How Students Think About a Reverse Problem of Derivative: A Study of Arnon's Coordination in Mental Mechanism

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Abstract: Mental mechanism of coordination is indispensable in the construction of certain objects, contributing to problem-solving. Although Arnon et al. hypothesized about how two processes are coordinated mentally, which is referred to as Arnon's Coordination, further analysis is required. Studies related to coordination had explored its emergence in reflective abstraction during problem-solving. Generally, descriptions of coordination focused exclusively on the outcomes and application to problem-solving, particularly direct problems. Therefore, this study aimed to describe how students performed coordination between two processes in reverse problem-solving, focusing mainly on derivative and antiderivative graphs. Data were collected from 4 of the 44 third-year mathematics education students through answer sheets, think-aloud, and semi-structured interviews to examine mental mechanisms. The results showed that students coordinated the two processes by adopting three cognitive actions, namely determining processes, transforming it into an object through encapsulation, and integration through assimilation or accommodation. This also included the challenges students experienced with each cognitive action. In conclusion, this study served as a reference for educators when designing mathematical problem-solving exercises and lessons.

Keywords: Coordination, Mental Mechanism, Derivative Functions, Reverse Problem

INTRODUCTION

The ability of an individual to solve problems is assessed based on four mental structures, namely action, process, object, and scheme (APOS). Action stage enables an individual to understand and identify information from the problems (Syarifuddin et al., 2020). According to Borji and Planell (2023) and Nagle et al. (2019), problems are solved by simply applying the procedures memorized, without understanding the reasons behind the application. Process stage entails the connection of basic and other related concepts (Borji & Martínez-Planell, 2023; Nagle et al., 2019), by using appropriate mathematical techniques to solve problems (Syarifuddin et al., 2020). At the object stage, certain concepts are understood as entities that can be manipulated or used in a broader context as well as applied to new situations (Borji & Martínez-Planell, 2023; Nagle et al., 2019). Meanwhile, the schema stage entails students constructing frameworks that integrate various mathematical concepts to solve problems in other disciplines (Borji & Martínez-Planell, 2023; Nagle et al., 2019). This enables the drawing of conclusions and organizing of data related to concepts into graphs or equations to show the relationships between concepts (Syarifuddin et al., 2020). An individual is able to connect different concepts by combining several varying processes (Kurniati et al., 2018; Syamsuri et al., 2017), organized through the mental mechanism of coordination.

In this context, coordination plays an important role in solving math problems. Dubinsky (2002) stated that the ability to perform the coordination needed to solve problems was based on built and reorganized schemas. Existing processes were coordinated to develop a new process that could be used in problem solving (Akgul & Yilmaz, 2023; Cetin & Dubinsky, 2017; Dubinsky, 2002; Paschos & Farmaki, 2006). This new process was further transformed into a mental object and used to understand the problems. According to Arnon et al. (2014), coordination was needed in constructing several objects that match the problems. Based on the study conducted by Syarifuddin et al. (2020), a coordination mechanism was used to determine the appropriate method to solve math problems. Syamsuri et al. (2017) also reported that it allowed students to develop a methodical approach in evaluating the relationship between mathematical objects, thereby improving problem analysis ability. In the truth-seeking process, coordination was applied through cross-concept checking to facilitate the identification of logical errors in the problem-solving process (Kurniati et al., 2018).

Previous research reported that many students cannot properly perform mental mechanisms of coordination. The results by Sopamena et al. (2016, 2018) show the need for reflection to be able to produce accurate generalizations. This was due to the results of incomplete coordination and the absence between the components of the cognitive structure that have been formed. Other studies related to reflective abstraction stated that students with impulsive cognitive styles tended to be imperfect in coordination, as well as unable to perform encapsulation and generalization mechanisms perfectly during problem-solving (Fuady et al., 2020). A similar circumstance was also observed in those with field-dependent cognitive style. The research subjects with this cognitive style are not yet able to coordinate the process of interiorization results but are able to change information into mathematical models. This led to the inability to develop their mental structure to be able to generalize (Fuady et al., 2019). The ability of students to coordinate is the reason for exploring how this mental mechanism (coordination) works in depth. Additionally, understanding how mental mechanism is conceptually implemented is crucial.

Based on the description, Arnon et al. (2014) proposed a hypothesis regarding mental actions applied during coordination. The present study referred to the coordination implementation hypothesized by Arnon et al. (2014). It is regarded as Arnon's Coordination, where there is a possibility to coordinate two processes, supposing students are able to first summarize one of the processes into an object, before the integration process. Coordination can be performed in the following way: either the object is assimilated and the process applied, or the process is accommodated to enable its application (Arnon et al., 2014). However, the hypothesis regarding the mechanism of coordination is still a reference for further analysis (Arnon et al., 2014). Mental mechanisms are difficult to ascertain directly and can only be inferred through observations when searching for solutions to problems or trying to understand certain phenomena (Dubinsky, 1991).

A particular task used to decipher mental construction of an individual is representation translation. The ability to interpret and translate various representations is a tool adopted to show individual mental construction of deep mathematical concepts (Cawley, 2016). This is because the representation translation of an individual enables the determination of the internal abstraction and cognitive schema. Pape & Tchoshanov (2001) stated that mathematical representations are explained as internal abstractions of related ideas or cognitive schemas. Therefore, representation translation is a factor that affects understanding of individual concepts. Hähkiöniemi (2004) and Roorda et al. (2009) stated that connections between physical applications and mathematical representations are one of the factors that influenced individual understanding. In this perspective, the understanding of mathematical and graphical representations is enabled by the abstraction performed during the construction process.

One of the important concepts related to the translation of representation is the concept associated with the graph of derivative and antiderivative functions. Learning about the derivative is fundamental in using different representations (Borji, Font, et al., 2018; Maharaj, 2013), which connects visualizations with respective symbolizations (Tall, 2011), providing meaningful insights related to formula symbolization and a deeper understanding of the concept (Borji, Font, et al., 2018; Tall, 2011; Voskoglou, 2017). Previous studies on the graphical representations of derivative concepts showed that students had difficulty in translating representations from to graphs, specifically during initial coordination activities (Ikram et al., 2020; Swastika et al., 2020). In constructing the process to determine the function whose derivative is the original, it is important to reverse the process of interiorization, which results in relation to the antiderivative issues (Arnon et al., 2014; Dubinsky, 2002). Considering this perspective, these types of issues are referred to as reverse problems.

Reverse problems are among the many mathematical tasks that foster active thinking in students (Ikram et al., 2021). Some studies reported that both direct and inverted problems are used to reconstruct mental processes of an individual, which is associated with reversibility (Ikram et al., 2021). Reverse or opposite problems can also be referred to as reversible and are presented in two ways (Maf'ulah & Juniati, 2020). According to Maf'ulah and Juniati (2020), the first way includes presenting the problems using two different methods relating to its opposite. The first problem is related to the inverse, while the second is not associated with the inverse. However, both problems are presented in the same context. The second way entails presenting the problems in a single form, focusing on the known and questionable aspects (Maf'ulah & Juniati, 2020). In this study, the problems solved by adopting the second presentation method were used. Other studies refer to this second method as dual problems, and when solving related issues, one must have reversibility (Ramful, 2015).

In problem-solving, reversibility helps in understanding the relationship between concepts. Fitmawati (2019) stated that this could be achieved by establishing a two-way correlation based on reversible problems. Reversibility is the ability to perform certain mental operations, including reversing the thought process in order to return to the starting point (Slavin, 2014). Piaget defined it as the ability to perform an action in two opposite directions while being aware of the fact that the actions are similar (Simon et al., 2016). Reversibility is the result of a second-level reflective abstraction of the original concept (Simon et al., 2016). Moreover, problems related to reversibility enables the in-depth analysis of reflective abstraction performed by an individual, specifically mental mechanism of coordination.

