A Framework for Mathematical Proof: A Combination of Deduction, Induction, and Abduction

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Abstract: This study explores the use of abductive reasoning by Mathematics Education students in mathematical proofs, integrating deduction, induction, and abduction. The proof task involved two functions: the first type was completed by 8 participants, and the second type by 13 participants, both focusing on number theory. Following the task, in-depth interviews were conducted with two participants from each problem type and two additional participants for source triangulation to validate the data. Data were analyzed through the reduction and presentation of relevant information to draw research conclusions. The results showed that deduction is the primary approach used by students because it produces definite conclusions. However, many students utilize a combination of deduction, induction, and abduction. Abduction is often used as the first step to formulate a hypothesis, deduction is applied to verify the truth, and induction helps identify patterns, although it is less dominant. These mixed strategies reflect the complex and varied process of mathematical proof. This study confirms the importance of a deep understanding of different types of reasoning in mathematical proof to foster critical and analytical thinking skills. Further studies are recommended to develop a theoretical model of the interaction among deduction, induction, and abduction in mathematical proof to strengthen pedagogy at the higher education level.

Keywords: Deductive reasoning, inductive reasoning, abductive reasoning, mathematical proof, mathematics education.

INTRODUCTION

Mathematical proofs provide a way of thinking that deepens an individual's understanding of mathematics. According to Szabo (2016), a proof is a collection of true and logical statements that serve as arguments to confirm the truth of mathematical statements. Proofs also function as an explanatory and discovery tool that helps individuals understand why a statement is true. In addition, proofs can be used as a discovery tool closely related to exploration activities. Exploration leads to further study of a definition to explore its full meaning. The role of proofs as exploration tools also becomes apparent when a proved theorem leads to the discovery of new ideas (Hanna, 1995).

According to NCTM (2000), proof is an important part of mathematical understanding, and every student should be able to recognize, develop, and apply various methods of proof. Successfully proving a statement confirms a truth and paves the way for further discovery and understanding. This aligns with the learning outcomes of mathematics education study programs in general, namely the ability to apply logical, critical, systematic, and innovative thinking in the context of developing or implementing knowledge. Therefore, many mathematical theorems are discovered through the proof process (Cellucci, 2017).

By understanding the proof process, Mathematics Education students can more easily access new theorems and concepts (Stylianides et al. 2016). Through the proof process, students develop critical thinking skills that are indispensable in everyday life. This implies that students must understand the role of proof in the overall development of mathematics, including how theorems are proved, discoveries are made, and how mathematical ideas develop over time. Thus, Mathematics Education students are expected to apply proof concepts in solving mathematical problems or in applying theorems in scientific or technical contexts (Hanna & De Villiers, 2012)

One type of proof studied by Mathematics Education students is algebraic proof (Hanna, 2020). Algebraic proof involves algebraic manipulations, such as expanding and factoring mathematical expressions to prove statements involving integers or algebraic terms. It is a fundamental aspect of mathematics because it allows for establishing the truth of mathematical statements with certainty and provides a strong basis for further mathematical development. An example of an algebraic proof problem is: "Prove that the product of two odd numbers will always be odd". This issue involves expressing even and odd numbers algebraically and using them to prove that the product of two odd numbers will always be odd.

Mathematical proofs require valid and accurate deductive reasoning (Komatsu & Jones, 2022). This emphasizes that mathematical proofs rely on valid deductive reasoning to ascertain conclusions from premises and the correctness of each logical step. These requirements guarantee the reliability and certainty of mathematical knowledge and findings. Deductive reasoning involves drawing logical conclusions based on premises (Sinaga et al. 2024). Premises are assumptions, ideas, and foundational statements considered true. If the premises are true, then the conclusion can be confirmed as valid.

In addition to deductive reasoning, inductive and abductive reasoning can be used in the process of mathematical proof when drawing conclusions (Salsabila et al. 2020). Inductive reasoning draws conclusions from general premises (observations, data, and facts) and then draws specific findings. Forms of inductive reasoning include generalization, analogy, and causal relationships. Through inductive reasoning and the observation of the phenomena, one can understand various patterns and regularities, the properties of mathematical objects, and the relationships between concepts based on specific examples. In addition, this reasoning helps to form mathematical conjectures or hypotheses and initial intuitions that form the basis for determining the direction of further proof (Conner et al. 2014).

The third type of reasoning in the mathematical proof process, abductive reasoning, is used to generate new ideas and innovative solutions. Abductive reasoning is a creative and explorative process that allows individuals to propose new ideas and innovative solutions by proposing plausible explanations for observed phenomena. Abductive reasoning encourages imaginative thinking and can lead to breakthroughs in problem-solving and decision-making (Durand-Guerrier et al. 2012). By using abductive reasoning, individuals can correctly describe the interconnectedness of the foundational principles of logic and mathematics, which are not universal generalizations of arithmetic truths or concrete branches of mathematics.

Deductive, inductive, and abductive reasoning are structural aspects of mathematical reasoning. These reasoning processes are activities that draw logically related conclusions. Figure 1 is the researcher's description of the relationship between deductive, inductive, and abductive reasoning.

