Editorial of the Winter 2025 Issue 62, Vol 17 No 6

Ryan L. Cerveza, Issue 62 Editor

Don Mariano Marcos Memorial State University, Philippines



What makes mathematics teaching truly meaningful and effective in today's diverse classrooms? In this issue, we invite you to explore fifteen insightful, research-based articles that tackle this question from multiple perspectives. Spanning educational levels, instructional contexts, and cultural settings, these studies illuminate the dynamic landscape of mathematics teaching, learning, and research. Organized into seven thematic categories—foundational studies on mathematical thinking and proof, teachers' knowledge and beliefs, curriculum design and instructional models, problem-solving and mathematical communication, integration of technology, mathematical representations and literacy, and assessment and evaluation—this collection offers readers a rich and multifaceted view of contemporary mathematics education.

Conducted by authors from Cyprus, Indonesia, the Philippines, Rwanda, Russia, Japan, Nepal, and South Korea, the research exemplifies MTRJ's dedication to scholarship that not only advances theory but also provides practical insights to inform and inspire classroom practice. Central to MTRJ's mission is the belief that meaningful mathematics education emerges from the intersection of theoretical insight, empirical research, and instructional practice.

The articles in this issue collectively reinforce this principle by demonstrating how mathematical teaching can be cultivated through purposeful task design, reflective teaching, thoughtful assessment, and the strategic use of technology. The editorial team sincerely thanks all the researchers who contributed their valuable articles to this issue. Your dedication, insights, and scholarly work have greatly enriched the content and purpose of this publication.

Taken as a whole, this issue offers a rich, research-grounded exploration of mathematical thinking and reasoning as lived experiences in real classrooms. Readers will gain insights into how students think, how teachers teach, how curricula are designed, and how assessment and technology can be leveraged to support deeper understanding. Each article contributes a distinct perspective, yet together they form a coherent narrative about the possibilities and challenges of advancing mathematics education through research.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

We invite readers—researchers, teacher educators, curriculum developers, and classroom practitioners alike—to engage deeply with the studies presented in this issue. By doing so, readers will not only encounter innovative ideas and empirical findings but also gain practical insights that can inform teaching, learning, and future research. This collection exemplifies the spirit of *MTRJ*: scholarship that **illuminates mathematical teaching and transforms mathematics classroom practice**.

Reasoning in Mathematics. The first set foregrounds the intellectual core of mathematics education: *how learners reason, coordinate ideas, and construct proofs*. **Aristidou's** study in Cyprus illustrates how mistakes and conjectural thinking in student projects can serve as catalysts for mathematical discovery. This work has significant implications for classroom practice, suggesting that teachers should create environments where exploration, error analysis, and conjecturing are valued as legitimate components of the learning process in mathematics.

Studies from Indonesia by **Pangadongan et al**. and **Widadah et al**. further deepen this perspective by examining students' cognitive coordination and reasoning strategies in advanced topics such as derivatives and proof. These findings imply that instruction should move beyond procedural fluency toward explicit support for abstraction, reflection, and reasoning, particularly in secondary and tertiary mathematics classrooms.

The Story of a "Theorem": From Mistakes to New Mathematics (pp. 7 - 22)

Michael Aristidou (Cyprus)

This article highlights the productive role of errors in mathematical discovery. Through a student project on quaternions, it shows how incorrect theorems and omissions can lead to new mathematical insights, reinforcing exploratory learning and authentic mathematical practice.

A Framework for Mathematical Proof: A Combination of Deduction, Induction, and Abduction (pp. 23 - 41)

Soffil Widadah, Tatag. Y. E. Siswono, and Rooselyna Ekawati (Indonesia)

This research examines how students use multiple forms of reasoning in constructing mathematical proofs. It emphasizes the complementary roles of abduction, deduction, and induction in developing rigorous yet flexible proof strategies.

How Students Think About a Reverse Problem of Derivative: A Study of Arnon's Coordination in Mental Mechanism (pp. 42 - 73)

Fara Virgianita Pangadongan, Toto Nusantara, and I Made Sulandra (Indonesia)

This study investigates students' mental coordination when solving reverse problems involving derivatives

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

and antiderivatives. It contributes to cognitive theory by detailing how encapsulation, assimilation, and accommodation operate in advanced mathematical thinking. This study served as a reference for educators when designing mathematical problem-solving exercises and lessons.

Teaching Knowledge and Practice. The studies by **Indartiningsih et al**. (Indonesia) and **Habiyaremye et al.** (Rwanda) emphasize that teachers' understanding of mathematical thinking directly shapes classroom opportunities for reasoning. Indartiningsih et al.'s research on primary mathematics teachers' conceptions of critical thinking shows that stronger conceptions are associated with more effective instructional design, highlighting the need for professional development focused on pedagogical reasoning. Similarly, Habiyaremye et al.'s quasi-experimental study demonstrates that integrating TPACK (Technological Pedagogical Content Knowledge) into lesson planning enhances tutors' mathematical knowledge for teaching, promoting more open-ended questioning and deeper student reasoning.

These findings suggest that teacher education programs and in-service training should prioritize pedagogical reasoning, reflective practice, and research-based instructional frameworks.

Classroom Practices Mirroring Mathematical Knowledge for Teaching in Rwandan Teacher-Training Colleges: TPACK-Informed Lesson Planning (pp. 74 - 95)

Hashituky Telesphore Habiyaremye, Celestin Ntivuguruzwa, and Philothere Ntawiha (Rwanda)

This quasi-experimental study demonstrates how integrating TPACK (Technological Pedagogical Content Knowledge) into lesson planning enhances tutors' mathematical knowledge for teaching. It documents shifts toward more open-ended questioning and deeper student reasoning.

Teachers' Conceptions of Critical Thinking in Mathematics Teaching (pp. 96 - 123)

Duhwi Indartiningsih, Tatag Yuli Eko Siswono, Wiryanto, and Suryanti (Indonesia)

This mixed-methods study examines how primary mathematics teachers conceptualize critical thinking. It shows that stronger conceptions are associated with better instructional design, highlighting the need for professional development focused on pedagogical reasoning.

Design-Based Instruction in Mathematics. Design research studies in this issue demonstrate the power of hypothetical learning trajectories, realistic mathematics education, and project-based learning in addressing students' learning obstacles. The works of **Zafirah et al**. and **Nurhikmayati et al**., conducted in Indonesian classrooms, show how carefully sequenced instructional designs can support conceptual understanding in social arithmetic and geometry.

For curriculum developers and classroom teachers, these studies highlight the importance of research-informed

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

curriculum planning that anticipates students' difficulties and scaffolds mathematical thinking over time. These studies emphasize the critical role of research-based instructional strategies in fostering both conceptual understanding and effective mathematical communication.

Design and Implementation of Hypothetical Learning Trajectory on the Geometry of Cubes and Cuboids using Project-Based Learning (pp. 124 - 150)

Iik Nurhikmayati, Sufyani Prabawanto, Darhim, Jarnawi Afgani Dahlan (Indonesia)

This article shows how HLT combined with project-based learning helps overcome epistemological, ontogenetic, and didactic learning obstacles in geometry.

Discovering the Power of Realistic Mathematics Education: A Hypothetical Learning Trajectory for Teaching Social Arithmetic (pp. 151 - 194)

Afifah Zafirah, Ahmad Fauzan, Yerizon, Randika Irwa Risky, and Fardatil Aini Agusti (Indonesia)

This design research study illustrates how a contextualized HLT grounded in realistic mathematics education supports students' understanding of profit and loss concepts.

Mathematical Discourse and Problem Solving. Several articles emphasize that mathematical thinking is made visible through problem-solving, problem-posing, and communication. A study from the Philippines reveals both the potential and the challenges faced by students and pre-service teachers in articulating mathematical ideas clearly and meaningfully. Complementing these findings, **Samura et al.'s** (Indonesia) quasi-experimental study demonstrates that problem-based learning significantly improves junior high school students' mathematical communication skills, enabling them to express ideas clearly and coherently.

The positive impact of problem-based learning suggests that classrooms should provide structured opportunities for discussion, justification, and critique, helping learners develop both mathematical understanding and communicative competence.

The Application of Problem-Based Learning in Improving Junior High School Students' Mathematical Communication Skills (pp. 195 - 222)

Asri Ode Samura, In Hi Abdullah, Muhammad Daut Siagian, Habibi Ratu Perwira Negara (Indonesia)

This quasi-experimental study demonstrates that problem-based learning significantly improves students' ability to communicate mathematical ideas clearly and coherently.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Well-Stated Problem: What is the Level of Problem-Posing Performance of Pre-Service Mathematics Teachers? (pp. 223 - 247)

John Rey Oficiar, Jerwin Reonisto, Anthony Bactol (Philippines)

This study evaluates the quality of problems posed by pre-service teachers and identifies common weaknesses such as ambiguity and misalignment of representations, underscoring the pedagogical importance of problem posing.

Technology-Enhanced Mathematics Learning. Technology-focused studies in this issue illustrate how digital tools can enhance conceptual understanding and learner engagement when grounded in sound pedagogical design. Research on augmented reality in Indonesia and Japan shows improvements in students' geometric understanding, while Moodle-based adaptation tools in Russia highlight the role of technology in supporting diverse and multilingual learners. Moreover, **Tristanti et al.**'s study demonstrates that digital worksheets are more effective than printed materials in developing students' mathematical argumentation.

These findings imply that technology integration in mathematics education should be intentional, research-driven, and inclusive, reinforcing rather than replacing mathematical reasoning. Read on!

Technology-Based Worksheets to Develop Students' Mathematical Argumentation Skills (pp. 248 - 277)

Lia Budi Tristanti, Syarifatul Maf'ulah, M.Bahrul Subkhi (Indonesia)

This research compares digital and printed worksheets, showing that technology-based materials significantly enhance students' mathematical argumentation using Toulmin's framework.

Boosting Junior High School Students' Mathematical Proficiency Utilizing the Integration of AR in the Classroom (pp. 278 - 298)

Wanda Nugroho Yanuarto, Elfis Suanto, Ira Hapsari, Masanori Fukui (Indonesia/ Japan)

This experimental study demonstrates that augmented reality applications improve students' understanding of geometry and foster positive attitudes toward learning mathematics.

Joint Research and Design of Educational Adaptation Tools in Mathematics for Foreign Students in the LMS Moodle (pp. 299 - 310)

Veronika I. Belousova, Alla A. Knysh, Kseniia S. Potorochina (Russia)

This article focuses on blended learning and digital adaptation tools designed to support foreign students in learning mathematics, emphasizing inclusivity and bilingual approaches.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Representation and Contextualization. The next set addresses the role of representations and real-world contexts in developing mathematical literacy. Studies from Nepal and Indonesia show that students often struggle to connect symbolic, graphical, and contextual representations.

The implication for classroom teaching is clear: mathematics instruction must deliberately foster representational flexibility and contextual reasoning, enabling students to move meaningfully between abstract symbols and real-world applications.

Jambi Batik Pattern: Developing PISA-Like Mathematics Problems (pp. 311 - 339)

Tria Gustiningsi, Feri Tiona Pasaribu, and Yelli Ramalisa (Indonesia)

This design research study develops culturally contextualized PISA-like problems, demonstrating their validity, practicality, and potential to enhance students' mathematical literacy.

Bridging the Gap: Understanding Students' Struggles with Algebraic and Graphical Representations of Functions (pp. 340 - 359)

Santosh Pathak (South Korea/ Nepal)

This study reveals students' strong preference for algebraic methods and limited awareness of graphical strategies, highlighting the need for balanced representational instruction.

Assessment in Mathematics Education. Assessment-related research in this issue highlights the importance of diagnostic and interpretive assessment practices. The Rasch analysis conducted by **Balasabas and Arnado** in the Philippines demonstrates how assessment data can reveal mismatches between test difficulty and student ability, as well as the influence of affective factors such as test anxiety.

For mathematics educators, this emphasizes the need to view assessment not merely as a grading tool but as a research-informed mechanism for improving instruction, identifying learning gaps, and supporting students' mathematical confidence.

Analyzing Mathematics 10 First Quarter Examination and Students' Test Taking Skill Through the Rasch Model (pp. 360 - 382)

Jay A. Balasabas, Alvic A. Arnado (Philippines)

Using the Rasch model, this study evaluates test item difficulty and student response patterns. It provides insights into test validity, student misfit behavior, and the role of affective factors such as test anxiety in mathematics assessment.

The Problem Corner (pp. 383 - 387)

Ivan Retamoso

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL 7
WINTER 2025
Vol 17 no 6

The Story of a "Theorem": From Mistakes to New Mathematics

Michael Aristidou

American University of Cyprus, Larnaca, Cyprus
michael.aristidou@aucy.ac.cy

Abstract: In this article, we look at the exploratory aspects of student projects. We discuss a specific project from a course on quaternions and see how incorrect theorems, student omissions, etc., led to some new mathematical results. In particularly, the fact that H/Z_p was not just another division algebra but simply a ring with zero-divisors opened new questions to be explored, such as finding non-trivial idempotents, tripotents, and exploring extensions to larger rings such as O/Z_p. This serves as a reminder that student projects occasionally could also have academic benefits for the students or the instructors. These are outcomes that could enhance the students' or the instructors' academic and professional development.

Keywords: project, theorem, quaternions, ring, zero divisors

INTRODUCTION

Student projects[1] within a college course are not only pedagogically beneficial to students but also could occasionally have *academic benefits*[2] for students or instructors too. The general benefits for the students and instructors were discussed in (Osburn & Karukstis, 2009) and more briefly in (Petrella & Jung, 2008). In both articles, the *initiation and production of research and new results* is identified as one of the benefits, and these fall into what I would call academic benefits. In this article, we show an instance of academic benefit for the instructor and how it actually occurred. In other words, we will see how engaging students in projects yielded research questions and new results which led to publications. Interestingly, in our case, the new results did not come from the students' produced results but from their omissions, mistakes, and from the material used and studied during their project work.[3]

---

[1] By "student projects" I mean traditional projects given to students (or initiated by them) in order to explore something during a course. I am not referring to "project-based learning" which is a bit different and I have criticized in (Aristidou, 2020).
[2] By "academic benefit" I mean any by-product of an activity that could boost the teacher's or the student's academic and professional development. For example, research that could lead to a publication in a scientific journal could be such a by-product. The term was elaborated further in (Aristidou, 2020).
[3] For some other examples of projects which produced academic benefit for students see (Aristidou, 2020). And for some that produced academic benefit for students and instructors see (Aristidou, 2020), (Aristidou & Hailemariam, 2021) and (Marciniak, 2018).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

More specifically, I will discuss a new result that came out of one of my courses at DigiPen Institute of Technology in 2008.[4] It was from a course on quaternion algebra and its applications in computer graphics in which the students had the option to work on a project of their liking in groups of two (see Appendix 1 for the Syllabus and more details). The students were strongly encouraged to come up with their own ideas for a project, but they were also told in class that a list of possible project suggestions would be available by the instructor at a later time. Three groups of two students accepted my suggestion to explore quaternions with coefficients not from the field of real numbers R but from the finite field Z_p.

In the following sections, we will describe how the assignment played out and what new came out from it. First, I will provide some basics on quaternions H and the quaternion ring H/Z_p (see also Appendix 2 for brief explanations of the key mathematical terms). Then, I will describe the mistake we found and its algebraic ramifications. Finally, in the last section, we will draw some conclusions.

THEORETICAL FRAMEWORK

The Quaternions H and the Ring H/Z_p

The quaternions, denoted by H, were first discovered by William R. Hamilton in 1843 as an extension of complex numbers into four dimensions (Remmert et al., 1991). Namely, a quaternion is an element of the form $x = a_0 + a_1 i + a_2 j + a_3 k$, where a_i are from R (real numbers) and i, j, k are such that $i^2 = j^2 = k^2 = ijk = -1$.

For example, $x = 2 + 3i - j + 7k$ is a quaternion. So are $y = 3k$, $z = i + 2j$ and $w = -2$.

The operations of addition "+" and multiplication "." in H are defined in (Herstein, 1975, p.124).[5] Basically, addition is done component-wise and multiplication distributive-wise resulting in the following formulas:

$$\begin{aligned} x + y &= (a_0 + a_1 i + a_2 j + a_3 k) + (b_0 + b_1 i + b_2 j + b_3 k) \\ &= (a_0 + b_0) + (a_1 + b_1) i + (a_2 + b_2) j + (a_3 + b_3) k \end{aligned} \quad (1)$$

$$\begin{aligned} xy &= (a_0 + a_1 i + a_2 j + a_3 k)(b_0 + b_1 i + b_2 j + b_3 k) \\ &= (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) + (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2) i + \\ &\quad (a_0 b_2 - a_1 b_3 + a_2 b_0 + a_3 b_1) j + (a_0 b_3 + a_1 b_2 - a_2 b_1 + a_3 b_0) k \end{aligned} \quad (2)$$

[4] I am referring to DigiPen's main campus in Redmond, WA.
[5] Multiplication is associative but not commutative, because ij = k and ji = -k.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

For example, adding $x = 2 + 3i - j + 7k$ and $z = i + 2j$ from the previous example above we get:

$$\begin{aligned} x + z &= (2 + 3i - j + 7k) + (i + 2j) \\ &= (2 + 3i - j + 7k) + (0 + i + 2j + 0k) \\ &= (2+0) + (3+1)i + (-1+2)j + (7+0)k \\ &= 2 + 4i + j + 7k \end{aligned} \qquad (3)$$

And, multiplying $x = 2 + 3i - j + 7k$ and $z = i + 2j$ we get:

$$\begin{aligned} xz &= (2 + 3i - j + 7k)(i + 2j) \\ &= (2 + 3i - j + 7k)(0 + i + 2j + 0k) \\ &= (0 - 3 + 2 - 0) + (2 + 0 + 0 - 14)i + \\ &\quad (4 - 0 + 0 + 7)j + (0 + 6 + 1 + 0)k \\ &= -1 - 12i + 11j + 7k \end{aligned} \qquad (4)$$

The ring H/Z_p contains finitely many elements of the form $x = a_0 + a_1 i + a_2 j + a_3 k$, where the a_i's come from Z_p. That is the set $Z_p = \{0, 1, 2, \ldots, p\text{-}1\}$, i.e., the set of remainders modulo p. In other words, the numbers resulting after we "subtract off" multiples of p. In regards to the ring H/Z_p, as $p = 0$ in Z_p, the operations are defined as in H but taking into consideration the modulo p operation (mod$p$). Notice also that as H is non-commutative, so is H/Z_p.

For example, in H/Z_5, adding $x = 2 + 3i + j + 4k$ and $z = 2i + 2k$ we get:

$$\begin{aligned} x + z &= (2 + 3i + j + 4k) + (2i + 2k) \\ &= 2 + 5i + j + 6k \\ &= 2 + j + k \end{aligned} \qquad (5)$$

(because $5 = 0$ in Z_5. Hence, $6 = 5 + 1 = 0 + 1 = 1$, etc.).

And, in H/Z_5, multiplying $x = 2 + 3i + j + 4k$ and $z = 2i + 2k$ we get:

$$\begin{aligned} xz &= (2 + 3i + j + 4k)(2i + 2k) \\ &= -14 + 6i + 2j + 2k \\ &= 1 + i + 2j + 2k \end{aligned} \qquad (6)$$

(because $5 = 0$ in Z_5. Hence, $15 = 14 + 1 \Rightarrow 5.3 = 14 + 1 \Rightarrow 0 = 14 + 1 \Rightarrow 14 = -1$ and $6 = 5 + 1 = 0 + 1 = 1$, etc.).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Algebraically speaking, H forms a non-commutative normed division algebra (skew field) over R of dimension 4 (Remmert et al., 1991, p.195-196). There are only four normed division algebras (Hurwitz, 1898), namely R, C, H and O, with the octonions O (which are non-commutative and non-associative) being the largest.

The structure and some of the properties of the finite ring H/Z_p, where p is a prime number, was done in Aristidou and Demetre (2009). A more detailed description of the structure H/Z_p was given by Miguel and Serodio (2011). Among others, they found the number of zero-divisors, the number of idempotent elements, and provided an interesting description of the zero-divisor graph.

METHOD

Students' Projects

The study of H/Z_p was exactly the project suggested to some students in the purposes of the course on quaternions that we described in the Introduction. The students were asked to explore H/Z_p, trying perhaps to define its operations, work out examples, look for some idempotents, solve simple equations in H/Z_p and possibly explore similar things in O/Z_p. At the time I was unaware that H/Z_p and some of the questions posed above were included as an exercise in (Herrnstein, 1975, p.136). What I found instead was an article by Kandasamy (2000) which I shared with the students as material to help them with their project. Certainly, Kandasamy's paper was a good source of information for them, due to its content and accessible level. But, unfortunately, a paper from which they used results at face value and imitated proofs without much caution.

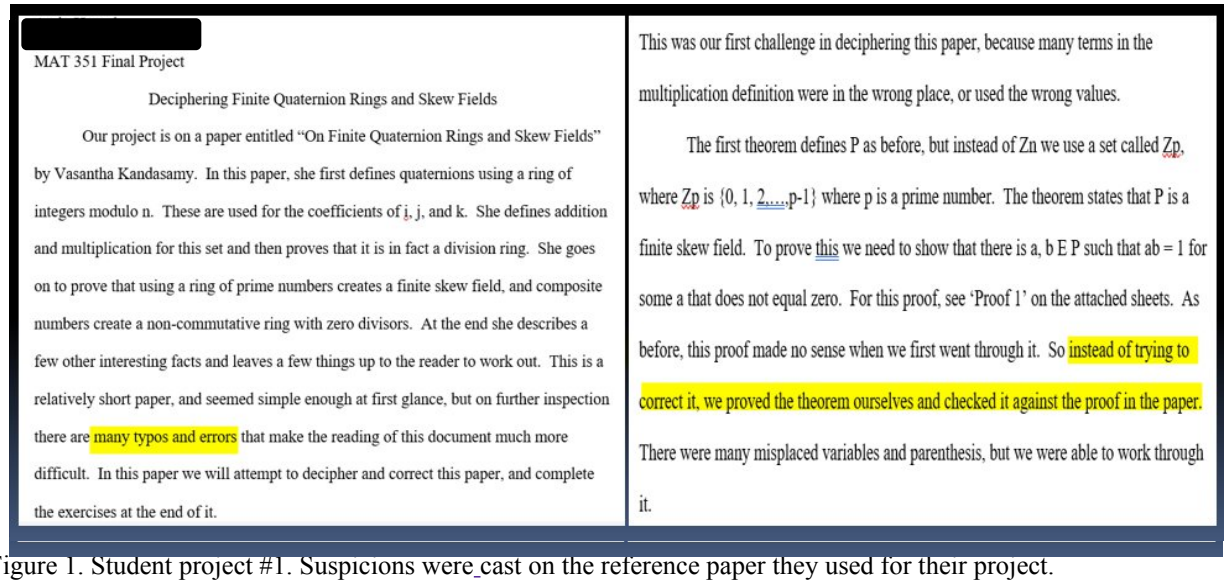
Then, a serious problem appeared while grading the students' projects. My colleague A. Demetre and I found a counterexample to what the students claimed and supposedly proved. So, we were convinced that there must have been a mistake somewhere in their work. Initially, we did not notice any mistake in the original article by Kandasamy which was given to the students as reference. After all, the latter was supposedly peer reviewed and published, so one did not have a reason to doubt it or double check it. But, since the students pretty much imitated a key theorem "proved"[6] in Kandasamy's article in their work, that raised a suspicion and gave us a reason to re-examine Kandasamy's result more carefully.

The students had also their own suspicions of Kandasamy's article. For example, in one project (see Figure 1), students already found several typos and errors in the article. Furthermore, they expressed suspicions for the proof of Kandasamy's Theorem 2 mentioned above (see footnote 6), which was relevant to their work. Nevertheless, they considered the result to be correct and played along trying to give their own proof of the same "fact," naturally making similar mistakes. It seems

[6] That was Theorem 2 in Kandasamy's article (2000, p.133), which falsely stated that H/Zp is a skew field.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

that from the four stages of creativity identified by Wallas (1926) (see also Marciniak, 2020), namely *preparation, incubation, illumination* and *verification*, the students missed the last stage.



[redacted]

MAT 351 Final Project

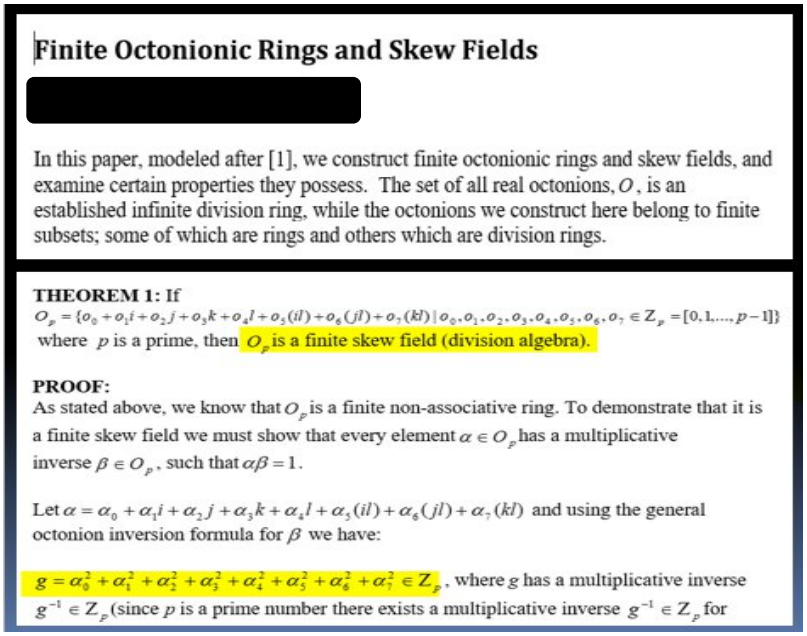
Deciphering Finite Quaternion Rings and Skew Fields

Our project is on a paper entitled "On Finite Quaternion Rings and Skew Fields" by Vasantha Kandasamy. In this paper, she first defines quaternions using a ring of integers modulo n. These are used for the coefficients of i, j, and k. She defines addition and multiplication for this set and then proves that it is in fact a division ring. She goes on to prove that using a ring of prime numbers creates a finite skew field, and composite numbers create a non-commutative ring with zero divisors. At the end she describes a few other interesting facts and leaves a few things up to the reader to work out. This is a relatively short paper, and seemed simple enough at first glance, but on further inspection there are many typos and errors that make the reading of this document much more difficult. In this paper we will attempt to decipher and correct this paper, and complete the exercises at the end of it.

This was our first challenge in deciphering this paper, because many terms in the multiplication definition were in the wrong place, or used the wrong values.

The first theorem defines P as before, but instead of Zn we use a set called Zp, where Zp is {0, 1, 2,...,p-1} where p is a prime number. The theorem states that P is a finite skew field. To prove this we need to show that there is a, b E P such that ab = 1 for some a that does not equal zero. For this proof, see 'Proof 1' on the attached sheets. As before, this proof made no sense when we first went through it. So instead of trying to correct it, we proved the theorem ourselves and checked it against the proof in the paper. There were many misplaced variables and parenthesis, but we were able to work through it.

Figure 1. Student project #1. Suspicions were cast on the reference paper they used for their project.



Finite Octonionic Rings and Skew Fields

[redacted]

In this paper, modeled after [1], we construct finite octonionic rings and skew fields, and examine certain properties they possess. The set of all real octonions, O, is an established infinite division ring, while the octonions we construct here belong to finite subsets; some of which are rings and others which are division rings.

THEOREM 1: If
$O_p = \{o_0 + o_1 i + o_2 j + o_3 k + o_4 l + o_5(il) + o_6(jl) + o_7(kl) \mid o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7 \in Z_p = [0, 1, \ldots, p-1]\}$
where p is a prime, then O_p is a finite skew field (division algebra).

PROOF:
As stated above, we know that O_p is a finite non-associative ring. To demonstrate that it is a finite skew field we must show that every element $\alpha \in O_p$ has a multiplicative inverse $\beta \in O_p$, such that $\alpha\beta = 1$.

Let $\alpha = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k + \alpha_4 l + \alpha_5(il) + \alpha_6(jl) + \alpha_7(kl)$ and using the general octonion inversion formula for β we have:

$g = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2 + \alpha_7^2 \in Z_p$, where g has a multiplicative inverse $g^{-1} \in Z_p$ (since p is a prime number there exists a multiplicative inverse $g^{-1} \in Z_p$ for

Figure 2. Student project #2. Attempt to generalize the main result from the ring H/Z_p to the ring O/Z_p

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

They simply did not check the validity of their ideas, even though they did express doubts on some of the information in the reference article.

In another project (see Figure 2) students also used Kandasamy's Theorem 2 (see footnote 6) and imitated her proof in order to extend the result from the quaternionic ring H/Z_p to the non-associative octonionic ring O/Z_p. The latter was also mistakenly "shown" by the students to be a skew field which caused a different kind of suspicions because the octonions, as Baez said, "*are the crazy old uncle nobody lets out of the attic*" (Baez, 2002). They are an "exceptional" structure that usually does not follow common algebraic expectations in the hyper-complex numbers' family (non-associative, not a Lie group, etc.). Which made the students' result sound a bit too good to be true and beg a careful examination. Clearly, this group of students missed Wallas' fourth stage of creativity (i.e., *verification*) too (Wallas, 1926; Marciniak, 2020). They did not check the validity of their results.

As we already mentioned, a counter-example was produced (see next section) which contradicted Kandasamy's supposed Theorem 2 (2000, p.133) which claimed that H/Z_p is supposedly a skew field. As one of the purposes of the student project was to promote exploration and inquiry-based learning[7], the specific counter-example was not revealed immediately to the students. Instead, the two groups of students were asked to produce some examples that confirm or disconfirm their work. Surprisingly, neither of the two groups included examples in their submitted articles even though Kandasamy's reference article includes two examples, namely H/Z_3 and H/Z_4, without the computational details (for H/Z_3 it incorrectly states that is a skew field). In a meeting after the students submitted their projects, the counter-example was shown to the students and their work was discussed. The students were quite surprised and almost in disbelief in how could that be since they were following a published and therefore correct result. One student protested, "Isn't a theorem something that is proved and is therefore true?". Another student simply said, "I knew it that something was wrong with this story!" I pointed out to the students of both groups that, perhaps, had they tried to do the calculations themselves even for the H/Z_3 case from the article they might have noticed that the example does not agree with the supposed theorem.

The students in the 1st group admitted that they were carried away and thought that the example was definitely correct because there was a general theorem proved for H/Z_p (i.e., for all primes) and for $p = 3$ one simply had a special case. The students in the 2nd group, the ones trying to extend the main result from H/Z_p to the octonionic ring O/Z_p, already disproved that O/Z_n is a skew field without providing a counter-example (they imitated Kandasamy's disproof and expanded the calculations). So, it was reasonable to expect at least from that group to produce some concrete counter-examples for O/Z_n, hoping that the calculations would lead them perhaps to some counter-

[7] Inquiry-based learning is a student-centered teaching method where students explore topics, investigate questions, conduct projects, etc., and obtain knowledge by self-study and discovery. Students are not passive recipients of in-formation from the teacher. Instead, they are encouraged to actively participate in the learning process which helps them strengthen their critical thinking and research skills.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

examples for O/Z_p. Their explanation was that they did not actually feel the need to check for counter-examples for O/Z_p as they believed that there were not any because in their "proof" that O/Z_p is a skew field, they simply imitated an established result from the reference article. Later that week, the 1st group of students produced one more counter-example of zero divisors in H/Z_p and sent it to me by email. Also, the 2nd group came by my office showing me a counterexample for O/Z_n , confirming their disproof that O/Z_n is a skew field. They did not produce a counterexample for O/Z_p .

Overall, upon reflection, I think that the project and the follow-up discussions had an impact on the students' learning experience and their understanding of quaternions, octonions, and abstract algebra in general. The students realized that one needs to be more careful when studying abstract and unfamiliar spaces. Furthermore, re-working some examples and doing the calculations is beneficial and gives more experience and insight. Finally, even though mathematics is a rigorous science, mistakes could happen, and one should always keep a skeptical eye on the various claims and results. Certainly, inquiry-based learning is not easy for the students or teachers. It is a pedagogical method that requires commitment and some degree of mathematical maturity from the student. But, it also requires time and good preparation from the teacher. The above-mentioned quaternion course was not specifically designed as an inquiry-based course. It was a lecture course that required a project in which students were free to explore topics related to the course content but not necessarily curriculum or standard topics. Nevertheless, some of the course objectives were "improving problem-solving skills", "develop the ability to answer questions and present material coherently", etc., which were observed and assessed especially via the project. Overall, I believe that the students got a small taste of inquiry-based learning and tried their best. I also believe that some improvements could be made from the instructor's side, in terms of project choices, closer monitoring, time management, etc. The latter are certainly crucial if the approach I just described is to be implemented in other mathematics courses too, especially to lower-level courses.

Given our counter-examples and our suspicions of some of the claims in Kandasamy's article, I wrote early on a polite email[8] to Kandasamy requesting some explanations on how to interpret these results, asking if we could possibly be mistaken, to which email unfortunately I never received a reply. In the meantime, as I mentioned above, more counter-examples were produced and while we were waiting for a reply, we were trying to find where Kandasamy's proof went wrong. I have to admit, we too missed the mistake the first time. Most likely just as the reviewers missed it during the review process before its publication. It was a subtle, yet simple, point which my colleague A. Demetre and I finally caught the second time around and it concerned a calculation in Z_p. The results were presented in (Aristidou & Demetre, 2009) which I summarize in the following section.

[8] Sadly, no record of that email remains as the specific work-email was deactivated after I moved to a different college in 2009.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

RESULTS

The "Theorem" and the Details

Kandasamy in Theorem 2 (2000, p.133) "shows" that H/Z_p, p prime, is a finite skew field. But this claim is *false* and hence its supporting proof *incorrect*. That is because the result contradicts well established results on finite division rings and because one can construct counterexamples to Kandasamy's "Theorem 2." The conjecture Kandasamy set out to prove was:

Conjecture: Let $H/Z_p = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_i \in Z_p, p \text{ prime}, i^2 = j^2 = k^2 = p - 1\}$. Then, H/$Z_p$ is a finite skew field.

The conjecture though is clearly *false*, as the following counterexample (Aristidou & Demetre, 2009) shows:

Counterexample: Let $p = 5$ and consider $q_1 = 1 + 3i + 1j + 3k$, $q_2 = i + 2j$ of H/Z_5. Multiplying the two, we get:

$$q_1 q_2 = (1 + 3i + 1j + 3k)(i + 2j) = 0 \qquad (7)$$

This shows that H/Z_5 has zero divisors, and hence H/Z_5 is *not* a skew field.

More specifically, in the "proof" Kandasamy constructs $g = a_0^2 + a_1^2 + a_2^2 + a_3^2 \in Z_p$, a quantity required for the inverse of $a = a_0 + a_1 i + a_2 j + a_3 k \in H/Z_p$ (see Figure 3). Then, she claims that g is non-zero, probably because it is a sum of squares of non-zero elements of Z_p. But even though this is true in R, it is *not* true in Z_p. For example, in Z_3 one has $g = 1^2 + 1^2 + 1^2 = 3 = 0$. So, an inverse cannot always be constructed in H/Z_p and therefore the "theorem" is false.

Another short proof that H/Z_p is not a skew field is the one we showed in (Aristidou & Demetre, 2009). Namely:

Theorem: H/Z_p, p prime, is *not* a skew field.

Proof: Assume that H/Z_p is a skew field. Since it is a finite skew field, by the Wedderburn's Theorem[9] we have that H/Z_p is also a field. Hence, this gives the following relations:

$$ij = ji,\ ij = k,\ ji = -k \overset{\text{mod } p}{=} (p - 1)k \qquad (8)$$

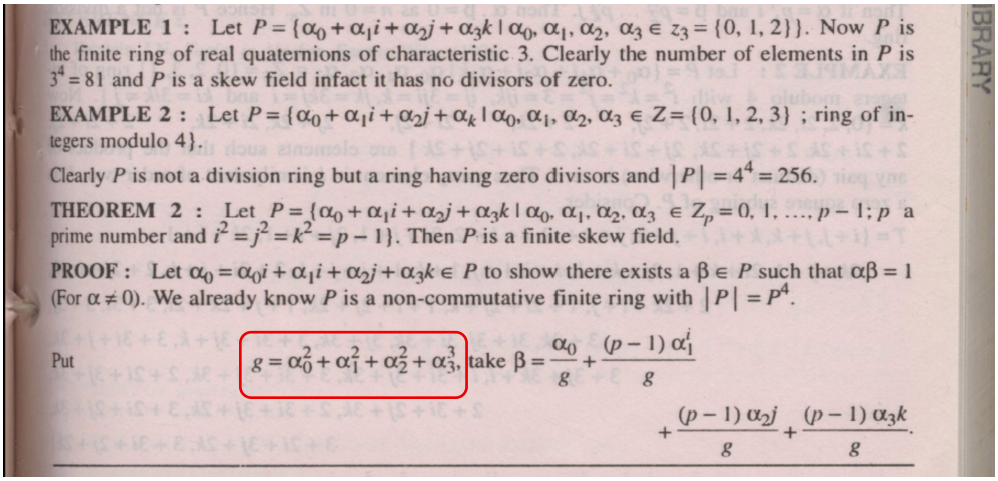
[9] In 1905, Wedderburn proved that "Every finite skew field is a field" (Herstein, 1975, p.361).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

But then, from (8), one also gets:

$$0 = ji - ij = (p-1)k - k = (p-2)k \qquad (9)$$

which is a contradiction if $p > 2$. The case $p = 2$ is trivial, since H/Z_2 is clearly not a skew field. (Just take $\alpha = 1 + i$, and notice that $\alpha \cdot \alpha = 0$, even though $\alpha \neq 0$).



EXAMPLE 1 : Let $P = \{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in z_3 = \{0, 1, 2\}\}$. Now P is the finite ring of real quaternions of characteristic 3. Clearly the number of elements in P is $3^4 = 81$ and P is a skew field infact it has no divisors of zero.

EXAMPLE 2 : Let $P = \{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_k \mid \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in Z = \{0, 1, 2, 3\}$; ring of integers modulo 4$\}$.

Clearly P is not a division ring but a ring having zero divisors and $|P| = 4^4 = 256$.

THEOREM 2 : Let $P = \{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in Z_p = 0, 1, \ldots, p-1; p$ a prime number and $i^2 = j^2 = k^2 = p - 1\}$. Then P is a finite skew field.

PROOF : Let $\alpha_0 = \alpha_0 i + \alpha_1 i + \alpha_2 j + \alpha_3 k \in P$ to show there exists a $\beta \in P$ such that $\alpha\beta = 1$ (For $\alpha \neq 0$). We already know P is a non-commutative finite ring with $|P| = P^4$.

Put $g = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^3$, take $\beta = \frac{\alpha_0}{g} + \frac{(p-1)\alpha_1^i}{g} + \frac{(p-1)\alpha_2 j}{g} + \frac{(p-1)\alpha_3 k}{g}$.

Figure 3. The main mistake in Kandasamy's proof. $g = a_0^2 + a_1^2 + a_2^2 + a_3^2$ could be 0 in Z_p.

CONCLUSIONS

Having students work on projects within a course is pedagogically important for students. Projects require students to move beyond mere memorization of the lessons taught. Engaging into deeper learning, research, and more serious problem solving ensures better understanding of the concepts involved and fosters critical thinking. Projects stimulate the students' curiosity, imagination, and enhances their creativity. Furthermore, projects promote important social skills such as good management of time, management of the material gathered, and organization and collaboration with other students and teachers.

In addition, student projects also occasionally have academic benefits for the students or the instructors. These are outcomes that could enhance the students' or the instructors' academic and professional development. As we mentioned earlier, a publication in a scientific journal would definitely be one such benefit. Academically it could certainly enhance creativity, knowledge, etc., but additionally it could also professionally improve one's C.V., open opportunities for

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

scholarships, funding, promotions, etc. Such benefits are benefits that could also motivate the teaching/learning process.

We discussed an example of a student project which resulted to some new mathematics. The new results did not come from the students' new positive ideas but from some of their mistakes, which in turn revealed deeper mistakes in an article they used as study material and as reference. The new results were presented in a conference and later published as a paper in a journal (Aristidou & Demetre, 2009). In particularly, the fact that H/Z_p was not just another division algebra but simply a ring with zero-divisors opened new questions to be explored, such as finding non-trivial idempotents, tripotents, and exploring extensions to larger rings such as O/Z_p.[10] Consequently, four more articles were published based on (Aristidou & Demetre, 2009) which interestingly started as a student project. This is a clear example of a student project resulting to academic benefit for the instructor. In addition, article (Aristidou & Demetre, 2009) yielded academic benefits for students who worked on subsequent related projects/papers (Aristidou & Hailemariam, 2021) and to other instructors who used the new published results of (Aristidou & Demetre, 2009) in their own research and publications.