The present study was conducted to review the hypothesis of Arnon et al. (2014) related to the coordination of two processes, or Arnon's Coordination. This study broadens the understanding of coordination in the context of mathematical problem-solving and fills the gaps in previous research, which has not deeply explored how students use coordination to solve reverse problems. Furthermore, it investigates how students perform the mechanism of coordination between two processes in solving a reverse problem, with a particular focus on concepts of derivative and antiderivative graphs. The exploration also focuses on analyzing the cognitive actions that students performed with respect to Arnon's Coordination framework. The analysis offers a more comprehensive guide for educators in designing teaching approaches that promote reverse thinking in calculus. The coordination observed by students was detected at each problem-solving step by Polya (1985). The method was selected because it enables the observation of mathematical problem-solving strategies that are unique to each individual, thereby providing insight into the coordination performed, including the challenges faced.

LITERATURE REVIEW

APOS and Reflective Abstraction

APOS theory focuses on how mathematical concepts can be learned, with the acronym depicting action, process, object, and scheme regarded as mental structure (Arnon et al., 2014). The description of the relationship between mental mechanism of reflective abstraction and mental structure is explained through APOS theory (Arnon et al., 2014). This framework is used to describe how individuals develop an understanding of the concept through APOS stages, as shown in Figure 1.

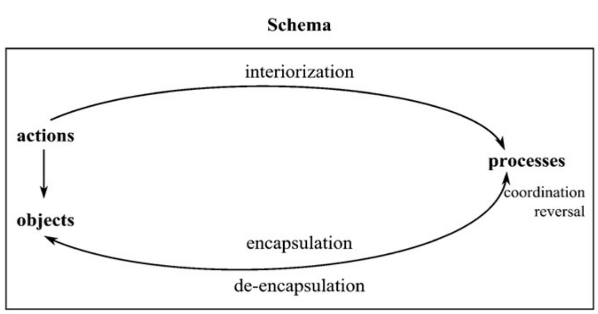


Figure 1: Mental structures and mechanisms for constructing mathematical knowledge. Source: Arnon et al. (2014)

The diagram shows the in-depth relationship between APOS and mental mechanism. The four mental structures are related to mental mechanisms in reflective abstraction, which supports and animates the construction of mathematical logic (Arnon et al., 2014). There are six mental construction types in reflective abstraction: interiorization, coordination, reversal, encapsulation, de-encapsulation, and generalization (Arnon et al., 2014). These types of construction are referred to as mental mechanisms, with the explanation reported as follows.

Interiorization

Interiorization refers to the awareness of an action, reflecting on it, and combining it with other actions. This mechanism allows for mental translation by imagining the execution of steps without having to skip and perform each explicitly (Arnon et al., 2014). Students engage in this process by reflecting on the procedure, including defining the concept (Cappetta & Zollman, 2013). For example, an individual might read, or become aware of the description of a function in terms of domain, range (perhaps implicit), and formula, enabling another to understand the problems based on what was written (Arnon et al., 2014). The interiorization process starts when an individual views a function as a type of transformation that pairs elements from a set, called the domain, with those from the second set, referred to as the range.

Coordination

Coordination is adopted to analyze the concepts, including constructing mathematical objects. It is defined as the composition of two or more processes to construct a new process (Dubinsky, 2002). In mathematics, this new process is constructed to analyze a mathematical concept by examining and integrating two or more processes (Cappetta & Zollman, 2013; Sopamena et al., 2018). A process is a mental structure that involves the same operations as an action but differs in terms of the transformations performed. According to Cetin and Dubinsky (2017), the operations performed in a process are the same as those performed in an action, but they occur entirely within the individual's mind. In actions, an individual must physically or mentally perform the transformation. Unlike an action, a process is internally driven, and the individual no longer needs external stimuli for guidance (Burns-Childers & Vidakovic, 2018). Additionally, external cues are not required at this stage (Arnon et al., 2014; Cetin & Dubinsky, 2017; Dubinsky, 2002), nor are explicit calculations necessary (Arnon et al., 2014). For example, in functions, individuals can imagine any element in the domain being transformed into an element in the range using the function's expression or another method (Arnon et al., 2014; Cetin & Dubinsky, 2017). Similarly, when observing a polynomial function graph, someone at the process stage can understand the type of function and determine its characteristics.

One example of coordination between two processes in function composition occurs when someone combines functions and to obtain. This process requires coordination between the two functions (Arnon et al., 2014). This coordination involves applying process to the elements obtained by applying process . Another example includes determining the square root of a non-quadratic positive integer between two natural numbers. Students are expected to coordinate processes: the process of determining the square root of a squared number and the process to carrying out the rational number in decimal. This also includes the processes of internalizing the square root of two positive integers (Akgul & Yilmaz, 2023). In the context of graphing a derivative functions, the process of determining the relationship between two points on a curve and its slope (as the points on the graph become closer together) are coordinated with the process of determining the average rate of change (as the difference in time intervals becomes smaller, that is as the length of the time intervals gets closer to zero) to relate the definition of the derivative to some other interpretations (Asiala et al., 1997).

Reversal

New mathematical ideas can be constructed by reversing the steps of the original ideas. For example, in derivative and antiderivative concepts, an individual may explore the act of determining the derivative of a function, including correctly solving problems related to the concept and then internalizing it into a process (Arnon et al., 2014). Reversal is performed to find the original function, a derivative of the known function (Arnon et al., 2014; Dubinsky, 1991). Another example is a problem related to square roots. At the process stage, students make progress by identifying the relationship between square numbers and their square roots as inverse operations, understood through the reversal application. (Akgul & Yilmaz, 2023).

Encapsulation

The constructed process is packed into an object through the mental mechanism of encapsulation. This is achieved when the individual becomes aware of the total process, realizing that a transformation could act upon the process, or its constructs (Cetin & Dubinsky, 2017). Furthermore, encapsulation occurs when an individual applies an action to a process, simply referring to when the dynamic structure (process) is considered static to which the action can be applied (Arnon et al., 2014). Cappetta & Zollman (2013) defined this as the act of personifying a concept, i.e. a collection of abstract that seemed meaningful to an individual. Related to mental mechanism, Sfard (1991) analyzed the inherent difficulty of reification (similar to encapsulation in APOS Theory) and suggests that the ability to view a familiar task in a completely new way is challenging to achieve. In addition, difficulty arises when the process is not transformed into an object (Arnon et al., 2014). A typical example of encapsulation in integral material is when an action applied to the Riemann sum process is performed to determine or calculate the existence of its limit or value (Arnon et al., 2014). Another example is the determination of the square root of natural numbers, shown by using its symbol outside the original context of geometry and square numbers. At the object stage, this is extended to non-square numbers, with the decimals between integers estimated in new situations by adding integers (Akgul & Yilmaz, 2023).

De-Encapsulation

The process leading to the object returns through the mental mechanism of de-encapsulation. These objects are de-encapsulated to regain the process when relevant (Arnon et al., 2014). For example, when working on problems related to the composition of two functions, the object function is de-encapsulated into its original process, and these processes are then coordinated to form the composition process, which is subsequently encapsulated into an object (Cetin & Dubinsky, 2017).

Generalization

Generalization is an existing schema represented and used in a new situation different from previous application (Dubinsky, 1991, 2002). Schemas are built as a coherent collection of structures from actions, processes, objects, and other schemas (Arnon et al., 2014; Nagle et al., 2019).

Arnon's Coordination in Mental Mechanism

In Arnon's Coordination, two processes, namely P_A and P_B as shown in Figure 2, are coordinated by applying P_A to P_B. This is realized by ensuring the learner encapsulates P_B into an Object, O_B, before applying P_A. Immediately after that is done, the coordination proceeds in the following way: O_B is assimilated and P_A applies, or P_A is accommodated enabling the learner to apply it to O_B. Previous studies stated that another alternative is the application of P_B to P_A in the same manner (Arnon et al., 2014). However, the hypothesis regarding the coordination implementation serves as a reference for further review (Arnon et al., 2014).

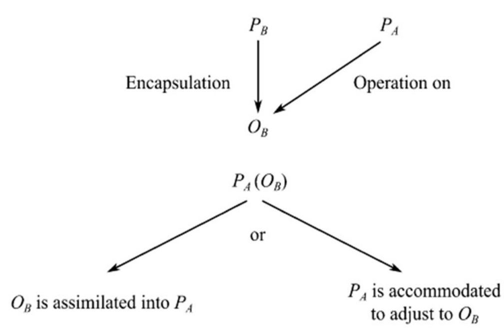


Figure 2: Coordination of two processes, P_A and P_B. Source: Arnon et al. (2014)

The diagram shows that a new process is constructed from the two processes during the coordination procedure by adopting several cognitive actions. One of these actions requires encapsulating the process before the integration with another predetermined process (Arnon et al., 2014). The subsequent action entails integrating the process and object, with Arnon's Coordination enabling the application to objects through assimilation or accommodation (Arnon et al., 2014; Goodson-Espy, 2014). The fit of the problem's structure to the individual's schema determines the use of assimilation or accommodation in problem-solving.