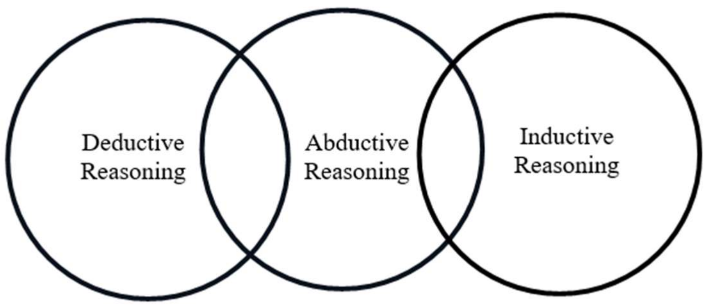


Figure 1. Relevance of Deductive, Inductive, and Abductive Reasoning

When individuals perform mathematical proofs, they may use deductive, inductive, or abductive reasoning. However, abductive reasoning can also occur when an individual (a Mathematics Education student) employs deductive or inductive reasoning. This happens because abductive reasoning is a form of logical inference based on the most likely explanation or hypothesis derived from a given set of observations or data. Unlike deductive reasoning, which draws conclusions from known premises, and inductive reasoning, which generalizes conclusions from observed patterns, abductive reasoning involves proposing plausible explanations or hypotheses to explain observed phenomena (Behfar & Okhuysen, 2018). These hypotheses may not be definitively proven but are chosen because they best fit the available evidence.

Within the realm of deductive and inductive reasoning, abductive reasoning offers a unique perspective, serving as a bridge between observed evidence and the generation of new evidence (Widadah, et al.2024). Abductive reasoning involves constructing plausible explanations or hypotheses to explain observed evidence or facts. Unlike deductive reasoning, which moves from general principles to specific conclusions, and inductive reasoning, which derives from general principles from specific observations, abductive reasoning operates in the realm of uncertainty. It is the art of identifying the best explanation among the available hypotheses.

In the context of proof, abductive reasoning serves as a starting point for logical exploration (Dong, et al. 2015). The best explanation identified through abductive reasoning becomes a hypothesis to be tested and validated. While abductive reasoning itself does not establish proof, it initiates the journey toward evidential support and logical validation. The interaction between abductive reasoning, deduction, and induction is crucial in the process of proving a statement. Abductive reasoning often leads to the formulation of hypotheses, which are then tested through deductive or inductive validation. Deductive reasoning verifies the logical consequences of the hypothesis, while inductive reasoning tests the generalization of the hypothesis to a broader context.

While not a stand-alone proof mechanism, abductive reasoning sets the stage for further exploration and validation. The relationship between abductive reasoning and proof underscores the dynamic and iterative nature of logical inquiry, where the journey from observed evidence to truth involves a complex state of hypothesis generation, testing, and refinement (Van Gelder & Wilcox, 2022). In the quest to understand and prove propositions, abductive reasoning serves as a valuable unit that guides the path toward logical validation and truth formation. Figure 2 shows the relationship between mathematical reasoning and abductive reasoning.

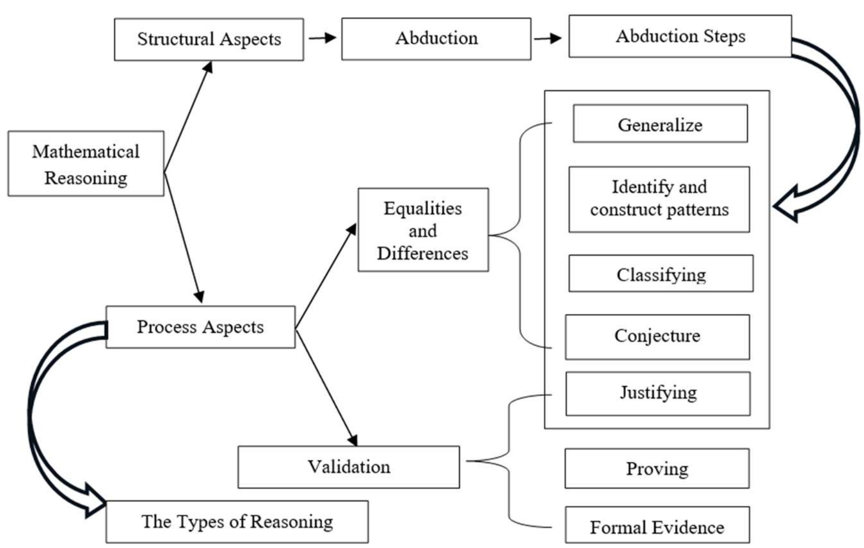


Figure 2. Mathematical Reasoning Relationship to Abductive Reasoning

Abduction is the structural aspect of mathematical reasoning, while the process aspect involves the search for similarities and differences, as well as the search for validation. The search for similarities and differences includes generalizing, identifying and building patterns, classifying, and conjecturing. The validation aspect comprises justifying, proving, and formal proof (Sommerhoff & Ufer, 2019). The search for similarities and differences is part of the abductive reasoning process, which involves observing, proposing hypotheses, evaluating hypotheses, identifying the best explanation, and justifying arguments. Justifying is the validation-seeking aspect of the reasoning process, while other validation-seeking aspects include proving and formal proof.

Based on these various explanations, the conceptual framework in this study is illustrated in Figure 3.

