

## The Story of a “Theorem”: From Mistakes to New Mathematics

Michael Aristidou

American University of Cyprus, Larnaca, Cyprus  
[michael.aristidou@aucy.ac.cy](mailto:michael.aristidou@aucy.ac.cy)

*Abstract: In this article, we look at the exploratory aspects of student projects. We discuss a specific project from a course on quaternions and see how incorrect theorems, student omissions, etc., led to some new mathematical results. In particular, the fact that  $H/Z_p$  was not just another division algebra but simply a ring with zero-divisors opened new questions to be explored, such as finding non-trivial idempotents, tripotents, and exploring extensions to larger rings such as  $O/Z_p$ . This serves as a reminder that student projects occasionally could also have academic benefits for the students or the instructors. These are outcomes that could enhance the students’ or the instructors’ academic and professional development.*

Keywords: project, theorem, quaternions, ring, zero divisors

### INTRODUCTION

Student projects<sup>1</sup> within a college course are not only pedagogically beneficial to students but also could occasionally have *academic benefits*<sup>2</sup> for students or instructors too. The general benefits for the students and instructors were discussed in (Osburn & Karukstis, 2009) and more briefly in (Petrella & Jung, 2008). In both articles, the *initiation and production of research and new results* is identified as one of the benefits, and these fall into what I would call academic benefits. In this article, we show an instance of academic benefit for the instructor and how it actually occurred. In other words, we will see how engaging students in projects yielded research questions and new results which led to publications. Interestingly, in our case, the new results did not come from the students’ produced results but from their omissions, mistakes, and from the material used and studied during their project work.<sup>3</sup>

<sup>1</sup> By “student projects” I mean traditional projects given to students (or initiated by them) in order to explore something during a course. I am not referring to “project-based learning” which is a bit different and I have criticized in (Aristidou, 2020).

<sup>2</sup> By “academic benefit” I mean any by-product of an activity that could boost the teacher’s or the student’s academic and professional development. For example, research that could lead to a publication in a scientific journal could be such a by-product. The term was elaborated further in (Aristidou, 2020).

<sup>3</sup> For some other examples of projects which produced academic benefit for students see (Aristidou, 2020). And for some that produced academic benefit for students and instructors see (Aristidou, 2020), (Aristidou & Hailemariam, 2021) and (Marciniak, 2018).

More specifically, I will discuss a new result that came out of one of my courses at DigiPen Institute of Technology in 2008.<sup>4</sup> It was from a course on quaternion algebra and its applications in computer graphics in which the students had the option to work on a project of their liking in groups of two (see Appendix 1 for the Syllabus and more details). The students were strongly encouraged to come up with their own ideas for a project, but they were also told in class that a list of possible project suggestions would be available by the instructor at a later time. Three groups of two students accepted my suggestion to explore quaternions with coefficients not from the field of real numbers  $\mathbb{R}$  but from the finite field  $\mathbb{Z}_p$ .

In the following sections, we will describe how the assignment played out and what new came out from it. First, I will provide some basics on quaternions  $\mathbb{H}$  and the quaternion ring  $\mathbb{H}/\mathbb{Z}_p$  (see also Appendix 2 for brief explanations of the key mathematical terms). Then, I will describe the mistake we found and its algebraic ramifications. Finally, in the last section, we will draw some conclusions.

## THEORETICAL FRAMEWORK

### The Quaternions $\mathbb{H}$ and the Ring $\mathbb{H}/\mathbb{Z}_p$

The quaternions, denoted by  $\mathbb{H}$ , were first discovered by William R. Hamilton in 1843 as an extension of complex numbers into four dimensions (Remmert et al., 1991). Namely, a quaternion is an element of the form  $x = a_0 + a_1i + a_2j + a_3k$ , where  $a_i$  are from  $\mathbb{R}$  (real numbers) and  $i, j, k$  are such that  $i^2 = j^2 = k^2 = ijk = -1$ .

For example,  $x = 2 + 3i - j + 7k$  is a quaternion. So are  $y = 3k$ ,  $z = i + 2j$  and  $w = -2$ .

The operations of addition “+” and multiplication “.” in  $\mathbb{H}$  are defined in (Herstein, 1975, p.124).<sup>5</sup> Basically, addition is done component-wise and multiplication distributive-wise resulting in the following formulas:

$$\begin{aligned} x + y &= (a_0 + a_1i + a_2j + a_3k) + (b_0 + b_1i + b_2j + b_3k) \\ &= (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k \end{aligned} \quad (1)$$

$$\begin{aligned} xy &= (a_0 + a_1i + a_2j + a_3k)(b_0 + b_1i + b_2j + b_3k) \\ &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)i + \\ &\quad (a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1)j + (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)k \end{aligned} \quad (2)$$

<sup>4</sup> I am referring to DigiPen’s main campus in Redmond, WA.

<sup>5</sup> Multiplication is associative but not commutative, because  $ij = k$  and  $ji = -k$ .

