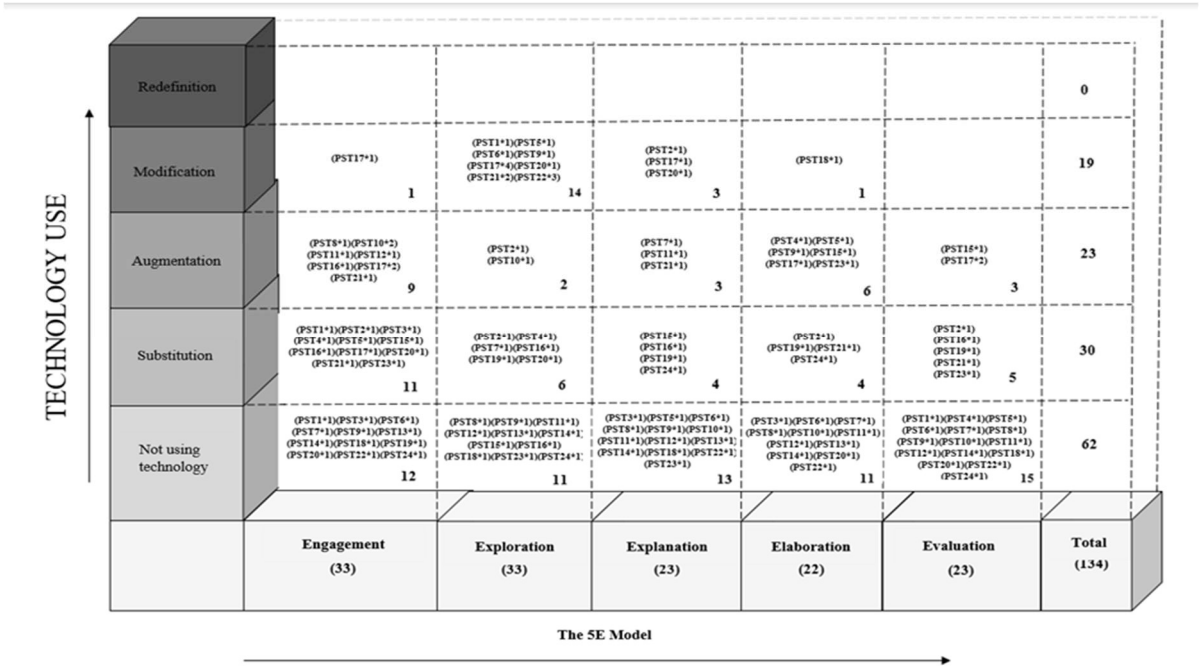


Figure 1. Techno-Pedagogical Integration Matrix (TPIM)

PMTs predominantly planned to integrate technology at the *substitution* level, with 30 instances recorded. Upon scrutinizing the distribution of the *substitution* level across the 5E phases, PMTs intended to employ this level most frequently during the *engagement* phase of the lesson, amounting to 11 instances. Specifically, they aimed to project lesson content onto the smart board. Figure 2 showcases a screenshot of the content projected onto the smart board by PMT17 at the onset of the lesson. PMT17 intended to project content featuring visuals illustrating the connection of the angle concept to daily life, alongside various related questions, to introduce the topic of angles at the beginning of the lesson.

Additionally, the *substitution* level demonstrates a relatively consistent distribution across the other four phases of the 5E model. PMTs aimed to incorporate *substitution*-level activities in six instances during the *exploration* phase, four during the *explanation* phase, four during the *elaboration* phase, and five during the *evaluation* phase. Throughout these phases, their primary approach involved projecting lesson content onto the smart board. Nevertheless, unlike others, PMTs intended to employ this level by administering virtual tests prepared in applications such as Desmos during the *elaboration* and *evaluation* phases.

PMTs incorporated a total of 23 activities at the *augmentation* level in their lesson plans concerning the utilization of technology. Upon examining the distribution of the *augmentation* level across the 5E phases, PMTs predominantly used technology in the *engagement* phase of the lesson. In this context, they intended to leverage technology by showcasing videos, graphing in GeoGebra,

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responding to questions in GeoGebra, conducting activities prepared in GeoGebra (without variable changes, providing a video-watching effect), or organizing the classroom using applications like Rakko (streamlining the grouping of students).

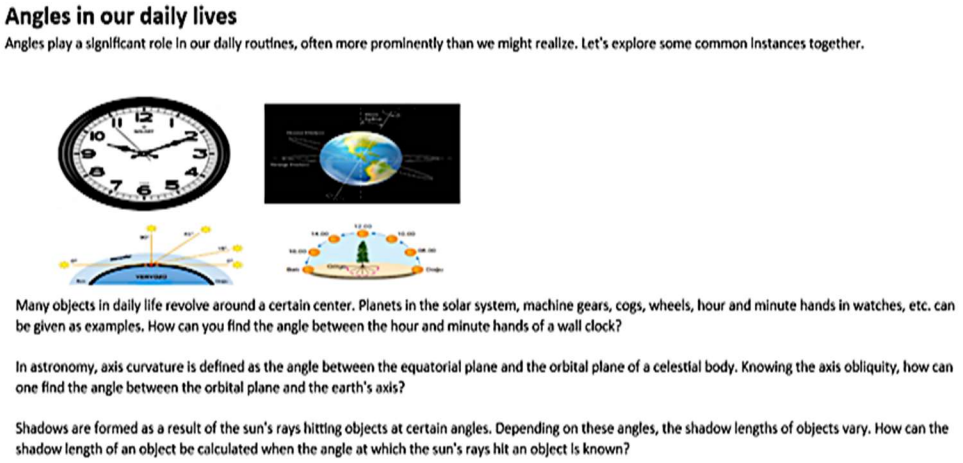


Figure 2. An example activity for the use of technology at the *substitution* level (PMT17)

Furthermore, PMTs strategically incorporated *augmentation*-level activities for various purposes across the other phases of the 5E model. They aimed to enhance students' inferential processes concerning the concept by utilizing tools such as GeoGebra's toolbox, graphing in Excel, generating new questions related to the concept using GeoGebra's random command, utilizing GeoGebra's calculator function, or solving questions related to the concept on various websites. As illustrated in Figure 3, PMT10 attempted to bolster students' inferential processes related to comprehension through graphing in GeoGebra. Consequently, PMT10 planned to implement technology at the *augmentation* level during the *exploration* phase of the 5E model.

There were 19 activities planned at the *modification* level. An examination of the distribution of activities at this level across the 5E model's phases shows that PMTs primarily aimed to utilize technology during the *exploration* phase (14). In doing so, they aimed to use GeoGebra's *slider* tool to enable students to make inferences based on different values . In this context, the planning at the *modification* level serves a single purpose, unlike other levels of technology. As seen in Figure 4, PMT21 planned to use GeoGebra's slider tool to introduce the concept to students during the *exploration* phase of the 5E model. The lesson prepared by PMT21 aimed to explore the formula for the area of a circle based on a polygon. In doing so, PMT21 focused on linking the two concepts by changing the number of sides of the polygon with the slider tool.

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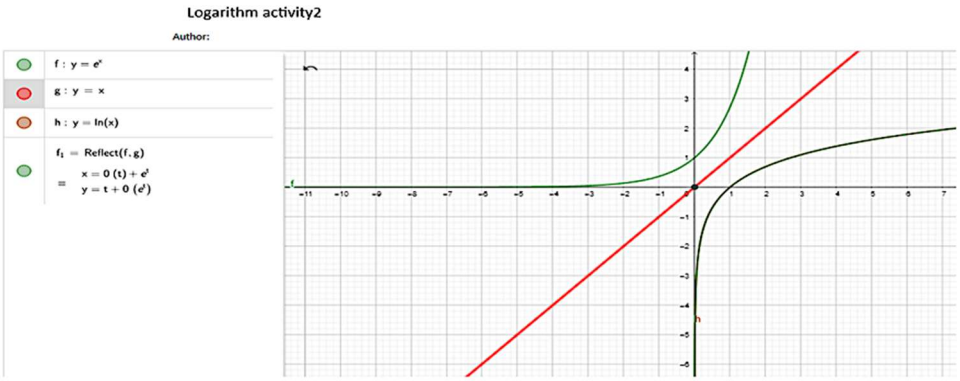


Figure 3. A sample activity section for the use of technology at the augmentation level (PMT10)

Additionally, as inferred from Figure 1, PMTs did not plan to use technology at the *redefinition* level. However, in the lesson plan of PMT18, the homework required designing an activity in GeoGebra similar to the activity used in the lesson. Although detailed information regarding this task was not offered in the plan, initially, the assignment might seem to be at the *redefinition* level. However, since the assignment solely entails technically designing a GeoGebra activity and is a repetition of an activity students have previously encountered, it does not meet the criteria for being classified at the *redefinition* level.

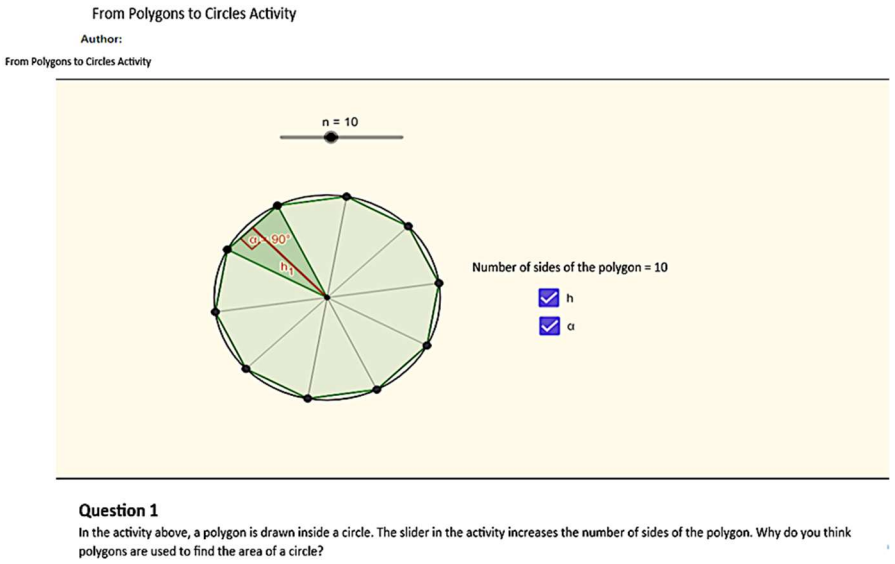


Figure 4. An example activity for the use of technology at the *modification* level (PMT21)

Finally, when evaluating the 5E model in general, PMTs have devised more activities in the *engagement* (33) and *exploration* (33) phases than in the other phases. In both phases, their plans primarily were at the level of *not using technology* or *substitution*. Particularly in the *exploration* phase, they aimed to employ GeoGebra to bolster students' inference-making (*modification level*).

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DISCUSSION AND CONCLUSION

The findings indicate that PMTs primarily focused on teacher-led rather than student-centered activities in their lesson plans, suggesting an emphasis on teaching rather than learning. Notably, most teacher and student activities were concentrated in the *engagement* phase of the 5E model, a pattern also reflected in the TPIM framework (see Figure 1). PMTs planned most of the activities during this phase with or without technology. The study primarily emphasized assessing students' readiness levels and establishing connections to real-life contexts. Concerning to the findings about enhancing student engagement, teacher educators should: (1) promote interactive tasks that require active student participation, (2) provide structured guidance to ensure lesson plans include observable and measurable student actions, and (3) model effective strategies for facilitating student-centered learning instead of direct instruction.

Across all phases, student participation in class discussions was a prominent activity. However, PMTs appeared to assume that active student engagement would occur automatically. For instance, the lesson plan section for student activities during the *engagement* phase often listed "being interested in the lesson," which does not constitute a measurable or observable action. PMTs should incorporate clearly defined student activities that involve observable participation to improve lesson planning.

"Initiating the process of discovery with the help of activities and questions to students." was the only identified teacher activity in the *exploration* phase. Key teacher activities in other phases included "formally presenting the concept" (*explanation*), "reinforcing the concept in different contexts" (*elaboration*), and "assessing students' understanding" (*evaluation*). These findings align with the literature's definitions of the 5E model phases (Bybee, 2014).

Among the 5E phases, the *elaboration* phase exhibited the most variety of teacher activities (seven types), while the *exploration* phase showed the most diverse student activities (ten types). This finding suggests that PMTs recognized multiple ways to deepen students' conceptual understanding, from addressing misconceptions to model construction.

Some PMTs misidentified or misplaced the phases of the 5E model in their lesson plans. The *exploration* phase was often confused with the preceding *engagement* phase and the subsequent *explanation* phase. Similarly, the *evaluation* and *elaboration* phases were frequently misinterpreted. These findings suggest that some PMTs struggle to distinguish the transitions between successive 5E phases, a challenge also reported by Enugu and Hokayem (2017). Findings indicate the importance of strengthening PMTs' comprehension of the 5E instructional model. Teacher educators should offer explicit instruction on the distinctions between its phases through specific examples and structured templates, facilitate lesson-planning workshops that provide feedback on activity sequencing, and incorporate classroom simulations that allow for real-time application and reinforcement of phase implementation. Analysis of lesson plans through the TPIM framework revealed that PMTs incorporated the highest number of activities in the *engagement* and *exploration* phases, with a notable focus on technology use. Specifically, technology was planned for 21 out of 33 activities in the *engagement* phase and 22 out of 33 activities in the *exploration* phase.

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In the *engagement* phase, they planned to use technology mainly at the *substitution* level by reflecting the relevant content on the smart board. PMTs employed technology at the *augmentation* level in the *exploration* phase, mainly through GeoGebra, to enhance students' mathematical discovery processes. To improve technology use in mathematics teaching, teacher educators should encourage PMTs to go beyond basic tools and use interactive digital tasks, like GeoGebra, for deeper learning. Since PMTs focus mostly on *engagement* and *exploration* while overlooking *elaboration* and *evaluation*, they should be guided to create more balanced lesson plans. This includes adding various activities in the *elaboration* phase and using better assessment methods in the *evaluation* phase.

Some PMTs aimed to integrate technology at the *modification* level for mathematical explanations. This finding is consistent with previous research (see Akkoç, et. al., 2022). However, no instances of *redefinition* were observed in their lesson plans, aligning with prior studies (Akkoç et. al., 2022; Akkoç & Hangül, 2023). Understanding and applying the TPACK and SAMR models can be challenging, even for experienced educators (Kirkland, 2017). Successful technology integration relies primarily on sound pedagogy rather than the technology itself (Dias & Atkinson, 2001). Effective technology integration requires more than technical skills; meaningful learning tasks must be thoughtfully designed. Therefore, a superficial comprehension of TPACK and SAMR models would be insufficient (Kirkland, 2017). However, educators should create opportunities for students to engage in innovative digital tasks, such as using augmented reality applications to design mathematical representations to foster this level of integration. Furthermore, teacher educators should incorporate research findings that effectively implemented the SAMR model (e.g., Adulyasas et al., 2021) to enhance instructional design and technology integration. The findings enhance the understanding of how TPIM and SAMR levels can be integrated into mathematics teaching. Puentedura's (2010) descriptions of SAMR levels tend to be general, lacking subject-specific applications, which may pose challenges for educators seeking to implement the model effectively. This study guides researchers and educators aiming to integrate the SAMR model into mathematics instruction, offering a more contextualized approach to its application.

These findings indicate that TPIM enhances the comprehensibility of the SAMR model's levels in the context of their integration into mathematics teaching. Analyzing the explanations of each level (see Puentedura, 2010) reveals their general nature, with notable limitations in domain-specific practices. This lack of specificity may pose challenges for researchers and educators seeking to implement the model effectively within their field. In this regard, the present study serves as a valuable resource for researchers and educators aiming to integrate the SAMR model into mathematics teaching, contributing to a clearer and more contextualized understanding of the model. Moreover, the TPIM framework helps PMTs better align their lesson planning with the 5E model and SAMR levels. By connecting phases such as *exploration* and *elaboration* with meaningful technology use, TPIM could guide PMTs to move beyond surface-level integration. This could contribute to more balanced, student-centered instructional design and support deeper learning in mathematics classrooms.

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In summary, effective use of the 5E model in the context of technology-enhanced mathematics teaching requires a structured approach that prioritizes student-centered learning. Providing targeted training, clear lesson-planning guidelines, and professional development opportunities will help PMTs could develop their instructional strategies and enhance their ability to integrate technology meaningfully. By addressing these gaps, teacher educators can better equip PMTs to foster deeper conceptual understanding and more engaging learning experiences in mathematics classrooms.

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Sustainable Development Awareness among Pre-Service Mathematics Teachers in Türkiye: Insights from a Case Study

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Abstract: This study examines the awareness of Sustainable Development Goals (SDGs) and Education for Sustainable Development (ESD) among pre-service mathematics teachers at Erciyes University, Türkiye. As future educators, these teachers play a vital role in raising students' awareness of sustainable practices and stimulating active engagement in sustainability initiatives. The study explores how sustainability awareness varies based on gender, academic year, and membership in environmental organizations. Findings indicate that over 60% of participants are familiar with the SDGs, though many expressed concerns about Türkiye's progress in achieving them. The results provide valuable insights for developing strategies to enhance sustainability training for future educators in Türkiye.

Keywords: Education for Sustainable Development (ESD), Environmental Awareness in Education, Sustainability in Mathematics Education, Case Study in Türkiye

INTRODUCTION

Over the past century, rapid industrialization has significantly increased human demand for natural resources, and that leads to unsustainable practices that have severely disrupted the planet's ecological balance. Critical levels of environmental degradation have been reached due to deforestation, overfishing, and the depletion of water resources (Mebratu, 1998). In response, Sustainable Development (SD) has emerged as a comprehensive framework that integrates economic, social, and environmental considerations. The aim was to ensure that present needs are

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met without compromising the ability of future generations to fulfill their own (Schaefer & Crane, 2005).

As awareness of sustainability challenges continues to grow, education has become a central tool in addressing these issues. Recognizing its transformative potential, the United Nations (UN) introduced the 17 Sustainable Development Goals (SDGs) to guide global sustainability efforts, with SDG 4 ("Quality Education") playing a pivotal role in fostering sustainability awareness and action. SDG 4 underscores the importance of inclusive, equitable, and high-quality education in promoting lifelong learning and enhancing global sustainability consciousness (UN General Assembly, 2015).

Education for Sustainable Development (ESD) serves as a fundamental approach to raising both individual and collective awareness of sustainability. Through curricula and pedagogy, education shapes behaviors, attitudes, and values that promote sustainable living (Tanriverdi, 2009). Teachers, particularly pre-service teachers, are central to this process, since they are responsible for imparting sustainability-related knowledge and values in future generations. To support this mission, the UNESCO Greening Curriculum Guidance provides a structured framework for integrating sustainability principles into education at all levels. The guide aims to cultivate environmental literacy and enhancing problem-solving skills among learners (UNESCO, 2024).

Recent international agreements, such as the COP28 Climate Change Conference, have reinforced the urgency of embedding climate change education into national policies. COP28 recognizes education as a key enabler of sustainability and calls for the integration of climate literacy in curricula, the mobilization of financial resources, and the advancement of climate-smart education systems (UNFCCC, 2023). Equipping future educators with the necessary knowledge, skills, and pedagogical approaches is essential for effectively teaching sustainability concepts and preparing students to address global challenges (Kioupi & Voulvoulis, 2019).

One of the most powerful tools for promoting sustainability within education is mathematics. Mathematics education provides students with essential analytical and problem-solving skills required to address complex sustainability issues (Guterres, 2020). Many critical environmental and societal challenges such as climate change, resource management, and economic sustainability require mathematical proficiency in data analysis, statistical modeling, and quantitative reasoning (Alsina, 2022). For instance, statistical methods are crucial for tracking climate change patterns, assessing carbon footprints, and evaluating the effectiveness of environmental policies. Similarly, optimization techniques contribute to improving resource allocation, renewable energy planning, and sustainable urban development. Mathematical modeling further supports sustainability efforts by predicting ecological trends and enabling policymakers to develop more effective environmental strategies (Guterres, 2020).

Recognizing the critical role of mathematical literacy in sustainability, SDG 4 emphasizes the need to strengthen mathematical education to enhance learning outcomes in both literacy and numeracy

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(Boeren, 2019). Integrating sustainability within mathematics education develops students' ability to interpret sustainability indicators. At the same time, it cultivates critical thinking and evidence-based decision-making, which are essential for addressing global sustainability challenges.

By embedding sustainability-focused topics into mathematics curricula, educators can enhance students' understanding of real-world issues, such as climate change, deforestation, energy consumption, and environmental resource management (Hui-Chuan & Tsung-Lung, 2020). Incorporating real-world data and interdisciplinary approaches in mathematics instruction helps bridge the gap between theoretical knowledge and practical applications. For instance, teachers can design lessons that involve analyzing carbon emission trends through statistical methods, applying optimization techniques for resource allocation, and modeling climate scenarios using differential equations. These strategies not only strengthen students' mathematical competencies but also cultivate a deep-rooted sustainability consciousness (Barwell, 2013; Bögeholz et al., 2014; Vásquez et al., 2021).

Despite the potential of mathematics education to promote sustainability awareness, limited research has explored how pre-service mathematics teachers perceive and integrate SDGs into their teaching practices, particularly in Türkiye. This study aims to address this knowledge gap by investigating how gender, academic level, and environmental club membership influence sustainability awareness among pre-service mathematics teachers. The findings will contribute to the development of strategic improvements in teacher training programs to enhance sustainability education in Türkiye and other similar countries.

MATERIALS AND METHODS

This section outlines the research model, participants, and methodology used to assess pre-service mathematics teachers' awareness of the SDGs at Erciyes University, Türkiye, during the 2023/2024 academic year.

Participants

The study involved a total of 154 pre-service mathematics teachers from the Department of Mathematics and Science Education at Erciyes University. The sample consisted of 132 female and 22 male participants at all four academic levels of Freshman, Sophomore, Junior and Senior. The distribution of participants by academic year is presented in Table 1.

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Class	Number
Freshman	14
Sophomore	47
Junior	56
Senior	37
Total	154

Table 1. Distribution of pre-service teachers by academic year

Study Objectives and Research Questions

This study aimed to:

1. Identify the primary sources of information on sustainable development among pre-service mathematics teachers.
2. Assess participants' awareness of ESD.
3. Examine variations in sustainability awareness based on gender, academic level, and environmental club membership.
4. Analyze participants' ability to match sustainable development concepts with development indicators.
5. Gather opinions on the perceived achievement levels of SDGs in Türkiye.

Accordingly, the study was guided by the following research questions:

- How do pre-service mathematics teachers in Türkiye acquire information on sustainable development?
- To what extent does their awareness of ESD align with sustainable development concepts, and how do gender, academic level, and environmental club membership influence their perceptions of SDG achievement?

These research questions seek to explore pre-service teachers' recognition of sustainability within their educational perspectives and how their awareness may shape their future teaching practices.

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Data Collection Procedure

The data collection tool was structured into four sections:

1. Personal Information Form – Collected demographic details, including gender, academic level, and environmental club membership, along with basic knowledge of SD and ESD.
2. Attitude Scale Towards ESD – A 22-item Likert scale measuring participants' perceptions of the individual and societal benefits of ESD and its significance in education.
3. Sustainability and Indicators Scale – A 32-item questionnaire assessing the perceived importance of various sustainability principles in relation to development indicators.
4. SDG Achievement Levels – A 17-item scale measuring participants' opinions on Türkiye's progress in achieving the 17 SDGs, rated on a scale from "Definitely Not Achieved" to "Definitely Achieved

Scales developed in Gökmen, 2014 were used to determine the awareness of pre-service mathematics teachers about sustainable development goals (Gökmen, 2014). Scales developed by Leiva-Brondo were used to learn pre-service mathematics teachers' opinions about the achievement levels of sustainable development goals (Leiva-Brondo et al., 2022). The study did not involve providing teachers with instructional materials on sustainability. Instead, the questionnaire was used solely to assess their pre-existing awareness and understanding of sustainability concepts.

Instrument Reliability

Cronbach's alpha coefficients were calculated for each scale to confirm the reliability and validity of the survey instruments.

- Attitude Scale Towards ESD: $\alpha = 0.927$
- Scale for Matching SD and Development Indicators: $\alpha = 0.953$
- SDG Achievement Level Scale: $\alpha = 0.868$

A Cronbach's alpha coefficient above 0.70 indicates good internal consistency and this confirms the reliability of the instruments (Kalaycı, 2017). Table 2 shows the distribution of data collection tool sections and number of questions.

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Section Number	Section Topic	Multiple Choice Questions	Open Ended Questions	Total
1	Personal information form	5	3	8
2	Attitude scale towards ESD	22	0	22
3	Scale for matching SD and development indicators	32	0	32
4	Achievement level of SDGs	17	0	17

Table 2. Distribution of data collection tool sections and number of questions

Data Analysis

Statistical analyses were conducted using IBM's SPSS software. Descriptive statistics, including frequencies and percentages, were calculated for each survey item to provide a graphical representation of the results. This initial step offered a comprehensive overview of participants' responses regarding their awareness of sustainability and ESD.

To determine the suitability of the data for parametric analysis, normality was assessed using skewness, kurtosis, and the skewness-to-standard error ratio. Following established guidelines, acceptable thresholds were defined as -2 to +2 for skewness and kurtosis, and -1.96 to +1.96 for the skewness-to-standard error ratio (George & Mallery, 2010).

The normality assessment for each scale is summarized as follows:

- **Attitude Scale Towards ESD:** Skewness (-0.13), kurtosis (1.39), and skewness-to-standard error ratio (0.627) were within acceptable limits. Despite a Kolmogorov-Smirnov test p-value of 0.007 ($p < 0.05$), the data were considered sufficiently normal for parametric analysis.
- **Matching Sustainable Development and Indicators Scale:** Skewness (0.10), kurtosis (0.54), and skewness-to-standard error ratio (0.512) indicated normality. Although the Kolmogorov-Smirnov test yielded $p = 0.000$, the data met the criteria for parametric testing.
- **SDG Achievement Level Scale:** Skewness (-0.25), kurtosis (0.36), and skewness-to-standard error ratio (-1.229) confirmed normality. The Kolmogorov-Smirnov test result ($p = 0.200$, $p > 0.05$) supported the use of parametric statistical methods.

For inferential analysis, independent t-tests were applied to examine gender-based differences, while One-Way ANOVA was used to analyze variations across academic levels. Table 3 presents the mean, skewness, and kurtosis values for each scale used in the study.

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Scale	Value	Statistics	Standard Error
Attitude scale towards ESD	Mean	86.93	0.49
	Skewness	-0.13	-0.20
	Kurtosis	1.39	0.40
Scale for matching SD and development indicators	Mean	127.87	1.14
	Skewness	0.10	0.20
	Kurtosis	0.54	0.40
Achievement level of SDGs	Mean	44.36	1.57
	Skewness	-0.25	0.20
	Kurtosis	0.36	0.40

Table 3. Mean, Skewness, and Kurtosis values of the scales

RESULTS AND DISCUSSION

The primary objective of this study was to investigate the sources of information on SD and ESD among pre-service mathematics teachers.

Sources of Information

Main sources through which pre-service mathematics teachers acquire knowledge about sustainability were identified. Responses were quantified, and frequencies were analyzed to determine percentage distributions. The results are presented in Figure 1.

The study achieved a 62% participation rate, which exceeds those reported in similar research (Leiva-Brondo et al., 2022). This higher response rate may be attributed to the administration of the survey during class time, a method known to improve participation rates (Diliman et al., 2009).

Findings indicate that the internet serves as the primary source of information on SD for pre-service mathematics teachers, playing a crucial role in disseminating sustainability-related knowledge. Lectures emerged as the second most significant source, contributing substantially to students' understanding of sustainability concepts. Meanwhile, traditional media—including television and newspapers—had a relatively minor influence in shaping sustainability awareness.

Although traditional educational channels remain relevant, the increasing reliance on digital resources highlights the internet's dominant role in shaping sustainability awareness among future educators.

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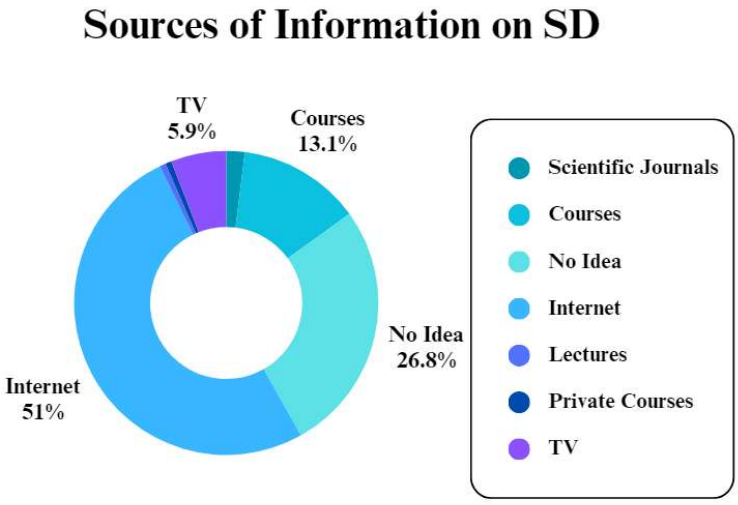


Figure 1. Sources of information on SD for pre-service mathematics teachers

As shown in Figure 1, 73.4% of pre-service mathematics teachers demonstrated awareness of sustainability issues, exceeding the awareness levels reported among Spanish students (28.1%) but slightly trailing behind Japanese students (73.7%) (García-Gonzalez et al., 2020; Ando et al., 2019). In contrast, SDG awareness among the general public remains below 20% (OECD, 2017), highlighting higher sustainability awareness among university students.

Sustainability awareness among mathematics teachers plays a crucial role in education, as mathematics serves as a powerful tool for analyzing real-world sustainability issues. Teachers who integrate sustainability into mathematics instruction can enhance student engagement by embedding real-world problem-solving scenarios into lessons. For instance, mathematics educators can use:

- Probability and statistics to analyze climate change trends,
- Optimization techniques to model resource allocation strategies, and
- Algebraic reasoning to explore financial planning for sustainable homes and workplaces (Silveira, 2018; Su, 2023; Renert, 2011).

Mathematics teachers who are conscious of sustainability can also promote critical thinking by assigning interdisciplinary problem-solving tasks. By incorporating sustainability-related challenges into mathematics education, teachers achieve two key objectives:

1. Developing students' analytical skills through real-world data interpretation.
2. Promoting responsible citizenship by raising environmental awareness and encouraging informed decision-making.

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Given these benefits, sustainability training should be a core component of teacher education to prepare future educators for the multidisciplinary teaching challenges they will encounter (Su, 2023). Integrating sustainability principles into mathematics education will enable students to develop solutions for current and future sustainability challenges. They are expected to understand the interconnections between natural and social systems, and apply critical thinking to real-world problems (Alsina, 2022; Bögeholz et al., 2014; Vásquez et al., 2021).

However, despite the high overall awareness levels, 26.8% of participants reported lacking knowledge of SD. Among respondents who were familiar with SD, the internet emerged as the dominant information source (51%), followed by lectures (13.1%) and television (5.9%), mirroring global trends where digital media serves as the primary channel for SDG-related information (Gangadharbatla et al., 2014).

Figure 2 illustrates pre-service mathematics teachers' knowledge of Education for Sustainable Development (ESD). The findings reveal that 39.7% of respondents identified the internet as their primary ESD information source, while 38.5% reported having no prior ESD knowledge. Table 4 presents moderate ESD understanding, with mean scores of 39.17 (SD = 3.79) for ESD practices and 32.29 (SD = 3.61) for individual and social benefits.

These findings emphasize the need for teacher education programs to further strengthen sustainability-related knowledge and pedagogical practices. While digital media plays a dominant role in shaping sustainability awareness, it also raises concerns about the reliability and depth of online information. This underscores the importance of structured educational interventions to ensure accuracy, critical evaluation, and comprehensive sustainability literacy among future educators.

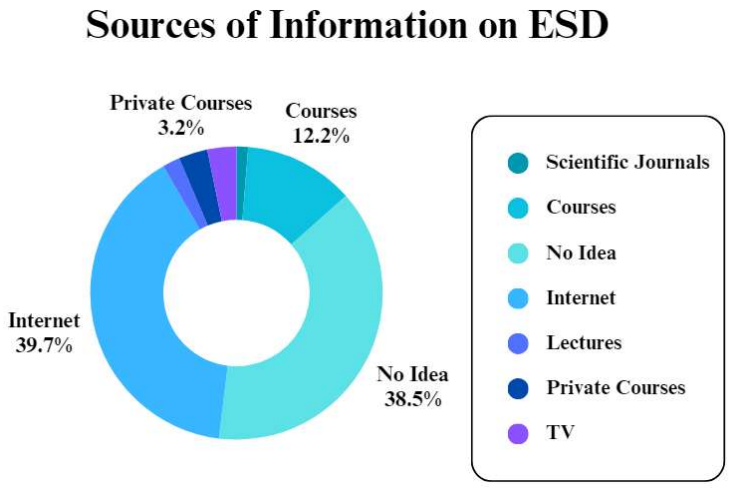


Figure 2. Knowledge of education for SD among pre-service mathematics teachers

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Table 4 shows descriptive statistics of pre-service mathematics teachers' attitudes towards ESD. It provides insights into pre-service mathematics teachers' attitudes toward ESD by categorizing their responses into three key dimensions: practices within schools, individual and social benefits.

A closer examination of the data reveals that pre-service teachers exhibit a solid grasp of both ESD practices in schools and its individual and societal benefits and this suggests a recognition of sustainability's role in education and broader social contexts. However, their responses indicate a weaker understanding of how sustainability goals translate into classroom applications. This shows a gap between theoretical awareness and practical implementation. The relatively lower mean score in the goals and applicability category suggests that while pre-service teachers value sustainability, they may lack the pedagogical tools or training necessary to integrate these concepts into their instructional strategies effectively.

These findings point to the need for targeted professional development initiatives aimed at bridging the disconnect between knowledge and application. Incorporating structured training on sustainability pedagogy within teacher education programs could enhance pre-service teachers' ability to translate their awareness into actionable classroom strategies. Strengthening this aspect of teacher preparation would not only enrich mathematics education but also qualify future educators to cultivate sustainability literacy among their students in a more meaningful and practical way.

Dimension	N	Min	Max	Mean	SD	Related Survey Questions
ESD practices in schools	154	25.00	45.00	39.17	3.79	1,4,7,10,13,16,18,20,22
Individual and social benefits of ESD	154	21.00	40.00	32.29	3.61	2,5,8,11,14,17,19,21
Goals and applicability of ESD	154	11.00	24.00	15.40		3,6,9,12,15
Valid N (listwise)	148					

Table 4. Descriptive statistics of pre-service mathematics teachers' attitudes towards ESD

Goals and Applicability of ESD

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The findings indicate that pre-service mathematics teachers demonstrate a baseline understanding of the goals and applicability of Education for SD, as reflected in a mean score of 15.40 (SD = 2.03). However, their grasp of sustainability extends unevenly across different dimensions, as summarized in Table 5, which shows descriptive statistics of pre-service mathematics teachers on the SD and indicators scale.

While awareness of ecological and social sustainability appears strong, as both dimensions share a mean score of 63.69 (SD = 7.66), knowledge of economic sustainability is notably lower, with a mean score of 15.43 (SD = 2.36). This incompability suggests that although pre-service teachers recognize the environmental and societal aspects of sustainability, they may lack a comprehensive understanding of its economic implications. Examples include resource management, financial planning for sustainability, and the long-term economic effects of environmental policies.

These results underscore the need for teacher education programs to incorporate a more balanced and integrative approach to sustainability. By strengthening economic sustainability education alongside ecological and social dimensions, pre-service teachers can develop a holistic perspective on sustainable development. This enables them to better integrate these principles into their future teaching practices. Addressing this gap would enhance their ability to prepare students for real-world sustainability challenges, ensuring that all three pillars (e.g. environmental, social, and economic sustainability) are given equal emphasis in education.

Survey dimension	N	Min.	Max.	Mean	SD	Related Survey Questions
Ecological Dimension	148	38.00	75.00	63.69	7.66	1,4,7,10,13,15,17,19,21,23,25,27, 29,31,32
Social Dimension	148	38.00	75.00	63.69	7.66	2,5,8,11,14,16,18,20,22,24,26, 28,30
Economical Dimension	148	8.00	20.00	15.43	2.36	3,6,9,12
Valid N (listwise)	148					

Table 5. Descriptive statistics of pre-service mathematics teachers on the SD and indicators scale

Achievement Level of SDGs

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Table 6 shows descriptive statistics of pre-service mathematics teachers' SDGs achievement level scale. The achievement levels of SDGs among pre-service mathematics teachers reveal incompabilities in their understanding of sustainability dimensions. The findings indicate that awareness of ecological sustainability remains limited, with a mean score of 10.73 (SD = 3.40). This suggests a lack of familiarity with ecological targets, including environmental sustainability issues such as climate action, biodiversity conservation, and responsible resource consumption.

In contrast, a higher mean score of 23.53 (SD = 7.44) suggests that the social dimension of sustainability is better understood by the participants. This indicates a stronger grasp of social justice, equity, and community well-being, suggesting that pre-service teachers are more attuned to the societal aspects of sustainable development.

The noticeable disparity between ecological and social dimensions highlights the need for targeted interventions in teacher education programs. While social sustainability awareness is relatively well-developed, enhancing ecological literacy is crucial. This will ensure that future educators can effectively address all aspects of sustainability within their teaching. Strengthening ecological knowledge will better equip pre-service mathematics teachers to integrate sustainability principles holistically, ensuring that students gain a comprehensive understanding of environmental, social, and economic sustainability.