Following the earlier description, knowledge assimilation refers to the mechanism by which an individual applies the schema owned without changing it due to the similarity with the structure of the problems. In assimilation, the structure of the problem faced is in accordance with the existing scheme, resulting in direct integration (Blake & Pope, 2008; Subanji & Nusantara, 2016). Therefore, the schema owned is applied directly without change by including cognitive objects that had never been encountered (Arnon et al., 2014; Blake & Pope, 2008; Subanji & Nusantara, 2016). The accommodation process is a change in the scheme due to lack of inappropriate schemes for the problems structure (Blake & Pope, 2008; Subanji & Nusantara, 2016). This process occurs through reconstructing or modifying existing schemas to address new situations (Arnon et al., 2014; Subanji, 2011). During the learning process, students face changes to the schema either by adding components, refining, developing, or changing the old concept (Arnon et al., 2014; Setyaningsih et al., 2018).

METHODS

Study Design and Subjects

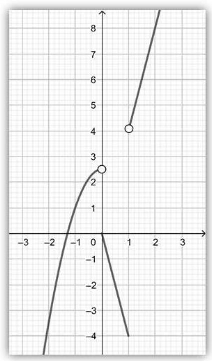
This study adopted a qualitative approach with a phenomenological method to provide an in-depth description of the experiences associated with various individuals related to certain phenomena (Creswell, 2015). Moreover, the adoption of the method was in accordance with the purpose of the analysis, which concentrated on exploring the coordination between the two processes carried out by students in solving a reverse problem, with a special focus on the concept of derivative and antiderivative functions. The phenomenological method enables the understanding of individual subjective experiences related to certain phenomena, revealing how meaning is given to these experiences (Burns et al., 2022). This study enabled an understanding of individual experiences related to actions carried out in coordination, including an exploration of the deeper meaning behind those actions. The phenomenological method aims to investigate the subject's experiences in depth, excluding generalized conclusions that apply broadly. As a result, this tended to have an impact on subject selection.

Participants consisted of 44 third-year mathematics education students who had taken advanced calculus courses, including derivative functions and antiderivative graphs. The subjects were selected based on a similar problem discussed in this study. The problem in the subject selection is stated in the appendix, with several criteria adopted during the selection process. Students who demonstrated problem understanding, coordination, and effective communication skills were selected as subjects. Furthermore, the understanding of the problem and coordination were viewed through the results written in the step of understanding the problem and devising a plan.

Data Collection

Data were obtained through student answers, thinking aloud, and semi-structured task-based interviews. The task sheet (Figure 3) gives instructions on how to solve the problem using Polya's problem-solving steps. As a result, students were asked to write down the processes used to solve the problem at each step.

Permasalahan:
Diketahui fungsi f kontinu pada $\mathbb{R}$ dengan $f(0) = 0$. Jika Gambar berikut merupakan grafik fungsi turunan dari fungsi f maka sketsalah grafik dari fungsi f.



TRANSLATION

Problem:
Suppose the function f is continuous on $\mathbb{R}$ with $f(0) = 0$.
If the following figure is the graph of the derivative function of the function f then sketch the graph of the function f.

Figure 3: Reverse problem related to graphs of derivative and antiderivative on the task sheet

Figure 3 was inspired by problems in studies related to the representation translation process (Swastika et al., 2023). In view of this perspective, the problem was selected because the translation is related to cognitive processes, specifically in terms of connecting one representation (source) with another (target); however, the notation remained unchanged (Bosse et al., 2014). Through these problems, the development of students' cognitive structure was associated with the construction type of reflective abstraction performed in solving derivative function graph. The problem could be solved using two alternative solutions, namely symbolic or verbal representation.

Analyzing Data

Data analysis was conducted in three stages, including reduction, presentation, and drawing conclusions (Miles et al., 2013). The reduction stage was carried out by transcribing think-aloud and interview data, as well as selecting, reducing, and exploring student coordination. This study is based on the mental mechanisms of Piaget and Dubinsky (Arnon et al., 2014), and Polya's problem-solving method (Polya, 1985). In addition, this method allowed for the observation of unique mathematical problem-solving strategies. This provided insight into the coordination used by students when applying respective strategies. The coordination performed was observed at each stage of the problem-solving process, as outlined by Polya, and shown in Table 1.

Polya's Concept Steps	Coordination in Every Step
Understanding the Problems	1. Coordinating the processes associated with determining the diverse parts of a function to identify the characteristics of derivative and antiderivative graphs. 2. Coordinating the processes related to determining the properties of derivative function to identify the characteristics of antiderivative graph.
Devising a Plan)	1. Coordinating the processes related to determining the characteristics of derivative and antiderivative graphs to identify the problem-solving steps
Carrying Out the Plan	1. Coordinating the processes associated with determining the information extracted from the problem, including the procedure for ascertaining the symbolic representation of a graph. This was aimed at determining the calculated result during problem-solving (Symbolic Translation). 2. Coordinating the processes related to determining the relationship between derivative and antiderivative graphs. This included identifying the characteristics of antiderivative functions (Verbal Translation). 3. Coordinating the processes associated with determining the characteristics of a function and how to sketch the graph.
Looking back	1. Coordinating the processes associated with determining the characteristics of derivative and antiderivative graphs to verify the accuracy of computations and answers in problem-solving.

Table 1: Coordination in Each Polya Step

RESULTS

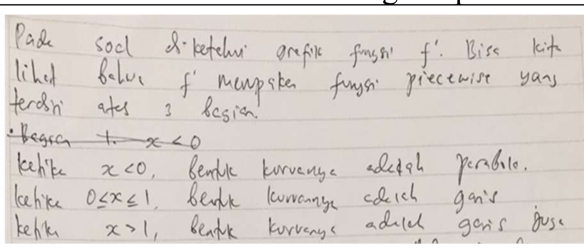
Based on the problem solved by the 44 participants, only four students who met the subject selection criteria were selected. This entailed understanding the problem (viewed from what was written regarding the given problem), coordination, and effective communication. Even though only one of the students could solve the problem correctly, the other three were selected. This was in line with the study's objective, which focused on how coordinating the two processes (Arnon's Coordination) enabled students to solve a reverse problem. The coordination procedure also included how assimilation and accommodation were done in this mental mechanism. Meanwhile, accommodation was realized through two procedures, namely modifying existing schemes or forming new structures and ensuring these were in line with new information (Subanji, 2011). The problem-solving process provided opportunities for the re-assessment of schema through reflection (Sopamena et al., 2018). This could also be achieved through questions, which were intended to access relevant parts of the scheme (Marshall, 2007).

The solving of the problem in Figure 3 provided an opportunity for the subject to reflect and provide intervention through questions. However, this intervention was carried out on three subjects, and one of the coordinations was made. Based on the results of the answers along with transcripts of think-aloud and interviews with the four students, one of them was able to complete the task independently without intervention (Subject 1 or S1). Two others solved the problem after receiving the intervention (Subjects 2 and 3 or S2 and S3, respectively), and the fourth student was unable to solve it despite the intervention (Subject 4 or S4). The presentation of the results related to how the coordination between the two processes was carried out in solving reverse problem is shown through three cognitive actions performed by students. Cognitive actions refer to mental activities used during coordination, and the definition is in line with a previous study (Dubinsky, 1991). The results are presented in two sub-sections, namely in relation to cognitive actions performed during the coordination of the two processes adopted in solving reverse problem and the challenges experienced by students associated with the intervention provided.

The Cognitive Actions in Coordination of Two Processes

Regarding the answers provided, all the subjects used symbolic translations to solve reverse problem that had to do with the graphs of derivative and antiderivative functions. The subjects determined the symbolic representation of the graphs of $f'(x)$ and $f(x)$. In addition, S1 was able to describe the characteristics of the graph of $f'(x)$ and relate it to the components of $f(x)$. S2, S3, and S4 were unable to describe the characteristics of the graph $f'(x)$ completely, nor did they determine the relationship between the two graphs. Based on this context, S2 and S3 were only able to identify the stationary point of the graph $f(x)$ by determining the value of $f'(x)$.