For example, adding  $x = 2 + 3i - j + 7k$  and  $z = i + 2j$  from the previous example above we get:

$$\begin{aligned}
 x + z &= (2 + 3i - j + 7k) + (i + 2j) \\
 &= (2 + 3i - j + 7k) + (0 + i + 2j + 0k) \\
 &= (2+0) + (3+1)i + (-1+2)j + (7+0)k \\
 &= 2 + 4i + j + 7k
 \end{aligned} \tag{3}$$

And, multiplying  $x = 2 + 3i - j + 7k$  and  $z = i + 2j$  we get:

$$\begin{aligned}
 xz &= (2 + 3i - j + 7k)(i + 2j) \\
 &= (2 + 3i - j + 7k)(0 + i + 2j + 0k) \\
 &= (0 - 3 + 2 - 0) + (2 + 0 + 0 - 14)i + \\
 &\quad (4 - 0 + 0 + 7)j + (0 + 6 + 1 + 0)k \\
 &= -1 - 12i + 11j + 7k
 \end{aligned} \tag{4}$$

The ring  $H/Z_p$  contains finitely many elements of the form  $x = a_0 + a_1i + a_2j + a_3k$ , where the  $a_i$ 's come from  $Z_p$ . That is the set  $Z_p = \{0, 1, 2, \dots, p-1\}$ , i.e., the set of remainders modulo  $p$ . In other words, the numbers resulting after we “subtract off” multiples of  $p$ . In regards to the ring  $H/Z_p$ , as  $p = 0$  in  $Z_p$ , the operations are defined as in  $H$  but taking into consideration the modulo  $p$  operation ( $\text{mod } p$ ). Notice also that as  $H$  is non-commutative, so is  $H/Z_p$ .

For example, in  $H/Z_5$ , adding  $x = 2 + 3i + j + 4k$  and  $z = 2i + 2k$  we get:

$$\begin{aligned}
 x + z &= (2 + 3i + j + 4k) + (2i + 2k) \\
 &= 2 + 5i + j + 6k \\
 &= 2 + j + k
 \end{aligned} \tag{5}$$

(because  $5 = 0$  in  $Z_5$ . Hence,  $6 = 5 + 1 = 0 + 1 = 1$ , etc.).

And, in  $H/Z_5$ , multiplying  $x = 2 + 3i + j + 4k$  and  $z = 2i + 2k$  we get:

$$\begin{aligned}
 xz &= (2 + 3i + j + 4k)(2i + 2k) \\
 &= -14 + 6i + 2j + 2k \\
 &= 1 + i + 2j + 2k
 \end{aligned} \tag{6}$$

(because  $5 = 0$  in  $Z_5$ . Hence,  $15 = 14 + 1 \Rightarrow 5.3 = 14 + 1 \Rightarrow 0 = 14 + 1 \Rightarrow 14 = -1$  and  $6 = 5 + 1 = 0 + 1 = 1$ , etc.).

Algebraically speaking,  $H$  forms a non-commutative normed division algebra (skew field) over  $\mathbb{R}$  of dimension 4 (Remmert et al., 1991, p.195-196). There are only four normed division algebras (Hurwitz, 1898), namely  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $H$  and  $O$ , with the octonions  $O$  (which are non-commutative and non-associative) being the largest.

The structure and some of the properties of the finite ring  $H/\mathbb{Z}_p$ , where  $p$  is a prime number, was done in Aristidou and Demetre (2009). A more detailed description of the structure  $H/\mathbb{Z}_p$  was given by Miguel and Serodio (2011). Among others, they found the number of zero-divisors, the number of idempotent elements, and provided an interesting description of the zero-divisor graph.

## METHOD

### Students' Projects

The study of  $H/\mathbb{Z}_p$  was exactly the project suggested to some students in the purposes of the course on quaternions that we described in the Introduction. The students were asked to explore  $H/\mathbb{Z}_p$ , trying perhaps to define its operations, work out examples, look for some idempotents, solve simple equations in  $H/\mathbb{Z}_p$  and possibly explore similar things in  $O/\mathbb{Z}_p$ . At the time I was unaware that  $H/\mathbb{Z}_p$  and some of the questions posed above were included as an exercise in (Herrnstein, 1975, p.136). What I found instead was an article by Kandasamy (2000) which I shared with the students as material to help them with their project. Certainly, Kandasamy's paper was a good source of information for them, due to its content and accessible level. But, unfortunately, a paper from which they used results at face value and imitated proofs without much caution.

Then, a serious problem appeared while grading the students' projects. My colleague A. Demetre and I found a counterexample to what the students claimed and supposedly proved. So, we were convinced that there must have been a mistake somewhere in their work. Initially, we did not notice any mistake in the original article by Kandasamy which was given to the students as reference. After all, the latter was supposedly peer reviewed and published, so one did not have a reason to doubt it or double check it. But, since the students pretty much imitated a key theorem "proved"<sup>6</sup> in Kandasamy's article in their work, that raised a suspicion and gave us a reason to re-examine Kandasamy's result more carefully.