Survey dimension	N	Min.	Max.	Mean	SD	Related survey questions
SDG ecological dimension	145	4.00	20.00	10.73	3.40	12,13,14,15
SDG social dimension	148	1.00	45.00	23.53	7.44	1,2,3,4,5,6,7,11,16
SDG economical dimension	146	3.00	15.00	7.61	2.70	8,9,10
Revitalizing the global partnership for the SDGs	145	1.00	5.00	2.88	0.97	
Valid N (listwise)	145					

Table 6. Statistics of pre-service mathematics teachers' SDGs achievement level scale

Economic Dimension and Overall Perceptions of SDG Achievement

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The findings indicate that pre-service mathematics teachers demonstrate moderate awareness of the economic dimension of SDGs, with an average score of 7.61 (SD = 2.70). However, this awareness remains lower compared to the social and ecological dimensions. This suggests that economic sustainability concepts, such as responsible consumption, financial literacy, and sustainable economic growth, may not be as well-integrated into their understanding of sustainable development.

Notably, the "Global Partnerships" dimension, which emphasizes international collaboration for SDG implementation, received the lowest average score of 2.88 (SD = 0.97). This reflects a significant gap in understanding global cooperation mechanisms. That might underscore the need for targeted educational initiatives to strengthen awareness of the interconnected and globalized nature of sustainability challenges.

As illustrated in Figure 3, perceptions of SDG achievement in Türkiye vary widely among participants, reflecting a mix of optimism and skepticism. The results show that 15.1% of pre-service teachers believe the SDGs have not been achieved, while 33.3% feel they are only marginally achieved. Meanwhile, 26.1% hold a neutral stance, and a smaller proportion—23% and 2.5%—consider the SDGs either achieved or highly achieved, respectively.

These findings show that while pre-service teachers recognize sustainable development goals, skepticism remains regarding Türkiye's progress in achieving them. This highlights the importance of integrating sustainability education within teacher training programs. That will ensure that future educators are equipped not only with knowledge of SDGs but also with the ability to advocate for and contribute to sustainable development initiatives effectively. Strengthening economic and global sustainability literacy within teacher education could foster a more comprehensive understanding of sustainability challenges and empower educators to instill these principles in their future classrooms.

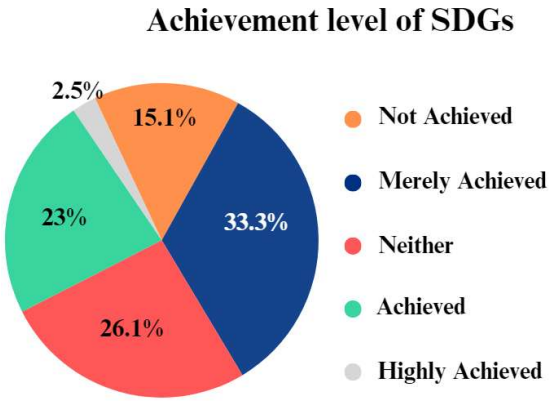


Figure 3. Average percentages of pre-service mathematics teachers' SDG achievement levels

SDG Achievement, Cross-Cultural Variations, and Implications for Math Education

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The analysis revealed that pre-service mathematics teachers most frequently identified SDG 4 (Quality Education), SDG 5 (Gender Equality), and SDG 8 (Decent Work and Economic Growth) as the most achieved goals. In contrast, SDG 1 (No Poverty) and SDG 6 (Clean Water and Sanitation) were perceived as the least achieved. These findings provide insights into how future educators assess sustainability progress and where they see the most pressing challenges.

Türkiye's 2023 Sustainable Development Report indicates a score of 70.8, ranking the country 72nd globally and falling below the regional average of 77.8 (Sachs et al., 2023). National data suggests that SDG 1, SDG 4, and SDG 13 (Climate Action) are among Türkiye's strongest-performing goals, which partially aligns with the perceptions of pre-service teachers in this study.

Comparisons with other countries highlight notable variations in SDG awareness and perceived achievement. Spanish students prioritized SDG 3 (Good Health), SDG 5, and SDG 6, whereas SDG 14 (Life Below Water) and SDG 15 (Life on Land) were ranked the lowest (Leiva-Brondo et al., 2022). In contrast, Japanese students viewed SDG 2 (Zero Hunger) and SDG 6 as the most fulfilled, while SDG 5 was perceived as the least achieved (Ando et al., 2019). The differences in SDG awareness among Turkish, Spanish, and Japanese pre-service teachers may stem from variations in cultural values, national policies, and sustainability education within teacher training programs. Additionally, the unique sustainability challenges faced by each country likely contribute to differing perceptions of SDG progress.

Countries that have explicitly integrated sustainability education into teacher training programs tend to produce pre-service teachers with a more developed understanding of ecological, social, and economic sustainability (Sachs et al., 2023; Biermann, 2022). Conversely, in education systems where sustainability is not an integral part of the curriculum, awareness often depends on personal interest rather than structured instruction. Socioeconomic factors and national sustainability priorities also play a critical role in shaping perceptions of sustainable development. Countries with strong environmental policies tend to foster greater sustainability awareness among teachers, which subsequently influences how sustainability is incorporated into their teaching. These variations underscore the need for internationally aligned and systematically developed frameworks to integrate sustainability into teacher education programs worldwide.

The inclusion of sustainability education in national curricula and teacher training programs is essential for developing students' sustainability competencies (Li, 2021). Cultural attitudes toward sustainability also influence awareness levels. Countries that prioritize environmental policymaking and maintain strong public discourse on sustainability often cultivate higher student engagement with SDGs (Edsand, 2020; Hajdukiewicz, 2020; Berglund, 2020). While Türkiye has made progress in integrating sustainability into education, SDG-focused curricula in teacher training remain underdeveloped. This emphasizes the urgent need for a more systematic and structured incorporation of sustainability content in mathematics teacher education.

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Sustainability awareness among teachers directly enhances the effectiveness of mathematics education, making lessons more interdisciplinary, relevant, and problem-solving oriented. Teachers equipped with sustainability knowledge can integrate real-world data and case studies into mathematics instruction. These teachers can help students understand climate trends, resource efficiency, and economic sustainability through quantitative reasoning (Vásquez et al., 2023). A sustainability-conscious mathematics teacher encourages critical thinking in the classroom, enabling students to engage with real-world exercises related to environmental and social challenges. Through interdisciplinary learning, mathematics can be effectively connected with environmental science and economics, encouraging a systems-thinking approach to problem-solving (Tesfamicael, 2024). Ultimately, students taught by sustainability-aware educators are expected to develop strong ethical and critical thinking skills, equipping them to become active and responsible global citizens.

Gender-Based Analysis

To assess the impact of gender on SDG awareness and perceptions, the data were analyzed using non-parametric statistical methods due to the unequal gender distribution within the sample (132 females, 22 males). Although all three scales exhibited normal distribution, the small male sample size ($n < 30$) necessitated the use of the Mann-Whitney U test to compare gender differences in sustainability awareness and perceptions of SDG achievement.

Gender Effects on Sustainable Development Awareness

Table 7 shows distribution of pre-service mathematics teachers' attitudes towards ESD by gender. The results highlight notable gender-based differences in attitudes toward ESD practices in schools. The findings indicate that female pre-service teachers demonstrated stronger awareness and understanding of ESD integration, with a mean score of 79.88, compared to male participants, who scored significantly lower at 65.18.

This disparity suggests that female students may benefit from different educational experiences, socialization patterns, or a stronger inclination toward environmental responsibility. It is possible that socio-cultural influences, educational exposure, or personal engagement with sustainability-related topics contribute to this variation.

Addressing this gender gap requires targeted strategies within teacher education programs to ensure that male pre-service teachers receive equal exposure to sustainability education. Promoting equitable awareness across genders will enable future educators to integrate sustainability principles more effectively into their teaching. That is expected to result in a more inclusive and comprehensive approach to sustainability education in mathematics classrooms.

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Survey Dimension	Gender	N	Mean
ESD practices in schools	Female	131	79.88
	Male	22	65.18
	Total	153	
Individual and social benefits of ESD	Female	131	76.61
	Male	22	79.34
	Total	**153**	
Goals and applicability of ESD	Female	131	74.38
	Male	22	92.59
	Total	**153**	

Table 7. Distribution of pre-service mathematics teachers' attitudes towards ESD by gender

Gender Differences in ESD Dimensions

The survey assessed pre-service mathematics teachers' perceptions of ESD practices in schools as well as its individual and social benefits, revealing variations in gender-based responses.

The findings indicate that both female and male participants demonstrated comparable recognition of ESD's personal and societal impacts. Female students score an average of 76.61 and male students score slightly higher at 79.34 in the individual and social benefits dimension.

However, a more noticeable difference emerged in perceptions of ESD goals and applicability, where male students scored significantly higher (92.59) compared to female students (74.38). This suggests that male participants held a more positive outlook on the applicability of ESD in educational contexts.

Table 8 summarizes dimensions of the attitude scale toward ESD. Despite these observed differences, Table 8 shows that p-values for all dimensions exceeded 0.05. This indicates that the gender-based variations in attitudes toward ESD were not statistically significant. These findings suggest that while subtle differences exist in perceptions of ESD dimensions, they do not reflect significant disparities in overall attitudes toward sustainability education. Further investigation is needed to explore the underlying factors contributing to these variations. Possible factors could include potential differences in exposure, engagement with sustainability topics, or broader socio-cultural influences.

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Test	ESD practices in schools	Individual and social benefits of ESD	Goals and applicability of ESD
Mann-Whitney U	1181.00	1389.50	1098.00
Wilcoxon W	1434.00	10035.50	9744.00
Z	-1.36	-0.27	-1.81
p	0.17	0.79	0.07

Table 8. Dimensions of the attitude scale toward ESD

3.6.3 Matching Sustainable Development and Development Indicators

Table 9 shows distribution of pre-service mathematics teachers' matching of SD and related indicators by gender. The alignment of sustainable development goals with relevant indicators among pre-service mathematics teachers was assessed, with results presented in Table 9. The findings reveal a slight gender-based variation in ecological awareness, as female students scored an average of 74.23, compared to 72.53 for male students. This suggests that female participants demonstrated marginally stronger ecological awareness.

Understanding the relationship between sustainable development and its indicators is essential for integrating sustainability principles into educational settings. Given the importance of ecological literacy in promoting sustainable practices, it is crucial that all pre-service teachers, regardless of gender, receive equitable exposure to sustainability education. The observed gender disparity, although minor, may be attributed to differences in educational experiences, engagement with sustainability topics, or societal influences.

These findings emphasize the need for comprehensive and inclusive training in teacher education programs. As a result, all future educators can develop a well-rounded understanding of ecological, social, and economic sustainability. Strengthening this knowledge base will better equip teachers to integrate sustainability concepts into mathematics instruction, ultimately achieving greater environmental consciousness among students.

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Survey dimension	Gender	N	Mean
Ecological dimension	Female	127	74.23
	Male	20	72.53
	Total	**147**	
Social dimension	Female	127	74.23
	Male	20	72.53
	Total	**147**	
Economical dimension	Female	127	74.61
	Male	20	70.10
	Total	**147**	

Table 9. Pre-service mathematics teachers' matching of SD and indicators by gender

Gender Differences in Social and Economic Dimensions

The analysis of pre-service mathematics teachers' understanding of sustainable development indicators revealed minimal gender-based variations across the social and economic dimensions. Table 10 shows pre-service mathematics teachers' sub-dimensions in the SD and related indicators scale. As shown in Table 10, female participants scored slightly higher than their male counterparts in both dimensions, though the differences were not statistically significant ($p > 0.05$).

In the social dimension, female students achieved an average score of 74.23, compared to 72.53 for males, indicating strong awareness among both genders. Similarly, in the economic dimension, female participants scored 74.61, while male students had a slightly lower mean of 70.10, reflecting moderate awareness overall.

These findings suggest that pre-service teachers, regardless of gender, demonstrate a comparable ability to link sustainable development goals to their corresponding dimensions. While females exhibited marginally higher scores, the absence of statistically significant differences implies that both groups possess similar levels of competency in understanding sustainability concepts.

Further research is needed to explore potential factors influencing these slight variations, such as differences in educational experiences, exposure to sustainability topics, or personal engagement with environmental and economic issues. To achieve equitable awareness across all dimensions, teacher education programs should continue integrating sustainability into training curricula. That will equip future educators with the necessary knowledge and pedagogical strategies to advocate for and effectively incorporate sustainability into their classrooms.

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Test	Ecological Dimension	Social Dimension	Economical Dimension
Mann-Whitney U	1240.50	1240.50	1192.00
Wilcoxon W	1450.50	1450.50	1402.00
Z	-0.17	-0.17	-0.45
p	0.87	0.87	0.65

Table 10. Pre-service mathematics teachers' sub-dimensions in the SD and indicators scale

Gender Differences in Perceptions of SDG Achievement

The analysis of pre-service mathematics teachers' perceptions of SDG achievement revealed gender-based differences in how sustainability progress is assessed. Table 11 shows distribution of pre-service mathematics teachers' views on SDG achievement by gender. As summarized in Table 11, female participants demonstrated more optimistic views regarding social and economic SDG achievements compared to their male counterparts.

In the social dimension, female pre-service teachers recorded an average score of 75.18, whereas male students scored 66.05, indicating that females perceive greater progress in social sustainability areas, such as gender equality, social justice, and community well-being. A similar trend was observed in the economic dimension, where female respondents scored 73.45, compared to 69.83 for males. This suggests a more positive outlook on economic sustainability advancements.

These findings suggest that female pre-service teachers exhibit greater optimism or heightened awareness regarding social and economic sustainability achievements. The reasons for these differences could stem from varying educational experiences, exposure to sustainability-related topics, or broader socio-cultural influences shaping gendered perspectives on SDGs.

To ensure balanced perspectives on SDG implementation, teacher training programs should incorporate targeted educational strategies that address these gender-based disparities. Providing equitable learning opportunities and fostering critical engagement with sustainability issues will better prepare all future educators to advocate for sustainability in their teaching practices, ensuring a comprehensive and inclusive approach to sustainability education.

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Survey dimension	Gender	N	Mean
SDG social dimension	Female	128	75.18
	Male	19	66.05
	Total	**147**	
SDG economical dimension	Female	127	73.45
	Male	18	69.83
	Total	**145**	
SDG ecological dimension	Female	126	74.19
	Male	18	60.69
	Total	**144**	
Revitalizing the global partnership for the SDGs	Female	126	73.79
	Male	18	63.44
	Total	**144**	

Table 11. Pre-service mathematics teachers' views on SDG achievement by gender

Gender Differences in Ecological Dimension and Global Partnerships

The findings reveal gender-based differences in pre-service mathematics teachers' perceptions of SDG achievements, particularly in the ecological dimension and global partnerships. Table 12 summarizes sub-dimensions of pre-service mathematics teachers' SDG achievement level. As shown in Table 12, female participants demonstrated more positive perceptions in both areas compared to their male counterparts.

In the ecological dimension, female pre-service teachers reported a mean score of 74.19, significantly higher than 60.69 for males, suggesting that female participants perceive greater progress in environmental sustainability efforts. Similarly, in the global partnerships dimension, female respondents scored 73.79, compared to 63.44 for males, indicating greater optimism about international collaboration in achieving SDGs.

Despite these observed gender-based variations, statistical analysis revealed no significant differences ($p > 0.05$), suggesting a general consensus among participants regarding SDG achievement. While disparities exist in perceptions of specific dimensions, they do not indicate fundamental differences in overall sustainability awareness.

These findings emphasize the need to provide pre-service teachers with more extensive training and resources related to ecological sustainability and global partnerships. A comprehensive

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trainings in these dimensions will equip them with the necessary knowledge and instrctional strategies to effectively integrate sustainability into mathematics education. By immersing pre-service teachers in these areas, teacher training programs can better support SDG implementation in Türkiye and beyond. This will result in a generation of educators who actively contribute to global sustainability efforts.

Test	Social dimension	Economical dimension	Ecological dimension	Revitalizing the global partnership for the SDGs
Mann-Whitney U	1065.00	1086.00	921.50	971.00
Wilcoxon W	1255.00	1257.00	1092.50	1142.00
Z	-0.873	-0.344	-1.29	-1.04
p	0.383	0.731	0.197	0.30

Table 12. Sub-dimensions of pre-service mathematics teachers' SDG achievement level

The analysis revealed no significant gender differences in sustainable development awareness, aligning with findings from previous studies (Leiva-Brondo et al., 2022). However, these results contrast with research suggesting higher awareness among males in certain contexts (Gough et al., 2020) or greater awareness among females in specific aspects of sustainability (Costello et al., 2009).

These discrepancies may stem from a range of factors, including knowledge exposure, personality traits, and political ideology, reflecting the complexity of environmental and sustainability awareness. As such, gender alone is insufficient to fully explain variations in SDG awareness. Future research should explore the broader influences shaping sustainability perceptions, incorporating demographic, psychological, and contextual factors. A deeper understanding of these variables will help develop more effective strategies for promoting sustainability education among pre-service teachers. As a result, a more comprehensive and inclusive approach to sustainability awareness in teacher training programs will be achieved.

Effects of Environmental Club Membership on Sustainability Awareness

The study examined the impact of environmental club membership on pre-service mathematics teachers' awareness and attitudes toward ESD. Among the participants, 20 individuals were members of environmental clubs, including 14 affiliated with TEMA (The Turkish Foundation for

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Combating Soil Erosion, for Reforestation, and the Protection of Natural Habitats) and 6 involved in other organizations, while the majority (132 participants) were non-members.

Table 13 shows distribution of pre-service mathematics teachers' survey responses by environmental club membership. As shown in Table 13, club membership was associated with higher awareness and more positive attitudes toward ESD, particularly among female participants. Female club members scored an average of 89.90, indicating strong engagement with sustainability concepts, whereas male club members scored lower, with an average of 74.47.

These findings suggest that involvement in environmental organizations enhances sustainability awareness, with female participants showing greater engagement compared to their male counterparts. Participation in such extracurricular activities provides valuable opportunities for pre-service teachers to deepen their understanding of sustainability, apply theoretical knowledge in real-world contexts, and develop a stronger commitment to environmental responsibility.

Encouraging greater participation in environmental clubs may serve as an effective strategy for strengthening sustainability education in teacher training programs. By integrating extracurricular sustainability initiatives with formal education, pre-service teachers can enhance their competencies in sustainability-related teaching practices, ultimately contributing to more effective integration of ESD in mathematics education.

Survey dimension	Member	N	Mean
Education for sustainable development attitude scale towards	Yes	20	89.90
	No	132	74.47
	Total	**152**	
Development with sustainable development matching indicators scale	Yes	20	90.75
	No	127	71.36
	Total	**147**	
Sustainable development goals achievement level	Yes	20	60.88
	No	128	76.63
	Total	**148**	

Table 13. Pre-service mathematics teachers' responses by environmental club membership

Environmental Club Membership and Sustainable Development Awareness

Table 14 shows environmental club membership of the study participants. The analysis examined the relationship between environmental club membership and sustainable development awareness,

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with results summarized in Table 14. The findings indicate differences in sustainability perceptions and understanding based on club membership, though these variations were not statistically significant due to the small sample size.

Regarding the alignment between SDGs and their indicators, female club members scored an average of 90.75, suggesting a strong understanding of the connections between sustainable development goals and their corresponding measures. In contrast, male club members scored lower, with an average of 71.36, indicating less familiarity with SDG-related indicators.

However, in terms of perceived SDG achievement, male participants demonstrated a more favorable outlook, scoring 76.63, compared to 60.88 among female club members. This suggests that male members of environmental clubs may perceive greater progress toward sustainability goals than their female counterparts.

Due to the small sample size of club members ($n < 30$), statistical analysis using the non-parametric Mann-Whitney U test did not reveal statistically significant differences ($p > 0.05$) across the scales. While environmental club membership appears to be associated with higher awareness and positive attitudes toward sustainability, the lack of statistical significance indicates that further research with larger sample sizes is needed to confirm these trends.

These findings underscore the potential value of extracurricular environmental engagement in fostering sustainability awareness among pre-service teachers. While club membership correlates with greater sustainability literacy, additional efforts are needed to ensure that all pre-service teachers—regardless of formal extracurricular involvement—develop a strong foundation in sustainable development principles.

Test	Education for sustainable development attitude scale	Development with sustainable development matching indicators scale	Sustainable development goals achievement level
Mann-Whitney U	1052.00	935.00	1007.50
Wilcoxon W	9830.00	9063.00	1217.50
Z	-1.47	-1.9	-1.53
p	0.14	0.06	0.126

Table 14. Environmental club membership

Although membership in environmental clubs is often linked to higher environmental literacy, the findings of this study revealed no significant differences in sustainable development awareness

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between members and non-members. This suggests that engagement with sustainability knowledge is not solely dependent on formal participation in environmental organizations.

One possible explanation for this lack of distinction is Generation Z's heavy reliance on digital platforms as primary sources of information. The widespread accessibility of online resources and social media allows young individuals to stay informed and actively engage with sustainability topics outside formal educational or extracurricular settings (Houman et al., 2023). Digital platforms provide immediate access to a vast array of environmental content, fostering sustainability awareness even among those not affiliated with formal environmental groups.

These findings underscore the importance of integrating digital resources into educational programs to align with the evolving information landscape. Leveraging online platforms, interactive learning tools, and digital media can enhance sustainability education, ensuring that pre-service mathematics teachers and other young learners develop comprehensive environmental literacy. By embracing the potential of digital engagement, teacher education programs can expand access to sustainability knowledge, equipping future educators with the tools to promote sustainable practices both inside and outside the classroom.

Academic-Year Effects on Sustainable Development Awareness

Table 15 shows distribution of pre-service mathematics teachers' attitudes towards ESD by academic years. The analysis of pre-service mathematics teachers' attitudes toward ESD by academic year, summarized in Table 15, reveals variations in sustainability awareness across different levels of study. These differences are likely influenced by variations in educational experiences, curricular exposure, and engagement with sustainability concepts over time.

Understanding these patterns is particularly valuable for designing targeted interventions within mathematics teacher education programs. By identifying how sustainability awareness develops over the course of teacher training, educators and policymakers can refine curriculum design, instructional strategies, and extracurricular opportunities. This will enable all pre-service teachers to acquire a comprehensive understanding of sustainable development.

Further curricular analysis, pedagogical improvements, and the integration of sustainability-focused activities can strengthen future teachers' ability to incorporate sustainability principles into their own classrooms. Developing a structured approach to embedding sustainability across different academic levels will better equip pre-service mathematics teachers to effectively promote sustainable development in education.

Survey dimension	Class	N	Mean

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ESD practices in school	Freshman	14	76.50
	Sophomore	47	79.46
	Junior	56	74.63
	Senior	37	79.73
	Total	**154**	
Individual and social benefits of ESD	Freshman	14	92.07
	Sophomore	47	73.22
	Junior	56	77.14
	Senior	37	77.96
	Total	**154**	
Goals and applica-bility of ESD	Freshman	14	66.86
	Sophomore	47	78.57
	Junior	56	76.63
	Senior	37	78.45
	Total	**154**	
Attitudes towards ESD	Freshman	14	84.25
	Sophomore	47	77.05
	Junior	56	75.57
	Senior	37	78.43
	Total	**154**	

Table 15. Distribution of pre-service math teachers' attitudes towards ESD by academic years

Academic-Year Differences in Attitudes Toward ESD

Table 16 presents the dimensions of the attitude scale toward ESD, summarizing the results of the Kruskal-Wallis test conducted to analyze variations in pre-service mathematics teachers' attitudes toward ESD across academic years. Given the limited number of freshman participants ($n < 30$), non-parametric methods were used for statistical analysis.

The findings indicate no statistically significant differences ($p > 0.05$) in attitudes toward ESD across academic years, suggesting a consistent level of sustainability awareness among pre-service teachers, regardless of their academic standing. This uniformity may reflect a standardized implementation of sustainability-related curriculum and experiences throughout the teacher training program.

Despite the absence of significant differences, further research is needed to examine curriculum content, instructional approaches, and extracurricular opportunities to assess their effectiveness in

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enhancing sustainability education. A deeper investigation into how sustainability is integrated at different academic levels can help identify areas for improvement. This will ensure that pre-service teachers develop strong sustainability competencies by the time they enter the profession.

Although no substantial variations across academic years were observed, the findings emphasize the need for continuous efforts to strengthen sustainability education. Enhancing curriculum depth, incorporating innovative teaching strategies, and expanding sustainability-focused extracurricular activities will further reinforce pre-service teachers' ability to integrate sustainable development concepts into their future classrooms.

Test	ESD practices in school	Individual and social benefits of ESD	Goals and applicability of ESD	Attitudes towards ESD
Chi-Square	0.43	1.97	0.91	0.45
df	3	3	3	3
p	0.94	0.58	0.82	0.93

Table 16. Dimensions of the attitude scale toward ESD

Academic-Year Effects on Matching Sustainable Development and Development Indicators

Table 17 shows distribution of pre-service mathematics teachers' matching of SD and related indicators by academic year. This analysis assesses whether academic progression influences the understanding of how sustainable development aligns with specific indicators. That will provide insights into the effectiveness of the curriculum in fostering sustainability literacy.

The findings serve as a valuable tool for evaluating the impact of curriculum design, instructional methods, and learning experiences on pre-service teachers' preparedness to integrate sustainability into their teaching. Identifying trends or inconsistencies across academic levels can help educators refine pedagogical approaches and adapt course content to ensure a more comprehensive and progressive development of sustainability competencies throughout teacher training programs.

By analyzing how sustainability awareness evolves over time, teacher education programs can implement targeted improvements that strengthen pre-service teachers' ability to apply sustainability principles in mathematics instruction. A structured approach to reinforcing the connection between sustainable development and its indicators will better equip future educators to incorporate sustainability concepts into their teaching practices.

Survey dimension	Class	N	Mean

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Ecological dimension	Freshman	14	76.21
	Sophomore	44	77.44
	Junior	54	74.80
	Senior	36	69.79
	Total	**148**	
Social dimension	Freshman	14	76.21
	Sophomore	44	77.44
	Junior	54	74.80
	Senior	36	69.79
	Total	**148**	
Economical dimension	Freshman	14	73.89
	Sophomore	44	74.82
	Junior	54	75.61
	Senior	36	72.68
	Total	**148**	
Development with Sustainable Development Matching Indicators Scale	Freshman	14	81.18
	Sophomore	44	73.11
	Junior	54	74.65
	Senior	36	73.38
	Total	**148**	

Table 17. Pre-service mathematics teachers' matching of SD and indicators by academic year

Academic-Year Differences in Matching Sustainable Development and Indicators

Table 18 shows dimensions of the SD and indicators matching scale, outlining the test values for pre-service mathematics teachers' understanding of the alignment between SD concepts and related indicators. The analysis found no statistically significant differences in scores across academic years, indicating a consistent level of understanding among participants, regardless of their academic progression.

This lack of variation suggests that sustainability awareness remains stable throughout teacher training, potentially reflecting a uniform curriculum structure or consistent exposure to sustainability-related topics. While this consistency may be beneficial in ensuring foundational sustainability knowledge, it also underscores the need for continuous curriculum assessment and enhancement to ensure progressive learning and deeper engagement with sustainability issues.

Future research should explore factors contributing to this uniformity, including teaching methods, content delivery strategies, and the availability of experiential learning opportunities. Investigating

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how practical engagement with sustainability concepts (e.g.project-based learning, case studies, and interdisciplinary approaches) affects student understanding could provide insights into ways to enhance sustainability education within teacher training programs.

Strengthening curriculum design to incorporate more dynamic, applied learning experiences will help pre-service mathematics teachers develop a more integrated and practical understanding of sustainability. That will make it possible that teachers are well-prepared to embed these concepts effectively in their future teaching practices.

Test	Ecological dimension	Social dimension	Economical dimension	Scale for matching SD and development indicators
Chi-Square	0.67	0.67	0.11	0.41
df	3	3	3	3
p	0.88	0.88	0.99	0.94

Table 18. Dimensions of the SD and indicators matching scale

Academic-Year Effects on Perceptions of SDG Achievement

Table 19 presents the distribution of pre-service mathematics teachers' perceptions of SDG achievement across academic years, aiming to assess whether perceptions of sustainability progress vary throughout teacher training. This analysis provides valuable insights into how awareness and understanding of sustainable development evolve as students advance in their academic journey.

Examining these differences is crucial for evaluating the effectiveness of the curriculum in shaping students' sustainability perspectives. If significant variations exist across academic years, this may indicate that exposure to sustainability concepts increases with academic progression. Conversely, a lack of differences could suggest a consistent, but potentially static, approach to sustainability education throughout the program.

The findings from this analysis will contribute to a better understanding of the relationship between academic progression and sustainability perceptions. This will offer guidance on how teaching strategies and curriculum design can be refined to enhance sustainability literacy. Identifying potential gaps or areas for improvement can help ensure that pre-service mathematics teachers develop a well-rounded, informed perspective on SDG achievements, equipping them to integrate sustainability more effectively into their future teaching practices.

Survey dimension	Class	N	Mean

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Ecological dimension	Freshman	14	57.29
	Sophomore	44	79.55
	Junior	55	65.71
	Senior	35	88.86
	Total	**148**	
Social dimension	Freshman	14	63.11
	Sophomore	44	75.08
	Junior	54	67.74
	Senior	34	84.88
	Total	**146**	
Economical dimension	Freshman	14	58.68
	Sophomore	44	72.88
	Junior	53	67.97
	Senior	34	86.90
	Total	**145**	
Development with Sustainable Development Matching Indicators Scale	Freshman	14	63.57
	Sophomore	44	76.93
	Junior	53	70.02
	Senior	34	76.44
	Total	**145**	
Level of achievement of Sustainable Development Goals	Freshman	14	58.57
	Sophomore	44	79.01
	Junior	55	65.96
	Senior	35	88.61
	Total	**148**	

Table 19. Pre-service mathematics teachers' SDG achievement opinions by academic year

Academic Progression and Perceptions of SDG Achievement

The analysis of pre-service mathematics teachers' perceptions of SDG achievement reveals that seniors demonstrate the highest awareness across the social, economic, and ecological dimensions. As shown in Table 19, SDG awareness scores increase progressively from the freshman to senior years, suggesting a positive correlation between academic progression and sustainability awareness.

For instance, scores in the SDG Social Dimension rise from 57.29 among freshmen to 88.86 among seniors, while the SDG Economic Dimension follows a similar pattern, increasing from 63.11 to

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84.88. This trend indicates that continued exposure to coursework, sustainability discussions, and teacher training experiences contributes to enhanced awareness and understanding of sustainable development over time.

Conversely, the lower scores observed among freshmen may reflect insufficient sustainability education during the early years of teacher training. This highlights the need for earlier integration of sustainability concepts within the curriculum to confirm that students develop a strong foundation in sustainable development from the beginning of their academic journey.

These findings emphasize the importance of progressive curriculum design that systematically reinforces sustainability education across all academic levels. By incorporating sustainability principles earlier in teacher training, education programs can foster deeper engagement with sustainability issues, equipping future mathematics teachers with the knowledge and skills to effectively integrate SDG concepts into their classrooms.

Academic-Year Analysis of SDG Achievement

Table 20 presents the SD dimensions and indicators matching scale. The data reveal significant disparities in SDG attainment perceptions among pre-service mathematics teachers, particularly by academic year. The results indicate that academic progression plays a key role in shaping students' views on sustainability achievements.

Statistical analysis revealed a p-value of 0.028 for the SDG Social Dimension and 0.038 for the overall scale, both of which fall below the 0.05 significance level. This finding suggests that academic year significantly influences how pre-service mathematics teachers perceive SDG achievement, with students at different academic levels demonstrating varying assessments of sustainability progress.

These differences may stem from variations in exposure to sustainability-related coursework, the depth of engagement with SDG topics, and the integration of sustainability discussions in teacher education programs. As students advance through their academic years, their increased exposure to sustainability education, research, and classroom discussions likely enhances their understanding and shapes their perceptions of SDG progress.

These findings highlight the importance of structuring sustainability education progressively across all academic years to ensure that students develop a well-rounded and informed perspective on sustainable development. By integrating comprehensive sustainability learning opportunities throughout teacher training programs, institutions can better prepare pre-service teachers to effectively incorporate SDG principles into their future teaching practices.

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Test	SDG social dimension	SDG eco-nomical di-mension	SDG ecological dimension	Revitalizing the global partnership for the SDGs	Level of achievement of SDGs
Chi-Square	9.13	4.43	6.17	1.75	8.40
df	3	3	3	3	3
p	0.028	0.218	0.104	0.625	0.038

Table 20. SD dimensions and indicators matching scale

The analysis revealed that senior pre-service mathematics teachers demonstrated higher awareness of the SDGs compared to students in earlier academic years. This finding is consistent with studies conducted in Spain (Leiva-Brondo et al., 2022) and Italy (Smaniotto et al., 2020), which similarly reported greater sustainability awareness among senior students. The increased awareness observed in senior students may be attributed to cumulative academic experiences, deeper coursework, and greater exposure to sustainability topics throughout their training.

However, some researchers, such as Kirby et al. (2021), caution against viewing academic progression or age alone as a reliable predictor of sustainability knowledge. Other factors, including the quality of educational content, engagement in sustainability-related activities, and personal motivation, likely play significant roles in shaping students' awareness and understanding of the SDGs.

These findings highlight the need for teacher education programs to adopt a structured, progressive approach to sustainability education, ensuring that students at all academic levels receive meaningful engagement with sustainability concepts. By enhancing course content, integrating sustainability-focused activities, and fostering student motivation, institutions can cultivate a more comprehensive and well-rounded understanding of sustainable development among future mathematics educators.

Implications for Teacher Education and Sustainability Integration

Studies have demonstrated that incorporating sustainability into teacher training significantly enhances educators' ability to integrate environmental topics into their teaching (Suralin, 2023; Karim et al., 2021). Given the findings of this study and the critical role of sustainability awareness in mathematics education, universities offering teacher training programs should consider the following strategic recommendations:

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- **Integrating real-world sustainability challenges into mathematics instruction:** Teacher education programs should incorporate sustainability-focused mathematical applications such as climate change modeling, statistical analysis of sustainability data, and resource management optimization. These topics can be embedded within mathematical reasoning and pedagogy courses, ensuring that future educators are equipped to connect sustainability concepts with core mathematics curricula.
- **Mandating sustainability training in teacher education programs:** Universities should consider requiring formal sustainability training for pre-service teachers. This could be achieved through dedicated ESD courses, interdisciplinary workshops, and sustainability-oriented pedagogical seminars. Each institution can develop its own tailored approach to effectively integrate sustainability into teacher training.
- **Establishing a national framework for sustainability education in Türkiye:** A structured, national model for integrating sustainability into mathematics teacher education could help standardize sustainability instruction across universities. A unified framework would ensure that all pre-service mathematics teachers receive consistent and systematic training, enabling them to effectively incorporate sustainability concepts into their future classrooms.

CONCLUSION

This study examined pre-service mathematics teachers' awareness of SDGs and ESD, focusing on how gender, academic level, and environmental club membership influence sustainability perceptions. While the findings indicate a general awareness of sustainability, they also reveal an uneven distribution of knowledge across different dimensions.

As a response to RQ1, the primary sources of information about sustainable development for pre-service math teachers in Türkiye are found to be educational and digital platforms. According to the study, more than half of the participants cited the internet as their primary source of SDG information, with university lectures coming in second. It was found that their awareness was only slightly shaped by traditional media, like newspapers and television. A similar trend was noted in relation to ESD, where 39.7% of respondents cited the internet as their primary source. There is a notable lack of structured exposure through formal education. On the other hand, a sizable percentage (38.5%) stated that they had no prior knowledge of ESD. These results demonstrate how pre-service teachers depend on internet resources for sustainability knowledge.

As a response to RQ2, pre-service math teachers in Türkiye showed uneven alignment with the three pillars of sustainable development, despite having a basic understanding of ESD. Social sustainability (e.g., quality education, gender equality) appears to be well understood, whereas ecological (e.g., climate action, clean water) and economic sustainability (e.g., responsible consumption, economic growth) receive less emphasis. This imbalance highlights the need for teacher education programs to strengthen instruction on environmental and economic sustainability to ensure a more comprehensive understanding.

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Results showed gender differences were minimal. While male participants scored slightly higher on perceptions of ESD applicability, female participants tended to be more optimistic about social and economic sustainability accomplishments and more aware of ESD practices in schools. Even though overall gender differences were not statistically significant, It was found that male and female awareness levels are generally comparable.

From freshman to senior years, there was a progressive increase in academic awareness of the SDGs and perceptions of their accomplishment. Seniors showed the greatest awareness of social, ecological, and economic aspects. This suggests that more academic exposure enhances understanding of sustainability. This trend demonstrated how exposure to academic materials and discussions shapes awareness and was statistically significant for both the SDG social dimension and overall achievement perceptions.

Members of environmental clubs, especially female participants, reported greater awareness and more positive attitudes toward sustainability than non-members. However, these differences were not statistically significant because of the small sample size. Nevertheless, the results imply that extracurricular activities can improve sustainability consciousness and strengthen links between ESD and sustainable development principles.

Recommendations for Strengthening Sustainability Education

Based on the findings, the following recommendations are proposed to enhance the integration of sustainability within mathematics teacher education:

- **Early Integration of Sustainability Concepts:** Sustainability education should be embedded in the early years of teacher training rather than being introduced in later academic years.
- **Real-Life Applications in Mathematics Instruction:** Mathematics courses should include real-world data analysis, case studies, and problem-solving exercises that connect mathematical concepts with sustainability challenges.
- **Development of National Frameworks:** Governments should establish clear national policies for sustainability education in teacher training programs to ensure a standardized and systematic approach across universities.
- **Enhancing Digital Literacy:** Teacher training should equip future educators with critical digital literacy skills to help them evaluate and apply sustainability-related information from electronic sources in an informed manner.