REFERENCES

[10] Surprisingly, both rings also have real life applications in Cryptography (some are mentioned in (Aristidou & De-metre, 2009).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[1] M. Aristidou and A. Demetre, "A Note on Quaternion Rings over Z_p", *International Journal of Algebra*, Vol.3, No.15, 2009, 725- 728.
[2] M. Aristidou, "Project Based Learning: Are there any Academic Benefits for the Teacher or Students?", *Journal of Humanistic Mathematics*, Vol.10, No.1, 2020, 458-471.
[3] M. Aristidou and K. Hailemariam, "Tripotent elements in quaternion rings over Z_p", *Acta Universitatis Sapientiae Mathematica*, Vol.13, No.1, 2021, 78−87.
[4] J Baez, "The Octonions", *Bulletin of the American Mathematical Society*, Vol.39, No.2, 2002, 145-205.
[5] I. N. Herstein, *Topics in Algebra*, 2nd ed., Wiley, 1975.
[6] A. Hurwitz, "Über die Komposition der quadratischen Formen von beliebig vielen Variablen", Nach. V. der Ges. der Wiss., Göttingen, *Math. Phys. Kl*., 1898, 309–316.
[7] W. B. V. Kandasamy, "On Finite Quaternion Rings and Skew Fields", *Acta Ciencia Indica*, Vol.XXVI, No.2, 2000, 133-135.
[8] M. A. Marciniak, "Creative Assignments in Upper-Level Undergraduate Courses Inspired By Mentoring Undergraduate Research Projects", *Journal of Humanistic Mathematics*, Vol.10, No.2, 2020, 465-483.
[9] M. A. Marciniak, "Approaches to creativity in undergraduate research projects", *Proceedings of the CUNY Research Summit: Creativity in STEM*, 2018, 34-36.
[10] C. J. Miguel and R. Serodio, "On the Structure of Quaternion Rings over Zp", *Int. J. Algebra*, Vol.5, No.27, 2011, 1313-1325.
[11] J. M. Osburn and K.K. Karukstis, "The Benefits of Undergraduate Research, Scholarship, and Creative Activity". (in *Broadening Participation in Undergraduate Research: Fostering Excellence and Enhancing the Impact*, by M. Boyd and J. Wesemann (eds.), Council on Undergraduate Research, Washington, DC, 2009, 41-53).
[12] J. K. Petrella and A. P. Jung, "Undergraduate Research: Importance, Benefits, and Challenges", *International Journal of Exercise Science*, Vol.1, No.3, 2008, 91-95.
[13] R. Remmert et al., *Numbers*, Springer, 1991.
[14] G. Wallas, *The Art of Thought*, New York: Harcourt Brace, 1926.

APPENDIX

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Appendix 1. Syllabus

SYLLABUS

MATH 351/551

9:30 - 10:50 T,Th
(Michelangelo)

Instructor: Dr. Michael Aristidou
Office Hours: 3:00-4:00 M,W

Phone: 425-895-4400, ext.4464
E-mail: maristidou@digipen.edu

Textbook: Class Notes (No textbook required) [1] .

Other sources: (1) *Visualizing Quaternions*, by A. J. Hanson, Morgan Kaufmann, 2006.
(2) *Quaternions and Rotation Sequences*, by J. B. Kuipers, Princeton Univ. Press, 1999.
(3) Other related papers and articles, such as:
(i) *Quaternions, Interpolation, and Animation*, by E. B. Dam, M. Koch, and M. Lillhom, Technical Report, Univ. of Copenhagen, 1998.
(ii) *Exploiting Quaternions to Support Expressive Interactive Character Motion*, by M. P. Johnson, PhD, MIT, 2003.
(iii) *Animating Rotation with Quaternion Curves*, by K. Shoemake, Siggraph Proceedings, 1985.
(iv) *Smooth Interpolation of Orientations with Angular Velocity Constraints Using Quaternions*, by A. H. Barr, B. Currin, S. Gabriel, and J. Hughes, Siggraph Proceedings, 1992.
(v) *The Octonions* (section 2), by J. C. Baez, Bullentin of American Math. Society, 2002.

Course Overview: This course gives an introduction to several mathematical topics of foundational importance in abstract algebra, and in particularly in the algebra of quaternions. Topics include:

Operations, Groups, Rings, Fields, vector Spaces, Algebras, Complex Numbers, Quaternions, Curves over the quaternionic sphere, Interpolation Techniques, Splines, Octonions, and Clifford Algebras.

Goals and objectives:
(1) Familiarize students with certain abstract discrete mathematical structures, such as groups, rings, algebras, etc, and most importantly the quaternion group, quaternion algebra, and parameterized curves, and encourage them to apply the latter in graphics

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

and other projects of their interest.
(2) Improve their problem–solving skills, develop their ability to present material and answer questions, coherently, completely and accurately, and enabling them to explain things to others clearly.
(3) Developing skills for teamwork by working in groups.

Grading: MATH 351**:** 40% - 2 Tests
40% - 2 Homework assignments (take-home)
20% - Project (15% paper + 5% 15-min presentation)
MATH 551**:** 40% - 2 Tests
30% - 2 Homework assignments (take-home)
30% - Project [(2)] (25% paper + 5% 15-min presentation)
(All test dates will be announced in class on due time and **NO** test scores will be dropped) [(3)]

Project: The project could be of practical or theoretical nature, but always related to the material and objectives of the course. Concepts from the material taught should be used within the project. Students should form groups of two individuals and cooperate. Equal contribution from both members of the group, in both the project and presentation, is expected. Each group should come up with its own project. The projects could be related to works from other courses, but they should always be relevant to the topics covered in this course.
Deadline: April 24th, 2008.

Grading Scale: The grade G for the course is then determined as follows:

A	$90\% \leq G \leq 100\%$
A-	$85\% \leq G < 90\%$
B+	$83\% \leq G < 85\%$
B	$77\% \leq G < 83\%$
B-	$75\% \leq G < 77\%$
C+	$73\% \leq G < 75\%$
C	$67\% \leq G < 73\%$
C-	$65\% \leq G < 67\%$
D	$50\% \leq G < 65\%$
F	$0\% \leq G < 50\%$.

Test Make-up Policy: Speak to me **before** the test or leave a telephone or an e-mail message. If you are not able to contact me before the test, contact me within the **next** couple of days. Documentation to verify the reason you missed the test is required. Only one make-up test will be allowed for the semester and that if there are extremely special circumstances

Special needs: DigiPen Institute of Technology will provide reasonable accommodations and academic adjustments for persons with documented disabilities, as indicated in the catalog. Students need to contact the Student Services Director at the beginning of the semester to ensure that classroom and academic ac-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

commodations are implemented in a timely fashion. All communication between students, the Director of Student Services, and the professor, concerning special needs will be strictly confidential.

Class Policy: Attendance in class is expected, although not mandatory. Keep in mind, though, that if you are absent for 2 weeks, or more, you are considered to have withdrawn from the course. Also, repeated absence might result to complications with financial support for some students. If you decide to drop the course it is your responsibility to do the paperwork and follow the correct procedures. Only pencil is to be used on tests. Calculators like TI-89, or TI-83, are useful to have, but are not required. All tests must be your work. Cell phones, pagers, laptops, ipods, etc, are expected to be turned off during class. No food or drinks are allowed in class.

Academic dishonesty: Academic dishonesty (including cheating and plagiarism) is a serious matter, and will not be tolerated. It will be dealt with appropriately as indicated in the student handbook. Homework and solutions of problems should be your own work, and not be copied. It is fine to consult others and discuss problems with each other, but the final solution should be your own work and in your own words, with your own drawings and your own explanations.

***Notes*:** (1) Some useful material regarding this course, or other courses, you may here: https://digipen.edu/~maristidou
(2) Students in MATH 551 will be expected to read at least one research paper, and show some degree of originality in the content of their papers.
(3) Changes, if any, to this syllabus will be announced in class.

Appendix 2. Key Terms

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Complex Number: a complex number is a number of the form $a + bi$, where a, b are real numbers. The i is such that $i^2 = -1$ and is called the "imaginary unit". For example, $2 + 3i$, 5, $-2i$, $\frac{p}{2} + \sqrt{2}i$ are complex numbers. Complex numbers extend the real numbers and they are 2-dimensional. Unlike real numbers complex numbers do not have natural ordering.

Quaternion: a quaternion is a number of the form $a + bi + cj + dk$, where a, b, c, d are real numbers. The i, j, k, are imaginary units, where that $i^2 = j^2 = k^2 = ijk = -1$. For example, $2 + 3i - j + 7k$, $-j$, $\sqrt{3} - k$ are quaternions. Quaternions extend the complex numbers and they are in 4-dimensional. (there is no complex number extension in 3-dimensions). Unlike real and complex numbers quaternions are not commutative.

Octonion: octonion is a number of the form $a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$, where a_i are reals and the e_i's are imaginary units. Namely, $e_i^2 = -1$ and $e_i\, e_j = e_k$, $e_j\, e_i = -e_k$. (see multiplication table below). For example, $-2 + e_5$, $\ln(2)e_4$, $\sqrt{10}e_1 - \frac{1}{100}e_6 + \frac{5}{3}e_7$ are octonions. Octonions extend the quaternions and they are 8-dimensional. Octonions are neither commutative nor associative.

e_ie_j	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Ring: a ring is an algebraic structure consisting of a set with two operations which obey the same basic laws as addition and multiplication of integers, which multiplication does not have to be commutative. Also, multiplicative inverses are not required to exist. For example, Z (integers), Z_4 (4-modulo space), R (reals), H (quaternions), $M_{2x2}(R)$ (2-by-2 matrices) are rings.
Field: a field is a commutative ring in which every nonzero element has a multiplicative inverse. For example, Z_5 (5-modulo space), R (reals), C (complex numbers) are fields.

Skew Field: A skew field (or division ring) is a ring (not necessarily commutative) in which every element has a multiplicative inverse. A commutative skew field is a field. For example, C (complex numbers), H (quaternions) are skew fields. Notice that H is a non-commutative skew field. A non-commutative, non-associative skew field is O (octonions).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Norm: the norm of a vector is basically its length. More generally, the norm of a mathematical object is a quantity that in some sense describes the object's size. It is usually denoted by || ||. For example, the norm of the number 7 is ||7|| = |7| = 7. The norm of the vector $\vec{u} = (1,3)$ is $\|\vec{u}\| = \vec{u} = \sqrt{1^2 + 3^2} = \sqrt{10}$.

Zero Divisors: zero divisors are non-zero elements a, b of a ring such that $ab = 0$. For example, in the ring Z_6 (6-modulo space) the elements 2 and 3 are zero divisors. Indeed, 2 and 3 are non-zero, but $2.3 = 6 = 0$. Notice that Z (integers) is a commutative ring with no zero divisors (such rings are called *integral domains*). Also. in a field (or a skew field) there are no zero divisors. In other words, in a field $ab = 0$ implies $a = 0$ or $b = 0$. The latter is crucial when students learn to solve equations by factoring.

Idempotent Element: an idempotent is an element a of a ring such that $a^2 = a$. For example, in the ring Z_6 (6-modulo space) the element 4 is an idempotent. Indeed, $4^2 = 16 = 4$. In the ring Z (integers) the only idempotents are the trivial elements 0 and 1. Any non-trivial idempotent a is a zero divisor because for a, b non-zero, where $b = 1 - a$, we have $ab = 0$. This means that integral domains, skew fields and fields do not have such idempotents.

Theorem: a theorem is a mathematical statement that has been proven to be true. A theorem is usually a more significant result than other propositions within a mathematical context.

Conjecture: a conjecture is a mathematical statement that is believed to be true, but for which a proof or disproof has not been found yet.

Counter-example: a counterexample is an example that disproves a conjecture.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL 23
WINTER 2025
Vol 17 no 6

A Framework for Mathematical Proof: A Combination of Deduction, Induction, and Abduction

Soffil Widadah, Tatag. Y. E. Siswono, Rooselyna Ekawati

Department of Mathematics Education, Surabaya State University, Indonesia,

soffil.22004@mhs.unesa.ac.id, tatagsiswono@unesa.ac.id, rooselynaekawati@unesa.ac.id

Abstract: This study explores the use of abductive reasoning by Mathematics Education students in mathematical proofs, integrating deduction, induction, and abduction. The proof task involved two functions: the first type was completed by 8 participants, and the second type by 13 participants, both focusing on number theory. Following the task, in-depth interviews were conducted with two participants from each problem type and two additional participants for source triangulation to validate the data. Data were analyzed through the reduction and presentation of relevant information to draw research conclusions. The results showed that deduction is the primary approach used by students because it produces definite conclusions. However, many students utilize a combination of deduction, induction, and abduction. Abduction is often used as the first step to formulate a hypothesis, deduction is applied to verify the truth, and induction helps identify patterns, although it is less dominant. These mixed strategies reflect the complex and varied process of mathematical proof. This study confirms the importance of a deep understanding of different types of reasoning in mathematical proof to foster critical and analytical thinking skills. Further studies are recommended to develop a theoretical model of the interaction among deduction, induction, and abduction in mathematical proof to strengthen pedagogy at the higher education level.

Keywords: Deductive reasoning, inductive reasoning, abductive reasoning, mathematical proof, mathematics education.

INTRODUCTION

Mathematical proofs provide a way of thinking that deepens an individual's understanding of mathematics. According to Szabo (2016), a proof is a collection of true and logical statements that serve as arguments to confirm the truth of mathematical statements. Proofs also function as an explanatory and discovery tool that helps individuals understand why a statement is true. In addition, proofs can be used as a discovery tool closely related to exploration activities. Exploration leads to further study of a definition to explore its full meaning. The role of proofs as exploration

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

tools also becomes apparent when a proved theorem leads to the discovery of new ideas (Hanna, 1995).

According to NCTM (2000), proof is an important part of mathematical understanding, and every student should be able to recognize, develop, and apply various methods of proof. Successfully proving a statement confirms a truth and paves the way for further discovery and understanding. This aligns with the learning outcomes of mathematics education study programs in general, namely the ability to apply logical, critical, systematic, and innovative thinking in the context of developing or implementing knowledge. Therefore, many mathematical theorems are discovered through the proof process (Cellucci, 2017).

By understanding the proof process, Mathematics Education students can more easily access new theorems and concepts (Stylianides et al. 2016). Through the proof process, students develop critical thinking skills that are indispensable in everyday life. This implies that students must understand the role of proof in the overall development of mathematics, including how theorems are proved, discoveries are made, and how mathematical ideas develop over time. Thus, Mathematics Education students are expected to apply proof concepts in solving mathematical problems or in applying theorems in scientific or technical contexts (Hanna & De Villiers, 2012)

One type of proof studied by Mathematics Education students is algebraic proof (Hanna, 2020). Algebraic proof involves algebraic manipulations, such as expanding and factoring mathematical expressions to prove statements involving integers or algebraic terms. It is a fundamental aspect of mathematics because it allows for establishing the truth of mathematical statements with certainty and provides a strong basis for further mathematical development. An example of an algebraic proof problem is: "Prove that the product of two odd numbers will always be odd". This issue involves expressing even and odd numbers algebraically and using them to prove that the product of two odd numbers will always be odd.

Mathematical proofs require valid and accurate deductive reasoning (Komatsu & Jones, 2022). This emphasizes that mathematical proofs rely on valid deductive reasoning to ascertain conclusions from premises and the correctness of each logical step. These requirements guarantee the reliability and certainty of mathematical knowledge and findings. Deductive reasoning involves drawing logical conclusions based on premises (Sinaga et al. 2024). Premises are assumptions, ideas, and foundational statements considered true. If the premises are true, then the conclusion can be confirmed as valid.

In addition to deductive reasoning, inductive and abductive reasoning can be used in the process of mathematical proof when drawing conclusions (Salsabila et al. 2020). Inductive reasoning draws conclusions from general premises (observations, data, and facts) and then draws specific findings. Forms of inductive reasoning include generalization, analogy, and causal relationships. Through inductive reasoning and the observation of the phenomena, one can understand various patterns and regularities, the properties of mathematical objects, and the relationships between concepts based on specific examples. In addition, this reasoning helps to form mathematical con-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

jectures or hypotheses and initial intuitions that form the basis for determining the direction of further proof (Conner et al. 2014).

The third type of reasoning in the mathematical proof process, abductive reasoning, is used to generate new ideas and innovative solutions. Abductive reasoning is a creative and explorative process that allows individuals to propose new ideas and innovative solutions by proposing plausible explanations for observed phenomena. Abductive reasoning encourages imaginative thinking and can lead to breakthroughs in problem-solving and decision-making (Durand-Guerrier et al. 2012). By using abductive reasoning, individuals can correctly describe the interconnectedness of the foundational principles of logic and mathematics, which are not universal generalizations of arithmetic truths or concrete branches of mathematics.

Deductive, inductive, and abductive reasoning are structural aspects of mathematical reasoning. These reasoning processes are activities that draw logically related conclusions. Figure 1 is the researcher's description of the relationship between deductive, inductive, and abductive reasoning.

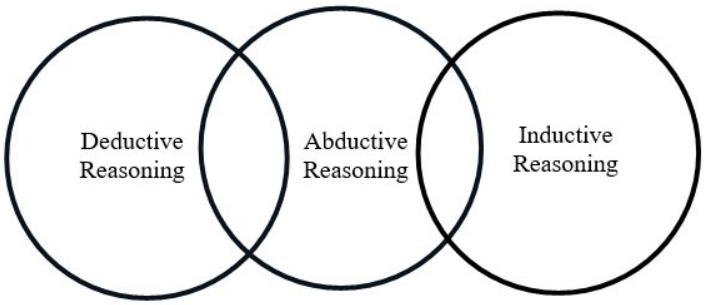


Figure 1. Relevance of Deductive, Inductive, and Abductive Reasoning

When individuals perform mathematical proofs, they may use deductive, inductive, or abductive reasoning. However, abductive reasoning can also occur when an individual (a Mathematics Education student) employs deductive or inductive reasoning. This happens because abductive reasoning is a form of logical inference based on the most likely explanation or hypothesis derived from a given set of observations or data. Unlike deductive reasoning, which draws conclusions from known premises, and inductive reasoning, which generalizes conclusions from observed patterns, abductive reasoning involves proposing plausible explanations or hypotheses to explain observed phenomena (Behfar & Okhuysen, 2018). These hypotheses may not be definitively proven but are chosen because they best fit the available evidence.

Within the realm of deductive and inductive reasoning, abductive reasoning offers a unique perspective, serving as a bridge between observed evidence and the generation of new evidence (Widadah, et al.2024). Abductive reasoning involves constructing plausible explanations or hypotheses to explain observed evidence or facts. Unlike deductive reasoning, which moves from general principles to specific conclusions, and inductive reasoning, which derives from general

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

principles from specific observations, abductive reasoning operates in the realm of uncertainty. It is the art of identifying the best explanation among the available hypotheses.

In the context of proof, abductive reasoning serves as a starting point for logical exploration (Dong, et al. 2015). The best explanation identified through abductive reasoning becomes a hypothesis to be tested and validated. While abductive reasoning itself does not establish proof, it initiates the journey toward evidential support and logical validation. The interaction between abductive reasoning, deduction, and induction is crucial in the process of proving a statement. Abductive reasoning often leads to the formulation of hypotheses, which are then tested through deductive or inductive validation. Deductive reasoning verifies the logical consequences of the hypothesis, while inductive reasoning tests the generalization of the hypothesis to a broader context.

While not a stand-alone proof mechanism, abductive reasoning sets the stage for further exploration and validation. The relationship between abductive reasoning and proof underscores the dynamic and iterative nature of logical inquiry, where the journey from observed evidence to truth involves a complex state of hypothesis generation, testing, and refinement (Van Gelder & Wilcox, 2022). In the quest to understand and prove propositions, abductive reasoning serves as a valuable unit that guides the path toward logical validation and truth formation. Figure 2 shows the relationship between mathematical reasoning and abductive reasoning.

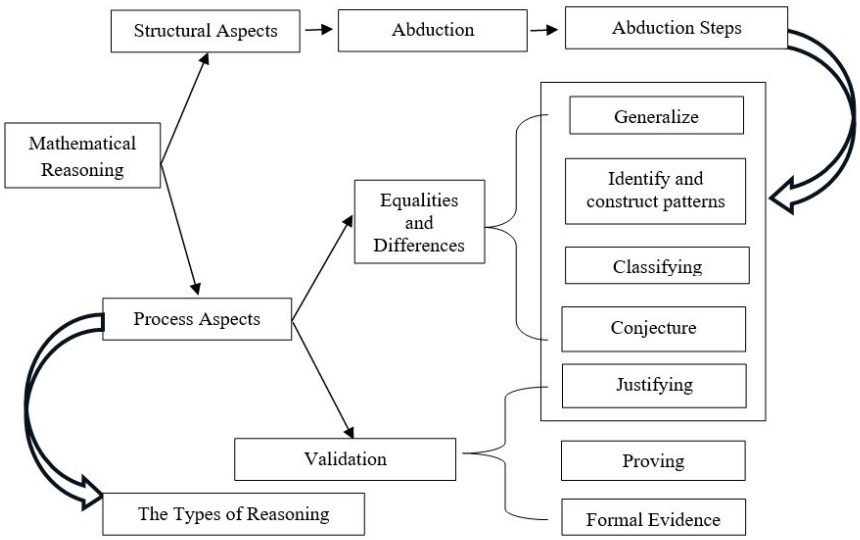


Figure 2. Mathematical Reasoning Relationship to Abductive Reasoning

Abduction is the structural aspect of mathematical reasoning, while the process aspect involves the search for similarities and differences, as well as the search for validation. The search for similarities and differences includes generalizing, identifying and building patterns, classifying, and

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

conjecturing. The validation aspect comprises justifying, proving, and formal proof (Sommerhoff & Ufer, 2019). The search for similarities and differences is part of the abductive reasoning process, which involves observing, proposing hypotheses, evaluating hypotheses, identifying the best explanation, and justifying arguments. Justifying is the validation-seeking aspect of the reasoning process, while other validation-seeking aspects include proving and formal proof.

Based on these various explanations, the conceptual framework in this study is illustrated in Figure 3.

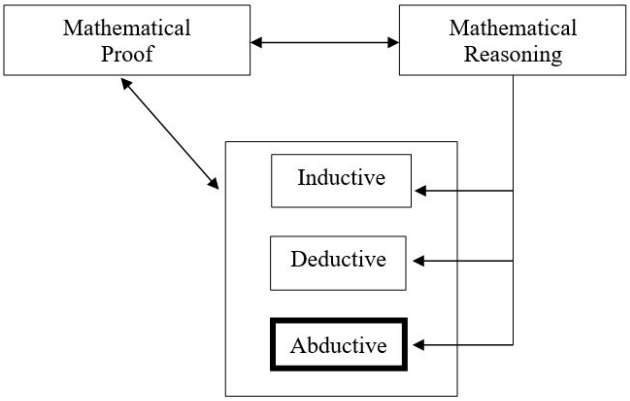


Figure 3. Conceptual Framework

Algebraic proof is part of mathematical proof, which is an integral aspect of the mathematical reasoning process aimed at establishing the truth or validity of statements or propositions. Proof is crucial to ensure the rigor and certainty of knowledge, including algebraic proof. Proofs involve a series of logically connected statements that follow from previous statements. The logical structure ensures that the conclusion is a necessary consequence of the premises (Tennant, 2015). Algebraic proof relies on logical deduction, and, in addition to this, algebraic proof provides a creative and exploratory foundation for conjectures and ideas.

While mathematical reasoning is a broader process of exploring, understanding, and reasoning about mathematical concepts, mathematical proof is a specific aspect of reasoning aimed at establishing the truth or validity of mathematical statements through logical deduction. The two are interconnected, with reasoning laying the foundation for the formulation of conjectures and proof providing formal verification of those conjectures. This statement confirms that the process of mathematical reasoning leads to the formulation of mathematical proofs. Proofs serve as formal representations of the logical reasoning process used to establish the truth of algebraic statements. An algebraic proof is a concrete and systematic presentation of the logical steps taken during the reasoning process (Goguen, 2021). In this way, an algebraic proof becomes a visible or tangible

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

confirmation of the reasoning. The act of proving mathematically is a tangible outcome of the broader cognitive activity of mathematical reasoning.

The urgency of this research lies in its potential to enrich both the theoretical and practical understanding of mathematical proof, improve learning effectiveness, and support the development of critical and creative thinking skills in mathematics education through proof construction. The questions in this study are as follows:

1) How is a framework that combines deductive, inductive, and abductive reasoning applied by students in mathematical proof?

2) What is the role of each type of reasoning (deductive, inductive, abductive) in mathematical proof construction according to the proposed framework?

3) How do students utilize deductive, inductive, and abductive reasoning in constructing mathematical proofs?

METHODS

This study explores the reasoning employed by mathematics education students in constructing proofs. This research is a mixed-method study, with a greater emphasis on the qualitative data section. Quantitatively, we only present the number of students who use abductive, inductive, and deductive reasoning in completing proof tasks. The subjects in this study were mathematics education students who had studied number theory and were at the second-year level. The students were only familiar with two types of reasoning: inductive and abductive. There was no specific training about types of reasoning; they learned about these types while studying number theory. A total of 21 students participated in this study, which corresponds to the number of sophomores at the research site. After completing the proof task, we interviewed two students from each task type. The interview aimed to deepen the data obtained from the proof task that had been completed. The criteria for selecting students who were interviewed were: 1) students who worked with at least two types of reasoning, and 2) students who were willing to be interviewed. The proof tasks were designed as follows:

Task type 1: "If a and b are odd numbers, show that $8|\left(a^2-b^2\right)$."
Task type 2: "Show that the square of an odd number results in an odd number."

The objectives of the two tasks are: 1) students can observe numerical patterns and results, and develop conjectures based on observations; 2) students can generalize numerical patterns using algebraic forms; and 3) students can develop formal proofs logically from conjectures and generalizations. We asked the 21 students to choose one of the proof tasks provided. Eight students chose type 1 tasks, while 13 students completed the type 2 task. In addition, we selected two

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

students to be interviewed for source triangulation purposes. The next step, after collecting data from the proof tasks and interviews, we analyzed the data by selecting task and interview data that were only needed in the research (data reduction), namely segments of data relevant to the use of deductive, inductive, and abductive reasoning; grouping data based on the types of reasoning used by students; and ignoring data that were not needed. Next, we presented the reduced data, namely by presenting data in the form of a table of proof task results showing the frequency of use of each type of reasoning; descriptive narratives explaining the patterns of reasoning that emerged; and direct quotes from interviews that support the findings. We then draw conclusions and conduct verification by interpreting the data to determine how the reasoning framework influenced students' understanding; comparing the findings with relevant literature or theories; and triangulating the data to ensure the validity of the findings. We used source triangulation, which involved checking the validity of the data by comparing one source with another, where students received the same treatment, namely completing assignments followed by interviews.

RESULTS

After working on the proof tasks, we interviewed two participants from each task type. The interviews aimed to deepen the data obtained from the completed proof tasks. In addition, we interviewed two participants for source triangulation.

Research Subject	Reasoning		
	Deductive	Inductive	Abductive
F	√	√	√
M	√	√	
I		√	√
S	√	√	√
U	√	√	√
N	√		√
A		√	√
Y	√	√	√

Table 1. Participants Reasoning in Constructing Evidence in Task Type 1

Table 2 shows the reasoning results of mathematics education students in constructing.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

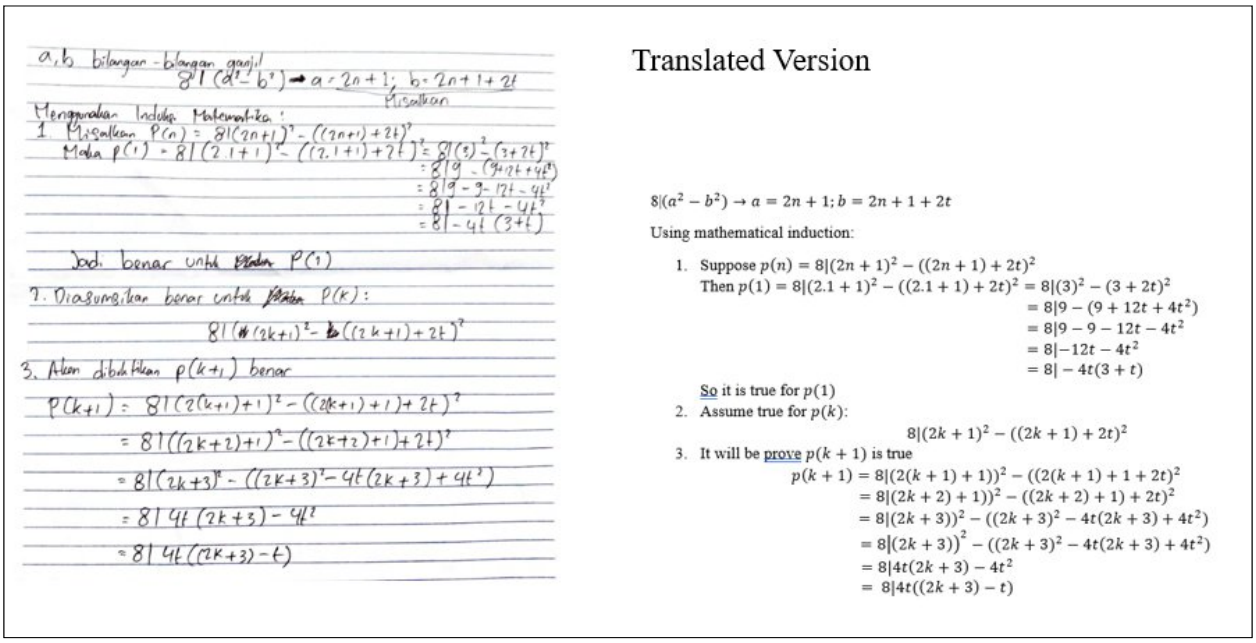
Table 2. Participants' Reasoning in Constructing Evidence in Task Type 2

Research Subject	Reasoning		
	Deductive	Inductive	Abductive
L	√		√
N	√		
J		√	√
H	√		√
C	√		√
D	√		√
K	√		√
M	√		√
R	√		√
T	√		√
S	√		√
B	√		
P		√	

Task Type 1

We describe this type 1 proof task based on participants' work. Out of 8 participants, we selected the work of two participants, namely F and S.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.



Translated Version

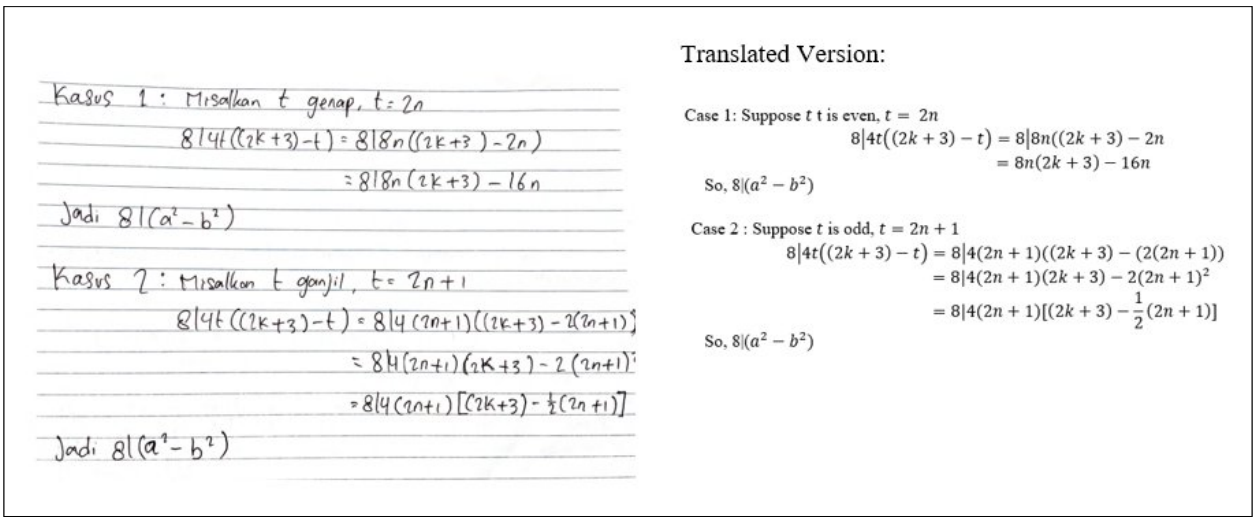
$8|(a^2 - b^2) \rightarrow a = 2n + 1; b = 2n + 1 + 2t$

Using mathematical induction:

1. Suppose $p(n) = 8|(2n+1)^2 - ((2n+1) + 2t)^2$
   Then $p(1) = 8|(2.1+1)^2 - ((2.1+1) + 2t)^2 = 8|(3)^2 - (3 + 2t)^2$
   $= 8|9 - (9 + 12t + 4t^2)$
   $= 8|9 - 9 - 12t - 4t^2$
   $= 8|-12t - 4t^2$
   $= 8| - 4t(3 + t)$
   So it is true for $p(1)$
2. Assume true for $p(k)$:
   $8|(2k+1)^2 - ((2k+1) + 2t)^2$
3. It will be prove $p(k+1)$ is true
   $p(k+1) = 8|(2(k+1)+1))^2 - ((2(k+1)+1+2t)^2$
   $= 8|(2k+2)+1))^2 - ((2k+2)+1)+2t)^2$
   $= 8|(2k+3))^2 - ((2k+3)^2 - 4t(2k+3) + 4t^2)$
   $= 8|(2k+3))^2 - ((2k+3)^2 - 4t(2k+3) + 4t^2)$
   $= 8|4t(2k+3) - 4t^2$
   $= 8|4t((2k+3) - t)$

Figure 4. F's Proof Process Part 1

Figure 4 shows that "F" solved the task using the principle of mathematical induction. In the first step, he assumed $P(n)$ with (1) . In the second step, "F" assumed it was true for $P(k)$. In the third step, the participant proves that it is true for $P\ (k + 1)$. F's proof step can be seen in part 2 (Figure 5) below.



Translated Version:

Case 1: Suppose t t is even, $t = 2n$

$8|4t((2k+3) - t) = 8|8n((2k+3) - 2n$
$= 8n(2k+3) - 16n$

So, $8|(a^2 - b^2)$

Case 2 : Suppose t is odd, $t = 2n + 1$

$8|4t((2k+3) - t) = 8|4(2n+1)((2k+3) - (2(2n+1))$
$= 8|4(2n+1)(2k+3) - 2(2n+1)^2$
$= 8|4(2n+1)[(2k+3) - \frac{1}{2}(2n+1)]$

So, $8|(a^2 - b^2)$

Figure 5. F's Proof Process Part 2

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

In part 1, "F" used deductive reasoning, while in part 2, "F" continued by using inductive reasoning, namely by sorting out whether t is even or t is odd. The reason why "F" did so can be seen in the following interview.

P : Why did you continue with the odd and even terms?
F : I thought that it had to be done, because if I only completed step 3 there was not enough reason to strengthen my proof.
P : Why did you equate the odd t with $2n + 1$? Is it possible to equate $t = 2n - 1$
F : You can, but most people use $2n + 1$ to represent odd numbers
P : Why did you visualize even and odd numbers?
F : Even and odd numbers are positive numbers and can also be negative integers. If I assume $t = 0$ it's not possible.

From the interview, it can be seen that "F" also used abductive reasoning by providing reasonable explanations. Based on the results of the proof and the interview, we found that when someone constructs a proof, they do not rely solely on deductive reasoning but also use inductive and abductive reasoning to obtain logical results. Here we present the proof results of "S".

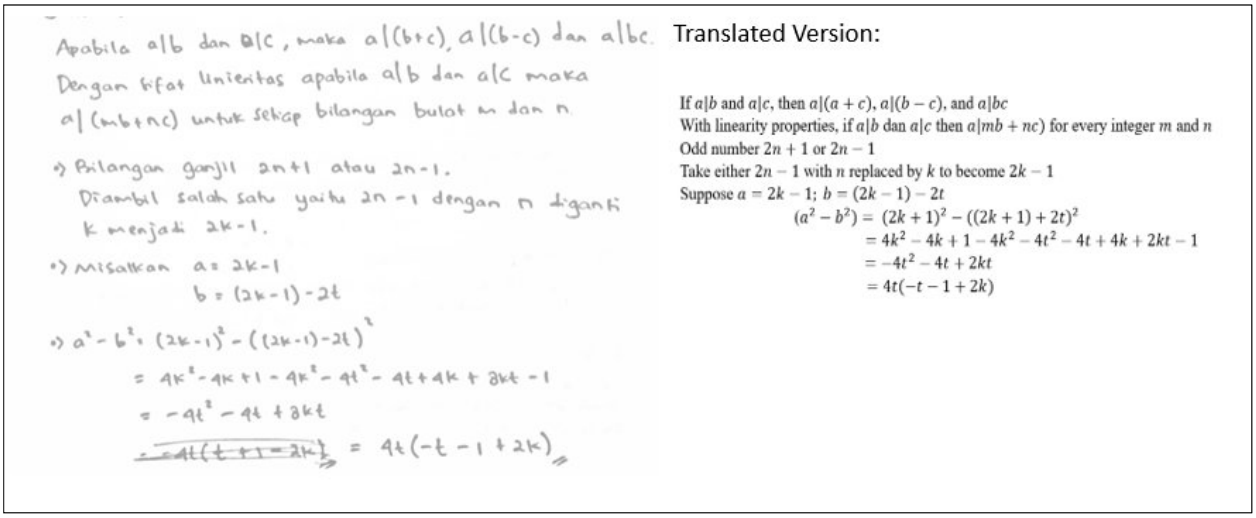


Figure 6. S's Proof Process Part 1

Figure 6 shows that "S" proved the statement using linearity properties, specifically "If $a|b$ and $a|c$, then $a|(a + c)$, $a|(b - c)$, and $a|bc$." With these properties, if $a|b$ and $a|c$ then $a|mbnc$for every integer m and n. This demonstrates that "S" performed an algebraic proof in a deductive manner. Deductive proof is not limited to principle of mathematical induction but also involves deriving existing theorems, definitions, or rules. In the next process, "S" initialized the odd numbers as

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

$2k - 1$ and $(2k - 1) - 2t$. Then they substitute these into $(a^2 - b^2)$. he following image (Figure 7) illustrates the process of S's proof construction.

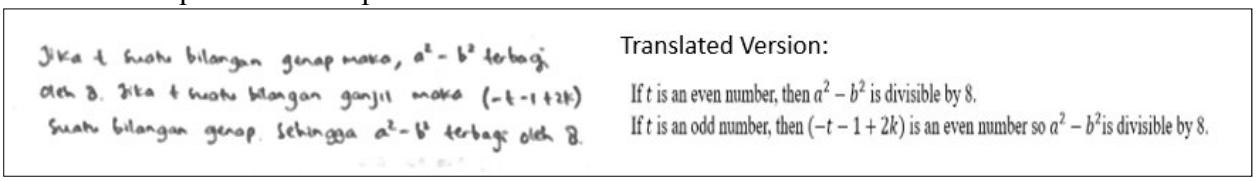


Figure 7. S's Proof Process Part 2

Figure 7 shows that "S" assumes t is an even number, stating, "If t is an even number, then $a^2 - b^2$ is divisible by 8". "S" also assumes t is an odd number, stating, "If t is an odd number, then $(-t - 1 + 2k)$ is an even number, so $a^2 - b^2$is divisible by 8 ". Thus, it is true that $8|(a^2 - b^2)$ for a,b are odd numbers. This reasoning involves both inductive and deductive approaches, as demonstrated in the following interview:

P : Why did you use the linearity property at the beginning of the proof?
S : In my opinion, formal proof can be done using existing rules
P : Why did you equate the odd t wi$2n - 1$? Is it possible to equate $t = 2n + 1$?
S : You can, most people use $2n + 1$ to represent odd numbers, but I want to try something else.
P : Why do you equate even and odd numbers?
S : I want to try one by one
P : Are you sure that $8|(a^2 - b^2)$?
S : Yes, I am sure, because I have tried one by one, even numbers and odd numbers and the result is divided by 8.

From the interview, it can be seen that "S" used inductive reasoning by stating "trying one by one". In addition, "S" employed abductive reasoning by providing reasonable explanations. Based on the proof results and the interview, it is clear that when someone constructs a proof, one uses not only deductive reasoning but also inductive and abductive reasoning to obtain logical conclusions.

Task Type 2

We describe this type 2 proof task based on the work of two participants out of the 13. These participants are L and N.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

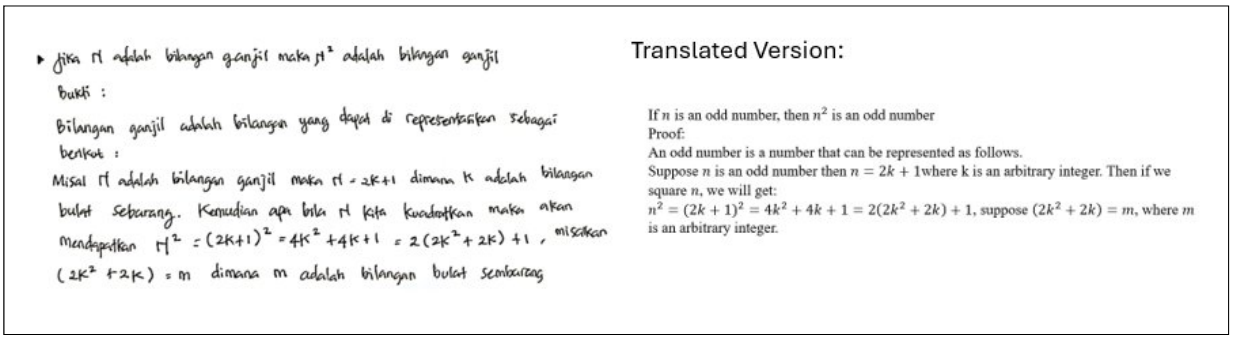


Figure 8: L's proof process part 1

Figure 8 shows that "L" defines odd numbers as $n = 2k + 1$ with k an arbitrary integer. This implies that "L" used deductive reasoning by stating the mathematical rules. Furthermore," L" squared n, i.e

$n^2 = (2k + 1)^2 = 4k^2 + 2k + 2k + 1 = 4k^2 + 4k + 1 = 4k^2 + 4k = 2(2k^2 + 2k)$ and explained that $(2k^2 + 2k) = m$ where m is an arbitrary integer. This implies that "L" also uses abductive reasoning.