The students had also their own suspicions of Kandasamy's article. For example, in one project (see Figure 1), students already found several typos and errors in the article. Furthermore, they expressed suspicions for the proof of Kandasamy's Theorem 2 mentioned above (see footnote 6), which was relevant to their work. Nevertheless, they considered the result to be correct and played along trying to give their own proof of the same "fact," naturally making similar mistakes. It seems

---

<sup>6</sup> That was Theorem 2 in Kandasamy's article (2000, p.133), which falsely stated that  $H/\mathbb{Z}_p$  is a skew field.

that from the four stages of creativity identified by Wallas (1926) (see also Marciniak, 2020), namely *preparation*, *incubation*, *illumination* and *verification*, the students missed the last stage.

<p>MAT 351 Final Project</p> <p>Deciphering Finite Quaternion Rings and Skew Fields</p> <p>Our project is on a paper entitled "On Finite Quaternion Rings and Skew Fields" by Vasantha Kandasamy. In this paper, she first defines quaternions using a ring of integers modulo <math>n</math>. These are used for the coefficients of <math>i</math>, <math>j</math>, and <math>k</math>. She defines addition and multiplication for this set and then proves that it is in fact a division ring. She goes on to prove that using a ring of prime numbers creates a finite skew field, and composite numbers create a non-commutative ring with zero divisors. At the end she describes a few other interesting facts and leaves a few things up to the reader to work out. This is a relatively short paper, and seemed simple enough at first glance, but on further inspection there are many typos and errors that make the reading of this document much more difficult. In this paper we will attempt to decipher and correct this paper, and complete the exercises at the end of it.</p>	<p>This was our first challenge in deciphering this paper, because many terms in the multiplication definition were in the wrong place, or used the wrong values.</p> <p>The first theorem defines <math>P</math> as before, but instead of <math>Z_n</math> we use a set called <math>Z_p</math>, where <math>Z_p</math> is <math>\{0, 1, 2, \dots, p-1\}</math> where <math>p</math> is a prime number. The theorem states that <math>P</math> is a finite skew field. To prove this we need to show that there is a, <math>b \in P</math> such that <math>ab = 1</math> for some <math>a</math> that does not equal zero. For this proof, see 'Proof 1' on the attached sheets. As before, this proof made no sense when we first went through it. So instead of trying to correct it, we proved the theorem ourselves and checked it against the proof in the paper. There were many misplaced variables and parenthesis, but we were able to work through it.</p>
---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------	------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

Figure 1. Student project #1. Suspicions were cast on the reference paper they used for their project.

### Finite Octonionic Rings and Skew Fields

In this paper, modeled after [1], we construct finite octonionic rings and skew fields, and examine certain properties they possess. The set of all real octonions,  $O$ , is an established infinite division ring, while the octonions we construct here belong to finite subsets; some of which are rings and others which are division rings.

**THEOREM 1:** If  $O_p = \{o_0 + o_1i + o_2j + o_3k + o_4l + o_5(il) + o_6(jl) + o_7(kl) \mid o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7 \in Z_p - \{0, 1, \dots, p-1\}\}$  where  $p$  is a prime, then  $O_p$  is a finite skew field (division algebra).

**PROOF:**  
As stated above, we know that  $O_p$  is a finite non-associative ring. To demonstrate that it is a finite skew field we must show that every element  $\alpha \in O_p$  has a multiplicative inverse  $\beta \in O_p$ , such that  $\alpha\beta = 1$ .

Let  $\alpha = \alpha_0 + \alpha_1i + \alpha_2j + \alpha_3k + \alpha_4l + \alpha_5(il) + \alpha_6(jl) + \alpha_7(kl)$  and using the general octonion inversion formula for  $\beta$  we have:

$g = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2 + \alpha_7^2 \in Z_p$ , where  $g$  has a multiplicative inverse  $g^{-1} \in Z_p$  (since  $p$  is a prime number there exists a multiplicative inverse  $g^{-1} \in Z_p$  for

Figure 2. Student project #2. Attempt to generalize the main result from the ring  $H/Z_p$  to the ring  $O/Z_p$

They simply did not check the validity of their ideas, even though they did express doubts on some of the information in the reference article.

In another project (see Figure 2) students also used Kandasamy's Theorem 2 (see footnote 6) and imitated her proof in order to extend the result from the quaternionic ring  $H/Z_p$  to the non-associative octonionic ring  $O/Z_p$ . The latter was also mistakenly "shown" by the students to be a skew field which caused a different kind of suspicions because the octonions, as Baez said, "*are the crazy old uncle nobody lets out of the attic*" (Baez, 2002). They are an "exceptional" structure that usually does not follow common algebraic expectations in the hyper-complex numbers' family (non-associative, not a Lie group, etc.). Which made the students' result sound a bit too good to be true and beg a careful examination. Clearly, this group of students missed Wallas' fourth stage of creativity (i.e., *verification*) too (Wallas, 1926; Marciniak, 2020). They did not check the validity of their results.