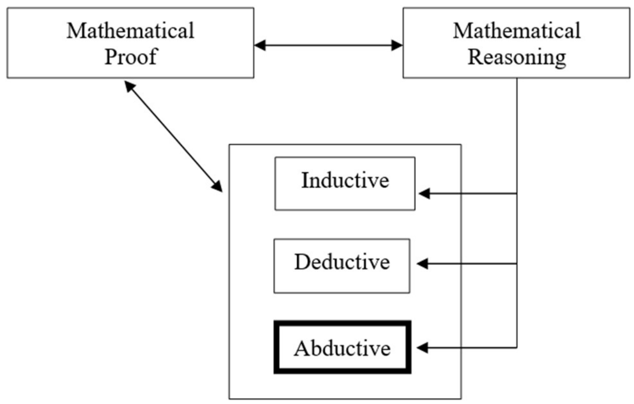


Figure 3. Conceptual Framework

Algebraic proof is part of mathematical proof, which is an integral aspect of the mathematical reasoning process aimed at establishing the truth or validity of statements or propositions. Proof is crucial to ensure the rigor and certainty of knowledge, including algebraic proof. Proofs involve a series of logically connected statements that follow from previous statements. The logical structure ensures that the conclusion is a necessary consequence of the premises (Tennant, 2015). Algebraic proof relies on logical deduction, and, in addition to this, algebraic proof provides a creative and exploratory foundation for conjectures and ideas.

While mathematical reasoning is a broader process of exploring, understanding, and reasoning about mathematical concepts, mathematical proof is a specific aspect of reasoning aimed at establishing the truth or validity of mathematical statements through logical deduction. The two are interconnected, with reasoning laying the foundation for the formulation of conjectures and proof providing formal verification of those conjectures. This statement confirms that the process of mathematical reasoning leads to the formulation of mathematical proofs. Proofs serve as formal representations of the logical reasoning process used to establish the truth of algebraic statements. An algebraic proof is a concrete and systematic presentation of the logical steps taken during the reasoning process (Goguen, 2021). In this way, an algebraic proof becomes a visible or tangible confirmation of the reasoning. The act of proving mathematically is a tangible outcome of the broader cognitive activity of mathematical reasoning.

The urgency of this research lies in its potential to enrich both the theoretical and practical understanding of mathematical proof, improve learning effectiveness, and support the development of critical and creative thinking skills in mathematics education through proof construction. The questions in this study are as follows:

1) How is a framework that combines deductive, inductive, and abductive reasoning applied by students in mathematical proof?

2) What is the role of each type of reasoning (deductive, inductive, abductive) in mathematical proof construction according to the proposed framework?

3) How do students utilize deductive, inductive, and abductive reasoning in constructing mathematical proofs?

METHODS

This study explores the reasoning employed by mathematics education students in constructing proofs. This research is a mixed-method study, with a greater emphasis on the qualitative data section. Quantitatively, we only present the number of students who use abductive, inductive, and deductive reasoning in completing proof tasks. The subjects in this study were mathematics education students who had studied number theory and were at the second-year level. The students were only familiar with two types of reasoning: inductive and abductive. There was no specific training about types of reasoning; they learned about these types while studying number theory. A total of 21 students participated in this study, which corresponds to the number of sophomores at the research site. After completing the proof task, we interviewed two students from each task type. The interview aimed to deepen the data obtained from the proof task that had been completed. The criteria for selecting students who were interviewed were: 1) students who worked with at least two types of reasoning, and 2) students who were willing to be interviewed. The proof tasks were designed as follows:

Task type 1: "If a and b are odd numbers, show that $8|(a^2 - b^2)$."
Task type 2: "Show that the square of an odd number results in an odd number."

The objectives of the two tasks are: 1) students can observe numerical patterns and results, and develop conjectures based on observations; 2) students can generalize numerical patterns using algebraic forms; and 3) students can develop formal proofs logically from conjectures and generalizations. We asked the 21 students to choose one of the proof tasks provided. Eight students chose type 1 tasks, while 13 students completed the type 2 task. In addition, we selected two students to be interviewed for source triangulation purposes. The next step, after collecting data from the proof tasks and interviews, we analyzed the data by selecting task and interview data that were only needed in the research (data reduction), namely segments of data relevant to the use of deductive, inductive, and abductive reasoning; grouping data based on the types of reasoning used by students; and ignoring data that were not needed. Next, we presented the reduced data, namely by presenting data in the form of a table of proof task results showing the frequency of use of each type of reasoning; descriptive narratives explaining the patterns of reasoning that emerged; and direct quotes from interviews that support the findings. We then draw conclusions and conduct verification by interpreting the data to determine how the reasoning framework influenced students' understanding; comparing the findings with relevant literature or theories; and triangulating the data to ensure the validity of the findings. We used source triangulation, which involved checking the validity of the data by comparing one source with another, where students received the same treatment, namely completing assignments followed by interviews.

RESULTS

After working on the proof tasks, we interviewed two participants from each task type. The interviews aimed to deepen the data obtained from the completed proof tasks. In addition, we interviewed two participants for source triangulation.

Research Subject	Reasoning		
	Deductive	Inductive	Abductive
F	√	√	√
M	√	√	
I		√	√
S	√	√	√
U	√	√	√
N	√		√
A		√	√
Y	√	√	√

Table 1. Participants Reasoning in Constructing Evidence in Task Type 1

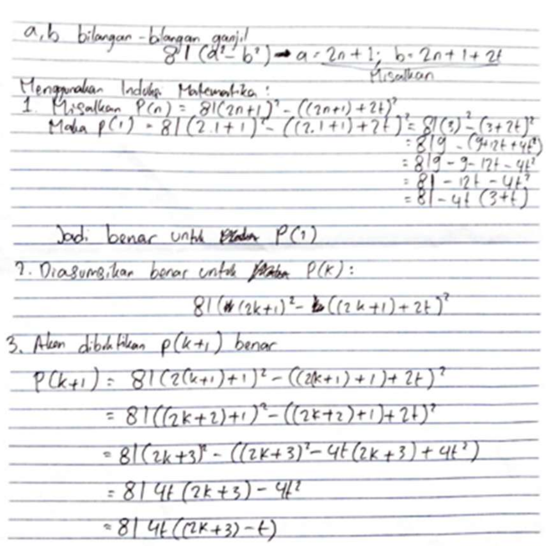
Table 2 shows the reasoning results of mathematics education students in constructing.