Figure 9 is part 2 of L's proof.

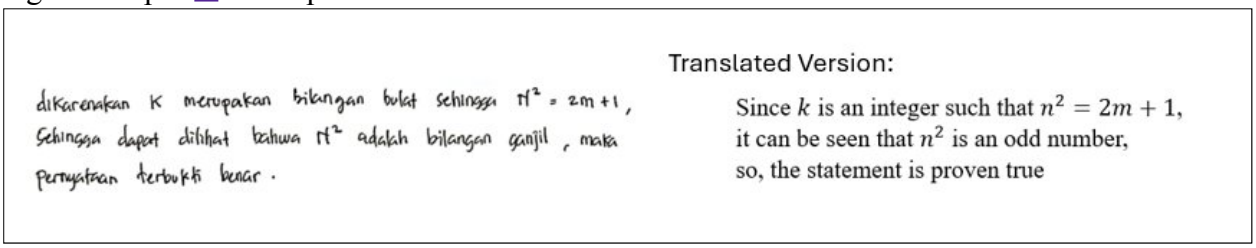


Figure 9: L's proof process part 2

Figure 9 shows that "L" provided the reason why n^2 is an odd number, namely $n^2 = (2m + 1)^2$. It can be seen that n^2 is an odd number, and it is proven that if n is odd, then n^2 is also odd. This is consistent with the following interview:

P : Why do you define odd numbers by $n = 2k + 1$?
S : I analogize with even numbers, if an even number is $n = 2k,$ then an odd number is defined $2k + 1$ or $2k - 1$ but I use $2k + 1$
P : Why do you initialize odd numbers with $2k + 1$?
S : I followed most people
P : Do you think you did the proof using the mathematical induction principle?
S : Yes, I think

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

P : Are you sure that your proof is a formal proof?
S : Yes, I'm sure, because I used the mathematical rule of the definition of odd numbers

From the interview, it can be seen that "L" constructed the proof using deductive or formal proof, even though the principle of mathematical induction was not employed. "L" realized that creating formal proof involves not only using the principle of mathematical induction but also applying mathematical rules. This is evident in the interview where "L" stated, "Yes, I am sure because I used a mathematical rule: the definition of odd numbers. This shows that "L" used deductive and abductive reasoning to construct the proof.

The following are the results of N's proof on Task Type 2.

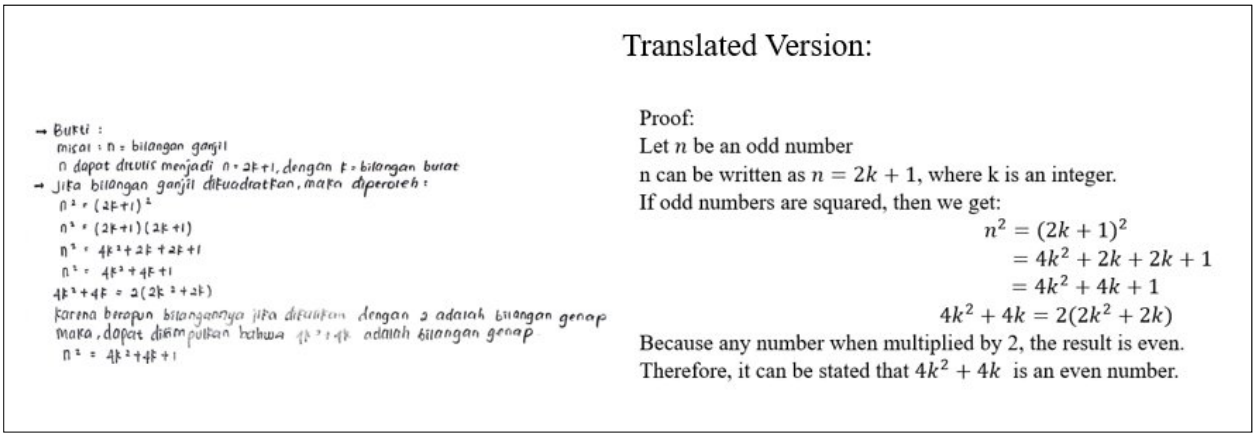


Figure 10: N's Proof Process Part 1

Figure 10 shows that "N" defined odd numbers as$n = 2k + 1$, where k is an arbitrary integer. This implies that subject N used deductive reasoning by writing down the rules in mathematics. Furthermore,"N" squared n, i.e $n^2 = (2k + 1)^2 = (2k + 1)\,(2k + 1) = 4k^2 + 2k + 2k + 1 = 4k^2 + 4k + 1 = 4k^2 + 4k =$

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

$2\left(2k^2+2k\right)$ and explained, "Any number when multiplied by 2 results in an even number, so it can be concluded that $4k^2+4k$ is an even number." This implies that "N" used abductive reasoning. N's next answer is shown in the Figure 11.

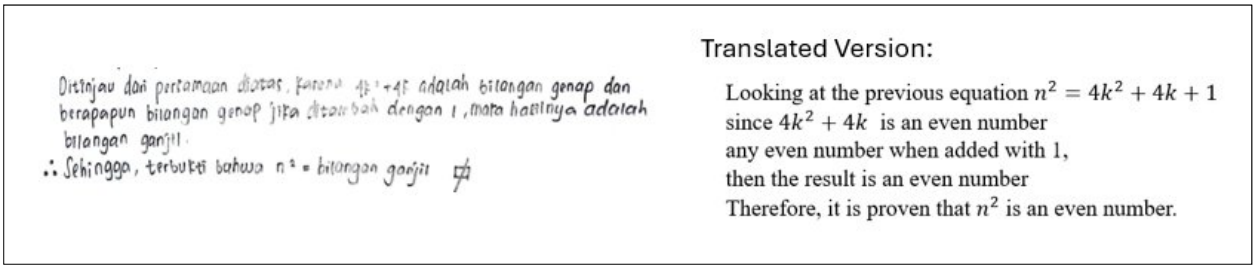


Figure 11: N's Proof Process Part 2

In part 2, "N" was seen giving reasons why n^2 is an odd number. "N" wrote, "Based on the equation ($n^2 = 4k^2 + 4k + 1$), $4k^2 + 4k$ is an even number, so any even number, if added with 1, results in an odd number. Furthermore, "N" concluded, "It is proven that n^2 is an odd number." "N" used abductive reasoning to construct the proof, namely by providing explanations for the steps of the proof. This is supported by the following interview results:

P : Why did you equate the odd n with $2k+1$? Is it okay to improvise $n = 2k-1$?
S : Actually, it is okay, most people use $2k+1$ to represent odd numbers, so I follow most people.
P : Why do you believe that if an even number is added to 1, the result is odd?
S : I tried one by one, but I didn't write it down.

The interview results illustrate that "N" also used abductive reasoning when doing the proof. This can be seen from the interview answer, stating, "I tried one by one, but I didn't write it down." This suggests that "N" implicitly used inductive reasoning.

DISCUSSION

Table 1 shows that 7 participants are using abductive reasoning in constructing proofs. However, these participants also employed a combination of inductive and deductive reasoning. This shows that abductive reasoning is often relied upon by participants in conducting proofs. The use of abduction in mathematical proofs reflects participants' efforts to identify interrelationships between mathematical rules, which were then further tested through induction and deduction. Only two participants did not use deductive reasoning, and one participant did not use inductive reasoning.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

This indicates that deduction, as a fundamental method in mathematical proof, remains a widely used technique in the proof process. Deduction offers a definite approach, where the conclusion is certain if the premises are true. Thus, deductive reasoning becomes the backbone of the proof (Attridge, 2013). In contrast, one participant did not use inductive reasoning. This indicates that not all participants felt the need to generalize from specific examples when proving a theorem. Those who did not use induction likely proceeded directly to deduction from the premises, without first identifying patterns from specific cases.

Table 2 shows that only two participants used inductive reasoning, while others employed a combination of deductive and abductive reasoning. However, one participant only used deductive reasoning and no other reasoning. This indicates that induction is not dominant in the process of mathematical proof among participants. This may be due to the probabilistic nature of induction, which contrasts with the certainty required in mathematical proofs, leading participants to favor deduction. This further emphasizes the important role of abduction in forming initial hypotheses, which are then confirmed through deduction. Interestingly, one participant used deductive reasoning, without combining of abduction or induction. This may illustrate a more rigorous, traditional, or formal approach, where proof begins directly with known premises, and conclusions are drawn deductively without the need for abduction or induction (Magnani, 2011).

Mathematical proofs are inherently dynamic, where participants often combine several types of reasoning to strengthen their proof (Harel & Sowder, 2007). The significant use of abductive reasoning suggests that participants rely on intuition or creative thinking to formulate the first step of the proof. In contrast, the dominance of deduction reaffirms that deduction remains the irreplaceable standard in mathematical proof. While inductive reasoning is used less frequently due to its less certain nature compared to deduction, it still plays a crucial role in finding patterns or trends.

Overall, this study provides important insights into how participants employ different types of reasoning in mathematical proofs, reflecting their approach to complex problem-solving.

Teacher Reflection and Didactic Impact on Learning Outcomes.

As part of a practical reflection on the findings of this study, several mathematics teachers in higher education, who also examined the deductive-inductive-abductive framework, noted that this approach holds significant potential for bridging the gap between students' initial intuition and the formal proof process. They observed that abductive reasoning, which had previously received less attention in the teaching of proofs, holds a strategic role in fostering students' confidence to initiate and explore possible mathematical arguments. The teacher observed that students often struggle when required to engage directly in deductive proofs. By providing space for students to develop initial hypotheses through abductive reasoning, they become more confident and actively involved

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

in the proof construction process. The integration of inductive reasoning is also seen as helpful in identifying patterns and strengthening intuition before transitioning to formal deduction. The impact of this didactic framework on learning outcomes was evident in two main aspects. First, in terms of process, students demonstrated improved ability to explain the reasoning behind each step of a proof, both in writing and orally. They did not merely imitate standard forms of proof but were able to construct coherent arguments based on observations and initial conjectures. Second, in terms of outcomes, students produced proofs that were more varied, creative, and demonstrated a deeper understanding of the nature of mathematical structures. Thus, this framework not only reinforces deductive logic but also fosters reflective and explorative thinking, which are essential components of modern mathematics learning. The teacher recommends integrating this approach into the proof curriculum at the university level, particularly in number theory and introductory proof courses.

Practical Guidelines and Materials for Replicating Classroom Studies.

This study can be replicated within the context of classroom mathematics learning by following these steps:

Learning Objectives

Learners are expected to identify and apply deductive, inductive, and abductive reasoning when proving the following two mathematical statements: 1) If a and b are odd numbers, then $8 \mid (a^2 - b^2)$ and 2) The square of an odd number is an odd number.

Learning Strategy

The learning activities are designed to integrate the three types of reasoning as follows:

- Deductive Reasoning: Formulating formal proofs using symbolic and algebraic representations.
- Inductive Reasoning: Identifying patterns through multiple numerical examples.
- Abductive Reasoning: Proposing conjectures based on observations and searching for plausible explanations.

Series of activities

1) *Concept Orientation*: An initial discussion on the general form of odd numbers $(2k + 1)$.
2) *Proof Exploration*: Guided proof of the two statements using a multi-reasoning approach (deductive, inductive, and abductive reasoning).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

3) *Reflection and Discussion*: Students evaluate the type of reasoning used and reflect on how each contributes to the validity and construction of the proof.

Supporting Materials

1) *Divisibility Notation*: An integer a completely divides an integer b is written as a|b if and only if there exists an integer k such that $b = ka$. If a does not completely divide b, it is written as $a \top b$.
2) *Definition of Odd Numbers*: An odd number is an integer that is not divisible by 2. It can be expressed in the form $2k + 1$, where $k \in \mathbb{Z}$ (the set of integers).

Assessment

Assessment is conducted formatively through class discussions and the observation of students' thought processes. Summative assessment may be provided through proof tasks that require the application of various reasoning methods, such as providing additional algebraic properties of odd and even numbers.

CONCLUSION

Based on the analysis of the presented research results, the majority of participants in this study primarily employed deductive reasoning in constructing proofs. This supports the assertion that deduction is the most trusted and reliable method in mathematical proofs, as it generates definitive conclusions from established premises. Additionally, many participants utilized a combination of deductive, inductive, and abductive reasoning. This demonstrates that the process of mathematical proof typically involves multiple types of reasoning. Abduction is employed as the initial step to formulate hypotheses, while deduction ensures the correctness and validity of the conclusions. Induction also plays a significant role in identifying patterns from specific examples, although it is less predominant. The widespread use of abductive reasoning among students suggests that it plays a key role in initiating the proof process. Students frequently relied on intuition or initial exploration to formulate conjectures, which were later verified through deduction. This suggests that while mathematical proof is grounded in logic, intuition and preliminary hypotheses play a crucial role in initiating the proof process. The study revealed variations in the proof strategies employed by students. Some students relied solely on deduction, while others adopted a mixed approach, incorporating deduction, induction, and abduction. This suggests that there is no single ‘correct’ method for constructing a mathematical proof; rather, multiple approaches may be employed, depending on individual preferences and the specific characteristics of the problem. This conclusion underscores the importance of a thorough understanding of various types of reasoning

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

and their integrated use in mathematical proofs, which contributes to the development of students' critical and analytical thinking skills. Analysis of students' reasoning process revealed that abductive reasoning plays a crucial role in initiating a proof, while deductive reasoning ensures logical consistency, and inductive reasoning aids in identifying patterns. Although this study relied minimally on quantitative approaches, the richness of the qualitative data facilitated an in-depth exploration of students' cognitive strategies in constructing proofs. The findings affirm that qualitative approaches are particularly effective in capturing the complexity and dynamics of mathematical reasoning processes in educational contexts.

We recommend that future research explore the development of a more detailed theoretical model illustrating how deductive, inductive, and abductive reasoning interact in mathematical proofs. Such a model could clarify the role of each type of reasoning in different proofs and offer guidance for more structured instruction in proof teaching.

ACKNOWLEDGEMENTS

We would like to express our gratitude to the students of Mathematics Education who participated in this research, as well as to the promoter and co-promoter for their invaluable guidance. We are also thankful to Surabaya State University for providing the facilities and opportunities, and to the lecturers and peers for their supportive contributions. Lastly, we extend our heartfelt thanks to our family and friends for their unwavering moral support throughout this research.

REFERENCES

[1] Szabo, M. E. (2016). *Algebra of proofs*. Elsevier.
[2] Hanna, G. (1995). Challenges to the importance of proof. *For the learning of Mathematics*, 15(3), 42-49.
[3] NCTM. (2000). Principles and Standards for School Mathematics. Reston, VA: NCTM.
[4] Cellucci, C. (2017). Is mathematics problem solving or theorem proving?. *Foundations of Science*, *22*(1), 183-199.
[5] Stylianides, A. J., Bieda, K. N., & Morselli, F. (2016). Proof and argumentation in mathematics education research. In *The second handbook of research on the psychology of mathematics education* (pp. 315-351).
[6] Hanna, G., & De Villiers, M. (2012). *Proof and proving in mathematics education: The 19th ICMI study* (p. 475). Springer Nature.
[7] Hanna, G. (2020). Mathematical proof, argumentation, and reasoning. *Encyclopedia of mathematics education*, 561-566.
[8] Komatsu, K., & Jones, K. (2022). Generating mathematical knowledge in the classroom through proof, refutation, and abductive reasoning. *Educational Studies in Mathematics*, *109*(3), 567-591.
[9] Sinaga, J. A. B., Sirait, M., Pasaribu, B. B., & Limbong, D. I. V. (2024). Moving from Premises to Conclusions: Exploring the Method of Drawing Conclusions in Deductive

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Reasoning. *EUROPEAN JOURNAL OF INNOVATION IN NONFORMAL EDUCATION*, *4*(1), 183-189.

[10] Salsabila, E., Rahayu, W., & Sampoerno, P. D. (2020). Performance Assessment to Measure Students' Mathematical Proving Ability Based on the Abductive-Deductive Approach. *KnE Social Sciences*, 343-354.

[11] Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Identifying kinds of reasoning in collective argumentation. *Mathematical Thinking and Learning*, *16*(3), 181-200.

[12] Durand-Guerrier, V., Boero, P., Douek, N., Epp, S. S., & Tanguay, D. (2012). Examining the role of logic in teaching proof. *Proof and proving in mathematics education: The 19th ICMI study*, 369-389.

[13] Behfar, K., & Okhuysen, G. A. (2018). Perspective—Discovery within validation logic: Deliberately surfacing, complementing, and substituting abductive reasoning in hypothetico-deductive inquiry. *Organization Science*, *29*(2), 323-340.

[14] Widadah, S., Siswono, T. Y. E., & Ekawati, R. (2024). Constructing mathematical proofs through abductive reasoning. *Review of Science, Mathematics and ICT Education*, *18*(2), 29-48.

[15] Dong, A., Lovallo, D., & Mounarath, R. (2015). The effect of abductive reasoning on concept selection decisions. *Design studies*, *37*, 37-58.

[16] Van Gelder, T., & Wilcox, J. (2022). Narrative Abduction. *Available at SSRN 4790721*.

[17] Sommerhoff, D., & Ufer, S. (2019). Acceptance criteria for validating mathematical proofs used by school students, university students, and mathematicians in the context of teaching. *Zdm*, *51*, 717-730.

[18] Tennant, N. (2015). The relevance of premises to conclusions of core proofs. *The Review Of Symbolic Logic*, *8*(4), 743-784.

[19] Goguen, J. A. (2021). Theorem proving and algebra. *arXiv preprint arXiv:2101.02690*.

[20] Attridge, N. (2013). *Advanced mathematics and deductive reasoning skills: testing the Theory of Formal Discipline* (Doctoral dissertation, © Nina Attridge).

[21] Magnani, L. (2011). *Abduction, reason, and science: Processes of discovery and explanation*. Springer Science & Business Media.

[22] Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. *Second handbook of research on mathematics teaching and learning*, *2*, 805-842.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

How Students Think About a Reverse Problem of Derivative: A Study of Arnon's Coordination in Mental Mechanism

Fara Virgianita Pangadongan [1,2], Toto Nusantara [1]*, I Made Sulandra [1]

[1]Universitas Negeri Malang, Indonesia

[2] Universitas Mulawarman, Indonesia

fara.virgianita.2103119@students.um.ac.id, toto.nusantara.fmipa@um.ac.id*, made.sulandra.fmipa@um.ac.id

Abstract: Mental mechanism of coordination is indispensable in the construction of certain objects, contributing to problem-solving. Although Arnon et al. hypothesized about how two processes are coordinated mentally, which is referred to as Arnon's Coordination, further analysis is required. Studies related to coordination had explored its emergence in reflective abstraction during problem-solving. Generally, descriptions of coordination focused exclusively on the outcomes and application to problem-solving, particularly direct problems. Therefore, this study aimed to describe how students performed coordination between two processes in reverse problem-solving, focusing mainly on derivative and antiderivative graphs. Data were collected from 4 of the 44 third-year mathematics education students through answer sheets, think-aloud, and semi-structured interviews to examine mental mechanisms. The results showed that students coordinated the two processes by adopting three cognitive actions, namely determining processes, transforming it into an object through encapsulation, and integration through assimilation or accommodation. This also included the challenges students experienced with each cognitive action. In conclusion, this study served as a reference for educators when designing mathematical problem-solving exercises and lessons.

Keywords: Coordination, Mental Mechanism, Derivative Functions, Reverse Problem

INTRODUCTION

The ability of an individual to solve problems is assessed based on four mental structures, namely action, process, object, and scheme (APOS). Action stage enables an individual to understand and identify information from the problems (Syarifuddin et al., 2020). According to Borji and Planell (2023) and Nagle et al. (2019), problems are solved by simply applying the procedures memorized, without understanding the reasons behind the application. Process stage entails the connection of basic and other related concepts (Borji & Martínez-Planell, 2023; Nagle et al., 2019), by using appropriate mathematical techniques to solve problems (Syarifuddin et al., 2020). At the object stage, certain concepts are understood as entities that can be manipulated or used in a broader context as well as applied to new situations (Borji & Martínez-Planell, 2023; Nagle et al., 2019). Meanwhile,

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

the schema stage entails students constructing frameworks that integrate various mathematical concepts to solve problems in other disciplines (Borji & Martínez-Planell, 2023; Nagle et al., 2019). This enables the drawing of conclusions and organizing of data related to concepts into graphs or equations to show the relationships between concepts (Syarifuddin et al., 2020). An individual is able to connect different concepts by combining several varying processes (Kurniati et al., 2018; Syamsuri et al., 2017), organized through the mental mechanism of coordination.

In this context, coordination plays an important role in solving math problems. Dubinsky (2002) stated that the ability to perform the coordination needed to solve problems was based on built and reorganized schemas. Existing processes were coordinated to develop a new process that could be used in problem solving (Akgul & Yilmaz, 2023; Cetin & Dubinsky, 2017; Dubinsky, 2002; Paschos & Farmaki, 2006). This new process was further transformed into a mental object and used to understand the problems. According to Arnon et al. (2014), coordination was needed in constructing several objects that match the problems. Based on the study conducted by Syarifuddin et al. (2020), a coordination mechanism was used to determine the appropriate method to solve math problems. Syamsuri et al. (2017) also reported that it allowed students to develop a methodical approach in evaluating the relationship between mathematical objects, thereby improving problem analysis ability. In the truth-seeking process, coordination was applied through cross-concept checking to facilitate the identification of logical errors in the problem-solving process (Kurniati et al., 2018).

Previous research reported that many students cannot properly perform mental mechanisms of coordination. The results by Sopamena et al. (2016, 2018) show the need for reflection to be able to produce accurate generalizations. This was due to the results of incomplete coordination and the absence between the components of the cognitive structure that have been formed. Other studies related to reflective abstraction stated that students with impulsive cognitive styles tended to be imperfect in coordination, as well as unable to perform encapsulation and generalization mechanisms perfectly during problem-solving (Fuady et al., 2020). A similar circumstance was also observed in those with field-dependent cognitive style. The research subjects with this cognitive style are not yet able to coordinate the process of interiorization results but are able to change information into mathematical models. This led to the inability to develop their mental structure to be able to generalize (Fuady et al., 2019). The ability of students to coordinate is the reason for exploring how this mental mechanism (coordination) works in depth. Additionally, understanding how mental mechanism is conceptually implemented is crucial.

Based on the description, Arnon et al. (2014) proposed a hypothesis regarding mental actions applied during coordination. The present study referred to the coordination implementation hypothesized by Arnon et al. (2014). It is regarded as Arnon's Coordination, where there is a possibility to coordinate two processes, supposing students are able to first summarize one of the processes into an object, before the integration process. Coordination can be performed in the following way: either the object is assimilated and the process applied, or the process is accommodated to enable its application (Arnon et al., 2014). However, the hypothesis regarding the mechanism of coordination is still a reference for further analysis (Arnon et al., 2014). Mental mechanisms are difficult to ascertain di-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

rectly and can only be inferred through observations when searching for solutions to problems or trying to understand certain phenomena (Dubinsky, 1991).

A particular task used to decipher mental construction of an individual is representation translation. The ability to interpret and translate various representations is a tool adopted to show individual mental construction of deep mathematical concepts (Cawley, 2016). This is because the representation translation of an individual enables the determination of the internal abstraction and cognitive schema. Pape & Tchoshanov (2001) stated that mathematical representations are explained as internal abstractions of related ideas or cognitive schemas. Therefore, representation translation is a factor that affects understanding of individual concepts. Hähkiöniemi (2004) and Roorda et al. (2009) stated that connections between physical applications and mathematical representations are one of the factors that influenced individual understanding. In this perspective, the understanding of mathematical and graphical representations is enabled by the abstraction performed during the construction process.

One of the important concepts related to the translation of representation is the concept associated with the graph of derivative and antiderivative functions. Learning about the derivative is fundamental in using different representations (Borji, Font, et al., 2018; Maharaj, 2013), which connects visualizations with respective symbolizations (Tall, 2011), providing meaningful insights related to formula symbolization and a deeper understanding of the concept (Borji, Font, et al., 2018; Tall, 2011; Voskoglou, 2017). Previous studies on the graphical representations of derivative concepts showed that students had difficulty in translating representations from to graphs, specifically during initial coordination activities (Ikram et al., 2020; Swastika et al., 2020). In constructing the process to determine the function whose derivative is the original, it is important to reverse the process of interiorization, which results in relation to the antiderivative issues (Arnon et al., 2014; Dubinsky, 2002). Considering this perspective, these types of issues are referred to as reverse problems.

Reverse problems are among the many mathematical tasks that foster active thinking in students (Ikram et al., 2021). Some studies reported that both direct and inverted problems are used to reconstruct mental processes of an individual, which is associated with reversibility (Ikram et al., 2021). Reverse or opposite problems can also be referred to as reversible and are presented in two ways (Maf'ulah & Juniati, 2020). According to Maf'ulah and Juniati (2020), the first way includes presenting the problems using two different methods relating to its opposite. The first problem is related to the inverse, while the second is not associated with the inverse. However, both problems are presented in the same context. The second way entails presenting the problems in a single form, focusing on the known and questionable aspects (Maf'ulah & Juniati, 2020). In this study, the problems solved by adopting the second presentation method were used. Other studies refer to this second method as dual problems, and when solving related issues, one must have reversibility (Ramful, 2015).

In problem-solving, reversibility helps in understanding the relationship between concepts. Fitmawati (2019) stated that this could be achieved by establishing a two-way correlation based on

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

reversible problems. Reversibility is the ability to perform certain mental operations, including reversing the thought process in order to return to the starting point (Slavin, 2014). Piaget defined it as the ability to perform an action in two opposite directions while being aware of the fact that the actions are similar (Simon et al., 2016). Reversibility is the result of a second-level reflective abstraction of the original concept (Simon et al., 2016). Moreover, problems related to reversibility enables the in-depth analysis of reflective abstraction performed by an individual, specifically mental mechanism of coordination.

The present study was conducted to review the hypothesis of Arnon et al. (2014) related to the coordination of two processes, or Arnon's Coordination. This study broadens the understanding of coordination in the context of mathematical problem-solving and fills the gaps in previous research, which has not deeply explored how students use coordination to solve reverse problems. Furthermore, it investigates how students perform the mechanism of coordination between two processes in solving a reverse problem, with a particular focus on concepts of derivative and antiderivative graphs. The exploration also focuses on analyzing the cognitive actions that students performed with respect to Arnon's Coordination framework. The analysis offers a more comprehensive guide for educators in designing teaching approaches that promote reverse thinking in calculus. The coordination observed by students was detected at each problem-solving step by Polya (1985). The method was selected because it enables the observation of mathematical problem-solving strategies that are unique to each individual, thereby providing insight into the coordination performed, including the challenges faced.

LITERATURE REVIEW

APOS and Reflective Abstraction

APOS theory focuses on how mathematical concepts can be learned, with the acronym depicting action, process, object, and scheme regarded as mental structure (Arnon et al., 2014). The description of the relationship between mental mechanism of reflective abstraction and mental structure is explained through APOS theory (Arnon et al., 2014). This framework is used to describe how individuals develop an understanding of the concept through APOS stages, as shown in Figure 1.

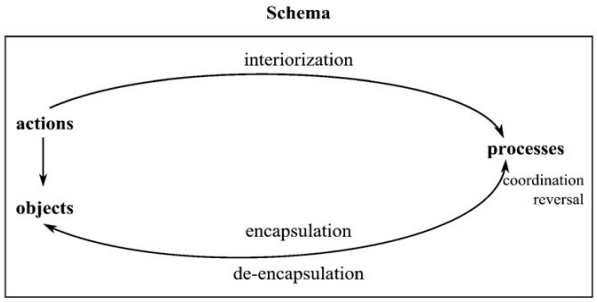


Figure 1: Mental structures and mechanisms for constructing mathematical knowledge. Source:

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Arnon et al. (2014)
The diagram shows the in-depth relationship between APOS and mental mechanism. The four mental structures are related to mental mechanisms in reflective abstraction, which supports and animates the construction of mathematical logic (Arnon et al., 2014). There are six mental construction types in reflective abstraction: interiorization, coordination, reversal, encapsulation, de-encapsulation, and generalization (Arnon et al., 2014). These types of construction are referred to as mental mechanisms, with the explanation reported as follows.

Interiorization

Interiorization refers to the awareness of an action, reflecting on it, and combining it with other actions. This mechanism allows for mental translation by imagining the execution of steps without having to skip and perform each explicitly (Arnon et al., 2014). Students engage in this process by reflecting on the procedure, including defining the concept (Cappetta & Zollman, 2013). For example, an individual might read, or become aware of the description of a function in terms of domain, range (perhaps implicit), and formula, enabling another to understand the problems based on what was written (Arnon et al., 2014). The interiorization process starts when an individual views a function as a type of transformation that pairs elements from a set, called the domain, with those from the second set, referred to as the range.

Coordination

Coordination is adopted to analyze the concepts, including constructing mathematical objects. It is defined as the composition of two or more processes to construct a new process (Dubinsky, 2002). In mathematics, this new process is constructed to analyze a mathematical concept by examining and integrating two or more processes (Cappetta & Zollman, 2013; Sopamena et al., 2018). A process is a mental structure that involves the same operations as an action but differs in terms of the transformations performed. According to Cetin and Dubinsky (2017), the operations performed in a process are the same as those performed in an action, but they occur entirely within the individual's mind. In actions, an individual must physically or mentally perform the transformation. Unlike an action, a process is internally driven, and the individual no longer needs external stimuli for guidance (Burns-Childers & Vidakovic, 2018). Additionally, external cues are not required at this stage (Arnon et al., 2014; Cetin & Dubinsky, 2017; Dubinsky, 2002), nor are explicit calculations necessary (Arnon et al., 2014). For example, in functions, individuals can imagine any element in the domain being transformed into an element in the range using the function's expression or another method (Arnon et al., 2014; Cetin & Dubinsky, 2017). Similarly, when observing a polynomial function graph, someone at the process stage can understand the type of function and determine its characteristics.

One example of coordination between two processes in function composition occurs when someone combines functions and to obtain. This process requires coordination between the two functions (Arnon et al., 2014). This coordination involves applying process to the elements obtained by ap-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

plying process . Another example includes determining the square root of a non-quadratic positive integer between two natural numbers. Students are expected to coordinate processes: the process of determining the square root of a squared number and the process to carrying out the rational number in decimal. This also includes the processes of internalizing the square root of two positive integers (Akgul & Yilmaz, 2023). In the context of graphing a derivative functions, the process of determining the relationship between two points on a curve and its slope (as the points on the graph become closer together) are coordinated with the process of determining the average rate of change (as the difference in time intervals becomes smaller, that is as the length of the time intervals gets closer to zero) to relate the definition of the derivative to some other interpretations (Asiala et al., 1997).

Reversal

New mathematical ideas can be constructed by reversing the steps of the original ideas. For example, in derivative and antiderivative concepts, an individual may explore the act of determining the derivative of a function, including correctly solving problems related to the concept and then internalizing it into a process (Arnon et al., 2014). Reversal is performed to find the original function, a derivative of the known function (Arnon et al., 2014; Dubinsky, 1991). Another example is a problem related to square roots. At the process stage, students make progress by identifying the relationship between square numbers and their square roots as inverse operations, understood through the reversal application. (Akgul & Yilmaz, 2023).

Encapsulation

The constructed process is packed into an object through the mental mechanism of encapsulation. This is achieved when the individual becomes aware of the total process, realizing that a transformation could act upon the process, or its constructs (Cetin & Dubinsky, 2017). Furthermore, encapsulation occurs when an individual applies an action to a process, simply referring to when the dynamic structure (process) is considered static to which the action can be applied (Arnon et al., 2014). Cappetta & Zollman (2013) defined this as the act of personifying a concept, i.e. a collection of abstract that seemed meaningful to an individual. Related to mental mechanism, Sfard (1991) analyzed the inherent difficulty of reification (similar to encapsulation in APOS Theory) and suggests that the ability to view a familiar task in a completely new way is challenging to achieve. In addition, difficulty arises when the process is not transformed into an object (Arnon et al., 2014). A typical example of encapsulation in integral material is when an action applied to the Riemann sum process is performed to determine or calculate the existence of its limit or value (Arnon et al., 2014). Another example is the determination of the square root of natural numbers, shown by using its symbol outside the original context of geometry and square numbers. At the object stage, this is extended to non-square numbers, with the decimals between integers estimated in new situations by adding integers (Akgul & Yilmaz, 2023).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

De-Encapsulation

The process leading to the object returns through the mental mechanism of de-encapsulation. These objects are de-encapsulated to regain the process when relevant (Arnon et al., 2014). For example, when working on problems related to the composition of two functions, the object function is de-encapsulated into its original process, and these processes are then coordinated to form the composition process, which is subsequently encapsulated into an object (Cetin & Dubinsky, 2017).

Generalization

Generalization is an existing schema represented and used in a new situation different from previous application (Dubinsky, 1991, 2002). Schemas are built as a coherent collection of structures from actions, processes, objects, and other schemas (Arnon et al., 2014; Nagle et al., 2019).

Arnon's Coordination in Mental Mechanism

In Arnon's Coordination, two processes, namely P_A and P_B as shown in Figure 2, are coordinated by applying P_A to P_B. This is realized by ensuring the learner encapsulates P_B into an Object, O_B, before applying P_A. Immediately after that is done, the coordination proceeds in the following way: O_B is assimilated and P_A applies, or P_A is accommodated enabling the learner to apply it to O_B. Previous studies stated that another alternative is the application of P_B to P_A in the same manner (Arnon et al., 2014). However, the hypothesis regarding the coordination implementation serves as a reference for further review (Arnon et al., 2014).

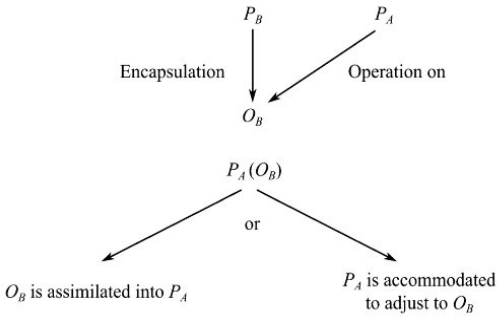


Figure 2: Coordination of two processes, P_A and P_B. Source: Arnon et al. (2014)

The diagram shows that a new process is constructed from the two processes during the coordination procedure by adopting several cognitive actions. One of these actions requires encapsulating the process before the integration with another predetermined process (Arnon et al., 2014). The subsequent action entails integrating the process and object, with Arnon's Coordination enabling the application to objects through assimilation or accommodation (Arnon et al., 2014; Goodson-Espy, 2014). The fit of the problem's structure to the individual's schema determines the use of assimilation or accommodation in problem-solving.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Following the earlier description, knowledge assimilation refers to the mechanism by which an individual applies the schema owned without changing it due to the similarity with the structure of the problems. In assimilation, the structure of the problem faced is in accordance with the existing scheme, resulting in direct integration (Blake & Pope, 2008; Subanji & Nusantara, 2016). Therefore, the schema owned is applied directly without change by including cognitive objects that had never been encountered (Arnon et al., 2014; Blake & Pope, 2008; Subanji & Nusantara, 2016). The accommodation process is a change in the scheme due to lack of inappropriate schemes for the problems structure (Blake & Pope, 2008; Subanji & Nusantara, 2016). This process occurs through reconstructing or modifying existing schemas to address new situations (Arnon et al., 2014; Subanji, 2011). During the learning process, students face changes to the schema either by adding components, refining, developing, or changing the old concept (Arnon et al., 2014; Setyaningsih et al., 2018).

METHODS

Study Design and Subjects

This study adopted a qualitative approach with a phenomenological method to provide an in-depth description of the experiences associated with various individuals related to certain phenomena (Creswell, 2015). Moreover, the adoption of the method was in accordance with the purpose of the analysis, which concentrated on exploring the coordination between the two processes carried out by students in solving a reverse problem, with a special focus on the concept of derivative and antiderivative functions. The phenomenological method enables the understanding of individual subjective experiences related to certain phenomena, revealing how meaning is given to these experiences (Burns et al., 2022). This study enabled an understanding of individual experiences related to actions carried out in coordination, including an exploration of the deeper meaning behind those actions. The phenomenological method aims to investigate the subject's experiences in depth, excluding generalized conclusions that apply broadly. As a result, this tended to have an impact on subject selection.

Participants consisted of 44 third-year mathematics education students who had taken advanced calculus courses, including derivative functions and antiderivative graphs. The subjects were selected based on a similar problem discussed in this study. The problem in the subject selection is stated in the appendix, with several criteria adopted during the selection process. Students who demonstrated problem understanding, coordination, and effective communication skills were selected as subjects. Furthermore, the understanding of the problem and coordination were viewed through the results written in the step of understanding the problem and devising a plan.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Data Collection

Data were obtained through student answers, thinking aloud, and semi-structured task-based interviews. The task sheet (Figure 3) gives instructions on how to solve the problem using Polya's problem-solving steps. As a result, students were asked to write down the processes used to solve the problem at each step.

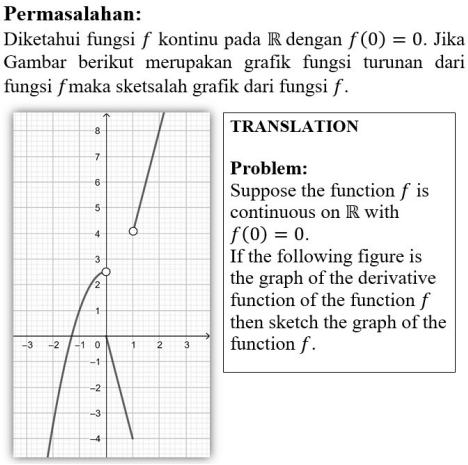


Figure 3: Reverse problem related to graphs of derivative and antiderivative on the task sheet

Figure 3 was inspired by problems in studies related to the representation translation process (Swastika et al., 2023). In view of this perspective, the problem was selected because the translation is related to cognitive processes, specifically in terms of connecting one representation (source) with another (target); however, the notation remained unchanged (Bosse et al., 2014). Through these problems, the development of students' cognitive structure was associated with the construction type of reflective abstraction performed in solving derivative function graph. The problem could be solved using two alternative solutions, namely symbolic or verbal representation.

Analyzing Data

Data analysis was conducted in three stages, including reduction, presentation, and drawing conclusions (Miles et al., 2013). The reduction stage was carried out by transcribing think-aloud and interview data, as well as selecting, reducing, and exploring student coordination. This study is based on the mental mechanisms of Piaget and Dubinsky (Arnon et al., 2014), and Polya's problem-solving method (Polya, 1985). In addition, this method allowed for the observation of unique mathematical problem-solving strategies. This provided insight into the coordination used by students when applying respective strategies. The coordination performed was observed at each stage of the problem-solving process, as outlined by Polya, and shown in Table 1.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Polya's Concept Steps	Coordination in Every Step
Understanding the Problems	1. Coordinating the processes associated with determining the diverse parts of a function to identify the characteristics of derivative and antiderivative graphs. 2. Coordinating the processes related to determining the properties of derivative function to identify the characteristics of antiderivative graph.
Devising a Plan)	1. Coordinating the processes related to determining the characteristics of derivative and antiderivative graphs to identify the problem-solving steps
Carrying Out the Plan	1. Coordinating the processes associated with determining the information extracted from the problem, including the procedure for ascertaining the symbolic representation of a graph. This was aimed at determining the calculated result during problem-solving (Symbolic Translation). 2. Coordinating the processes related to determining the relationship between derivative and antiderivative graphs. This included identifying the characteristics of antiderivative functions (Verbal Translation). 3. Coordinating the processes associated with determining the characteristics of a function and how to sketch the graph.
Looking back	1. Coordinating the processes associated with determining the characteristics of derivative and antiderivative graphs to verify the accuracy of computations and answers in problem-solving.

Table 1: Coordination in Each Polya Step

RESULTS

Based on the problem solved by the 44 participants, only four students who met the subject selection criteria were selected. This entailed understanding the problem (viewed from what was written regarding the given problem), coordination, and effective communication. Even though only one of the students could solve the problem correctly, the other three were selected. This was in line with the study's objective, which focused on how coordinating the two processes (Arnon's Coordination) enabled students to solve a reverse problem. The coordination procedure also included how assimilation and accommodation were done in this mental mechanism. Meanwhile, accommodation was realized through two procedures, namely modifying existing schemes or forming new structures and ensuring these were in line with new information (Subanji, 2011). The problem-solving process provided opportunities for the re-assessment of schema through reflection (Sopamena et al., 2018). This could also be achieved through questions, which were intended to access relevant parts of the scheme (Marshall, 2007).

The solving of the problem in Figure 3 provided an opportunity for the subject to reflect and provide intervention through questions. However, this intervention was carried out on three subjects, and one of the coordinations was made. Based on the results of the answers along with transcripts of think-

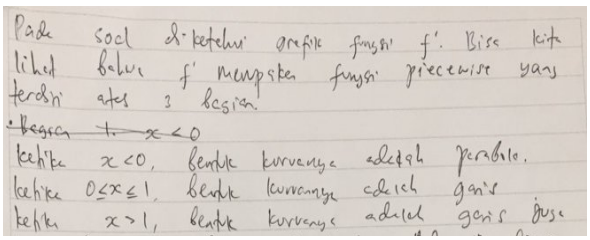
This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

aloud and interviews with the four students, one of them was able to complete the task independently without intervention (Subject 1 or S1). Two others solved the problem after receiving the intervention (Subjects 2 and 3 or S2 and S3, respectively), and the fourth student was unable to solve it despite the intervention (Subject 4 or S4). The presentation of the results related to how the coordination between the two processes was carried out in solving reverse problem is shown through three cognitive actions performed by students. Cognitive actions refer to mental activities used during coordination, and the definition is in line with a previous study (Dubinsky, 1991). The results are presented in two sub-sections, namely in relation to cognitive actions performed during the coordination of the two processes adopted in solving reverse problem and the challenges experienced by students associated with the intervention provided.