As we already mentioned, a counter-example was produced (see next section) which contradicted Kandasamy's supposed Theorem 2 (2000, p.133) which claimed that  $H/Z_p$  is supposedly a skew field. As one of the purposes of the student project was to promote exploration and inquiry-based learning<sup>7</sup>, the specific counter-example was not revealed immediately to the students. Instead, the two groups of students were asked to produce some examples that confirm or disconfirm their work. Surprisingly, neither of the two groups included examples in their submitted articles even though Kandasamy's reference article includes two examples, namely  $H/Z_3$  and  $H/Z_4$ , without the computational details (for  $H/Z_3$  it incorrectly states that is a skew field). In a meeting after the students submitted their projects, the counter-example was shown to the students and their work was discussed. The students were quite surprised and almost in disbelief in how could that be since they were following a published and therefore correct result. One student protested, "Isn't a theorem something that is proved and is therefore true?". Another student simply said, "I knew it that something was wrong with this story!" I pointed out to the students of both groups that, perhaps, had they tried to do the calculations themselves even for the  $H/Z_3$  case from the article they might have noticed that the example does not agree with the supposed theorem.

The students in the 1st group admitted that they were carried away and thought that the example was definitely correct because there was a general theorem proved for  $H/Z_p$  (i.e., for all primes) and for  $p = 3$  one simply had a special case. The students in the 2nd group, the ones trying to extend the main result from  $H/Z_p$  to the octonionic ring  $O/Z_p$ , already disproved that  $O/Z_n$  is a skew field without providing a counter-example (they imitated Kandasamy's disproof and expanded the calculations). So, it was reasonable to expect at least from that group to produce some concrete counter-examples for  $O/Z_n$ , hoping that the calculations would lead them perhaps to some counter-

---

<sup>7</sup> Inquiry-based learning is a student-centered teaching method where students explore topics, investigate questions, conduct projects, etc., and obtain knowledge by self-study and discovery. Students are not passive recipients of information from the teacher. Instead, they are encouraged to actively participate in the learning process which helps them strengthen their critical thinking and research skills.

examples for  $O/Z_p$ . Their explanation was that they did not actually feel the need to check for counter-examples for  $O/Z_p$  as they believed that there were not any because in their “proof” that  $O/Z_p$  is a skew field, they simply imitated an established result from the reference article. Later that week, the 1st group of students produced one more counter-example of zero divisors in  $H/Z_p$  and sent it to me by email. Also, the 2nd group came by my office showing me a counterexample for  $O/Z_n$ , confirming their disproof that  $O/Z_n$  is a skew field. They did not produce a counterexample for  $O/Z_p$ .

Overall, upon reflection, I think that the project and the follow-up discussions had an impact on the students' learning experience and their understanding of quaternions, octonions, and abstract algebra in general. The students realized that one needs to be more careful when studying abstract and unfamiliar spaces. Furthermore, re-working some examples and doing the calculations is beneficial and gives more experience and insight. Finally, even though mathematics is a rigorous science, mistakes could happen, and one should always keep a skeptical eye on the various claims and results. Certainly, inquiry-based learning is not easy for the students or teachers. It is a pedagogical method that requires commitment and some degree of mathematical maturity from the student. But, it also requires time and good preparation from the teacher. The above-mentioned quaternion course was not specifically designed as an inquiry-based course. It was a lecture course that required a project in which students were free to explore topics related to the course content but not necessarily curriculum or standard topics. Nevertheless, some of the course objectives were “improving problem-solving skills”, “develop the ability to answer questions and present material coherently”, etc., which were observed and assessed especially via the project. Overall, I believe that the students got a small taste of inquiry-based learning and tried their best. I also believe that some improvements could be made from the instructor’s side, in terms of project choices, closer monitoring, time management, etc. The latter are certainly crucial if the approach I just described is to be implemented in other mathematics courses too, especially to lower-level courses.

Given our counter-examples and our suspicions of some of the claims in Kandasamy’s article, I wrote early on a polite email<sup>8</sup> to Kandasamy requesting some explanations on how to interpret these results, asking if we could possibly be mistaken, to which email unfortunately I never received a reply. In the meantime, as I mentioned above, more counter-examples were produced and while we were waiting for a reply, we were trying to find where Kandasamy’s proof went wrong. I have to admit, we too missed the mistake the first time. Most likely just as the reviewers missed it during the review process before its publication. It was a subtle, yet simple, point which my colleague A. Demetre and I finally caught the second time around and it concerned a calculation in  $Z_p$ . The results were presented in (Aristidou & Demetre, 2009) which I summarize in the following section.

---

<sup>8</sup> Sadly, no record of that email remains as the specific work-email was deactivated after I moved to a different college in 2009.

## RESULTS

### The “Theorem” and the Details

Kandasamy in Theorem 2 (2000, p.133) “shows” that  $H/Z_p$ ,  $p$  prime, is a finite skew field. But this claim is *false* and hence its supporting proof *incorrect*. That is because the result contradicts well established results on finite division rings and because one can construct counterexamples to Kandasamy’s “Theorem 2.” The conjecture Kandasamy set out to prove was:

Conjecture: Let  $H/Z_p = \{a_0 + a_1i + a_2j + a_3k \mid a_i \in Z_p, p \text{ prime}, i^2 = j^2 = k^2 = p-1\}$ . Then,  $H/Z_p$  is a finite skew field.