In conclusion, while pre-service mathematics teachers in Türkiye exhibit moderate awareness of SDGs, their knowledge remains fragmented and is strongly influenced by their academic level. A structured and multidimensional approach to sustainability education is essential to bridging these gaps. Strengthening sustainability training within teacher education will empower future educators to integrate mathematics and sustainability effectively. A well trained teachers are expected to

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prepare students to engage with global challenges through analytical and sustainability-oriented thinking.

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Enhancing Teachers' Performance and Students' Problem-Solving Abilities through Constructivist-Based Lesson Study: Developing a Study Lesson on the Sum of Consecutive Natural Numbers

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Abstract: This qualitative research article examines the role of Lesson Study, a collaborative professional development method based on constructivist learning theory, in enhancing teachers' instructional practices and students' problem-solving abilities. Despite efforts by educational policymakers to promote modern teaching methods in mathematics, many teachers still rely on traditional approaches that prioritize rote memorization over conceptual understanding. This study focuses on a Lesson Study group consisting of four mathematics teachers and twelve seventh-grade students from an international school in Malaysia. The group designed and implemented a Study Lesson centered on the summation of consecutive natural numbers from one to n. Through data collected via observations and interviews, the study demonstrates that Lesson Study not only improves teachers' instructional methods but also enhances students' problem-solving skills within the context of the study lesson, utilizing constructivist learning theory.

Keywords: Lesson Study, Study Lesson, Mathematics problem

INTRODUCTION

Mathematics holds a prominent position in the curriculum, with students needing to acquire appropriate mathematical knowledge for practical application in their daily lives. However, the current state of mathematics education necessitates a paradigm shift due to the prevailing perception among students at various educational levels that mathematics is a challenging subject (Fritz et al., 2019; Li & Schoenfeld, 2019). In order to promote students' mathematical thinking abilities, educators must underscore the humanistic nature of mathematics education, wherein students actively engage in group discussions to explore problem-solving ideas rather than passively acquiring static knowledge and skills (Li & Schoenfeld, 2019). By shifting the focus towards mathematical inquiry and collaborative learning, teachers can foster a deeper understanding and appreciation of mathematics among students. Active engagement through independent speculation,

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exploration, and analysis of problems fosters meaningful mathematical learning, empowering students to discover alternative problem-solving methods and enhancing their understanding of mathematical concepts (Lin et al., 2017). Furthermore, promoting effective group dynamics entails ensuring that all group members possess the ability to explain and discuss their collaborative work.

In mathematics education, a problem is distinguished from an exercise by its inherent challenge, demanding higher-order thinking skills, whereas exercises entail routine practice and application of established procedures (Schoenfeld, 1985). Traditional mathematics instruction predominantly centers on solving exercises, resulting in students memorizing formulas, theorems, shortcuts, axioms, and methods without cultivating a profound comprehension of the fundamental concepts (Gholami et al., 2019; Khalid, 2017; Mon et al., 2016). In the realm of mathematics education, researchers grapple with the inquiry of how students can attain a deep understanding of mathematical concepts and appreciate their inherent beauty without actively engaging in problem-solving activities. It appears that educators' involvement in innovative professional development approaches, such as Lesson Study, can enhance their instructional effectiveness, consequently improving students' problem-solving skills and critical thinking abilities.

This study explores the impact of lesson study on teachers' ability to create educational content about the sum of natural numbers, ranging from one to any given natural number and related problems. Through collaborative group work, teachers can refine each other's teaching methods while fostering professional growth. The primary outcome of this process is the significant enhancement of students' problem-solving skills. This dual benefit underscores the value of lesson study in improving both teaching quality and student learning outcomes.

LITERATURE REVIEW

The creation and sequencing of mathematical content are crucial in lesson preparation, as they shape the structure and coherence of instruction, promoting a logical progression that enhances student understanding and engagement. Aligning content with pedagogical goals enables teachers to deliver clearer and more effective lessons, fostering meaningful learning experiences. This principle was exemplified in a study by Fernandez et al. (2003), where sixteen mathematics teachers from New Jersey collaboratively planned a sixth-grade math lesson. Initially following the textbook sequence by teaching rectangle area before triangle area, the group reconsidered their approach after a Japanese educator suggested starting with right triangles before scalene triangles, based on the conceptual link between triangles and rectangles. Although some American teachers struggled with this structured knowledge due to limited content familiarity, the cross-cultural collaboration ultimately enriched their pedagogical strategies, underscoring the value of diverse perspectives in lesson design.

The collaborative planning of lessons involves determining the topic and content order, which relies on the collective knowledge and experience of the teachers involved. Their expertise shapes

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the decision-making process for effective instructional design. According to Marjanovic (2017), a group of mathematics teachers proposed an instructional approach during a discussion meeting. Their suggestion involved utilizing Figure 1 as a central component of the lesson to demonstrate the proof of the mathematical relationship $1+3+5+\ldots+(2n-1)=n^2$. Subsequently, the teachers identified the proof of the relationship $1+2+3+\ldots+n=(n^2+n)\div 2$ as a problem to be addressed within the lesson, utilizing the aforementioned relationship as a means of solving it.

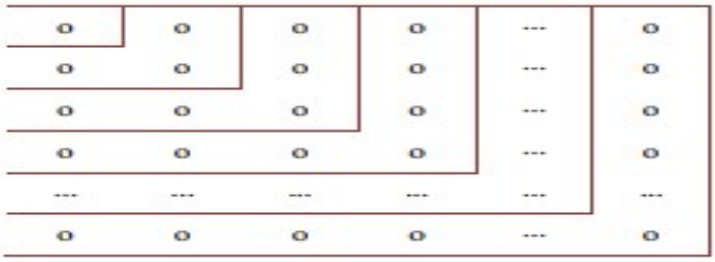


Figure 1: Summation of Odd Natural Numbers from 1 to $(2n\text{-}1)$ (Marjanovic, 2017)

Figure 2 illustrates the solution method proposed by mathematics teachers.

$$1+3+5+\cdots+(2n-1)=n^2$$
$$\Rightarrow (2\times 1-1)+(2\times 2-1)+(2\times 3-1)+\cdots+(2n-1)=n^2$$
$$\Rightarrow 2(1+2+3+\cdots+n)-n=n^2$$
$$\Rightarrow 1+2+3+\cdots+n=(n^2+n)\div 2$$

Figure 2: Summation of Natural Numbers from 1 to n (Marjanovic, 2017)

In fact, Marjanovic (2017) initially established a relationship for the sum of odd natural numbers (see Figure 1) and subsequently derived the formula for the sum of natural numbers from 1 to n based on this relationship (see Figure 2). In contrast, the present study takes the sum of natural numbers as the foundational concept and explores numerous novel problems by applying this relation. This approach facilitates students' understanding of fundamental mathematical tools and enhances their ability to apply these tools effectively in solving related problems.

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Lesson Study

Lesson Study has been employed by Japanese teachers as a method for professional development in mathematics education since the 1950s (Abiko, 2011). The Japanese term "Jugyo Kenkyu" was translated into English as "Lesson Study" by Yoshida in 1999 (Doig & Groves, 2011). Lesson Study has become increasingly popular worldwide, with widespread adoption among mathematics teachers in diverse educational settings. However, Lesson Study is known by different names in other countries, such as "Learning Study" in Hong Kong (Ling et al., 2005), "Teacher Research Group" in China (Yang, 2009), and "Learning Communities" in Singapore (Chua, 2009). Muarata and Takahashi (2002) found that 91.3% of Japanese mathematics teachers believe Lesson Study is effective in enhancing problem-solving and higher-order thinking skills. The Singapore Ministry of Education's school cluster system, based on geographical location, facilitates collaborative Lesson Study among mathematics educators, allowing them to harness the contextual benefits of this approach (Chua, 2009).

In recent decades, Lesson Study has emerged as a notable educational approach, attracting attention globally. This collaborative method involves a team of educators who jointly concentrate on a specific mathematics topic, investing considerable time in planning, delivering, and discussing a lesson. Lesson Study aims to optimize students' mathematical learning by focusing on the development of problem-solving and critical thinking skills through the refinement of teaching practices (Matanluk et al., 2013). During the Lesson Study process, educators develop and implement lessons known as Research Lessons (Fujii, 2014) or Study Lessons (Yoshida, 1999), which serve as integral components of the pedagogical cycle in teaching (Lewis, 2002; McDonald, 2009). McDonald (2009) emphasizes the significance of the Study Lesson in the professional development of mathematics teachers and highlights its key role in facilitating their growth and improvement. Lesson Study is an educational methodology that empowers educators to teach mathematical concepts through a problem-solving approach, fostering deep learning and enhancing students' critical thinking skills, aligning with cognitivist learning theory and applicable across multiple subjects and educational levels (Cheng & Yee, 2011).

In Japan's educational system, Lesson Study is categorized into four main types: School-based, District-based, National-level, and Association-sponsored. Each type follows a similar cycle but differs in terms of the intended audience for the Study Lessons (Fujii, 2016; Takahashi & McDougal, 2016). School-based Lesson Study focuses on improving teaching practices within a specific school, targeting learners within that school. District-based Lesson Study targets learners within a district, often involving teachers from multiple schools in the area. National-level Lesson Study aims to benefit learners across the entire country, typically conducted in national lab schools or through subject matter organizations. Association-sponsored Lesson Study is supported by national subject matter associations, allowing teachers to focus on specific subjects or areas of interest. These types of Lesson Study work together to enhance instructional quality and curriculum development across different levels of the educational system. Among these, School-based Lesson Study is considered the most robust as it allows mathematics educators to tailor

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Study Lessons with appropriate materials and open-ended problems that cater to the specific abilities of students in their school, resulting in more effective mathematics learning outcomes.

Alamri (2022) conducted a study in Saudi Arabia where mathematics teachers explored the Lesson Study approach and reported positive outcomes in teaching mathematics. Teachers gained autonomy in developing their teaching schedules and objectives, improved content knowledge and pedagogical content knowledge through collaborative work, and enhanced students' problem-solving abilities and critical thinking skills through rich Study Lessons. Lesson Study showed promising effects on teacher development and student learning outcomes. According to the results of a Lesson Study program for middle school mathematics teachers in the context of Turkey, Lesson Study not only encourages mathematics teachers to conduct reflection activities effectively but also it encourages them for self-evaluation and improving their teaching by increasing their awareness of their own teaching practices using rich Study Lessons (Bozkurt & Yetkin-Ozdemir, 2018).

The Stages of Lesson Study

Fujii (2014) proposed a framework consisting of five distinct phases for the successful implementation of the Lesson Study program, which are outlined as follows: Firstly, during the Goal Setting phase, teachers establish long-term objectives designed to create learning environments that promote learners' problem-solving abilities and higher-order thinking skills. Secondly, during the Lesson Planning phase, teachers collaborate to design a Study Lesson, using mathematics materials that are tailored to meet the specific needs and abilities of the learners. Next, during the Study Lesson phase, a designated member of the Lesson Study group teaches the prepared lesson, while other members observe the teaching process and gather relevant data to improve the quality of the Study Lesson. After that, during the Post-lesson Discussion phase, teachers engage in discussions about students' problem-solving abilities, observed misconceptions, disciplinary mathematical content, various solution methods for the given problems, and suggestions for enhancing the Study Lesson. Lastly, during the Reflection phase, teachers introduce new mathematical problems to be addressed in the next cycle of Lesson Study. They also engage in discussions to solve these problems and compile a comprehensive report that documents the findings and outcomes of the Study Lesson. Figure 3 illustrates the Lesson Study approach cycle, as informed by Fujii's work in 2014.

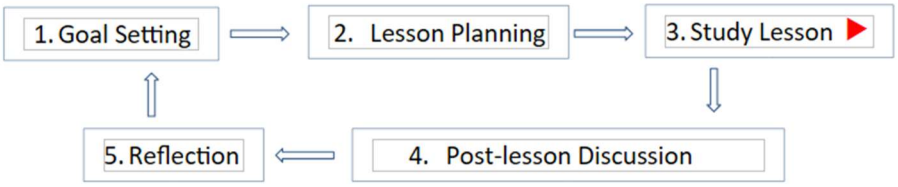


Figure 3: Phases of the Lesson Study Program (Fujii, 2014)

The theoretical framework, based on constructivist theory and Lesson Study, highlights the active role of learners in teaching the sum of natural numbers from 1 to n. This method posits that

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students actively build mathematical understanding by engaging with their educational environment (Hoover, 1996). Lesson Study offers a collaborative approach for teachers to create engaging lessons that promote meaningful mathematical discovery and effective problem-solving (Fujii, 2016). The constructivist approach highlights the crucial role of problem-solving activities, discussions, and reflection in facilitating students' active construction of knowledge (Gergen, 1995). Lesson Study enables teachers to craft a vibrant learning environment that encourages student engagement, promotes critical thinking, and cultivates a profound understanding of the concept of summing natural numbers from 1 to n and associated problems. The iterative nature of Lesson Study facilitates continuous refinement and adaptation of teaching strategies, aligning them with students' evolving responses and needs, while reinforcing the constructivist approach by emphasizing student-centered learning and active knowledge construction (Alamri, 2020; Bozkurt & Yetkin-Ozdemir, 2018). Through this cycle, educators can tailor their methods to better support students grasping complex concepts, such as summing natural numbers from 1 to n.

RESEARCH QUESTION

This article seeks to familiarize middle school mathematics teachers with the Lesson Study approach, aiming to refine instructional strategies and boost student achievement. In particular, it highlights the development of a Study Lesson focused on the concept of summing natural numbers from one to n. Consequently, the research question for this study is formulated as follows:

Research Question: How does the implementation of Lesson Study influence teachers' instructional skills and students' learning outcomes in the context of teaching the summation of natural numbers?

METHOD

This qualitative case study was conducted at one of the international schools of Kuala Lumpur, a prominent non-profit institution known for its learner-centered approach, providing an ideal environment for implementing innovative teaching practices like Lesson Study. The participants consisted of four mathematics teachers, namely A, B, C, and D, three of whom were male and one female. These teachers as Lesson Study group members possessed a minimum of 12 years of experience in teaching mathematics to middle school students. Additionally, twelve students from an existing Grade 7 class participated in the study. These students were divided into six groups, each consisting of two members.

Before participating, all mathematics teachers and student participants provided informed consent by signing a consent letter. The researcher convened a meeting with the teacher participants to introduce the general topic of the Study Lesson for the Lesson Study group. Following this, the teachers collaborated to develop a Study Lesson focused on the concept of summing natural numbers from one to any natural number. They refined the lesson to address various related problems. The selection of the topic and the sequence of problems within the Study Lesson were determined through collaborative decision-making, drawing on the collective expertise and experiences of the Lesson Study group members. In the textbook, this concept is presented within the

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chapter on introducing various strategies for solving mathematical problems. Consequently, this chapter provides teachers with opportunities to engage students in diverse problem-solving activities, thereby enhancing their critical thinking and problem-solving skills.

During the initial meeting, members of the Lesson Study group collaboratively planned, discussed, and designed the Study Lesson. In the subsequent meeting, one teacher delivered the lesson in a regular classroom while the remaining group members observed the instructional process. In the final meeting, the mathematics teachers conducted a post-lesson discussion to reflect on their observations and collaboratively refine the Study Lesson. Figure 4 illustrates the process of preparing the Study Lesson.

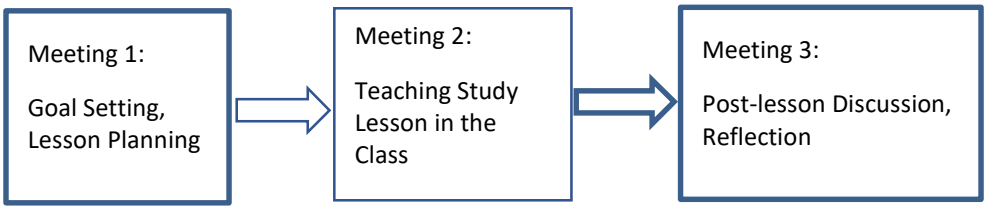


Figure 4: The Process of Preparing the Study Lesson (Author)

For example, during the initial discussion meeting, one of the teachers proposed including the following problem in the Study Lesson as a homework assignment. The group members reached a consensus that this particular problem would be suitable for enhancing students' critical thinking skills due to its level of difficulty.

Problem: Determine the value of the sum $3+7+11+\ldots+199$.

The teachers presented two potential solution methods for this problem, which are illustrated in Figure 5. In solution method 1, the teachers employed a technique where they divided each term of the series into pairs of consecutive natural numbers. By combining these pairs, they effectively converted the problem into finding the sum of the natural numbers ranging from 1 to 100. Utilizing the relevant formula for such sums, they determined that the sum of the natural numbers from 1 to 100 equals 5050, which represents the sum of the given series. In solution method 2, the Lesson Study group broke down each term of the series into pairs of consecutive natural numbers. They split the series into odd and even sequences. By applying the formulas for the sum of consecutive odd and even natural numbers, they calculated the sum of odd numbers from 1 to 99 as 2500 and the sum of even numbers from 2 to 100 as 2550. Combining these sums, they obtained the total sum of the given series as 5050. Both solution methods share a common approach of identifying the pattern within the series and transforming it into a more manageable form. These methods exhibit creativity and effectively connect to the concepts discussed in the Study Lesson.

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Solution method 1:	**Solution method 2:**
$3+7+11+\ldots+199$ $=(1+2)+(3+4)+\ldots+(99+100)$ $=1+2+3+\ldots+100=\frac{100(100+1)}{2}=5050.$	$3+7+11+\ldots+199$ $=(1+2)+(3+4)+\ldots+(99+100)$ $=(1+3+\ldots+99)+(2+4+\ldots+100)$ $=50\times 50+50\times 51=5050.$

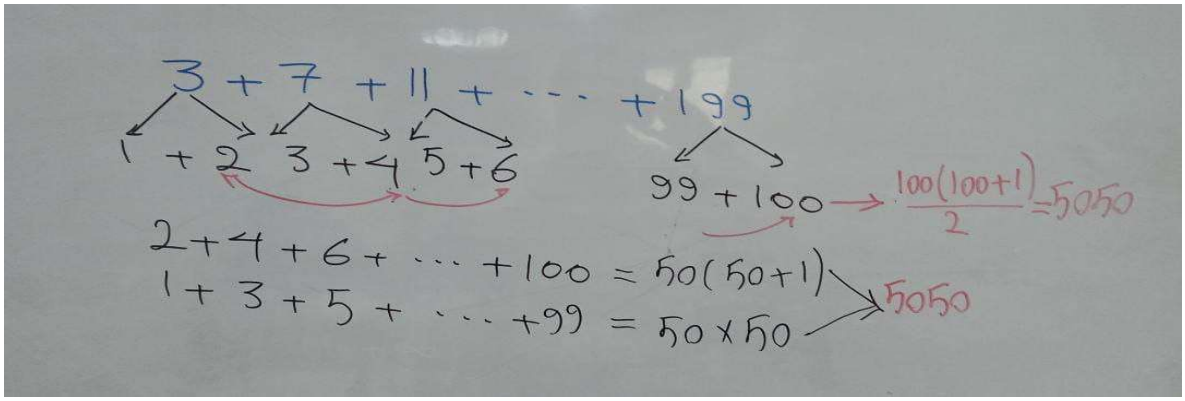


Figure 5: Two Solution Methods for the Given Problem

This study collected data from observations of the discussion meetings, the teaching process in the classroom, and interviews with teachers and students. The collected data were subjected to a descriptive analysis, which involved a detailed examination of the observations, discussions, and interview responses. This approach allowed for a comprehensive understanding of the Lesson Study process and its impact on teaching and learning outcomes.

RESULTS

The primary outcome of this study entails the development of a comprehensive Study Lesson aimed at seventh-grade students, focusing on the concept of the sum of consecutive natural numbers. The Study Lesson incorporates various materials, beginning with a problem presented to the Lesson Study group, namely "find the value of $A=1+2+3+\ldots+100$." The intention behind this problem is to engage students in critical thinking, prompting them to explore the core aspects of the lesson. Initial attempts by the students to solve the problem proved challenging, leading the teacher to share the story of Carl Friedrich Gauss, a renowned German mathematician from the 18th century. Gauss's remarkable solution to determining the sum of consecutive natural numbers

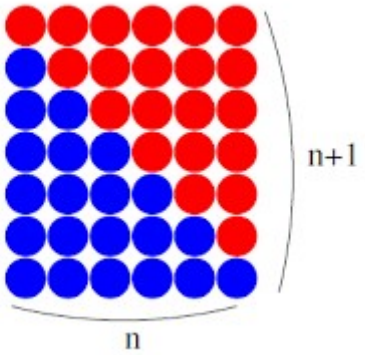
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from 1 to 100 served as inspiration for the students. Gauss cleverly organized the numbers into fifty pairs, summing the first and last numbers, the second and second-to-last numbers, and so forth. Each pair exhibited a sum of 101, resulting in a total sum of the natural numbers from 1 to 100 equating to 50×101, which simplifies to 5050.

Furthermore, the teacher introduced an additional creative strategy to compute the sum of the natural numbers from 1 to 100, aiming to foster students' creativity. This alternative method involved juxtaposing the original list of numbers with its reverse order. By adding the corresponding terms of the two series, a novel series emerged, consisting of 100 terms each equaling 101. Recognizing that the original list was duplicated, the total sum of 100 multiplied by 101 was divided by 2, yielding the sum of 5050.

The teacher provided a comprehensive explanation regarding the calculation of the sum for the series $1+2+3+\ldots+n$, considering two cases based on the parity of n. In the case where n is even, the teacher demonstrated the pairing of numbers, such that each term from 1 to n is paired with its corresponding counterpart, resulting in $\frac{n}{2}$ pairs. Each pair sums to $n+1$, thereby yielding a total sum of $\frac{n(n+1)}{2}$ for the series. Conversely, when n is odd, the teacher illustrated the pairing of numbers where each term from 1 to n-1 is paired with its counterpart, generating $\frac{n-1}{2}$ pairs with a sum of n for each pair. Consequently, the total sum of the series in this case is given by $\frac{n(n-1)}{2}+n=\frac{n(n+1)}{2}$. The classroom discussion surrounding these two cases facilitated a generalization of the concept, leading to the algebraic relation $1+2+3+\ldots+n=\frac{n(n+1)}{n}$.

The teacher emphasized the significance of geometrical shapes in enhancing students' reasoning abilities. To facilitate their understanding of the above relation, the teacher employed Figure 6 as a visual aid, encouraging students to explore and discover this relation using the shape presented. During the lesson, the students examined Figure 6 and engaged in a discussion centered around the relation $(1+2+3+\ldots+n)+(1+2+3+\ldots+n)=n(n+1)$. Through their analysis, the students made an important discovery: they realized that the sum of the natural numbers from 1 to n, denoted as $1+2+3+\ldots+n$, is equivalent to $\frac{n(n+1)}{2}$. This result was derived from their observation and reasoning based on the given figure.



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Figure 6: A proof without word for the sum of natural numbers from 1 to n (**Source:** https://www.quora.com/How-do-I-find-the-sum-of-all-natural-numbers-from-1-to-n)

Each student demonstrated their proficiency in utilizing the formula for calculating the sum of natural numbers by successfully determining the sum of $1+2+3+\ldots+200$. Through the application of the formula,$1+2+3+\ldots+200=(200\times 201)\div 2=20100$, they efficiently computed the sum, showcasing their understanding and competence in solving such series. This exemplifies their ability to find the sum of a series of natural numbers from 1 to any given arbitrary number using the formula.

During this segment of the Study Lesson, the Lesson Study group engaged in extensive discussions and rearrangements to present a series of mathematical problems aimed at enhancing students' problem-solving abilities. These problems were strategically organized in increasing complexity, introducing novel techniques for problem resolution to the students.

> *Problem 1:* Determine the sum of even natural numbers ranging from 2 to 300, represented as $2+4+6+\ldots+300$.
>
> *Problem 2:* Find the sum of odd natural numbers from 1 to 499, represented as $1+3+5+\ldots+499$.
>
> *Problem 3:* Calculate the sum of natural numbers from 201 to 600, represented as $201+202+203+\ldots+600$.
>
> *Problem 4:* Determine the sum of even natural numbers from 302 to 800, represented as $302+304+\ldots+800$.
>
> *Problem 5:* Find the sum of odd natural numbers from 101 to 599, represented as $101+103+105+\ldots+599$.

The Lesson Study group recognized the challenging nature of the presented problems and sought to foster students' problem-solving skills by dividing them into groups of two. This collaborative approach aimed to enhance their ability to tackle these problems effectively. The inclusion of the aforementioned problems within the Study Lesson was strategically designed to introduce new concepts and techniques to students, while also reinforcing the application of the formula for finding the sum of natural numbers from 1 to n in solving them.

By providing an appropriate framework for students to approach and solve each problem, their problem-solving capabilities and creativity were further developed. During class session, the teacher presented the problems to the students sequentially, allowing the six groups to engage in solving them. This active participation and collaborative effort among students fostered a dynamic learning environment that encouraged critical thinking and problem-solving skills.

Each student in the class successfully demonstrated their problem-solving skills by solving problem 1. Four groups of students employed solution method 1, as depicted in Figure 7, while the remaining two groups utilized solution method 2. This diversity in approaches emphasizes the

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students' proficiency in logical reasoning and their ability to adapt different strategies to arrive at the correct solution. It showcases their flexibility in thinking and problem-solving, contributing to a dynamic and enriching learning environment.

Solution 1:	Solution 2:
$2+4+6+\ldots+300$ $=2(1+2+3+\ldots+150)$ $=2\left(\frac{150\times 151}{2}\right)=150\times 151$ $=22650.$	$2+4+6+\ldots+300$ $=(1+1)+(2+2)+\ldots+(150+150)$ $=(1+2+\ldots+150)+(1+2+\ldots+150)$ $=\frac{150\times 151}{2}+\frac{150\times 151}{2}=22650.$

Figure 7: Comparative Analysis of Solution Methods for Problem 1

Solution 1 demonstrates a methodical approach by recognizing patterns, simplifying the series, and leveraging existing mathematical formula ($1+2+3+\ldots+n=\frac{n(n+1)}{2}$). In fact, this method showcases creativity by employing techniques that reduce complexity and facilitate more straightforward calculations. Also, Solution 2 showcases an alternative approach to the problem by pairing each term with itself and simplifying the expression into identical sums. It cleverly utilizes the relation $1+2+3+\ldots+n=\frac{n(n+1)}{2}$ to calculate the sum of all given natural numbers. This approach demonstrates creative thinking and adaptability in finding different perspectives to solve the same problem.

During the discussion, the teacher introduced the relation $2+4+6+\ldots+2n=n(n+1)$ with the help of Figure 8 to efficiently find the sum of even natural numbers from 2 to 2n. This pedagogical approach incorporates visual aid to enhance understanding, demonstrating the application of mathematical concepts and logical reasoning in deriving a concise formula for the sum. Such teaching methods encourage critical thinking, exploration of different techniques, and foster a deeper understanding of mathematical principles among students.

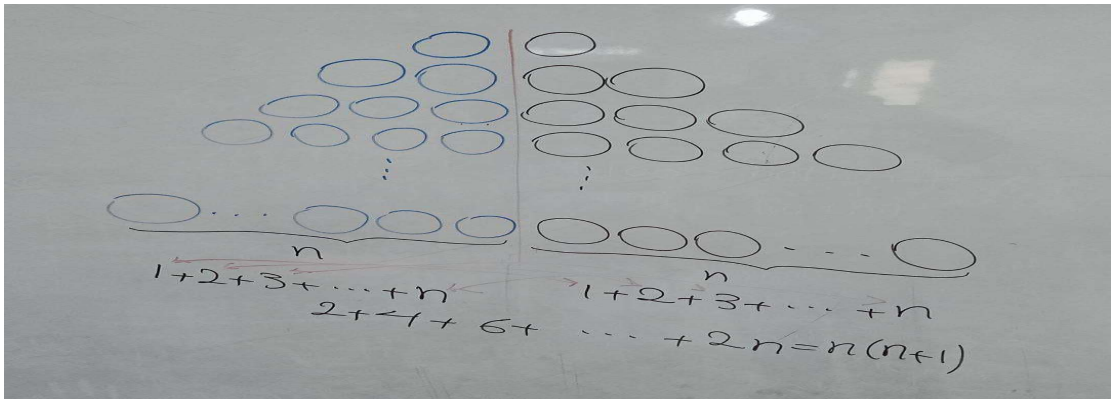


Figure 8: An Approach for Summing Even Natural Numbers from 2 to $2n$

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Problem 2 served as an effective task to enhance critical thinking skills among the students. Since the task involved finding the sum of odd numbers from 1 to 499, it required students to employ logical reasoning and devise appropriate methods to arrive at a solution. Notably, only three groups managed to successfully solve the problem, as depicted in Figure 9. Two groups proposed Solution 1, while the remaining group presented Solution 2 as their approach to solving the problem. These findings highlight the students' ability to think critically and creatively when confronted with mathematical challenges.

Solution 1:	Solution 2:
$1+3+5+\ldots+49$ $=(1+2+\ldots+500)-(2+4+\ldots+500)$ $=(1+2+\ldots+500)-2(1+2+\ldots+250)$ $=\frac{500\times 501}{2}-2\times\frac{250\times 251}{2}$ $=250\times 501-250\times 251=62500.$	$1+3=4=2\times 2$ $1+3+5=9=3\times 3$ $1+3+5+7=16=4\times 4$ . . . $1+3+5+\ldots+499=250\times 250=62500.$

Figure 9: Comparative Analysis of Solution Methods for Problem 2

Solution 1 demonstrates the students' creative approach in finding the sum of odd natural numbers from 1 to 499 indirectly. Rather than directly calculating the sum of the odd numbers, they devised a method by calculating the sum of all natural numbers from 1 to 500 and the sum of all even natural numbers from 2 to 500. By subtracting the sum of even numbers from the sum of all numbers, they were able to determine the sum of odd natural numbers from 1 to 499. This inventive solution showcases the students' ability to think critically and devise alternative methods to solve complex problems. Solution 2 displays the students' ingenuity by recognizing the pattern of consecutive odd numbers and their relationship to perfect squares. They creatively observed that the sum of the first n odd natural numbers is equal to n squared, enabling them to determine the sum of odd numbers from 1 to 499. This method demonstrates their analytical thinking and ability to identify mathematical patterns to solve problems efficiently.

The teacher facilitated the calculation of the sum of odd natural numbers from 1 to an arbitrary odd number by utilizing Figure 10. This geometrical shape provided students with a straightforward method for determining the sum. By examining specific examples, such as $1+3+5+7=4\times 4$ and $1+3+5+7+9=5\times 5$, students were able to discern a general pattern. Consequently, they derived a relation for calculating the sum of all odd natural numbers from 1 to $2n$-1 as $n\times n$. This approach showcases the teacher's effective use of visual aids and the students' ability to identify patterns and generalize them into a mathematical relation.

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Figure 10: The sum of odd natural numbers
(**Source:** https://www.studysmarter.co.uk/explanations/math/pure-maths/sum-of-natural-numbers/)

Problem 3 solved by four groups of students, who demonstrated their creativity through the development of two distinct approaches. Figure 11 showcases these solutions, with Solution 1 being successfully implemented by three groups, while Solution 2 was devised and executed by a single group.

Solution 1:	Solution 2:
$201+202+203+\ldots+600$ $=(1+2+\ldots+600)-(1+2+\ldots+200)$ $=\frac{600\times 601}{2}-\frac{200\times 201}{2}$ $=300\times 601-100\times 201$ $=180300-20100=160200.$	$201+202+203+\ldots+600$ $=(200+1)+(200+2)+\ldots+(200+400)$ $=400\times 200+(1+2+3+\ldots+400)$ $=80000+\frac{400\times 401}{2}$ $=80000+80200=160200.$

Figure 11: Comparative Analysis of Solution Methods for Problem 3

In Solution 1, the students cleverly approached the problem of finding the sum of numbers from 201 to 600. They utilized the concept of subtracting the sum of numbers from 1 to 200 from the sum of numbers from 1 to 600. By applying the formula for the sum of consecutive natural numbers from 1 to n, they simplified the expression and obtained the final result of 160200. This solution demonstrates the students' ability to decompose complex problems and apply mathematical principles to efficiently find the solution.

In Solution 2, the students subtracted 200 from each number in the given series and applied the formula for the sum of natural numbers from 1 to n to find the solution. By grouping the terms as $(200+1)+(200+2)+\ldots+(200+400)$, they simplified the expression and calculated the sum of the modified series. Utilizing the relation for the sum of natural numbers, they efficiently obtained the result of 160200. This approach demonstrates their strategic thinking in manipulating the series and utilizing mathematical concepts to arrive at the solution.

Four groups of students demonstrated their success in solving Problem 4 by employing the solu-

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tion method depicted in Figure 12. In this solution method, the students employed the sum formula for even natural numbers from 2 to $2n$. By subtracting the sum of even numbers from 2 to 300 from the sum of even numbers from 2 to 800, they were able to find the sum of the given series. This approach showcases the students' ability to utilize mathematical formulas and techniques to efficiently solve the problem.

Solution method:

$$302+304+306+\ldots+800=(2+4+6+\ldots+800)-(2+4+6+\ldots+300)$$
$$=400(400+1)-150(150+1)=160400-22650=137750.$$

Figure 12: A solution method for problem 4

Two groups of students exhibited their proficiency by successfully presenting a comprehensive solution to Problem 5 within the designated time-frame, as illustrated in Figure 13. By subtracting the sum of odd numbers from 1 to 99 from the sum of odd numbers from 1 to 599, they effectively eliminated the terms up to 99 and obtained the desired sum. Utilizing the property of the sum of consecutive odd numbers, the students calculated the result as 87500. This approach demonstrates their ability to decompose the problem, apply mathematical properties, and devise an efficient solution strategy.

Solution method:

$$101+103+105+\ldots+599=(1+3+5+\ldots+599)-(1+3+5+\ldots+99)$$
$$=300\times 300-50\times 50=87500.$$

Figure 13: A solution method for problem 5

The Lesson Study group members recognized the significance of engaging and captivating students in their learning process. Consequently, teachers incorporated the following enjoyable puzzles related to the topic in the Study Lesson to foster student participation and enthusiasm in problem-solving activities.

> *Puzzle 1:* Given a set of 50 bags, each containing 100 coins, all but one bag contains genuine gold coins, while a single bag holds counterfeit coins. The counterfeit coins weigh 3 grams each, whereas the genuine gold coins weigh 4 grams each. Your task is to identify the bag containing the counterfeit coins, with the constraint that you can use a digital scale for measurement only once.

The teacher commenced by presenting the puzzle to the students and elucidating its solution through the utilization of concepts introduced during the session. The teacher proceeded to resolve the puzzle in the following manner: Firstly, the teacher instructed the students, "I will assign a label to each bag from 1 to 50 and extract a number of coins from each bag corresponding to its label." In other words, one coin was extracted from bag number 1, two coins from bag number 2, three coins from bag number 3, and so forth, until a total of 50 coins were obtained from bag number 50. The cumulative count of coins extracted from these 50 bags amounted to $1+2+3+\ldots+50=\frac{50\times 51}{2}=1275$. If all the coins were genuine, their combined weight would be 5100 grams. However, since it was known that one bag contained counterfeit coins, the disparity be-

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tween 5100 grams and the actual weight of the coins determined the bag harboring the counterfeits. For instance, if the actual weight of the coins measured 5090 grams, then the fake coins resided in bag number 10 (5100-5090=10).

The teacher presented the students with Puzzle 2, challenging them to employ the concepts covered in the current session to find the solution. The puzzle is as follows:

Puzzle 2: The natural numbers are grouped into categories as follows:

$$(1), (2,3), (4, 5, 6), \ldots$$

In the first category, there is only one number. In the second and third categories, there are two and three numbers respectively, and so on. The task is to determine the sum of all the numbers in the twentieth category.

The teacher provided an explanation to the students on how to approach and solve this puzzle. It was emphasized that finding the first number of the twentieth category would suffice. Consequently, the number of natural numbers classified in categories one to nineteen was determined to be $1+2+3+\ldots+19=\frac{19\times20}{2}=190$. Hence, the first number in the twentieth category was found to be 191. To calculate the sum of all numbers in this category, the approach involved subtracting the sum of numbers from 1 to 190 from the sum of numbers from 1 to 210. Thus, the sum was calculated as $(1+2+3+\ldots+210)-(1+2+3+\ldots+190)=4010$.

The students displayed great enthusiasm in response to these logical explanations. Through engaging in discussions surrounding these puzzles, they had the opportunity to witness the real-world application of the properties taught in the Study Lesson multiple times. As a result, they were able to solidify their conceptual understanding of the topic, benefiting from the incorporation of enjoyable materials and activities throughout the learning process.

Research studies have indicated that students who engage in homework assignments containing suitable mathematical problems exhibit enhanced conceptual understanding compared to those who solely rely on conventional textbook tasks (Jacobse & Harskamp, 2009; Ozcan & Erktin, 2015). In fact, the inclusion of challenging mathematical problems in homework assignments contributes to the holistic process of mathematics learning among students. Thus, the Lesson Study group members deemed it essential to incorporate the following problems as part of the homework for this particular Study Lesson.

Problem 1: Determine the value of the sum $1001+1003+1005+\ldots+2023$.

Problem 2: Calculate the value of the sum $3+7+11+\ldots+199$.

Problem 3: Find the value of the sum $215+219+223+\ldots+611$.

Problem 4: Consider a lively gathering consisting of 50 attendees, where each guest is assigned a card bearing a randomly selected natural number from 1 to 50. The number written on the card corresponds to the quantity of chocolates received

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by the cardholder, with each individual receiving three times the value indicated on their card. What is the total number of chocolates distributed among the guests at this event?