The Cognitive Actions in Coordination of Two Processes

Regarding the answers provided, all the subjects used symbolic translations to solve reverse problem that had to do with the graphs of derivative and antiderivative functions. The subjects determined the symbolic representation of the graphs of $f'(x)$ and $f(x)$. In addition, S1 was able to describe the characteristics of the graph of $f'(x)$ and relate it to the components of $f(x)$. S2, S3, and S4 were unable to describe the characteristics of the graph $f'(x)$ completely, nor did they determine the relationship between the two graphs. Based on this context, S2 and S3 were only able to identify the stationary point of the graph $f(x)$ by determining the value of $f'(x)$.

The subjects' coordinations at each step of problem-solving were in line with the method proposed by Polya. In the step of understanding the problems, the subjects' coordinations focused on determining the characteristics $f'(x)$ and $f(x)$ graphs. Coordination related to determining the characteristics of $f'(x)$ was viewed concretely when S1 divided the graph into several intervals, paying attention to the points passed through and their shape to understand the characteristics. The results of S1's written test about understanding the problems are shown in Figure 4, and following the think



TRANSLATION:

In the problem, the graph of the function f, was known as the piecewise function consisting of 3 parts.

When $x < 0$, the shape of the curve is a parabola
When $0 < x < 1$, the shape of the curve is a line
When $x > 1$, the shape of the curve is also a line

Figure 4: S1's Written test results in step of understanding the problems

S1: In relation to the problem, the graph of the function f' is known as a piecewise function, consisting of three parts. The first step is to identify the respective functions by dividing the problems' work order into three. The first part is for the interval that, why not less than or

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

equal to? Because this point *(pointing to the coordinate point that is the boundary of the graph),* what is the name of this point? The curve obtained was not included (*in the interval being considered)*. For the first part, the negative aspect, excluding zero, was considered. In this context, zero was included in the second part.

Following the work mentioned earlier, it was realized that S1 determined the characteristics of graph starting with the intervals. Based on the think-aloud excerpt, it was found that S1 studied the graph and realized the meaning of the symbols. In addition, S1 explained the reason for writing the interval, implying an understanding of the problem from what was written. This showed that S1 interiorized the action into a process related to determining the interval. S1 also paid attention to the shape of the graph in each interval, as shown in the following think-aloud excerpt:

S1: For this first part, (*determined while looking at the image of the graph*), depicted by the parabolic curve. It was evident that even though the parabolic curve did not pass through the zero point, the zero was undefined. The point served as a reference used to determine the parabolic curve. When is zero, the parabola is at this point *(pointing to a point)*, and the -value is . Similarly, for the other points, when is , the -value is . …

Based on the think-aloud excerpt, S1 observed the graph and mentally remembered the diverse forms and characteristics. S1 explained the reason behind the shape of the graph. This shows that S1 understands the problem situation from what is written. It indicated S1 has interiorized the actions into processes related to determining the form of the graph. S1 remembered, internalized, and mentally processed the procedure of determining the shape of the graph without the help of explicit steps**.** These took place in S1's mind, showing the student was at the process stage of understanding the concept of polynomial graph forms.

Considering the two processes, S1 determined the characteristics of $f'(x)$ by coordinating. In this coordination, S1 initially ascertained the two processes and then encapsulated the process of determining the intervals. This encapsulation was evidenced by the ability of S1 to take action on the process, including determining the coordinate points at intervals. After that, S1 integrated it into the process of determining the graph shape. The following is an excerpt from a related interview.

R: What does one intend to know about it? How did you come to know these characteristics? (*It is necessary to ask about the characteristics of* $f'(x)$*as presented by the subject*)

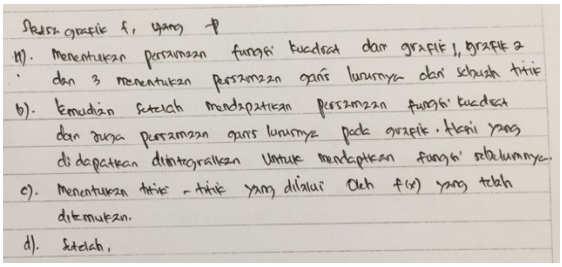
S1: ...Therefore, the shape of the graph was analyzed (*pointing to the graph in the interval*), for example, if this were analyzed, the curve sketch would be familiar. As a result, when this problem was initially encountered, I mainly focused on what would happen if the function were reflected. Assuming it was reflected later, the result would be a parabola, leading to the inference that this must be a quadratic function ...

S1 directly determined the characteristics of $f'(x)$ based on the shape of the graph in each interval. The repeated observations were made only to pay close attention to every detail of the graph image.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The procedure showed that the coordination was performed without the need to change the existing scheme or simply by assimilating the object (intervals on the graph) into the process (the procedure for determining its shape). This was due to the familiarity of S1 with the condition. During the coordination, S1 performed cognitive actions determining the processes, encapsulating one into objects, while integrating it through assimilation. S2, S3, and S4 adopted a similar method during coordination in the step of understanding the problems.

In terms of devising a plan, all subjects made a solution plan without giving a detailed explanation of the design. S1 explained the plan related to determining the integration constant but did not explain the plan to sketch the graph. In this context, S2 and S4 explained the plan briefly. Meanwhile, S3 explained in detail the plan for determining the symbolic representation of the graphs $f'(x)$ and $f(x)$, including the procedure before sketching $f(x)$. Similar to S2 and S4, S3 did not explain how to determine the constant of integration; only S1 was able to give a detailed explanation. The results of S3's written test on steps adopted in devising a plan are shown in Figure 5, and the following think-aloud excerpt:


Sketsa grafik f, yang ...
a). Menentukan persamaan fungsi kuadrat dari grafik 1, grafik 2 dan 3 menentukan persamaan garis lurusnya dari sebuah titik
b). kemudian setelah mendapatkan persamaan fungsi kuadrat dan juga persamaan garis lurusnya pada grafik tersebut di dapatkan diintegralkan untuk mendapatkan fungsi sebelumnya.
c). Menentukan titik - titik yang dilalui oleh f(x) yang telah ditentukan.
d). Setelah,

TRANSLATION:
Sketch the graph of f
a) Determine the equation of the quadratic function of graphs 1, 2, and 3. Determine the equation of a straight line from a point.
b) Then, after getting the quadratic function and straight line equations on the graph. The results obtained were integrated to get the previous function.
c) Determining the points that pass through $f(x)$ was found.
d) After...

Figure 5: S3's Written Test Results on the Step of Devising a Plan

S3: First, the quadratic function equation was determined from graphs 1, 2, and 3, while the equation of the straight line was obtained from the point (*stops writing and pays attention to the problem*). Furthermore, after obtaining the equation, from graphs 1, 2 and 3, it was integrated. *(Goes back to writing)* to determine the quadratic function and straight-line equations on the graph. The results obtained were integrated to get the previous function (*pauses and pays attention to the problem and the graph image*). After obtaining the graph equation, the points that passed through f(x) were determined *(pauses for a moment)* and then immediately sketched.

In line with Figure 5 and think-aloud excerpts, S3 coordinates to construct the process of sketching the graph of $f(x)$. This included connecting graphical, symbolic, and conceptual processes to understand the sketching procedure of graph $f(x)$. In determining this procedure, S3 related various representations and mathematical operations such as quadratic functions, straight lines, points, and integrals. This showed the coordination between various processes, including the identification of

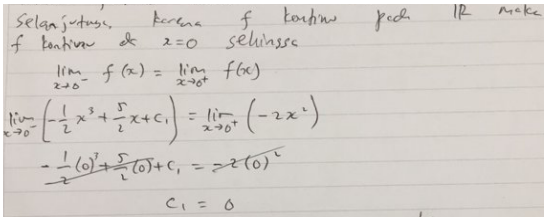
This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

graphs, formation of equations, integration, sketching graphs based on the integration results, and point analysis.

A typical example is the coordination related to determining the symbolic representation of $f(x)$. Regarding the think aloud excerpt above, S3 stated "...that after getting the quadratic function and straight line equations...the results obtained were integrated to get the previous function." S3 started to perform interiorization, which involved trying to understand information related to the characteristics of $f'(x)$, including the relationship between graphs $f'(x)$ and $f(x)$ mentally.

Regarding the two interiorization processes, S3 determined how to find a symbolic representation of $f(x)$ by coordinating. In this context, S1 encapsulated the process of identifying the graph into objects (types of polynomial functions). S3 applied the object to the process of determining the relationship between the two graphs. S3 was able to directly integrate this process and object due to being familiar with the problem. It was inferred that integrating the process and object in coordination was performed through assimilation. In devising a plan, the other subjects also carried out the same cognitive action related to coordination.

In the step of carrying out the plan, all subjects coordinated with the same mechanism as in the previous steps. In this step, the coordination included accommodation in the act of integrating. In coordination to determine the integration constant, S1, S2, and S3 integrate processes and objects through accommodation. S2 and S3 required intervention during the coordination, while S1 performed the coordination without intervention. The results of S1's written test on the step of carrying out the plan to determine the integral constant is shown in Figure 6 and the following think aloud excerpt:



TRANSLATION:

Then since f is continuous on $\mathbb{R}$, f is continuous at $x = 0$, therefore

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

$$\lim_{x\to 0^-}\left(-\frac{1}{2}x^3+\frac{5}{2}x+C_1\right) = \lim_{x\to 0^+}(-2x^2)$$

$$-\frac{1}{2}(0)^3+\frac{5}{2}(0)+C_1 = -2(0)^2$$

Figure 6: S1's Written Test Results on the Step of Carrying Out the Plan to Determine the Constant of Integration

S1: ...Here, the left and right continuous were used to find C_1 and C_3. Since f is continuous at $x = 0$, the limit x approaches 0 from the left of f must be equal to the limit x approaches 0 from the right of f, which is equal to the value of $f(x)$...
For x approaches 0 from the left, the value of f is unknown; therefore, it can be written di-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

rectly,$-\frac{1}{2}x^3+\frac{5}{2}x+C_1$, equivalent to the limit x approaches 0 from the right of $-2x^2+C_2$. Already C_2 is 0; therefore, it was unnecessary to write it again. Since this polynomial function is continuous, there is a need to substitute $x=0$ immediately. It was directly stated that $-\frac{1}{2}(0)^3+\frac{5}{2}(0)+C_1$ equals $-2(0)^2$, which could be crossed out, ensuring that C_1 is also 0. ...

Based on the think-aloud excerpt, S1 coordinated two processes to determine the integration constant. This is the coordination between symbolic procedures and calculus concepts**.** S1 combined procedural results (integral) with analytic concepts (limit and continuity) to develop a method for determining the constant of integration. S1 used information related to the algebraic representation of $f(x)$ at each interval. In determining C_1 and C_2, S1 only needed to use information related to $f(0)=0$ and the concept of function continuity (related to the concept of limits) to determine C_3. S1's statement that *"For x approaches 0 from the left, the value of f is unknown, therefore it can be written directly,*$-\frac{1}{2}x^3+\frac{5}{2}x+C_1$*..."* shows that S1 has internalized the process of determining the characteristics of the function associated with its algebraic expression at each interval. This process is done mentally without the aid of explicit steps. Additionally, S1 remembered and internalized the conditions for the continuity of the function $f(x)$ without explicit steps.

Regarding the two interiorization processes, S1 determined how to find the constant of integration through coordination, by encapsulating the adopted process. For example, in determining C_3, S1 encapsulated the process related to the requirements for obtaining continuous functions (the function's left-hand and right-hand limit values were the same). This was evident when S1 performed certain actions, replacing $f(x)$ with a particular equation as in Figure 6. Furthermore, S1 performed cognitive actions to integrate the process and the object. The following is the think-aloud excerpt related to the cognitive action performed:

S1: ...Well, aside from that, f is continuous at $x=1$ *(pauses for a moment)*. The limit of x approaching 1 from the left of $f(x)$ would be equivalent to the limit x approaching 1 from the right of $f(x)$ *(pauses for a moment)*. When f approaches 1 from the left, the function value is $-2x^2$... therefore, the limit of x approaches 1from the left, and $-2x^2$ would be the same as from the right, and $2x^2+C_3$. When x approaches 1 from the left, then the argument that these two functions are polynomials was applied to enable direct substitution, equivalent to $-2(1)^2$ and $2(1)^2+C_3$. As a result, the value of C_3 is the same as -4.

The think-aloud excerpt shows that S1 could not directly integrate the process into the object. Unlike the coordination to determine C_1 and C_2, S1 needed to pause for a moment to rethink the use of continuous function concepts, even though the possibility of its use in the solution plan had been mentioned. This was because S1 was still not sure of their thoughts. The following is an excerpt from the interview:

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

R: ...After integrating the equation $f'(x)$ why was the concept of a limit considered in this part? *(The subject's work related to determining the constant of integration was reviewed).*

S1: ...But what about C_1 and C_3, considering that f is a continuous function? Therefore, the value of the left-hand limit must be equal to the right-hand limit for $x < 0$. The limit of x approaching 1 from left must also be equal to the value approaching 1 from right. It enabled the immediate carrying out of the task, alongside getting used to solving similar problems.

R: So, you can directly determine the use of the left-hand and right-hand limits**.**

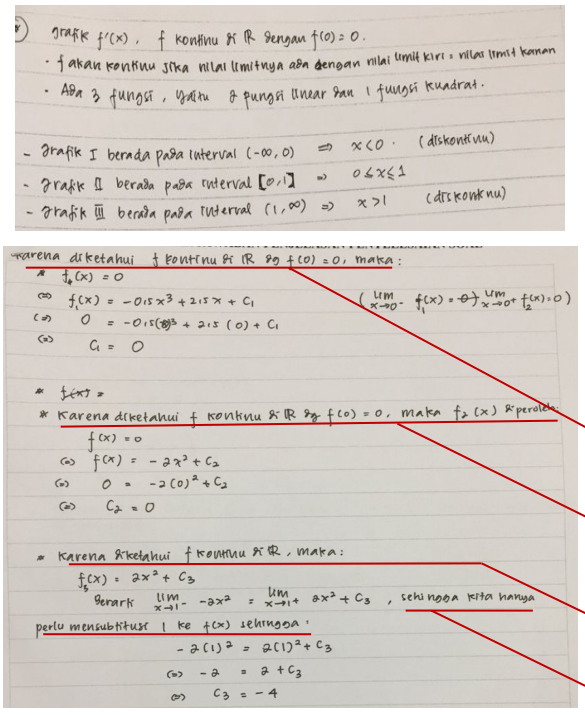
S1: Yes, I thought of it right away...

R: Were any difficulties encountered in coordinating what was in mind concerning the concept of limit, and integral including the sketching process?

S1: Yes, occasionally I was confused about when to use this and that *(while pointing to the information on the picture related to the characteristics of the graph f(x), which is f(x) continuous in ℝ)*, leading to confusion in terms of usage *(pointing to the determination of C_3)*.

The excerpt shows that S1 needed to make accommodation in the integration action. The accommodation was performed by students in terms of trying to build a new understanding that the constant of integration could be determined, although not arbitrarily. This was based on the continuity of the function, with the adaptation process performed without any intervention.

S2 and S3 needed intervention to coordinate the processes. The results of S2's written test in the steps of understanding the problems and carrying out the plan to determine the integration constant are shown in Figure 7 and the following think-aloud excerpt:



TRANSLATION:

S2's Written Test Results in the Step of Understanding the Problem

Graph ,$f'(x)f$ is continuous in ℝ with .$f(0) = 0$

- f is continuous if its limit values exist, with the left-hand limit value equal to the right.
- There were three functions, two linear and one quadratic.

graph I is on the interval$(-\infty,0) \Rightarrow x < 0$ (discontinue)
graph II is on the interval $[0,1] \Rightarrow 0 \leq x \leq 1$
graph III is on the interval$(1,\infty) \Rightarrow x > 1$ (discontinue)

S2's Written Test Results in the Step of Carrying Out the Plan

Since f is continuous in ℝ with$f(0) = 0$ then:
...
Since f is continuous in ℝ with$f(0) = 0$ then$f_2(x)$ is obtained:
...
Since f is continuous in ℝ, then:
....
Therefore, 1 only needs to be substituted into $f(x)$ so that

This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Figure 7: S1's Written test results in the steps of understanding the problems and carrying out the plan

(The intervention for S2 was realized by asking some questions. The first intervention to determine the processes to be coordinated.)

R: In this context, it was written that $f'(x)$ was discontinuous in this interval, while it was continuous for $f(x)$, give reasons?
S2: Oh yes, this implies that a description of f in $\mathbb{R}$ is continuous. (*Writes down the determination of* C_1)...
R: What does continuous mean?
S2: The limit and value of the function were the same.
R: Is there any other condition for a continuous function?
S2: A function is continuous if its limit value exists. In addition, it should be equal to the function value.
R: Do you remember the left-hand and right-hand limits?
S2: The limit exists if the limit from the left is equal to the one from the right....
R: In this interval *(pointing to the intervals* $x < 0$*)*, 0 was excluded, but the left-hand limit should be included.
S2: It should be equal to the right-hand limit.
R: Therefore, the limit should be the same as this one *(pointing to the intervals* $x < 0$ *and* $0 \leq x \leq 1$*)*? Try to write it down.
S2: So $\lim_{x \to 0^-} f_1(x) = \lim_{x \to 0^+} f_2(x)$ (*writes on the answer sheet next to the determination of* C_1).

In line with Figure 7 and the think-aloud excerpt, several instructions including a question, were given as an intervention to S2. The first intervention was needed to understand the information used in coordination. This aimed to construct the process related to the procedure for determining the integration constant. S2 understood the information associated with the problem. However, S2 had difficulty ascertaining the concepts to use in unfamiliar conditions. S2 also needed a second intervention to determine the integral result of $f'(x)$ through the point $x = 1$. This showed the inability to encapsulate the process related to continuous function requirements. The result was also evident when S2 was unable to take action to determine the equation for x approaching 1. The following think-aloud excerpt also proved the need for intervention:

(The intervention for S2 is to ask some questions. Second Intervention to encapsulate a process into an object)

S2: And then the third (*pauses for a moment*).
R: Notice this *(pointing to the graph of* $x > 1$ *),* because in the graph at, $x < 0$, $x = 1$ was not in-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

cluded, right?

S2: Yes, ma'am *(S2 writes back)*. Since f is continuous in $\mathbb{R}$, then for $f_3(x) = 2x^2 + C_3$. Ma'am, this is different from the previous calculations (*points to* C_1 *and* C_2*, pauses, and stops working*)

R: What did you write earlier *(pointing to the writing above in relation to* f *continuous in* $\mathbb{R}$*)*, therefore, for this function at $x = 1$, how? Based on what was initially written, what are the left-hand and right-hand limits?

S2: As a result, the limit approaches 1 both from the left of $f(x)$ and from the right... (*and then speechless*)

Considering the excerpt mentioned earlier, S2 was unable to determine the value of the integration constant in the interval $x > 1$. This was due to the inability to enter the value of $x = 1$ in the interval directly. S2 felt that although f is continuous, in the interval (for $x = 1$) it did not enter the result area of the function. The process is clearly explained in the following interview excerpt:

R: What other difficulties were encountered while working on this problem?

S2: When trying to find $\boldsymbol{C_3}$ and plotting the graph, including the third power

R: What difficulty was encountered in the part about finding $\boldsymbol{C_3}$?

S2: Because there was no circle sign, Ma'am *(pointing to the symbol of boundary point of graph in the interval* $\boldsymbol{0 \le x \le 1}$*, closed circle sign),* I had to rethink the process. This implied that at that point *(point* $\boldsymbol{x = 1}$ *in the interval* $\boldsymbol{0 \le x \le 1}$*)*, the limit as x approaches 1 exists. However, I had a rethink that here the interval started from 0 to 1 *(pointing to the graph in the interval* $\boldsymbol{0 \le x \le 1}$ *, which is from point* $\boldsymbol{(0,0)}$ *to* $\boldsymbol{(1, -4)}$*)*. I was unsure whether to substitute $\boldsymbol{x = 1}$ from the left or the right. This led to the assumption of equating the values *(pointing to the graph in the interval* $\boldsymbol{0 \le x \le 1}$*, which is* $\boldsymbol{x = 1, y = -4}$*)* and then finding x again *(pointing to the graph in the interval* $\boldsymbol{x > 1}$*)*.

Think-aloud and interview excerpts show that S2 was unable to perform the action of calculating the integration constant associated with the continuous function requirement and limit concept in the interval $x > 1$. Meanwhile, in the interval $0 \le x \le 1$, S2 was able to perform this action, resulting in the conclusion that S2 had difficulty performing the same action in different situations, including diverse intervals. The entire scenario showed S2's inability to encapsulate the process into an object (the result of integration). It was also reported that S2 required an intervention to reconstruct their understanding related to determining the integration constant. The result was evident in the following think-aloud and interview excerpts related to the determination of C_3:

(The intervention for S2 is to ask some questions. Third intervention to integrate a process and an object)

R: It was presumed that at $x = 1$, which part was the $f(x)$?

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

S2: The ones in $f_2(x)$ and $f_3(x)$.
R: Yes, therefore, which equations are $-2x^2$ and $2x^2 + C_3$?
S2: Yes, ma'am.
R: As a result, one only needs to enter the value of x, which is equal to 1?
S2: *(S2 remained silent momentarily before looking at the graph of f'(x))* Oh, yes. *(writes back)* It means that $\lim_{x\to 1^-} -2x^2 = \lim_{x\to 1^+} 2x^2 + C_3$, substituting 1 into $f(x)$, so $-2(1)^2 = 2(1)^2 + C_3$…

Interview excerpt explaining this was realized after S2 solved the problem:
R: At first, did you know whether you would need to use the concept of continuity and the information that $\boldsymbol{f(0) = 0}$? Did you immediately think of using them?
S2: Perhaps it was because the problems I usually work on never contain anything like this (pointing to the information stating that f is continuous and f(0) = 0). So, actually, there was information on this problem that could be used to determine the value of C. But I forgot it at the time, Ma'am.

The think-aloud excerpt and interview shows that S2 performed cognitive actions to integrate a process and an object through accommodation. This process involved determining the characteristics of the function *f* in relation to its algebraic expression at each interval. It led to an understanding that the integration constant is not arbitrary but rather based on the continuity of the function (as an object). This demonstrates that the process was reconstructed to accommodate this understanding.

In the looking-back step, the three subjects (those who were able to solve the problem correctly) did not carry it out. However, during the interview, the researcher attempted to ask the subjects for their thoughts on the results of their problem-solving process and the possibility of alternative methods to solve the problem. All three subjects shared the same perspective that the evaluation of their work was based on the accuracy of the calculations and the conformity of the graph's shape with the typical forms of quadratic and cubic functions. Moreover, the three subjects were unable to identify alternative ways to solve the problem other than the method they had used (i.e., determining the symbolic representation from the graphical representation).

Following Arnon's Coordination, the procedure started with two processes that had been determined. The procedure was then continued with the cognitive action of encapsulation and integration of process and object through assimilation or accommodation as shown in Figure 2. Considering the results of the study, the subjects determined the appropriate processes from several existing concepts in order to perform coordination. This was in relation with the schema selection, and during problem-solving, an individual was confronted with the selection process. Therefore, in this study, the coordination of the two processes was described as in Figure 8.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

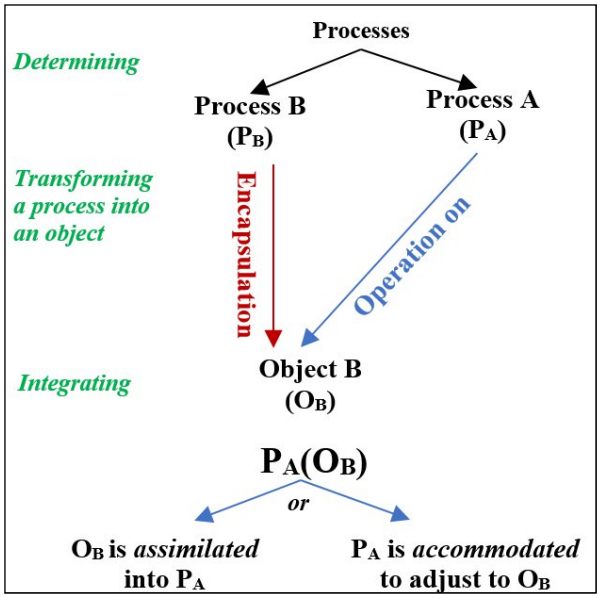


Figure 8: Mechanism of Coordination Based on Study Results

The analysis produced results depicting that the coordination of the two processes consisted of three cognitive actions, namely determining the processes, transforming a process to an object or encapsulation, and integration into another process. In addition, the integration of a process and an object was realized through assimilation or accommodation, pertaining to the conformity of the problems' structure with individual mental structure. The mechanism of coordination was described from the subject's selection of the processes to their integration, as shown in Figure 8.

The Challenges of Each Cognitive Action

During each cognitive action, the subjects faced certain challenges. S1 experienced challenges in determining the process and handled it without intervention. Meanwhile, S2, S3, and S4 required intervention in one of the coordination performed. Only S2 and S3 were able to coordinate after the intervention. The following interview excerpt showed how the challenges were handled by S2 and S3:

R: At the beginning were you aware that you could use the concept of continuity and information related to the value of $f(0) = 0$? Did you immediately think it could be used?

S2: ...because the problem I had worked on was not like this (*pointing to the problem description regarding the continuity of f and the information $f(0)$*). When solving the problem, the information extracted could be used to determine the value of C, but I did not seem to remember that idea (*pointing to what was written about what was known from the problem*).

Interview excerpt of S3:

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

R: So, did you have time to think about something to be able to determine the C value? This led to an explanation associated with continuous f in relation to the limit and $f(0) = 0$ *(points to S3's writing about what is known from the problem)*. Did you think this would be used?
S3: no *(while shaking his head)*
...
R: So, you were confused when connecting the concepts of continuous function, limit and integral?
S3: Yes, and because I already knew about the nature of continuous functions *(pointing to what was written in relation to the information extracted from the problem)*. But I did not think I would use it to find the value of the constant of integration. I thought there was a known f that is continuous after a question about the known information from the problem was given.

Based on the interview excerpts above, it was found that S2 and S3 encountered difficulties in determining the processes to be used in coordination. S4 experienced the same issue. Furthermore, S2, S3, and S4 also struggled with the action of encapsulation. This was evident in the previous think-aloud excerpt from S2, which related to the intervention given to support the encapsulation process within coordination for determining the integration constant. A similar difficulty was experienced by S4, leading the researcher to provide an intervention. However, even after the intervention, S4 could still not solve the problem. The following are the results of the think-aloud session and the intervention provided:

S4: ... because we have got the integral, then we draw this... *(speechless)*
R: Try to pay attention to what is known
(S4 was silent and only paid attention to the description part and pointed to f(0)=0 and f is continuous in $\mathbb{R}$*)*
R: It is continuous on $\mathbb{R}$ and $f(0) = 0$, what do you think you know?
S4: If it is continuous, the process implies that the value of the left-hand and right-hand limits would be similar.
R: Well, try to think about whether it can be used?
S4: Then this is $2x^2 + C_1$ on the interval $x > 1$ and $-2x^2 + C_2$ on the interval $0 \leq x \leq 1$. This depicts that C_1 and C_2 are similar. *(stops writing and falls silent)*
R: How can they be the same?
S4: This is because the left-hand and right-hand limits are the same. (*pauses*) Oh no, what should I do, Ma'am? I'm confused.

The think-aloud excerpt proves that S4 was unable to properly use one of the conditions for a continuous function to obtain an integration constant. S4 stated that because the limit values from left and right were the same, the integration constant in the intervals $x > 1$ and $0 \leq x \leq 1$ would be similar. This showed that S4 could determine the process used but was unable to engage in encapsulation in relation to the continuous function condition. The result was evident when S4 was unable to perform the action of calculating the integration constant. Regarding this perspective, intervention

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

was stopped because of the inability of S4. The study results showed that each cognitive action presented certain challenges, as shown in Table 2.

Cognitive Actions	Challenges
Determining the Processes	Subjects had difficulty deciding which process to select. This was due to unfamiliar conditions.
Transforming a Process to an Object	Subjects had difficulty encapsulating the process into an object. This was because students were unable to view familiar situations in a completely new way or under conditions different from before.
Integrating an Object into Another Process	Subjects had difficulty making accommodations through the reconstruction of their respective schemes. This was due to students' inability to build relationships between the concepts they possessed.

Table 2: The Challenges of each cognitive action

In accordance with Table 2, each cognitive action was characterized by the challenges faced by the subjects. These had causes that were related to the understanding of the basic calculus concepts, including the coherence of schemes owned by subjects.

DISCUSSION

Coordination is a mental mechanism used to construct new processes in APOS theory. This is defined as the composition of two or more processes used to construct a new one (Dubinsky, 2002). The new processes are constructed to be adopted in solving problems (Akgul & Yilmaz, 2023; Cetin & Dubinsky, 2017; Planell & Trigueros, 2019; Syarifuddin et al., 2020) or to learn a concept (Arnon et al., 2014). In this context, Arnon et al. (2014) hypothesized the coordination mechanism (Arnon's Coordination), which explains how individuals mentally coordinate two processes. The hypothesis is still recommended for future study. Therefore, an educator needs to understand how the mechanism functions. This is related to the focus of APOS theory, which provides a framework for understanding mental processes that occur when individuals learn mathematical concepts. Considering the description, the mechanism serves as a basis for developing lessons and evaluating student performance in solving mathematical problems (Arnon et al., 2014). In this study, a reverse problem was selected to determine the mental processes that occurred when students coordinated the processes in problem solving. This was because in some studies, direct and reverse problems were used to reconstruct individual mental processes associated with reversibility (Ikram et al., 2021).

The results describe how students coordinated two processes to solve the reverse problem. These results were proven through three cognitive actions performed by students in each coordination. The

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

actions include determining the processes, transforming one of them into an object, and integrating with other processes that have been determined. This suggests that the hypothesis proposed by Arnon et al. (2014) excluded the act of determining processes. In Arnon et al.'s hypothesis (Arnon's Coordination), the coordination implementation mechanism is described by a predetermined process shown in Figure 2. However, as this is a crucial factor in the success of coordination, the description of the mechanism in the present study started with the cognitive action of determining the processes, as in Figure 8. The result was in line with the theory proposed by Dubinsky (1991) that in coordination, there is a cognitive action of taking two or more processes and using them to build a new process. However, in this study, we term taking processes as determining processes.

The results show that the first cognitive action of determining processes plays an important role in coordination. Students did this cognitive act by determining the necessary processes based on the adopted concepts. This finding is consistent with those of Dubinsky (2002), who stated that coordination entails the evaluation of cognitive actions aimed at identifying pre-existing processes in order to construct new concepts. However, not all students can easily do it. According to the research results description, three students could not determine the integration constant. They were unable to do so because they could not construct a new process related to determining the integration constant through coordination. They were unfamiliar with the given problem. This is evident from the results of their work in the stage of devising a plan, in which they failed to devise a plan related to determining the integration constant. However, when determining the algebraic representation of, they realized a method was needed to determine the integration constant. Based on interviews, a significant challenge encountered in problem-solving was associated with determining the concepts used. In this study, three students were given interventions, in the form of questions that helped them reflect on what had been done. This was in accordance with the findings of Sopamena et al. (2018), which emphasized the need to provide students with opportunities to re-access their respective schemas through reflection.

The intervention enabled the three students to determine the processes to be coordinated. The intervention, in the form of questions, helped students review what had been written in relation to the step of understanding the problems. After realizing that the written information had not been used, the three subjects started to think about the possibility of its adoption. The response was in accordance with the theory proposed by Marshall (2007), which states that questions can be asked with the aim of accessing relevant parts of the scheme. After determining the processes used, the coordination process was continued by performing the second cognitive action.

The second cognitive action focused on transforming a process into an object through the encapsulation mechanism. Furthermore, encapsulation occurs when an individual performs actions on a process, realizing it as a totality and entity (Arnon et al., 2014; Cetin & Dubinsky, 2017). The results showed that cognitive action was evidenced by the ability of students to perform certain actions in a process. These included dividing the graph based on the interval, determining the location of a point, and the shape of the graph, performing integration and limit operations on a function, as well as sketching a function graph and others.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The cognitive act of transforming processes into objects played a crucial role in coordination. Students who encapsulated the process were able to continue the next cognitive action on coordination. Based on the results obtained, as well as in the first cognitive action, three students experienced obstacles. This also occurred during the coordination to construct new processes related to the determination of integration constants. The three students needed intervention. However, despite receiving intervention, one of the three students was unable to transform the process into an object. This occurred while she was determining the integration constant. She could not encapsulate the process related to obtaining continuous functions. This was in accordance with Borji & Planell (2023), who stated that students perform coordination by applying one process to an encapsulated process to construct a new process. The result was also in line with the hypothesis proposed by Arnon et al. (2014), that in the coordination of two processes, one of the processes was encapsulated into an object, enabling the other to be applied.

The challenge encountered in this second cognitive action is associated with the ability of an individual to summarize a concept by constructing its meaning. As generally understood, encapsulation entails the personification of a concept, specifically, a set of abstract ideas to make it meaningful (Cappetta & Zollman, 2013). In this study, students were unable to interpret the results of integration to perform limit operations. Although students had successfully performed similar actions on the integration, results were obtained at different intervals. This is consistent with the theory proposed by Sfard (1991), which focuses on the inherent difficulty of reification (similar to encapsulation in APOS Theory). The theory stated that the ability to detect familiarity in a completely new way was difficult to achieve. This difficulty arose when a process was not converted into an object (Arnon et al., 2014). The object that had been generated was integrated with the other process that had been determined.

The third cognitive action centered on integrating objects into a process that had been determined. The results show that when students coordinated the two previously owned processes under the condition of a familiar problem, the cognitive action could be performed directly, without intervention. The hypothesis proposed by Arnon et al. (2014) relates to how an individual assimilates the object, including applying the process. The study reported that students could directly determine how an action was applied to the object. The response was consistent with expert findings, which stated that the assimilation of knowledge referred to the mechanism by which an individual can apply the schema owned without alteration. This was because the problems' structure was in line with the schema already owned (Arnon et al., 2014; Blake & Pope, 2008; Subanji & Nusantara, 2016). However, not all of these cognitive actions were realized through assimilation.

In this context, coordination included the impact of accommodation on the cognitive action of integration. The results show that this occurred when students coordinated two processes they already possessed, but in a new or unfamiliar situation. In addition, students obtained accommodation by reconstructing the process they owned, adjusting the cognitive object, and applying it to the process. This response supported the result of Arnon et al. (2014) that in accommodation, the scheme is reconstructed and modified to address new situations. The reconstruction of this process was done

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

when students built a new understanding that the constant of integration can be determined, based on the continuity of the function. In this study, some students received interventions that enabled the reconstruction process. This challenge was encountered because subjects had difficulty obtaining accommodation through the reconstruction of the schemes. The difficulty was also caused by students' inability to build relationships between the concepts. Therefore, the intervention was realized by questions based on what had been written in terms of building relationships between concepts. This result supports the view of Corral et al. (2019), who points out that the core difficulty in learning mathematics lies in grasping the interrelationships among concepts within a problem, rather than merely executing solution strategies. Previous studies from different perspectives have demonstrated challenges in solving reverse problems associated with understanding the relationships between concepts.

The challenges students encounter in solving reverse problems related to the graphs of derivative and antiderivative functions have been reported in previous studies from different perspectives. For example, in studies focused on mathematical connections (García & Flores, 2021; Nieto et al., 2023), schema formation relied on derivative concepts (Fuentealba et al., 2019), representation translation (Swastika et al., 2023), and reversibility (Nieto et al., 2023). In line with this study, a proper understanding of the basic calculus concepts played a crucial role in reverse problem-solving. Therefore, educators are recommended to provide conceptual learning that focused on the relationship between concepts, including graphical-symbolic translation. This is because the connection processes enabled solving calculus tasks related to reversibility and translations of representations, such as exercises related to drawing and without symbolic expression (García & Flores, 2020, 2021; Nieto et al., 2023).

The results of this study show how the coordination of the two processes was associated with Arnon's hypothesis, and the challenges faced by students in each cognitive action when solving the reverse problem. The three challenges affected students' ability to construct a new process through coordination in each problem-solving step. For example, in terms of devising a plan, only one subject identified the plan for determining the constant of integration, alongside the step adopted during the execution process without intervention. Therefore, the results should be considered by educators when designing lessons that focus on the cultivation of deep concepts. This is related to the coherence of how actions, processes, and objects were interconnected and organized in larger mental structures called schemas. Regarding the description, schema coherence allow an individual to decide whether a schema can be used in a particular mathematical situation (Cetin & Dubinsky, 2017).

Following the description, educators are advised to design lessons that explicitly motivated students to analyze and reflect on the actions, processes, and mathematical objects used, as well as connect new schemas with existing concepts. This learning enables students to build a deeper and more applicable understanding of mathematical concepts through APOS-based learning. Borji et al. (2018) proved that APOS-ACE-based learning effectively helped students develop a deeper understanding through reflection and schema connection. Another possible solution is the provision of calculus-related problem-solving exercises with different contexts, using similar mathematical concepts. It

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

also helps students to recognize patterns of problems in diverse forms. The result is in line with the finding of Setyaningsih et al. (2018), which states that an individual's schema can be developed through problems given. These problems can include reverse problems involving the translation process between representations, such as the one provided to students in this study. In addition, educators are expected to evaluate the learning process by asking reflective questions to determine the extent to which students understand a concept. The results of such evaluations can inform the design of instruction that supports the development of students' conceptual understanding.

CONCLUSIONS

In conclusion, the mechanism of coordination between two processes carried out by students in solving the reverse problem associated with the graph of derivatives and antiderivative functions is explained through three cognitive actions. These include determining processes, transforming a process into an object through encapsulation, and integration through assimilation or accommodation. This result shows differences in line with the hypothesis proposed by Arnon regarding the coordination (Arnon's Coordination). The difference is the first cognitive action, namely the determination of the process. Based on the results, this cognitive action affects the success of coordination.

Other results are related to challenges encountered during each cognitive action and its causes. In the first cognitive action, students had difficulty determining the process to be selected. This was because the conditions of the problem were unfamiliar or had not been encountered. In the second cognitive action, students had difficulty encapsulating the process into an object due to the inability to view familiar tasks in a completely new way or in different conditions. Meanwhile, in the third cognitive action, students encountered difficulty making accommodations through the reconstruction of schemes owned because of the inability to build relationships between the concepts. These three challenges depended on students' understanding of the basic concepts of calculus and the coherence of their respective schemes.

Regarding the results obtained, students' success in the mental mechanism of coordination in solving reverse problems is influenced by conceptual understanding. Therefore, educators are recommended to design lessons that explicitly motivate analysis, alongside the reflection of an action, process, and object, including the connection of new and existing schemas. Another possible solution is to provide calculus-related problem-solving exercises with different contexts using similar mathematical concepts. The exercises are expected to be direct and reverse problems, including the translation process of representation. In addition, educators are expected to evaluate learning through reflective questions to determine the understanding level of students' concepts. This study recommends future analyses related to the effectiveness of learning strategies designed based on the results associated with improving students' understanding of mathematical concepts and solving reverse problems.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

ACKNOWLEDGEMENTS

The authors would like express gratitude to all the study participants and the State University of Malang, which provided many experimental experiences. The authors are also grateful to Mulawarman University, particularly the mathematics education study program, for helping to coordinate the subject selection. This analysis is part of doctoral students' research in mathematics education at the State University of Malang.

REFERENCES

[1] Akgul, E. N., & Yilmaz, R. (2023). Secondary School Students' Construction Processes of Square Root Concept with Realistic Problems: An APOS Perspective. Mathematics Teaching-Research Journal, 15(1), 174–197. https://files.commons.gc.cuny.edu/wp-content/blogs.dir/34462/files/2024/07/10YILMAZ-1.pdf

[2] Arnon, I., Dubinsky, E., Fuentes, S. R., Weller, K., Cottrill, J., Oktac, A., & Trigueros, M. (2014). APOS Theory. Springer.