The conjecture though is clearly *false*, as the following counterexample (Aristidou & Demetre, 2009) shows:

Counterexample: Let  $p = 5$  and consider  $q_1 = 1 + 3i + 1j + 3k$ ,  $q_2 = i + 2j$  of  $H/Z_5$ .  
Multiplying the two, we get:

$$q_1q_2 = (1 + 3i + 1j + 3k)(i + 2j) = 0 \quad (7)$$

This shows that  $H/Z_5$  has zero divisors, and hence  $H/Z_5$  is *not* a skew field.

More specifically, in the “proof” Kandasamy constructs  $g = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \in Z_p$ , a quantity required for the inverse of  $\alpha = \alpha_0 + \alpha_1i + \alpha_2j + \alpha_3k \in H/Z_p$  (see Figure 3). Then, she claims that  $g$  is non-zero, probably because it is a sum of squares of non-zero elements of  $Z_p$ . But even though this is true in  $\mathbb{R}$ , it is *not* true in  $Z_p$ . For example, in  $Z_3$  one has  $g = 1^2 + 1^2 + 1^2 = 3 = 0$ . So, an inverse cannot always be constructed in  $H/Z_p$  and therefore the “theorem” is false.

Another short proof that  $H/Z_p$  is not a skew field is the one we showed in (Aristidou & Demetre, 2009). Namely:

Theorem:  $H/Z_p$ ,  $p$  prime, is *not* a skew field.

Proof: Assume that  $H/Z_p$  is a skew field. Since it is a finite skew field, by the Wedderburn’s Theorem<sup>9</sup> we have that  $H/Z_p$  is also a field. Hence, this gives the following relations:

$$ij = ji, ij = k, ji = -k \equiv (p-1)k \pmod{p} \quad (8)$$

<sup>9</sup> In 1905, Wedderburn proved that “Every finite skew field is a field” (Herstein, 1975, p.361).

But then, from (8), one also gets:

$$0 = ji - ij = (p-1)k - k = (p-2)k \quad (9)$$

which is a contradiction if  $p > 2$ . The case  $p = 2$  is trivial, since  $H/Z_2$  is clearly not a skew field. (Just take  $\alpha = 1 + i$ , and notice that  $\alpha \cdot \alpha = 0$ , even though  $\alpha \neq 0$ ).

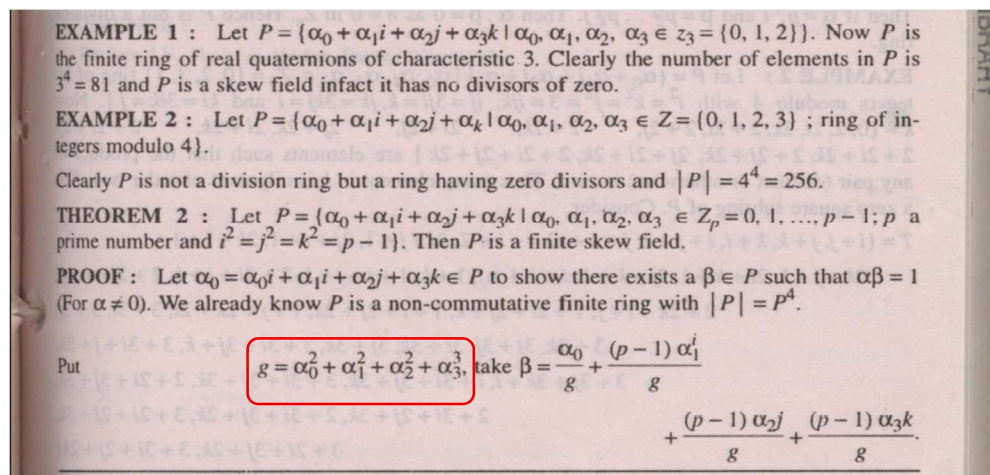


Figure 3. The main mistake in Kandasamy's proof.  $g = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2$  could be 0 in  $Z_p$ .

## CONCLUSIONS

Having students work on projects within a course is pedagogically important for students. Projects require students to move beyond mere memorization of the lessons taught. Engaging into deeper learning, research, and more serious problem solving ensures better understanding of the concepts involved and fosters critical thinking. Projects stimulate the students' curiosity, imagination, and enhances their creativity. Furthermore, projects promote important social skills such as good management of time, management of the material gathered, and organization and collaboration with other students and teachers.

In addition, student projects also occasionally have academic benefits for the students or the instructors. These are outcomes that could enhance the students' or the instructors' academic and professional development. As we mentioned earlier, a publication in a scientific journal would definitely be one such benefit. Academically it could certainly enhance creativity, knowledge, etc., but additionally it could also professionally improve one's C.V., open opportunities for

scholarships, funding, promotions, etc. Such benefits are benefits that could also motivate the teaching/learning process.