The subjects' coordinations at each step of problem-solving were in line with the method proposed by Polya. In the step of understanding the problems, the subjects' coordinations focused on determining the characteristics $f'(x)$ and $f(x)$ graphs. Coordination related to determining the characteristics of $f'(x)$ was viewed concretely when S1 divided the graph into several intervals, paying attention to the points passed through and their shape to understand the characteristics. The results of S1's written test about understanding the problems are shown in Figure 4, and following the think aloud


Pada soal diketahui grafik fungsi f'. Bisa kita lihat bahwa f' merupakan fungsi piecewise yang terdiri atas 3 bagian.
~~Bagian 1: x<0~~
ketika x<0, bentuk kurvanya adalah parabola.
ketika 0≤x≤1, bentuk kurvanya adalah garis
ketika x>1, bentuk kurvanya adalah garis juga

TRANSLATION:
In the problem, the graph of the function f' was known as the piecewise function consisting of 3 parts.
When $x < 0$, the shape of the curve is a parabola
When $0 < x < 1$, the shape of the curve is a line
When $x > 1$, the shape of the curve is also a line

Figure 4: S1's Written test results in step of understanding the problems

S1: In relation to the problem, the graph of the function f' is known as a piecewise function, consisting of three parts. The first step is to identify the respective functions by dividing the problems' work order into three. The first part is for the interval that, why not less than or equal to? Because this point *(pointing to the coordinate point that is the boundary of the* *graph*), what is the name of this point? The curve obtained was not included (*in the interval being considered)*. For the first part, the negative aspect, excluding zero, was considered. In this context, zero was included in the second part.

Following the work mentioned earlier, it was realized that S1 determined the characteristics of graph starting with the intervals. Based on the think-aloud excerpt, it was found that S1 studied the graph and realized the meaning of the symbols. In addition, S1 explained the reason for writing the interval, implying an understanding of the problem from what was written. This showed that S1 interiorized the action into a process related to determining the interval. S1 also paid attention to the shape of the graph in each interval, as shown in the following think-aloud excerpt:

S1: For this first part, (*determined while looking at the image of the graph*), depicted by the parabolic curve. It was evident that even though the parabolic curve did not pass through the zero point, the zero was undefined. The point served as a reference used to determine the parabolic curve. When is zero, the parabola is at this point *(pointing to a point)*, and the -value is . Similarly, for the other points, when is , the -value is . …

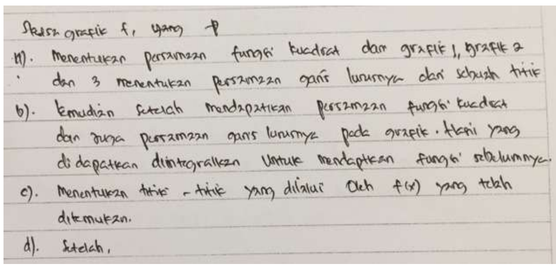
Based on the think-aloud excerpt, S1 observed the graph and mentally remembered the diverse forms and characteristics. S1 explained the reason behind the shape of the graph. This shows that S1 understands the problem situation from what is written. It indicated S1 has interiorized the actions into processes related to determining the form of the graph. S1 remembered, internalized, and mentally processed the procedure of determining the shape of the graph without the help of explicit steps. These took place in S1's mind, showing the student was at the process stage of understanding the concept of polynomial graph forms.

Considering the two processes, S1 determined the characteristics of $f'(x)$ by coordinating. In this coordination, S1 initially ascertained the two processes and then encapsulated the process of determining the intervals. This encapsulation was evidenced by the ability of S1 to take action on the process, including determining the coordinate points at intervals. After that, S1 integrated it into the process of determining the graph shape. The following is an excerpt from a related interview.

R: What does one intend to know about it? How did you come to know these characteristics? (*It is necessary to ask about the characteristics of $f'(x)$as presented by the subject*)
S1: ...Therefore, the shape of the graph was analyzed (*pointing to the graph in the interval*), for example, if this were analyzed, the curve sketch would be familiar. As a result, when this problem was initially encountered, I mainly focused on what would happen if the function were reflected. Assuming it was reflected later, the result would be a parabola, leading to the inference that this must be a quadratic function ...

S1 directly determined the characteristics of $f'(x)$ based on the shape of the graph in each interval. The repeated observations were made only to pay close attention to every detail of the graph image. The procedure showed that the coordination was performed without the need to change the existing scheme or simply by assimilating the object (intervals on the graph) into the process (the procedure for determining its shape). This was due to the familiarity of S1 with the condition. During the coordination, S1 performed cognitive actions determining the processes, encapsulating one into objects, while integrating it through assimilation. S2, S3, and S4 adopted a similar method during coordination in the step of understanding the problems.

In terms of devising a plan, all subjects made a solution plan without giving a detailed explanation of the design. S1 explained the plan related to determining the integration constant but did not explain the plan to sketch the graph. In this context, S2 and S4 explained the plan briefly. Meanwhile, S3 explained in detail the plan for determining the symbolic representation of the graphs $f'(x)$ and $f(x)$, including the procedure before sketching $f(x)$. Similar to S2 and S4, S3 did not explain how to determine the constant of integration; only S1 was able to give a detailed explanation. The results of S3's written test on steps adopted in devising a plan are shown in Figure 5, and the following think-aloud excerpt:


Sketsa grafik f, yang f
a). Menentukan persamaan fungsi kuadrat dari grafik 1, grafik 2 dan 3 menentukan persamaan garis lurusnya dari sebuah titik
b). Kemudian setelah mendapatkan persamaan fungsi kuadrat dan juga persamaan garis lurusnya pada grafik. Hasil yang didapatkan diintegralkan untuk mendapatkan fungsi sebelumnya.
c). Menentukan titik - titik yang dilalui oleh f(x) yang telah ditemukan.
d). Setelah,

TRANSLATION:
Sketch the graph of f
a) Determine the equation of the quadratic function of graphs 1, 2, and 3. Determine the equation of a straight line from a point.
b) Then, after getting the quadratic function and straight line equations on the graph. The results obtained were integrated to get the previous function.
c) Determining the points that pass through $f(x)$ was found.
d) After...

Figure 5: S3's Written Test Results on the Step of Devising a Plan

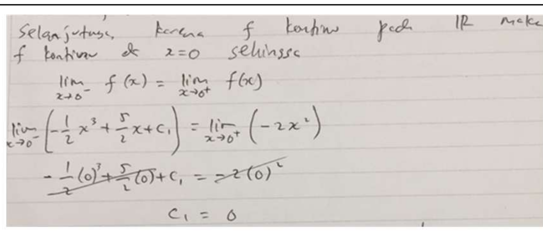
> S3: First, the quadratic function equation was determined from graphs 1, 2, and 3, while the equation of the straight line was obtained from the point (*stops writing and pays attention to the problem*). Furthermore, after obtaining the equation, from graphs 1, 2 and 3, it was integrated. *(Goes back to writing)* to determine the quadratic function and straight-line equations on the graph. The results obtained were integrated to get the previous function (*pauses and pays attention to the problem and the graph image*). After obtaining the graph equation, the points that passed through f(x) were determined *(pauses for a moment)* and then immediately sketched.

In line with Figure 5 and think-aloud excerpts, S3 coordinates to construct the process of sketching the graph of $f(x)$. This included connecting graphical, symbolic, and conceptual processes to understand the sketching procedure of graph $f(x)$. In determining this procedure, S3 related various representations and mathematical operations such as quadratic functions, straight lines, points, and integrals. This showed the coordination between various processes, including the identification of graphs, formation of equations, integration, sketching graphs based on the integration results, and point analysis.

A typical example is the coordination related to determining the symbolic representation of $f(x)$. Regarding the think aloud excerpt above, S3 stated "...that after getting the quadratic function and straight line equations...the results obtained were integrated to get the previous function." S3 started to perform interiorization, which involved trying to understand information related to the characteristics of $f'(x)$, including the relationship between graphs $f'(x)$ and $f(x)$ mentally.