DISCUSSIONS

During the development of this Study Lesson, teachers introduced captivating topics related to the summation of natural numbers from 1 to n, resulting in enhanced classroom engagement and performance. They prioritized conceptual learning through a problem-solving approach, rooted in constructivist learning theory. When designing the problems for this lesson, the teachers intentionally structured them to facilitate valuable insights and discussions for the students with each new problem. While there was initially disagreement among group members regarding the order of the problems in the Study Lesson, they reached a consensus after thorough deliberation, ultimately deciding to present the problems to the students in the prescribed order outlined in this Study Lesson.

Furthermore, the implementation of the prepared Study Lesson in an actual classroom setting contributed to the refinement of teaching methodologies among the Lesson Study group members. Post-discussion meetings allowed the group to identify areas for improvement in the Study Lesson's quality. For instance, during the class, it was observed that some students encountered difficulties in grasping the concept of summing odd natural numbers. To address this, the teacher recommended the utilization of Figure 10 as a visual aid to enhance the understanding of the concept. The class was divided into six groups of two students each, and the teacher dedicated approximately ten minutes to introducing the main topic of the Study Lesson: the method for finding the sum of consecutive natural numbers from 1 to an arbitrary value. Subsequently, the class adopted a student-centered approach, wherein the teacher assigned problems to the students and monitored the progress of the different groups. At times, the teacher provided gentle guidance to assist the students in devising suitable problem-solving strategies. The performance of the students during this session indicated that the prepared Study Lesson was well-suited to their needs, although certain problems may require modification based on the students' proficiency levels in various contexts. Figure 14 illustrates the performance of each group in solving five assigned problems. Specifically, it highlights that only Groups 1 and 5 successfully solved Problem 5 in its entirety. In the Results section, the detailed performance of each group on all five problems is described, indicating which problems were fully solved and which were not. The pie charts in Figure 14 provide a visual summary of these group outcomes, showing the proportion of groups that completely solved each problem versus those that did not. This representation helps to quickly convey the overall success rates across the different problems.

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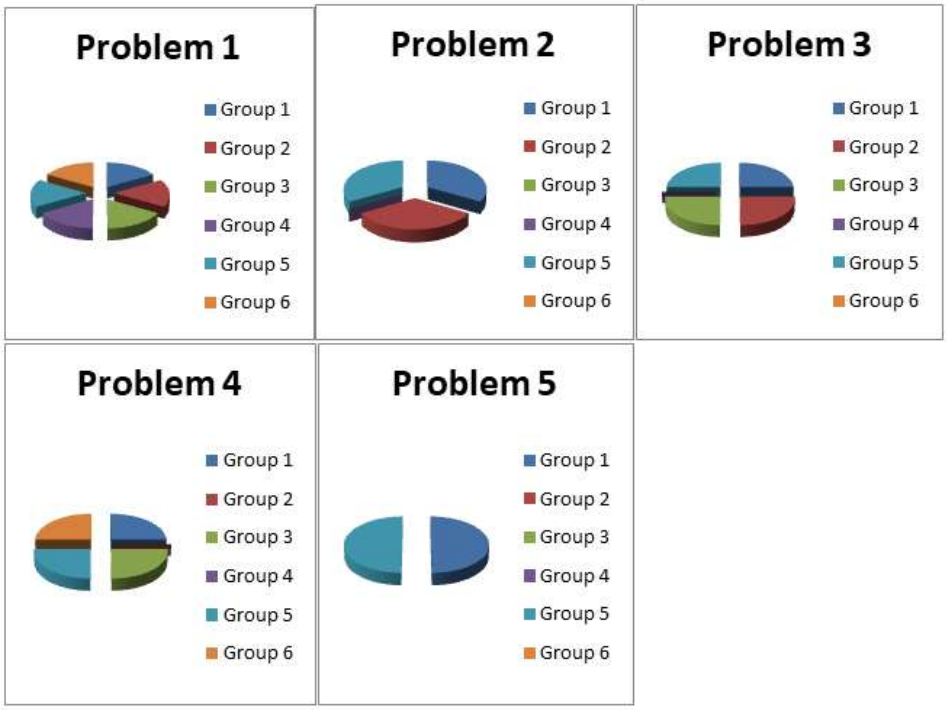


Figure 14: Comparative Analysis of Student Performance in Problem Solving

The Lesson Study program played a significant role in enhancing students' proficiency in mathematics problem-solving. Through the implementation of the problem-solving approach based on the model of Polya (1945), students were able to engage in a systematic process encompassing problem understanding, strategizing, solution implementation, and solution verification. The study lesson adopted a student-centered teaching model, fostering the development of students' problem-solving skills and critical thinking abilities in accordance with the principles of constructivist learning theory. As a result, students experienced considerable growth in their problem-solving aptitude and critical thinking capabilities. A student provided an explanation, stating:

> *During this session, I engaged in contemplation on various mathematical problems, all centered on the summation of series of natural numbers. Each problem necessitated the exploration of innovative solution methods, leading to the acquisition of new and creative techniques through subsequent discussions. Additionally, I found great enjoyment in participating in classroom discussions centered on two intriguing mathematical puzzles.*

In traditional teaching approaches, the classroom environment tends to be teacher-centered, with students dedicating substantial class time to copying and transcribing materials presented by the teacher (Gholami, Yunus, et al., 2021). However, the members of the study group adopted an alternative method that eliminated the need for students to invest time in note-taking. Instead, class

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time was effectively utilized for engaging students in optimal problem-solving activities. A student expressed their perspective, stating:

> *During this session, the teaching method employed eliminated the need for me to engage in the task of copying mathematical notes. Instead, I actively engaged with a diverse range of problems, making optimal use of the class time. This approach proved to be both captivating and enjoyable, allowing me to fully immerse myself in the problem-solving process.*

All members of the Lesson Study group reached a consensus that this instructional approach not only enhanced their pedagogical expertise but also actively engaged students through carefully selected mathematics problems. This engagement fostered the development of students' problem-solving abilities and supported a constructivist learning environment. Within the framework of the Lesson Study program, teachers collaboratively identified appropriate problems and deliberated on the most effective strategies for conveying mathematical concepts. By emphasizing the importance of problem sequencing, they examined a variety of solution methods to stimulate students' creativity. The Study Lesson functioned as a repository of diverse solution strategies and pedagogical techniques, thereby serving as a valuable resource to inform and improve future teaching practices. As an example, Figure 15 showcases three additional solution methods incorporated by the teachers for problem 4 in the Study Lesson.

Solution 1: $302+304+306+\ldots+800=2(151+152+153+\ldots+400)$ $=2[(1+2+3+\ldots+400)-(1+2+3+\ldots+150)]$ $=2\left(\frac{400\times 401}{2}-\frac{150\times 151}{2}\right)$ $=2(80200-11325)=137750.$
Solution 2: $302+304+306+\ldots+800=(300+2)+(300+4)+(300+6)+\ldots+(300+500)$ $=250\times 300+(2+4+6+\ldots+500)$ $=75000+2(1+2+3+\ldots+250)$ $=75000+2\left(\frac{250\times 251}{2}\right)$ $=75000+62750=137750.$
Solution 3: $302+304+306+\ldots+800=(2+4+6+\ldots+800)-(2+4+6+\ldots+300)$ $=2(1+2+3+\ldots+400)-2(1+2+3+\ldots+150)$ $=2\left(\frac{400\times 401}{2}\right)-2\left(\frac{150\times 151}{2}\right)$ $=160400-22650=137750.$

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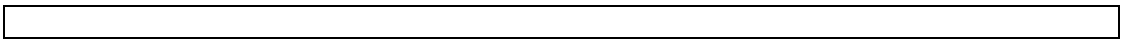


Figure 15: Three Added Solution Methods for Problem 4 by Teachers in the Study Lesson

Teacher B described that:

> *In the teaching session, the teacher actively interacted with various groups, carefully evaluating their work and providing guidance when necessary. This instructional approach underscores the significance of adopting a problem-solving approach in mathematics education. In one particular case, a group of students presented an interesting strategy to solve Problem 2. The group classified the numbers in the series* $1+3+5+\ldots+499$ *into pairs such as* $(1,499), (3,497), (5,495)$*, and so on, up to* $(249,251)$*. They correctly identified that the sum of the two numbers in each pair is always 500. Utilizing this insight, they calculated the sum of all the numbers in the series as* $250\times500=125000$*. While acknowledging the group's commendable strategy, the teacher pointed out an error in their counting of the number of pairs. The group struggled to identify the correct count. Recognizing this, the teacher provided additional guidance to help them rectify their mistake and complete the solution.*

Despite having taught this concept to students for several years, the participating teachers expressed that they acquired a new understanding of mathematics teaching through the utilization of a problem-solving approach. Teacher D reflected on the teaching approach, explaining that:

> *Within the Lesson Study program, the participating teachers directed their attention towards enhancing problem-solving and critical thinking skills in the Study Lesson, thereby refining their ability to design effective lesson plans for their students. While the classroom discussions primarily revolved around the following mathematical relationships:*
>
> $$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$
>
> $$2+4+6+\ldots+2n=n(n+1)$$
>
> $$1+3+5+\ldots+2(n-1)=n\times n.$$
>
> *Both teachers and students actively engaged with numerous new problems that could be solved by utilizing these relationships. Thus, the Lesson Study program presented a valuable opportunity for me to enhance my teaching methodology through the application of a problem-solving approach.*

CONCLUSION

Based on the stated curriculum aims, mathematics educators commonly engage in formal professional development programs as a means to meet institutional requirements (Friedman & Phillips, 2010) and enhance students' mathematical learning outcomes (Bautista & Ortega-Ruiz,

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2015; Jacobs et al., 2017). However, educational studies in the field of mathematics teaching have shown that educators may only incorporate newly acquired methods, skills, and knowledge from these programs into their teaching on a limited basis, reverting back to their previous approaches, or not implementing the new knowledge and methods at all (Supriatna, 2011). Soebari (2012) explained that this phenomenon occurs among mathematics teachers due to the inapplicability of the materials, skills, methods, and concepts obtained from professional development programs to their specific classes, which can be attributed to variations in class size, differences in students' problem-solving abilities in mathematics, and inadequate resources. Thus, continuous professional development programs for mathematics educators are imperative and should be carried out alongside their regular teaching responsibilities (Jalal et al., 2009). It appears that Lesson Study holds promise in assisting mathematics educators in obtaining reliable evaluations of their teaching practices, thereby aiding their future instructional endeavors (Bozkurt & Yetkin-Ozdemir, 2018).

Creating and designing innovative problems is one of the essential skills for mathematics educators, which play an important role in improving the quality of their teaching (Burgos et al., 2024). Hence, Lesson Study emerges as a valuable approach for supporting educators in optimizing their classroom performance and fostering learners' problem-solving abilities. By engaging in Lesson Study, educators actively work towards improving the design and implementation of study lessons tailored to the specific context of their classrooms, thereby continuously enhancing their content knowledge and pedagogical content knowledge (Fujii, 2016). Indeed, a well-designed study lesson with a strong mathematical content equips teachers with a deeper understanding of students' conceptions and misconceptions, enabling them to identify effective mathematical problems and models that promote enhanced mathematical thinking (Kuennen & Beam, 2020). Numerous studies have underscored the pivotal role of study lessons within the Lesson Study program (Fujii, 2016; Lewis, 2002). Through the preparation of appropriate materials and activities, such as math problems and engaging puzzles, mathematics educators strive to cultivate learners' problem-solving skills and critical thinking abilities. Furthermore, at the conclusion of the Lesson Study program, mathematics teachers compile reports on their study lessons, facilitating the sharing of insights and experiences among their colleagues. Consequently, each Lesson Study group can iteratively enhance the quality of their study lessons based on their students' problem-solving proficiency.

Study lessons in mathematics teaching offer a fundamental advantage by fostering deep conceptual learning among students, in alignment with the principles of constructivist learning theory. This approach emphasizes the active construction of knowledge and understanding through students' meaningful engagement with mathematical concepts and problem-solving tasks. Therefore, considering suitable mathematical problems in the study lessons enhance students' interest in problem solving activities (Gholami, Ayub, et al., 2021). By actively participating in study lessons, students are encouraged to explore mathematical ideas, construct their own mental models, and make connections between different mathematical concepts and procedures. In line with constructivist principles, study lessons provide opportunities for students to engage in authentic mathematical tasks, encouraging them to think critically, collaborate with peers, and reflect on

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their own learning processes. Through this process, students develop a deeper understanding of the underlying principles and relationships within mathematics, rather than simply relying on rote memorization or formulaic approaches.

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Exploring Writing in Mathematics: A Case Study analyzing Grade 10 Classroom Practice through Four Types of Writing

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Abstract: Writing is an integral component of classroom instruction. This qualitative ***case study*** *examined the significance of teachers' writing as a critical component of mathematics instruction, focusing on how different types of writing contribute to effective classroom practices. Three video-recorded function lessons from one Grade 10 class were purposively sampled and analyzed to investigate their integration of the four writing forms during teaching and learning using thematic analysis. The teacher's writing was categorized into four distinct types, namely, exploratory, explanatory, argumentative, and mathematically creative writing. Utilizing classroom observations, we explored how these writing practices facilitate the communication of mathematical concepts, foster learner engagement, and support the development of critical thinking skills. Our findings indicate that varied writing approaches, enhancing teachers' ability to convey complex ideas and promoting a more interactive and dynamic learning environment were underutilized. This research underscores the necessity of incorporating diverse and structured writing strategies into mathematics teaching to elevate instructional quality and enhance students' understanding. The study concludes with recommendations for teachers and curriculum designers to prioritize development in writing as an essential element in mathematics education, ultimately aiming to enrich the overall learning experience for students.*

Keywords: Exploratory, Informative, Argumentative, Mathematically creative writing, Critical thinking.

INTRODUCTION

Writing is an important aspect of communication, especially in formal environments like schools and workplaces. In schools, teachers and students interact with one another verbally, but also in the form of writing. Currently, people prefer to use social media to communicate in writing form. This demonstrates that writing does not only take place formally but also informally at home and other places.

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The concept of writing is not new in research; in the United States in particular, many authors published works on students' writing in mathematics and its influence on learning. However, its incorporation in teaching is still problematic in many countries. In mathematics most teachers rely more on the writing of symbols and diagrams (images), with limited use of words. It is, therefore, essential that such concepts resurface in research to get a better understanding of how they are incorporated into mathematics teaching and their contribution to learning. Generally, writing is a powerful tool for learning, enabling students to articulate ideas, reason through problems, and explore concepts. However, there is still a lack of empirical research in writing (Herbert & Powell, 2016). Through writing, teachers can see the level of students' understanding, knowledge gaps, and misconceptions, thus providing opportunities for remedial work or intervention (Graham et al., 2020). Therefore, writing provides evidence of students' thinking and understanding of learning (NCTM, 2014).

Researchers support writing in classrooms. For instance, Silva and Limongi (2019) contend that writing about the subject matter aids the learning process, whereas Galbraith and Baaijen (2018) created a framework indicating that writing prompts learning and stimulates cognitive processes. School students still depend on their teachers' guidance to be proficient in mathematics. To do this, teachers need to shift from conventional ways of teaching and embrace teaching strategies that enhance learner engagement and subsequently, understanding (Zeeshan, 2022). By modeling good and effective writing, teachers can influence students to engage in writing activities to enhance their understanding of mathematical concepts. However, some teachers are still reluctant to move away from traditional teaching methods and embrace new or different ways of teaching (Engelbrecht & Borba, 2024). Writing provides affordances of multimodal representation of mathematical ideas and allows students to think beyond talk (Casa et al., 2016).

Graham et al. (2020) argue that,

> "*A variety of different types of writing should support academic learning, including using writing to summarize information, compare and contrast ideas, connect new and old information, describe one or more processes, explain how something works, create a story or poem to illustrate or extend ideas, construct analogies, and build an argument*" (p. 181).

The action words used (summarize, compare and contrast, explain, and create or extend), link directly with Casa et al.'s (2016) writing types namely, exploratory, informative/explanatory, argumentative, and mathematically creative writing. These writing types offer distinct yet complementary approaches to engaging with mathematics in the sense that using one unconsciously leads to another. This study analysed the extent to which these writing types were evident in observed lessons on functions and explored their potential to deepen mathematical understanding and expression. Although they may not all be used at once, however, the integration of these types of writing in mathematics classrooms has a powerful influence on students' ability to express ideas to enhance understanding.

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Casa et al. (2016) describe the four writing types as follows:

1. Exploratory Writing**:** Involves brainstorming, questioning, and reflecting to make sense of mathematical concepts.
2. Informative/Explanatory Writing**:** Focuses on clearly explaining mathematical processes, definitions, or solutions.
3. Argumentative Writing**:** Encourages students to justify solutions, critique reasoning, and engage in mathematical debate.
4. Mathematically Creative Writing: Involves generating original problems, exploring patterns, and creating unique solutions.

Teachers' instructional strategies have a big impact on how mathematical topics, primarily functions, are taught and learnt (Yang & Kaiser, 2022). With an emphasis on functions, this study intends to investigate the different forms of writing that mathematics teachers in grade 10 employ in their instruction. Knowing the kinds of writing that are frequently incorporated into mathematics classes helps teachers explain difficult ideas and involve students in worthwhile learning activities. Additionally, we explore the ways in which Grade 10 mathematics teachers encourage their students to write. Writing in mathematics is more than just an extracurricular activity; it is an essential tool for students to express their comprehension, make sense of their ideas, and evaluate their learning (Shen & Chong, 2023). We also emphasize the value of writing as a crucial part of mathematics instruction by identifying teachers' strategies used to encourage writing. Finally, the study looks at how Grade 10 maths teachers' writing techniques improve students' understanding of functions. Teachers can model mathematical reasoning, dispel misconceptions, and promote student interest in the material by using good writing. The goal of this research is to better understand how writing and mathematics instruction interact by examining how teachers' writing affects students' learning outcomes related to functions. This will help teachers improve their pedagogical strategies and promote better content comprehension and mathematical achievement.

To achieve these objectives, the study seeks to answer the following research questions:

- What types of writing do Grade 10 mathematics teachers commonly use in their teaching of functions?
- What opportunities do Grade 10 mathematics teachers create to promote writing for their students?
- How do Grade 10 mathematics teachers' writing enhance the learning of functions?

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LITERATURE REVIEW

Writing in mathematics has garnered increasing attention as a pedagogical tool for deepening understanding and fostering communication. Unlike other writing genres, mathematical writing necessitates a combination of general writing skills, mathematical content knowledge, and discipline-specific abilities like using mathematical representations, using precise mathematical language, and combining equations and expressions with words (Colonnese, 2023). Research highlights that incorporating writing in mathematics classrooms helps students clarify their thinking, make connections between concepts, and express reasoning (Burns, 2004). Writing-to-learn frameworks emphasize that writing enables students to move beyond rote memorization and engage with ideas on a conceptual and analytical level (Minasyan & Supriatna, 2024). Incorporating writing in mathematics can be a panacea for poor achievement which is a continuing problem in schools because when writing, students become engaged with content, and subsequently learning is supported (Graham et al., 2020).

Exploratory Writing

Exploratory writing allows students to reflect on their understanding, pose questions, and brainstorm ideas. It enables students to use their own words when trying to understand the problem (Casa et al., 2016). Students can also use their mother tongue to help them explore the problem (Colonnese, 2023). This form of writing aligns with constructivist theories of learning, which emphasize active engagement and the co-construction of knowledge (Vygotsky, 1978). Therefore, in exploratory writing, students can make drafts and represent a problem in different ways as they try to answer the "what" questions in a mathematical problem. In this way, exploratory writing assists in learning and promotes understanding. Studies suggest that reflective journals and informal prompts encourage students to engage in metacognition, helping them identify gaps in their understanding and articulate new insights (Lavi et al., 2019).

Informative/Explanatory Writing

Informative writing involves describing mathematical processes, definitions, and solutions clearly (Arsenault et al., 2024), and it is crucial for building procedural fluency and ensuring students can communicate their understanding. Explanatory (informative) writing deals with the "how" questions that require students to describe procedures. Because these questions require students to regurgitate procedures already learned from their teachers, if used excessively it can limit their ability to use writing to reason mathematically (Colonnese, 2023). Explanatory tasks such as summarizing steps to solve an equation or defining a mathematical concept help reinforce learning by requiring students to organize their thoughts logically.

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Argumentative Writing

A mathematical argument is a description of how one arrived at a certain conjecture (Sriraman & Umland, 2020). Argumentative writing encourages students to justify their solutions and critique the reasoning of others in a mathematics discourse where the audience is fellow students and their teacher (Colonnese, 2023). Incorporating arguments in teaching aligns with the Standards for Mathematical Practice outlined by the National Council of Teachers of Mathematics (NCTM), which emphasizes constructing viable arguments and critiquing the reasoning of others (NCTM, 2000). Research shows that engaging in mathematical argumentation fosters critical thinking and helps students develop deeper conceptual understanding (Knipping, 2008). Colonnese (2019) proposed that teachers should ask students "why" questions to encourage them to explain mathematical ideas to their peers. Toker (2021) supports this view by stating that teachers can benefit from students' explanations of their thinking because they can recognize the sources of students' misconceptions. Therefore, prompting students to reason develops their cognitive thinking skills and fluency in communication. It also builds a foundation for mathematical proofs required in higher classes (Colonnese, 2023).

Mathematically Creative Writing

Creativity in mathematics often involves problem generation, pattern exploration, and the formulation of novel solutions. Mathematically creative writing encourages students to approach problems from multiple perspectives and express their unique ideas, extending them beyond what they have learned (Colonnese, 2023). Research by Nilimaa (2023) underscores the importance of fostering creativity in mathematics to cultivate innovation and adaptability. Furthermore, students who understand the learning content can create mental representations of a concept, which leads to good problem-solving skills (Toker, 2021). Therefore, giving students tasks that require the creation of their problems and solving them would not only incorporate mathematically creative writing but also assist teachers in seeing if learning has occurred.

CONCEPTUAL FRAMEWORK

This study is grounded in a framework that integrates Casa et al. (2016) four writing types namely, exploratory, informative/explanatory, argumentative, and mathematically creative writing as essential components of mathematical pedagogy. The conceptual framework is guided by the following principles:

Writing as a Tool for Learning

Researchers (Santos et al., 2019; Van Schalkwyk et al., 2019) encourage active learning and assert that students learn better when actively involved in the learning process. However, active learning seems to be a problem and is insufficiently incorporated in most classrooms (Reinholz et al., 2022). One way of actively engaging students in learning is by writing. Writing allows students to process

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and internalize mathematical concepts. Exploratory and informative writing helps students build a foundational understanding of mathematical concepts (Kooli, 2023). With exploratory writing, students are allowed to explore on their own or in groups to make sense of the problem before teaching commences, while informative writing assists in describing the problem (Casa et al., 2016). Therefore, informative writing activities require students to use words to describe mathematical concepts or situations. In argumentative writing, students should provide justifications for their reasoning (Colonnese, 2019).

Communication and Reasoning

Communicating effectively in mathematics is significant to learning (Sari & Fauzi, 2025). Mathematics is a language of logic and structure (Coregliano & Razborov, 2020), consisting of semiotic and grammatical aspects. Semiotics involves the use of signs (words, images, and objects), and grammatical aspects which entail words, phrases, clauses, and sentences (Cholili & Lapele, 2020). These (signs and words/sentences) are then used by teachers and students to communicate mathematical ideas. Mathematics contains technical vocabulary that may not carry the same meaning as ordinary words (Lin, 2021). This is where symbols are integrated to distinguish technical vocabulary from words used in everyday language. It is impossible to teach or study mathematics without carefully utilizing mathematical tools. These tools include words, symbols, and images.

Writing provides students with a medium to communicate their reasoning, critique arguments, and justify solutions, fostering critical thinking and collaborative learning. Teachers are more likely to spark lively discussions involving cognitive thinking when they ask questions that challenge students' knowledge retention (Gentaz & Richards, 2022; Slate & Charlesworth, 1988).

Creativity and Exploration

Mathematics is inherently creative, involving pattern recognition, problem-solving, and innovation. These characteristics are at play in individuals who have reached a level of conceptual understanding. Mathematically creative writing allows students to explore these aspects, encouraging flexibility and originality in their thinking. Mathematically creative writing involves students' creations to demonstrate a deeper understanding of mathematical content. Therefore, argumentative and creative writing both encourage deeper engagement and application, which are traits of cognitive engagement.

Holistic Engagement by Integrating the Four Writing Types

Holistic engagement holds a multi-dimensional perspective in learning (Shen & Chong, 2023). Among others, it constitutes learner engagement in perspectives such as cognitive, affective, and social (Wong & Liem, 2022). The cognitive aspect requires students to engage with the subject content through being actively involved in learning, that is, by reading, writing activities, and responding to questions. Wong and Liem further state that in the affective aspect, students learn to

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appreciate knowledge when it is contextual and relatable, while the social aspect develops students into creating relationships when learning, and this can be developed through cooperative learning. The cognitive dimension can be developed by integrating the four writing types in mathematics teaching and learning. This holistic approach ensures that students develop technical skills, confidence, and curiosity when communicating mathematical ideas.

Effective mathematics instruction integrates multiple writing types to address diverse learning goals. Studies by Pugalee (2004) demonstrate that combining explanatory, argumentative, and creative writing tasks promotes both procedural fluency and conceptual understanding. Additionally, providing students with varied writing opportunities ensures that they can engage with mathematics as both a logical and creative discipline (Bicer, 2021). Figure 1 is a framework showing how teachers can interact with students, giving them tasks utilizing the four writing types.

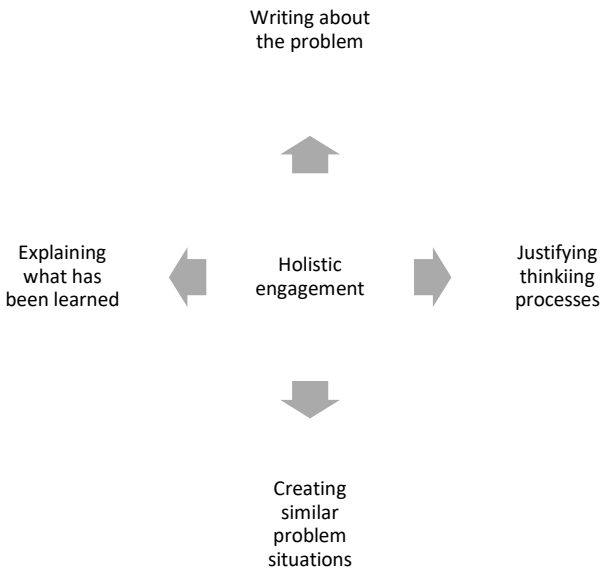


Figure 1: Holistic engagement of students utilizing the four writing types

Figure 1 shows that utilizing all four writing types assists in the holistic engagement of students. The four aspects in the holistic engagement framework are unpacked as follows:

> ***Writing about the problem***: Teachers can allow students to work in groups as they write about how they see and understand the problem, and the strategies they can apply in solving it. For example, they can be asked to explain how a linear function graph will be affected if its slope changes. When asking such questions, teachers can provide visuals/diagrams to assist students in exploring the problem. This allows students to put their thoughts into

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words when exploring different facets of a problem (Sa'adah, 2020). They combine exploratory and explanatory writing to express ideas that involve organization of information and reflection, which leads to critical thinking and greater insights about the problem (Zulyusri, 2023). Therefore, students are not only cognitively engaged, but they also develop a personal connection with the study material.

Justifying thinking processes: This can be prompted by asking open-ended questions (Kim et al., 2021) to allow students the opportunity to critically evaluate their thinking and consider alternative perspectives to a problem using argumentative writing. Teachers should encourage them to "think aloud" so that they can be helped to phrase their arguments in collaborative environments. Prompting students to justify their reasoning assists in developing open-mindedness in problem-solving (Huang & Sang, 2023), including an understanding that problems may have different solutions and various factors might influence these solutions.

Creating similar problem situations: This is a higher level of thinking in mathematically creative writing that involves the application of learned content, showing mathematical understanding. It also shows relevance and applicability of the learned content beyond the original problem, thereby solidifying the learning process. Prompting students to generate problems similar to those given in teaching activities and solve them (Nilimaa, 2023) can be a clear demonstration of their understanding. This can be done in informal assessments where teachers can ask students to generate problems related to those given.

Explaining what has been learned: This is a reflective practice that utilizes explanatory writing and requires students to share their personal experiences with the learning material (Marshall et al., 2021). This can be done by asking students to write journals or mathematics essays where they reflect on their learning. When teachers prompt students to write about what they have learned, they reinforce their engagement with the study material by synthesizing information and connecting various parts of the content (Graham et al., 2020). They are compelled to think as they write about strategies that worked or did not work, and how they can approach such problems in their future learning.

When these four aspects are applied in teaching and learning, students engage holistically and naturally, allowing them to engage with the study material in a deep and interconnected manner, enhancing understanding and knowledge retention.

It shows that each form of writing does not necessarily lead to another, but can be utilized on its own. However, if all are utilized, students can be mathematically proficient.

The framework emphasizes the interplay between writing types and their collective potential to enhance mathematical proficiency. In analyzing classroom practice, this study seeks to identify how these writing types are employed in classrooms and how they can be better integrated to support diverse students.

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METHOD

Research Approach, Design and Sampling

The study utilized qualitative data collection methods to get deeper insights into the teacher's writing activities and how his writing enhanced the learning of functions in general. A qualitative case study was conducted in one of the South African secondary schools in 2024 to get deeper insights about teachers' writing in Grade 10. There was only one Grade 10 mathematics teacher at the selected school, therefore, the teacher was selected as the only participant in the case studied. The participant was selected based on the researchers' judgment that he would generate the required data for the study. The teacher offered five linear function lessons to **42 learners**, and from the five, the first three were sampled to get a continuation of how the teacher utilized writing in his teaching. In addition, there was only one participant, therefore, we needed sufficient data to corroborate the discussion. Moreover, the researchers felt that the first lesson would serve as the introduction to the linear functions topic and would not provide clear and sufficient evidence of the teacher's writing.

Data Collection and Analysis

Video-recorded lesson observations were conducted because videos capture real-time context of the classroom environment (Cheung et al., 2021). Additionally, they provide visual and auditory evidence that complements qualitative data. Data were transcribed verbatim from the three lessons observed. To attain the study's credibility, researchers had a prolonged engagement with data by observing three lessons from the same class, transcribing the recorded lessons, and reading the data several times during transcription and analysis. The data from the transcribed lesson observations included stammering, pauses, and repetition of words as spoken by the teacher (Moser & Korstjens, 2018). Thematic analysis was used to analyze data in the study because it is flexible and can be adapted to a wide range of research questions and data types (Helou, 2020). Furthermore, it allows for the inclusion of diverse voices and perspectives, hence it is suitable for studies aiming at understanding complex social phenomena (Yamamoto & Keogh, 2018). The analysis outlined the importance of including the different types of writing in teaching functions, and how they connect to enhance learning (Jossey et al., 2021). The lessons were analyzed to identify instances where the four writing types were either explicitly incorporated or could have been utilized to enhance learning. Data analysis was specifically focused on emerging patterns that were associated with pre-determined themes, namely:

- The teacher's instructional strategies.
- Opportunities for student writing and verbal expression.
- The alignment of discourse with each writing type.

The pre-determined themes guided the researchers to focus only on the teacher's writing activities and their alignment with Casa et al.'s (2016) four types of writing. Steps followed in analyzing data were: 1) Reading and transcribing data for familiarization and immersion, 2) Data coding, 3)

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Comparing codes with predetermined themes, 4) Naming and articulating what each theme represented, and 5) Reporting the findings

To enhance the trustworthiness of the findings, raw data transcripts from lesson observations were included. **The researchers gave the teacher participant the pseudonym of "Mike" to ensure confidentiality.**

RESULTS

In answering the research questions, data were coded according to the intended objective of the words spoken and a sample of these are presented in Table 1.

Codes Identified

Code	Strategy/technique used	Supporting Text
Explanation/ definition:	Informative writing	"*So, when it comes to functions, the functions can be, we can have many, er, many to one, where more than one x-value will have more than one y-values."* *For, for as long as it will be able to touch the graph ga one fela (only once), it means the graph satisfy the condition of being a function.*
Student engagement	Question and answer	*At how many points am I touching the graph?* *"What do you think will happen to the mother graph, compared to the, the graph that we are going to draw next? Will it go up when we add 2, or will it go down?"*
Critical thinking prompts	Argumentative/ Exploratory writing	*So you think this graph passes the vertical line test? Does it pass?*

Table 1: Codes identified from data transcripts

Coding used did not focus on the frequency of words, but on meaning (Hennink & Kaiser, 2020); therefore, they were later synthesized into findings using predetermined themes listed in the data collection and analysis of the methodology section.

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Exploratory Writing

Exploratory writing was minimally evident in the observed lessons. The teacher occasionally posed reflective questions, such as, "*So, I want us to check, what do you think will happen to the mother graph, compared to the, the graph that we are going to draw next? Will it go up when we add 2, or will it go down?"* Students rarely had opportunities to respond to such questions and explore these ideas through written reflection. The teacher prompted students to think, however, he guided them toward the correct answer by saying, "*Will it go up when we add 2, or will it go down?"* In this way, students were allowed limited opportunities to explore. Although exploratory writing opportunities were limited during teaching, some activities did include some form of exploration, see Figure 2.

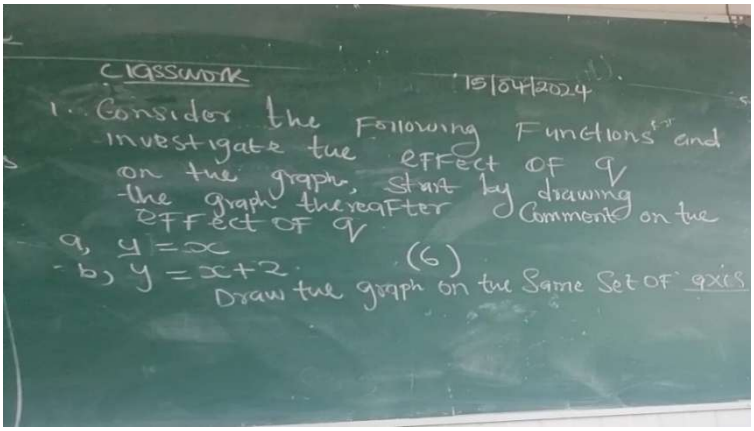


Figure 2: Mike's first activity on linear functions

In the activity, students were asked to explore or investigate the effect of q in linear function graphs. The activity required students to draw two graphs on the same axis before comparing them. It combines exploration with graph-drawing procedures (informative writing) that students should have in their activity books, showing that types of writing complement one another to develop students holistically. However, the exploratory writing form incorporated in the activity is minimal and at a foundational stage since students' answers could be in the form of a word or short phrase. For example, after sketching the two graphs, students could respond by saying, q increased or the graph shifted up. Moreover, exploration could have been part of the teacher's activities during lessons to encourage critical thinking before assessing the students. Therefore, integrating exploratory journals or quick writing before teaching a concept could encourage students to brainstorm and connect ideas more deeply.

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Informative/Explanatory Writing

Informative writing was a dominant feature of the lessons. The teacher often explained mathematical concepts, such as defining a function: "A function is a mathematical relationship where each input has exactly one output" and "*So, when it comes to functions, the functions can be, we can have many, er, many to one, where more than one x-value will have more than one y-values."* Students were prompted to record these definitions and explanations in their notes, reinforcing procedural fluency. Furthermore, the teacher gave more explanations instead of allowing for exploration, hence limiting students' opportunities to engage critically or explore. For instance, Mike explained:

> *Mike: So what do we do to check the vertical line test, you can take a ruler and check at one point or at one x-value how many times will the vertical line touch the graph. The line that you have drawn in broken lines, how many times is it going to touch the graph?*
>
> *Students: Two*
>
> *Mike: Two? How many times? You can take a ruler, and someone won't give you a ruler. This is my graph let's say this is f(x), how many times am I touching it here at this point? Am I touching this graph?*
>
> *Students: No*
>
> *Mike: no neh? And then here? Am I touching?*
>
> *Students: Yes (chorus).*
>
> *Mike: How many times? At how many points am I touching the graph?*
>
> *Students: Two. Am I touching it here? If the ruler is like this, am I touching it here? Here, here?*
>
> *Students: Mhm (chorus).*
>
> *Mike: Only here neh?*
>
> *Students: Yes.*
>
> *Mike: At one point. I am shifting now to a different x-value of a negative one. How many times am I touching the graph at this point?*
>
> *Students: One*

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In the lesson observation transcript above, students were asked yes or no questions, while the teacher explained the procedure for conducting the vertical line test. Furthermore, the activities given also reinforced procedural fluency which is linked to informative writing, see Figure 3.

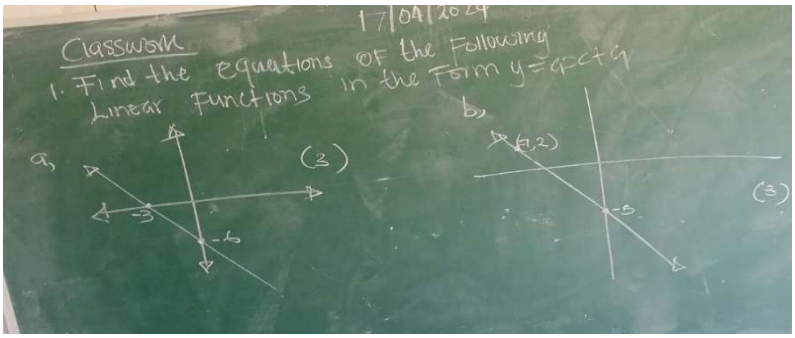


Figure 3: Mike's second activity on linear functions

Both the transcript and the activity show that the teacher relied more on explanatory (informative) writing and students had limited opportunities to create their explanations, which could have strengthened their conceptual understanding and assisted in identifying gaps and misconceptions.