We discussed an example of a student project which resulted to some new mathematics. The new results did not come from the students' new positive ideas but from some of their mistakes, which in turn revealed deeper mistakes in an article they used as study material and as reference. The new results were presented in a conference and later published as a paper in a journal (Aristidou & Demetre, 2009). In particular, the fact that  $H/Z_p$  was not just another division algebra but simply a ring with zero-divisors opened new questions to be explored, such as finding non-trivial idempotents, tripotents, and exploring extensions to larger rings such as  $O/Z_p$ .<sup>10</sup> Consequently, four more articles were published based on (Aristidou & Demetre, 2009) which interestingly started as a student project. This is a clear example of a student project resulting to academic benefit for the instructor. In addition, article (Aristidou & Demetre, 2009) yielded academic benefits for students who worked on subsequent related projects/papers (Aristidou & Hailemariam, 2021) and to other instructors who used the new published results of (Aristidou & Demetre, 2009) in their own research and publications.

---

<sup>10</sup> Surprisingly, both rings also have real life applications in Cryptography (some are mentioned in (Aristidou & Demetre, 2009)).

## REFERENCES

- [1] M. Aristidou and A. Demetre, “A Note on Quaternion Rings over  $Z_p$ ”, *International Journal of Algebra*, Vol.3, No.15, 2009, 725- 728.
- [2] M. Aristidou, “Project Based Learning: Are there any Academic Benefits for the Teacher or Students?”, *Journal of Humanistic Mathematics*, Vol.10, No.1, 2020, 458-471.
- [3] M. Aristidou and K. Hailemariam, “Tripotent elements in quaternion rings over  $Z_p$ ”, *Acta Universitatis Sapientiae Mathematica*, Vol.13, No.1, 2021, 78–87.
- [4] J Baez, “The Octonions”, *Bulletin of the American Mathematical Society*, Vol.39, No.2, 2002, 145-205.
- [5] I. N. Herstein, *Topics in Algebra*, 2<sup>nd</sup> ed., Wiley, 1975.
- [6] A. Hurwitz, “Über die Komposition der quadratischen Formen von beliebig vielen Variablen”, *Nach. V. der Ges. der Wiss., Göttingen, Math. Phys. Kl.*, 1898, 309–316.
- [7] W. B. V. Kandasamy, “On Finite Quaternion Rings and Skew Fields”, *Acta Ciencia Indica*, Vol.XXVI, No.2, 2000, 133-135.
- [8] M. A. Marciniak, “Creative Assignments in Upper-Level Undergraduate Courses Inspired By Mentoring Undergraduate Research Projects”, *Journal of Humanistic Mathematics*, Vol.10, No.2, 2020, 465-483.
- [9] M. A. Marciniak, “Approaches to creativity in undergraduate research projects”, *Proceedings of the CUNY Research Summit: Creativity in STEM*, 2018, 34-36.
- [10] C. J. Miguel and R. Serodio, “On the Structure of Quaternion Rings over  $Z_p$ ”, *Int. J. Algebra*, Vol.5, No.27, 2011, 1313-1325.
- [11] J. M. Osburn and K.K. Karukstis, “The Benefits of Undergraduate Research, Scholarship, and Creative Activity”. (in *Broadening Participation in Undergraduate Research: Fostering Excellence and Enhancing the Impact*, by M. Boyd and J. Wesemann (eds.), Council on Undergraduate Research, Washington, DC, 2009, 41-53).
- [12] J. K. Petrella and A. P. Jung, “Undergraduate Research: Importance, Benefits, and Challenges”, *International Journal of Exercise Science*, Vol.1, No.3, 2008, 91-95.
- [13] R. Remmert et al., *Numbers*, Springer, 1991.
- [14] G. Wallas, *The Art of Thought*, New York: Harcourt Brace, 1926.

## APPENDIX

### Appendix 1. Syllabus

# SYLLABUS

MATH 351/551

9:30 - 10:50 T,Th  
(Michelangelo)

**Instructor:** Dr. Michael Aristidou  
**Office Hours:** 3:00-4:00 M,W

**Phone:** 425-895-4400, ext.4464  
**E-mail:** [maristidou@digipen.edu](mailto:maristidou@digipen.edu)

**Textbook:** Class Notes (No textbook required)<sup>(1)</sup>.

- Other sources:
- (1) *Visualizing Quaternions*, by A. J. Hanson, Morgan Kaufmann, 2006.
  - (2) *Quaternions and Rotation Sequences*, by J. B. Kuipers, Princeton Univ. Press, 1999.
  - (3) Other related papers and articles, such as:
    - (i) *Quaternions, Interpolation, and Animation*, by E. B. Dam, M. Koch, and M. Lillhom, Technical Report, Univ. of Copenhagen, 1998.
    - (ii) *Exploiting Quaternions to Support Expressive Interactive Character Motion*, by M. P. Johnson, PhD, MIT, 2003.
    - (iii) *Animating Rotation with Quaternion Curves*, by K. Shoemake, Siggraph Proceedings, 1985.
    - (iv) *Smooth Interpolation of Orientations with Angular Velocity Constraints Using Quaternions*, by A. H. Barr, B. Currin, S. Gabriel, and J. Hughes, Siggraph Proceedings, 1992.
    - (v) *The Octonions* (section 2), by J. C. Baez, Bulletin of American Math. Society, 2002.