Regarding the two interiorization processes, S3 determined how to find a symbolic representation of $f(x)$ by coordinating. In this context, S1 encapsulated the process of identifying the graph into objects (types of polynomial functions). S3 applied the object to the process of determining the relationship between the two graphs. S3 was able to directly integrate this process and object due to being familiar with the problem. It was inferred that integrating the process and object in coordination was performed through assimilation. In devising a plan, the other subjects also carried out the same cognitive action related to coordination.

In the step of carrying out the plan, all subjects coordinated with the same mechanism as in the previous steps. In this step, the coordination included accommodation in the act of integrating. In coordination to determine the integration constant, S1, S2, and S3 integrate processes and objects through accommodation. S2 and S3 required intervention during the coordination, while S1 performed the coordination without intervention. The results of S1's written test on the step of carrying out the plan to determine the integral constant is shown in Figure 6 and the following think aloud excerpt:



TRANSLATION:
Then since f is continuous on $\mathbb{R}$, f is continuous at $x = 0$, therefore

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

$$\lim_{x\to 0^-}\left(-\frac{1}{2}x^3 + \frac{5}{2}x + C_1\right) = \lim_{x\to 0^+}(-2x^2)$$

$$-\frac{1}{2}(0)^3 + \frac{5}{2}(0) + C_1 = -2(0)^2$$

Figure 6: S1's Written Test Results on the Step of Carrying Out the Plan to Determine the Constant of Integration

S1: ...Here, the left and right continuous were used to find C_1 and C_3. Since f is continuous at $x = 0$, the limit x approaches 0 from the left of f must be equal to the limit x approaches 0 from the right of f, which is equal to the value of $f(x)$...
For x approaches 0 from the left, the value of f is unknown; therefore, it can be written directly, $-\frac{1}{2}x^3 + \frac{5}{2}x + C_1$, equivalent to the limit x approaches 0 from the right of $-2x^2 + C_2$. Already C_2 is 0; therefore, it was unnecessary to write it again. Since this polynomial function is continuous, there is a need to substitute $x = 0$ immediately. It was directly stated that

$-\frac{1}{2}(0)^3 + \frac{5}{2}(0) + C_1$ equals $-2(0)^2$, which could be crossed out, ensuring that C_1 is also 0. ...

Based on the think-aloud excerpt, S1 coordinated two processes to determine the integration constant. This is the coordination between symbolic procedures and calculus concepts**.** S1 combined procedural results (integral) with analytic concepts (limit and continuity) to develop a method for determining the constant of integration. S1 used information related to the algebraic representation of $f(x)$ at each interval. In determining C_1 and C_2, S1 only needed to use information related to $f(0) = 0$ and the concept of function continuity (related to the concept of limits) to determine C_3. S1's statement that *"For x approaches 0 from the left, the value of f is unknown, therefore it can be written directly,$-\frac{1}{2}x^3 + \frac{5}{2}x + C_1$..."* shows that S1 has internalized the process of determining the characteristics of the function associated with its algebraic expression at each interval. This process is done mentally without the aid of explicit steps. Additionally, S1 remembered and internalized the conditions for the continuity of the function $f(x)$ without explicit steps.

Regarding the two interiorization processes, S1 determined how to find the constant of integration through coordination, by encapsulating the adopted process. For example, in determining C_3, S1 encapsulated the process related to the requirements for obtaining continuous functions (the function's left-hand and right-hand limit values were the same). This was evident when S1 performed certain actions, replacing $f(x)$ with a particular equation as in Figure 6. Furthermore, S1 performed cognitive actions to integrate the process and the object. The following is the think-aloud excerpt related to the cognitive action performed:

S1: ...Well, aside from that, f is continuous at $x = 1$ *(pauses for a moment)*. The limit of x approaching 1 from the left of $f(x)$ would be equivalent to the limit x approaching 1 from the right of $f(x)$ *(pauses for a moment)*. When f approaches 1 from the left, the function value is $-2x^2$... therefore, the limit of x approaches 1from the left, and $-2x^2$ would be the same as from the right, and $2x^2 + C_3$. When x approaches 1 from the left, then the argument that these two functions are polynomials was applied to enable direct substitution, equivalent to $-2(1)^2$ and $2(1)^2 + C_3$. As a result, the value of C_3 is the same as -4.

The think-aloud excerpt shows that S1 could not directly integrate the process into the object. Unlike the coordination to determine C_1 and C_2, S1 needed to pause for a moment to rethink the use of continuous function concepts, even though the possibility of its use in the solution plan had been mentioned. This was because S1 was still not sure of their thoughts. The following is an excerpt from the interview:

R: ...After integrating the equation $f'(x)$ why was the concept of a limit considered in this part? *(The subject's work related to determining the constant of integration was reviewed)*.

S1: ...But what about C_1 and C_3, considering that f is a continuous function? Therefore, the value of the left-hand limit must be equal to the right-hand limit for $x < 0$. The limit of x approaching 1 from left must also be equal to the value approaching 1 from right. It enabled the immediate carrying out of the task, alongside getting used to solving similar problems.

R: So, you can directly determine the use of the left-hand and right-hand limits**.**

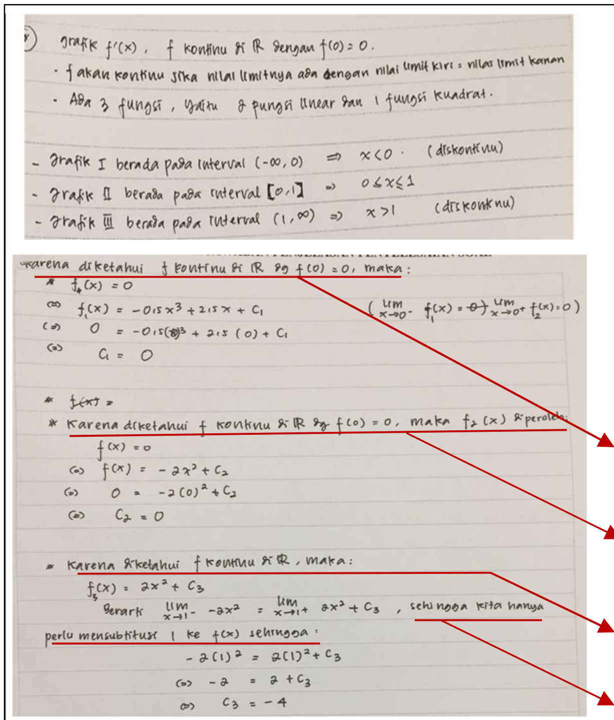
S1: Yes, I thought of it right away...

R: Were any difficulties encountered in coordinating what was in mind concerning the concept of limit, and integral including the sketching process?

S1: Yes, occasionally I was confused about when to use this and that *(while pointing to the information on the picture related to the characteristics of the graph f(x), which is f(x) continuous in ℝ)*, leading to confusion in terms of usage *(pointing to the determination of C_3)*.

The excerpt shows that S1 needed to make accommodation in the integration action. The accommodation was performed by students in terms of trying to build a new understanding that the constant of integration could be determined, although not arbitrarily. This was based on the continuity of the function, with the adaptation process performed without any intervention.

S2 and S3 needed intervention to coordinate the processes. The results of S2's written test in the steps of understanding the problems and carrying out the plan to determine the integration constant are shown in Figure 7 and the following think-aloud excerpt:



TRANSLATION:

S2's Written Test Results in the Step of Understanding the Problem

Graph $,f'(x)f$ is continuous in $\mathbb{R}$ with $.f(0) = 0$

- f is continuous if its limit values exist, with the left-hand limit value equal to the right.
- There were three functions, two linear and one quadratic.

graph I is on the interval$(-\infty, 0) \Rightarrow x < 0$ (discontinue)
graph II is on the interval $[0,1] \Rightarrow 0 \leq x \leq 1$
graph III is on the interval$(1, \infty) \Rightarrow x > 1$ (discontinue)

S2's Written Test Results in the Step of Carrying Out the Plan

Since f is continuous in $\mathbb{R}$ with$f(0) = 0$ then:
...
Since f is continuous in $\mathbb{R}$ with$f(0) = 0$ then$f_2(x)$ is obtained:
...
Since f is continuous in $\mathbb{R}$, then:
....
Therefore, 1 only needs to be substituted into $f(x)$ so that...