[3] Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. E. (1997). The development of students' graphical understanding of the derivative. Journal of Mathematical Behavior, 16(4), 399–431. https://doi.org/10.1016/s0732-3123(97)90015-8

[4] Blake, B., & Pope, T. (2008). Developmental Psychology: Incorporating Piaget's and Vygotsky's Theories in Classrooms. Journal of Cross-Disciplinary Perspectives in Education, 1(1), 59–67. https://mxtsch.people.wm.edu/Teaching/JCPE/Volume1/JCPE_2008-01-09.pdf

[5] Borji, V., Alamolhodaei, H., & Radmehr, F. (2018). Application of the APOS-ACE Theory to Improve Students' Graphical Understanding of Derivative. Eurasia Journal of Mathematics, Science and Technology Education, 14(7), 2947–2967. https://doi.org/10.29333/ejmste/91451

[6] Borji, V., Font, V., Alamolhodaei, H., & Sánchez, A. (2018). Application of the Complementarities of Two Theories, APOS and OSA, for the Analysis of the University Students' Understanding on the Graph of the Function and Its Derivative. Eurasia Journal of Mathematics, Science and Technology Education, 14(6), 2301–2315. https://doi.org/10.29333/ejmste/89514

[7] Borji, V., & Martínez-Planell, R. (2023). On Students' Understanding of Volumes of Solids of Revolution: An APOS analysis. Journal of Mathematical Behavior, 70, 1–20. https://doi.org/10.1016/j.jmathb.2022.101027

[8] Bosse, M. J., Adu-Gyamfi, K., & Cheetham, M. (2014). Students' Differentiated Translation Processes. International Journal for Mathematics Teaching and Learning, *828*, 1–28. https://www.cimt.org.uk/journal/bosse5.pdf

[9] Burns, M., Bally, J., Burles, M., Holtslander, L., & Peacock, S. (2022). Constructivist Grounded Theory or Interpretive Phenomenology? Methodological Choices Within Specific Study Contexts. International Journal of Qualitative Methods, 21, 1–13. https://doi.org/10.1177/16094069221077758

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[10] Cappetta, R. W., & Zollman, A. (2013). Agents of Change in Promoting Reflective Abstraction : A Quasi-Experimental Study on Limits in College Calculus. REDIMAT - Journal of Research in Mathematics Education, 2(3), 343–357. https://doi.org/10.4471/redimat.2013.35

[11] Cawley, A. (2016). Developmental Mathematics Students' Use of Representations to Describe the Intercepts of Linear Functions Anne Cawley University of Michigan. Developmental Mathematics Students' Use of Representations to Describe the Intercepts of Linear Functions. http://sigmaa.maa.org/rume/crume2016/Papers/RUME_19_paper_24.pd

[12] Cetin, I., & Dubinsky, E. (2017). Reflective Abstraction in Computational Thinking. Journal of Mathematical Behavior, 47, 70–80. https://doi.org/10.1016/j.jmathb.2017.06.004

[13] Corral, D., Quilici, J. L., & Rutchick, A. M. (2019). The effects of early schema acquisition on mathematical problem solving. Psychological Research, 84(6), 1495–1506. https://doi.org/10.1007/s00426-019-01164-8

[14] Creswell, J. W. (2015). 30 Essential Skills for the Qualitative Researcher. SAGE Publications Inc.

[15] Dubinsky, E. (1991). Constructive Aspects of Reflective Abstraction in Advanced Mathematics. In L. P. Steffe (Ed.), Epistemological Foundations of Mathematical Experience (pp. 160–202). Springer – Verleg. https://doi.org/10.1007/978-1-4612-3178-3_9

[16] Dubinsky, E. (2002). Reflective Abstraction in Advanced Mathematical Thinking. In D. Tall (Ed.), Advanced Mathematical Thinking (pp. 95–126). Kluwer Academic Publishers.

[17] Fitmawati, E. E., Siswono, T. Y. E., & Lukito, A. (2019). Student's Reversibility in Solving Algebraic Problem. International Journal of Trends in Mathematics Education Research, 2(4), 188–192. https://doi.org/10.33122/ijtmer.v2i4.98

[18] Fuady, A., Purwanto, Susiswo, & Rahardjo, S. (2019). Abstraction Reflective Student in Problem Solving of Mathematics Based Cognitive Style. International Journal of Humanities and Innovation, 2(4), 103–107. https://doi.org/10.33750/ijhi.v2i4.50

[19] Fuady, A., Purwanto, Susiswo, & Rahardjo, S. (2020). Student Reflective Abstraction of Impulsive and Reflective in Solving Mathematical Problem. International Journal of Scientific and Technology Research, 9(2), 122–125. http://www.ijstr.org/final-print/feb2020/Student-Reflective-Abstraction-Of-Impulsive-And-Reflective-In-Solving-Mathematical-Problem.pdf

[20] Fuentealba, C., Badillo, E., Sánchez-Matamoros, G., & Cárcamo, A. (2019). The Understanding of the Derivative Concept in Higher Education. Eurasia Journal of Mathematics, Science and Technology Education, 15(2), 1–15. https://doi.org/10.29333/ejmste/100640

[21] García, J. G., & Flores, C. D. (2021). Pre-university Students' Mathematical Connections when Sketching the Graph of Derivative and Antiderivative Functions. Mathematics Education Research Journal, 33(1), 1–22. https://doi.org/10.1007/s13394-019-00286-x

[22] García, J. G., & Flores, C. D. (2020). Exploring Pre-University Students' Mathematical Connections when Solving Calculus Application Problems. International Journal of Mathematical Education in

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Science and Technology, 52(6), 912–936. https://doi.org/10.1080/0020739X.2020.1729429

[23] Goodson-Espy, T. (2014). Reflective Abstraction as an Individual and Collective Learning Mechanism. Constructivist Foundations, 9(3), 381–383. http://constructivist.info/9/3/381

[24] Hähkiöniemi, M. (2004). Perceptual and Symbolic Representations As a Starting Point of the Acquisition of the Derivative. Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 3, 73–80. https://eric.ed.gov/?id=ED489558

[25] Ikram, M., Purwanto, Nengah Parta, I., & Susanto, H. (2020). Mathematical Reasoning Required when Students Seek the Original Graph from a Derivative Graph. Acta Scientiae, 22(6), 45–64. https://doi.org/10.17648/acta.scientiae.5933

[26] Ikram, M., Purwanto, & Parta, I. N. (2021). Analysis of the Occurrence of Reversible Reasoning for Inverse Cases: A Case Study on the Subject Adjie. International Journal of Progressive Mathematics Education, 1(1), 1–15. https://doi.org/10.22236/ijopme.v1i1.6635

[27] Kurniati, D., Purwanto, As'ari, A. R., & Dwiyana. (2018). Exploring the Mental Structure and Mechanism: How the Style of Truth-Seekers in Mathematical Problem-Solving?. Journal on Mathematics Education, 9(2), 311–326. https://doi.org/10.22342/jme.9.2.5377.311-326

[28] Maf'ulah, S., & Juniati, D. (2020). The Effect of Learning with Reversible Problem-Solving Approach on Prospective Math Teacher Students' Reversible Thinking. International Journal of Instruction, 13(2), 329–342. https://doi.org/10.29333/iji.2020.13223a

[29] Maharaj, A. (2013). An APOS Analysis of Natural Science Students' Understanding of Derivatives. South African Journal of Education, 33(1), 1–19. https://doi.org/10.15700/saje.v33n1a458

[30] Marshall, S. P. (2007). Schemas in Problem Solving. Cambridge University Press.

[31] Miles, M. B., Huberman, A. M., & Saldaña, J. (2013). Qualitative Data Analysis: A Methods Sourcebook (Third Edit). SAGE Publications, Inc.

[32] Nagle, C., Martínez-planell, R., & Moore-russo, D. (2019). Using APOS Theory as a Framework for Considering Slope Understanding. Journal of Mathematical Behavior, 54, 1–14. https://doi.org/https://doi.org/10.1016/j.jmathb.2018.12.003

[33] Nieto, C. A. R., Vásquez, F. M. R., & Moll, V. F. (2023). Combined Use of the Extended Theory of Connections and the Onto-semiotic Approach to Analyze Mathematical Connections by Relating the Graphs of f and f'. Educational Studies in Mathematics, 114, 63–88. https://doi.org/10.1007/s10649-023-10246-9

[34] Pape, S. J., & Tchoshanov, M. A. (2001). The Role of Representation(s) in Developing Mathematical Understanding. IEEE Microwave Magazine, 40(2), 118–127. https://doi.org/10.1109/MMM.2019.2922800

[35] Paschos, T., & Farmaki, V. (2006). the Reflective Abstraction in the Construction of the Concept of the Definite Integral: a Case Study. Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education, 4, 337–344. https://files.eric.ed.gov/full-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

text/ED496934.pdf

[36] Planell, R. M., & Trigueros, M. (2019). Using Cycles of Research in APOS: The Case of Functions of Two Variables. Journal of Mathematical Behavior, 55, 1–22. https://doi.org/10.1016/j.jmathb.2019.01.003

[37] Polya, G. (1985). How to Solve it: A New Aspect of Mathematical Method. Princeton University Press.

[38] Ramful, A. (2015). Reversible Reasoning and the Working Backwards Problem Solving Strategy. Australian Mathematics Teacher, 71(4), 28–32. http://ezproxy.umuc.edu/login?url=http://search.ebscohost.com/login.aspx?direct=true&db=a9h&AN=111722462&site=eds-live&scope=site

[39] Roorda, G., Vos, P., & Goedhart, M. (2009). Derivatives and applications; development of one student's understanding. Proceedings of CERME, 2296–2305. http://www.inrp.fr/editions/cerme6

[40] Setyaningsih, N., Juniati, D., & Suwarson. (2018). Student 's Scheme in Solving Mathematics Problems. International Conference on Mathematics: Pure, Applied and Computation, 1–8. https://doi.org/10.1088/1742-6596/974/1/012012

[41] Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin. Educational Studies in Mathematics, 22(1), 1–36. https://doi.org/https://doi.org/10.1007/BF00302715

[42] Simon, M. A., Kara, M., Placa, N., & Sandir, H. (2016). Categorizing and Promoting Reversibility of Mathematical Concepts. Educational Studies in Mathematics, 93, 137–153. https://doi.org/10.1007/s10649-016-9697-4

[43] Slavin, R. E. (2014). Educational Psycology: Theory and Practice (10th ed.). Pearson.

[44] Sopamena, P., Nusantara, T., Irawan, E. B., & Sisworo, S. (2016). Characteristics of Thinking Processes abstraction Reflective of Students in Solving limits Problems. IOSR Journal of Mechanical and Civil Engineering, 13(05), 13–21. https://doi.org/10.9790/1684-1305041321

[45] Sopamena, P., Nusantara, T., Irawan, E. B., Sisworo, S., & Wahyu, K. (2018). Student's Thinking Path in Mathematics Problem-Solving Referring to the Construction of Reflective Abstraction. Beta: Jurnal Tadris Matematika, 11(2), 156–166. https://doi.org/10.20414/betajtm.v11i2.230

[46] Subanji. (2011). Teori Berpikir Pseudo Penalaran Kovariasional. UM Press

[47] Subanji, S., & Nusantara, T. (2016). Thinking Process of Pseudo Construction in Mathematics Concepts. International Education Studies, 9(2), 17. https://doi.org/10.5539/ies.v9n2p17

[48] Swastika, G. T., Nusantara, T., Subanji, & Irawati, S. (2020). Alteration Representation in the Process of Translation Graphic to Graphic. Humanities & Social Sciences Reviews, 8(1), 334–343. https://doi.org/10.18510/hssr.2020.8144

[49] Swastika, G. T., Nusantara, T., Subanji, S., & Irawati, S. (2023). How Do Assimilation and Accommodation Occur in the Translation Process of Representation?. Participatory Educational Research,

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

10(3), 167–190. https://doi.org/10.17275/per.23.50.10.3

[50] Syamsuri, S., Purwanto, P., Subanji, S., & Irawati, S. (2017). Using APOS Theory Framework: Why Did Students Unable To Construct a Formal Proof?. International Journal on Emerging Mathematics Education, 1(2), 135. https://doi.org/10.12928/ijeme.v1i2.5659

[51] Syarifuddin, S., Nusantara, T., Qohar, A., & Muksar, M. (2020). Students' Thinking Processes Connecting Quantities In Solving Covariation Mathematical Problems In High School Students of Indonesia. Participatory Educational Research, 7(3), 59–78. https://doi.org/10.17275/per.20.35.7.3

[52] Tall, D. (2011). A Sensible approach to the calculus. El Cálculo Y Su Enseñanza, 3, 40–67. https://doi.org/10.61174/recacym.v3i1.139

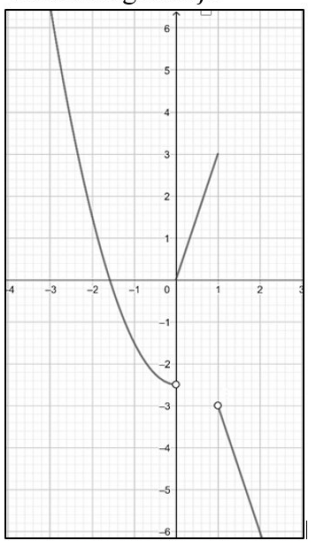
[53] Voskoglou, M. G. (2017). A Note on The Graphical Representation of the Derivatives. Journal of Physical and Mathematical Education, 2(12), 9–16. http://nbuv.gov.ua/UJRN/fmo_2017_2_3

APPENDIX

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Appendix 1. Pre-Test Problems

Permasalahan:
Grafik $f'(x)$ disajikan pada Gambar 1. Jika f kontinu pada $\mathbb{R}$ dengan $f(0) = 0$, sketsalah grafik f.



TRANSLATION

Problem:
The graph of $f'(x)$ is shown in Figure 1. If f is continuous on $\mathbb{R}$ with $f(0) = 0$, sketch the graph of f.

Note: The pre-test problem is only used to select students who will be used as research subjects

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Classroom Practices Mirroring Mathematical Knowledge for Teaching in Rwandan Teacher-Training Colleges: TPACK-Informed Lesson Planning.

Hashituky Telesphore Habiyaremye*[1], Celestin Ntivuguruzwa[2], Philothere Ntawiha[2]

[1]African Center of Excellence for Innovative Teaching and Learning of Mathematics and Science (ACEITLMS), University of Rwanda College of Education (URCE), Kayonza, Rwanda

[2]School of Education, University of Rwanda College of Education (URCE), Kayonza, Rwanda

hhtelesphore@gail.com

Abstract: Persistent challenges in effectively integrating technology, pedagogy, and content knowledge into mathematics instruction in Rwandan Teacher Training Colleges highlight the need for targeted professional development. This study employed a quasi-experimental design to address this gap by enhancing teaching practices by integrating Technological Pedagogical Content Knowledge into lesson planning. Six tutors from randomly selected schools participated, with half assigned to an experimental group receiving training, while the control group received no specific instruction. The six-month intervention aimed to enhance tutors' mathematical knowledge for teaching through lesson planning that explicitly integrated technological, pedagogical, and content knowledge components. The data collection utilized a customized Lesson Observational Analysis Protocol to assess classroom dynamics. Findings revealed notable shifts in tutors' practices post-intervention. The experimental group shifted away from predominantly using closed-ended questions, favoring more open-ended 'why' questions that promote deeper student reasoning. Additional changes were observed in how tutors confirmed understanding, structured group activities, and student justifications. The study contributes valuable insights into the impact of TPACK-informed training on teaching practices in Rwandan TTCs, offering a foundation for refining instructional strategies and improving mathematics education. The study was limited to tutors' practices and called for future direction in diving into students' practices.

Keywords: Technological Pedagogical Content Knowledge (TPACK), Mathematical Knowledge for Teaching (MKT), Teacher Professional Development, Lesson Observation Analysis Protocol (LOAP), Quasi-Experimental Design, Rwandan Teacher Training Colleges, Mathematics Instructional Practices

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

INTRODUCTION

Rwanda experienced a curriculum transformation from knowledge-based to competency-based in 2016. Implementing a competency-based curriculum (CBC) and a national continuous professional development (CPD) framework for teachers involves a significant shift in teaching fundamental concepts (REB, 2020; Habiyaremye et al., 2023). However, mathematics tutors in Rwandan Teacher Training Colleges (TTCs) continue to rely on traditional teaching practices, with limited evidence of practical shifts in line with CBC expectations. The gap between intended curriculum reform and real instructional change presents a critical issue that this study addresses. Educators are charged under CBC with transferring knowledge and abilities and assisting students in their learning journey to gain practical knowledge and skills indicated in the new curriculum. Furthermore, given the goals of the National Teacher CPD Framework, it is critical to construct and communicate a description of teaching excellence that is both evidence-based and context-sensitive at all levels (REB, 2019). Effective teaching of mathematics necessitates a firm understanding of key concepts, astute pedagogical abilities, and the incorporation of Information and Communication Technology (ICT) tools to enhance learning experiences and create a better comprehension of mathematical principles (Serin, 2023). The curriculum in Rwandan mathematical instruction requires the participation of teachers. Their job is critical in supporting students in grasping mathematical concepts, allowing them to properly apply and integrate these principles into a variety of areas such as science, mathematics, and technology (REB, 2020). The goal is to improve knowledge and abilities by fostering critical thinking and problem-solving skills, while promoting methodical, creative, and independent learning..

Taking a global perspective on mathematics education, scholars have developed a variety of frameworks for measuring teachers' mathematical knowledge for teaching (MKT) (Hill et al., 2008). These frameworks consider including knowledge in both subject content knowledge (SCK) and pedagogical content knowledge (PCK). Krauss et al. (2008) used this paradigm to investigate knowledge of mathematical tasks as instructional tools, interpretation of students' thinking, familiarity with different representations, and explanations of mathematical difficulties. Integrating Technological Pedagogical Content Knowledge (TPACK) into these frameworks enhances the understanding of how technology can effectively support teaching and learning in mathematics (Habiyaremye et al., 2022b). The findings confirmed two key issues in the context of mathematics education: the challenge of designing items that target PCK alone, as Specialized Content Knowledge often serves as a foundation for PCK; and the identification of teachers who lack sufficient SCK, thereby impeding the development of a robust TPACK framework. This updated TPACK framework includes an additional component known as "horizon knowledge," which builds upon subject-specific content knowledge by offering insights into curricular progression and the sequencing of concepts across grade levels (Jacinto & Jakobsen, 2020).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

While the MKT model appears to demonstrate only minor differences between domains, it has the ability to display the complete range of interactions between MKT domains (Koponen et al., 2019). As specified in the Official Gazette (2020), teachers in Rwanda are required to improve their skills and knowledge through capacity-building programs. This policy emphasizes the importance of teachers honing their skills, particularly in mathematics instruction, by tailoring instruction to learners' levels, incorporating creativity and innovation, connecting mathematics with other subjects and real-life situations, and possessing a comprehensive mastery of mathematical content (Habiyaremye et al., 2023; REB, 2020). Despite the requirement, obstacles remain in delivering traditional knowledge and skills in Rwandan schools (Habiyaremye et al., 2022a, 2023). Traditional knowledge transmission faces challenges that must be addressed. These obstacles may impede the optimal transfer of critical cultural and practical knowledge. Addressing these difficulties is critical to improving Rwanda's overall educational experience (Tabaro, 2018).

Despite implementing a CBC, Rwanda lacks a mathematical knowledge framework for assessing teachers' proficiency in teaching mathematics (Sugawara & Nikaido, 2014). The implementation of CBC anticipates teachers not only imparting knowledge and skills, but also facilitating students' learning processes for developing practical knowledge and abilities. However, it is unclear whether teachers' teaching approaches have changed post-shift. While existing research highlights the theoretical potential of TPACK to enhance instruction [7], [18], few empirical studies have investigated its practical impact on observable teaching behaviors in African teacher-training contexts. The National Teacher CPD Framework attempts to provide evidence-based descriptions of high-quality teaching at various levels within the Rwandan context. Despite this emphasis, there are still issues in providing instructors with the requisite information for teaching skills in Rwandan schools (Dorimana et al., 2022; Habiyaremye et al., 2022a), as traditional teaching techniques fall short of building a profound comprehension of academic subject directed toward societal applications. Due to the emphasis on the notion of MKT in the CBC, Rwanda has begun measuring teachers' proficiency in teaching mathematics. As Mathematics specialists on the JICA Project, we noted a recurring issue among educators in interpreting the knowledge and abilities indicated in the new curriculum (JICA, 2020). Despite meeting with instructors regularly, it became clear that many failed to understand the mathematical ideas stated in CBC while aiding students in building the necessary competencies.

The literature review indicates various interpretations of MKT and differing perspectives on mathematics teachers' responsibilities in education. There are numerous opinions on how educators understand and express mathematical topics, reflecting the complexities of their teaching responsibilities (Scheiner et al., 2023). This investigation emphasizes the importance of a comprehensive understanding of the teacher's role in conveying mathematical knowledge. The many interpretations in the literature necessitate a more in-depth assessment of effective teaching tactics and pedagogical approaches in mathematics education. In the same vein, Rwandan mathematics teachers have difficulty understanding their students' thinking processes across a range of mathematical

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

topics (Dorimana et al., 2022). These challenges underscore the need for additional study to bridge the gap between teachers' expertise and effective classroom instruction, emphasizing the importance of aligning instructional techniques with the intricacies of mathematics learning (Habiyaremye et al., 2023). The present assessment system for mathematics tutors in Rwandan teacher-training colleges stresses their ability to perform pedagogical activities connected to the subject knowledge required for mathematics instruction. However, there is a need for a more complete examination of the junction of educators' subject matter competence and their ability to implement successful teaching approaches. To address these challenges, this study explores the effectiveness of TPACK-guided lesson planning training to improve tutors' classroom practices that reflect MKT mastery. By integrating technology with pedagogical and content knowledge, the intervention is designed to help tutors plan and deliver mathematics lessons that convey subject matter and support active learning, conceptual understanding, and student engagement—core aspects of MKT. By focusing on the specific instructional challenges within mathematics education in Rwanda, this research offers insights that can inform the refinement of teacher training programs, particularly in countries with similar educational reforms, curriculum structures, or resource constraints..

This extensive investigation has major implications for improving mathematics instruction in Rwanda. The study provides actionable recommendations to better teacher training programs by identifying the strengths and areas for growth in mathematics tutors' performance. Addressing specific pedagogical issues in mathematics instruction should contribute to developing more effective teaching methodologies. Finally, this study's findings can potentially promote a more robust and impactful mathematics education system in Rwanda, aligning with broader aims of educational excellence and equipping students for success in an increasingly complicated terrain of mathematical knowledge.

Theoretical Framework

Classroom observation stands as a valuable and dynamic tool for assessing the effectiveness of teachers' instructional practices. By systematically documenting classroom activities, observers can gain insights into the nuances of teaching, providing a comprehensive view of pedagogical strategies, student-teacher interactions, and the overall learning environment (Nkundabakura et al., 2023). This firsthand approach offers a real-time perspective on implementing instructional methods, allowing evaluators to gauge the alignment between planned curriculum and actual classroom delivery. The effectiveness of classroom observation lies in its ability to capture the richness of teaching practices, including the utilization of diverse teaching aids, classroom management techniques, and responsiveness to students' individual needs (Delgado-Rebolledo & Zakaryan, 2020). Moreover, this method fosters a reflective and collaborative culture among educators, as

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

feedback derived from observations can inform professional development initiatives and contribute to continuous improvement in teaching methodologies (Bognar et al., 2024).

The study used the TPACK paradigm as its guiding framework to assess tutors' ability to smoothly integrate technology, pedagogy, and subject matter knowledge into their teaching approaches (Nzaramyimana & Umugiraneza, 2023). The TPACK model provides a complete evaluation of the teachers' competency in navigating the intricate junction of technology tools, pedagogical tactics, and mathematical topic understanding (Adipat, 2021; Uwurukundo & Tusiime, 2022). This analytical technique is expected to give useful insights into the overall effectiveness of mathematics teaching within Rwandan teacher-training institutions, perhaps informing adjustments and upgrades in pedagogical practices and instructional strategies. The TPACK model, established as a theoretical framework, enables a nuanced assessment of the delicate interplay of technological, pedagogical, and content knowledge in the context of mathematics education (Schmidt et al., 2014). It provides a systematic lens through which to evaluate how teachers use their expertise across different domains to improve their students' learning experiences (Santos & Castro, 2021). The researchers' goal in using the TPACK model in this study is not only to assess the current state of mathematics education in Rwandan teacher-training colleges but also to contribute to the broader discourse on effective pedagogical approaches and strategies for teaching mathematics, with an emphasis on the integration of technology and pedagogy to optimize the conveyance of mathematical concepts.

Research Questions

Two research questions have guided this study:

(1) How did tutors' practices evolve following TPACK-informed training, comparing the period before and after the intervention?
(2) Which specific aspects of tutors' practices were significantly affected by TPACK-informed lesson planning, as observed in the changes before and after the training?

METHODS

Research Design

Using a quasi-experimental research design, the research intervention improved Rwandan Teacher Training Colleges' (TTCs) tutors' MKT through progressive lesson plan guidance incorporating TPACK elements. A quasi-experimental design was appropriate for this study due to practical limitations in random assignment and the need to maintain authentic school settings. This approach is commonly used in education research to assess interventions under naturalistic conditions (Cresswell, 2014). Six teachers from 6 out of 16 schools were chosen at random for the study. The

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

participants were divided into two groups: the experimental group received training based on the TPACK paradigm, whereas the control group received no specific instruction. The first author worked hand in hand with tutors (teachers in the context of TTC) in an experimental group to improve their lesson planning by incorporating technology, pedagogy, and content aspects. A classroom observation tool was used to capture pertinent data from both groups. This method allowed for a thorough analysis of many elements in tutors' instructional methods, ensuring an objective and measurable assessment of their capacity to blend pedagogical tactics with mathematical subject mastery. The use of a quantitative observational methodology in this study improved the rigor and precision of the evaluation, giving a solid quantitative foundation for drawing conclusions about the quality of mathematics instruction in the selected teacher-training institutes.

Intervention Delivered

The intervention took a dual strategy, integrating TPACK in lesson planning and evaluating tutors' practices in delivering a lesson plan. As a guide, the researchers created an extensive training program for instructors to boost their effectiveness in teaching mathematics. The participants were divided into two groups: the experimental group received training based on the TPACK paradigm, whereas the control group received no specific instruction. This setting allowed for a controlled assessment of the impact of the intervention versus its absence. An observation checklist was used to evaluate the intervention's effectiveness. This checklist methodically collected data throughout classroom sessions, allowing for a detailed analysis of numerous factors in the teachers' educational approaches. This technique ensured a full and objective evaluation of their abilities to mix educational tactics with a strong comprehension of mathematical subjects. The study aimed to improve the precision and rigor of the evaluation by employing a quantitative observational design. This quantitative foundation offered a solid foundation for making conclusions about the overall quality of mathematics education in the TTC, providing useful insights for refining teaching approaches and improving the training of future mathematics educators.

Data Collection Procedures

Before embarking on data collection, ethical permission was obtained from the Unit of Research and Innovation at the University of Rwanda College of Education, and ethical clearance was obtained from the Directorate of Science, Technology, and Research at the Ministry of Education. This clearance was presented to TTC's head teachers. Reaching the classroom, we clearly explained the study's purpose and ensured that both tutors and students were voluntarily willing to participate. We used the S-T graph Lesson Observational Analysis Protocol (LOAP) tool to collect data during classroom observations (Habiyaremye et al., 2025). Developed through an iterative process grounded in instructional design principles and educator feedback, LOAP aligns

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

with Rwanda's Competence-Based Curriculum (CBC) by emphasizing clearly defined objectives, teacher-student interaction, and structured lesson phases. The tool includes six key icons: Start (real-time activity tracking), Log (organized data storage), Analysis (timing summaries), S-T Graph (teacher-student talk ratio), Activity (frequency and duration of actions), and Observation (structured lesson flow, currently under development). These features were refined over multiple trials to ensure clarity, usability, and validity.

We used LOAP to observe tutors in both control and experimental groups. We observed one classroom for 80 minutes with each tutor before intervention. Three tutors in the experimental group were guided to prepare an effective lesson plan (Ndihokubwayo et al., 2022) over a period of six months or two semesters (January to July 2023), while those in the control group continued their usual routine. Throughout this period, we observed each tutor in the experimental group four times, served a post-intervention (we call this after intervention), and observed each tutor in the control group at the end of this period. Therefore, we recorded 21 lessons (we call this a classroom observation), including six lessons in control and 15 in the experimental group.

Data Analysis Procedures

We evaluated 14 activities:

(1) While closed questions can serve a purpose in certain contexts, they tend to elicit specific answers and may not inherently promote deep critical thinking. This activity can lean towards a more teacher-centered approach.
(2) Forming groups for a specific purpose promotes collaborative learning, teamwork, and the exchange of ideas among students. It is aligned with active learning.
(3) Modifying tasks based on difficulty levels allows for differentiation, catering to individual student needs. It supports effective teaching practices.
(4) Drawing attention to important details is a good teaching practice that helps students focus on critical aspects of the content. It supports effective teaching practices.
(5) Connecting a topic to prior knowledge enhances understanding and helps students build on what they already know. It supports effective teaching practices.
(6) Addressing students' claims or concerns fosters a supportive learning environment and ensures that individual needs are acknowledged. It supports effective teaching practices.
(7) Providing encouragement is a positive teaching practice that motivates students and contributes to a positive learning atmosphere. It supports effective teaching practices.
(8) Using examples to illustrate a concept provides concrete instances for understanding abstract ideas. It supports effective teaching practices.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

(9) Asking and Responding to 'Why' Questions: This activity encourages critical thinking and deeper understanding by prompting students to explore the reasons behind concepts or solutions. It is aligned with active learning.
(10) Make Groups for a Specific Purpose: Forming groups for a specific purpose promotes collaborative learning, teamwork, and the exchange of ideas among students. It is aligned with active learning.
(11) Confirming understanding is a good practice to ensure that students comprehend the material, providing an opportunity for clarification and reinforcement. It supports effective teaching practices.
(12) Allowing students to justify their ideas promotes active participation, communication, and reasoning skills development. It is aligned with active learning.
(13) Demonstrate a Task (Teacher-led): Teacher-led demonstrations involve the instructor showcasing a task to the students. While valuable, this can be more teacher-centered, especially if students are passive observers rather than actively participating.
(14) Demonstrate a Task (Student-led): Student-led demonstrations encourage peer learning, collaboration, and a hands-on approach to understanding tasks or concepts. It is aligned with active learning.

The selection of the 14 evaluated teaching activities was grounded in established instructional and theoretical frameworks, particularly those underpinning MKT [5], TPACK [21], and principles of active learning and formative assessment [27]. These activities were chosen to reflect observable behaviors that align with core components of pedagogical content knowledge (e.g., using representations, checking understanding), cognitive engagement (e.g., asking "why" questions, justifying ideas), and instructional differentiation (e.g., modifying task difficulty). Together, they form a structured lens through which teaching effectiveness can be evaluated in accordance with competency-based education principles.

The recorded data obtained at preset intervals (in seconds) was thoroughly analyzed using Excel, ensuring a full assessment of quantitative and visual aspects. The obtained data was then effectively displayed through clear and informative graphs for both the control and experimental groups. This visual depiction made the findings more understandable and allowed for a more detailed assessment of performance trends over time. This analytical method improved the interpretation of data in the context of the mathematics classroom, providing useful insights into the success of the intervention in enhancing teaching techniques.

We used a confidence interval to compute statistical significance between groups on specific tutor practice. We followed these general steps to compute confidence intervals for our data: We calculated the mean and standard deviation for each group (Experimental and Control) and measurement (Before and After). These descriptive statistics were used to estimate the center and spread of the data. We chose 95% as the confidence level for our interval. We computed Standard Error

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[Standard Deviation/sqrt (sample size)]; our sample size was 3 for the control before and after intervention because we observed each of the three tutors once before and after intervention. Similarly, the sample size was three observations in the experimental group before and 12 observations after intervention, since we observed each teacher once and four times during the intervention. We then averaged the before and after mean, standard deviation, and standard error (SE) scores for a repeated measurement. We computed a pooled standard error using SE for control and experimental groups, as $SE_{pooled} = \sqrt{[(SE_{control}^2 + SE_{exp}^2) / 2]}$

We finally computed lower bound [$CI^- = (Mean_{control} - Mean_{exp}) - 1.96 * SE_{pooled}$ and upper bound ($CI^+ = (Mean_{control} - Mean_{exp}) + 1.96 * SE_{pooled}$ of confidence interval. The z-score for a 95% confidence interval is approximately 1.96. When the CI- and CI+ interval does not include zero, the interpretation is that there is a statistically significant difference between the compared groups on a certain practice.

RESULTS

In this section, we present the findings related to the first research question, how tutors' practices evolve following TPACK-informed training, and then the second research question on specific aspects of tutors' practices that are significantly affected by TPACK-informed lesson planning.

RQ1: How did tutors' practices evolve following TPACK-informed training, comparing the period before and after the intervention?

The described differences in practices before the intervention reflect similarities in the frequency and emphasis on certain instructional approaches between tutors in the control and experimental groups. These findings contribute to understanding the baseline practices, laying the groundwork for assessing the impact of the intervention within the TPACK framework (see Figure 1). More pronounced practices were "asking the closed question," "providing encouragement," and "use examples to illustrate a concept" which took an average of more than 600 seconds in an 80-minute class.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

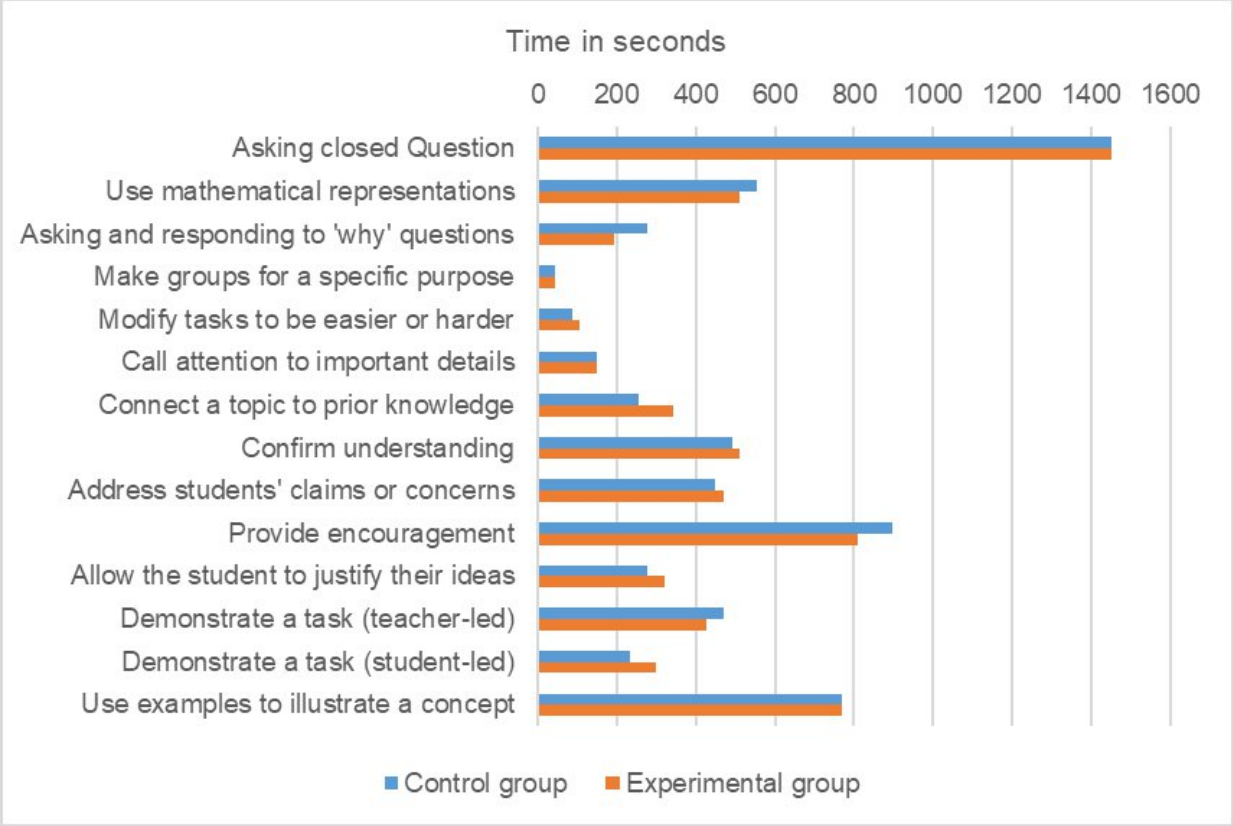


Figure 1. Tutor's classroom practices before teaching intervention

Asking Closed Questions: In both groups, tutors spent a substantial 1451 seconds asking closed questions. This big portion of time in a class suggests a potential similarity in the frequency and emphasis on closed questioning as an instructional approach.

Provide Encouragement: Both groups allocated substantial time to providing encouragement, with the control group spending 896 seconds and the experimental group dedicating 811 seconds. This similarity suggests a shared recognition of the significance of positive reinforcement in motivating and supporting students.

Use examples to illustrate a concept: Tutors in both groups allocated the same time (768 seconds) to use examples to illustrate a concept. This similarity suggests a shared recognition of the significance of positive reinforcement in the usage of concrete from abstract to improve students understanding.

Use of Mathematical Representations: While tutors in both groups used mathematical representations, those in the control group invested slightly more time (555 seconds) than the experimen-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

tal group (512 seconds). This subtle difference indicates a shared recognition of the importance of visual aids in conveying mathematical concepts.

Confirm Understanding: Both groups demonstrated a commitment to confirming understanding, with the experimental group dedicating 512 seconds and the control group spending 491 seconds. While the difference is marginal, it indicates a shared emphasis on assessing comprehension within both instructional settings.

Demonstrate a task (teacher-led): Tutors in the control group allocated more time (469 seconds) to connect a topic to prior knowledge compared to the experimental group (427 seconds). This difference suggests a potential variation in how tutors in the control group sought to demonstrate a teacher-led task.

Address Students' Claims or Concerns: Both groups demonstrated a commitment to addressing students' claims or concerns, with the experimental group dedicating 469 seconds and the control group spending 448 seconds. While the difference is marginal, this shared focus underscores the importance placed on attending to individual students' needs and fostering a supportive learning environment across both groups.

Connect a Topic to Prior Knowledge: Tutors in the experimental group allocated more time (341 seconds) to connect a topic to prior knowledge than the control group (256 seconds). This difference suggests a potential variation in how tutors in the experimental group sought to build on students' existing understanding through connections to prior knowledge.

Post-Intervention Observations

The post-intervention observations revealed nuanced changes in teaching practices within both the control and experimental groups. These shifts include a decreased reliance on closed questions, an increased emphasis on 'why' questions in the experimental group, modifications in the confirmation of understanding, notable adjustments in group activities, student justifications, and a reduction of teacher-led task demonstrations. These findings provide valuable insights into the evolving teaching practices following the intervention within the TPACK framework (see Figure 2).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

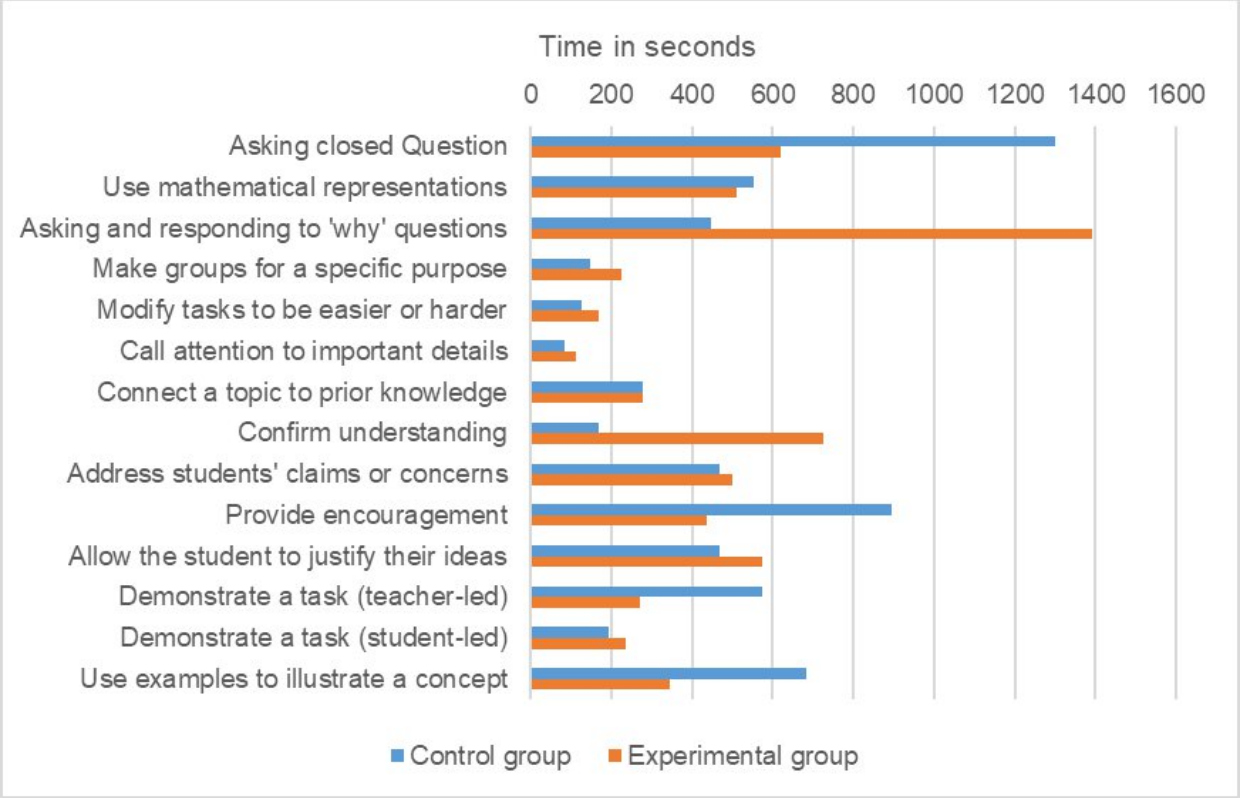


Figure 2. Tutor's classroom practices after the teaching intervention

Asking Closed Questions: Following the intervention, the control group showed a decrease in asking closed questions, spending 1301 seconds compared to the pre-intervention period. In contrast, the experimental group experienced a more substantial decrease, allocating only 619 seconds to this practice. This suggests a significant shift away from relying on closed questioning, with the experimental group demonstrating the effectiveness of helping tutors improve their lesson planning through the TPACK framework.