Table 2. Participants' Reasoning in Constructing Evidence in Task Type 2

Research Subject	Reasoning		
	Deductive	Inductive	Abductive
L	√		√
N	√		
J		√	√
H	√		√
C	√		√
D	√		√
K	√		√
M	√		√
R	√		√
T	√		√
S	√		√
B	√		
P		√	

Task Type 1

We describe this type 1 proof task based on participants' work. Out of 8 participants, we selected the work of two participants, namely F and S.



Translated Version

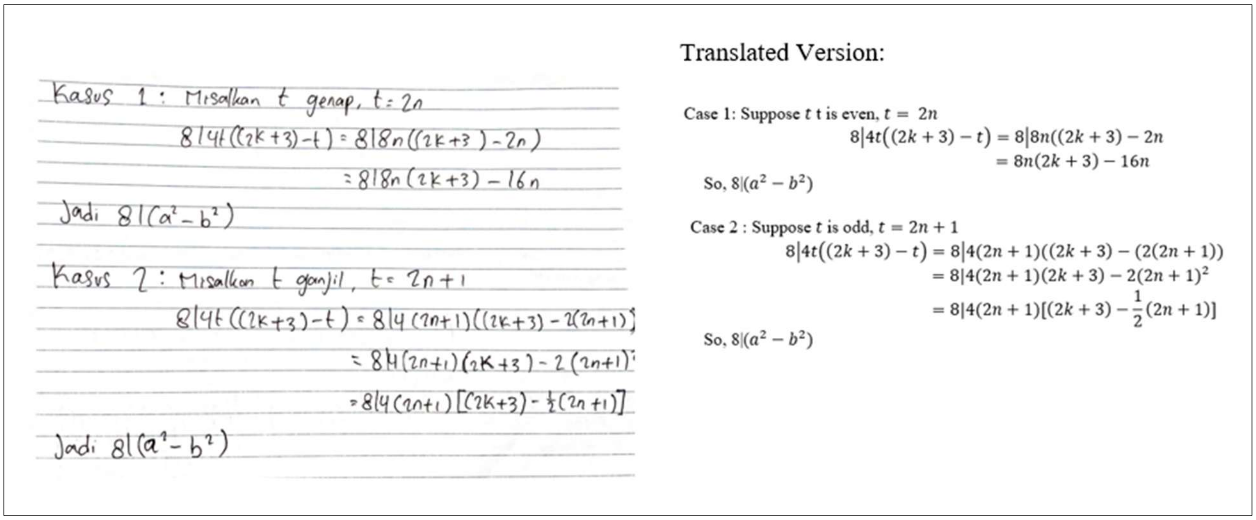
$8|(a^2 - b^2) \to a = 2n + 1; b = 2n + 1 + 2t$

Using mathematical induction:

1. Suppose $p(n) = 8|(2n+1)^2 - ((2n+1)+2t)^2$
   Then $p(1) = 8|(2.1+1)^2 - ((2.1+1)+2t)^2 = 8|(3)^2 - (3+2t)^2$
   $= 8|9 - (9 + 12t + 4t^2)$
   $= 8|9 - 9 - 12t - 4t^2$
   $= 8|-12t - 4t^2$
   $= 8| - 4t(3+t)$
   So it is true for $p(1)$
2. Assume true for $p(k)$:
   $8|(2k+1)^2 - ((2k+1)+2t)^2$
3. It will be prove $p(k+1)$ is true
   $p(k+1) = 8|(2(k+1)+1))^2 - ((2(k+1)+1+2t)^2$
   $= 8|(2k+2)+1))^2 - ((2k+2)+1)+2t)^2$
   $= 8|(2k+3))^2 - ((2k+3)^2 - 4t(2k+3)+4t^2)$
   $= 8|(2k+3))^2 - ((2k+3)^2 - 4t(2k+3)+4t^2)$
   $= 8|4t(2k+3) - 4t^2$
   $= 8|4t((2k+3) - t)$

Figure 4. F's Proof Process Part 1

Figure 4 shows that “F” solved the task using the principle of mathematical induction. In the first step, he assumed $P(n)$ with (1) . In the second step, “F” assumed it was true for $P(k)$. In the third step, the participant proves that it is true for $P\ (k+1)$. F's proof step can be seen in part 2 (Figure 5) below.