Argumentative Writing

Argumentative writing, where students justify their reasoning, was occasionally encouraged but not systematically integrated. For example, during discussions on the vertical line test, the teacher provided a reason why the graph failed the vertical line test, saying:

> *Mike: Ga one (once). I am only touching it here, you see? For, for as long as it will be able to touch the graph ga one fela (only once), it means the graph satisfy the condition of being a function.*

Mike used another example and after using a ruler for the vertical line test, he said:

> *Mike: So you think this graph passes the vertical line test? Does it pass?*
>
> *Students: No.*
>
> *Mike: The vertical line test you only pass it when the ruler touches the graph once.*

Instead of asking students to explain why the graphs failed or passed the vertical line tests in both instances, Mike provided the reasoning behind his questions, which in a way, encouraged arguments. However, this demonstrates that the teacher provided limited opportunities for students to be exposed to argumentative writing because students were not allowed to respond to the "why"

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questions. Instead of asking why the graph does not pass the vertical line test, the teacher immediately substantiates it by providing the reason. Moreover, these instances lacked written justifications as they were only verbal utterances. Therefore, structured activities like writing mathematical proofs or defending solutions could enhance students' argumentative skills.

Mathematically Creative Writing

Mathematically creative writing was largely absent in all the lessons. The latter focused more on procedural tasks, such as plotting graphs and solving equations, with limited opportunities for students to pose original problems or explore patterns. Students were given activities and not encouraged to create their functions or problems. The activities given also tested students' skills in drawing graphs and procedures, see Figures 2 and 3 under exploratory and informative writing respectively. Interactive engagements during lessons were also about procedures when doing calculations.

Encouraging mathematically creative writing, by incorporating activities like asking the students to create a function that meets certain criteria or write a story involving a linear function was absent during lessons and in the activities. Such questions could foster students' creativity and effective engagement with the study material.

DISCUSSION OF THE FINDINGS

The Alignment of Discourse With Each Writing Type

In answering the first research question, *What types of writing do Grade 10 mathematics teachers commonly use in their teaching of functions?,* the results of this study revealed that students were exposed to explanatory (informative) writing more than any other type of writing when they described and defined functional concepts (Casa & Choppin, 2020; Colonnese, 2023). They demonstrated the sketching of graphs, calculations involving substituting in formulas, and testing whether graphs were functions or not. This type of writing focuses on talking more about facts and details and is not necessarily about understanding. It encourages regurgitation of procedures, which in turn promotes rote learning. The teacher's strategy of talking and explaining more during lessons reflects the traditional way of teaching where they transmit knowledge and students become passive recipients (Spiteri & Chang Rundgren, 2020). The attempt to incorporate exploratory writing was limited to certain sections of the content whereas informative writing was incorporated throughout the lessons, covering every section of the functions topic. Similarly, argumentative and mathematically creative writing was absent. Therefore, students were allowed limited opportunities for writing in function lessons because they were exposed to explanatory writing more than

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other types of writing. The National Council of Teachers of Mathematics (NCTM) (2000) recommends that students be encouraged to communicate in writing form during teaching and learning to allow them to express ideas. Learning environments incorporating extensive writing provide opportunities for practical engagement where students express ideas and reasoning behind their thinking verbally and in writing. This crucial learning feature is barely present in the observed lessons.

Opportunities for Student Writing and Verbal Expression

For the second research question *"What opportunities do Grade 10 mathematics teachers create to promote writing for their students"?* the results revealed that the students' responses to questions were mostly yes or no answers, confirming the teacher's actions and procedures. Additionally, questions were about algorithms and procedures, also encouraging informative writing. However, some sections of work as prescribed in the CAPS required students to compare graphs and investigate the effect of a and q on linear and quadratic functions. Questions asked in these sections enabled students to respond in words or phrases to describe the behavior of the graphs. In these sections, the type of questions used encouraged students to utilize foundational explanatory writing where they described mathematical situations in a few words, such as decrease or increase. Therefore, limited opportunities were provided to promote writing for students. The results support Powell et al. (2020) when they said many educators regard writing as important but still struggle to incorporate it into mathematics teaching.

The Teacher's Instructional Strategies

In addressing the third research question, *"How do Grade 10 mathematics teachers' writing practices enhance the learning of functions?"* Our findings revealed that the emphasis on explanatory writing in teachers' instructional materials led to an informative approach to teaching functions. This type of writing primarily involves the reproduction of factual knowledge, which, in turn, promotes rote learning and memorization of procedures. Although students remained engaged throughout the lessons, the questions posed focused mainly on procedures and algorithms. Unfortunately, a holistic approach incorporating all four writing types was not observed. The results highlight a lack of proficiency among teachers in integrating mathematical writing into their instructional practices, thereby supporting Powell et al.'s (2020) argument that teachers continue to struggle with incorporating mathematical writing into their teaching.

On the one hand, exploratory writing sparks curiosity, enhances engagement with content, and develops an individual's investigative and explorative skills. On the other hand, students involved in argumentative writing can interact with peers to advance arguments and justify their reasoning.

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These kinds of environments support learning and conceptual understanding because teachers can use peer assessment to critique students' work and provide feedback. Therefore, instead of providing detailed conceptual explanations, teachers could lead inquiry-based teaching to allow students to explore, argue, and justify their thinking (Evans & Dietrich, 2022). Peer assessment is another form of student engagement that enhances self-regulated learning (Segaran & Hasim, 2021). Depriving students of the opportunity to engage in arguments limits their communication and critical thinking skills. It also inhibits their ability to interact effectively with the learning material and their peers, which is an aspect of socio-cognitive theory. Lastly, mathematically creative writing displays a high level of conceptual understanding because students create problems and solve them using previous knowledge. However, in this study, the teacher provided limited opportunities for students to create their own problems. Leikin and Sriraman (2022) advocate for the advancement of creativity in mathematics learning. Without creativity, mathematics teaching and learning would be a mere reproduction of facts and procedures that encourage rote learning. Therefore, the creative aspect of mathematics learning was under-developed. Teachers utilized informative writing rather than exploratory, argumentative, and mathematically creative writing, placing holistic engagement in a compromised state. When one writing type is dominant over others, as is the case with the observed lessons, students' communication and critical thinking skills, and creativity become underdeveloped.

CONCLUSION AND RECOMMENDATIONS

This paper examined the use of four writing types—exploratory, informative/explanatory, argumentative, and mathematically creative writing within a Grade 10 mathematics classroom. Sure, here's a refined version of your paragraph in clearer and more polished English:

The findings revealed a strong emphasis on explanatory writing, often at the expense of other forms of writing. This focus was evident in the teacher's explanations and definitions of concepts. Other types of writing such as exploratory, argumentative, and mathematically creative writing, were largely underutilized. As a result, learners were not given adequate opportunities to explore or engage in critical discussions about mathematical concepts.

Drawing on transcription data from lessons on functions, the study identified how each writing type could enhance mathematical understanding, communication, and creativity. The paper emphasizes the importance of integrating diverse writing practices to foster holistic engagement in mathematics learning.

The four types of writing offer a rich framework for enhancing mathematical learning. While the observed lessons provided a strong foundation in explanatory (informative) practices, integrating exploratory, argumentative, and creative writing could promote more profound conceptual understanding and student engagement. By embracing a multi-faceted approach to writing, mathematics

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teachers can empower students to develop mathematical thinking and problem-solving skills, and communicate effectively.

To integrate the four writing types more effectively, the study recommends:

1. Introduction of reflective journals or prompts encouraging students to speculate and connect ideas in exploratory writing.
2. Provision of opportunities for informative writing by asking students to explain concepts in their own words, both verbally and in writing
3. Implementing tasks requiring students to provide written justifications, proofs, or debates about solutions.
4. Encouraging students to generate original problems, explore patterns, and create narratives involving mathematical concepts.

Although we provide specific recommendations for each writing type, it is important to recognise that all four types of writing complement one another and can be integrated into a single task. However, to do this, teachers need to invest more time in planning and designing challenging tasks incorporating the four writing types. Additionally, students should be guided through similar tasks prior to the implementation of the designed tasks to achieve optimal results. Lastly, policy-makers should include mathematical writing in mathematics educational policies to promote its integration into teaching.

LIMITATIONS

The findings were derived from three function lessons taught by a single Grade 10 teacher; therefore, they cannot be generalized to a greater population. Future research should consider a broader pool of participants that includes different grades and topics to either affirm the findings or yield different results.

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Characterizing the Reasoning of Moroccan Students in Problem-solving Tasks: the Case of Inductive Reasoning

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Abstract: This study provides a detailed characterization of inductive reasoning among Moroccan secondary school students, which remains underexplored in national and international contexts. Using an original analysis grid based on structural and processual reasoning steps, the research offers insight into how students engage with inductive thought when solving mathematical problems, revealing cognitive tendencies and curricular influences.

A test composed of five carefully designed problems was administered to elicit inductive reasoning through familiar concepts and representations. The results show that most students successfully engaged with the initial phases of reasoning, particularly observation and model identification. However, generalization was less frequently achieved. The study's originality lies in its fine-grained, step-by-step analysis of students' reasoning, highlighting the potential of inductive thinking in a context where it is often overshadowed by a dominant focus on deduction.

Keywords: Inductive reasoning, problem-solving, qualifying secondary cycle, analysis grid.

INTRODUCTION

In the Moroccan high school mathematics curriculum, several types of reasoning are introduced as foundational principles: reasoning by equivalence, contrapositive, disjunction of cases, contradiction, and mathematical induction (Pedagogical Orientations of Mathematics, Secondary Qualifying, 2014, p. 17). These reasoning modes are typically presented as formal tools for proof, often without sustained attention to how students develop them through problem-solving. Among these, deductive reasoning remains dominant, shaping the structure of both teaching and assessment (Nhiry et al., 2023).

Yet mathematical reasoning is not limited to deduction. Inductive reasoning plays a crucial role in helping learners identify patterns, make conjectures, and construct generalizations from specific

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cases. Despite its centrality in mathematical discovery and early learning, inductive reasoning often remains underrepresented in instructional practice, limiting students' capacity to explore and justify mathematical ideas flexibly (Brousseau, 2000; Duval, 1995; Lithner, 2008).

This study addresses a significant gap in literature. While existing research has explored mathematical reasoning broadly, very few studies have offered a detailed empirical analysis of how Moroccan students engage in inductive reasoning, especially through the lens of their written problem-solving processes. The originality of this study lies in its use of a custom-designed analysis grid—based on structural and processual dimensions of reasoning—to characterize how inductive thinking unfolds across multiple stages of problem-solving.

Although this work focuses primarily on student productions, it is situated within a broader reflection on the influence of curriculum and teaching practices. The didactic environment—heavily oriented toward formal proof—may shape how students approach or neglect inductive reasoning when required. Understanding how students reason inductively therefore offers key insights into cognitive processes and the curricular framework in which they operate.

This research contributes new empirical data to the field by analyzing the responses of first-year baccalaureate students to five carefully selected mathematical problems. It aims to reveal both the potential and the limits of inductive reasoning among secondary students and to open future pathways for research and teaching practices that value exploratory reasoning in mathematics.

THEORETICAL BACKGROUND

Inductive Reasoning: Historical, Epistemological, and Didactic Anchoring

David Hume (1748) challenged the validity of induction, suggesting it depends on habit rather than logical certainty. Observing events repeatedly — like the sun rising daily — doesn't logically ensure they will happen again, as the uniformity of nature cannot be proven. The assumption that the future will mirror the past is rooted purely in psychological habit.

Polya (1957) characterizes induction as "a way of reasoning that leads to combination." Induction deals with the transition from the particular to the general, leading us to the knowledge of general truth (Arnaud & Nicole, 1965).

Osherson et al. (1990) and Sloman (1993) recognize the influence of perceptual similarity in their main theories of induction in adults, specifically in terms of overlapping structural characteristics between an inductive base and a target, which plays a major role in determining the strength of inductive inferences.

In mathematics education, inductive reasoning is defined as an *ampliative inference* leading to the construction of new knowledge from the observation of cases, generalized to a broader set (Pedemonte, 2002). Often applied locally, especially in experimental contexts, induction alone cannot account for all processes at play. A process-oriented perspective—considering actions such as

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generalizing, conjecturing, and classifying—helps identify two problem-solving approaches that are neither purely deductive nor purely inductive: the experimental approach and the successive test-adjustment approach. The experimental approach articulates phases of experimentation—conducting tests, observing results, drawing conclusions—together with the formulation and attempted proof of conjectures (Gardes, 2013).

Inductive reasoning follows the pattern-data-result-rule (Pedemonte, 2002; Cabassut, 2005; Jeannotte, 2015). The process begins with observing data, recognizing patterns, and drawing conclusions or creating a rule. Duquesne (2003) states, "Inductive inference is exercised on observed regularities from which more general conclusions can be drawn." Therefore, it involves deriving a rule from data and making assertions about that data. It moves from the specific to the general. According to Jeannotte (2015), generalization, conjunction, and the identification of regularities are essential for the induction process. It is also helpful to consider the dynamic aspect of reasoning in teaching and learning mathematical reasoning (MR). Jeannotte (2015) proposes a distinction between two dimensions to better understand the components of inductive reasoning in mathematics education.

- Structural aspect: This refers to the overall organization of the reasoning process, how a student moves from data to rule, and how different stages are ordered and connected.
- Processual aspect: This refers to the cognitive processes involved in constructing that reasoning, which Jeannotte categorizes as:
    - Processes of exploration, such as identifying regularities, comparing, classifying, conjecturing, and generalizing.
    - Processes of validation, where learners seek to confirm or justify the reasoning behind their conjectures.

In terms of problem-solving using inductive reasoning:

The study by Cañadas (2007) proposes a more precise categorization of inductive reasoning based on the steps outlined by Polya (1967) and Reid (2002), specifically: 1) observation of cases; 2) organization of cases; 3) search for and prediction of models; 4) formulation of conjectures; 5) validation of conjectures; 6) generalization of conjectures; and 7) justification of general conjectures. This framework was used to study the inductive reasoning of twelve Spanish secondary school students.

Cañadas et al. (2009) used a model for the characterization of inductive reasoning composed of seven steps: 1) working on particular cases; 2) organizing particular cases; 3) searching for and predicting models; 4) formulating conjectures; 5) justifying (validating the conjecture based on particular cases); 6) generalizing; and 7) justifying the generalization (formal proof) (Cañadas & Castro, 2007). This model was employed to analyze the responses of Spanish students in grades 9 and 10 when working on linear and quadratic sequence problems.

Papageorgiou (2009) proposes and evaluates a training program that integrates problem-solving through inductive reasoning with the development of mathematical concepts among sixth-grade students in elementary schools in an urban area of Cyprus.

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Sosa Moguel et al. (2019) proposed three processes to characterize secondary mathematics teachers' inductive reasoning when working with a quadratic model to obtain a general rule: 1) observing regularities; 2) establishing a model; and 3) formulating a generalization. The results showed that some teachers encountered difficulties in transitioning from observing regularities to formulating a generalization.

Inductive Reasoning: Characterization and Detection of Similarities and Differences With Deductive Reasoning

Inductive reasoning is a fundamental cognitive process in which we observe data (judged to be true) to deduce a conclusion that allows us to create generalizations.

This type of reasoning is often characterized as allowing one to move from the particular to the general (Polya, 1965). An inductive reasoning step can be characterized as data-rule-affirmation, as shown in the diagram (Figure 1). However, according to Jeannotte (2015), this characterization is insufficient to describe RM; the processual aspects—generalization, conjunction, and identification of a regularity—are also necessary for the induction process.

A disadvantage of inductive reasoning is that inferences are made from specific situations, which may not have any meaning in the real world.

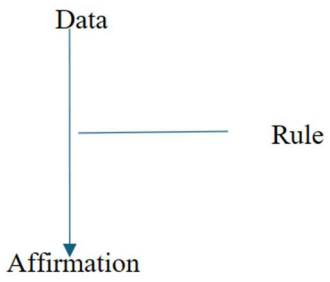


Figure 1: Model of an inductive reasoning step (Favier, Stéphane & Chanudet, Maud 2021, p.90)

On the other hand, deductive reasoning works in the opposite direction from inductive reasoning. It is a process that allows obtaining a statement from data and a rule. It involves the use of general hypotheses and logical premises to arrive at a logical conclusion.

Therefore, both reasoning types are considered alternatives; inductive reasoning produces generalizations, while deductive reasoning produces particularizations. Additionally, there is a difference in the production of proof, as the two reasons are at different and non-linear times.

> *In a proof approach, the deductive and inductive points of view are located at different and non-linear times. The inductive point of view is present in the development of a conjecture or the study of hypotheses when the conclusion is to be constructed. The deductive point of view is predominant in the writing of the proof. (Denise GRENIER, 2012, p.107)*

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On the other hand, there are also similarities. For example, the degree of certainty that allows deduction and induction to be placed on a common degree of argument strength (Aidan Feeney, Evan Heit, 2007).

Inductive Reasoning: Detection of the Place it Occupies in the Moroccan Program

Reasoning is the foundation of mathematics, as Ross wrote in 1998:

> *It is essential to emphasize that the foundation of mathematics is reasoning. While science verifies through observation, mathematics verifies through logical reasoning. Thus, the essence of mathematics lies in proofs, and the distinction among illustrations, conjectures, and proofs should be emphasized (Ross, 1998, p.254)*

In Moroccan secondary school programs (Men, 2007), mathematical reasoning is central, appearing explicitly and implicitly in activities, problem-solving exercises, and problems. Explicit instruction on reasoning mainly occurs in the "Logic" course, specifically within the Mathematical Reasoning section, which covers direct reasoning (deductive), disjunction of cases, proof by absurdity, contrapositive, counterexample, and induction. This study follows the reference model of mathematical reasoning proposed by Jeannotte (2015), used in Nhiry et al. (2023), which examines the structural aspects of reasoning. Their findings highlight the dominance of the deductive step, with the inductive step also strongly represented in the curriculum—hence its inclusion here, especially in problem-solving contexts. Although inductive reasoning is rarely named in textbooks and appears only briefly in official teacher guidelines (Nhiry et al., 2023), it is more often encountered in preparatory activities than in exercises and problems designed to build understanding of mathematical concepts (see Appendix 2).

Structural Aspect in Inductive Reasoning

The structural aspect of inductive reasoning involves observing particular data, identifying regularities (also referred to as assertions), and ultimately inferring a rule that allows the passage of data to assertions (Jeannotte, 2015). In addition to its structural aspect, the grid presented (Nhiry and al. 2023) also described its procedural aspects: identifying regularities, conjecturing, and generalizing.

To describe inductive reasoning, four stages were considered, Pólya (1967):

- Observation of cases.
- Formulation of a conjecture based on previous particular cases.
- Generalization.
- Verification of the conjecture with new special cases.

For uncertain cases, Reid (2002) proposed a conjecture formulation with five detailed steps:

- Work on specific cases.
- Analysis model.

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- Information analysis.
- Analysis of information on specific cases (with doubts).
- Generalization, use of generalization to prove.

We utilized his writings in our study to develop an analysis grid (Table 1) that characterizes the inductive reasoning of secondary school students when they solve problems that can be solved using this type of reasoning.

Analysis criterion (C)	**C1: Case observation**	**C2: Identification of a model**	**C3: Identification of regularities**	**C4: Formulation of a conjecture**	**C5: Generalization**
Steps to follow in the generalization process	Examine several specific examples to identify patterns or regularities.	Observe subsequent cases and infer a pattern.	Infer a recursive relationship between mathematical objects by analyzing their similarities and differences.	Infer a statement about a regularity that has potential for mathematical theorization	Infer a property of a set of mathematical objects, or of the relationships within it, from a more restricted subset.

Table 1: Inductive Reasoning Analysis Grid.

METHODOLOGY

Participants

The participants were 130 Moroccan students from the first year of the baccalaureate: 1 bac (2nd year of high school) distributed as follows:

- 80 students in the first year of the experimental sciences baccalaureate (abbreviated as 1sc.exp) are divided into three classes, ages 16 to 17.
- 50 students in the first year of the baccalaureate in mathematical sciences (1SM), divided into two classes (ages 16-17).

The choice of school level was based on the idea that students at this level, with these two branches, often engage in mathematical reasoning during the "notion of logic" lesson, where they explore different types of mathematical reasoning.

Data Collection Tools

We developed a test consisting of five problems to document and characterize students' inductive reasoning. The test was designed not only to collect answers but also to elicit different stages of inductive reasoning, ranging from observation and model identification to conjecture and generalization. It thus served as both a research instrument and a didactic tool, enabling us to capture the ways in which students mobilize inductive reasoning when confronted with mathematical tasks.

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The full version of the test is provided in Appendix 2, while Appendix 3 summarizes the objectives, task types, and detailed reasoning steps associated with each problem. For clarity, Table 2 presents a summary of the five problems, indicating their registers and main objectives.

problem	Register	Objective of the problem
Prob1	Digital + Algebraic	Use inductive reasoning to identify the sequence and induce its explicit formula.
Prob2	Digital + Algebraic	Use inductive reasoning to determine the units digit of powers (e.g., 13^n) and generalize the pattern.
Prob3	Digital	Use inductive reasoning to observe the data of n^2-n+11 and predict its behavior for all natural numbers.
Prob4	Digital + Algebraic	Use inductive reasoning to examine the behavior of functions x^n as x reverses -∞ and generalize the rule for their limits.
Prob5	Geometric + Algebraic	Use inductive reasoning to determine the number of sticks in a growing pattern and conjecture a general rule.

Table 2: Summary of Test Problems and Their Main Objectives

The Test Process

We administered a test of five problems to students individually in class at the end of the school year in May 2024, aiming to gather more results for analysis, given the particularity of the chosen school level (first year of the baccalaureate). The student has taken a reasonable basis of the acquired knowledge and new concepts at the level of the common core science and in the first year of the baccalaureate he is ready to analyze, think and use his knowledge and especially reason and also by the reason that in this level the student becomes more familiar with mathematical reasoning and logical thinking through the lessons of the notion of logic which is located at the beginning of the program. The estimated time to answer the test was 90 minutes. To collect more criteria on the students' inductive reasoning, we asked the students to write what they thought and answer in detail, explaining that it was a matter of providing reasoning rather than an immediate result of the questions. We also asked them to draft their answers on a copy.

Analysis Instrument

The analysis of student reasoning was guided by three main dimensions: (1) the structure of inductive steps, following Jeannotte's (2015) model of data, assertion, and rule; (2) the registers of semiotic representation mobilized (Duval, 1995); and (3) the curricular location of each task in the Moroccan secondary mathematics program. A complementary a priori analysis, presented in

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Appendix 4, illustrates how the analytical grid was applied and ensures transparency in the coding process.

Method of Analysis

We used a study based on the grid to analyze students' inductive reasoning. It aims to characterize the inductive reasoning developed by the student (the structural step of the reasoning) and to document the different stages of inductive reasoning (criteria of the grid) which also describes the processual step of the reasoning and this through a descriptive analysis: numbers, percentages and averages for each problem of the test.

- ❖ Example of the analysis of the reasoning given by a student according to the grid:

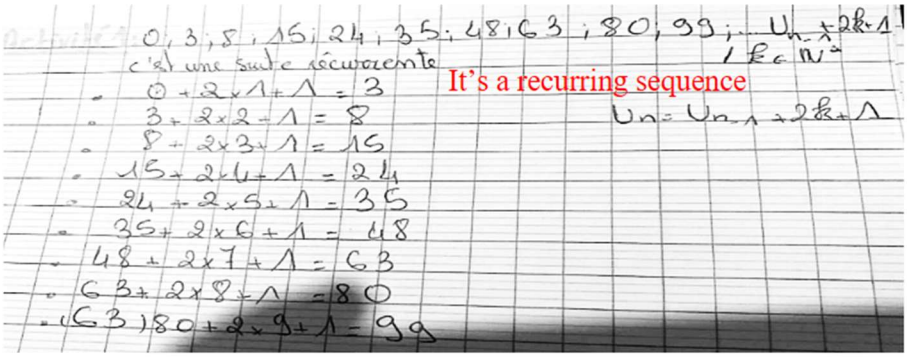


Figure 2: Example of inductive reasoning given by a student on problem

Analysis criterion (C)	C1: Case observation	C2: Identification of a model	C3: Identification of regularities	C4: Formulation of a conjecture	C5: Generalization
Approach to the reasoning process	F	F	NF	F	NF

Table 3: Criteria for analyzing students' reasoning; F: done; NF: not done; FE: done with error

This student employed inductive reasoning effectively to answer the question posed in Problem 1 of the test. We then analyzed the steps of this reasoning according to the grid, as shown in Table 4. The results indicated that the student followed inductive reasoning and completed most of the steps mentioned in the grid; however, he did not provide a generalization at the end. Inductive reasoning was considered successful if the student observed and formulated a conjecture, provided a generalization, or completed most of the steps and formulated a conjecture without generalizing. Conversely, inductive reasoning was considered unsuccessful if the student limited himself to simple observation or produced only a very brief conjecture. In this illustrative case, the student can therefore be qualified as having successfully employed inductive reasoning.

After presenting this example of student reasoning, it was important to clarify how the analytical grid was applied more broadly. For this reason, an a-priori analysis of the test tasks is provided in

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Appendix 4. This analysis presents one possible solution method for each problem and shows in advance how the grid was used to code reasoning steps. By including this complementary analysis, the coding procedure is made explicit and methodological transparency is ensured.

While both the student example and the a-priori analysis illustrate the application of the grid at the level of individual cases, the methodological framework extended well beyond these illustrations. The approach was not limited to a descriptive presentation of the sample, tasks, and procedures; rather, it was deliberately designed to investigate whether and how students were able to mobilize inductive reasoning. The test composed of five carefully selected problems, the explicit instructions emphasizing justification, and the analytical grid for coding reasoning steps, all constituted methodological interventions that directly shaped the outcomes. This integrated design allowed us to go beyond simple description and to characterize the presence and quality of inductive reasoning across the entire student sample.

RESULTS

Frequencies of the students' inductive structural step according to each problem.

Table 4 below shows the frequency with which students identified inductive reasoning in solving the 5 problems proposed in the test.

The first phase of our analysis involved reviewing the students' copies. We found students who did not answer all the problems and students who, despite answering the problems, provided a direct answer to certain questions, that is, an absence of a reasoning approach.

In Table 4, "used" refers to the number of students who employed the inductive step (observation, statement, or rule), while "not used" or "did not answer" refers to the number of students who either did not reason or did not answer the problem directly (resulting in an absence of response).

	Frequency of inductive step presence	
	Used	*Not used or not answered the problem*
Prob1	*50 (38.46%)*	*80 (61.53%)*
Prob2	*28 (21.53%)*	*102 (78.46%)*
Prob3	*102 (78.46%)*	*27 (19.23%)*
Prob4	*111 (85.38%)*	*19 (14.61%)*
Prob5	*100 (76.92%)*	*30 (23.07%)*
Total	***391 (60.15%)***	***259(39.84%)***

Table 4: Frequency of use of inductive step

For the five proposed problems and the 130 student scripts collected, a total of 650 effective resolutions were expected for analysis. However, 259 resolutions (39.84%) were excluded. Nearly half were removed because no solution appeared in the student's script (blank space or an explicit statement of inability to solve the problem), because only a brief answer was provided, or because the responses combined correct and incorrect elements without demonstrating any evidence of inductive reasoning.

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Regarding the rate of inductive reasoning use, the academic track had a significant impact on the percentage observed. Table 5 presents a visualization of the difference between the two tracks in the first year of the Baccalaureate.

	Frequency of the presence of inductive step according to the branches			
	Used		*Not used or not answered the problem*	
	1SM	*1EXP*	*1SM*	*1EXP*
Total reasoning according to the total copies analyzed	***230 (58.82%)***	***161 (41.11%)***	***20 (07.72%)***	***239 (92.27%)***
Total reasoning according to copies of the same branch	***230 (92%)***	***113 (28.25%)***	***20 (05.11%)***	***239 (59.75%)***

Table 5: Frequency of use of the inductive step by branch at the 1st-grade school level.

Among the two branches studied, students in the 1SM branch showed a significant use of inductive reasoning (92%). This can be explained by the fact that students in the SM stream, being specialized in mathematics, develop more in-depth skills in this area. They also demonstrate a more attentive and methodical attitude, which can be interpreted as a better mobilization of methodological and cognitive skills, allowing them to approach tasks more efficiently than their counterparts in the 1EXP branch.

Subsequently, the analysis focused exclusively on the **391 responses (60.15%)** that exhibited evidence of inductive reasoning, in accordance with the previously developed analysis grid.

Frequency of Steps Taken by Students for Each Problem

To determine the frequencies of the steps performed by students on the test problems, we identified the steps taken by each student in the 391 reasonings related to each test problem by filling in the previously presented grid. We present these frequencies in Table 6 and Figure 3:

	C1	*C2*	*C3*	*C4*	*C5*
Prob1	*50 (100%)*	*31 (62%)*	*13 (26%)*	*20 (04%)*	*7 (14%)*
Prob2	*25 (89.28%)*	*19 (67.85%)*	*6 (21.42%)*	*10 (35.71%)*	*2 (07.14%)*
Prob3	*102 (100%)*	*93 (91.17%)*	*55 (53.92%)*	*100 (98.03%)*	*95 (93.13%)*
Prob4	*111 (100%)*	*106 (95.49%)*	*68 (61.26%)*	*63 (56.75%)*	*62 (55.85%)*
Prob5	*100 (100%)*	*90 (90%)*	*64 (64%)*	*34 (34%)*	*26 (26%)*
TOTAL	***388 (99.23%)***	***339 (86.70%)***	***206 (52.68%)***	***227 (58.05%)***	***192 (49.10%)***

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Table 6: Frequency of inductive reasoning steps

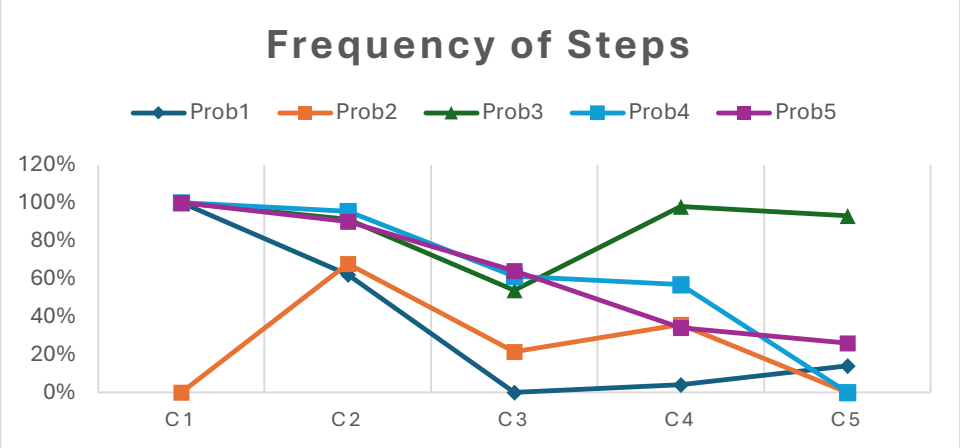
For criterion C1: Observation of cases, the results show that students who used the inductive step in their resolutions succeeded well in this initial step, with a success rate of **99.23%.** This shows their ability to make remarks and observations.

For criterion C2: Identification of a model, students were able to identify a model in 86.70% of the reasonings, with high percentages in problems 3, 4, and 5 and moderately high percentages in problems 1 and 2. In all the resolutions where the model identification step appears, there is already a case observation step, which allowed us to conclude that accurate data observation leads to the detection of the model.

For criterion C3: Identification of regularities, half of the solutions contain the identification of regularities, with most of these instances appearing in problems 3, 4, and 5, compared to lower percentages in problems 1 and 2. This is due, according to a brief interview conducted with students after the test, to the fact that problem 3 contains more cases, and problems 4 and 5 include, respectively, a table and a figure, which made the regularity more noticeable. So here, the identification is linked to the structure of the problem.

For criterion C4: Formulation of a Conjecture, the reasoning in which the formulation of conjectures was 58.05%, with slight moderation in problems 1, 2, and 5. According to the students' comments, these problems were slightly more challenging for them to identify compared to problems 3 and 4. Most students were able to find it, especially in the resolution of problem 3, with a large percentage achieving a score of **98.03%.**

For criterion C5: Generalization, we observed a low weighting, particularly in problems 1, 2, and 5, during the final step of inductive reasoning. However, in general, fewer than half of the reasons contain generalization, even in resolutions where students have already made a conjecture. So it is clear that the students have a confusion between making a conjecture and giving the generalization that is the expected rule. On the other hand, there are instances where students have reached generalization without going through the other steps, solely from observation (30.09%).



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Figure 3: Frequency of inductive reasoning steps per problem.

The frequency analysis of the steps across the five problems reveals a consistent trend, with similar increases and decreases from one problem to another. The most frequently employed steps (Figure 4) are: (C1) observation of given cases, (C2) model identification, and (C3) conjecture formulation.

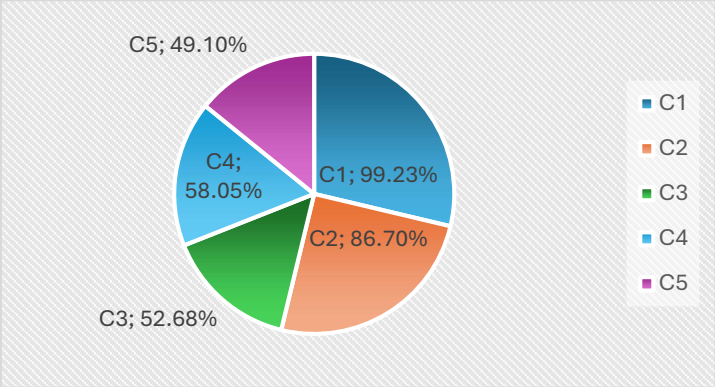


Figure 4. Frequency of inductive reasoning steps in the five problems.

The results previously obtained concern the students' copies where traces of inductive reasoning were identified. However, our study was conducted on a sample of 650 resolutions. To better visualize the results, Figure 5 presents the frequencies of the steps used by all students.

- **C1 (59%):** This category is the most dominant, accounting for more than half of the occurrences. This could indicate that step C1 is essential or very frequently used by students in their reasoning or resolutions.
- **C2 (52.15%)**: This step is also very frequent, although less than C1. It can play a complementary or secondary role in resolutions.
- **C3 (31.69%):** This step is used by approximately a third of students, which could indicate that it is relevant but less central than the previous steps.
- **C4 (34.92%):** Its frequency is similar to that of C3, suggesting it is also used occasionally or in specific cases.
- **C5 (29.53%):** This category is the least frequent, accounting for approximately a quarter of the occurrences. It could correspond to a more specific or advanced stage, used in particular situations.

These data show an uneven distribution in steps or categories, with a marked preponderance of steps C1 and C2. This suggests that students focus more on key steps in their solutions, while others, such as C4 and C5, are less systematically used.

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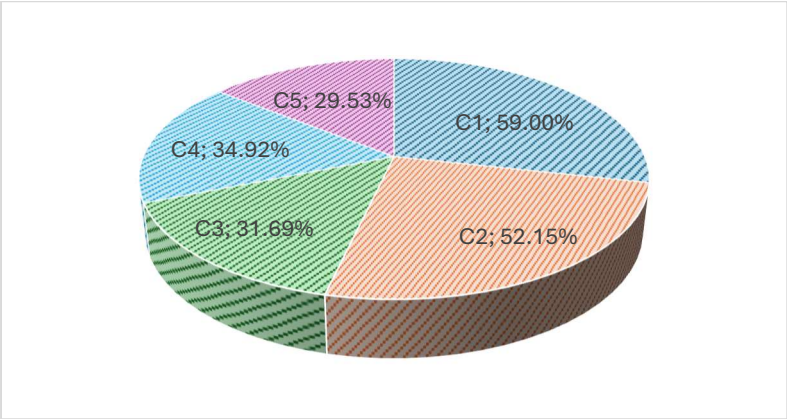


Figure 5: Frequency of steps in inductive reasoning.

These percentages reveal a clear hierarchy in the accessibility of the different stages of inductive reasoning. Nearly all students were able to observe cases and most could identify a model, but the proportion decreased significantly when moving to more advanced steps such as identifying regularities, formulating conjectures, and especially generalizing. This distribution suggests that while students are generally capable of initiating inductive reasoning, they struggle to bring it to completion. In particular, the transition from conjecture to generalization appears to represent a critical difficulty. These findings indicate that inductive reasoning is often only partially mobilized, with students demonstrating strengths in the initial phases but encountering obstacles in the later, more abstract stages.