Figure 7: S1's Written test results in the steps of understanding the problems and carrying out the plan

(The intervention for S2 was realized by asking some questions. The first intervention to determine the processes to be coordinated.)

R: In this context, it was written that $f\ '(x)$ was discontinuous in this interval, while it was continuous for $f(x)$, give reasons?
S2: Oh yes, this implies that a description of f in $\mathbb{R}$ is continuous. (*Writes down the determination of* C_1)...
R: What does continuous mean?
S2: The limit and value of the function were the same.
R: Is there any other condition for a continuous function?
S2: A function is continuous if its limit value exists. In addition, it should be equal to the function value.
R: Do you remember the left-hand and right-hand limits?
S2: The limit exists if the limit from the left is equal to the one from the right....
R: In this interval *(pointing to the intervals $x < 0$)*, 0 was excluded, but the left-hand limit should be included.
S2: It should be equal to the right-hand limit.
R: Therefore, the limit should be the same as this one *(pointing to the intervals $x < 0$ and $0 \leq x \leq 1$)*? Try to write it down.
S2: So $\lim_{x \to 0^-} f_1(x) = \lim_{x \to 0^+} f_2(x)$ (*writes on the answer sheet next to the determination of* C_1).

In line with Figure 7 and the think-aloud excerpt, several instructions including a question, were given as an intervention to S2. The first intervention was needed to understand the information used in coordination. This aimed to construct the process related to the procedure for determining the integration constant. S2 understood the information associated with the problem. However, S2 had difficulty ascertaining the concepts to use in unfamiliar conditions. S2 also needed a second intervention to determine the integral result of $f'(x)$ through the point $x = 1$. This showed the inability to encapsulate the process related to continuous function requirements. The result was also evident when S2 was unable to take action to determine the equation for x approaching 1. The following think-aloud excerpt also proved the need for intervention:

(The intervention for S2 is to ask some questions. Second Intervention to encapsulate a process into an object)

S2: And then the third (*pauses for a moment*).
R: Notice this *(pointing to the graph of $x > 1$),* because in the graph at, $x < 0$, $x = 1$ was not included, right?
S2: Yes, ma'am *(S2 writes back)*. Since f is continuous in $\mathbb{R}$, then for $f_3(x) = 2x^2 + C_3$.
Ma'am, this is different from the previous calculations (*points to C_1 and C_2, pauses, and stops working*)

R: What did you write earlier *(pointing to the writing above in relation to* f *continuous in* $\mathbb{R}$*)*, therefore, for this function at $x = 1$, how? Based on what was initially written, what are the left-hand and right-hand limits?
S2: As a result, the limit approaches 1 both from the left of $f(x)$ and from the right... (*and then speechless*)

Considering the excerpt mentioned earlier, S2 was unable to determine the value of the integration constant in the interval $x > 1$. This was due to the inability to enter the value of $x = 1$ in the interval directly. S2 felt that although f is continuous, in the interval (for $x = 1$) it did not enter the result area of the function. The process is clearly explained in the following interview excerpt:

R: What other difficulties were encountered while working on this problem?
S2: When trying to find $\boldsymbol{C_3}$ and plotting the graph, including the third power
R: What difficulty was encountered in the part about finding $\boldsymbol{C_3}$?
S2: Because there was no circle sign, Ma'am *(pointing to the symbol of boundary point of graph in the interval* $0 \leq x \leq 1$*, closed circle sign),* I had to rethink the process. This implied that at that point *(point* $x = 1$ *in the interval* $0 \leq x \leq 1$*)*, the limit as x approaches 1 exists. However, I had a rethink that here the interval started from 0 to 1 *(pointing to the graph in the interval* $0 \leq x \leq 1$ *, which is from point* $(0,0)$ *to* $(1,-4)$*)*. I was unsure whether to substitute $x = 1$ from the left or the right. This led to the assumption of equating the values *(pointing to the graph in the interval* $0 \leq x \leq 1$*, which is* $x = 1, y = -4$*)* and then finding x again *(pointing to the graph in the interval* $x > 1$*)*.

Think-aloud and interview excerpts show that S2 was unable to perform the action of calculating the integration constant associated with the continuous function requirement and limit concept in the interval $x > 1$. Meanwhile, in the interval $0 \leq x \leq 1$, S2 was able to perform this action, resulting in the conclusion that S2 had difficulty performing the same action in different situations, including diverse intervals. The entire scenario showed S2's inability to encapsulate the process into an object (the result of integration). It was also reported that S2 required an intervention to reconstruct their understanding related to determining the integration constant. The result was evident in the following think-aloud and interview excerpts related to the determination of C_3:

(The intervention for S2 is to ask some questions. Third intervention to integrate a process and an object)

R: It was presumed that at $x = 1$, which part was the $f(x)$?
S2: The ones in $f_2(x)$ and $f_3(x)$.
R: Yes, therefore, which equations are $-2x^2$ and $2x^2 + C_3$?
S2: Yes, ma'am.
R: As a result, one only needs to enter the value of x, which is equal to 1?
S2: *(S2 remained silent momentarily before looking at the graph of f'(x))* Oh, yes. *(writes back)*

It means that $\lim_{x \to 1^-} -2x^2 = \lim_{x \to 1^+} 2x^2 + C_3$, substituting 1 into $f(x)$, so $-2(1)^2 = 2(1)^2 + C_3 \ldots$

Interview excerpt explaining this was realized after S2 solved the problem:

R: At first, did you know whether you would need to use the concept of continuity and the information that $f(0) = 0$? Did you immediately think of using them?

S2: Perhaps it was because the problems I usually work on never contain anything like this (pointing to the information stating that f is continuous and f(0) = 0). So, actually, there was information on this problem that could be used to determine the value of C. But I forgot it at the time, Ma'am.

The think-aloud excerpt and interview shows that S2 performed cognitive actions to integrate a process and an object through accommodation. This process involved determining the characteristics of the function *f* in relation to its algebraic expression at each interval. It led to an understanding that the integration constant is not arbitrary but rather based on the continuity of the function (as an object). This demonstrates that the process was reconstructed to accommodate this understanding.

In the looking-back step, the three subjects (those who were able to solve the problem correctly) did not carry it out. However, during the interview, the researcher attempted to ask the subjects for their thoughts on the results of their problem-solving process and the possibility of alternative methods to solve the problem. All three subjects shared the same perspective that the evaluation of their work was based on the accuracy of the calculations and the conformity of the graph's shape with the typical forms of quadratic and cubic functions. Moreover, the three subjects were unable to identify alternative ways to solve the problem other than the method they had used (i.e., determining the symbolic representation from the graphical representation).

Following Arnon's Coordination, the procedure started with two processes that had been determined. The procedure was then continued with the cognitive action of encapsulation and integration of process and object through assimilation or accommodation as shown in Figure 2. Considering the results of the study, the subjects determined the appropriate processes from several existing concepts in order to perform coordination. This was in relation with the schema selection, and during problem-solving, an individual was confronted with the selection process. Therefore, in this study, the coordination of the two processes was described as in Figure 8.

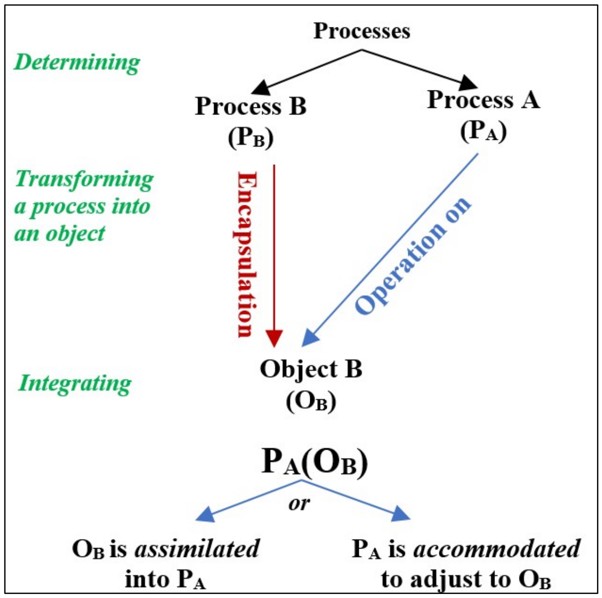


Figure 8: Mechanism of Coordination Based on Study Results

The analysis produced results depicting that the coordination of the two processes consisted of three cognitive actions, namely determining the processes, transforming a process to an object or encapsulation, and integration into another process. In addition, the integration of a process and an object was realized through assimilation or accommodation, pertaining to the conformity of the problems' structure with individual mental structure. The mechanism of coordination was described from the subject's selection of the processes to their integration, as shown in Figure 8.