Kasus 1 : Misalkan t genap, $t = 2n$

$8|4t((2k+3)-t) = 8|8n((2k+3)-2n)$

$= 8|8n(2k+3) - 16n$

Jadi $8|(a^2-b^2)$

Kasus 2 : Misalkan t ganjil, $t = 2n+1$

$8|4t((2k+3)-t) = 8|4(2n+1)((2k+3)-2(2n+1))$

$= 8|4(2n+1)(2k+3)-2(2n+1)^2$

$= 8|4(2n+1)[(2k+3)-\frac{1}{2}(2n+1)]$

Jadi $8|(a^2-b^2)$

Translated Version:

Case 1: Suppose t t is even, $t = 2n$

$8|4t((2k+3)-t) = 8|8n((2k+3)-2n$

$= 8n(2k+3) - 16n$

So, $8|(a^2-b^2)$

Case 2 : Suppose t is odd, $t = 2n+1$

$8|4t((2k+3)-t) = 8|4(2n+1)((2k+3)-(2(2n+1))$

$= 8|4(2n+1)(2k+3) - 2(2n+1)^2$

$= 8|4(2n+1)[(2k+3) - \frac{1}{2}(2n+1)]$

So, $8|(a^2-b^2)$

Figure 5. F’s Proof Process Part 2

In part 1, “F” used deductive reasoning, while in part 2, “F” continued by using inductive reasoning, namely by sorting out whether t is even or t is odd. The reason why “F” did so can be seen in the following interview.

P : Why did you continue with the odd and even terms?

F : I thought that it had to be done, because if I only completed step 3 there was not enough reason to strengthen my proof.

P : Why did you equate the odd t with $2n+1$? Is it possible to equate $t = 2n-1$

F : You can, but most people use $2n+1$ to represent odd numbers

P : Why did you visualize even and odd numbers?

F : Even and odd numbers are positive numbers and can also be negative integers. If I assume $t = 0$ it's not possible.

From the interview, it can be seen that “F” also used abductive reasoning by providing reasonable explanations. Based on the results of the proof and the interview, we found that when someone constructs a proof, they do not rely solely on deductive reasoning but also use inductive and abductive reasoning to obtain logical results. Here we present the proof results of “S”.

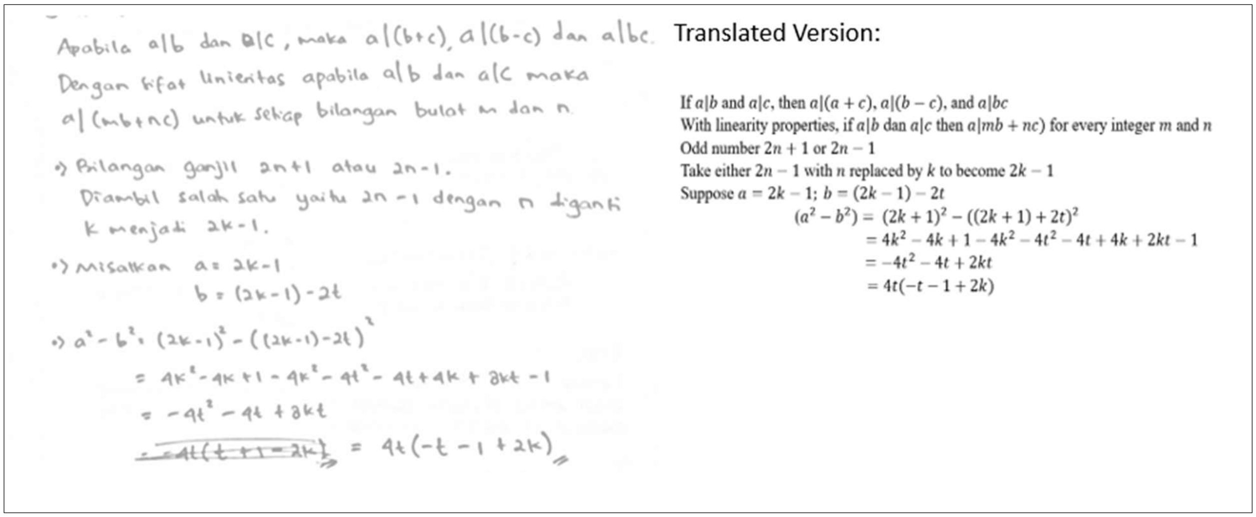


Figure 6. S's Proof Process Part 1

Figure 6 shows that "S" proved the statement using linearity properties, specifically "If $a|b$ and $a|c$, then $a|(a+c)$, $a|(b-c)$, and $a|bc$." With these properties, if $a|b$ and $a|c$ then $a|mbnc$for every integer m and n. This demonstrates that "S" performed an algebraic proof in a deductive manner. Deductive proof is not limited to principle of mathematical induction but also involves deriving existing theorems, definitions, or rules. In the next process, "S" initialized the odd numbers as $2k-1$ and $(2k-1)-2t$. Then they substitute these into (a^2-b^2). he following image (Figure 7) illustrates the process of S's proof construction.

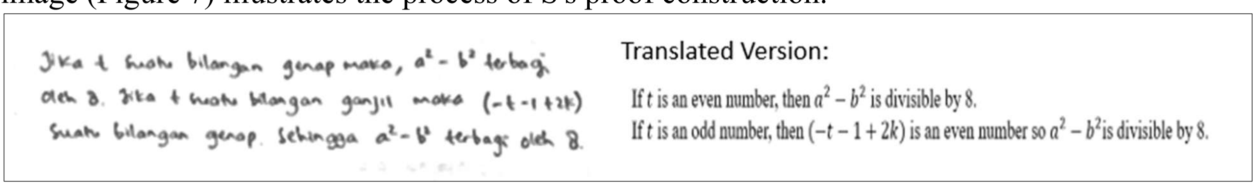


Figure 7. S's Proof Process Part 2

Figure 7 shows that "S" assumes t is an even number, stating, "If t is an even number, then $a^2 - b^2$ is divisible by 8". "S" also assumes t is an odd number, stating, "If t is an odd number, then $(-t-1+2k)$ is an even number, so $a^2-b^2$is divisible by 8 ". Thus, it is true that $8|(a^2-b^2)$ for a,b are odd numbers. This reasoning involves both inductive and deductive approaches, as demonstrated in the following interview:

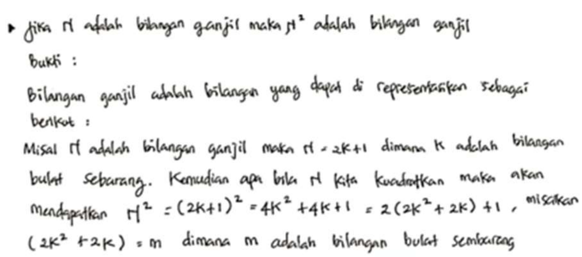
P : Why did you use the linearity property at the beginning of the proof?
S : In my opinion, formal proof can be done using existing rules
P : Why did you equate the odd t wi$2n-1$? Is it possible to equate $t = 2n+1$?

S : You can, most people use $2n + 1$ to represent odd numbers, but I want to try something else.
P : Why do you equate even and odd numbers?
S : I want to try one by one
P : Are you sure that $8|(a^2 - b^2)$?
S : Yes, I am sure, because I have tried one by one, even numbers and odd numbers and the result is divided by 8.

From the interview, it can be seen that "S" used inductive reasoning by stating "trying one by one". In addition, "S" employed abductive reasoning by providing reasonable explanations. Based on the proof results and the interview, it is clear that when someone constructs a proof, one uses not only deductive reasoning but also inductive and abductive reasoning to obtain logical conclusions.

Task Type 2

We describe this type 2 proof task based on the work of two participants out of the 13. These participants are L and N.



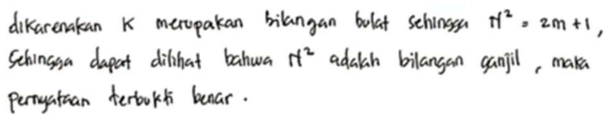
Translated Version:

If n is an odd number, then n^2 is an odd number
Proof:
An odd number is a number that can be represented as follows.
Suppose n is an odd number then $n = 2k + 1$where k is an arbitrary integer. Then if we square n, we will get:
$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, suppose $(2k^2 + 2k) = m$, where m is an arbitrary integer.

Figure 8: L's proof process part 1

Figure 8 shows that "L" defines odd numbers as $n = 2k + 1$ with k an arbitrary integer. This implies that "L" used deductive reasoning by stating the mathematical rules. Furthermore," L" squared n, i.e $n^2 = (2k + 1)^2 = 4k^2 + 2k + 2k + 1 = 4k^2 + 4k + 1 = 4k^2 + 4k = 2(2k^2 + 2k)$ and explained that $(2k^2 + 2k) = m$ where m is an arbitrary integer. This implies that "L" also uses abductive reasoning.

Figure 9 is part 2 of L's proof.



Translated Version:

Since k is an integer such that $n^2 = 2m + 1$,
it can be seen that n^2 is an odd number,
so, the statement is proven true

Figure 9: L's proof process part 2

Figure 9 shows that "L" provided the reason why n^2 is an odd number, namely $n^2 = (2m + 1)^2$. It can be seen that n^2 is an odd number, and it is proven that if n is odd, then n^2 is also odd. This is consistent with the following interview:

P : Why do you define odd numbers by $n = 2k + 1$?
S : I analogize with even numbers, if an even number is $n = 2k$, then an odd number is defined $2k + 1$ or $2k - 1$ but I use $2k + 1$
P : Why do you initialize odd numbers with $2k + 1$?
S : I followed most people
P : Do you think you did the proof using the mathematical induction principle?
S : Yes, I think
P : Are you sure that your proof is a formal proof?
S : Yes, I'm sure, because I used the mathematical rule of the definition of odd numbers

From the interview, it can be seen that "L" constructed the proof using deductive or formal proof, even though the principle of mathematical induction was not employed. "L" realized that creating formal proof involves not only using the principle of mathematical induction but also applying mathematical rules. This is evident in the interview where "L" stated, "Yes, I am sure because I used a mathematical rule: the definition of odd numbers. This shows that "L" used deductive and abductive reasoning to construct the proof.

The following are the results of N's proof on Task Type 2.