Examples of Illustrations of Students' Reasoning

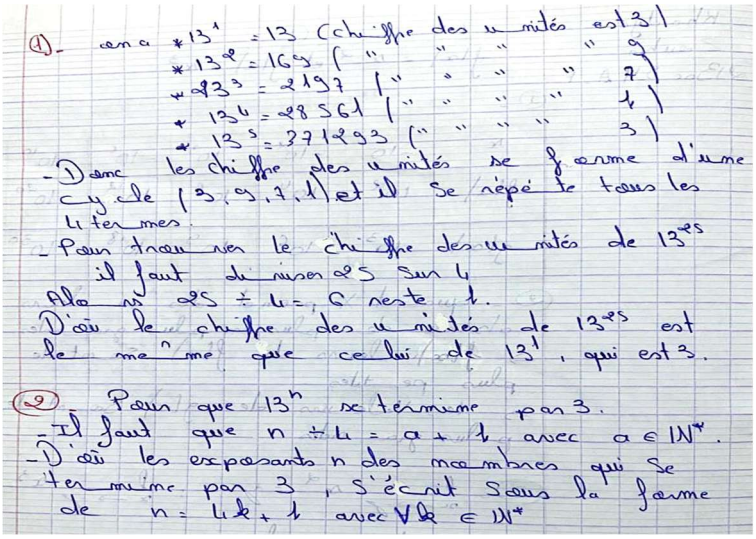
We present some of the students' productions that relate to inductive reasoning, as illustrated by the given problems (10 examples). We illustrate successful and unsuccessful inductive reasoning:

Inductive Reasoning Succeeds

- Student production 1 of branch 1SM on problem 02 succeeds without doing all the steps presented in the grid:

The student here has respected the structural step of inductive reasoning: data - statement - rule. For the steps of reasoning, we find that the student has done the step of observing the data well (we have ... The units digit is ...), then the identification of the model has been produced well (the unit's digits are repeated in cycles of 4), then the student has reformulated the conjecture. To find the units digit, we must divide by 4 and observe the remainder. Since the remainder of 25 divided by 4 is 1, the units digit is 3. Towards the end, the student provided a generalization of the power formula to have a unit digit equal to 1 ($4K + 1 /\forall K \in N$). In this production, we find that the student has not completed the step of identifying regularities; despite this, his reasoning is considered successful.

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Translation:

so, the unit's digits are formed by a cycle (3,9,7,1) and they repeat every 4 terms.

to find the units digit of... you have to divide 25 by 4

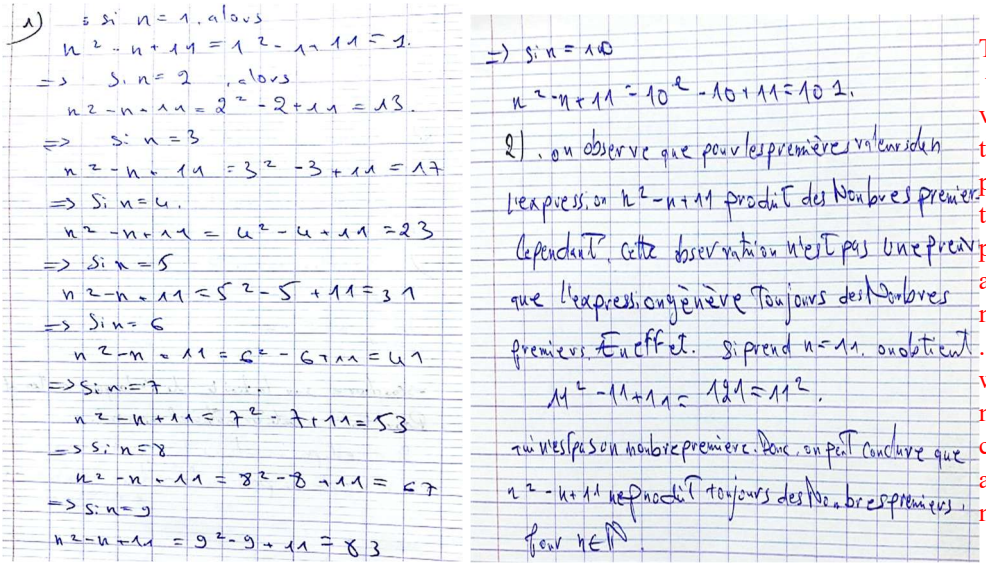
hence the units digit of... is the same

so that... ends in 3
it is necessary that ... with..
hence the exponents n of numbers that end in 3 are written in the form of...

Figure 6: Student 1 SM's successful production on test problem 2.

- Student production 2 of branch 1SM on problem 03, respecting all the steps and considering the epistemological value of the reasoning:

In problem 3, the student applied inductive reasoning in full accordance with the analysis grid—progressing through observation, model identification, and recognition of regularities—to formulate the generalization "product of prime numbers." Demonstrating an understanding of the epistemological limitation of induction, he acknowledged that conclusions drawn from limited data are only probable and accordingly refined his generalization while supplying a counterexample. This production is therefore considered successful in inductive reasoning and of high quality in terms of mathematical logic.



Translation:
we observe that for the first values of n
the expression ... produces prime numbers. However, this observation is not a proof that the expression always generates prime numbers. indeed, if we take .... we obtain ....
which is not a prime number. therefore, we can conclude that ... does not always produce prime numbers.

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Figure 7: Student 2 SM's successful solution to test problem 3.

- **Student production 3 (1EXP branch, problem 3):**

In this production, the student employed inductive reasoning by observing several cases of n, conjecturing that the expression n n^2-n+11 yields prime numbers, and generalizing that it is always prime for any n. Although the steps of model identification and recognition of regularities were not explicitly stated, the reasoning is deemed successful. Unlike student 2's production for the same problem, this one does not address the epistemological value of induction; the generalization is thus considered valid for the observed cases, but the reasoning as a whole remains merely plausible.

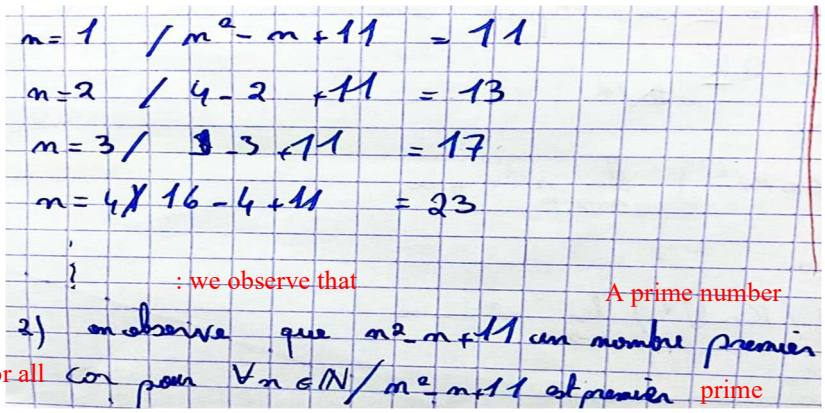
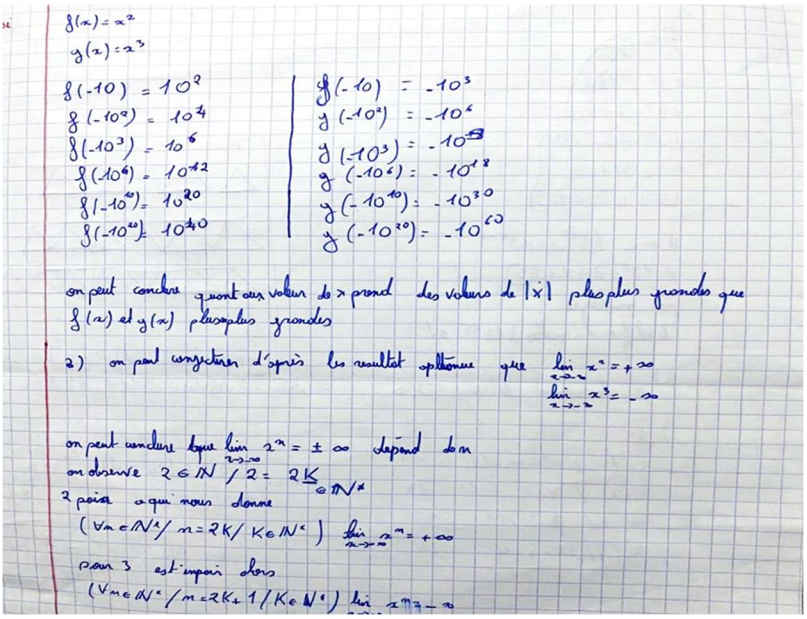


Figure 8: Student's successful production of EXP 3 on problem 3 of the test.

- Student production 4 of branch 1EXP on problem 04:

In identifying the usual limit, the student successfully applied inductive reasoning to generalize the rule. By calculating the values for each function, we observe and identify the model that emerges. In addition to the regularities between the two functions, the formulation of the conjectures related to each function was established as a result of the generalization (the expected rule).

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Translation:

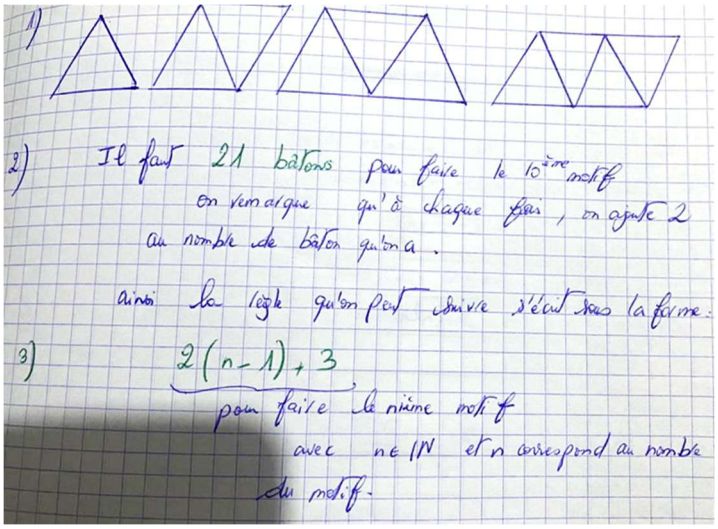
we can conclude that the value of x takes on values of .... and .... increasingly larger.

we can conjecture from the results that their....

Figure 9: Student's successful production of EXP 4 on problem 4 of the test.

- Student production 5 of branch 1SM on problem 05: successful inductive reasoning on a geometric problem.

The student illustrated the observation phase on the drawn patterns, then identified the pattern and regularities of adding 2 sticks each time. Then, he conjectured the number of sticks needed to create the 10th pattern and the nth pattern as part of a generalization process.



Translation:

2) it takes 21 sticks to make the 10th pattern

we notice that each time, we add 2 to the number of sticks we have

so the rule that we can follow is written in the form...

3) to make the nth pattern with .... and ... corresponds to the number of the pattern

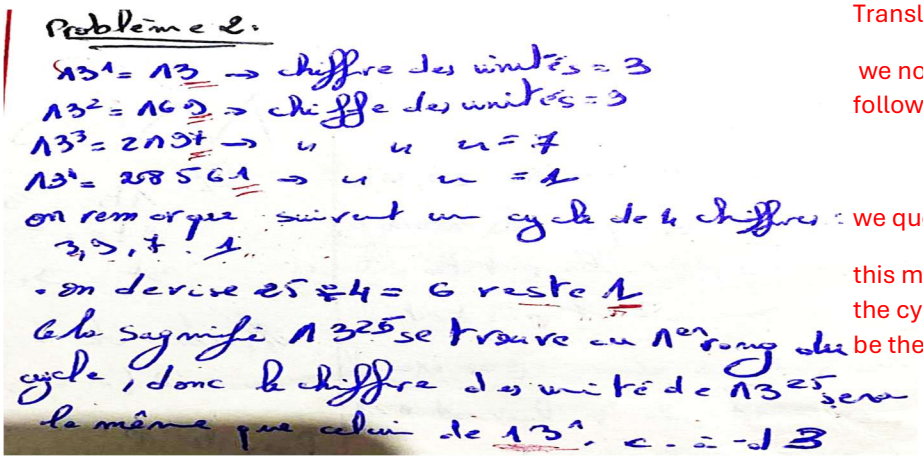
Figure 10: Student 5 SM's successful solution to problem 5 of the test.

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Inductive reasoning failed:

- Student production 6 of branch 1EXP on problem 02:

The student used inductive reasoning to answer question 1 of the problem. He completed all the steps, observing and analyzing the powers of 13. Then, he identified the patterns and regularities, revealing a conjecture about the units digit of 13 raised to the power of 25. Since division by 4 gives a remainder of 1, it follows that the units digit of 13 raised to the power of 1 is also 1. However, the student was unable to generalize or devise a sufficient rule for determining the exponent that ends in 3. This reasoning is considered unsuccessful.



Translation:

we notice that a cycle of 4 digits follows

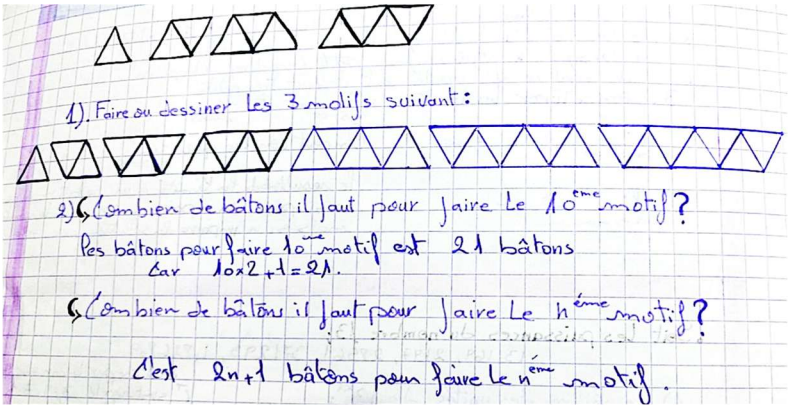
we quote ...

this means .... is at the 1st rank of the cycle, so the unit digit of .... will be the same as that of ...

Figure 11: Unsuccessful production of student 6's 1EXP on problem 2 of the test.

- Student production 7 of branch 1EXP on problem 05:

In this reasoning, the student was satisfied with two steps: drawing the patterns without detailing the observations and the conjecture itself, without specifying which model was followed or even the regularities detected that led to this conjecture. He lacks generality, and he did not give any information on the unknown for the conjectured expression. It is for these reasons that his reasoning is unsuccessful.



Translation:

1- make or draw the following 3 patterns:

2- how many sticks do you need to make the 10th pattern?

the sticks to make the 10th pattern are 21 sticks.

how many sticks do you need to make the nth pattern

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Figure 12: Unsuccessful production of student 7's EXP on problem 5 of the test.

- Student production 8 of branch 1EXP on problem 01:

In this case, the student's use of inductive reasoning was deemed unsuccessful. While the reasoning began appropriately—with accurate observations, correct completion of the numerical sequence through model detection, and identification of a regularity explained in the "analysis" section—it failed at the stages of conjecture and generalization (FE: formulated with error).

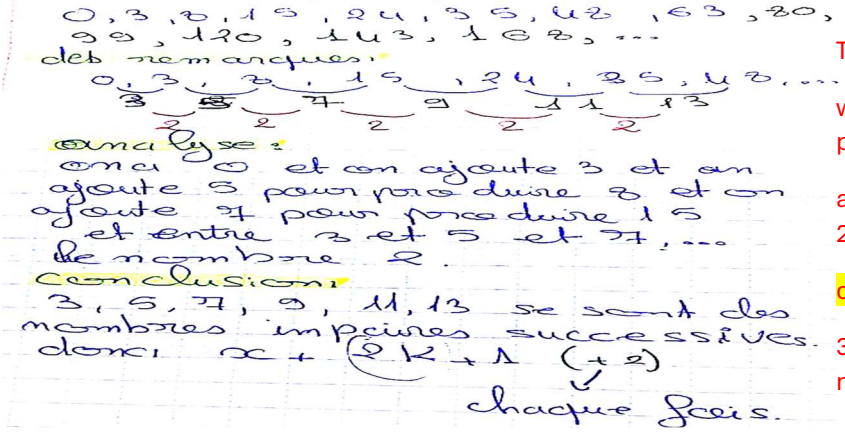


Figure 13: Failed production of student 8 EXP on test problem 1.

- Student production 9 from branch 1SM on problem 03:

Conversely, although this student's inductive reasoning followed all steps of the analysis grid, it was deemed unsuccessful. After performing calculations and observations, the student formulated a conjecture and proposed a concluding rule; however, the conjecture was incorrect. The problem was treated as a recurrence sequence, but the student failed to generalize correctly (FE), particularly concerning the primality of the values of n^2-n+11.

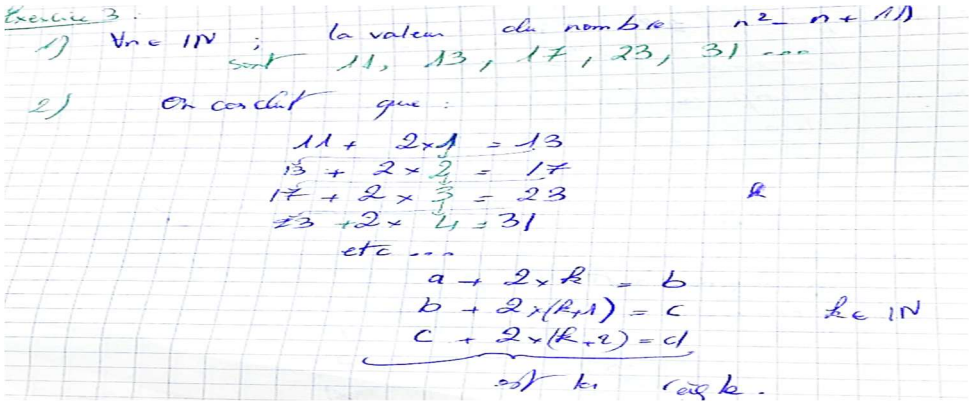


Figure 14: Unsuccessful production of student 9 SM on problem 3 of the test.

- Student production 10 of branch 1EXP on problem 02:

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Unsuccessful production: The student completed the observation step by circling the identified unit digits but failed to carry out model identification and recognition of regularities. Consequently, the generalization was flawed. Although the structural step of inductive reasoning was followed, the omission of the initial phases hindered the understanding and formulation of conjectures.

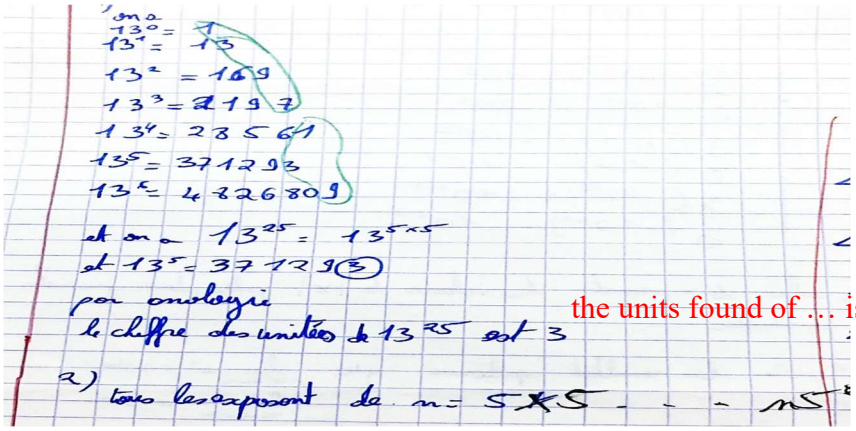


Figure 15: Student failed to produce 10 EXP on problem 2 of the test.

DISCUSSION

This study aimed to characterize the inductive reasoning of secondary school students, specifically those enrolled in the first year of the baccalaureate program. Drawing on the framework developed by Nhiry et al. (2023), which distinguishes between structural and processual dimensions of inductive reasoning, we applied a tailored analytical grid to examine students' responses to five mathematical problems. The test was administered to 130 students, and the results revealed both strengths and limitations in how inductive reasoning was mobilized in this context.

The analysis showed a clear hierarchy in the accessibility of inductive steps. Almost all students (99.23%) were able to make relevant observations, and the majority (86.70%) identified a model, demonstrating their capacity to recognize structure in specific cases. However, fewer students moved forward to more complex stages: 52.68% identified regularities, 58.05% formulated conjectures, and only 49.10% reached generalization. These results suggest that while students are generally comfortable initiating inductive reasoning, they encounter significant challenges in completing the reasoning cycle. The tendency to stop at conjecture without moving to generalization reflects both cognitive difficulties and didactic influences. In contexts where deduction dominates, inductive exploration often lacks explicit scaffolding, leaving students with limited opportunities to consolidate the later stages of inductive reasoning.

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The partial nature of inductive reasoning observed here is consistent with international findings. Jeannotte (2015) highlighted that the transition from conjecture to generalization remains a persistent obstacle in secondary mathematics, with many students formulating conjectures that are never generalized. Similarly, Cañadas and Castro (2007) reported that students often succeed in producing conjectures but fail to extend them to broader generalizations. Our results thus corroborate these findings and extend them to the Moroccan context, showing that the challenge of completing the inductive process is not context-specific but resonates with international evidence. This alignment reinforces the idea that inductive reasoning, while fundamental to mathematical thinking, requires explicit support in order to develop fully.

Importantly, these results were not the product of descriptive observation alone but emerged from a methodological design intentionally structured to elicit inductive reasoning. The five problems were chosen to stimulate students' capacity to identify patterns and generalize. Instructions emphasized explanation and justification, encouraging students to articulate their reasoning rather than merely provide answers. The analytical grid offered a systematic framework for coding reasoning steps, ensuring consistency in interpretation. Together, these interventions constituted methodological choices that directly shaped the findings, allowing us to characterize inductive reasoning across the entire student sample. Thus, the study contributes not only empirical results but also a replicable methodological approach for analyzing reasoning.

From a pedagogical standpoint, the findings highlight the need to rebalance classroom practices. In the current Moroccan curriculum, as in many international contexts, deduction is strongly emphasized, often at the expense of inductive reasoning. To make more room for inductive reasoning, teachers could implement several strategies. For instance, beginning new topics with open-ended exploratory tasks rather than formal definitions would invite students to observe and conjecture before rules are introduced. Encouraging students to make and test conjectures before formal proofs can help them see the value of induction as a pathway to deduction. Classroom discussions structured around observed patterns, counterexamples, and student reasoning would promote collaborative exploration and critical reflection. In addition, scaffolding prompts—such as asking "How might this pattern continue?" or "Can you propose a rule that fits all these cases?"—can explicitly guide students toward generalization.

These pedagogical implications are directly connected to the empirical findings. The fact that nearly all students succeeded in observation (99.23%) and most in model identification (86.70%) suggests that exploratory tasks can serve as effective entry points for inductive reasoning. At the same time, the difficulty observed in moving from conjecture (58.05%) to generalization (49.10%) highlights the need for scaffolding prompts and structured guidance to help students complete the reasoning cycle. By explicitly aligning teaching strategies with these observed strengths and weaknesses, instruction can better support the development of inductive reasoning.

Beyond the immediate findings, this study makes a theoretical and methodological contribution to the literature on mathematical reasoning. It demonstrates how an analytical grid, based on structural and processual dimensions, can be used to systematically identify and evaluate inductive reasoning in student work. This tool has the potential to be applied or adapted in other instructional

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contexts, enabling cross-national comparisons and fostering dialogue on how inductive reasoning is fostered in different educational systems. By connecting the Moroccan case to broader international research, this study contributes to a deeper understanding of the challenges and possibilities of inductive reasoning in mathematics education worldwide.

Overall, the study shows that inductive reasoning among secondary students is frequently partial: students readily engage in early stages such as observation and model identification but struggle to reach full generalization. This finding is consistent with international research and highlights a pressing didactic challenge. Addressing this challenge requires both methodological tools, such as the one proposed in this study, and pedagogical practices that deliberately scaffold inductive reasoning. In this sense, the study bridges theoretical analysis, empirical data, and practical implications, offering a contribution that is both contextually grounded and internationally relevant.

CONCLUSION

This study provides a detailed characterization of how Moroccan secondary school students engage in inductive reasoning when solving mathematical problems. Using a structured analysis grid grounded in recent theoretical work (Nhiry et al., 2023), we evaluated student responses across key reasoning stages: observation, model identification, regularity recognition, conjecture, and generalization. The findings showed that while many students demonstrated strong observational skills and could identify models, fewer advanced to formulating valid conjectures and generalizations.

One of this study's main contributions lies in its methodological approach. The use of a fine-grained, step-by-step analytical grid enabled a nuanced understanding of reasoning processes, moving beyond binary notions of success or failure. Rather than merely evaluating correctness, the study captured how students constructed generalizations from specific cases—an essential feature of inductive reasoning. This approach provides a replicable tool that can be adapted in other contexts to assess reasoning more systematically.

The results further revealed that although nearly 60% of students demonstrated inductive reasoning, the remaining 40% did not, particularly among the 1SEXP group. This finding raises concerns about the limited visibility of inductive reasoning in the Moroccan mathematics curriculum, which heavily emphasizes formal deduction. Similar challenges have been reported internationally: Jeannotte (2015) noted that the transition from conjecture to generalization remains a persistent obstacle, while Cañadas and Castro (2007) highlighted that students often formulate conjectures without extending them to valid generalizations. Our study therefore corroborates these findings and situates them within the Moroccan context, contributing to the international dialogue on mathematical reasoning.

While this research did not directly observe classroom teaching, the results point toward clear pedagogical implications. Inductive reasoning could be better supported by integrating open-ended exploratory tasks at the beginning of topics rather than starting with formal definitions,

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encouraging students to make and test conjectures before introducing proofs, structuring classroom discussions around observed patterns and counterexamples, and providing scaffolding prompts that explicitly support generalization. Such strategies could help students progress beyond observation to higher levels of inductive reasoning, complementing deductive approaches and fostering a more balanced mathematical experience.

This research thus establishes a foundation for future studies on the role of teaching methods and task design in advancing inductive reasoning. It emphasizes the importance of pedagogical strategies that recognize induction as both a precursor to proof and a central aspect of mathematical thinking. By bridging theoretical insights with instructional considerations, the study contributes to academic discussions on reasoning and offers guidance for mathematics educators seeking to strengthen students' reasoning skills.

Nevertheless, the study has limitations. The sample size was restricted, which may limit generalizability to other student populations or educational settings. In addition, relying solely on written assessments, without triangulation through classroom observations or student interviews, constrained the depth of analysis regarding the dynamics of reasoning. Future research could adopt longitudinal approaches to monitor the development of inductive reasoning over time, investigate how specific teacher interventions foster generalization, or analyze classroom conversations to capture the evolving nature of reasoning during collaborative work.

CONFLICTS OF INTEREST STATEMENT

All authors declare that they have no conflicts of interest.

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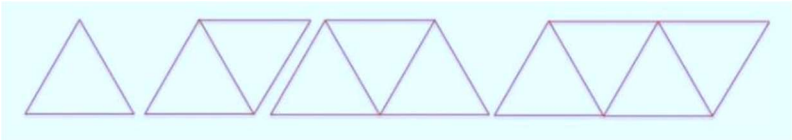
APPENDIX

Appendix 01: Distribution of Task Types in the Moroccan Program and Manuals for the First Year of the Baccalaureate: 1SC.EXP and 1SM.

1sc.exp	1SM
- Preparatory activities (introductory, reminder, and constructive activities). - Exercises: Invest in my knowledge - Exercises: integrating my learning - Application exercises - Exercises to reinforce acquired skills. - Improvement exercises.	-Preparatory activities (reminder activities; constructive activities). -Exercises: applications; reinforcement of acquired knowledge; improvement. -Problems.

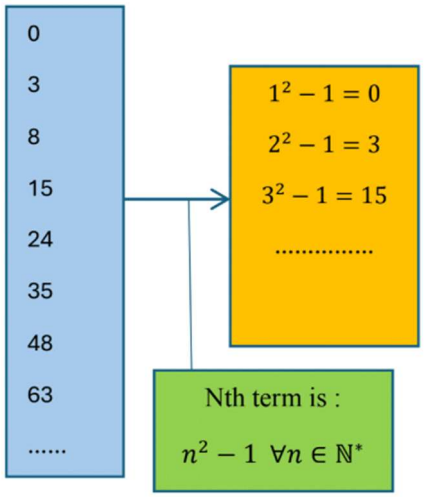
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Appendix 02: Task Testing

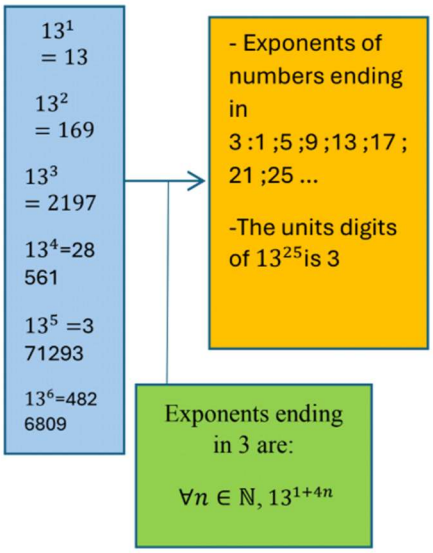
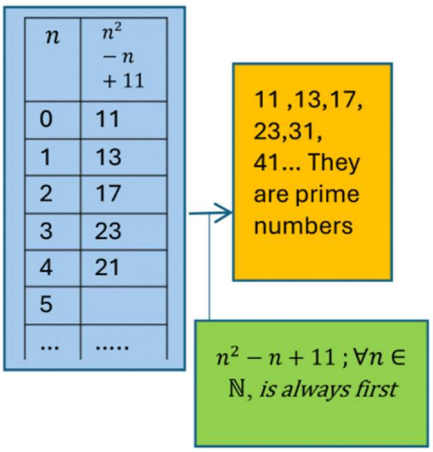
<table>
<tr><th>Codification</th><th>Problems</th></tr>
<tr><td>Prob1</td><td>Let the following sequence be:
0;3;8;15;24;35;48; 63….
Give the following 5 terms, then determine the nth term.</td></tr>
<tr><td>Prob2</td><td>Let the powers of the number 13 be: 13,169,2197, 28561, 371293, 4826809…
1) Determine the units digit of 13^{25}
2) Determine all the exponents of numbers ending in 3.</td></tr>
<tr><td>Prob3</td><td>1) Give $\forall n \in N$,the values of the number:
n^2-n+11
2) By justifying, what can we conclude about:n^2-n+11</td></tr>
<tr><td>Prob4</td><td>Let two numerical functions be defined by:
$f(x)=x^2$ And$g(x)=x^3$
1) Complete the following table:
<table>
<tr><td>x</td><td>-10</td><td>-10^2</td><td>-10^3</td><td>-10^6</td><td>-10^{10}</td><td>-10^{20}</td></tr>
<tr><td>$f(x)$</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>$g(x)$</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>
2) What can we conclude about the values of $f(x)$and $g(x)$when x takes on increasingly larger absolute values?
3) Guess the limits: $\lim\limits_{x\to-\infty} x^2$and $\lim\limits_{x\to-\infty} x^3$
4) What can we conclude about $\lim\limits_{x\to-\infty} x^n$; $\forall n \in N^*$</td></tr>
<tr><td>Prob5</td><td> 
1) Make or draw the following 3 patterns
2) How many sticks does it take to make the $10^{éme}$pattern?
3) How many sticks? if it is necessary to make the $n^{éme}$pattern?</td></tr>
</table>

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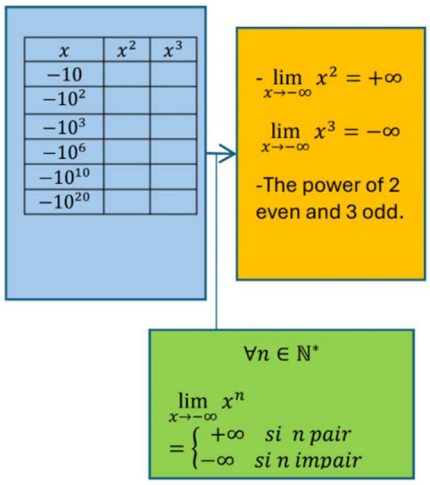
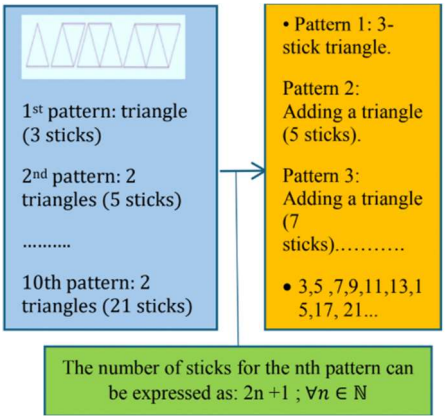
Appendix 03: Characteristics of Test Problems

	Inductive step structure	Register	Location in the 1EXP & 1SM program	Objective of the problem	Reason for choice
Prob1		Digital + Algebraic	Digital sequences (The 2 branches)	- Use inductive reasoning to determine the sequence of elements in the given list and induce its explicit form.	This problem is widely used for the introduction of numerical sequences in its explicit form. -an activity that allows the student to use inductive reasoning.

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Prob2		Digital + Algebraic	digital sequences (The 2 branches) Arithmetic for 1SM	- Determine the units digit of numbers raised to a power, here the example of 13 and the exponents which end with a precise digit using inductive reasoning.	-A task that uses inductive reasoning to determine the units digit of numbers with power. - Application exercise on the notion of divisibility and explicit sequence and also of congruence for 1SM.
Prob3		Digital	-Notion of logic (both branches) - Arithmetic for 1SM	-Make good observations of the data and arrive at a prediction about the nature of the expression n^2-n+11 for all natural integers n.	-An activity in which the use of inductive reasoning is necessary to determine the nature of the numbers in an expression, and which students encounter in the logic concept course.

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Prob4		Digital + Algebraic	Limits of numerical functions (both branches)	-See the behavior of the two type functions $x^n$when x reverses -∞. -Generalize the general rule for calculating the limit of these functions.	-An introductory activity on the usual limits in-∞ which allows students to use inductive reasoning.
Prob5		Geometric + Algebraic	Digital sequences (The 2 branches)	-Determine the number of sticks to construct a pattern. -Conjecture a general rule to calculate this number.	This problem allows the use of inductive reasoning in the context of numerical sequences using a geometric figure.

Appendix 04: ***A Priori Analysis of One Method for Solving Test Tasks According to the Grid.***

- ***Problem 01***

Problem 01	C1: Case observation	C2: Identifying a model	C3: Identification of regularities	C4: Formulation of a conjecture	C5: Generalization

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Let the following sequence be: 0;3;8;15;24;35;48 ; 63…. Give the following 5 terms then determine the nth term.	- The student must observe the first given terms. -The student adds more terms (5 terms or more) According to the observation that makes it.	-The student should observe that these terms are related to perfect squares. 1^2=1 2^2=4 3^2=9 4^2=16 5^2=25 6^2=36 -The student must compare this remark with the terms of the sequence and note that each term is expressed in the form of a square minus 1: 1^2-1=0 2^2-1=3 3^2-1=8 4^2-1=15 5^2-1=24	-The student must ensure that there is a regularity that each term is formed by a square minus 1, by observing other terms: 6^2-1=35 7^2-1=48 8^2-1=63 9^2-1=80 ….	-The student must formulate a conjecture for the nth term of the sequence isn^2-1	-The student must infer a general rule, returning to the first term he must notice that: 0^2-1=-1 and which is not the first term of the sequence (0), so the rule is true for all terms starting with 1 hence the rule is: $\forall\, n \in N^*; n^2$-1

- ***Problem 02***

Problem 02	**C1: Case observation**	**C2: Identification of a model**	**C3: Identification of regularities**	**C4: Formulation of a conjecture**	**C5: Generalization**
Let the powers of the number 13 be: 13,169,2197,28561 … 1) Determine the units digit of13^{25} 2) Determine all the exponents of the numbers that end in 3.	-The student must observe the powers given and at the same time observe the units digit: 13^1=13, the units digit is 3. 13^2=169, the units digit is 9. 13^3=2197 , ≤ *chiffre des unités es*	-The student must notice that the unit digits of the powers follow a cycle of 4 numbers:	-The student must identify the repeating units-digit cycle in powers of 13 (3, 9, 7, 1) and uses the remainder	Formulates a conjecture about the units digit of $13^n$by identifying the cyclic pattern (3, 9, 7, 1)	-The student must generalize that the units digit of $13^{25}$is 3. - The student must generalize that the

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	13^4=28561, the units digit is 7. 13^5=371293, the units digit is 3. 13^6=4826809, the units digit is 9.	3,9,7,1 and then it starts again.	when dividing the exponent by 4 to determine the units digit.	using modulo 4.	exponent ending in 3 is: $\forall n \in N$, 13^{1+4n}

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- ***Problem 03***

Problem 03	C1: Case observation	C2: Identification of a model	C3: Identification of regularities	C4: Formulation of a conjecture	C5: Generalization
1) Give $\forall n \in N$,the values of the number: n^2-n+11 **2) In justifying, what can we conclude about:n^2-n+11**	-The student must calculate and observe the particular cases of n^2-n+11, in answering question 1 For n=0 0^2-0+11=11 For n=1 1^2-1+11=11 For n=2 2^2-2+11=13 For n=3 3^2-3+11=17 For n=4 4^2-4+11=23 For n=5 5^2-5+11=31 …….	-The student must observe the results and other cases: n=0→ 11 n=1→ 11 n=2→ 13 n=3→ 17 n=4→ 23 n=5→ 31 n=6→ 41 n=7→ 53 … -The student must identify that The expression n^2-n+11produces prime numbers.	- The student must see the regularities that for n=0→ 11 n=1→ 11 n^2-n+11 gives 11 which is a prime number and for n ≥ 2 the values of are also prime numbers	-The student must guess that: 11, 13, 17, 23, 31, 41…these are prime numbers therefore n^2-n+11product of prime numbers seems to be true for the observed cases	-The student must generalize that$\forall n \in N$; n^2-n+11, *is always first*

- ***Problem 04***

Problem 04	C1: Case observation	C2: Identification of a model	C3: Identification of regularities	C4: Formulation of a conjecture	C5: Generalization

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Let two numerical functions be defined by: $f(x)=x^2$ and $g(x)=x^3$ **1) Complete the following table:** x | -10 | -10 | -10 | -10 | -10 $f(x$ | | | | | | $g(x$ | | | | | | **2) What can we conclude about the values of $f(x)$ and $g(x)$ when x takes on increasingly larger absolute values?** **3)Conjecture the limits:** $\lim_{x\to-\infty} x^2$ **And** $\lim_{x\to-\infty} x^3$ **4) What can we conclude about** $\lim_{x\to-\infty} x^n$; $\forall n \in N^*$	- Identifies that for large $\|x\|$, the function grows rapidly and reaches large values, and recognizes that $g(x)$ exhibits similar behavior while preserving its sign.	- Identifies that both functions are powers of x; $f(x)$ remains positive and increases rapidly, whereas $g(x)$ retains the sign of x and grows faster in magnitude.	-The student must identify that the more x increases in absolute value, the more $f(x)$ and $g(x)$ reach very large values. However, $f(x)$ it always remains positive while $g(x)$ the sign of follows x.	- Observes that as $\|x\|$ increases, both $f(x)$ and $g(x)$ grow significantly; however, $f(x)$ is always positive, whereas $g(x)$ retains the sign of x.	-The student must generalize that: $\forall n \in N^*$ $\lim_{x\to-\infty} x^n =$ $\begin{cases} +\infty & si\ n\ pair \\ -\infty & si\ n\ impair \end{cases}$

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- ***Problem 05***

Problem 05	C1: Case observation	C2: Identification of a model	C3: Identification of regularities	C4: Formulation of a conjecture	C5: Generalization
 1) Make or draw the following 3 patterns 2) How many sticks do you need to make the 10th pattern? 3) How many sticks does it take to make the same pattern?	- Observes that the triangles are equilateral and that each new pattern adds one triangle sharing a side with the previous figure; counts the number of sticks to identify the growing sequence (3, 5, …, 21).	-The student must identify that for each additional pattern, 2 sticks are added for the reason that each triangle shares a side with the previous triangle.	-The student must observe the regularity of the number of sticks: for each additional pattern there is an addition of 2 sticks.	-The student must conjecture that: Pattern 1: 3-stick triangle. Pattern 2: Adding a triangle (5 sticks). Pattern 3: Adding a triangle (7 sticks). ……. -The student must guess the formula 2n +1 which expresses the numbers of sticks for each pattern: 3,5,7,9,11,13,15,17, 21...	The student should generalize that the number of sticks for the $n^{éme}$ pattern can be expressed as: 2n +1 ; $\forall n \in N$

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Achievement Emotions among Indonesian Senior High School Students in Conducting Mathematics Activities: Differences of Gender, Age, Major, and Demographics

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Abstract: Achievement emotions significantly influence students' mathematics performance, yet many students struggle with emotional instability in learning environments. Exactly, the condition causes their mathematics achievement. Various factors, including gender, age, major, and demographics, predictively influence these emotional experiences. A quantitative survey was conducted to investigate achievement emotions in mathematics activities among Indonesian students. The Achievement Emotions Questionnaire in Mathematics was administered to 198 public upper secondary students at West Java, Indonesia, and data was analyzed using mean, standard deviation, skewness, kurtosis, independent sample t-test, and one-way MANOVA. Results of this study showed different achievement emotions by gender, major, and demographics in doing mathematics activities. For instance, girls exhibited greater pride, boys more frustration; science students felt prouder, social science peers more anxious; and urban students reported positivity, rural counterpart greater despair. Age, however, proved insignificant, underscoring the predominant influence of contextual and environmental determinants. Consequently, these findings highlight the need for targeted interventions, such as implementing problem-based learning and gamification for boy students, providing mentorship programs for girl students, using mathematics' real-world applications for social science students, and providing additional teacher training and fostering a more interactive classroom environment in rural schools, to support students' stable achievement emotions in mathematics learning.