Asking and Responding to 'Why' Questions: A noteworthy change occurred in the experimental group's practice of asking and responding to 'why' questions. Tutors in the control group spent 448 seconds on this, while those in the experimental group significantly increased this practice post-intervention, dedicating 1392 seconds. This suggests a deliberate effort within the experimental group to foster deeper understanding and critical thinking through the exploration of 'why' questions.

Confirm Understanding: Tutors in the control group dramatically decreased the time spent on confirming understanding (from 491 to 171 seconds), while the experimental group experienced

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

a more significant increase from 512 to 725 seconds. This suggests a collective effort in both groups to place a heightened emphasis on assessing and ensuring comprehension.

Provide Encouragement: There was a considerable decrease in the time allocated to encouraging the experimental group after the intervention, dropping from 811 to 437 seconds. In contrast, the control group maintained a similar level, spending 896 seconds before and after the intervention. This divergence suggests a notable and negative shift in the experimental group's approach to providing encouragement.

Use of Mathematical Representations: Both groups exhibited consistent results in the use of mathematical representations after the intervention. The control group spent 555 seconds before and after the intervention, indicating a consistent approach. However, the experimental group showed a consistent reduction before and after, both at 512 seconds.

Use Examples to Illustrate a Concept: Tutors in the experimental group demonstrated a substantial decrease in using examples to illustrate a concept after the intervention (from 768 to 347 seconds), while those in the control group showed a more moderate decrease (from 768 to 683 seconds). This implies a notable reevaluation of the role of examples in the instructional practices of the experimental group.

Allow the Student to Justify Their Ideas: Tutors in the experimental group showed an increase in allowing students to justify their ideas, spending 576 seconds after the intervention compared to 320 seconds before. This indicates a deliberate effort within the experimental group to promote student reasoning and expression of ideas.

Address Students' Claims or Concerns: All tutors in both groups demonstrated increased practice, with more in the experimental group (from 469 to 501 seconds).

Demonstrate a Task (Teacher-led) and (Student-led): Both groups experienced changes in demonstrating tasks after the intervention. The experimental group showed a decrease in teacher-led demonstrations (from 576 to 272 seconds), while tutors in the control group kept increasing this practice (from 469 to 576). However, tutors in both groups consistently decreased results in student-led demonstrations, though control demonstrated a greater decline (from 235 to 192 seconds).

Make Groups for a Specific Purpose: Both groups increased the time spent making groups for a specific purpose after the intervention. The control group went from 43 to 149 seconds, while the experimental group increased from 43 to 224 seconds. This indicates a shared recognition of the value of purposeful group formations, with the experimental group demonstrating a more substantial increase.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Connect a Topic to Prior Knowledge: Tutors in the experimental group failed to connect a topic to prior knowledge, demonstrated by a decrease in time (from 341 to 277 seconds) dedicated to this practice after the intervention.

Modify Tasks to be Easier or Harder: Tutors in both groups showed an increasing dedicated time but with the same trend.

Call Attention to Important Details: Tutors in both groups decreased the time dedicated to this practice after instruction.

RQ2: Which specific aspects of tutors' practices were significantly affected by TPACK-informed lesson planning, as observed in the changes before and after the training?

Based on the mean, SE_{pooled}, and confidence intervals (CI- and CI+), 6 out of 14 tutors' practices revealed statistically significant differences after intervention, 4 in favor of tutors in the experimental group and 2 in favor of tutors in the control group. Other groups did not show any statistically significant difference before and after intervention.

The mean time spent on (1) *asking closed questions* in the control group (1376 seconds) is statistically significantly different from the mean time in the experimental group (1034.67 seconds). The 95% confidence interval ranged from 180.73 to 501.93. The estimated difference in the mean time for asking closed questions between the control and experimental groups is statistically significant, with the control group spending more time and the experimental group spending less time. Therefore, this practice translated into the mean time spent on (2) *asking and responding to 'why' questions* in the control group (362.67 seconds), which is statistically significantly different from the mean time spent in the experimental group (792 seconds). The 95% confidence interval ranged from -553.70 to -304.97. Thus, the estimated difference in the time spent asking and responding to 'why' questions between the control and experimental groups is statistically significant, with the experimental group spending more time.

The mean time spent on (3) *confirming understanding* in the control group (330.67 seconds) is statistically significantly different from the meantime in the experimental group (618.67 seconds). Confidence Interval (95%): [-384.25, -191.75]. The estimated difference in the mean time for this practice was significant, with the experimental group spending more time. However, for the (4) *"provide encouragement"* activity, the mean time spent by the control group was 896 seconds, while the experimental group spent 624 seconds (with confidence intervals for the difference in means (CI- and CI+) of 198.35 and 345.65, respectively). This indicates that the experimental group, on average, spent less time providing encouragement compared to the control group. The

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

negative values in the confidence intervals suggest that the experimental group spent less time while the control group spent enough time on this activity.

The mean time spent on (5) *demonstrating a task in a teacher-led* approach in the control group (522.67 seconds) is statistically significantly different from the meantime in the experimental group (349.33 seconds). Confidence Interval (95%): [62.10, 284.56]. The estimated difference in the meantime for this practice is significant, with the control group spending more time and the experimental group reducing time. However, the mean time spent on (6) *using examples to illustrate a concept* in the control group (725.33 seconds) is statistically significantly different from the meantime in the experimental group (557.33 seconds). Confidence Interval (95%): [40.88, 295.12]. Thus, based on the statistical analysis, the time spent on using examples to illustrate a concept differs significantly between these groups, favoring the control group.

DISCUSSION

Interestingly, both the control and experimental groups started with similar baseline practices, suggesting a common foundation in their instructional approaches. The similarities observed in these practices may reflect the existing pedagogical norms or training received by the tutors before the intervention. In delving deeper into the baseline practices of tutors within Teacher-Training Colleges (TTCs) in Rwanda, a notable pattern emerges—a commonality in instructional approaches. The tutors, equipped with a university background in mathematics and education, have undergone Continuing Professional Development (CPD) training facilitated by the Rwanda Education Board (REB) and developmental partners (Ndihokubwayo et al., 2021; REB, 2018). This shared educational foundation among the tutors lays the groundwork for understanding their instructional practices.

The post-intervention phase illuminates a canvas of transformative dynamics within the teaching practices of tutors in TTCs in Rwanda. The observed changes reflect a conscious effort to depart from conventional instructional methods and embrace a more nuanced, student-centric pedagogical approach. The intervention consisted of a structured training program designed to guide tutors in incorporating TPACK elements into lesson planning. This included workshops on aligning learning objectives with CBC standards, selecting and integrating digital tools (e.g., GeoGebra, PowerPoint, and videos) for mathematical concepts, and designing student-centered tasks that foster reasoning, group interaction, and justification. Tutors received individualized support through planning discussions and reflective sessions after each lesson implementation. These targeted supports are likely responsible for the significant post-intervention shifts observed in the experimental group—such as reduced reliance on closed questions, increased use of ‘why’ questions, and improved methods for confirming understanding. The training emphasized a shift from procedural instruction to conceptual understanding, reinforcing MKT's pedagogical and content

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

dimensions within a TPACK framework. Some practices showed statistically significant differences between the control and experimental groups post-intervention, providing insights into the effectiveness of the training.

The mean time spent asking closed questions emerges as a significant difference between the control and experimental groups post-intervention. The experimental group invested less time in this practice and advocated more time for asking a 'why' question to induce deep understanding. This shift suggests a move away from rote questioning toward practices that encourage reasoning and justification—an essential aspect of student-centered learning and MKT development. The realm of 'why' questions becomes a focal point, underscoring the experimental group's commitment to nurturing a culture of inquiry and critical thinking. These changes suggest a shift towards more student-centered and inquiry-based instruction (Habiyaremye et al., 2022a). The observed shifts in post-intervention teaching practices hold profound implications for the development of MKT in the context of TPACK-informed lesson planning (Hill et al., 2008). The decreased reliance on closed questions suggests a move towards more interactive and open-ended pedagogical approaches (Ukobizaba & Byukusenge, 2023). The increased emphasis on 'why' questions in the experimental group indicates a conscious effort to foster deeper understanding and critical thinking, aligning with the goals of TPACK integration.

The mean time spent on teacher-led task demonstrations unravels a significant difference, with the control group surpassing the experimental group. Tutors in the experimental group declined investment in teacher-led demonstrations, reflecting a pedagogical preference induced by the effective intervention of TPACK-led lesson planning, a hint at fully embracing student-led approaches. Recognizing the interdependence of content knowledge and pedagogical skills, TTCs can play a critical role in developing highly effective mathematics educators who not only have a deep understanding of the subject matter but also excel at communicating complex concepts in a way that resonates with their students, ultimately improving the overall quality of mathematics education at the collegiate level (Kleickmann et al., 2013). The landscape of confirming understanding unveils a significant transformation post-intervention, with the experimental group dedicating more time than the control group. This distinction prompts inquiry into the underlying dynamics. This heightened focus on confirming understanding indicates a more deliberate approach to assessing comprehension and the ripple effects it creates in the learning environment (Uwurukundo & Tusiime, 2022). This, in turn, can lead to more targeted and tailored instructional strategies, addressing individual student needs effectively. Furthermore, adjustments in the confirmation of understanding highlight a nuanced approach to assessing comprehension, emphasizing the importance of formative assessment strategies (Ukobizaba & Byukusenge, 2023). This aligns with the principles of MKT development (Arnal-Bailera & Oller-Marcén, 2020), emphasizing what is taught and how students comprehend it.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The statistical analysis sheds light on the practice of using examples to illustrate a concept, favoring tutors in the control group. This prompts a deeper inquiry, depending on how these examples resonate with student comprehension and what factors contribute to the observed variations in approach. Likewise, encouraging takes center stage, revealing a nuanced divergence. The control group maintained a consistent level, while the experimental group witnessed a reduction. This reduction signifies a recalibration of motivational strategies, which negatively impacts student motivation and confidence in mathematical exploration. The complexity of conceptualizing teacher knowledge involves understanding teaching processes, the nature of knowledge, and practical application in the classroom (Habiyaremye et al., 2022a). Empirical research highlights that teacher quality significantly influences student achievement, emphasizing the need for a comprehensive understanding of teacher knowledge that contributes most to positive learning outcomes (Filgona et al., 2020). The decrease in encouragement and the use of examples in the experimental group warrants further investigation into potential challenges in implementing positive reinforcement strategies post-training. This is critical for ensuring that educators have the necessary skills and knowledge to effectively teach mathematics in accordance with the CBC standards, thereby contributing to the success and efficacy of the educational system (Aksu & Kul, 2016).

In practical terms, these changes signify a potential enhancement in teaching effectiveness, fostering a more dynamic and interactive learning environment. Students may benefit from increased engagement, deeper understanding, and improved critical thinking skills. The observed shifts, therefore, contribute to the theoretical framework of MKT and hold tangible implications for the day-to-day teaching practices and overall learning experiences within the teacher-training context. These implications underscore the effectiveness of TPACK-informed training in shaping teaching practices that resonate with the objectives of MKT development (Nzaramyimana & Umugiraneza, 2023). Integrating technological, pedagogical, and content knowledge in lesson planning emerges as a catalyst for transformative shifts in instructional strategies, ultimately contributing to the enhancement of MKT among TTC tutors.

All core intervention materials were developed to support replication and validation of this study and can be shared upon request. These include the TPACK-based lesson planning framework used during training, observation checklists, planning templates aligned with the CBC, example lesson plans co-developed with tutors, and the customized LOAP tool. The training module focused on guiding tutors through structuring learning objectives, selecting digital resources (e.g., GeoGebra, videos, and PowerPoint), integrating technology during instruction, and embedding student-centered questioning strategies. While not all these materials are embedded in this paper, they are available via the project repository and can be provided to interested researchers for replication or adaptation.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

CONCLUSIONS

The analysis of tutors' classroom practices before and after the intervention, which involved training teachers in TPACK-informed lesson planning, offers valuable insights into the impact of the training on their instructional approaches. The findings reveal nuanced shifts in teaching practices within the control and experimental groups. Before the intervention, both groups demonstrated similarities in the frequency and emphasis on certain instructional approaches, such as asking closed questions, providing encouragement, and using examples to illustrate a concept. The observed practices reflected a baseline understanding of pedagogical methods, laying the groundwork for assessing the intervention's impact. Post-intervention, notable changes were observed in both groups. There was a collective decrease in the reliance on closed questions, an increased emphasis on 'why' questions in the experimental group, modifications in the confirmation of understanding, adjustments in group activities, student justifications, and a reduction in teacher-led task demonstrations. These shifts provide valuable insights into the evolving teaching practices within the TPACK framework. After the intervention, the statistical analysis revealed significant differences in certain practices between the control and experimental groups. Notably, the control group spent more time asking closed questions and demonstrating a task in a teacher-led approach, while the experimental group exhibited a more pronounced decrease in asking closed questions and an increase in asking and responding to 'why' questions. Tutors in the experimental group also reduced time spent on teacher-led demonstrations and increased time for confirming understanding. However, there was a loophole in providing encouragement and using examples to illustrate a concept among the tutors of the experimental group. These findings suggest that the TPACK-informed training has influenced specific teaching practices, promoting a shift towards more interactive and student-engaged methods. The observed changes align with the goal of enhancing teachers' Mathematical Knowledge for Teaching through effective lesson planning within the TPACK framework. Therefore, it is recommended that teachers receive training in order to deliver mathematical content competently.

LIMITATIONS

This study only focused on tutors' practices. Therefore, more studies are needed to reveal the students' practices in mathematics and other subjects. We randomly picked six schools (one teacher from each) to implement our intervention extensively. Due to the long period of intervention that took six months, we could not involve many teachers or schools; however, we acknowledge this as a limitation of generalization.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

ACKNOWLEDGEMENTS

We acknowledge the tutors and students who participated in this study and the financial support from the African Center of Excellence for Innovative Teaching, Mathematics and Science (ACEITLMS) offered to the first author.

DECLARATIONS

Ethics Statement

This study was approved by the Ethics Committee of the Research and Innovation at the University of Rwanda College of Education, with ethics approval reference 03/DRI-CE/066/EN/gi/2020. The INFORMED CONSENT was obtained from all participants for our study.

Conflict of Interest

The authors declare no conflict of interest regarding the publication of this paper.

Data Availability

The datasets analyzed during the current study are available from the MKT DATASET COLLECTED FOR PHD PROGRAM (figshare.com).

REFERENCES

[1] Adipat, S. (2021). Developing Technological Pedagogical Content Knowledge (TPACK) through Technology-Enhanced Content and Language-Integrated Learning (T-CLIL) Instruction. *Education and Information Technologies*, 26(5), 6461–6477. https://doi.org/10.1007/s10639-021-10648-3

[2] Aksu, Z., & Kul, Ü. (2016). Exploring mathematics teachers' pedagogical knowledge of students. *Journal of Education and Practice*, 7(30), 35–42. www.iiste.org

[3] Arnal-Bailera, A., & Oller-Marcén, A. M. (2020). Prospective secondary mathematics teachers read Clairaut: professional knowledge and original sources. *Educational Studies in Mathematics*, 105(2), 237–259. https://doi.org/10.1007/s10649-020-09988-7

[4] Bognar, B., Matić, L. J., & Sablić, M. (2024). Professional Development Interventions for Mathematics Teachers: A Systematic Review. *Mathematics Teaching-Research Journal*, 15(6), 39–58.

[5] Cresswell, J. . (2014). *Research Design: Quantitative Qualitative and Mixed methods* (5th Edition). SAGE Publications, Inc.

[6] Delgado-Rebolledo, R., & Zakaryan, D. (2020). Relationships Between the Knowledge of

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Practices in Mathematics and the Pedagogical Content Knowledge of a Mathematics Lecturer. *International Journal of Science and Mathematics Education*, 18(3), 567–587. https://doi.org/10.1007/s10763-019-09977-0

[7] Dorimana, A., Uworwabayeho, A., & Nizeyimana, G. (2022). Enhancing Upper Secondary Learners ' Problem - solving Abilities using Problem-based Learning in Mathematics. *International Journal of Learning, Teaching and Educational Research*, 21(8), 235–252. https://doi.org/https://doi.org/10.26803/ijlter.21.8.14

[8] Filgona, J., Jacob, F., John, S., & Gwany, D. M. (2020). Teachers' Pedagogical Content Knowledge and Students' Academic Achievement: a Theoretical Overview. *Journal of Global Research in Education and Social Science*, 14(2)(2454–1834), 14–44.

[9] Habiyaremye, H. T., Ntivuguruzwa, C., & Ntawiha, P. (2022a). Assessment of Teaching Methods in Mathematical Simplicity and Complexity in Rwandan Schools via Pedagogical Content Knowledge. *Mathematics Teaching-Research Journal*, 14(4), 129–147.

[10] Habiyaremye, H. T., Ntivuguruzwa, C., & Ntawiha, P. (2022b). From pedagogical toward technological pedagogical content knowledge frameworks and their effectiveness in teaching mathematics: A mapping review. *F1000Research*, 11, 1029. https://doi.org/10.12688/f1000research.125073.1

[11] Habiyaremye, H. T., Ntivuguruzwa, C., & Ntawiha, P. (2023). Rwandan teacher training college's mathematics teachers' pedagogical content knowledge for teaching: assessment toward competency-based curriculum. *Frontiers in Education*, 8 (November). https://doi.org/10.3389/feduc.2023.1214396

[12] Habiyaremye, H. T., Ntivuguruzwa, C., & Ntawiha, P. (2025). Bridging Knowledge to Practice: Development of Lesson Observation Analysis Protocol (LOAP) Tool. *Parabolum Education*, 1(1), 1–11. https://doi.org/10.58197/tckbpk23

[13] Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, 26(4), 430–511. https://doi.org/10.1080/07370000802177235

[14] Jacinto, E. L., & Jakobsen, A. (2020). Mathematical Knowledge for Teaching: How do Primary Pre-service Teachers in Malawi Understand it? *African Journal of Research in Mathematics, Science and Technology Education*, 24(1), 31–40. https://doi.org/10.1080/18117295.2020.1735673

[15] JICA. (2020). *The Project for Supporting Institutionalizing and Improving Quality of SBI Activity (SIIQS): Project Completion Report* (Number January). https://openjicareport.jica.go.jp/pdf/12327383.pdf

[16] Kleickmann, T., Richter, D., Kunter, M., Elsner, J., Besser, M., Krauss, S., & Baumert, J. (2013). Teachers' Content Knowledge and Pedagogical Content Knowledge: The Role of Structural Differences in Teacher Education. *Journal of Teacher Education*, 64(1), 90–106. https://doi.org/10.1177/0022487112460398

[17] Koponen, M., Asikainen, M. A., Viholainen, A., & Hirvonen, P. E. (2019). Using network analysis methods to investigate how future teachers conceptualize the links between

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

the domains of teacher knowledge. *Teaching and Teacher Education*, 79(January), 137–152. https://doi.org/10.1016/j.tate.2018.12.010

[18] Krauss, S., Baumert, J., & Blum, W. (2008). Secondary mathematics teachers' pedagogical content knowledge and content knowledge: Validation of the COACTIV constructs. *ZDM - International Journal on Mathematics Education*, 40(5), 873–892. https://doi.org/10.1007/s11858-008-0141-9

[19] Ndihokubwayo, K., Byukusenge, C., Byusa, E., Habiyaremye, H. T., Mbonyiryivuze, A., & Mukagihana, J. (2022). Lesson plan analysis protocol (LPAP): A useful tool for researchers and educational evaluators. *Heliyon*, 8(1), e08730. https://doi.org/10.1016/j.heliyon.2022.e08730

[20] Ndihokubwayo, K., Nyirigira, V., Murasira, G., & Munyensanga, P. (2021). Is Competence-Based Curriculum well-monitored? Learning from Rwandan sector education officers. *Rwandan Journal of Education*, 5(1), 82–95. https://www.ajol.info/index.php/rje/article/view/202577

[21] Nkundabakura, P., Nsengimana, T., Nyirahabimana, P., Nkurunziza, J. B., Mukamwambali, C., Dushimimana, J. C., Uwamariya, E., Batamuliza, J., Byukusenge, C., Nsabayezu, E., Twahirwa, J. N., Iyamuremye, A., Mbonyiryivuze, A., Ukobizaba, F., & Ndihokubwayo, K. (2023). Usage of modernized tools and innovative methods in teaching and learning mathematics and sciences: A case of 10 districts in Rwanda. *Education and Information Technologies*, 28(9), 11379–11400. https://doi.org/10.1007/s10639-023-11666-z

[22] Nzaramyimana, E., & Umugiraneza, O. (2023). Investigating teachers' technological pedagogical content knowledge in teaching mathematics in Rwanda secondary schools. *Education and Information Technologies*. https://doi.org/10.1007/s10639-023-12403-2

[23] REB. (2018). Training Module for TTC Tutors, Student-Teacher Leavers and Teachers from Demonstration Schools on the Competence-Based Curriculum. *Ministry of Education*.

[24] REB. (2019). The national teacher CPD framework. *In The Republic of Rwanda*. https://www.eenet.org.uk/resources/docs/THE_NATIONAL_TEACHER_CPD_FRAMEWORK-Booklet-B5Size.pdf

[25] REB. (2020). MATHEMATICS SYLLABUS FOR TTCs OPTION: SCIENCE AND MATHEMATICS EDUCATION YEAR 1, 2 & 3. *Rwanda Basic Education Board*. https://elearning.reb.rw/pluginfile.php/15486/mod_resource/content/1/Mathematics Syllabus_SME.pdf

[26] Santos, J. M., & Castro, R. D. R. (2021). Technological Pedagogical content knowledge (TPACK) in action: Application of learning in the classroom by pre-service teachers (PST). *Social Sciences and Humanities Open*, 3(1), 100110. https://doi.org/10.1016/j.ssaho.2021.100110

[27] Scheiner, T., Buchholtz, N., & Kaiser, G. (2023). Mathematical knowledge for teaching and mathematics didactic knowledge: a comparative study. *Journal of Mathematics Teacher Education*, 0123456789. https://doi.org/10.1007/s10857-023-09598-z

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[28] Schmidt, D. A., Baran, E., Thompson, A. D., Mishra, P., Koehler, M. J., & Shin, T. S. (2014). Technological Pedagogical Content Knowledge (TPACK): The Development and Validation of an Assessment Instrument for Preservice Teachers. *Journal of Research on Technology in Education*, 42(2), 2531p.

[29] Serin, H. (2023). Teaching Mathematics: Strategies for Improved Mathematical Performance. *International Journal of Social Sciences & Educational Studies,* 10(3). https://doi.org/10.23918/ijsses.v10i3p146

[30] Sugawara, E., & Nikaido, H. (2014). Properties of AdeABC and AdeIJK efflux systems of Acinetobacter baumannii compared with those of the AcrAB-TolC system of Escherichia coli. *Antimicrobial Agents and Chemotherapy,* 58(12), 7250–7257. https://doi.org/10.1128/AAC.03728-14

[31] Tabaro, C. (2018). The Rwandan Secondary School Competence-Based Curriculum: Knowledge, Skills And Attitudes To Incorporate In The University Of Rwanda-College Of Education Programs To Align Them With The Current Curriculum. *International Journal of Education and Practice*, 6(2), 64–75. https://doi.org/10.18488/journal.61.2018.62.64.75

[32] Ukobizaba, F., & Byukusenge, C. (2023). Investigating Effective Formative Assessment Applications to Enhance Students' Performance and Interest in Learning Mathematics. *Journal of Classroom Practices*, 2(2), 1–11. https://doi.org/10.58197/prbl/zauj4780

[33] Uwurukundo, M. S., & Tusiime, M. (2022). Enhancing Students' Attitudes in Learning 3-Dimension Geometry using GeoGebra. *International Journal of Learning, Teaching and Educational Research*, 21(6), 286–303. https://doi.org/https://doi.org/10.26803/ijlter.21.6.17

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Teachers' Conceptions of Critical Thinking in Mathematics Teaching

Duhwi Indartiningsih, Tatag Yuli Eko Siswono*, Wiryanto, Suryanti

State University of Surabaya, Indonesia

duhwi.22011@mhs.unesa.ac.id, *tatagsiswono@unesa.ac.id, wiryanto@unesa.ac.id, suryanti@unesa.ac.id

Abstract: The aim of this research is to describe teachers' conceptions of critical thinking in mathematics teaching. It also aims to evaluate critical thinking conceptions of teachers with high, moderate, and low conceptions. The research method used was Explanatory Sequential Mixed Methods with quantitative data collected sequentially through questionnaires distributed to 200 primary school mathematics teachers in in Sidoarjo District, East Java, Indonesia. Furthermore, qualitative data were collected through interviews and observations in the high, medium, and low categories. The results showed that teachers with high critical thinking conceptions had appropriate conceptions and were better able to design mathematics instruction. Teachers with medium and low critical thinking skills only had appropriate conceptions of the indicators, which makes them unable to design mathematics teaching activities.

Keywords: Teachers' Conceptions, Critical Thinking, Mathematical Critical Thinking, Mathematics Teaching

INTRODUCTION

Critical thinking is an indispensable skill in the 21st century (Tang et al., 2020; Toheri et al., 2020) because it improves the quality of thought (Prado-Arenas et al., 2022). This skill need to be incorporated into learning due to the relation with daily activities, which can affect the progress level (Nurdiana et al., 2023). According to Julianto et al. (2023), the skill also triggers self-actualization procedures in social, educational, and work life. An individual who possess this skill is confident that the decisions made were properly analyzed (Gass & Seiter, 2019). Therefore, students must utilize critical thinking at all levels of education, particularly for all elementary school. Some experts state that the skill is focused on resolving inferences on what to accept or execute (Ennis, 2015). The skill is a thought process aimed to solve problems, interpret meaning, and obtain proofs (Facione, 2015). It also refers to the capacity to assess and interpret a particularity by searching for relevant data or evidence before deciding to address the problems, assess circumstances, and take appropriate action (Hunaepi et al., 2020). This skill is adopted by developing or practicing

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

ways to solve problems (Sutama et al., 2022), including exploring existing or relevant information, as well as conducting analysis, synthesis, evaluations, and drawing inferences (Ulfiana et al., 2019). Critical thinking can also be adopted through mathematics teaching training. Previous research reported that mathematics was a science that evaluated critical reasoning, requiring careful thought and observation procedures due to its usefulness in daily activities (Munawaroh et al., 2018).

In Indonesia, the students' mathematics learning outcomes are currently unsatisfactory. Based on several research studies, students exhibit poor critical thinking skills in mathematics learning at all levels of education, including elementary school (Sarwanto et al., 2020). These students also experience many obstacles in solving related problems, resulting in unsatisfactory learning outcomes. This simply proves that critical thinking skill have not been properly adopted (Delfia et al., 2020; Maulidiya & Nurlaelah, 2019). The skill is lacking in elementary school students due to teachers' inability to understand the concept (Nurbaeti et al., 2019). It is one of the objectives achieved in learning mathematics, hence the need for teachers to consider the use of learning conceptions to obtain the desired objectives (Scristia et al., 2018).

Regarding this description, Thompson (1992) stated that teachers' perception of desired objectives play a role in the adoption of appropriate classroom activities, as well as preferred instructional methods, legitimate mathematical procedures, and acceptable outcomes (Rodriguez-Muniz et al., 2022). Critical thinking can be adopted in mathematics teaching by designing learning activities that direct students according to the stipulated objectives (Lusiana et al., 2024). In addition, teachers' conceptions of the skill in mathematics teaching are based on its understanding, including mathematical critical thinking relevant indicators, and teaching design alongside related learning activities.

The lack of conception by teachers, including associated learning activities, leads to difficulty in designing lessons that can develop students' critical thinking skills. Prior research reports the need to understand and practice teaching designs that develop critical thinking (Gunawan et al., 2020). Moreover, adopted teaching design requires specific learning strategies, basic questioning tactics, the ability to draw inferences, conduct analysis, synthesis, and evaluation (Acharya, 2018). Teachers' conceptions are relevant attributes that must be considered in teaching. In respect to critical thinking, it is essential to design mathematics teaching that fosters the skill through interactive and student-centered teaching methods (Moodley & Chetty, 2024).

Recent research focused on investigating secondary school teachers' conceptions of critical thinking. Data was collected by distributing questionnaires to 77 teachers, and the results showed the perception of an advanced reasoning competency, including reflective and careful thoughts alongside other mind related habits, processes, or strategies, such as curiosity, exploration, information gathering, and intellectual caution (Ma et al., 2023). The skill focused on the need for teachers to develop critical thinking with training programs used to improve teachers understanding and practice in the classroom (Oner & Aggul, 2022).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Based on the description above, several research studies are focused on elementary school teachers' conceptions of critical thinking in mathematics teaching, leading to the formulation of the following questions:

1. What is teacher's conception of critical thinking in mathematics teaching?
2. Why are teachers with high conceptions able to design mathematics teaching that fosters critical thinking compared to those with low or moderate conceptions?

The current research focuses on elementary school teachers of mathematics, investigating in depth three teachers who had high, medium, and low conceptions in designing related teaching activities that can develop critical thinking. The critical thinking conceptions are used to determine the methods used by teachers to design mathematics instruction and facilitate future practice.

LITERATURE REVIEW

Teacher Conception

Conception refers to personal understanding or interpretation of a concept, resulting in differing experience. This is outlined by Philips' perception of conception as a mental framework consisting of beliefs, meanings, concepts, propositions, rules, mental representations, and preferences (Rodriguez-Muniz et al., 2022). Conceptions are conscious and unconscious cognitive and affective beliefs that indicate the abilities and experiences that one has. (Amirali, 2010). According to Thompson (1992), conceptions include considering the desired objectives of mathematics teaching, roles played by teachers, and students, appropriate classroom activities, methods, and preferred instructional emphases, as well as valid procedures and acceptable results (Rodriguez-Muniz et al., 2022).

Based on previous research, teacher's conception refers to the understanding or interpretation of a concept. In respect to mathematics teaching, it is closely related to the understanding of critical thinking, mathematical critical thinking, indicators, and teaching design as shown in Figure 1.

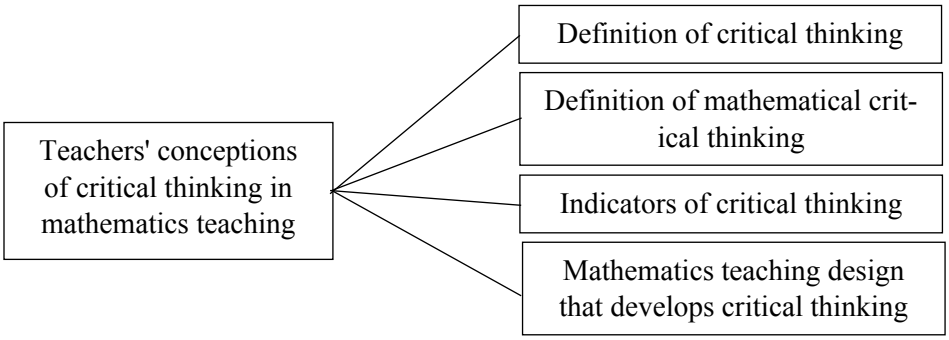


Figure 1: Teachers' conceptions of critical thinking in mathematics teaching

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Critical Thinking and Critical Thinking Indicators

Critical thinking is a reflective thought that focuses on making decisions about what to believe or execute (Ennis, 2016). This thought process aims to solve problems, interpret the meaning of a circumstance, and prove drawn inferences (Facione, 2015). Previous research states that it also referred to as the capacity to accurately assess and explain phenomena by collecting related data or evidence before making choices, solving problems, assessing circumstances, and taking action (Hunaepi et al., 2020). Critical thinking is a clear and structured mental activity that can involve processes such as solving problems, analyzing existing assumptions, and making decisions based on evaluations (Fitriani et al., 2019; Munawaroh et al., 2018). It is a metacognitive process that includes the ability to analyze, synthesize, evaluate, and summarize ideas or solutions effectively to solve problems (Fitriana et al., 2023). Critical thinking is a skill that analyzes, evaluates, assesses, and justifies misinformation (Saputra et al., 2019). This includes the ability to examine assumptions, distinguish hidden values, assess, evaluate evidence, and draw inferences (Bayuningsih et al., 2018).

The possession of critical thinking skill can be viewed based on the following indicators: interpreting, analyzing, evaluating, concluding, explaining, and self-regulating (Facione, 2015). According to Ennis (2016), these indicators include basic clarification, giving reasons for a decision, drawing inferences, and further clarification. Critical thinking indicators consisted of clarification, judgment, conclusion, and strategy (Corich et al., 2007). Facione (2015) reported that these indicators were selected because individuals who think critically must be able to have self-regulation and re-examine the resolution of problem, ensuring appropriate decisions were taken. Self-regulation also enables the ability to think strongly and critically (Facione, 2015). An individual with great self-regulation is grouped in the high critical thinking category (Bayuningsih et al., 2018). Considering the definition by experts, it is inferred that critical thinking is a thought process in making reasoned decisions through analysis, evaluation, and conclusion based on existing and relevant evidence. The indicators include interpreting, analyzing, evaluating, concluding, explaining, and self-regulating.

Mathematical Critical Thinking

Mathematical critical thinking is a thought process in making reasoned decisions in solving related problems through analyzing, synthesizing and evaluating skills based on existing and relevant evidences (Saragih & Zuhri, 2019). Abilities and dispositions combined with knowledge, mathematical reasoning skills, and cognitive strategies to solve relevant problems (Nurbaeti et al., 2019). Mathematical critical thinking refers to the capability to solve related problems with the ability to understand, recall, distinguish, analyze, provide reasons, reflect, interpret, search for relationships, evaluate, and make temporary conjectures (Sahidun et al., 2023). This skill was used to analyze, evaluate or assess, and summarize opinions or solutions appropriately to solve a problem (Saputra et al., 2019). Based on the diverse definitions from experts, it was inferred that math-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

ematical critical thinking is a reasoned thought process in solving related problems by analyzing, evaluating, and drawing inferences on existing and relevant evidence.

Mathematics Teaching Design that Develops Critical Thinking

Critical thinking can be developed early, starting from elementary school (Siswono, 2016). In mathematics, effective teaching practices and student engagement play a significant role (Yanuari & Turmudi, 2023). This skill can also be absorbed by giving problems that develop one's ability to analyze and evaluate (O'Reilly et al., 2022). Teachers should give non-routine problems and allow students to argue or express respective opinions, analyze, or evaluate it, while drawing inferences (Willingham, 2019). The provided questions in discussions and role play in problem solving also aid in developing this skill (Toheri et al., 2020).

Teachers use learning methods or models in designing math lessons in accordance with critical thinking indicators. Previous research states that teaching activities using Realistic Mathematics Education (Cahyaningsih & Nahdi, 2021; Hikayat et al., 2020), scientific methods (Nurhikmayati & Jatisunda, 2018; Sukarna & Iman, 2019), problem-based (Aini et al., 2019; Maulidiya & Nurlaelah, 2019; Tanjung et al., 2020), Cooperative (Afrilianto et al., 2022; Siswono et al., 2019), and Discovery learning (Chusni et al., 2020, 2022; Putri et al., 2020) aid in the development process. Teaching procedure is designed to train certain abilities in line with critical thinking indicators. These include interpretation, analysis, evaluation, explanatory conclusions, and self-regulation (Seventika et al., 2018). The following steps based on the indicators are adopted in teaching mathematics.

A. Interpretation refers to the ability to understand and explain the meaning of information or an event. Teachers propose math problems at the beginning of learning activities, guiding students to identify and organize relevant information, as well as ensuring these are understood by correctly writing the known information and question asked.
B. Analysis is the ability to identify the relationship between questions, statements, concepts, descriptions, or other representations intended to express beliefs, judgments, experiences, reasons, information, or opinions. Teachers direct students with guided questions to find the relationship between statements and concepts presented, ensuring problem solving strategies are determined.
C. Evaluation is described as the ability to assess the credibility of questions or other representations in the form of reports or descriptions of perceptions, experiences, situations, judgments, beliefs, or opinions. This includes interpreting the logical strength of referential correlations or other intended representations. Teachers guide students to make judgments regarding the value of an idea, creation, or method used in solving problems.
D. Inference refers to the ability to draw logical conclusions, make assumptions, and generate hypotheses, while considering relevant information, consequences, situations, questions, and other representations. The students are guided to make reasonable inferences by considering relevant information in problem solving.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

E. Explanation is the capability to articulate results, as well as reasoning clearly and concisely. Students are facilitated to explain the results, procedures, and arguments related to the solution of the problem.
F. Self-regulation depends on the ability to monitor one's cognitive activities and elements used in problem solving, particularly analysis and evaluation. Teachers ask students to re-examine the working process and engage in self-reflection.

METHODS

This research adopted the Explanatory Sequential Mixed Method, a combination of quantitative and qualitative methods to provide a comprehensive understanding of complex phenomena. The methods consist of two phases namely the collection of quantitative, and qualitative data to describe quantitative results (Sofiyana et al., 2022). In this regard, quantitative data collection was conducted by distributing critical thinking conception questionnaires to elementary school teachers in Sidoarjo district, East Java, Indonesia. Meanwhile, Sidoarjo district was selected as the research sample because it is one of the most populous districts in East Java that is characterized by a rich and diverse culture. The region is closer to the city center of Surabaya, the capital of East Java province, a large area with a diverse population in terms of education and economic factors. The criteria for selecting the research sample were teachers with three years of experience teaching mathematics in grades III, IV, V, or VI. The number of elementary schools in Sidoarjo district amount to 585 schools. Of the 585 schools, 200 teachers met the sample selection criteria. Quantitative data were collected by distributing questionnaires consisting of 14 statements prepared in Bahasa, Indonesia. These were based on a grid related to conception of critical thinking, mathematical critical thinking, indicators, and teaching design. The questionnaire was assessed using a Likert scale, namely strongly agree, agree, disagree, and strongly disagree, as shown in Table 1.

Aspect	Description	Statement	
		Positive	Negative
Conception of critical thinking	Statements related to teachers' conceptions of critical thinking.	3	2
Conception of mathematical critical thinking	Statements related to teachers' conceptions of mathematical critical thinking.	2	1
Conception of critical thinking indicators	Statements related to teachers' conceptions of critical thinking indicators.	1	1
Conception of mathematics teaching design that develops critical thinking	Statements related to teachers' conceptions of teaching design that can develop critical thinking.	2	2

Table 1: Questionnaire grid for critical thinking conceptions

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Each positive and negative statement show an appropriate and inappropriate conception, respectively. For positive statements, a larger scale shows a more appropriate conception of critical thinking in mathematics teaching. Meanwhile, for negative statements, a larger scale implies an inappropriate or incompatible conception. For example, a participant's answer agreeing with a negative statement implies a teacher has an incompatible conception of critical thinking in mathematics teaching. The critical thinking conception questionnaire was validated by two experts in mathematics education, a language expert, and two mathematics teachers.

The responses of the 200 teachers were converted to scores, for positive statements, the answers "strongly agree", "agree", "disagree", and "strongly disagree" were converted to the following scores 4, 3, 2, and 1. For negative statements, the answers "strongly agree", "agree", "disagree", and "strongly disagree" were converted to scores 1, 2, 3, and 4. The scores of each participant were then summed up, with a typical example of the calculated critical thinking questionnaire response score shown in Table 2.

Respondent 1	Response	Score
Statement 1 (Positive statement)	Strongly Agree	4
Statement 2 (Positive statement)	Agree	3
Statement 3 (Negative statement)	Disagree	3
etc	....	...
Total score		35

Table 2: Example of response score calculation

The formula for determining the percentage value of the questionnaire statements answered by respondents is shown in Figure 2 as follows:

$$\text{Percentage value} = \frac{the\ score\ obtained}{maximum\ score} \text{ x } 100\%$$

Figure 2: Percentage value formula

The percentage values obtained were then categorized for critical thinking conceptions as in Table 3.

Percentage value interval (%)	Category
$81{,}25 < x \leq 100$	Very high
$71{,}50 < x \leq 81{,}25$	High
$62{,}50 < x \leq 71{,}50$	Medium
$43{,}75 < x \leq 62{,}50$	Low
$0 < x \leq 43{,}75$	Very low

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Table 3: Category of the percentage value of the results of critical thinking conception questionnaire

The subjects were selected according to the research objectives (Purposive sampling), namely teachers with very high or high critical thinking conceptions, another from the medium, and very low or low categories. Qualitative data were collected through semi-structured interviews and observation of lesson plans prepared by the three selected teachers. Additionally, to reduce the possibility of information sharing, each teacher was interviewed separately. The initial step was the confirmation of willingness to be interviewed. The interviews focused on exploring teachers' conceptions of critical thinking in mathematics teaching, alongside respective instructional designs. The results of the interview were transcribed and analyzed to show each teacher's conception of critical thinking and design in mathematics teaching.

RESULTS

Quantitative Data Analysis

The results of the questionnaire filled out by 200 teachers, showed 37 (18.5%), 159 (79.5%), and 4 (2%) had critical thinking conceptions in teaching mathematics with low categories, medium, and high categories, respectively. This implied that out of 200 teachers who filled out the questionnaire, 4 had appropriate critical thinking conceptions, while the remaining 196 had inappropriate conceptions in teaching mathematics. Based on the results obtained, one teacher from the high, medium, and low categories were selected to be interviewed regarding conception of critical thinking and design of mathematics teaching that enables the development of this skill.

Qualitative Data Analysis

Semi-structured interviews were conducted with three teachers selected as research subjects to collect qualitative data on conceptions of critical thinking, mathematical critical thinking, indicators, and mathematics teaching designs. More importantly, teaching design made by the three teachers to investigate learning activities that exhibited the development of critical thinking was observed. The results of the interviews were transcribed and coded R, S1, S2 and S3 for the research, teachers with low, medium, and high conceptions of critical thinking, respectively.