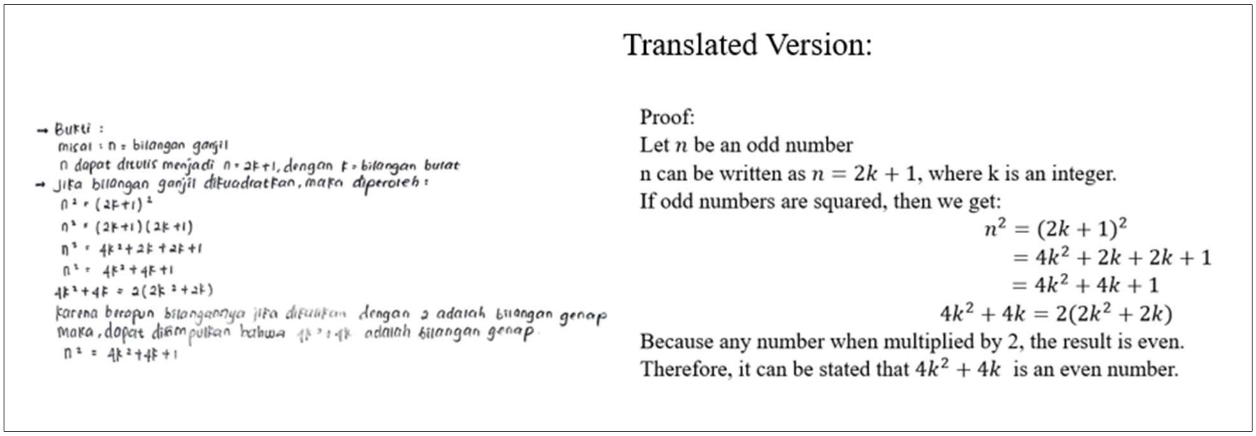


Figure 10: N's Proof Process Part 1

Figure 10 shows that "N" defined odd numbers as$n = 2k + 1$, where k is an arbitrary integer. This implies that subject N used deductive reasoning by writing down the rules in mathematics. Furthermore,"N" squared n, i.e $n^2 = (2k+1)^2 = (2k+1)(2k+1) = 4k^2 + 2k + 2k + 1 = 4k^2 + 4k + 1 = 4k^2 + 4k =$

$2(2k^2 + 2k)$ and explained, "Any number when multiplied by 2 results in an even number, so it can be concluded that $4k^2 + 4k$ is an even number." This implies that "N" used abductive reasoning. N's next answer is shown in the Figure 11.

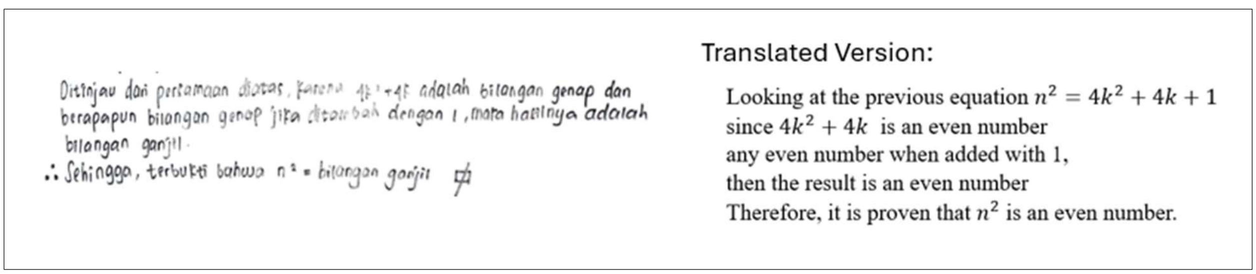


Figure 11: N's Proof Process Part 2

In part 2, "N" was seen giving reasons why n^2 is an odd number. "N" wrote, "Based on the equation ($n^2 = 4k^2 + 4k + 1$), $4k^2 + 4k$ is an even number, so any even number, if added with 1, results in an odd number. Furthermore, "N" concluded, "It is proven that n^2 is an odd number." "N" used abductive reasoning to construct the proof, namely by providing explanations for the steps of the proof. This is supported by the following interview results:

P : Why did you equate the odd n with $2k + 1$? Is it okay to improvise $n = 2k - 1$?

S : Actually, it is okay, most people use $2k + 1$ to represent odd numbers, so I follow most people.
P : Why do you believe that if an even number is added to 1, the result is odd?
S : I tried one by one, but I didn't write it down.

The interview results illustrate that "N" also used abductive reasoning when doing the proof. This can be seen from the interview answer, stating, "I tried one by one, but I didn't write it down." This suggests that "N" implicitly used inductive reasoning.

DISCUSSION

Table 1 shows that 7 participants are using abductive reasoning in constructing proofs. However, these participants also employed a combination of inductive and deductive reasoning. This shows that abductive reasoning is often relied upon by participants in conducting proofs. The use of abduction in mathematical proofs reflects participants' efforts to identify interrelationships between mathematical rules, which were then further tested through induction and deduction. Only two participants did not use deductive reasoning, and one participant did not use inductive reasoning. This indicates that deduction, as a fundamental method in mathematical proof, remains a widely used technique in the proof process. Deduction offers a definite approach, where the conclusion is certain if the premises are true. Thus, deductive reasoning becomes the backbone of the proof (Attridge, 2013). In contrast, one participant did not use inductive reasoning. This indicates that not all participants felt the need to generalize from specific examples when proving a theorem. Those who did not use induction likely proceeded directly to deduction from the premises, without first identifying patterns from specific cases.

Table 2 shows that only two participants used inductive reasoning, while others employed a combination of deductive and abductive reasoning. However, one participant only used deductive reasoning and no other reasoning. This indicates that induction is not dominant in the process of mathematical proof among participants. This may be due to the probabilistic nature of induction, which contrasts with the certainty required in mathematical proofs, leading participants to favor deduction. This further emphasizes the important role of abduction in forming initial hypotheses, which are then confirmed through deduction. Interestingly, one participant used deductive reasoning, without combining of abduction or induction. This may illustrate a more rigorous, traditional, or formal approach, where proof begins directly with known premises, and conclusions are drawn deductively without the need for abduction or induction (Magnani, 2011).

Mathematical proofs are inherently dynamic, where participants often combine several types of reasoning to strengthen their proof (Harel & Sowder, 2007). The significant use of abductive reasoning suggests that participants rely on intuition or creative thinking to formulate the first step of the proof. In contrast, the dominance of deduction reaffirms that deduction remains the irreplaceable standard in mathematical proof. While inductive reasoning is used less frequently due to its less certain nature compared to deduction, it still plays a crucial role in finding patterns or trends.