Keywords: achievement emotions, age, demographics, gender, Indonesia, major, mathematics activities

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INTRODUCTION

Achievement emotions play a crucial role in students' success in mathematics. Stable achievement emotions, such as confidence, enjoyment, and pride, significantly contribute to higher mathematics achievement by fostering persistence, engagement, and motivation in learning (Suparman, Juandi, & Turmudi., 2025; Suparman, Juandi, et al., 2024). Students with positive and stable emotions are more likely to solve problems effectively. They also participate more actively in class and approach math tasks with a positive mindset. In contrast, students with fluctuating emotions may struggle to maintain focus and motivation, leading to inconsistent academic performance (Hollenstein et al., 2024). Stable emotions help students build resilience, allowing them to overcome challenges and persevere through difficult mathematical concepts. Furthermore, stable emotions create a supportive cognitive environment where students can process and retain mathematical knowledge effectively (Zhang, 2022). The ability to regulate emotions also reduces test anxiety, which is a common barrier to mathematical success (Lazarides & Raufelder, 2021). Without emotional stability, students may experience a fear of failure, leading to avoidance behaviors that hinder their learning progress. Therefore, fostering stable achievement emotions is essential for students to reach their full potential in mathematics and achieve high academic performance.

Many students, however, experience unstable achievement emotions in mathematics learning environments, leading to significant challenges in their academic performance (Liverani et al., 2023; Putwain et al., 2021). A common issue is that students' positive emotions, such as enjoyment and confidence, are often low, while their negative emotions, such as anxiety, frustration, and boredom, are high (Suparman, La'ia, et al., 2024). Several factors contribute to this instability, including teaching methods, assessment pressure, prior experiences, and peer influence. A rigid or unengaging teaching style may fail to create a stimulating and encouraging environment, causing students to feel disconnected from the subject (Putwain & Wood, 2023). Additionally, frequent high-stakes assessments increase students' stress levels, making them more prone to performance anxiety. Students with previous negative experiences in mathematics may develop a fixed mindset, believing they lack the ability to succeed in the subject, which further intensifies their emotional instability (Mata et al., 2022). Social comparison among peers also plays a role, as students who perceive themselves as less capable may experience a decline in self-confidence (Broda et al., 2023). These factors collectively contribute to unstable achievement emotions, ultimately affecting students' ability to perform well in mathematics.

Students' unstable achievement emotions in mathematics learning are also influenced by individual differences, such as gender, age, major, and demographic background (Basarkod et al., 2023; Chen, 2025; Hanin & Gay, 2023; Holm et al., 2020; Stilin et al., 2024). These differences can shape students' experiences and emotional responses to various mathematics-related activities, including attending mathematics classes, solving mathematics tests, and completing mathematics homework. For instance, gender differences in confidence and attitudes toward mathematics have been widely observed, with boy and girl students often demonstrating varying emotional reactions to mathematical challenges (Holm et al., 2020). Additionally, age plays a role, as younger students

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may lack the emotional regulation skills needed to manage frustration, while older students might experience increased pressure due to academic expectations (Hanin & Gay, 2023). Similarly, students' academic major and demographic background influence their perceptions and emotional responses to mathematics, with those from different educational tracks or socio-economic backgrounds facing unique challenges (Chen, 2025; Stilin et al., 2024). These variations suggest that understanding and addressing students' unstable achievement emotions require consideration of these individual factors.

Many empirical studies have explored how factors, such as gender, age, grade level, major, and demographics influence students' unstable achievement emotions in mathematics learning (Chen, 2025; Erturan & Jansen, 2015; Frenzel et al., 2007; Hanin & Gay, 2023; Holm et al., 2020; Stilin et al., 2024). Research conducted in various countries has consistently shown that these factors play a crucial role in shaping students' emotional responses to mathematical activities. Several studies consistently show that gender differences in mathematics-related emotions are common, with girls experiencing more anxiety and boys displaying more confidence (Erturan & Jansen, 2015; Holm et al., 2020). These patterns align with age-related findings (Hanin & Gay, 2023), suggesting that both gender and developmental factors influence emotional responses in similar ways. Additionally, students from different academic majors and socio-economic backgrounds demonstrate varying levels of emotional stability in mathematics learning (Chen, 2025; Stilin et al., 2024). While these studies provide valuable insights, there is a need for further research in the Indonesian context, particularly among Indonesian senior high school students regarding achievement emotions. Understanding how these factors influence achievement emotions of Indonesian students can contribute to a more comprehensive understanding of students' emotional experiences in mathematics and inform the development of culturally relevant interventions.

The present study aims to investigate the influence of gender, age, major, and demographic differences on achievement emotions among Indonesian senior high school students in mathematics learning. By examining how these factors contribute to students' emotional stability or instability, this research seeks to provide valuable insights into the emotional challenges faced by Indonesian students in mathematical settings. The findings of this study will contribute to the existing literature by offering a culturally specific perspective on achievement emotions in mathematics. Additionally, this research can inform educators and policymakers about the need for targeted interventions that address students' emotional needs, ultimately enhancing their mathematical achievement. By considering the role of gender, age, major, and demographics, present study supports the development of strategies that promote stable achievement emotions, leading to improved learning outcomes in mathematics education.

THEORETICAL FRAMEWORK

Achievement Emotions

Achievement emotions refer to the emotions experienced by students in academic settings, particularly in response to learning activities, performance outcomes, and academic experiences. Pekrun (2024) defines *achievement emotions* as emotions that are directly related to achievement

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activities or achievement outcomes, influencing students' motivation, engagement, and overall academic success. These emotions include both positive emotions, such as enjoyment and pride, and negative emotions, such as anxiety and frustration. According to Pekrun et al. (2023), achievement emotions play a crucial role in shaping students' cognitive processes, as they can either enhance or hinder concentration, memory retention, and problem-solving abilities. Researches emphasize that these emotions are not just momentary reactions but are shaped by personal beliefs, learning experiences, and environmental factors (Loderer et al., 2020; Zhang, 2022). Additionally, Lichtenfeld et al. (2023) highlight that achievement emotions can be stable over time or fluctuate depending on situational factors, such as classroom dynamics and assessment conditions. When students experience positive achievement emotions, they are more likely to engage deeply with the subject matter, develop a sense of competence, and persist in overcoming challenges. Conversely, negative achievement emotions can lead to avoidance behaviors, decreased motivation, and lower academic performance (Suparman et al., 2021). Synthesizing these perspectives, achievement emotions can be understood as fundamental psychological responses that influence students' attitudes, behaviors, and academic achievements in both direct and indirect ways. Recognizing the importance of achievement emotions is crucial for educators and researchers in designing learning environments that promote emotional stability and academic success.

Achievement emotions can be categorized based on three dimensions, including: valence, object focus, and activation (Pekrun, 2024). The first dimension, *valence*, distinguishes between positive and negative achievement emotions. Positive emotions, such as enjoyment, pride, and hope, enhance motivation and learning, while negative emotions, such as anxiety, frustration, and shame, often lead to disengagement and academic difficulties (Juandi & Suparman, 2024). The second dimension, *object focus*, classifies achievement emotions based on their connection to specific aspects of academic activities. Activity-related emotions, such as enjoyment in solving a mathematics problem, arise during the learning process, whereas outcome-related emotions, such as pride in achieving a high-test score or disappointment in failing, emerge after an academic result (Suparman et al., 2021). The third dimension, *activation*, refers to the level of physiological and psychological arousal associated with achievement emotions. Activating emotions, such as excitement and anxiety, increase alertness and readiness to act, whereas deactivating emotions, such as relaxation and boredom, reduce cognitive engagement and effort (Pekrun & Linnenbrink-Garcia, 2012). Pekrun (2024) emphasize that these dimensions interact dynamically, shaping students' emotional experiences in academic settings. For instance, a student who enjoys mathematics (positive, activity-related, activating) may remain highly engaged, whereas a student who feels frustrated with mathematics (negative, activity-related, activating) may struggle with motivation. Particularly in the present study, referring to the valence dimension, five positive emotions, including *enjoyment, pride, hope, contentment, and relaxation*, are involved, as well as five negative emotions, consisting of *anger, anxiety, hopelessness, shame, and boredom*. Understanding these types of achievement emotions allows educators to develop strategies to promote positive emotions while minimizing the impact of negative emotions on students' academic experiences.

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Mathematics Activities-related to Achievement Emotions

Achievement emotions, which include both positive and negative emotions, significantly influence students' experiences in mathematics learning (Dettmers et al., 2011; Schukajlow et al., 2017). Positive achievement emotions, such as enjoyment, pride, hope, contentment, and relaxation, can enhance students' motivation and engagement in mathematical activities. Conversely, negative emotions, such as anger, anxiety, hopelessness, shame, and boredom, can hinder students' ability to focus and perform well in mathematics (Di Leo et al., 2019; Goetz et al., 2012). These emotions emerge in various mathematics-related contexts, including attending mathematics classrooms, solving mathematics tests, and doing mathematics homework (Suparman, Juandi, et al., 2024). For instance, students who enjoy mathematics are more likely to actively participate in class discussions and confidently attempt problem-solving tasks. In contrast, students who experience high anxiety during mathematics tests may struggle with time management and question interpretation. A sense of pride can develop when students successfully complete challenging mathematical problems, reinforcing their motivation to continue learning. However, boredom can arise when students find mathematical content uninteresting or repetitive, leading to disengagement. Additionally, shame may occur when students feel embarrassed about making mistakes, preventing them from seeking help when needed. Therefore, understanding how achievement emotions influence students' mathematics experiences is essential for improving learning outcomes.

Students' achievement emotions in mathematics activities serve as critical predictors of their academic success in the subject. When students experience positive emotions, they are more likely to engage deeply with mathematical concepts and develop strong problem-solving skills. Enjoyment and hope encourage students to persist in learning even when faced with challenging problems. Pride in mathematical success reinforces self-efficacy, motivating students to take on more complex mathematical tasks (Camacho-Morles et al., 2021; Pekrun et al., 2017). On the other hand, negative emotions such as anxiety and hopelessness can impair cognitive functioning, making it difficult for students to concentrate and perform calculations accurately. Research suggests that mathematics anxiety is linked to working memory deficits, which reduce students' ability to retain and process mathematical information efficiently. Boredom can lead to decreased effort and avoidance behaviors, further hindering academic achievement in mathematics. Additionally, students who experience shame about their mathematics performance may develop a fear of failure, preventing them from seeking assistance or attempting challenging problems (Lichtenfeld et al., 2023; Raccanello et al., 2018). Motivation and emotional regulation play a crucial role in shaping students' learning behaviors, directly affecting their mathematics achievement (Suparman, La'ia, et al., 2024). Therefore, fostering positive achievement emotions and reducing negative emotions is essential for improving students' performance in mathematics.

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Differences of Gender, Age, Major, and Demographics in Achievement Emotions

Gender differences play a significant role in shaping students' achievement emotions in mathematics activities (Erturan & Jansen, 2015; Frenzel et al., 2007; Holm et al., 2020). In the present study, gender is categorized into boy and girl students, as these groups often exhibit different emotional responses toward mathematics. Research has shown that boy students generally report higher levels of confidence and enjoyment in mathematics, while girl students tend to experience higher levels of anxiety and self-doubt (Holm et al., 2020). Societal stereotypes that associate mathematical ability with boys can contribute to these emotional differences, influencing students' self-perceptions and attitudes toward the subject. Girl students may feel more pressure to prove their competence in mathematics, leading to increased anxiety and a fear of making mistakes. In contrast, boy students may be more likely to take risks in problem-solving and participate actively in mathematics discussions (Erturan & Jansen, 2015). Studies have also found that girl students are more likely to experience mathematics-related shame and hopelessness, particularly when facing complex mathematical problems (Erturan & Jansen, 2015; Frenzel et al., 2007). Additionally, teacher expectations and classroom environments can reinforce gender differences in achievement emotions, either encouraging or discouraging students' confidence in their abilities. Addressing these gender disparities is essential to creating a more inclusive mathematics learning environment (Dettmers et al., 2011). By understanding how boy and girl students differ in their emotional responses to mathematics, educators can develop targeted interventions to support all students' emotional well-being and academic success.

Age differences also influence students' achievement emotions in mathematics learning (Bekker et al., 2023; Hanin & Gay, 2023; Sachisthal et al., 2021). In the present study, age is categorized into four groups, including 15-year-old, 16-year-old, 17-year-old, and 18-year-old students, each of whom experiences mathematics-related emotions differently. Younger students, such as those aged 15 and 16, may exhibit higher levels of enjoyment and curiosity in mathematics due to their enthusiasm for learning new concepts. However, as students grow older, academic pressure and increased difficulty in mathematical content can lead to higher levels of anxiety and frustration (Hanin & Gay, 2023). Research has shown that 17- and 18-year-old students are more likely to experience mathematics-related hopelessness and boredom, particularly if they struggle with complex mathematical concepts or feel unprepared for standardized tests (Bekker et al., 2023). Emotional regulation skills also develop with age, meaning older students may be better equipped to manage their emotions during stressful mathematical activities. However, external factors such as college entrance examinations and career expectations can increase anxiety levels among older students. Younger students may be more likely to seek help from teachers and peers, whereas older students may hesitate due to concerns about self-image (Sachisthal et al., 2021). Understanding the emotional experiences of students across different age groups can help educators tailor instructional strategies to support students' mathematics learning at various stages of their academic development.

Academic major also significantly influences students' achievement emotions in mathematics activities (Stilin et al., 2024). In the present study, major is categorized into natural sciences and

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social sciences, as these fields involve different levels of engagement with mathematics. Students in the natural sciences often experience higher levels of enjoyment and pride in mathematics, as their field of study requires frequent mathematical applications. These students are more likely to develop positive emotions toward mathematics due to their familiarity and confidence in problem-solving (Adigun et al., 2024). Conversely, students in the social sciences may view mathematics as less relevant to their future careers, leading to lower levels of motivation and higher anxiety (Jenßen et al., 2023). The perception of mathematics as a difficult subject can contribute to frustration and hopelessness among social sciences students, particularly when they struggle to understand abstract concepts. Research suggests that students in mathematics-intensive fields, such as physics and engineering, tend to exhibit greater emotional resilience in mathematics compared to those in humanities-based fields (Suparman, Juandi, et al., 2024). Social sciences students may also be more prone to mathematics boredom, especially if they do not see direct applications of mathematical concepts in their studies (Suparman et al., 2021). The different emotional responses between natural sciences and social sciences students highlight the importance of addressing discipline-specific needs in mathematics education. By understanding how students' academic majors influence their achievement emotions, educators can design instructional approaches that cater to diverse learning needs.

Demographic differences, particularly school location, play a crucial role in shaping students' achievement emotions in mathematics learning (Chen, 2025). In the present study, demographics are categorized into urban and rural schools, as students in these environments face different educational challenges. Students in urban schools often have access to better educational resources, qualified teachers, and extracurricular activities that enhance their mathematics learning experience. As a result, urban students may exhibit higher levels of confidence and enjoyment in mathematics (Merrick & Fyfe, 2023). In contrast, students in rural schools may face limited access to quality mathematics instruction, leading to higher levels of anxiety and hopelessness. Research indicates that students from rural backgrounds are more likely to experience mathematics-related boredom due to a lack of engaging learning opportunities and advanced coursework (Gildehaus & Jenßen, 2023). Additionally, socioeconomic disparities between urban and rural schools can impact students' emotional responses to mathematics, with rural students experiencing greater frustration due to financial and infrastructural constraints. Cultural attitudes toward mathematics also vary between urban and rural settings, influencing students' self-perceptions and emotional engagement with the subject (Gur et al., 2023). Addressing these demographic disparities is essential for promoting equal access to quality mathematics education. By understanding how school location influences students' achievement emotions, educators and policymakers can develop strategies to support students from diverse backgrounds in achieving success in mathematics.

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METHOD

Research Design

The present study employed a quantitative survey approach to investigate achievement emotions among Indonesian senior high school students in mathematics-related activities. This method was chosen to systematically analyze students' emotional experiences while attending mathematics classrooms, solving mathematics tests, and doing mathematics homework. The study specifically examined how gender, age, major, and demographic background influenced students' achievement emotions, allowing for a comprehensive understanding of emotional differences across various student groups. A quantitative survey was deemed appropriate because it enables researchers to collect large-scale, standardized data, ensuring objective measurement and statistical analysis of students' emotional responses (Cohen et al., 2018). Additionally, the survey method allowed for efficient data collection from a broad sample of students, enhancing the study's generalizability to the wider population of Indonesian senior high school students. The structured format of a quantitative survey also ensured high reliability and consistency, reducing potential biases in data interpretation (Creswell, 2013). Therefore, a quantitative survey was the most suitable approach to achieve the study's objective of systematically examining achievement emotions in mathematics learning.

Participant

The participants were 198 Indonesian students at public upper secondary schools in West Java selected by purposive sampling. They consisted of 55.05% girl students aged 15 to 17 years (M = 16.23 year, SD = 2.13) and 44.95% boy students aged 15 to 18 years (M = 16.89, SD = 1.74). Moreover, 51.52% students took courses focusing on the natural sciences (e.g., physics, chemistry, and biology), whereas 48.48% students took courses focusing on the social sciences (e.g., economics, sociology, and geography) but all of them were absolutely required to take mathematics as a fundamental course. Additionally, they were differentiated to be 18.18% students aged 15 years, followed by 51.01% students aged 16 years, 28.79% students aged 17 years, and 2.02% students aged 18 years. Subsequently, 48.99% of them lived in the urban area and 51.01% of students lived in the rural area.

Instrument

Achievement Emotions Questionnaire in Mathematics (AEQ-M) was used to measure students' achievement emotions in following mathematics activities. The AEQ-M used in this study was specifically adapted and developed to measure the mathematics achievement emotions of senior high school students. The questionnaire was based on the work of Bieleke et al. (2022) and Pekrun et al. (2023) and reworked by Suparman, Juandi, Turmudi et al. (2025), consisting of 68 items measured on a 4-point Likert scale, ranging from 1 (strongly disagree) to 4 (strongly agree) (See Appendices). Of these, 52 items covered seven categories of achievement emotions: enjoyment, anxiety, pride, shame, hopelessness, anger, and boredom. The remaining 16 items focused on three additional categories: hope, contentment, and relaxation. Following the framework of Pekrun

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(2016), the emotions were grouped into five positive categories (enjoyment, pride, hope, contentment, relaxation) and five negative ones (anxiety, anger, shame, hopelessness, boredom). The questionnaire was designed to assess students' emotional experiences during mathematics-related activities, such as attending classes (e.g., I find the material discussed in math class very interesting; I am proud of my contributions in math class), taking exams (e.g., I am optimistic that I can complete the math test; I am satisfied with my performance in math class), and completing homework (e.g., I would rather give up on math homework; I do not really care about the math homework given).

Procedure

The AEQ-M was initially translated into Indonesian by two graduate students fluent in English and Indonesian with expertise in mathematics education. The 68 items were adjusted to align with the Indonesian cultural and educational context. To ensure content validity, the translated questionnaire was reviewed by three experts specializing in educational psychology, and guidance and counseling. After revisions, the finalized Indonesian version was administered to senior high school students in West Java, Indonesia.

Data Analysis

Descriptive statistics, including mean, standard deviation, skewness, and kurtosis, were used to summarize and describe students' achievement emotions in mathematics activities, considering gender, age, major, and demographics. These descriptive statistics allowed for a clear overview of students' achievement emotions, facilitating comparisons between different groups (Creswell & Creswell, 2018). By analyzing the distributional properties of the data, researchers could determine whether further inferential statistical techniques were appropriate for examining group differences. To examine differences in students' achievement emotions based on gender, age, major, and demographics, the study employed inferential statistical techniques, specifically the independent sample t-test and one-way multivariate analysis of variance (MANOVA). The independent sample t-test was used to compare the mean achievement emotions between two groups, such as boy and girl students or urban and rural schools or natural and social sciences, determining whether significant differences existed. One-way MANOVA was conducted to assess the combined effect of age categories (15, 16, 17, and 18 years old) on multiple dependent variables, including positive and negative achievement emotions (Rutherford, 2011). By employing both descriptive and inferential statistical methods, this study provided a comprehensive examination of achievement emotions among Indonesian senior high school students in mathematics activities. All of the calculations in this study were promoted by a few software (e.g., SPSS ver. 26 and Ms. Excel).

RESULTS

To describe positive and negative emotions among students in following mathematics activities, including: attending mathematics classroom, solving mathematics test, and doing mathematics homework, some descriptive statistics, such as mean, deviation standard, skewness, and kurtosis,

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were applied (See Table 1). As seen in Table 1, positive emotions (enjoyment, pride, hope, contentment, and relaxation) are generally higher in classroom and test settings, with homework showing the lowest enjoyment (M = 2.358) and relaxation (M = 2.502), suggesting that students may find homework more stressful or less engaging. In contrast, negative emotions (anger, anxiety, shame, hopelessness, and boredom) are more prominent during tests and homework, with the highest anxiety levels during homework (M = 2.493) and the highest anger levels during tests (M = 2.164), indicating that students experience significant pressure in these settings. The skewness values show that emotions are mostly symmetrically distributed, though hopelessness and boredom exhibit slight right-skewness, meaning some students experience particularly high negative emotions. Negative kurtosis across most emotions suggests a broad range of emotional responses rather than a strong concentration around the mean. Overall, students experience moderate positive emotions but also significant negative emotions, particularly in high-pressure activities like tests and homework, emphasizing the need for strategies to enhance emotional well-being in mathematics learning.

To examine gender difference on students' achievement emotions in following mathematics activities, independent sample t-test was applied (See Table 2). The results in Table 2 indicate significant gender differences in students' achievement emotions during mathematics activities. Girl students reported higher positive emotions, particularly enjoyment in classrooms ($p < 0.05$), pride in homework ($p < 0.05$), and contentment in homework and overall ($p < 0.05$), suggesting greater engagement and satisfaction. Boy students exhibited significantly higher negative emotions, including anger in homework, tests, and overall ($p < 0.05$) and boredom in homework ($p < 0.05$), indicating frustration and disengagement. Girl also experienced greater relaxation in classrooms ($p < 0.05$), implying that boy students may feel more stress in this setting. While no significant gender differences were found in overall anxiety, shame, or hopelessness, the findings suggest that emotional disparities are most pronounced in homework and classroom settings, emphasizing the need for gender-responsive strategies to improve students' emotional well-being in mathematics learning.

To examine major difference on students' achievement emotions in following mathematics activities, independent sample t-test was applied (See Table 3). The results in Table 3 indicate significant differences in achievement emotions between natural sciences and social sciences students in mathematics activities. Natural sciences students reported significantly higher pride in tests ($p < 0.05$) and overall ($p < 0.05$), as well as greater relaxation during tests ($p < 0.05$), suggesting they feel more confident and at ease in mathematics-related tasks. Social sciences students exhibited significantly higher shame overall ($p < 0.05$), implying they may experience greater self-doubt and discomfort in mathematics activities. While there were no significant differences in enjoyment, hope, contentment, or anxiety, natural sciences students consistently showed slightly higher positive emotions, whereas social sciences students had slightly elevated negative emotions, particularly in test situations. These findings suggest that students' academic major influences their emotional experiences in mathematics, with natural sciences students generally displaying more confidence and engagement, while social sciences students report greater emotional struggles, particularly in test-related contexts.

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To examine demographics difference on students' achievement emotions in following mathematics activities, independent sample t-test was applied (See Table 4). The results in Table 4 reveal significant differences in achievement emotions between students from urban and rural schools during mathematics activities. Urban students reported significantly higher positive emotions, including enjoyment in classrooms and overall ($p < 0.05$), pride in homework, tests, and overall ($p < 0.05$), hope in homework, tests, and overall ($p < 0.05$), and contentment in classrooms, homework, and overall ($p < 0.05$), indicating greater confidence and engagement in mathematics. Conversely, rural students experienced significantly higher negative emotions, such as anger in tests ($p < 0.05$), anxiety in classrooms, tests, and overall ($p < 0.05$), shame in tests ($p < 0.05$), hopelessness in classrooms, homework, and overall ($p < 0.05$), and boredom in all activities ($p < 0.05$), suggesting greater emotional struggles with mathematics. Urban students also exhibited lower overall negative emotions ($p < 0.05$), while rural students displayed more frustration and disengagement, particularly in test-taking and classroom settings. These findings suggest that school location influences students' emotional experiences in mathematics, with urban students generally showing higher motivation and confidence, while rural students may require additional support to manage negative emotions and enhance engagement in mathematics learning.

To examine age difference on students' achievement emotions in following mathematics activities, one-way MANOVA was applied (See Table 5). The results in Table 5 show that age differences do not significantly impact achievement emotions in mathematics activities, as indicated by the lack of statistically significant F-values across all emotions. Younger students (15 and 16 years old) reported slightly higher enjoyment, hope, and relaxation, particularly in classroom and homework settings, while older students (17 and 18 years old) showed slightly higher pride and contentment, especially during tests. Negative emotions, such as anxiety, boredom, and hopelessness, were relatively stable across all age groups, with minor variations in different activities. Although 18-year-old students exhibited the highest levels of pride and contentment in tests, their sample size was very small ($N = 4$), limiting the reliability of these differences. Overall, the findings suggest that students' emotional responses to mathematics activities remain consistent across different ages, indicating that other factors, such as gender, major, and demographics, may have a stronger influence on achievement emotions.

DISCUSSION

The findings reveal that there are significant gender differences in achievement emotions among students in mathematics activities, aligning with previous research on gender disparities in emotional experiences related to mathematics. Girl students exhibited higher levels of enjoyment, pride, contentment, and relaxation, particularly in classroom and homework settings, suggesting a greater sense of engagement and motivation. This supports the findings of Goetz et al. (2012), who reported that girl students tend to display stronger positive emotions in structured learning environments, such as classrooms, where social interaction and teacher support are present. However, boy students demonstrated higher levels of anger and boredom, especially in homework and test situations, indicating greater frustration and disengagement. These results are consistent with the study by Frenzel et al. (2007), which found that boy students are more likely to experience

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negative emotions due to lower perceived relevance and interest in mathematics homework. The differences in relaxation levels, with girl students feeling more at ease in classroom settings, suggest that boys may experience greater pressure or stress in structured learning environments. Additionally, previous research has shown that mathematics anxiety is more common among girls (Ali & Das, 2024; Erturan & Jansen, 2015), but in the present study, no significant gender differences were found in overall anxiety levels, suggesting that both boy and girl students experience similar levels of anxiety in mathematics learning. These findings imply that gender-specific interventions are necessary, particularly to help boy students manage frustration and to provide girl students with continued support in maintaining positive emotions. Addressing these emotional differences may contribute to a more balanced and inclusive mathematics learning environment.

Students' academic major significantly influences their achievement emotions in mathematics, particularly in terms of pride and relaxation. Natural sciences students reported higher levels of pride in test-taking and overall mathematics activities, which aligns with the findings of Stilin et al. (2024), who suggested that students in STEM-related fields develop greater confidence in mathematics due to its direct relevance to their studies. This also supports research by Pekrun et al. (2023), which found that students with strong intrinsic motivation for mathematics tend to experience more positive emotions, including pride, when performing well in assessments. In contrast, social sciences students displayed higher levels of shame, particularly in overall mathematics activities, indicating a lack of confidence or discomfort with the subject. These findings are consistent with the work of Rodríguez et al. (2020), which found that students who do not view mathematics as essential for their academic or career goals tend to experience higher levels of negative emotions. The significant differences in relaxation during tests, where natural sciences students felt more at ease than their social sciences counterparts, suggest that frequent exposure to mathematics in STEM-related fields may help students develop emotional resilience in high-stakes situations. Although no major differences were observed in overall anxiety and enjoyment, the trend suggests that social sciences students may require additional support to reduce negative emotions and enhance engagement in mathematics activities. These findings highlight the need for curriculum adjustments and targeted interventions to help social sciences students develop a more positive emotional connection to mathematics.

Students' demographic backgrounds, particularly their school location (urban vs. rural), significantly impact achievement emotions in mathematics. Urban students reported higher enjoyment, pride, hope, contentment, and positive emotions overall, particularly in classroom and homework settings. These results are in line with the findings of Chen (2025), who noted that students in urban schools often have better access to resources, experienced teachers, and supportive learning environments, which contribute to higher positive emotions. Conversely, rural students exhibited higher levels of anger, anxiety, shame, hopelessness, and boredom, particularly in test situations, suggesting greater emotional struggles with mathematics. This aligns with research by Schukajlow et al. (2023), which found that students from underprivileged educational backgrounds often experience higher levels of mathematics anxiety due to limited instructional support and fewer academic opportunities. The lower relaxation levels among rural students in

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classroom settings indicate that they may experience greater stress or discomfort in structured learning environments, possibly due to a lack of confidence or prior negative experiences. Furthermore, the higher levels of boredom and hopelessness in rural students support previous research showing that students in resource-limited schools may struggle with motivation due to less engaging teaching methods and a lack of advanced learning opportunities (Ahmed et al., 2013). These findings emphasize the need for targeted interventions in rural schools, such as improved teaching strategies, access to supplementary learning resources, and emotional support programs to help students manage negative emotions in mathematics learning.

Age does not play a significant role in shaping students' achievement emotions in mathematics activities, as there were no statistically significant differences across the four age groups (15, 16, 17, and 18 years old). This contrasts with previous studies, such as the work of Hanin and Gay (2023), which suggested that younger students tend to exhibit higher enjoyment and positive emotions in mathematics, while older students experience increased anxiety and frustration due to growing academic pressures. However, the present study found that enjoyment, pride, hope, contentment, and relaxation remained relatively stable across all age groups, indicating that students' emotional experiences in mathematics learning may be influenced more by external factors such as gender, academic major, or school demographics rather than age alone. This aligns with research by (Pekrun, 2024), who proposed that achievement emotions are shaped by students' personal experiences and environmental factors rather than solely by age-related development. Although 18-year-old students showed slightly higher levels of pride and contentment in test situations, their small sample size limits the generalizability of this finding. Similarly, negative emotions such as anxiety, shame, and hopelessness remained relatively consistent across different age groups, contradicting the assumption that older students necessarily experience more mathematics-related stress. While the study did not find statistically significant differences across age groups, the limited representation of older students suggests caution in interpreting these results as conclusive. Future research should explore whether other psychological or contextual factors contribute to students' emotional experiences in mathematics learning across different age groups.

Implications to Psychology in Mathematics Education

The findings and discussion above have several theoretical implications for understanding students' achievement emotions in mathematics activities, considering gender, age, major, and demographics. First, the results support Control-Value Theory of Achievement Emotions of Pekrun (2024), which suggests that students' emotions are shaped by their perceived control over learning and the value they place on mathematics. The significant gender differences in emotions align with previous research indicating that boy and girl students experience mathematics differently, with boys showing more frustration and girls exhibiting higher engagement (Goetz et al., 2012). The academic major differences further reinforce the idea that students who perceive mathematics as relevant to their future careers (e.g., natural sciences students) are more likely to experience pride and confidence, as suggested by Chen (2025). Moreover, the demographic disparities in emotions, where rural students reported more negative emotions than urban students,

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highlight the importance of educational environments in shaping achievement emotions, consistent with Schukajlow et al. (2023). Interestingly, the lack of significant differences across age groups challenges the assumption that achievement emotions naturally evolve over time, suggesting that emotional stability in mathematics learning is influenced more by external factors. These findings contribute to the broader theoretical understanding of how personal and contextual factors shape students' emotional experiences in mathematics learning. Future studies should explore additional psychological and motivational factors that may interact with gender, major, and demographics to influence achievement emotions. Integrating emotional, cognitive, and motivational frameworks can provide a more holistic model for explaining students' mathematics learning experiences.

The practical implications of these findings highlight the need for targeted interventions to support students' achievement emotions in mathematics learning, particularly based on gender, major, and demographics. Given that boy students experience more frustration and boredom, teachers should implement engaging instructional strategies, such as problem-based learning and gamification, to enhance boy students' interest in mathematics. For girl students, who report higher engagement but also experience mathematics anxiety, teachers should create confidence-building activities, such as growth mindset interventions and mentorship programs, to help sustain their positive emotions. The differences between natural sciences and social sciences students suggest teachers to implement mathematics instruction, tailored to increase social sciences students' confidence and reduce their feelings of shame, potentially through real-world applications of mathematics in non-STEM fields. Since rural students report higher anxiety, hopelessness, and boredom, policymakers and educators should focus on improving access to quality mathematics resources, providing additional teacher training, and fostering a more interactive classroom environment in rural schools. Additionally, the lack of significant emotional differences by age implies that emotional support strategies, conducted by teachers should be implemented at all grade levels, rather than assuming that younger students require less emotional regulation support than older students. Schools can incorporate emotional awareness programs that help students identify and manage their achievement emotions effectively. Finally, given that achievement emotions strongly predict mathematics success, integrating emotional well-being support into mathematics curricula can help create a more inclusive and supportive learning environment for all students.

Limitations and Suggestions

This study has several limitations that should be acknowledged, particularly regarding its quantitative survey approach to examining students' achievement emotions in mathematics activities across gender, age, major, and demographics. First, the reliance on self-reported data may introduce social desirability bias, where students may not accurately report their emotions due to personal or cultural influences. Second, while the study examined achievement emotions in three mathematics contexts (classroom, homework, and tests), it did not account for other learning environments, such as group projects, tutoring sessions, or informal study settings, which may also shape students' emotional experiences. Third, the study only focused on senior high school students, limiting the generalizability of findings to other educational levels, such as junior high school or university students, who may exhibit different emotional patterns. Fourth, emotions are

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dynamic and context-dependent, but this study used a cross-sectional design, meaning it captured students' emotions at a single point in time, rather than tracking changes over time. Fifth, while the study considered gender, age, major, and demographics, it did not explore other individual factors, such as personality traits, motivation, or prior mathematics performance, which may also influence achievement emotions. Sixth, the sample size of 18-year-old students was relatively small, making it difficult to draw firm conclusions about age-related differences. Seventh, cultural influences on achievement emotions were not explicitly analyzed, despite research suggesting that cultural attitudes toward mathematics can shape students' emotional responses (Pekrun, 2024). Eighth, qualitative insights were not included, meaning the study lacks in-depth explanations of why students experience certain emotions in mathematics learning. Ninth, the study was conducted in Indonesia, limiting its applicability to students from other countries or educational systems with different mathematics curricula. Lastly, teacher-related factors, such as teaching style, classroom climate, and teacher-student relationships, were not examined, despite their potential role in shaping students' achievement emotions.