I. Conceptions of Critical Thinking

R: What do you mean by critical thinking?

S1: Critical thinking is a thought process associated with solving problems. An individual who can solve problems in several ways.

S2: Critical thinking refers to the ability to solve problems quickly using various methods.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

S3: Critical thinking describes a meaningful thought process applied when solving problems based on accountable answers.

Building upon the description above, S1, S2, and S3 had different conceptions of critical thinking. S1 and S2 had similar conception related to critical thinking, namely the ability to solve problems in various ways. According to S3, critical thinking is a meaningful thought process applied when solving problems based on available evidence and the accountability of the answer.

II. Conceptions of Mathematical Critical Thinking

R: What is meant by mathematical critical thinking?
S1: The ability to answer questions or solve mathematical problems quickly.
S2: The ability to solve mathematical problems using different methods.
S3: Thinking reasonably when solving mathematical problems and using evidence to determine the appropriate answer.

Conceptions of S1, S2, and S3 differ in respect to mathematical critical thinking. According to S1, mathematical critical thinking refers to the ability to solve related problems quickly. Regarding this perspective, S2 described it as the ability to solve problems by using different methods. In this context, S3 stated that mathematical critical thinking is the process of applying meaningful reason and evidence to account for the right answer.

III. Conception of Critical Thinking Indicators

R: Describe the indicators of critical thinking?
S1: An individual who can give an explanation of the problem solution, understood and recognized by others.
S2: One who is able to convey reasons related to decision making from problem solving, clearly and appropriately.
S3: A person who is able to correctly identify answers based on facts, as well as evaluate which solutions were appropriate, an indication of critical thinking. In addition, critical people can also explain the reasons behind the selected answers.

Certain differences were detected in conceptions of S1, S2, and S3 regarding critical thinking indicators. According to S1 and S2, the indicators refer to the ability to explain the reasons for decision making related to problem solving. S3 defined it as being able to determine the correct answer based on evidence obtained from the analysis precisely conducted, including explaining the reasons behind the decisions taken.

R: Based on critical thinking indicators, are you a critical thinker? Explain the reason for your answer?
S1: I am critical thinker because I always prioritize quick thinking when solving problems.
S2: I am not critical thinker because I solve problems using the same method.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

S3: I am critical thinker because I am able to determine which solution is appropriate for the problem based on existing evidence. Prior to solving the problem, I understand it, while deciding on the right solution.

In line with the description above, it was reported that S1 and S3 were believed to be critical thinkers. According to S1, being a critical thinker entails the prioritization of thinking quickly when solving problems. S3 stated that critical thinkers solve problems by determining the right solution based on existing evidence. In addition, they believe problems are solved by understanding and thinking of the right way to address the issue. S2 considers himself not a critical thinker because he uses the same method when solving problems.

R: What steps were adopted when working on math problems?
S1: I try to understand the problem, then find information on Google...including whether there any similar problems? ...this resulted in the modelling of the solution.
S2: I read and understood the problem then tried to solving it by adopting commonly used methods.
S3: I read and understood the problem, before writing the relevant information. I try to think of a way to answer it, and search for the solution.

The problem-solving steps adopted by S1, S2, and S3 were different. S1 solved the problem by searching for examples of existing or common solutions, then modifying the procedure. However, S2 solved the problem by understanding it and adopting commonly used methods. S3 solved the problem by understanding and writing relevant information, as well as trying to think of a solution or answer.

IV. Mathematics Teaching Design that Develops Critical Thinking

R: What are the learning activities associated with math teaching that can develop critical thinking?
S1: I give practical problems to students to solve and these were discussed together. The frequent practicing of math problems, helped develop students' critical thinking skills.
S2: Giving mathematical problems and providing guided triggering questions, that would enable students understand it and find solutions.
S3: Each group was given a problem and then asked to engage in discussions when solving the question. The students were guided by reading the problem together to understand it, followed by compiling relevant information regarding what is known and asked. This also included finding a way to solve the problem ... with what formula ... etc. The students were also asked to write the inferences drawn from the answer.

S1 stated that learning activities associated with developing critical thinking include the practice of mathematical problems and engaging in discussions. S2 provided mathematical problems and triggering questions that helped in understanding and finding a solution. S3 divided the students

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

into groups and each were given a math problem, which was discussed together. The students were further guided by giving trigger questions that enabled an understanding of the problem. This also included writing relevant information, the formula used, and inferences drawn from the answers.

R: What teaching models or methods were applied to design mathematics teaching that developed critical thinking? Explain the reasons for your response?
S1: Inquiry method. The students were invited to find formulas or answers to math problems guided by teacher. This included asking questions that invited students to think about the stages of problem solving.
S2: A scientific method was adopted by giving students mathematical problems to practice.
S3: The problem-based model was used to design learning that develops critical thinking. Mathematical problems were given and students motivated to discuss the questions in groups.
R: Have you ever designed a lesson using this learning method or model?
S1: Never. I use existing teaching plans or copy those on Google.
S2: I have made one.
S3: I have also made teaching plans.

S1's answer was related to the learning model used to design lessons that develop critical thinking, namely the inquiry method. However, S1 never made a mathematics teaching design using an inquiry method. S2 stated that the use of a scientific method, as well as giving mathematical problems, develops critical thinking. Building upon the discussion, S2 prepared a mathematics teaching design using a scientific method. S3 stated that problem-based learning can develop critical thinking by starting with problems students needed to solve. Mathematics teaching design was made using a problem-based learning model.

R: Do you have any problems in designing mathematics teaching activities that develops critical thinking?
S1: I do not know how to make the designs, occasionally I search for it on Google. But I do not know if its right or wrong.
S2: I do not understand learning activities that develop critical thinking.
S3: I am still confused about the steps adopted, despite having made the designs, I do not know if it is right.

Similar obstacles were experienced by S1, S2, and S3 in designing mathematics teaching that enabled the development of critical thinking skills. S1 did not understand how to design lessons and tried to find examples on Google. However, S1 could not decipher the appropriateness of the examples on Google. S2 lacked an understanding of learning activities that developed critical thinking. Considering this perspective, S3 made a mathematics teaching design that developed critical thinking but doubts the accuracy of the steps adopted.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

S1, S2, and S3 were assigned with the task of designing a mathematics teaching plan that could develop critical thinking.

R: Make a math lesson plan that develops critical thinking!

The following is a diagram of the math lesson plan made by S1 shown in Figure 3, using a project-based learning model with material on the characteristics of flat shapes.

Material: Characteristics of flat shapes with project-based learning model.
1. Understand the main problem
   a. The teacher divided students into three groups, with each consisting of three to four students.
   b. Students were shown regularly used objects with flat shapes such as books, triangular rulers and origami paper.
   c. The teacher asks, what shape is this origami paper? This included the shape of other objects.
2. Planning the project
   a. Students were shown a video by the teacher, regarding the characteristics of flat shapes.
   b. Discussion activities were carried out in each group, with students directed to complete the tasks on the worksheet.
   c. Instructions or signs was listened to in preparing products, namely making various flat shapes from folding paper.
3. Presenting project results
   Students present the results of the project, namely the products they have made.
4. Evaluating and reflecting on the process and results of the project
   Students from other groups give responses to the group that is presenting.

Figure 3: S1's mathematics teaching design

S1 designed the learning process into four phases. First, the teacher shows the students pictures of different objects, and asks them to identify the shapes, grouping those that are similar. The teacher asks questions related to the characteristics of the surface shapes. In the second phase of project planning, a video is played regarding the characteristics of flat surfaces, as well as instructing students to plan the task of making various flat shapes using origami paper and arranging it in a creative manner. The third phase focuses on the provision of opportunities for students to present the results of respective projects. The fourth phase focuses on evaluating and reflecting on the project results, with the teacher offering each group an opportunity to review the results of others.

The following is a diagram of the mathematics teaching design made by S2 (Figure 4). Based on this description, teaching design was realized using scientific method for numbers 1001 – 5000.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

1. The teacher showed the number flashcards 1001 to 5000.
2. Learners mentioned the numbers according to the cards shown by the teacher.
3. The teacher distributed ice cream sticks and student worksheets to each group.
4. Learners were asked to count and write the number symbols.
5. While the learners were working on the LKPD, the teacher assessed the observation sheet of the students.
6. Learners presented the executed tasks.
7. This included working independently on numbers 1001 to 5000.
8. While the work was ongoing, the teacher assisted mostly those in the low groups.

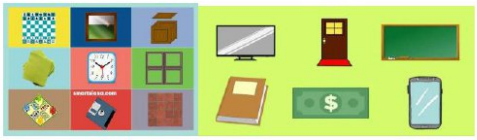
Figure 4: S2's mathematics teaching design

The learning activities designed by S2 were in accordance with the scientific method of observing, questioning, reasoning, attempting, and communicating. The teacher shows the number cards and students read the written number symbols. Meanwhile, students are grouped and assigned with the task of counting the number of ice cream sticks and writing the symbols on the available worksheets.

The following is a diagram of the mathematics teaching design that develops critical thinking made by S3 (Figure 5). In addition, a problem-based learning model focused on the characteristics of shapes, particularly rectangles and squares, was adopted.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Learning activities with problem-based learning model
Phase 1: Orientation of learners concerning the problem
a. Students observe the pictures presented by the teacher.



b. The teacher inquires about the shapes of the object surfaces in the picture. The students were assigned to group the surfaces with similar shapes.
c. The teacher inquired about names of flat shapes similar to the object surfaces in the picture.

Phase 2: Organizing students to learn
a. The teacher divided students into two groups
b. Students joined respective groups
c. The guidance of the teacher led to the discussion of the differences between square and rectangular shapes.

Phase 3: Guiding individual and group investigations
a. Each groups started to investigate the characteristics of squares and rectangles, writing it on the worksheet.

No	Shape	Characteristics
1		
2		

Phase 4: Develop and present work
a. After the discussion on the properties of the square and rectangle, students continue to conclude and write on the worksheet.
b. Each group presents the conclusion of the properties of the square and rectangle.

Phase 5: Analyze and evaluate the problem solving process
a. Other groups evaluate the presentation results and the teacher provides confirmation regarding the results of all groups' presentations.
b. Students discuss the differences and similarities of the square and the rectangle.

Figure 5: S3's mathematics teaching design

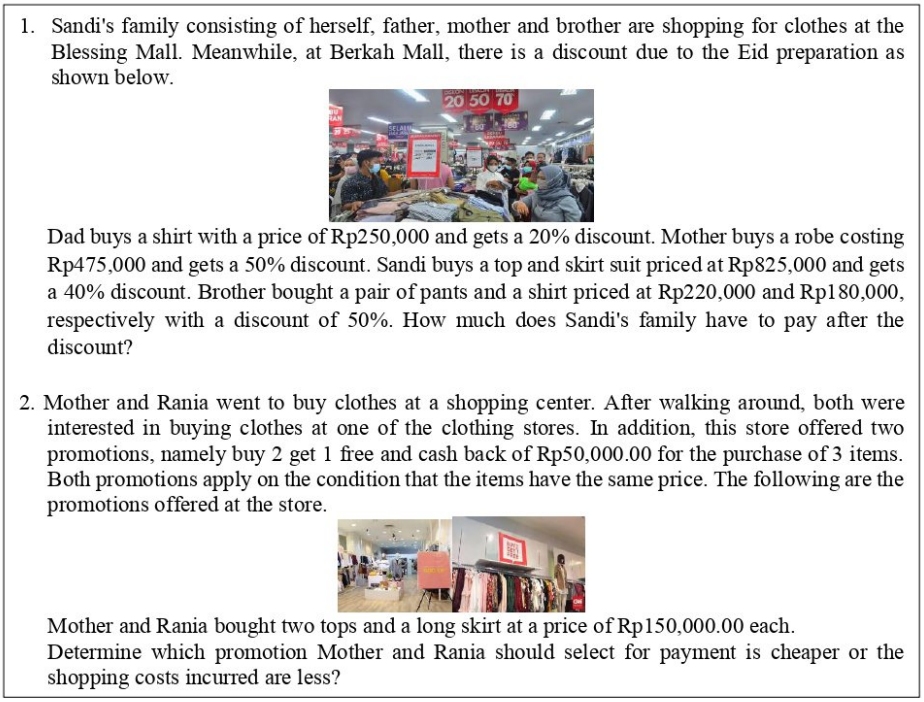
The math teaching design made by S3 is divided into five phases. In the first phase, the teacher provides pictures of objects with rectangular and square surfaces, assigning students to group it based on similar surface shapes. In the second phase, groups are formed to complete the task of observing the surfaces of rectangular and square objects. However, in the third phase, the teacher guides and provides the groups with worksheets to write relevant information related to the properties of rectangles and squares based on its sides and angles. For the fourth phase, students write respective results and draw inferences from the difference between a rectangle and a square. In the fifth phase, each group presents the results of analyzing and evaluating the properties of rectangles and squares from the surrounding objects and images.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

R: Describe the characteristics of mathematical problems or issues that can develop critical thinking?
S1: Problems can be solved using diverse formulas, particularly those with a high level of difficulty.
S2: Problems that can train the ability to analyze and evaluate.
S3: Higher order thinking problems.

S1 stated mathematical problems that aid in the development of critical thinking are those with high level of difficulty and can be solved in various ways, using more than one formula. According to S2 and S3, high-level thinking mathematical problems enable the development of critical thinking.

R: Look carefully at the questions in Figure 6 and choose the one that can develop critical thinking! Give reasons why you selected a particular problem!


1. Sandi's family consisting of herself, father, mother and brother are shopping for clothes at the Blessing Mall. Meanwhile, at Berkah Mall, there is a discount due to the Eid preparation as shown below.

Dad buys a shirt with a price of Rp250,000 and gets a 20% discount. Mother buys a robe costing Rp475,000 and gets a 50% discount. Sandi buys a top and skirt suit priced at Rp825,000 and gets a 40% discount. Brother bought a pair of pants and a shirt priced at Rp220,000 and Rp180,000, respectively with a discount of 50%. How much does Sandi's family have to pay after the discount?

2. Mother and Rania went to buy clothes at a shopping center. After walking around, both were interested in buying clothes at one of the clothing stores. In addition, this store offered two promotions, namely buy 2 get 1 free and cash back of Rp50,000.00 for the purchase of 3 items. Both promotions apply on the condition that the items have the same price. The following are the promotions offered at the store.

Mother and Rania bought two tops and a long skirt at a price of Rp150,000.00 each.
Determine which promotion Mother and Rania should select for payment is cheaper or the shopping costs incurred are less?

Figure 6: Math problems

S1: Problem number 1 due to the discount. The students were expected to calculate each of the question to determine the shopping payment, and this was quite difficult. This led to the need for students to think harder to solve the problem.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

S2: Problem number 2 because students must determine the shopping promotion that is cheaper. Therefore, students must assess which of the two promotions was the most profitable and cheaper.
S3: In my opinion, critical thinking question is number 2 because the problem required determining which promo should be selected for cheaper shopping payments. First, calculate the purchase of clothes with both promos individually. Second, compare which is cheaper and finally determine the promo to select.

S1 selected problem number 1 as a mathematical problem that can develop critical thinking. This was because problem number 1 included many steps and must be calculated individually, which is quite difficult. Meanwhile, S2 and S3 selected problem number 2 as a mathematical problem capable of developing critical thinking because students determine which promotion to take to enable cheaper shopping payments. This also implies evaluating and drawing inferences about the selected promotion.

R: What are the difficulties faced by teachers in terms of designing mathematics teaching that develops critical thinking? What lesson plan should be prepared to develops students' critical thinking?
S1: I do not understand critical thinking, and I am confused about the steps that should be adopted during teaching. Based on the result, teachers need to possess this skill, when preparing lesson ideas and challenges that foster critical thinking. I have never participated in training and do not read enough books thereby lacking the relevant knowledge.
S2: I do not know the steps associated with teaching math that develop critical thinking. Moreover, there is need for teachers to possess adequate understanding, of how to impact the learning activities related to the development of students' critical thinking. I have never attended training for teachers, particularly regarding critical thinking, so I search for information on YouTube, while reading several books.
S3: I still do not understand the steps related to teaching math that develops critical thinking. Teachers must have proper knowledge of the skill, because it cannot be taught if lacking. Moreover, teachers also need to be prepared, including possessing fundamental thinking skills. I had to attend teacher trainings and reading books to improve personal knowledge and teaching skills.

The answers provided by the three teachers were related to the readiness in teaching mathematics to develops students' critical thinking. S2 and S3 stated the relevance for teachers to possess a strong understanding of the skills and related steps. According to S1, teachers must also have the ability to think critically before impacting students. S3 improved personal knowledge and teaching skills by attending training sessions and reading books. S2 had never attended training but sought out information from YouTube and reading books. In line with this discussion, S1 read less books and had never attended training.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

DISCUSSION

I. Conceptions of Critical Thinking

S1, S2, and S3 had different conceptions of critical thinking, based on their respective responses. According to S3, critical thinking is a thought process that enables solving problems and results in accountable answers. This conception was in accordance with the statements made by several experts. Facione (2015) reports that critical thinking focuses on proving a point, meaningful interpretation, and solving problems. It is also defined as a reflective thinking process that focuses on making decisions about beliefs or activities executed (Ennis, 2015). However, S1 and S2 stated that critical thinking is the ability to solve problems in various ways using different methods. This conception of critical thinking is inappropriate due to the direction towards the notion of creative thinking, namely a person's ability to solve problems in more than one way or solution (Nasution et al., 2021; Wijayanti et al., 2021).

II. Conceptions of Mathematical Critical Thinking

Differing conceptions of mathematical critical thinking was observed based on responses of S1, S2, and S3. According to S3, this factor refers to thinking reasonably when solving mathematical problems, including using evidence to determine the right answer. This conception is appropriate and in line with the result that mathematical critical thinking is a thought process in making reasoned decisions when solving related problems through analyzing, synthesizing, and evaluating skills based on existing and relevant evidence (Saragih & Zuhri, 2019). However, S1's conception is not appropriate because solving problems quickly does not necessarily mean that the solution is precise and accurate. The problem-solving process requires mathematical reasoning and solution strategies (Nurbaeti et al., 2019), as well as higher-order thinking procedures that use the capacity to investigate, assess, and make provisional predictions of the solution (Sahidun et al., 2023). S2's conception is also not appropriate due to the similarity with mathematical creative thinking, namely an individual's ability to create new ways of solving problems (Suherman & Vidákovich, 2022).

III. Conception of Critical Thinking Indicators

S1 and S2 reported that one of the indicators of critical thinking is the ability to explain or argue about certain solutions. However, S3 gave a detailed explanation of this factor, as well as the reasons for making decisions, analyzing, and evaluating relevant evidence by determining the appropriate solution or answer to the problem. Inference is drawn after evaluation to decide the final result of the decision made. Critical thinking indicators consist of interpretation, analysis, evaluation, inferences, explanation, and self-regulation (Cruz et al., 2020; Seventika et al., 2018; Tohir et al., 2021). Based on the analysis of S1, S2, and S3 interview answers, S1 and S2 were able to explain one of critical thinking indicators. S3 provided an in-depth explanation of critical thinking indicators.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

S1 is believed to be a critical thinker because of the ability to solve problems quickly. This reasoning does not portray the characteristics of mathematical critical thinkers. Previous research stated that critical thinker can be viewed from the ability to solve problems based on critical thinking indicators (Facione, 2015). S2 is not a critical thinker due to the inability to use methods different from the ways adopted by others. This result is not in accordance with the characteristics of critical thinkers. S3 is believed to be a critical thinker because they solve problems based on understanding the problem and thinking of the right solution depending on existing evidence. S3 is able to conclude accurately and is able to convey strong reasons and evidence as a critical thinker, while the opposite occurs in S1.

Personal beliefs are thinking dispositions necessary for one to become critical thinker (O'Reilly et al., 2022). This is characterized by curiosity, accurate information, trust in reason, open- and fair-minded evaluation, flexibility, honesty in facing personal biases, thoughtfulness during judgments, willingness to reconsider, clear understanding about the problem, and organization in complex matters (Cruz et al., 2020). However, in deciding what to believe or do, a person is assisted by a set of dispositions and critical thinking abilities (Ennis, 2015). S3 is believed to be critical thinker because they showed a similar disposition of explaining the steps related to solving mathematical problems. S1's belief in the ability to think critically tends to affect personal views on teaching mathematics (Kusaeri & Aditomo, 2019), knowledge, as well as filtering all new information to form a particular mindset (Rodriguez-Muniz et al., 2022).

The responses of S1, S2, and S3 regarding the problem-solving steps differed. S1 solves problems by searching for examples of existing or common problem solutions, followed by its modification. However, S2 understands the problem, which is then solved using a common method. For S3, the problem is understood, then important information is written, linking the problem with the answer and drawing inferences. In addition, S3 provided a better explanation of the problem-solving process in respect to critical thinking indicators, namely interpretation, analysis, and inferences.

IV. Mathematics Teaching Design that Develops Critical Thinking

S1, S2, and S3 provided similar responses related to mathematics teaching activities that can develop critical thinking. S1 did not discuss in detail the abilities required by students to develop critical thinking. On the other hand, S2 and S3 provided a detailed explanation of teaching process adopted by teachers, namely the ability to interpret, analyze, and evaluate information derived from problems, which includes drawing inferences. These abilities can be acquired, improved, and refined through instructions with clear objectives (Lorencová et al., 2019). Students are trained to solve mathematical problems by finding relevant information and understanding the question. Teacher should also provide a triggering question to connect the problem and adopted strategy. The problem-solving process presented by S2 and S3 shows the diverse stages of training students to think critically. The ability to analyze, evaluate, and draw appropriate conclusions when solving problems is a component of the critical thinking process (Fitriana et al., 2023). This skill can also be developed through the support of appropriate ideas and learning exercises. An example is

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

the use of learning models and giving related questions (Julianto et al., 2023). Based on the description above, S2 and S3 had a better understanding of teaching activities that help students develop critical thinking. S1 was unable to provide a detailed explanation of teaching activities that develop this skill.

In line with the discussion, S1 reported that teaching activities associated with the inquiry method can develop students' critical thinking in learning mathematics. Students are guided to find answers to math problems with directed questions to prepare or foster critical thinking. This was in accordance with finding that that the adoption of an inquiry method can improve critical thinking skills (Chususiyah et al., 2020; Yasin et al., 2019). S2 stated that teaching activities requiring the use of a scientific method and given mathematical problems develop students' critical thinking. This was in line with the results of research that states the adoption of the steps could impact the skill (Nurhikmayati & Jatisunda, 2018; Sukarna & Iman, 2019). According to S3, the problem-based learning model, in which teachers provide mathematical problems to be solved both individually and in groups, helps students learn and develop critical thinking. The given problems were high-level thinking questions in the field of analysis and evaluation (Tanjung et al., 2020). S3's statement was in line with the results of research that show the effectiveness of problem-based learning in developing students' critical thinking in mathematics (Aini et al., 2019; Amin et al., 2020; Maulidiya & Nurlaelah, 2019; Saputra et al., 2019).

Responses from S1, S2, and S3 concerning the application of learning strategies or models that foster critical thinking differed. S1 had never made a mathematics learning design that developed critical thinking due to lack of knowledge about designs and lack of understanding about the diverse stages fostering critical thinking skills. S2 and S3 had designed mathematics teaching that developed critical thinking but were hesitant or confused about the right steps in designing related teaching activities. Both hoped that training sessions are organized for teachers.

The mathematics teaching design made by S1 used a project-based learning model in planning discoveries that could create critical thinking. The designed learning model was subjected to four phases, namely 1) understanding the main problem, 2) planning the project, 3) presenting the results, as well as 4) evaluating and reflecting on the project process and results. S1 lacked an understanding of teaching stages contained in each phase of model. Project-based learning consists of six phases, such as 1) determining the main question, 2) designing the project, 3) preparing the schedule, 4) observing, 5) testing the results, and 6) assessment. The design made by S1 was not explained in detail in each phase. Moreover, teaching activities that were in line with critical thinking indicators, include interpretation (common in the phase of determining the main problem), analyzation of the project planning, explanation of the presented work results, evaluation of the project, and reflection phases. The teaching design created by S1 is not able to develop critical thinking because it only develops two indicators of critical thinking, namely interpretation and self-regulation.

The mathematics teaching design made by S2 entailed the use of a scientific method. Teaching activities were in line with the learning practices in the scientific method, namely observation,

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

questioning, reasoning, attempting, and communicating. Learning activities and worksheets focus only on developing 2 indicators of critical thinking, namely interpretation and explanation, so that the teaching design created by S2 has not been able to develop critical thinking.

The mathematics teaching design made by S3 used a problem-based learning model. The teaching activities designed are based on the phases of the problem-based learning model, including 1) orienting students to the problem, 2) organizing learning, 3) guiding both individual and group investigations, 4) developing and presenting work, and 5) analyzing and evaluating the problem-solving processes. Learning activities using problem-based learning tend to impact critical thinking (Yu & Zin, 2023). Additionally, teaching activities and student worksheets designed by S3 are able to have an impact on these abilities based on critical thinking indicators. Discussions are held in groups to solve the problems given by teacher, including grouping objects with square and rectangular surfaces. The students must observe and discuss with friends to conclude the characteristics of square and rectangular shapes. Inferences are presented, with given assessment and input to improve the conclusions drawn.

S1 stated that mathematical problems associated with the development of critical thinking are those with answers realized using more than one interrelated formula and several methods. The response that critical thinking problems can be solved in many ways is not in line with the characteristics of mathematical problems responsible for developing this skill. S1 selected problem number 1 which was inappropriate in terms of developing critical thinking because it does not train the ability to evaluate. Meanwhile, S2 stated that mathematical problems that develop critical thinking are those able to impact the ability to analyze and evaluate. S3 reported that math problems responsible for developing this skill are higher order thinking problems. Building upon this result, problem number 2 was selected by S2 and S3 due to its capability. The problem can develop critical thinking because it is a high-level reasoning question based on Bloom's taxonomy in the analysis and evaluation fields (Tanjung et al., 2020). Higher Order Thinking Skills (HOTS) category questions made significant contributions to improving critical thinking skills including problem solving using various solutions, decision making, and inferences (Thornhill-Miller et al., 2023). Regarding non-routine questions with criteria requiring solution justification trains students to think critically according to the indicators (Khusna et al., 2024). Based on the description above, S2 and S3 were able to explain the characteristics of mathematics questions that developed critical thinking.

The difficulties experienced by the three teachers were consistent with the results of research that identified the difficulties faced by Indonesian elementary school mathematics teachers in integrating critical thinking into teaching. This includes deficiencies in pedagogical knowledge, self-confidence, and resources which hindered the design of effective mathematics teaching (Trisnani et al., 2024). The difficulties experienced are due to the lack of training in line with the needs of teachers in developing knowledge of critical thinking and skills in designing mathematics teaching. Therefore, there is a need to organize special training for teachers to understand critical thinking and teaching design (Badescu & Stan, 2020). Of the three teachers, S3 had attended teacher

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

training and read books related to mathematics teaching knowledge and skills, resulting in adequate understanding. Meanwhile, S1 and S2 had never attended training, with S2 obtaining information from watching YouTube and reading books.

Following the discussion above, teachers must have a strong conception of critical thinking, critical mathematical thinking, indicators, and teaching designs. Strong teacher conceptions provide a firm understanding of designing and implementing mathematics teaching that develop critical thinking. Teacher conceptions also play an important role in the implementation of strong educational experiences (Bezanilla et al., 2023). Additionally, these conceptions and skills in designing mathematics teaching can be impacted through training (Gunawan et al., 2020).

A comparison of teacher conceptions of critical thinking for the high (S3), medium (S2), and low (S1) categories is shown in Table 4 as follows:

The concept of critical thinking in mathematics teaching	Teachers with high critical thinking concepts	Teachers with medium critical thinking concepts	Teachers with low critical thinking concepts
Conception of critical thinking	Correct	Incorrect	Incorrect
Conception of mathematical critical thinking	Correct	Incorrect	Incorrect
Conception of critical thinking indicators	Correct	Correct	Correct
Mathematics teaching design	Teachers were able to explain mathematics teaching activities ensuring the design was able to develop critical thinking.	Teacher was able to explain mathematics teaching activities but the design was unable to develop critical thinking.	Teachers were unable to explain mathematics teaching activities, therefore the design was not able to develop critical thinking.

Table 4: Comparison of teachers' conceptions of critical thinking for high, low, and medium categories

This research obtained samples and subjects from a particular district in East Java, limiting its applicability to a broader context. The challenges faced by mathematics teachers in the future in terms of designing teaching activities that develop critical thinking outline the need for related knowledge and the integration of active methods into teaching practices (Shyshenko et al., 2022). Therefore, future research should be conducted on a broad scale to determine elementary school teachers' conceptions of critical thinking, alongside the ability to design and practice classroom teaching.

CONCLUSIONS

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

In conclusion, teachers with low critical thinking conceptions had inappropriate conceptions of critical thinking and mathematical critical thinking. Conceptions related to critical thinking indicators were appropriate but unable to explain mathematical problem-solving procedures and related teaching activities.

Teachers with medium critical thinking conceptions had inappropriate conceptions related to critical thinking and mathematical critical thinking. However, these teachers had appropriate conceptions related to critical thinking indicators and were able to explain mathematical problem-solving procedures and teaching activities.

Following the description above, teachers with high critical thinking conceptions had appropriate conceptions related to critical thinking, mathematical critical thinking, indicators, teaching steps, and characteristics of mathematics problems that develop critical thinking. These teachers were able to explain mathematical problem-solving procedures and teaching activities. One of the factors that aided teachers with high critical thinking conceptions was the ability to attend training, actively seeking information by reading books and practicing in the classroom.

Following the description, teachers must have high conception of critical thinking to be able to design and teach mathematics that developed critical thinking. The difficulty in designing mathematics teaching is due to the misconception of critical thinking and relevant steps. These teachers require regular training and guidance to be able to develop students' critical thinking in the future.

ACKNOWLEDGMENTS

The authors are grateful to the Principal and Surabaya State University for the permission to conduct this research, and the three teachers who contributed as subjects of this research.

REFERENCES

[1] Acharya, K. P. (2018). Exploring Critical Thinking For Secondary Level Students In Chemistry: From Insight To Practice. *Journal of Advanced College of Engineering and Management*, 3, 31. https://doi.org/10.3126/jacem.v3i0.18812

[2] Afrilianto, M., Rosyana, T., Linda, & Wijaya, T. T. (2022). Project-Activity-Cooperative Learning-Exercise Model in Improving Students' Creative Thinking Ability in Mathematics. *Infinity Journal*, *11*(2), 285–296. https://doi.org/10.22460/infinity.v11i2.p285-296

[3] Aini, N. R., Syafril, S., Netriwati, N., Pahrudin, A., Rahayu, T., & Puspasari, V. (2019).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Problem-Based Learning for Critical Thinking Skills in Mathematics. *Journal of Physics: Conference Series*, *1155*(1). https://doi.org/10.1088/1742-6596/1155/1/012026

[4] Amin, S., Utaya, S., Bachri, S., Sumarmi, & Susilo, S. (2020). Effect of problem-based learning on critical thinking skills and environmental attitude. *Journal for the Education of Gifted Young Scientists*, *8*(2), 743–755. https://doi.org/10.17478/jegys.650344

[5] Amirali, M. (2010). Students' conceptions of the nature of mathematics and attitudes towards mathematics learning. *Journal of Research and Reflections in Education*, *4*(1), 27–41. https://ecommons.aku.edu/pakistan_ied_pdck/8

[6] Badescu, O., & Stan, C. (2020). Training Mathematics Teachers for Developing Critical Thinking Skills in VIGrade Pupils. *Acta Didactica Napocensia*, *13*(2), 186–195. https://doi.org/10.24193/adn.13.2.13

[7] Bayuningsih, A. S., Usodo, B., & Subanti, S. (2018). Critical thinking level in geometry based on self-regulated learning. *Journal of Physics: Conference Series*, *983*(1), 1–7. https://doi.org/10.1088/1742-6596/983/1/012143

[8] Bezanilla, M. J., Galindo-Domínguez, H., Campo, L., Fernández-Nogueira, D., Ruiz, M. P., María, *, & Bezanilla, J. (2023). Student and Teacher perceptions and experiences: How do they align? Understanding critical thinking: A comparative analysis between university students' and teachers' conception Tuning Journal for Higher Education Understanding critical thinking: A compar. *Tuning Journal for Higher Education* •, *10*(2), 223–244. https://doi.org/https://doi.org/10.18543/tjhe1022023

[9] Cahyaningsih, U., & Nahdi, D. S. (2021). The Effect of Realistic Mathematics Education on Elementary Students' Critical Thinking Skills. *Journal of Physics: Conference Series*, *1764*(1). https://doi.org/10.1088/1742-6596/1764/1/012127

[10] Chusni, M. M., Saputro, S., Surant, S., & Rahardjo, S. B. (2022). Enhancing Critical Thinking Skills of Junior High School Students through Discovery-Based Multiple Representations Learning Model. *International Journal of Instruction*, *15*(1), 927–945. https://doi.org/10.29333/iji.2022.15153a

[11] Chusni, M. M., Saputro, S., Suranto, & Rahardjo, S. B. (2020). The potential of discovery learning models to empower students' critical thinking skills. *Journal of Physics: Conference Series*, *1464*(1). https://doi.org/10.1088/1742-6596/1464/1/012036

[12] Chususiyah, S., Dafik, & Prastiti, T. D. (2020). The analysis of application of learning materials based on inquiry based learning and its effect on critical thinking skills of students in solving fractions problems. *Journal of Physics: Conference Series*, *1563*(1). https://doi.org/10.1088/1742-6596/1563/1/012072

[13] Corich, S., Norris, S., McPeck, J., & Paul, R. (2007). Using an automated tool to measure evidence of critical thinking of individuals in discussion forums . *Digital Age*, *Celda*, 163–171. https://citrenz.org.nz/citrenz/conferences/2009/15-21.pdf

[14] Cruz, G., Payan-Carreira, R., Dominguez, C., Silva, H., & Morais, F. (2020). What critical thinking skills and dispositions do new graduates need for professional life? Views from Portuguese employers in different fields. *Higher Education Research and Development*, *0*(0), 1–17. https://doi.org/10.1080/07294360.2020.1785401

[15] Delfia, S., Irwan, I., & Yerizon, Y. (2020). Preliminary Research of Model Eliciting Ac-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

tivities learning Based to Improve Junior High School Students' Mathematical Critical Thinking Ability. *Journal of Physics: Conference Series*, *1554*(1), 1–7. https://doi.org/10.1088/1742-6596/1554/1/012019

[16] Ennis. (2015). The Nature of Critical Thinking: Outlines of General Critical Thinking Disposition and Abilities. *Sixth International Conference on Thinking at MIT*, *2013*, 1–8. http://criticalthinking.net/wp-content/uploads/2018/01/The-Nature-of-Critical-Thinking.pdf.

[17] Ennis, R. H. (2016). Critical Thinking Across the Curriculum: A Vision. *Topoi*, *37*(1), 165–184. https://doi.org/10.1007/s11245-016-9401-4

[18] Facione, P. a. (2015). Critical Thinking : What It Is and Why It Counts. In *Insight assessment*. Measured Reasons LLC. https://www.insightassessment.com/CT-Resources/Teaching-For-and-About-Critical-Thinking/Critical-Thinking-What-It-Is-and-Why-It-Counts/Critical-Thinking-What-It-Is-and-Why-It-Counts-PDF

[19] Fitriana, L., Tiyanto, T., Wiraya, A., Hendriyanto, A., Sahara, S., & Muhaimin, L. H. (2023). Implementation of the Jigsaw Model to Improve Critical-Thinking Skills. *Journal of Higher Education Theory and Practice*, *23*(15), 11–20. https://doi.org/10.33423/jhetp.v23i15.6402

[20] Fitriani, H., Asy'ari, M., Zubaidah, S., & Mahanal, S. (2019). Exploring the prospective teachers' critical thinking and critical analysis skills. *Jurnal Pendidikan IPA Indonesia*, *8*(3), 379–390. https://doi.org/10.15294/jpii.v8i3.19434

[21] Gass, R. H., & Seiter, J. S. (2019). The Nature of Critical Thinking. *Arguing, Reasoning, and Thinking Well*, 62–81. https://doi.org/10.4324/9781351242493-4

[22] Gunawan, D., Sutrisno, S., & Muslim, M. (2020). Pengembangan Perangkat Pembelajaran Matematika Berdasarkan TPACK untuk Meningkatkan Kemampuan Berpikir Kritis. *Jurnal Pendidikan Matematika*, *11*(2), 249. https://doi.org/10.36709/jpm.v11i2.11518

[23] Hikayat, C., Suparman, Hairun, Y., & Suharna, H. (2020). Design of realistic mathematics education approach to improve critical thinking skills. *Universal Journal of Educational Research*, *8*(6), 2232–2244. https://doi.org/10.13189/ujer.2020.080606

[24] Hunaepi, Firdaus, L., Samsuri, T., Susantini, E., & Raharjo. (2020). Biology prospective teachers' critical thinking disposition and critical thinking skills of IKIP Mataram. *Journal of Physics: Conference Series*, *1464*(1). https://doi.org/10.1088/1742-6596/1464/1/012031

[25] Julianto, J., Wiryanto, W., Suprayitno, S., Susetyo R, A., Hidayati, F., & Rahmawati, E. (2023). Analysis of Critical Thinking Skills of Prospective Elementary School Teacher Student. *IJORER : International Journal of Recent Educational Research*, *4*(3), 286–295. https://doi.org/10.46245/ijorer.v4i3.302

[26] Khusna, A. H., Siswono, T. Y. E., & Wijayanti, P. (2024). Mathematical Problem Design to Explore Students' Critical Thinking Skills in Collaborative Problem Solving. *Mathematics Teaching-Research Journal*, *16*(3), 217–240. https://files.eric.ed.gov/fulltext/EJ1442273.pdf

[27] Kusaeri, & Aditomo, A. (2019). Pedagogical beliefs about Critical Thinking among Indonesian mathematics pre-service teachers. *International Journal of Instruction*, *12*(1),

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

573–590. https://doi.org/10.29333/iji.2019.12137a

[28] Lorencová, H., Jarošová, E., Avgitidou, S., & Dimitriadou, C. (2019). Critical thinking practices in teacher education programmes: a systematic review. *Studies in Higher Education*, *44*(5), 844–859. https://doi.org/10.1080/03075079.2019.1586331

[29] Lusiana, R., Sa, C., Subanji, S., & Chandra, T. D. (2024). Elementary Teachers ' Noticing of Students ': How to Stimulate Students ' Critical and Creative Thinking. *TEM Journal*, *13*(2), 1319–1330. https://doi.org/10.18421/TEM132

[30] Ma, S., Tiruneh, D. T., & Spector, J. M. (2023). Critical thinking conceptualization in K-12: A case study of middle school teachers. *Social Sciences and Humanities Open*, *8*(1), 100517. https://doi.org/10.1016/j.ssaho.2023.100517

[31] Maulidiya, M., & Nurlaelah, E. (2019). *The effect of problem based learning on critical thinking ability in mathematics education The effect of problem based learning on critical thinking ability in mathematics education*. https://doi.org/10.1088/1742-6596/1157/4/042063

[32] Moodley, D. E., & Chetty, R. (2024). Conceptualizing Critical Thinking Pedagogy in Teacher Education. *Childhood and Philosophy*, *20*(May). https://doi.org/10.12957/childphilo.2024.81799

[33] Munawaroh, H., Sudiyanto, S., & Riyadi, R. (2018). Teachers' Perceptions of Innovative Learning Model toward Critical Thinking Ability. *International Journal of Educational Methodology*, *4*(3), 153–160. https://doi.org/10.12973/ijem.4.3.153

[34] Nasution, E. Y. P., Yulia, P., Anggraini, R. S., Putri, R., & Sari, M. (2021). Correlation between mathematical creative thinking ability and mathematical creative thinking disposition in geometry. *Journal of Physics: Conference Series*, *1778*(1). https://doi.org/10.1088/1742-6596/1778/1/012001

[35] Nurbaeti, E., Sugiharti, M., & Maya, R. (2019). Improving Critical Thinking Ability And Mathematical Disposition Of High School Students Through Integrated Saintific Approach To Brain Based Learning. *Journal of Innovative Mathematics Learning*, *2*(3), 112–120. https://doi.org/https://doi.org/10.22460/jiml.v2i3.p112-120

[36] Nurdiana, N., Hunaepi, H., Ikhsan, M., Suwono, H., & Sulisetijono, S. (2023). Exploring curiosity and critical thinking skills for prospective biology teacher. *International Journal of Evaluation and Research in Education*, *12*(1), 131–138. https://doi.org/10.11591/ijere.v12i1.23302

[37] Nurhikmayati, I., & Jatisunda, M. G. (2018). SCIENTIFIC LEARNING TO IMPROVE CRITICAL THINKING ABILITY. *EduMa*, *7*(2), 1–10. https://doi.org/http://dx.doi.org/10.24235/eduma.v7i2.3398.g2061

[38] O'Reilly, C., Devitt, A., & Hayes, N. (2022). Critical thinking in the preschool classroom - A systematic literature review. *Thinking Skills and Creativity*, *46*(101110), 1–20. https://doi.org/10.1016/j.tsc.2022.101110

[39] Oner, D., & Aggul, Y. G. (2022). *Critical Thinking for Teachers*. *May*, 319–336. https://doi.org/10.1007/978-3-031-15963-3_18

[40] Prado-Arenas, D., Junyent, M., & Oliveras, B. (2022). Concepciones iniciales de Pensamiento Crítico y Creativo del profesorado de ciencias. *Profesorado, Revista de Cur-*

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

rículum y Formación Del Profesorado, *26*(3). https://doi.org/10.30827/profesorado.v26i3.21445

[41] Putri, A., Kartini, K., & Yuanita, P. (2020). The Effectiveness of Learning Tools Based on Discovery Learning That Integrates 21st Century Skills to Mathematical Critical Thinking Ability in Trigonometric Materials in High School. *Journal of Physics: Conference Series*, *1655*(1). https://doi.org/10.1088/1742-6596/1655/1/012071

[42] Rodriguez-Muniz, L. J., Alguilar-Gonzales, A., Lindorff, A., & Muniz-Rodriguez, L. (2022). Undergraduates conceptions of mathematics teaching and learning: an emprirical study. *Educational Studies in Mathematics*, *109*, 523–547. https://doi.org/10.1007/s10649-021-10105-5

[43] Sahidun, M., Suyitno, A., & Pujiastuti, E. (2023). Mathematical critical thinking ability through Brain based learning model in view of self-regulated learning. *International Journal of Trends in Mathematics Education Research*, *6*(2), 177–185. https://doi.org/10.33122/ijtmer.v6i2.213

[44] Saputra, M. D., Joyoatmojo, S., Wardani, D. K., & Sangka, K. B. (2019). Developing critical-thinking skills through the collaboration of Jigsaw model with problem-based learning model. *International Journal of Instruction*, *12*(1), 1077–1094. https://doi.org/10.29333/iji.2019.12169a

[45] Saragih, S., & Zuhri, D. (2019). Teacher Behavior in Students' Critical Thinking Ability Development. *Journal of Physics: Conference Series*, *1320*(1). https://doi.org/10.1088/1742-6596/1320/1/012006

[46] Sarwanto, Fajari, L. E. W., & Chumdari. (2020). Open-Ended Questions to Assess Critical-Thinking Skills in Indonesian Elementary School. *International Journal of Instruction*, *14*(1), 615–630. https://doi.org/10.29333/IJI.2021.14137A

[47] Scristia, C. H., Hiltrimartin, C., & Santoso, B. (2018). Analisis Konsepsi Guru-Guru Matematika Sekolah Menengah Tentang Hakikat Masalah Matematika Dalam Pendekatan Students …. *Jurnal Pendidikan Matematika*, *12*(1), 79–88. https://core.ac.uk/download/pdf/267822099.pdf

[48] Seventika, S. Y., Sukestiyarno, Y. L., & Mariani, S. (2018). Critical thinking analysis based on Facione (2015) - Angelo (1995) logical mathematics material of vocational high school (VHS). *Journal of Physics: Conference Series*, *983*(1). https://doi.org/10.1088/1742-6596/983/1/012067

[49] Shyshenko, I., Lukashova, T., & Chkana, Y. (2022). Professional Training of Future Teachers of Mathematics: the Problem of Forming Critical Thinking. *Scientific Bulletin of Uzhhorod University. Series: «Pedagogy. Social Work»*, *1*(1(50)), 319–323. https://doi.org/10.24144/2524-0609.2022.50.319-323

[50] Siswono, T. Y. E. (2016). Berpikir Kritis dan Berpikir Kreatif sebagai Fokus Pembelajaran Matematika. *Seminar Nasional Matematika Dan Pendidikan Matematika*, *5*(2), 11–26.