**Course Overview:** This course gives an introduction to several mathematical topics of foundational importance in abstract algebra, and in particular in the algebra of quaternions. Topics include:

Operations, Groups, Rings, Fields, vector Spaces, Algebras, Complex Numbers, Quaternions, Curves over the quaternionic sphere, Interpolation Techniques, Splines, Octonions, and Clifford Algebras.

#### Goals and objectives:

- (1) Familiarize students with certain abstract discrete mathematical structures, such as

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International ([CC BY-NC-SA 4.0](https://creativecommons.org/licenses/by-nc-sa/4.0/)). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.



groups, rings, algebras, etc, and most importantly the quaternion group, quaternion algebra, and parameterized curves, and encourage them to apply the latter in graphics and other projects of their interest.

- (2) Improve their problem-solving skills, develop their ability to present material and answer questions, coherently, completely and accurately, and enabling them to explain things to others clearly.
- (3) Developing skills for teamwork by working in groups.

**Grading:** MATH 351: 40% - 2 Tests  
40% - 2 Homework assignments (take-home)  
20% - Project (15% paper + 5% 15-min presentation)

MATH 551: 40% - 2 Tests  
30% - 2 Homework assignments (take-home)  
30% - Project<sup>(2)</sup> (25% paper + 5% 15-min presentation)

(All test dates will be announced in class on due time and **NO** test scores will be dropped)<sup>(3)</sup>

**Project:** The project could be of practical or theoretical nature, but always related to the material and objectives of the course. Concepts from the material taught should be used within the project. Students should form groups of two individuals and cooperate. Equal contribution from both members of the group, in both the project and presentation, is expected. Each group should come up with its own project. The projects could be related to works from other courses, but they should always be relevant to the topics covered in this course.

**Deadline: April 24<sup>th</sup>, 2008.**

**Grading Scale:** The grade  $G$  for the course is then determined as follows:

A	$90\% \leq G \leq 100\%$
A-	$85\% \leq G < 90\%$
B+	$83\% \leq G < 85\%$
B	$77\% \leq G < 83\%$
B-	$75\% \leq G < 77\%$
C+	$73\% \leq G < 75\%$
C	$67\% \leq G < 73\%$
C-	$65\% \leq G < 67\%$
D	$50\% \leq G < 65\%$
F	$0\% \leq G < 50\%$

**Test Make-up Policy:** Speak to me **before** the test or leave a telephone or an e-mail message. If you are not able to contact me before the test, contact me within the **next** couple of days. Documentation to verify the reason you missed the test is required. Only one make-up test will be allowed for the semester and that if there are extremely special circumstances.

**Special needs:** DigiPen Institute of Technology will provide reasonable accommodations and academic adjustments for persons with documented disabilities, as indicated in the catalog. Students need to contact the Student Services Director at the beginning of the semester to ensure that classroom and academic accommodations are implemented in a timely fashion. All communication between students, the Director of Student Services, and the professor, concerning special needs will be strictly confidential.

**Class Policy:** Attendance in class is expected, although not mandatory. Keep in mind, though, that if you are absent for 2 weeks, or more, you are considered to have withdrawn from the course. Also, repeated absence might result to complications with financial support for some students. If you decide to drop the course it is your responsibility to do the paperwork and follow the correct procedures. Only pencil is to be used on tests. Calculators like TI-89, or TI-83, are useful to have, but are not required. All tests must be your work. Cell phones, pagers, laptops, ipods, etc, are expected to be turned off during class. No food or drinks are allowed in class.

**Academic dishonesty:** Academic dishonesty (including cheating and plagiarism) is a serious matter, and will not be tolerated. It will be dealt with appropriately as indicated in the student handbook. Homework and solutions of problems should be your own work, and not be copied. It is fine to consult others and discuss problems with each other, but the final solution should be your own work and in your own words, with your own drawings and your own explanations.

**Notes:** (1) Some useful material regarding this course, or other courses, you may here:

<https://digipen.edu/~maristidou>

- (2) Students in MATH 551 will be expected to read at least one research paper, and show some degree of originality in the content of their papers.
- (3) Changes, if any, to this syllabus will be announced in class.

## Appendix 2. Key Terms

**Complex Number:** a complex number is a number of the form  $a + bi$ , where  $a, b$  are real numbers.

The  $i$  is such that  $i^2 = -1$  and is called the “imaginary unit”. For example,  $2 + 3i$ ,  $5$ ,  $-2i$ ,  $\frac{\pi}{2} + \sqrt{2}i$  are complex numbers. Complex numbers extend the real numbers and they are 2-dimensional. Unlike real numbers complex numbers do not have natural ordering.

**Quaternion:** a quaternion is a number of the form  $a + bi + cj + dk$ , where  $a, b, c, d$  are real numbers. The  $i, j, k$ , are imaginary units, where that  $i^2 = j^2 = k^2 = ijk = -1$ . For example,  $2 + 3i - j + 7k$ ,  $-j$ ,  $\sqrt{3} - k$  are quaternions. Quaternions extend the complex numbers and they are in 4-dimensional. (there is no complex number extension in 3-dimensions). Unlike real and complex numbers quaternions are not commutative.