The Challenges of Each Cognitive Action

During each cognitive action, the subjects faced certain challenges. S1 experienced challenges in determining the process and handled it without intervention. Meanwhile, S2, S3, and S4 required intervention in one of the coordination performed. Only S2 and S3 were able to coordinate after the intervention. The following interview excerpt showed how the challenges were handled by S2 and S3:

R: At the beginning were you aware that you could use the concept of continuity and information related to the value of $f(0) = 0$? Did you immediately think it could be used?

S2: ...because the problem I had worked on was not like this (*pointing to the problem description regarding the continuity of f and the information $f(0)$*). When solving the problem, the information extracted could be used to determine the value of C, but I did not seem to remember that idea (*pointing to what was written about what was known from the problem*).

Interview excerpt of S3:

R: So, did you have time to think about something to be able to determine the C value? This led to an explanation associated with continuous f in relation to the limit and $f(0) = 0$ *(points to S3's writing about what is known from the problem)*. Did you think this would be used?
S3: no *(while shaking his head)*
...
R: So, you were confused when connecting the concepts of continuous function, limit and integral?
S3: Yes, and because I already knew about the nature of continuous functions *(pointing to what was written in relation to the information extracted from the problem)*. But I did not think I would use it to find the value of the constant of integration. I thought there was a known f that is continuous after a question about the known information from the problem was given.

Based on the interview excerpts above, it was found that S2 and S3 encountered difficulties in determining the processes to be used in coordination. S4 experienced the same issue. Furthermore, S2, S3, and S4 also struggled with the action of encapsulation. This was evident in the previous think-aloud excerpt from S2, which related to the intervention given to support the encapsulation process within coordination for determining the integration constant. A similar difficulty was experienced by S4, leading the researcher to provide an intervention. However, even after the intervention, S4 could still not solve the problem. The following are the results of the think-aloud session and the intervention provided:

S4: ... because we have got the integral, then we draw this... *(speechless)*
R: Try to pay attention to what is known
(S4 was silent and only paid attention to the description part and pointed to f(0)=0 and f is continuous in $\mathbb{R}$*)*
R: It is continuous on $\mathbb{R}$ and $f(0) = 0$, what do you think you know?
S4: If it is continuous, the process implies that the value of the left-hand and right-hand limits would be similar.
R: Well, try to think about whether it can be used?
S4: Then this is $2x^2 + C_1$ on the interval $x > 1$ and $-2x^2 + C_2$ on the interval $0 \leq x \leq 1$. This depicts that C_1 and C_2 are similar. *(stops writing and falls silent)*
R: How can they be the same?
S4: This is because the left-hand and right-hand limits are the same. (*pauses*) Oh no, what should I do, Ma'am? I'm confused.

The think-aloud excerpt proves that S4 was unable to properly use one of the conditions for a continuous function to obtain an integration constant. S4 stated that because the limit values from left and right were the same, the integration constant in the intervals $x > 1$ and $0 \leq x \leq 1$ would be similar. This showed that S4 could determine the process used but was unable to engage in encapsulation in relation to the continuous function condition. The result was evident when S4 was unable to perform the action of calculating the integration constant. Regarding this perspective, intervention was stopped because of the inability of S4. The study results showed that each cognitive action presented certain challenges, as shown in Table 2.

Cognitive Actions	Challenges
Determining the Processes	Subjects had difficulty deciding which process to select. This was due to unfamiliar conditions.
Transforming a Process to an Object	Subjects had difficulty encapsulating the process into an object. This was because students were unable to view familiar situations in a completely new way or under conditions different from before.
Integrating an Object into Another Process	Subjects had difficulty making accommodations through the reconstruction of their respective schemes. This was due to students' inability to build relationships between the concepts they possessed.

Table 2: The Challenges of each cognitive action

In accordance with Table 2, each cognitive action was characterized by the challenges faced by the subjects. These had causes that were related to the understanding of the basic calculus concepts, including the coherence of schemes owned by subjects.

DISCUSSION

Coordination is a mental mechanism used to construct new processes in APOS theory. This is defined as the composition of two or more processes used to construct a new one (Dubinsky, 2002). The new processes are constructed to be adopted in solving problems (Akgul & Yilmaz, 2023; Cetin & Dubinsky, 2017; Planell & Trigueros, 2019; Syarifuddin et al., 2020) or to learn a concept (Arnon et al., 2014). In this context, Arnon et al. (2014) hypothesized the coordination mechanism (Arnon's Coordination), which explains how individuals mentally coordinate two processes. The hypothesis is still recommended for future study. Therefore, an educator needs to understand how the mechanism functions. This is related to the focus of APOS theory, which provides a framework for understanding mental processes that occur when individuals learn mathematical concepts. Considering the description, the mechanism serves as a basis for developing lessons and evaluating student performance in solving mathematical problems (Arnon et al., 2014). In this study, a reverse problem was selected to determine the mental processes that occurred when students coordinated the processes in problem solving. This was because in some studies, direct and reverse problems were used to reconstruct individual mental processes associated with reversibility (Ikram et al., 2021).

The results describe how students coordinated two processes to solve the reverse problem. These results were proven through three cognitive actions performed by students in each coordination. The actions include determining the processes, transforming one of them into an object, and integrating with other processes that have been determined. This suggests that the hypothesis proposed by Arnon et al. (2014) excluded the act of determining processes. In Arnon et al.'s hypothesis (Arnon's Coordination), the coordination implementation mechanism is described by a predetermined process shown in Figure 2. However, as this is a crucial factor in the success of coordination, the description of the mechanism in the present study started with the cognitive action of determining the processes, as in Figure 8. The result was in line with the theory proposed by Dubinsky (1991) that in coordination, there is a cognitive action of taking two or more processes and using them to build a new process. However, in this study, we term taking processes as determining processes.

The results show that the first cognitive action of determining processes plays an important role in coordination. Students did this cognitive act by determining the necessary processes based on the adopted concepts. This finding is consistent with those of Dubinsky (2002), who stated that coordination entails the evaluation of cognitive actions aimed at identifying pre-existing processes in order to construct new concepts. However, not all students can easily do it. According to the research results description, three students could not determine the integration constant. They were unable to do so because they could not construct a new process related to determining the integration constant through coordination. They were unfamiliar with the given problem. This is evident from the results of their work in the stage of devising a plan, in which they failed to devise a plan related to determining the integration constant. However, when determining the algebraic representation of, they realized a method was needed to determine the integration constant. Based on interviews, a significant challenge encountered in problem-solving was associated with determining the concepts used. In this study, three students were given interventions, in the form of questions that helped them reflect on what had been done. This was in accordance with the findings of Sopamena et al. (2018), which emphasized the need to provide students with opportunities to re-access their respective schemas through reflection.

The intervention enabled the three students to determine the processes to be coordinated. The intervention, in the form of questions, helped students review what had been written in relation to the step of understanding the problems. After realizing that the written information had not been used, the three subjects started to think about the possibility of its adoption. The response was in accordance with the theory proposed by Marshall (2007), which states that questions can be asked with the aim of accessing relevant parts of the scheme. After determining the processes used, the coordination process was continued by performing the second cognitive action.

The second cognitive action focused on transforming a process into an object through the encapsulation mechanism. Furthermore, encapsulation occurs when an individual performs actions on a process, realizing it as a totality and entity (Arnon et al., 2014; Cetin & Dubinsky, 2017). The results showed that cognitive action was evidenced by the ability of students to perform certain actions in a process. These included dividing the graph based on the interval, determining the location of a point, and the shape of the graph, performing integration and limit operations on a function, as well as sketching a function graph and others.