Overall, this study provides important insights into how participants employ different types of reasoning in mathematical proofs, reflecting their approach to complex problem-solving.

Teacher Reflection and Didactic Impact on Learning Outcomes.

As part of a practical reflection on the findings of this study, several mathematics teachers in higher education, who also examined the deductive-inductive-abductive framework, noted that this approach holds significant potential for bridging the gap between students' initial intuition and the formal proof process. They observed that abductive reasoning, which had previously received less attention in the teaching of proofs, holds a strategic role in fostering students' confidence to initiate and explore possible mathematical arguments. The teacher observed that students often struggle when required to engage directly in deductive proofs. By providing space for students to develop initial hypotheses through abductive reasoning, they become more confident and actively involved in the proof construction process. The integration of inductive reasoning is also seen as helpful in identifying patterns and strengthening intuition before transitioning to formal deduction. The impact of this didactic framework on learning outcomes was evident in two main aspects. First, in terms of process, students demonstrated improved ability to explain the reasoning behind each step of a proof, both in writing and orally. They did not merely imitate standard forms of proof but were able to construct coherent arguments based on observations and initial conjectures. Second, in terms of outcomes, students produced proofs that were more varied, creative, and demonstrated a deeper understanding of the nature of mathematical structures. Thus, this framework not only reinforces deductive logic but also fosters reflective and explorative thinking, which are essential components of modern mathematics learning. The teacher recommends integrating this approach into the proof curriculum at the university level, particularly in number theory and introductory proof courses.

Practical Guidelines and Materials for Replicating Classroom Studies.

This study can be replicated within the context of classroom mathematics learning by following these steps:

Learning Objectives

Learners are expected to identify and apply deductive, inductive, and abductive reasoning when proving the following two mathematical statements: 1) If a and b are odd numbers, then $8 \mid (a^2 - b^2)$ and 2) The square of an odd number is an odd number.

Learning Strategy

The learning activities are designed to integrate the three types of reasoning as follows:

- Deductive Reasoning: Formulating formal proofs using symbolic and algebraic representations.
- Inductive Reasoning: Identifying patterns through multiple numerical examples.
- Abductive Reasoning: Proposing conjectures based on observations and searching for plausible explanations.

Series of activities

1) *Concept Orientation*: An initial discussion on the general form of odd numbers $(2k + 1)$.
2) *Proof Exploration*: Guided proof of the two statements using a multi-reasoning approach (deductive, inductive, and abductive reasoning).
3) *Reflection and Discussion*: Students evaluate the type of reasoning used and reflect on how each contributes to the validity and construction of the proof.

Supporting Materials

1) *Divisibility Notation*: An integer a completely divides an integer b is written as a|b if and only if there exists an integer k such that $b = ka$. If a does not completely divide b, it is written as $a \nmid b$.
2) *Definition of Odd Numbers*: An odd number is an integer that is not divisible by 2. It can be expressed in the form $2k + 1$, where $k \in \mathbb{Z}$ (the set of integers).

Assessment

Assessment is conducted formatively through class discussions and the observation of students' thought processes. Summative assessment may be provided through proof tasks that require the application of various reasoning methods, such as providing additional algebraic properties of odd and even numbers.

CONCLUSION

Based on the analysis of the presented research results, the majority of participants in this study primarily employed deductive reasoning in constructing proofs. This supports the assertion that deduction is the most trusted and reliable method in mathematical proofs, as it generates definitive conclusions from established premises. Additionally, many participants utilized a combination of deductive, inductive, and abductive reasoning. This demonstrates that the process of mathematical proof typically involves multiple types of reasoning. Abduction is employed as the initial step to formulate hypotheses, while deduction ensures the correctness and validity of the conclusions. Induction also plays a significant role in identifying patterns from specific examples, although it is less predominant. The widespread use of abductive reasoning among students suggests that it plays a key role in initiating the proof process. Students frequently relied on intuition or initial exploration to formulate conjectures, which were later verified through deduction. This suggests that while mathematical proof is grounded in logic, intuition and preliminary hypotheses play a crucial role in initiating the proof process. The study revealed variations in the proof strategies employed by students. Some students relied solely on deduction, while others adopted a mixed approach, incorporating deduction, induction, and abduction. This suggests that there is no single 'correct' method for constructing a mathematical proof; rather, multiple approaches may be employed, depending on individual preferences and the specific characteristics of the problem. This conclusion underscores the importance of a thorough understanding of various types of reasoning and their integrated use in mathematical proofs, which contributes to the development of students' critical and analytical thinking skills. Analysis of students' reasoning process revealed that abductive reasoning plays a crucial role in initiating a proof, while deductive reasoning ensures logical consistency, and inductive reasoning aids in identifying patterns. Although this study relied minimally on quantitative approaches, the richness of the qualitative data facilitated an in-depth exploration of students' cognitive strategies in constructing proofs. The findings affirm that qualitative approaches are particularly effective in capturing the complexity and dynamics of mathematical reasoning processes in educational contexts.

We recommend that future research explore the development of a more detailed theoretical model illustrating how deductive, inductive, and abductive reasoning interact in mathematical proofs. Such a model could clarify the role of each type of reasoning in different proofs and offer guidance for more structured instruction in proof teaching.

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