To address these limitations, future research should adopt a mixed-methods approach, incorporating qualitative methods such as interviews, open-ended surveys, or classroom observations to gain deeper insights into students' emotional experiences. Additionally, longitudinal studies should be conducted to track changes in students' achievement emotions over time, providing a more comprehensive understanding of emotional development in mathematics learning. Future studies should also expand the scope by including students from different educational levels, such as junior high school, university, or vocational education, to examine how achievement emotions evolve across different academic stages. Furthermore, researchers should investigate the role of additional individual factors, such as self-efficacy, learning strategies, and academic resilience, to better understand how students regulate their emotions in mathematics. A larger and more balanced sample for each age group is also necessary to strengthen the validity of age-related findings. Moreover, cultural influences on mathematics emotions should be explored through cross-cultural studies, comparing students from different countries or educational backgrounds to identify potential cultural differences in emotional responses to mathematics. Future studies could also examine how teacher-related factors, such as instructional methods and classroom climate, contribute to students' emotional stability in mathematics learning. Finally, researchers should explore intervention strategies that help students manage negative emotions, such as anxiety and hopelessness, while promoting positive emotions like pride and enjoyment, ultimately fostering a more emotionally supportive mathematics learning environment.

CONCLUSIONS

The findings of this study highlight the significant role of achievement emotions in students' mathematics learning, influenced by gender, academic major, demographics, and to a lesser extent, age. Gender differences were evident, with girl students experiencing higher positive emotions such as enjoyment and pride, while boy students reported greater frustration and boredom, particularly in homework and test settings. Similarly, academic major differences showed that natural sciences students exhibited higher pride and relaxation, whereas social sciences students

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reported more shame and anxiety, indicating the need for subject-specific emotional support strategies. Demographic differences were also significant, with urban students demonstrating higher positive emotions, while rural students experienced greater anxiety, boredom, and hopelessness, emphasizing the importance of educational equity. In contrast, age differences did not significantly affect students' achievement emotions, suggesting that external factors such as school environment and subject relevance play a more crucial role in shaping students' emotional experiences. These findings align with Control-Value Theory of Achievement Emotions of Pekrun et al. (2023), reinforcing that students' emotions are shaped by their perceived control over learning and the value they place on mathematics. Given that achievement emotions strongly influence students' engagement and performance, addressing emotional disparities in mathematics education is essential for fostering a more supportive learning environment. Educational interventions should be tailored to gender-specific needs, support social sciences students in building confidence, and provide rural students with improved access to quality mathematics education. Future research should explore cultural and motivational influences on achievement emotions, adopt longitudinal approaches, and consider teacher-related factors in shaping students' emotional responses. By integrating these insights into educational policies and teaching practices, educators can help students develop positive achievement emotions, ultimately leading to greater success and enjoyment in mathematics learning.

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Author Contribution

All of authors were involved in writing this article. Particularly, Suparman and Nana Diana translated and re-designed the instrument, and administrated it to students and collected the data. Bambang Avip Priatna Martadiputra organized, analyzed, and interpreted the data using SPSS 26.0 and Ms. Excel. Dadang Juandi and Turmudi reviewed and supervised this study overall.

Institutional Review Board Statement

The higher educational authority of Universitas Pendidikan Indonesia has been permitted us to do the survey and involve students as human participant in this study (Letter Number: B-11412/UN40.A4.1/PK.01.03/2024).

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APPENDICES

Table 1: Achievement emotions among students in following mathematics activities

Type of Emotions	Mathematics Activities	Mean	Deviation Standard	Skewness	Kurtosis
Enjoyment	Mathematics Classroom	2.694	0.664	-0.416	-0.422
	Mathematics Homework	2.358	0.780	0.294	-0.746
	Mathematics Test	2.522	0.776	-0.011	-0.501
	Overall	2.581	0.634	-0.122	-0.745
Pride	Mathematics Classroom	2.691	0.809	-0.017	-0.629
	Mathematics Homework	2.969	0.860	-0.763	-0.293
	Mathematics Test	2.909	0.848	-0.631	-0.201
	Overall	2.856	0.731	-0.486	-0.546
Hope	Mathematics Classroom	2.568	0.794	-0.096	-0.568
	Mathematics Homework	2.573	0.782	-0.053	-0.648
	Mathematics Test	2.613	0.806	-0.104	-0.676
	Overall	2.584	0.703	-0.095	-0.787
Contentment	Mathematics Classroom	2.641	0.853	0.002	-0.718
	Mathematics Homework	2.767	0.921	-0.504	-0.494
	Mathematics Test	2.833	0.911	-0.599	-0.321
	Overall	2.721	0.757	-0.328	-0.784
Relaxation	Mathematics Classroom	2.553	0.698	-0.315	-0.165
	Mathematics Homework	2.502	0.766	0.102	-0.460
	Mathematics Test	2.494	0.796	0.108	-0.571
	Overall	2.516	0.654	0.078	-0.769
Positive Emotions	Mathematics Classroom	2.629	0.650	-0.069	-0.753
	Mathematics Homework	2.634	0.680	-0.293	-0.744
	Mathematics Test	2.674	0.661	-0.367	-0.585
	Overall	2.641	0.625	-0.193	-0.744
Anger	Mathematics Classroom	2.310	0.803	0.146	-0.575
	Mathematics Homework	1.979	0.748	0.601	0.024
	Mathematics Test	2.164	0.847	0.606	-0.530
	Overall	2.151	0.707	0.470	-0.459
Anxiety	Mathematics Classroom	2.271	0.697	0.333	-0.580

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	Mathematics Homework	2.493	0.746	-0.107	-0.787
	Mathematics Test	2.496	0.755	-0.163	-0.654
	Overall	2.413	0.660	-0.088	-0.876
Shame	Mathematics Classroom	2.399	0.758	-0.051	-1.048
	Mathematics Homework	2.111	0.785	0.341	-0.302
	Mathematics Test	2.116	0.818	0.372	-0.409
	Overall	2.236	0.676	0.224	-0.772
Hopelessness	Mathematics Classroom	2.315	0.791	0.206	-0.919
	Mathematics Homework	2.179	0.848	0.545	-0.288
	Mathematics Test	2.356	0.815	0.176	-0.890
	Overall	2.283	0.729	0.381	-0.574
Boredom	Mathematics Classroom	2.276	0.806	0.277	-0.517
	Mathematics Homework	2.159	0.901	0.532	-0.555
	Mathematics Test	2.247	0.889	0.339	-0.681
	Overall	2.234	0.793	0.408	-0.598
Negative Emotions	Mathematics Classroom	2.314	0.626	0.118	-0.798
	Mathematics Homework	2.184	0.640	0.472	-0.308
	Mathematics Test	2.276	0.669	0.345	-0.514
	Overall	2.282	0.618	0.257	-0.522

Table 2: The results of independent sample t-test for gender difference

Mathematics Activities-related Achievement Emotions	Gender	N	Mean	Deviation Standard	T-value
Enjoyment-Classroom	Male	89	2.583	0.707	-2.162*
	Female	109	2.785	0.615	
Enjoyment-Homework	Male	89	2.269	0.787	-1.453
	Female	109	2.431	0.771	
Enjoyment-Test	Male	89	2.455	0.861	-1.109
	Female	109	2.578	0.697	
Enjoyment-Overall	Male	89	2.484	0.675	-1.956
	Female	109	2.660	0.590	
Pride-Classroom	Male	89	2.657	0.880	-0.542
	Female	109	2.720	0.749	
Pride-Homework	Male	89	2.831	0.835	-2.060*
	Female	109	3.082	0.867	
Pride-Test	Male	89	2.814	0.809	-1.419

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	Female	109	2.986	0.875	
Pride-Overall	Male	89	2.767	0.758	
	Female	109	2.929	0.704	-1.556
Hope-Classroom	Male	89	2.471	0.870	
	Female	109	2.646	0.721	-1.546
Hope-Homework	Male	89	2.488	0.775	
	Female	109	2.642	0.784	-1.375
Hope-Test	Male	89	2.511	0.801	
	Female	109	2.697	0.805	-1.621
Hope-Overall	Male	89	2.490	0.734	
	Female	109	2.661	0.670	-1.709
Contentment-Classroom	Male	89	2.550	0.879	
	Female	109	2.715	0.828	-1.356
Contentment-Homework	Male	89	2.584	0.914	
	Female	109	2.917	0.903	-2.566*
Contentment-Test	Male	89	2.707	0.955	
	Female	109	2.935	0.863	-1.760
Contentment-Overall	Male	89	2.598	0.776	
	Female	109	2.821	0.728	-2.077*
Relaxation-Classroom	Male	89	2.432	0.773	
	Female	109	2.651	0.618	-2.213*
Relaxation-Homework	Male	89	2.427	0.781	
	Female	109	2.564	0.751	-1.255
Relaxation-Test	Male	89	2.443	0.850	
	Female	109	2.536	0.750	-0.815
Relaxation-Overall	Male	89	2.434	0.709	
	Female	109	2.583	0.601	-1.602
Positive Emotions-Classroom	Male	89	2.538	0.714	-1.785

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	Female	109	2.703	0.586	
Positive Emotions-Homework	Male	89	2.520	0.680	-2.152*
	Female	109	2.727	0.669	
Positive Emotions-Test	Male	89	2.586	0.706	-1.703
	Female	109	2.746	0.616	
Positive Emotions-Overall	Male	89	2.545	0.667	-1.957
	Female	109	2.719	0.579	
Anger-Classroom	Male	89	2.348	0.854	0.596
	Female	109	2.279	0.762	
Anger-Homework	Male	89	2.162	0.775	3.181*
	Female	109	1.830	0.694	
Anger-Test	Male	89	2.309	0.930	2.195*
	Female	109	2.045	0.756	
Anger-Overall	Male	89	2.273	0.777	2.210*
	Female	109	2.051	0.631	
Anxiety-Classroom	Male	89	2.278	0.751	0.120
	Female	109	2.266	0.653	
Anxiety-Homework	Male	89	2.427	0.807	-1.131
	Female	109	2.547	0.692	
Anxiety-Test	Male	89	2.396	0.834	-1.693
	Female	109	2.578	0.677	
Anxiety-Overall	Male	89	2.361	0.724	-1.006
	Female	109	2.456	0.604	
Shame-Classroom	Male	89	2.307	0.756	-1.552
	Female	109	2.474	0.755	
Shame-Homework	Male	89	2.191	0.810	1.296
	Female	109	2.045	0.762	
Shame-Test	Male	89	2.196	0.848	1.253

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	Female	109	2.050	0.790	
Shame-Overall	Male	89	2.245	0.704	
	Female	109	2.230	0.656	0.122
Hopelessness-Classroom	Male	89	2.286	0.793	
	Female	109	2.339	0.793	-0.467
Hopelessness-Homework	Male	89	2.297	0.922	
	Female	109	2.082	0.774	1.785
Hopelessness-Test	Male	89	2.348	0.860	
	Female	109	2.362	0.781	-0.120
Hopelessness-Overall	Male	89	2.311	0.783	
	Female	109	2.261	0.684	0.473
Boredom-Classroom	Male	89	2.359	0.849	
	Female	109	2.208	0.767	1.316
Boredom-Homework	Male	89	2.314	0.995	
	Female	109	2.032	0.800	2.214*
Boredom-Test	Male	89	2.376	0.957	
	Female	109	2.142	0.819	1.854
Boredom-Overall	Male	89	2.350	0.855	
	Female	109	2.138	0.729	1.880
Negative Emotions-Classroom	Male	89	2.316	0.658	
	Female	109	2.313	0.601	0.025
Negative Emotions-Homework	Male	89	2.278	0.701	
	Female	109	2.107	0.577	1.883
Negative Emotions-Test	Male	89	2.325	0.734	
	Female	109	2.235	0.611	0.936
Negative Emotions-Overall	Male	89	2.314	0.673	
	Female	109	2.256	0.570	0.658

* < 0.05

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Table 3: The results of independent sample t-test for major difference

Mathematics Activities-related Achievement Emotions	Major	N	Mean	Deviation Standard	T-value
Enjoyment-Classroom	Natural Sciences	102	2.758	0.605	1.420
	Social Sciences	96	2.625	0.718	
Enjoyment-Homework	Natural Sciences	102	2.323	0.705	-0.650
	Social Sciences	96	2.395	0.855	
Enjoyment-Test	Natural Sciences	102	2.598	0.731	1.411
	Social Sciences	96	2.442	0.817	
Enjoyment-Overall	Natural Sciences	102	2.626	0.556	1.030
	Social Sciences	96	2.533	0.707	
Pride-Classroom	Natural Sciences	102	2.784	0.749	1.662
	Social Sciences	96	2.593	0.862	
Pride-Homework	Natural Sciences	102	3.068	0.802	1.675
	Social Sciences	96	2.864	0.910	
Pride-Test	Natural Sciences	102	3.102	0.679	3.401*
	Social Sciences	96	2.703	0.958	
Pride-Overall	Natural Sciences	102	2.985	0.621	2.581*
	Social Sciences	96	2.720	0.814	
Hope-Classroom	Natural Sciences	102	2.578	0.730	0.187
	Social Sciences	96	2.557	0.861	
Hope-Homework	Natural Sciences	102	2.504	0.719	-1.268
	Social Sciences	96	2.645	0.842	
Hope-Test	Natural Sciences	102	2.632	0.783	0.336
	Social Sciences	96	2.593	0.834	
Hope-Overall	Natural Sciences	102	2.572	0.656	-0.263
	Social Sciences	96	2.598	0.753	
Contentment-Classroom	Natural Sciences	102	2.686	0.832	0.761
	Social Sciences	96	2.593	0.877	
Contentment-Homework	Natural Sciences	102	2.745	0.852	-0.355
	Social Sciences	96	2.791	0.993	
Contentment-Test	Natural Sciences	102	2.951	0.883	1.885
	Social Sciences	96	2.708	0.928	
Contentment-Overall	Natural Sciences	102	2.767	0.716	0.885
	Social Sciences	96	2.671	0.798	
Relaxation-Classroom	Natural Sciences	102	2.578	0.612	0.526
	Social Sciences	96	2.526	0.782	
Relaxation-Homework	Natural Sciences	102	2.514	0.683	0.230
	Social Sciences	96	2.489	0.849	
Relaxation-Test	Natural Sciences	102	2.617	0.679	2.257*
	Social Sciences	96	2.364	0.889	

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Relaxation-Overall	Natural Sciences	102	2.570	0.544	1.180
	Social Sciences	96	2.460	0.753	
Positive Emotions-Classroom	Natural Sciences	102	2.677	0.590	1.061
	Social Sciences	96	2.579	0.708	
Positive Emotions-Homework	Natural Sciences	102	2.631	0.613	-0.063
	Social Sciences	96	2.637	0.748	
Positive Emotions-Test	Natural Sciences	102	2.780	0.555	2.341*
	Social Sciences	96	2.562	0.745	
Positive Emotions-Overall	Natural Sciences	102	2.692	0.536	1.200
	Social Sciences	96	2.586	0.705	
Anger-Classroom	Natural Sciences	102	2.392	0.729	1.477
	Social Sciences	96	2.224	0.870	
Anger-Homework	Natural Sciences	102	1.955	0.710	-0.462
	Social Sciences	96	2.005	0.789	
Anger-Test	Natural Sciences	102	2.137	0.817	-0.459
	Social Sciences	96	2.192	0.880	
Anger-Overall	Natural Sciences	102	2.161	0.649	0.212
	Social Sciences	96	2.140	0.768	
Anxiety-Classroom	Natural Sciences	102	2.281	0.584	0.216
	Social Sciences	96	2.260	0.803	
Anxiety-Homework	Natural Sciences	102	2.500	0.692	0.131
	Social Sciences	96	2.486	0.803	
Anxiety-Test	Natural Sciences	102	2.482	0.689	-0.256
	Social Sciences	96	2.510	0.823	
Anxiety-Overall	Natural Sciences	102	2.414	0.576	0.019
	Social Sciences	96	2.412	0.743	
Shame-Classroom	Natural Sciences	102	2.333	0.729	-1.252
	Social Sciences	96	2.468	0.785	
Shame-Homework	Natural Sciences	102	1.980	0.670	-2.444
	Social Sciences	96	2.250	0.873	
Shame-Test	Natural Sciences	102	2.019	0.667	-1.720
	Social Sciences	96	2.218	0.945	
Shame-Overall	Natural Sciences	102	2.143	0.573	-
	Social Sciences	96	2.334	0.762	2.005*
Hopelessness-Classroom	Natural Sciences	102	2.240	0.773	-1.385
	Social Sciences	96	2.395	0.807	
Hopelessness-Homework	Natural Sciences	102	2.102	0.784	-1.307
	Social Sciences	96	2.260	0.908	
Hopelessness-Test	Natural Sciences	102	2.294	0.818	-1.102
	Social Sciences	96	2.421	0.812	
Hopelessness-Overall	Natural Sciences	102	2.334	0.762	-1.429
	Social Sciences	96	2.212	0.692	

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Boredom-Classroom	Natural Sciences	102	2.189	0.735	-1.563
	Social Sciences	96	2.368	0.871	
Boredom-Homework	Natural Sciences	102	2.093	0.831	-1.061
	Social Sciences	96	2.229	0.970	
Boredom-Test	Natural Sciences	102	2.142	0.790	-1.726
	Social Sciences	96	2.359	0.975	
Boredom-Overall	Natural Sciences	102	2.148	0.712	-1.576
	Social Sciences	96	2.325	0.865	
Negative Emotions-Classroom	Natural Sciences	102	2.287	0.555	-0.634
	Social Sciences	96	2.343	0.695	
Negative Emotions-Homework	Natural Sciences	102	2.126	0.541	-1.318
	Social Sciences	96	2.246	0.728	
Negative Emotions-Test	Natural Sciences	102	2.215	0.567	-1.320
	Social Sciences	96	2.340	0.760	
Negative Emotions-Overall	Natural Sciences	102	2.238	0.523	-1.013
	Social Sciences	96	2.328	0.704	

* < 0.05

Table 4: The results of independent sample t-test for demographics difference

Mathematics Activities-related Achievement Emotions	Demographics	N	Mean	Deviation Standard	T-value
Enjoyment-Classroom	Urban School	97	2.859	0.514	3.542*
	Rural School	101	2.534	0.749	
Enjoyment-Homework	Urban School	97	2.417	0.786	1.041
	Rural School	101	2.302	0.774	
Enjoyment-Test	Urban School	97	2.577	0.754	0.970
	Rural School	101	2.470	0.796	
Enjoyment-Overall	Urban School	97	2.698	0.534	2.582*
	Rural School	101	2.468	0.702	
Pride-Classroom	Urban School	97	2.773	0.803	1.387
	Rural School	101	2.613	0.812	
Pride-Homework	Urban School	97	3.113	0.792	2.329*
	Rural School	101	2.831	0.903	
Pride-Test	Urban School	97	3.030	0.759	1.995*
	Rural School	101	2.792	0.914	
Pride-Overall	Urban School	97	2.972	0.646	2.201*
	Rural School	101	2.745	0.793	
Hope-Classroom	Urban School	97	2.680	0.715	1.962
	Rural School	101	2.460	0.853	
Hope-Homework	Urban School	97	2.757	0.729	3.333*
	Rural School	101	2.396	0.794	
Hope-Test	Urban School	97	2.768	0.753	2.681*

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	Rural School	101	2.465	0.831	
Hope-Overall	Urban School	97	2.735	0.619	3.009*
	Rural School	101	2.440	0.750	
Contentment-Classroom	Urban School	97	2.768	0.790	2.062*
	Rural School	101	2.519	0.897	
Contentment-Homework	Urban School	97	2.907	0.842	2.106*
	Rural School	101	2.633	0.976	
Contentment-Test	Urban School	97	2.896	0.783	0.962
	Rural School	101	2.772	1.018	
Contentment-Overall	Urban School	97	2.835	0.658	2.096*
	Rural School	101	2.611	0.829	
Relaxation-Classroom	Urban School	97	2.670	0.616	2.336*
	Rural School	101	2.440	0.755	
Relaxation-Homework	Urban School	97	2.505	0.737	0.047
	Rural School	101	2.500	0.796	
Relaxation-Test	Urban School	97	2.515	0.751	0.354
	Rural School	101	2.475	0.840	
Relaxation-Overall	Urban School	97	2.563	0.584	0.984
	Rural School	101	2.471	0.716	
Positive Emotions-Classroom	Urban School	97	2.750	0.560	2.593*
	Rural School	101	2.513	0.710	
Positive Emotions-Homework	Urban School	97	2.740	0.609	2.165*
	Rural School	101	2.532	0.731	
Positive Emotions-Test	Urban School	97	2.757	0.562	1.738
	Rural School	101	2.595	0.739	
Positive Emotions-Overall	Urban School	97	2.750	0.523	2.439*
	Rural School	101	2.536	0.695	
Anger-Classroom	Urban School	97	2.273	0.803	-0.641
	Rural School	101	2.346	0.805	
Anger-Homework	Urban School	97	1.876	0.753	-1.919
	Rural School	101	2.079	0.733	
Anger-Test	Urban School	97	2.030	0.789	-
	Rural School	101	2.292	0.883	2.189*
Anger-Overall	Urban School	97	2.059	0.701	-1.795
	Rural School	101	2.239	0.706	
Anxiety-Classroom	Urban School	97	2.149	0.641	-
	Rural School	101	2.388	0.731	2.442*
Anxiety-Homework	Urban School	97	2.594	0.766	-1.949
	Rural School	101	2.373	0.670	
Anxiety-Test	Urban School	97	2.613	0.815	-
	Rural School	101	2.336	0.755	2.259*
Anxiety-Overall	Urban School	97	2.296	0.591	

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	Rural School	101	2.526	0.705	-2.476*
Shame-Classroom	Urban School	97	2.459	0.759	-1.135
	Rural School	101	2.005	0.768	
Shame-Homework	Urban School	97	2.212	0.791	-1.872
	Rural School	101	1.984	0.761	
Shame-Test	Urban School	97	2.242	0.853	-2.241*
	Rural School	101	2.175	0.729	
Shame-Overall	Urban School	97	2.141	0.637	-1.946
	Rural School	101	2.327	0.703	
Hopelessness-Classroom	Urban School	97	2.450	0.829	-2.477*
	Rural School	101	1.958	0.779	
Hopelessness-Homework	Urban School	97	2.391	0.861	-3.697*
	Rural School	101	2.304	0.789	
Hopelessness-Test	Urban School	97	2.405	0.841	-0.877
	Rural School	101	2.099	0.769	
Hopelessness-Overall	Urban School	97	2.146	0.659	-2.646*
	Rural School	101	2.416	0.770	
Boredom-Classroom	Urban School	97	2.445	0.808	-3.079*
	Rural School	101	1.984	0.817	
Boredom-Homework	Urban School	97	2.328	0.949	-2.712*
	Rural School	101	2.082	0.782	
Boredom-Test	Urban School	97	2.405	0.958	-2.595*
	Rural School	101	2.207	0.565	
Boredom-Overall	Urban School	97	2.061	0.714	-3.063*
	Rural School	101	2.400	0.833	
Negative Emotions-Classroom	Urban School	97	2.417	0.666	-2.385*
	Rural School	101	2.042	0.565	
Negative Emotions-Homework	Urban School	97	2.320	0.679	-3.128*
	Rural School	101	2.155	0.571	
Negative Emotions-Test	Urban School	97	2.392	0.736	-2.523*
	Rural School	101	2.698	0.534	
Negative Emotions-Overall	Urban School	97	2.159	0.534	-2.783*
	Rural School	101	2.400	0.671	

* < 0.05

Table 5: The results of one-way MANOVA for age difference

Mathematics Activities-related Achievement Emotions	Demographics	N	Mean	Deviation Standard	F-value
Enjoyment-Classroom	15th Years Old	36	2.705	0.789	0.073
	16th Years Old	101	2.701	0.665	

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	17th Years Old	57	2.684	0.603	
	18th Years Old	4	2.550	0.341	
Enjoyment-Homework	15th Years Old	36	2.693	0.664	
	16th Years Old	101	2.555	1.005	1.562
	17th Years Old	57	2.376	0.766	
	18th Years Old	4	2.201	0.632	
Enjoyment-Test	15th Years Old	36	2.375	0.478	
	16th Years Old	101	2.358	0.780	1.285
	17th Years Old	57	2.736	0.989	
	18th Years Old	4	2.500	0.764	
Enjoyment-Overall	15th Years Old	36	2.678	0.807	
	16th Years Old	101	2.584	0.637	0.496
	17th Years Old	57	2.524	0.519	
	18th Years Old	4	2.445	0.206	
Pride-Classroom	15th Years Old	36	2.447	0.645	
	16th Years Old	101	2.250	0.288	1.051
	17th Years Old	57	2.522	0.776	
	18th Years Old	4	2.847	0.939	
Pride-Homework	15th Years Old	36	2.633	0.818	
	16th Years Old	101	2.728	0.713	1.347
	17th Years Old	57	2.250	0.500	
	18th Years Old	4	2.691	0.809	
Pride-Test	15th Years Old	36	2.805	0.965	
	16th Years Old	101	2.940	0.911	0.201
	17th Years Old	57	3.087	0.688	
	18th Years Old	4	3.500	0.408	
Pride-Overall	15th Years Old	36	2.828	0.903	
	16th Years Old	101	2.829	0.755	0.222
	17th Years Old	57	2.923	0.584	
	18th Years Old	4	2.831	0.362	
Hope-Classroom	15th Years Old	36	2.969	0.860	
	16th Years Old	101	2.833	1.041	1.896
	17th Years Old	57	2.915	0.839	
	18th Years Old	4	2.956	0.751	
Hope-Homework	15th Years Old	36	2.750	0.645	
	16th Years Old	101	2.909	0.848	0.583
	17th Years Old	57	2.777	0.921	
	18th Years Old	4	2.599	0.784	
Hope-Test	15th Years Old	36	2.394	0.717	
	16th Years Old	101	2.375	0.478	0.302
	17th Years Old	57	2.568	0.794	

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	18th Years Old	4	2.652	0.868	
Hope-Overall	15th Years Old	36	2.717	0.854	
	16th Years Old	101	2.595	0.741	0.826
	17th Years Old	57	2.493	0.523	
	18th Years Old	4	2.417	0.315	
Contentment-Classroom	15th Years Old	36	2.833	1.021	
	16th Years Old	101	2.628	0.814	1.208
	17th Years Old	57	2.578	0.822	
	18th Years Old	4	2.125	0.250	
Contentment-Homework	15th Years Old	36	2.666	0.925	
	16th Years Old	101	2.782	0.975	0.245
	17th Years Old	57	2.789	0.839	
	18th Years Old	4	3.000	0.816	
Contentment-Test	15th Years Old	36	2.694	1.009	
	16th Years Old	101	2.811	0.935	1.157
	17th Years Old	57	2.912	0.808	
	18th Years Old	4	3.500	0.577	
Contentment-Overall	15th Years Old	36	2.756	0.899	
	16th Years Old	101	2.712	0.756	0.034
	17th Years Old	57	2.714	0.693	
	18th Years Old	4	2.687	0.314	
Relaxation-Classroom	15th Years Old	36	2.611	0.879	
	16th Years Old	101	2.544	0.707	0.240
	17th Years Old	57	2.517	0.574	
	18th Years Old	4	2.750	0.288	
Relaxation-Homework	15th Years Old	36	2.763	0.929	
	16th Years Old	101	2.480	0.767	1.960
	17th Years Old	57	2.377	0.635	
	18th Years Old	4	2.500	0.408	
Relaxation-Test	15th Years Old	36	2.652	0.977	
	16th Years Old	101	2.509	0.790	1.068
	17th Years Old	57	2.359	0.679	
	18th Years Old	4	2.625	0.629	
Relaxation-Overall	15th Years Old	36	2.675	0.840	
	16th Years Old	101	2.511	0.643	1.180
	17th Years Old	57	2.418	0.540	
	18th Years Old	4	2.625	0.372	
Positive Emotions-Classroom	15th Years Old	36	2.755	0.835	
	16th Years Old	101	2.621	0.648	0.707
	17th Years Old	57	2.580	0.530	
	18th Years Old	4	2.410	0.249	

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Positive Emotions-Homework	15th Years Old	36	2.688	0.821	
	16th Years Old	101	2.637	0.720	0.210
	17th Years Old	57	2.586	0.513	
	18th Years Old	4	2.750	0.443	
Positive Emotions-Test	15th Years Old	36	2.727	0.878	
	16th Years Old	101	2.663	0.667	0.106
	17th Years Old	57	2.657	0.507	
	18th Years Old	4	2.725	0.221	
Positive Emotions-Overall	15th Years Old	36	2.725	0.818	
	16th Years Old	101	2.636	0.636	0.310
	17th Years Old	57	2.599	0.470	
	18th Years Old	4	2.582	0.247	
Anger-Classroom	15th Years Old	36	2.083	0.874	
	16th Years Old	101	2.351	0.795	1.227
	17th Years Old	57	2.386	0.773	
	18th Years Old	4	2.250	0.645	
Anger-Homework	15th Years Old	36	1.930	0.829	
	16th Years Old	101	2.059	0.800	0.855
	17th Years Old	57	1.886	0.597	
	18th Years Old	4	1.750	0.500	
Anger-Test	15th Years Old	36	2.166	0.918	
	16th Years Old	101	2.193	0.883	0.240
	17th Years Old	57	2.096	0.764	
	18th Years Old	4	2.375	0.478	
Anger-Overall	15th Years Old	36	2.059	0.815	
	16th Years Old	101	2.201	0.740	0.399
	17th Years Old	57	2.122	0.589	
	18th Years Old	4	2.122	0.415	
Anxiety-Classroom	15th Years Old	36	2.250	0.908	
	16th Years Old	101	2.264	0.683	0.159
	17th Years Old	57	2.280	0.584	
	18th Years Old	4	2.500	0.540	
Anxiety-Homework	15th Years Old	36	2.342	0.931	
	16th Years Old	101	2.458	0.706	1.375
	17th Years Old	57	2.631	0.674	
	18th Years Old	4	2.750	0.741	
Anxiety-Test	15th Years Old	36	2.395	0.843	
	16th Years Old	101	2.438	0.804	1.302
	17th Years Old	57	2.644	0.599	
	18th Years Old	4	2.750	0.353	
Anxiety-Overall	15th Years Old	36	2.328	0.825	0.855

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	16th Years Old	101	2.380	0.675	
	17th Years Old	57	2.508	0.517	
	18th Years Old	4	2.657	0.396	
Shame-Classroom	15th Years Old	36	2.445	0.976	
	16th Years Old	101	2.330	0.711	0.670
	17th Years Old	57	2.474	0.703	
	18th Years Old	4	2.665	0.386	
Shame-Homework	15th Years Old	36	2.083	1.003	
	16th Years Old	101	2.108	0.740	0.164
	17th Years Old	57	2.114	0.744	
	18th Years Old	4	2.375	0.250	
Shame-Test	15th Years Old	36	2.222	0.865	
	16th Years Old	101	2.108	0.859	0.482
	17th Years Old	57	2.043	0.739	
	18th Years Old	4	2.375	0.250	
Shame-Overall	15th Years Old	36	2.277	0.866	
	16th Years Old	101	2.203	0.662	0.329
	17th Years Old	57	2.248	0.590	
	18th Years Old	4	2.500	0.180	
Hopelessness-Classroom	15th Years Old	36	2.361	0.975	
	16th Years Old	101	2.207	0.768	1.604
	17th Years Old	57	2.447	0.705	
	18th Years Old	4	2.750	0.288	
Hopelessness-Homework	15th Years Old	36	2.291	0.951	
	16th Years Old	101	2.248	0.884	0.273
	17th Years Old	57	2.157	0.738	
	18th Years Old	4	2.250	0.500	
Hopelessness-Test	15th Years Old	36	2.347	1.005	
	16th Years Old	101	2.302	0.800	0.425
	17th Years Old	57	2.447	0.723	
	18th Years Old	4	2.500	0.707	
Hopelessness-Overall	15th Years Old	36	2.333	0.929	
	16th Years Old	101	2.219	0.733	0.592
	17th Years Old	57	2.350	0.583	
	18th Years Old	4	2.502	0.471	
Boredom-Classroom	15th Years Old	36	2.277	0.874	
	16th Years Old	101	2.208	0.794	1.115
	17th Years Old	57	2.420	0.792	
	18th Years Old	4	1.917	0.571	
Boredom-Homework	15th Years Old	36	2.152	1.047	0.335
	16th Years Old	101	2.113	0.882	

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	17th Years Old	57	2.254	0.861	
	18th Years Old	4	2.000	0.707	
Boredom-Test	15th Years Old	36	2.333	1.007	
	16th Years Old	101	2.222	0.876	0.172
	17th Years Old	57	2.228	0.866	
	18th Years Old	4	2.375	0.629	
Boredom-Overall	15th Years Old	36	2.258	0.907	
	16th Years Old	101	2.184	0.780	0.406
	17th Years Old	57	2.318	0.761	
	18th Years Old	4	2.072	0.600	
Negative Emotions-Classroom	15th Years Old	36	2.283	0.828	
	16th Years Old	101	2.272	0.617	0.576
	17th Years Old	57	2.401	0.498	
	18th Years Old	4	2.415	0.395	
Negative Emotions-Homework	15th Years Old	36	2.160	0.836	
	16th Years Old	101	2.177	0.639	0.054
	17th Years Old	57	2.208	0.516	
	18th Years Old	4	2.225	0.371	
Negative Emotions-Test	15th Years Old	36	2.293	0.841	
	16th Years Old	101	2.253	0.685	0.174
	17th Years Old	57	2.292	0.533	
	18th Years Old	4	2.475	0.357	
Negative Emotions-Overall	15th Years Old	36	2.263	0.819	
	16th Years Old	101	2.255	0.624	0.262
	17th Years Old	57	2.334	0.465	
	18th Years Old	4	2.405	0.356	

* < 0.05

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ACHIEVEMENT EMOTIONS QUESTIONNAIRE-MATHEMATICS (AEQ-M)

Full Name :
Age :
Gender :
Class :
School Origin :

Questionnaire Instructions

1. The achievement emotions questionnaire consists of ten emotional types, namely enjoyment, pride, hope, contentment, relaxation, anxiety, boredom, shame, hopelessness, and anger.
2. Each statement from all emotional types can be rated on a scale of 1 – 4 where each scale is described as follows:
   (1) = (Strongly disagree)
   (2) = (Disagree)
   (3) = (Agree
   (4) = (Strongly Agree)
3. Please select one scale between 1 – 4 for each statement by checking (✓) in the space provided.
4. Please fill out this questionnaire in a way that reflects achievement emotions that you feel or experience when doing math activities, such as: attending math classes, doing math homework, and taking math exams.

No	Statements	Scale			
		SD	D	A	SA
1	I look forward to math class every week				
2	I enjoy attending math class				
3	I find the material discussed in math class very interesting				
4	I am happy because I can understand math material well				
5	I smile happily at the teacher while learning in math class				
6	I am in a good mood when doing math homework				
7	I really enjoy doing math homework				

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8	I feel happy when taking a math test				
9	I smile joyfully when a math test goes well				
10	I am proud of my knowledge in mathematics				
11	I am proud of my contributions in math class				
12	I am proud of my own work after completing math homework				
13	I feel proud when I can do my math homework well				
14	I am proud of myself after completing a math test				
15	I feel proud when I can do a math test well				
16	I am optimistic about learning mathematics				
17	I believe I can master math material				
18	I am optimistic that I can complete my math homework				
19	I choose not to give up on math homework				
20	I believe that the math test will go well				
21	I am optimistic that I can complete the math test				
22	I am satisfied with my knowledge in mathematics				
23	I am satisfied with my performance in math class				
24	I am satisfied with my own work after completing math homework				
25	I feel satisfied after being able to complete a math test independently				
26	I follow math lessons in a relaxed manner				
27	I express my math ideas in front of the class calmly				
28	I do my math homework calmly				
29	I do not feel burdened when doing difficult math homework				
30	I take math tests calmly				
31	I do not feel burdened when taking a difficult math test				
32	I feel upset to the point of wanting to leave math class				
33	I feel frustrated because math material is too difficult				
34	I am annoyed because math homework takes up most of my playtime				
35	I want to throw my math homework in the trash				
36	I want to tear my math test paper into pieces				
37	I am frustrated because the teacher asks difficult questions in the math test				
38	I feel nauseous when thinking about math class				
39	I am worried if the math material is too difficult				
40	I choose not to go to school because math is scary				
41	I am worried about my ability to fully understand math material				
42	I am worried that I will not complete my math homework on time				
43	My heart races when I cannot finish my math homework				
44	I would rather not do difficult math homework				
45	I feel tense and nervous when taking a math test				
46	I worry about getting a bad grade on a math test				

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47	I am afraid of failing even before the math test starts				
48	I feel nauseous when thinking about an upcoming math test				
49	My face turns red when expressing ideas in math class				
50	I feel embarrassed when I cannot answer a math question well				
51	I feel ashamed of my lack of knowledge in mathematics				
52	I do not want to tell anyone when I do not understand my math homework				
53	I avoid eye contact with classmates when discussing math homework				
54	My face turns red after taking a math test				
55	I do not want to talk about anything after taking a math test				
56	I feel hopeless about math class				
57	I keep thinking that I cannot understand the math material being taught				
58	I would rather give up on math homework				
59	I no longer try to do my math homework				
60	I feel hopeless during a math test				
61	I keep thinking that I will never get a good grade on a math test				
62	I find math lessons boring				
63	I do not focus when the teacher explains math material				
64	I feel so bored that I do not want to study math anymore				
65	I do not really care about the math homework given				
66	Math homework bores me to death				
67	Math tests make me very bored				
68	I do not really care about the math tests given				

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