The cognitive act of transforming processes into objects played a crucial role in coordination. Students who encapsulated the process were able to continue the next cognitive action on coordination. Based on the results obtained, as well as in the first cognitive action, three students experienced obstacles. This also occurred during the coordination to construct new processes related to the determination of integration constants. The three students needed intervention. However, despite receiving intervention, one of the three students was unable to transform the process into an object. This occurred while she was determining the integration constant. She could not encapsulate the process related to obtaining continuous functions. This was in accordance with Borji & Planell (2023), who stated that students perform coordination by applying one process to an encapsulated process to construct a new process. The result was also in line with the hypothesis proposed by Arnon et al. (2014), that in the coordination of two processes, one of the processes was encapsulated into an object, enabling the other to be applied.

The challenge encountered in this second cognitive action is associated with the ability of an individual to summarize a concept by constructing its meaning. As generally understood, encapsulation entails the personification of a concept, specifically, a set of abstract ideas to make it meaningful (Cappetta & Zollman, 2013). In this study, students were unable to interpret the results of integration to perform limit operations. Although students had successfully performed similar actions on the integration, results were obtained at different intervals. This is consistent with the theory proposed by Sfard (1991), which focuses on the inherent difficulty of reification (similar to encapsulation in APOS Theory). The theory stated that the ability to detect familiarity in a completely new way was difficult to achieve. This difficulty arose when a process was not converted into an object (Arnon et al., 2014). The object that had been generated was integrated with the other process that had been determined.

The third cognitive action centered on integrating objects into a process that had been determined. The results show that when students coordinated the two previously owned processes under the condition of a familiar problem, the cognitive action could be performed directly, without intervention. The hypothesis proposed by Arnon et al. (2014) relates to how an individual assimilates the object, including applying the process. The study reported that students could directly determine how an action was applied to the object. The response was consistent with expert findings, which stated that the assimilation of knowledge referred to the mechanism by which an individual can apply the schema owned without alteration. This was because the problems' structure was in line with the schema already owned (Arnon et al., 2014; Blake & Pope, 2008; Subanji & Nusantara, 2016). However, not all of these cognitive actions were realized through assimilation.

In this context, coordination included the impact of accommodation on the cognitive action of integration. The results show that this occurred when students coordinated two processes they already possessed, but in a new or unfamiliar situation. In addition, students obtained accommodation by reconstructing the process they owned, adjusting the cognitive object, and applying it to the process. This response supported the result of Arnon et al. (2014) that in accommodation, the scheme is reconstructed and modified to address new situations. The reconstruction of this process was done when students built a new understanding that the constant of integration can be determined, based on the continuity of the function. In this study, some students received interventions that enabled the reconstruction process. This challenge was encountered because subjects had difficulty obtaining accommodation through the reconstruction of the schemes. The difficulty was also caused by students' inability to build relationships between the concepts. Therefore, the intervention was realized by questions based on what had been written in terms of building relationships between concepts. This result supports the view of Corral et al. (2019), who points out that the core difficulty in learning mathematics lies in grasping the interrelationships among concepts within a problem, rather than merely executing solution strategies. Previous studies from different perspectives have demonstrated challenges in solving reverse problems associated with understanding the relationships between concepts.

The challenges students encounter in solving reverse problems related to the graphs of derivative and antiderivative functions have been reported in previous studies from different perspectives. For example, in studies focused on mathematical connections (García & Flores, 2021; Nieto et al., 2023), schema formation relied on derivative concepts (Fuentealba et al., 2019), representation translation (Swastika et al., 2023), and reversibility (Nieto et al., 2023). In line with this study, a proper understanding of the basic calculus concepts played a crucial role in reverse problem-solving. Therefore, educators are recommended to provide conceptual learning that focused on the relationship between concepts, including graphical-symbolic translation. This is because the connection processes enabled solving calculus tasks related to reversibility and translations of representations, such as exercises related to drawing and without symbolic expression (García & Flores, 2020, 2021; Nieto et al., 2023).

The results of this study show how the coordination of the two processes was associated with Arnon's hypothesis, and the challenges faced by students in each cognitive action when solving the reverse problem. The three challenges affected students' ability to construct a new process through coordination in each problem-solving step. For example, in terms of devising a plan, only one subject identified the plan for determining the constant of integration, alongside the step adopted during the execution process without intervention. Therefore, the results should be considered by educators when designing lessons that focus on the cultivation of deep concepts. This is related to the coherence of how actions, processes, and objects were interconnected and organized in larger mental structures called schemas. Regarding the description, schema coherence allow an individual to decide whether a schema can be used in a particular mathematical situation (Cetin & Dubinsky, 2017).

Following the description, educators are advised to design lessons that explicitly motivated students to analyze and reflect on the actions, processes, and mathematical objects used, as well as connect new schemas with existing concepts. This learning enables students to build a deeper and more applicable understanding of mathematical concepts through APOS-based learning. Borji et al. (2018) proved that APOS-ACE-based learning effectively helped students develop a deeper understanding through reflection and schema connection. Another possible solution is the provision of calculus-related problem-solving exercises with different contexts, using similar mathematical concepts. It also helps students to recognize patterns of problems in diverse forms. The result is in line with the finding of Setyaningsih et al. (2018), which states that an individual's schema can be developed through problems given. These problems can include reverse problems involving the translation process between representations, such as the one provided to students in this study. In addition, educators are expected to evaluate the learning process by asking reflective questions to determine the extent to which students understand a concept. The results of such evaluations can inform the design of instruction that supports the development of students' conceptual understanding.

CONCLUSIONS

In conclusion, the mechanism of coordination between two processes carried out by students in solving the reverse problem associated with the graph of derivatives and antiderivative functions is explained through three cognitive actions. These include determining processes, transforming a process into an object through encapsulation, and integration through assimilation or accommodation. This result shows differences in line with the hypothesis proposed by Arnon regarding the coordination (Arnon's Coordination). The difference is the first cognitive action, namely the determination of the process. Based on the results, this cognitive action affects the success of coordination.

Other results are related to challenges encountered during each cognitive action and its causes. In the first cognitive action, students had difficulty determining the process to be selected. This was because the conditions of the problem were unfamiliar or had not been encountered. In the second cognitive action, students had difficulty encapsulating the process into an object due to the inability to view familiar tasks in a completely new way or in different conditions. Meanwhile, in the third cognitive action, students encountered difficulty making accommodations through the reconstruction of schemes owned because of the inability to build relationships between the concepts. These three challenges depended on students' understanding of the basic concepts of calculus and the coherence of their respective schemes.

Regarding the results obtained, students' success in the mental mechanism of coordination in solving reverse problems is influenced by conceptual understanding. Therefore, educators are recommended to design lessons that explicitly motivate analysis, alongside the reflection of an action, process, and object, including the connection of new and existing schemas. Another possible solution is to provide calculus-related problem-solving exercises with different contexts using similar mathematical concepts. The exercises are expected to be direct and reverse problems, including the translation process of representation. In addition, educators are expected to evaluate learning through reflective questions to determine the understanding level of students' concepts. This study recommends future analyses related to the effectiveness of learning strategies designed based on the results associated with improving students' understanding of mathematical concepts and solving reverse problems.

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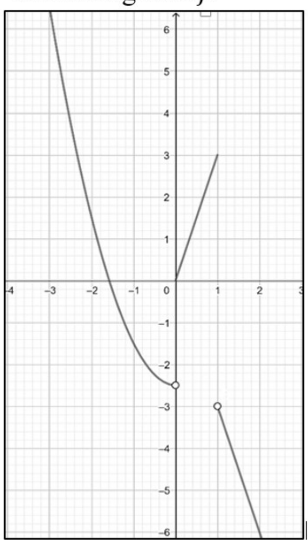
APPENDIX

Appendix 1. Pre-Test Problems

Permasalahan:
Grafik $f'(x)$ disajikan pada Gambar 1. Jika f kontinu pada $\mathbb{R}$ dengan $f(0) = 0$, sketsalah grafik f.



TRANSLATION

Problem:
The graph of $f'(x)$ is shown in Figure 1. If f is continuous on $\mathbb{R}$ with $f(0) = 0$, sketch the graph of f.

Note: The pre-test problem is only used to select students who will be used as research subjects