**Octonion:** octonion is a number of the form  $a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$ , where  $a_i$  are reals and the  $e_i$ ’s are imaginary units. Namely,  $e_i^2 = -1$  and  $e_i e_j = e_k$ ,  $e_j e_i = -e_k$ . (see multiplication table below). For example,  $-2 + e_5$ ,  $\ln(2)e_4$ ,  $\sqrt{10}e_1 - \frac{1}{100}e_6 + \frac{5}{3}e_7$  are octonions. Octonions extend the quaternions and they are 8-dimensional. Octonions are neither commutative nor associative.

$e_i e_j$	$e_0$	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
$e_0$	$e_0$	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
$e_1$	$e_1$	$-e_0$	$e_3$	$-e_2$	$e_5$	$-e_4$	$-e_7$	$e_6$
$e_2$	$e_2$	$-e_3$	$-e_0$	$e_1$	$e_6$	$e_7$	$-e_4$	$-e_5$
$e_3$	$e_3$	$e_2$	$-e_1$	$-e_0$	$e_7$	$-e_6$	$e_5$	$-e_4$
$e_4$	$e_4$	$-e_5$	$-e_6$	$-e_7$	$-e_0$	$e_1$	$e_2$	$e_3$
$e_5$	$e_5$	$e_4$	$-e_7$	$e_6$	$-e_1$	$-e_0$	$-e_3$	$e_2$
$e_6$	$e_6$	$e_7$	$e_4$	$-e_5$	$-e_2$	$e_3$	$-e_0$	$-e_1$
$e_7$	$e_7$	$-e_6$	$e_5$	$e_4$	$-e_3$	$-e_2$	$e_1$	$-e_0$

**Ring:** a ring is an algebraic structure consisting of a set with two operations which obey the same basic laws as addition and multiplication of integers, which multiplication does not have to be commutative. Also, multiplicative inverses are not required to exist. For example,  $\mathbb{Z}$  (integers),  $\mathbb{Z}_4$  (4-modulo space),  $\mathbb{R}$  (reals),  $\mathbb{H}$  (quaternions),  $M_{2 \times 2}(\mathbb{R})$  (2-by-2 matrices) are rings.

**Field:** a field is a commutative ring in which every nonzero element has a multiplicative inverse. For example,  $\mathbb{Z}_5$  (5-modulo space),  $\mathbb{R}$  (reals),  $\mathbb{C}$  (complex numbers) are fields.

**Skew Field:** A skew field (or division ring) is a ring (not necessarily commutative) in which every element has a multiplicative inverse. A commutative skew field is a field. For example,  $\mathbb{C}$  (complex numbers),  $\mathbb{H}$  (quaternions) are skew fields. Notice that  $\mathbb{H}$  is a non-commutative skew field. A non-commutative, non-associative skew field is  $\mathbb{O}$  (octonions).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International ([CC BY-NC-SA 4.0](https://creativecommons.org/licenses/by-nc-sa/4.0/)). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.



**Norm:** the norm of a vector is basically its length. More generally, the norm of a mathematical object is a quantity that in some sense describes the object's size. It is usually denoted by  $\| \cdot \|$ . For example, the norm of the number 7 is  $\|7\| = |7| = 7$ . The norm of the vector  $\vec{u} = (1, 3)$  is  $\|\vec{u}\| = \sqrt{1^2 + 3^2} = \sqrt{10}$ .

**Zero Divisors:** zero divisors are non-zero elements  $a, b$  of a ring such that  $ab = 0$ . For example, in the ring  $Z_6$  (6-modulo space) the elements 2 and 3 are zero divisors. Indeed, 2 and 3 are non-zero, but  $2 \cdot 3 = 6 = 0$ . Notice that  $Z$  (integers) is a commutative ring with no zero divisors (such rings are called *integral domains*). Also, in a field (or a skew field) there are no zero divisors. In other words, in a field  $ab = 0$  implies  $a = 0$  or  $b = 0$ . The latter is crucial when students learn to solve equations by factoring.

**Idempotent Element:** an idempotent is an element  $a$  of a ring such that  $a^2 = a$ . For example, in the ring  $Z_6$  (6-modulo space) the element 4 is an idempotent. Indeed,  $4^2 = 16 = 4$ . In the ring  $Z$  (integers) the only idempotents are the trivial elements 0 and 1. Any non-trivial idempotent  $a$  is a zero divisor because for  $a, b$  non-zero, where  $b = 1 - a$ , we have  $ab = 0$ . This means that integral domains, skew fields and fields do not have such idempotents.

**Theorem:** a theorem is a mathematical statement that has been proven to be true. A theorem is usually a more significant result than other propositions within a mathematical context.

**Conjecture:** a conjecture is a mathematical statement that is believed to be true, but for which a proof or disproof has not been found yet.

**Counter-example:** a counterexample is an example that disproves a conjecture.