[51] Siswono, T. Y. E., Hartono, S., Kohar, A. W., Karim, K., & Lastiningsih, N. (2019). How do prospective teachers manage students' learning of mathematics? *TEM Journal*, *8*(2),

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

677–685. https://doi.org/10.18421/TEM82-49

[52] Sofiyana, M. S., Sukhoiri, Aswan, N., Munthe, B., W, L. A., Jannah, R., Juhara, S., SK, T., Laga, E. A., Sinaga, J. A. B., Suparman, A. R., Suaidah, I., Fitrisari, N., & Herman, H. (2022). Buku Metode Penelitian Campuran. In *Yayasan Penerbit Muhammad Zaini* (Issue November).

[53] Suherman, S., & Vidákovich, T. (2022). Assessment of mathematical creative thinking: A systematic review. *Thinking Skills and Creativity*, *44*(January). https://doi.org/10.1016/j.tsc.2022.101019

[54] Sukarna, N., & Iman, N. (2019). The Implementation Of Scientific Approach With Group Setting To Improve The 11th Grade Science 6 Students' Mathematical Critical Thinking Skills At Sman 22 Bandung In The Learning Of Derivatives And Its Applications. *Journal of Innovative Mathematics Learning*, *2*(4), 160–168. https://doi.org/https://doi.org/10.22460/jiml.v2i4.p160-168

[55] Sutama, S., Fuadi, D., Narimo, S., Hafida, S. H. N., Novitasari, M., Anif, S., Prayitno, H. J., Sunanih, S., & Adnan, M. (2022). Collaborative mathematics learning management: Critical thinking skills in problem solving. *International Journal of Evaluation and Research in Education*, *11*(3), 1015–1027. https://doi.org/10.11591/ijere.v11i3.22193

[56] Tang, T., Vezzani, V., & Eriksson, V. (2020). Developing critical thinking, collective creativity skills and problem solving through playful design jams. *Thinking Skills and Creativity*, *37*(March), 100696. https://doi.org/10.1016/j.tsc.2020.100696

[57] Tanjung, H. S., Nababan, S. A., Sa'dijah, C., & Subanji. (2020). Development of assessment tools of critical thinking in mathematics in the context of hots. *Advances in Mathematics: Scientific Journal*, *9*(10), 8659–8667. https://doi.org/10.37418/amsj.9.10.91

[58] Thornhill-Miller, B., Camarda, A., Mercier, M., Burkhardt, J. M., Morisseau, T., Bourgeois-Bougrine, S., Vinchon, F., El Hayek, S., Augereau-Landais, M., Mourey, F., Feybesse, C., Sundquist, D., & Lubart, T. (2023). Creativity, Critical Thinking, Communication, and Collaboration: Assessment, Certification, and Promotion of 21st Century Skills for the Future of Work and Education. *Journal of Intelligence*, *11*(3), 1–32. https://doi.org/10.3390/jintelligence11030054

[59] Toheri, Winarso, W., & Haqq, A. A. (2020). Where exactly for enhance critical and creative thinking: The use of problem posing or contextual learning. *European Journal of Educational Research*, *9*(2), 877–887. https://doi.org/10.12973/eu-jer.9.2.877

[60] Tohir, M., Maswar, M., Mukhlis, M., Sardjono, W., & Selviyanti, E. (2021). Prospective teacher's expectation of students' critical thinking process in solving mathematical problems based on Facione stages. *Journal of Physics: Conference Series*, *1832*(1). https://doi.org/10.1088/1742-6596/1832/1/012043

[61] Trisnani, N., Retnawati, H., & Wuryandani, W. (2024). Challenges of Indonesian elementary school mathematics teachers in integrating critical thinking into the classroom. *Journal on Mathematics Education*, *15*(3), 905–924. https://doi.org/10.22342/jme.v15i3.pp905-924

[62] Ulfiana, E., Mardiyana, M., & Triyanto, T. (2019). Determining ways to improve critical thinking skills in the math mathematics in student style. *Journal of Physics: Conference*

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Series, *1321*(2), 3–8. https://doi.org/10.1088/1742-6596/1321/2/022098

[63] Wijayanti, K., Khasanah, A. F., Rizkiana, T., Mashuri, Dewi, N. R., & Budhiati, R. (2021). Mathematical creative thinking ability of students in treffinger and brain-based learning at junior high school. *Journal of Physics: Conference Series*, *1918*(4). https://doi.org/10.1088/1742-6596/1918/4/042085

[64] Willingham, D. T. (2019). How to Teach Critical Thinking. *Education: Future Fronties*. https://doi.org/10.2753/RES1060-9393330549

[65] Yanuari, N. F., & Turmudi, T. (2023). Critical thinking in mathematics education: A bibliometric analysis. *International Journal of Trends in Mathematics Education Research*, *6*(2), 191–197. https://doi.org/10.33122/ijtmer.v6i2.241

[66] Yasin, M., Jauhariyah, D., Madiyo, M., Rahmawati, R., Farid, F., Irwandani, I., & Mardana, F. F. (2019). The guided inquiry to improve students mathematical critical thinking skills using student's worksheet. *Journal for the Education of Gifted Young Scientists*, *7*(4), 1345–1360. https://doi.org/10.17478/jegys.598422

[67] Yu, L., & Zin, Z. M. (2023). The critical thinking-oriented adaptations of problem-based learning models: a systematic review. *Frontiers in Education*, *8*. https://doi.org/10.3389/feduc.2023.1139987

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Design and Implementation of Hypothetical Learning Trajectory on The Geometry of Cubes and Cuboids using Project-Based Learning

Iik Nurhikmayati[1,2], Sufyani Prabawanto[2], Darhim[2], Jarnawi Afgani Dahlan[2]

[1]Universitas Majalengka, Indonesia

[2]Universitas Pendidikan Indonesia, Indonesia

sufyani@upi.edu

Abstract: The geometry of cubes and cuboids serves as a crucial foundation for learning other spatial shapes in mathematics education. Students often experience difficulties achieving objectives due to various learning obstacles. Mathematics teachers also struggle to design learning activities that address students' needs and effectively overcome learning obstacles. This research aimed to investigate the impact of Hypothetical Learning Trajectory (HLT) using project-based learning to overcome students learning obstacles on the topics of cubes and cuboids. Students collaborate in groups to create miniature building projects that are used to analyze the concepts of elements, nets, surface area, and volume. This study uses a design research method consisting of 2 cycles, each with 3 stages: preliminary design, teaching experiments, and retrospective analysis. 64 eighth-grade students from a secondary school in Indonesia were involved in the implementation of 8 HLT across 9 teaching experiments. Data were collected through classroom observations, interviews, tests, and problem-solving exercises. There are 21 types of characteristic learning obstacles that have been identified on the topic of cubes and cuboids. HLT combined with project-based learning successfully addressed 87% of students' learning obstacles related to this the topic, including epistemological, ontogenetic, and didactic learning obstacles.

Keywords: design research, hypothetical learning trajectory, project-based learning, geometry, cubes and cuboids.

INTRODUCTION

Accurate drawing and understanding of a drawing is of utmost importance to effectively support mathematical performances (Rellensmann et al., 2017), especially in geometry of cubes and cuboids (Gal & Linchevski, 2010; Hoffmann & Németh, 2021). As part of geometry, the topic of

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

cubes and cuboids remains challenging in mathematics education (Dimla, 2018; Snyders, 1995). Many students still struggle to solve problems related to cubes and cuboids (Priskila et al., 2023). Students sometimes find it difficult to solve problems due to learning obstacles, including concept limitations, such as errors in using surface area and volume formulas, limitations in prerequisite material such as the Pythagorean Theorem and area unit conversion, and student limited experience in solving non-routine problems. This difficulty is also caused by low motivation and fear of mathematics (Saadati & Celis, 2023).

The results of previous research have shown that the identified learning obstacles cause students to have difficulty solving cubes and cuboids problems (Priskila et al., 2023; Safitri & Dasari, 2022). Students cannot achieve learning objectives optimally. This fact leads to the task of teachers to overcome problems in learning mathematics. However, current education systems are rarely equipped to tackle complex and thorny problems that emerge in schools and classrooms, which include a wide range of academic, social, mental, and emotional issues that students struggle with, along with distinct achievement gaps that have persisted and accelerated in recent years (Dorn et al., 2020). This has led to higher levels of stress, burnout, and depression among students and teachers (Baker et al., 2021; Henriksen et al., 2022). Teachers should understand the learning obstacles that students have and design a learning process that is able to overcome the various difficulties that students face in the classroom. Teachers need to study the phenomena that underlie the process of building hypothetical learning designs by, examining concept images and learning obstacles during the prospective analysis stage to serve as an analysis of didactic context before learning (Nurhikmayati et al., 2022).

HLT is a learning process design that considers students' responses based on their obstacles (Kuncoro et al., 2023a; Simon, 1995a). The conventional wisdom in the mathematics education community holds that HLTs are an important tool in improving mathematics education (Baroody et al., 2022). A HLT framework has the potential to help teachers develop knowledge about students, their conceptions and misconceptions, and their thinking and learning strategies (Bardsley & ellen, 2006). It is an ongoing challenge for teachers to integrate their teaching and learning goals with trajectories of students' thinking and learning; HLT may help in addressing these challenges (Simon, 1995a). This research focuses on developing HLT using Project-Based Learning (PBL) which is part of the 3 models recommended in the implementation of the 2013 curriculum in Indonesia (MoEC, 2016). These models allow teachers to select the most suitable learning method to achieve specific objectives (Lainufar et al., 2021). PBL provides more extensive problems than other models by offering more contextual and authentic experience (Capraro et al., 2013). It enhances the effectiveness of teaching and learning geometry (Chin, 2014), enabling students to construct knowledge through project experiences on a team. The criteria of one group was more successful in solving geometry problems in collaboration, such as through project experiences, while the other could have been more successful (Farida et al., 2023).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

In geometry, PBL allows students to visualize and manipulate geometric shapes and structures, thereby deepening the understanding of abstract concepts. It is widely accepted that mathematics instruction should prioritize developing students' abilities to apply mathematical concepts to solve real-world problems (NCTM, 2000). Projects made by students about building space will provide a meaningful experience. Students can analyze each part of the project to generalize a new understanding. This research introduces the development of HLT through PBL as a novel method. Previous efforts to develop HLT for cubes and cuboids by Sari et al (2024a) and Surya et al (2021a) relied on problem-based learning. HLT developed in this research aims to enhance students' conceptual understanding of materials through projects contextualized with real-world applications.

The importance of cubes and cuboids in geometry as a basis for learning other spatial shapes alongside the identification of students' learning obstacles emphasizes a gap in this research. The development of HLT using PBL stages is expected to overcome students' learning obstacles on the topics of cube and cuboid geometry. This study is important as it aims to provide an overview to educators in an effort to improve student understanding based on the problems in the learning obstacles that students have.

This study aims to investigate the impact of HLT implementation using PBL in overcoming students' learning obstacles on the topics of cube and cuboid. To achieve this objective, three specific research questions have been established as follows:

1. What are the characteristics of students' learning obstacles in cubes and cuboids?
2. How is the HLT designed and implemented through PBL activities to address students' learning obstacles in understanding cubes and cuboids?
3. What is the impact of HLT using PBL in overcoming students' learning obstacles?

LITERATURE REVIEW

Student's Learning Obstacles

Learning obstacles is a condition where a person cannot follow the learning process properly, which is characterised by certain obstacles preventing individuals from achieving learning outcomes (Moru, 2009). Brousseau (2002) categorises learning obstacles into three types: ontogenic obstacles, epistemological obstacles, and didactical obstacles. Ontogenic obstacles are learning obstacles related to mental readiness. This mental readiness is related to the level of student development that is not in accordance with their age (Cesaria & Herman, 2019). If the obstacle arises only because of slow mental development and not because of an inherited disease, the obstacle will disappear by itself along with the growth of the student (Cesaria & Herman, 2019). Brousseau (2002b) noted that epistemological obstacles are related to the limited ability of students' concepts. Epistemological obstacles occur when students experience obstacles in concepts, procedures, and calculation techniques. Epistemological understanding is essentially a person's knowledge, which

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

was only limited to certain contexts (Duval, 2006). If the person is faced with different contexts, the possessed knowledge becomes unusable, or she or he undergoes difficulty using it. In this case, students' views on other concepts are disjointed and not comprehensive. Finally, Brousseau (2002b) noted that didactical obstacles are obstacles related to teaching materials or learning presentations by teachers that are not appropriate to the conditions of students.

Hypothetical Learning Trajectory

The term Hypothetical Learning Trajectory (HLT) in mathematics learning was first introduced by Simon in 1995. HLT is a learning process design that pays attention to learner responses consisting of three components, namely learning objectives, learning activities, and learning process conjectures predictions about how students' thinking and understanding will develop in the context of learning activities (Simon, 1995b). HLT is a hypothesis or prediction of how learners' thinking and understanding develop in learning activities. HLT also refers to a hypothetical learning path as a conjecture about the series of activities that children go through in solving a problem or understanding a concept. The design of HLT in the learning process is based on students' learning obstacles (Kuncoro et al., 2023b). Each activity and task in the HLT must aim to overcome learning obstacles that are in accordance with the suspected way of thinking for students when performing the given task. The presumption of students' way of thinking that has been designed by the teacher will make it easier for the teacher to provide didactical anticipation during the learning process. When the HLT designed is not in accordance with the students' expected learning process, the HLT is revised. This activity continues until the HLT is in accordance with the alleged learning process and, can overcome students' learning obstacles to achieve optimal learning objectives.

Project-Based Learning

Project-Based Learning (PBL) has gained so much recognition mainly because it was the first method where teachers at the time could see students gaining new knowledge while working on activities. It allows students to construct their knowledge at their own pace (Gay, 2022a). PBL is one mode of learning that creates an active, collaborative atmosphere and can increase confidence in students(Hudson et al., 2015; Pablos et al., 2017). It is a learning model that focuses on the main concepts and principles of a discipline, involves students in problem-solving and reasoning activities, provides opportunities for students to work autonomously to construct their own learning, and produces products of student work (Fisher et al., 2020). In this study, we used PBL stages which include reflection, research, discovery, application, and communication (Laboy-Rush, 2011a). These five stages allow students to explore mathematical understanding openly, conduct scientific investigations in the learning classroom, and validate the process through application activities. ¶

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Relevant Research on Geometry Learning

Various studies on geometry learning have been conducted with various methods and education levels. A literature review from 2017-2021 shows that most geometry learning interventions are effective, especially on van Hiele stage-based approaches, technological media, and concrete manipulative media (Waluya et al., 2022). The study recommended the development of more structured geometry scaffolding. Student-centered learning approaches were shown to be more effective than traditional methods (Juman et al., 2022), with activity-based learning also showing good results (Noreen & Rana, 2019). In addition, it is important for teachers to relate geometry concepts to the real world (Nazarovich & Kurudirek, 2024).

In addition to PBL, other approaches such as PBL are also effective in understanding geometry concepts. A literature review and meta-analysis showed that 86% of studies noted the positive impact of PBL supported by GeoGebra and Cabri 3D on students' geometry understanding (Ramadhanti & Juandi, 2022), as well as knowledge improvement and retention (Ahamad et al., 2017; Haji et al., 2023; Herawati & Sidik, 2023). The inquiry learning model can also improve students' ability to identify geometry concepts (Kedikli & Katranci, 2022; Ratmokohadi, 2022; Triwahyuningtyas & Suastika, 2021), through skills such as observation, measurement, and data presentation (Kedikli & Katranci, 2022). PBL and inquiry models are excellent at developing higher order thinking skills, but for highly visual geometry concepts, the real-world experience offered by PBL is needed. PBL excels in providing geometry understanding because it integrates contextual, collaborative learning and relevant real products. When combined with technology, this approach can provide a more immersive, engaging, and meaningful learning experience.

METHOD

Research Design

This research aimed to enhance the presentation of mathematics learning methods on the topic of cubes and cuboids by developing HLT based on students' identified learning obstacles. To achieve this objective, the design research method was adopted which focused on developing theories about learning process (Plomp, 2006). The design method was defined as the systematic analysis and evaluation of educational interventions with the dual purpose of producing research-based solutions to complex problems in educational practice. This design also advanced the knowledge of the characteristics of the interventions and the process of designing and developing the innovations (Plomp, 2006).

In implementing this research, the design method was applied which included 3 stages, namely preliminary design, teaching experiment, and retrospective analysis (Akker et al., 2006). The design method was a cyclical process of testing a series of learning activities. Part of the key char-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

acteristics of design research was the cyclic-iterative nature which could be divided into macro- and micro-cycles (Gravemeijer & Eerde, 2009a). In this research, 2 cycles were conducted as stages of HLT development on the topic of cubes and cuboids as presented in Figure 1.

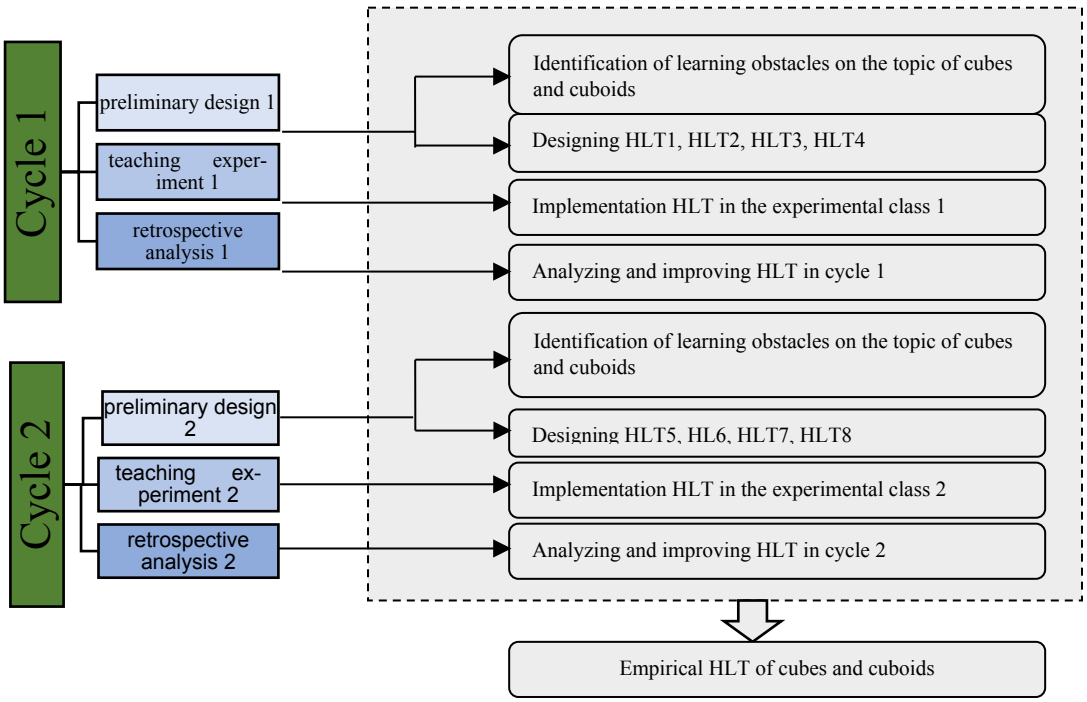


Figure 1. Flow of Research Design for Cubes and Cuboids

HLT in this research was divided into universal and partial. Universal HLT represented the final design derived from the partial HLT in each cycle. The HLT in each cycle was designed according to the stages of PBL, starting with project assigned to students during the reflection and research stages. HLT implementation scheme which incorporated the stages of PBL is presented in Figure 2.

The sub-topic of cubes and cuboids was studied in each HLT while ensuring adherence to the material hierarchy. Projects given to students were reviewed and developed at HLT1 and HLT5. Furthermore, the analysis of each sub-topic on cubes and cuboids was carried out based on projects completed by students which were reviewed during the discovery, application, and communication stages in HLT2, HLT3, HLT4, HLT6, HLT7, and HLT8.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Sample and Data Collection

Participants of this research included 64 eighth-grade students from a public junior high school in Majalengka, Indonesia. In the first cycle of the experiment, 29 students participated while 31 took part in the second cycle. The first cycle served as the initial didactic design stage, and the second focused on improving the design from the initial cycle. Although all participants came from the same school, students were in different grades and participated at different stages of the cycle examining the same material. Participants engaged in a 3-month teaching experiment. During the first cycle, students used HLT1, HLT2, HLT3, and HLT4. In the second cycle, students used HLT5, HLT6, HLT7, and HLT8, which served as improvements on the first cycle.

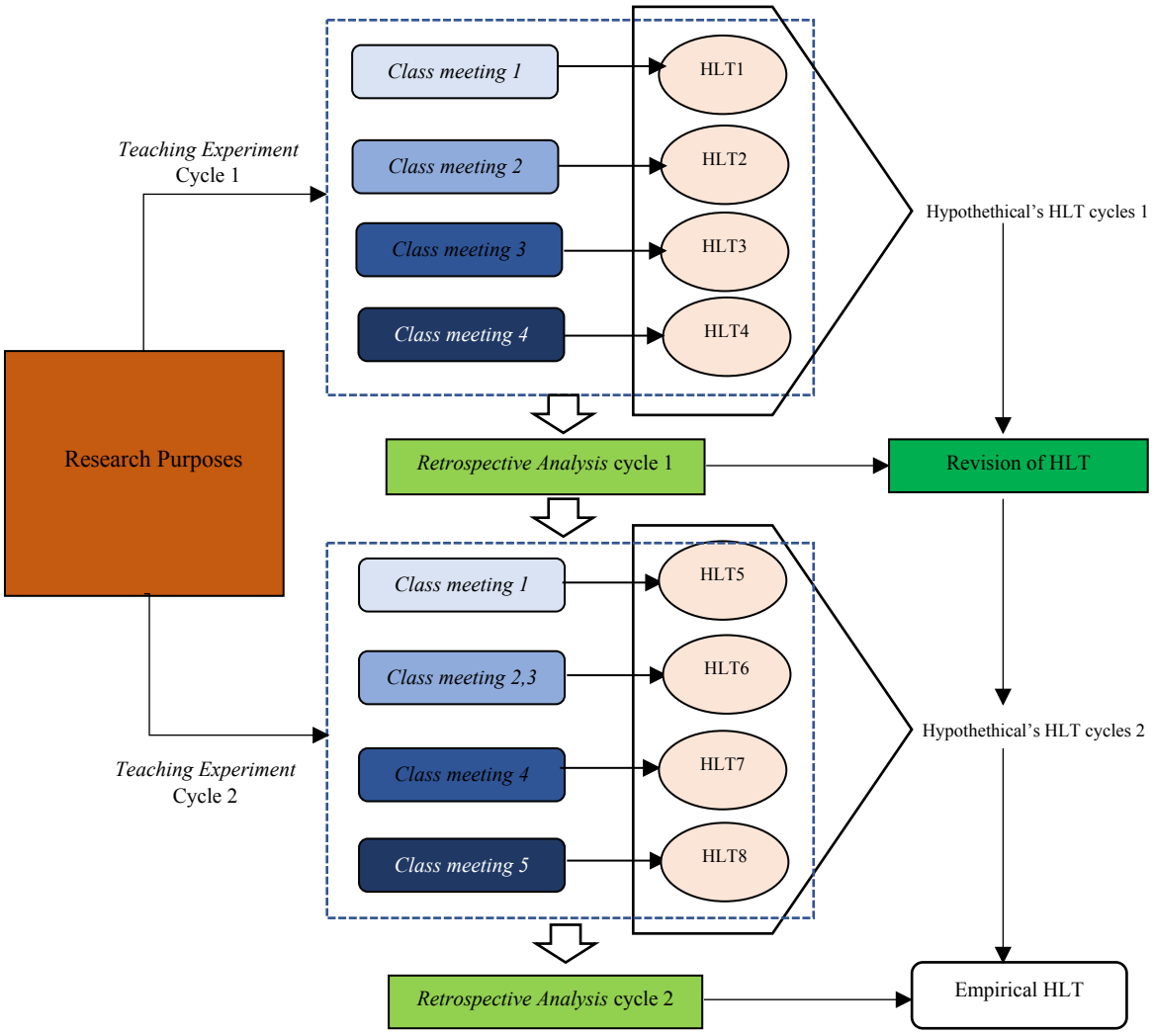


Figure 2. HLT Implementation Scheme for Cubes and Cuboids Topics with PBL

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Learning obstacles data were collected through interviews with teachers and students, classroom observations and tests. This test used questions on the topic of cubes and cuboids that were descriptive in nature. At the end of the HLT implementation, We collected students' test data through practice questions to examine how the HLT helped reduce their learning obstacles and increased their understanding of the geometry concepts of cubes and cuboids.

Analyzing of Data

HLT design on cubes and cuboids was developed based on the identified learning obstacles. The process of identifying students' learning obstacles on the topics of cubes and cuboids was analyzed using the qualitative analysis method from (Miles et al., 2014a). This analysis consists of three flows, namely data condensation, data display and conclusion. This research further used data triangulation which was a method for testing the validity of data by cross-checking the accuracy of information from different perspectives. Interview data, classroom observations, and tests were used to identify learning obstacles on the topic of cubes and cuboids. The interviews were semi-structured as it did not follow systematic and complete interview guidelines but focused on key issues to be addressed.

RESULTS

Characteristics of Learning Obstacles in Cubes and Cuboids

Learning obstacles in cubes and cuboids were identified through 3 methods namely learning observation notes (SO), interview questionnaire notes (SI), and student test results (ST). The analysis of learning obstacles was conducted using the data analysis model by (Miles et al., 2014a). The identified learning obstacles in cubes and cuboids were presented in Table 1.

Didactic situation	Source	Coding
Context limitations in understanding the types of elements of cubes and cuboids	SO, ST	EO-1
Context limitations in understanding the properties of cubes and cuboids	SO, ST	EO-2
Limitations in depicting 3-dimensional space	SI, ST	EO-3
Context limitations in understanding the shape of the nets of cubes and cuboids	ST	EO-4
Context limitations in understanding cubes and cuboids skeletons	ST	EO-5
Errors in identifying the value of elements in the formula for the surface area of cubes and cuboids	ST	EO-6
Errors in formula usage	ST	EO-7
Limited understanding of the concept of the surface area of cubes and cuboids	ST	EO-8
Misconceptions on the meaning of volume of cubes and cuboids as the	SI, ST	EO-9

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

exact number of unit cubes stored on a cube or cuboid		
Context limitations on new problems in problem-solving	SO,SI,ST	EO-10

Table 1: Epistemological learning obstacles

Epistemological learning obstacles identified in this research referred to students' conceptual limitations in understanding cubes and cuboids. These conceptual limitations were closely connected to contextual limitations and difficulties in comprehending mathematical symbols. Students' limited understanding of the concepts related to cubes and cuboids contributed to the low performance in solving related problems. When students experienced a problem related to calculating the surface area of unit cuboids of different sizes, errors occurred in determining the correct formula (EO-7) and in assigning values in the surface area formula (EO-6). Epistemological learning obstacles in cubes and cuboids were often interrelated and connected to students' conceptual understanding. Several learning obstacles were identified from a single students' answer, each related to different concepts. Ontogenic learning obstacles identified in cubes and cuboids are further presented in Table 2.

Didactic situation	Source	Coding
Errors in understanding the sentences in the question	ST	OO-1
Limited knowledge of the prerequisite material of the Pythagorean theorem	SO, ST	OO-2
Operational learning obstacles	SO, ST	OO-3
Limited students' prior knowledge related to comparison material	SO, ST	OO-4
Students' low academic awareness and interest in learning mathematics	SO	OO-5

Table 2. Ontogenic Learning Obstacles

The preliminary analysis further showed several ontogenic learning obstacles faced by students. Many of these learning obstacles originated from psychological and instrumental factors. Observations showed that students exhibited low interest in mathematics with some deliberately arriving late to class despite knowing it had already started. The test results also showed that many students had insufficient knowledge of prerequisite materials, such as the Pythagorean theorem and ratio concepts. Furthermore, a significant number of students struggled with basic calculation skills, which were essential for solving mathematical problems. Didactic learning obstacles identified in cubes and cuboids were further presented in Table 3.

Didactic situation	Source	Coding
Misconception between the known volume formula and the application in the figure with the unit cube	ST	DO-1
Context limitation on unfamiliar problems	SO, ST	DO-2
Limited use of teaching aids and technology in learning	SO, SI	DO-3
Limited discussion and question and answer between teachers and students as well as students and students	SO	DO-4
Limited opportunities, experiences, and opportunities for students to think as a whole in constructing knowledge	SO	DO-5

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Errors in conveying volume concepts	SO, ST	DO-6

Table 3. Didactical Learning Obstacles

Didactic learning obstacles identified in this research were related to the teaching methods used by teachers when instructing students on cubes and cuboids. Didactical obstacles arise from the didactic system, encompassing factors such as sequencing and curriculum stages, including their presentation in classroom instruction (Fauzi & Suryadi, 2020), such as the use of learning media. The observations outlined several didactic situations that contributed to the learning obstacles including the limited use of teaching aids such as models of cubes and cuboids as well as the skeletal structures. There was also a lack of integration of learning technologies such as GeoGebra, Maple, or Matlab mathematical applications. Teachers primarily relied on textbooks as visual aids for teaching cubes and cuboids. Additionally, the observations showed that students had limited opportunities and experiences in constructing individual knowledge, as students mainly focused on verifying the concepts presented by teachers.

Design of HLT for Cubes and Cuboids

HLT was designed based on the identified learning obstacles to achieve the objectives on the topic of cubes and cuboids which was aimed to address and overcome the learning obstacles. Following the research design, HLT was developed during Cycles 1 and 2. The final HLT of each cycle became an empirical reference for teaching cubes and cuboids using PBL. It could be used by teachers and other scholars as a guide. In this research, there were 2 learning cycles namely Cycle 1 which included HLT1, HLT2, HLT3, and HLT4, as well as Cycle 2, comprising HLT5, HLT6, HLT7, and HLT8, as improvements from the initial HLT.

Each HLT was specifically designed for the 2 cycles. In Cycle 1, four HLT (HLT1, HLT2, HLT3, and HLT4) were created, following the stages of PBL model. According to HLT implementation scheme as shown in Figure 2, the design was not organized by material sub-topics but was centered around project initiated at the initial stage of learning. This project which was conducted during the reflection and research stages served as a basis for learning cubes and cuboids comprehensively in each HLT. The four HLT in Cycle 1 were refined based on the teaching experiment process, leading to the creation of HLT5, HLT6, HLT7, and HLT8 in Cycle 2.

Implementation of HLT on Cubes and Cuboids

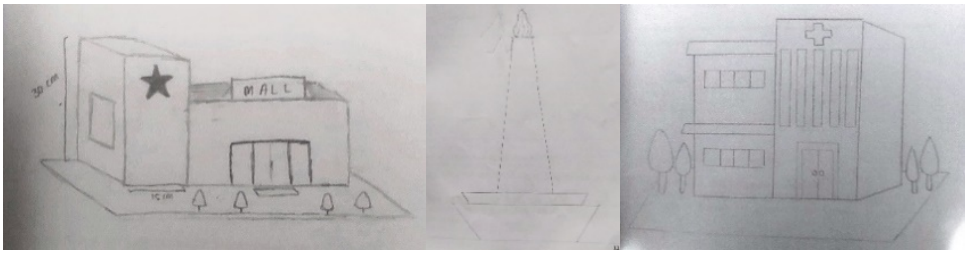
HLT for cubes and cuboids was designed in correlation with the stages of PBL model, which included reflection, research, discovery, application, and communication (Laboy-Rush, 2011b). The implementation of HLT included a teaching experiment based on the designed HLT.

Teaching Experiment Cycle 1

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Implementation of HLT 1

HLT 1 focused on designing learning activities on cubes and cuboids, addressing learning obstacles EO1–EO and DO4–DO5. The stages followed reflection and research. Students were introduced to related problems and encouraged to link prior knowledge to the project. They identified surrounding objects to recall concepts; some gave accurate examples, while others erred or didn't respond. In the research stage, students prepared a mini-proposal for a project involving building miniatures from cubes and cuboids using recycled materials. Several designs were illustrated through sketches and project outcomes (see Figure 1).



(a) (a) (a)

Figure 1: Students' Cubes and cuboids Building Sketches in Cycle 1

The sketches of the miniature building project designs develop by students mostly depicted structures encountered in daily living such as malls (Figure 1a), Monas (Figure 1b) as iconic buildings in Indonesia, and hospitals (Figure 1c). Students relied on imagination, drawing from group discussions and school textbooks. The results of these projects were shown in Figure 2.



(a) (b) (c)

Figure 2: Students' cubes and cuboids building project in cycle 1

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Students had gained an understanding of the shapes of cubes and cuboids as reflected in the buildings constructed. This process helped students develop a deeper understanding through observation and discussion with peers.

Implementation of HLT 2

HLT 2 focused on the discovery stage of learning. Projects that students had previously created were then used as a foundation for exploring, analyzing, understanding, and generalizing concepts of the materials. Students analyzed projects to answer questions on the student worksheet which was designed to guide students toward a meaningful understanding of cubes and cuboids. The worksheet covered subtopics such as the elements, nets, surface area, and volume. Some student groups were able to correctly identify all the elements, while others could only identify some. Some student groups successfully described the nets with several different shapes. Some groups still made errors in the general formula for the surface area and volumes of cubes and cuboids. An example of knowledge construction about cubes and cuboids from a project that has been created is presented in Figure 3. Each group of students gave different results in constructing knowledge about cubes and cuboids. Figure 3 is one of the construction results from the student group. The process of finding your own understanding of cubes and cuboids is a very meaningful way for students.

Implementation of HLT 3

The HLT focused on the application stage of learning. At this stage, students worked to understand all concepts related to cubes and cuboids using GeoGebra application and validated the understanding through practice problems. As students practiced the understanding through GeoGebra application, most were able to create accurate drawings of cubes and cuboids. Some students still struggled to completely identify all elements. Students also attempted to create nets of the shapes constructed to visualize how the nets formed complete spaces. In the surface area and volume subtopic, students were guided to calculate these values based on the dimensions of the spaces created and to match the spaces with the formulas previously learned.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

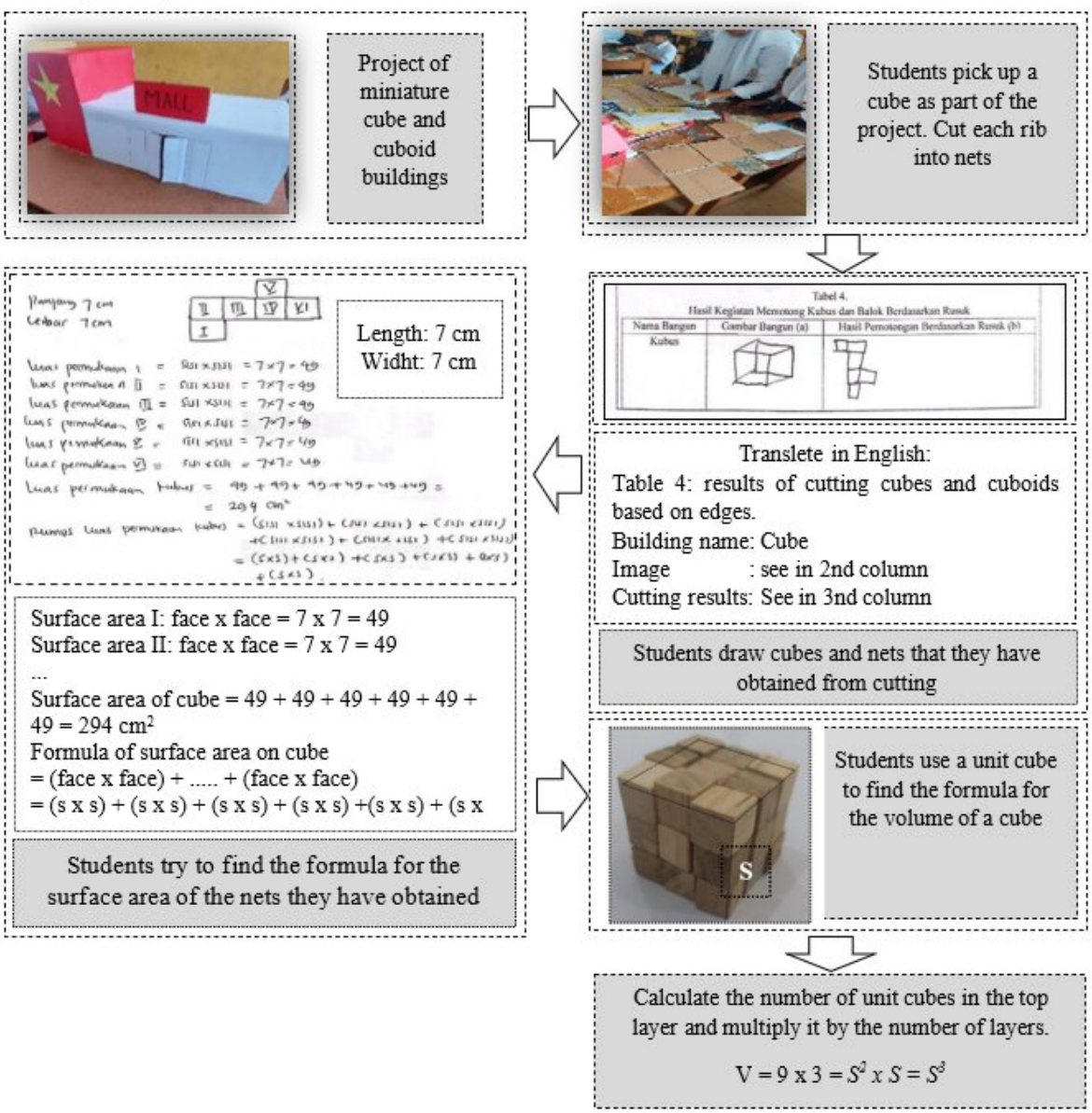


Figure 3: Students constructed knowledge on the topic of cubes through the discovery stage

Implementation of HLT 4

HLT 4 focused on the communication stage of learning. Students were asked to present and explain the results of projects and to verbally communicate their comprehension. Due to time constraints, only group representatives participated in this activity. The scholar provided verbal validation when misunderstandings arose. Practice problems were given to assess students' application of the concepts which was learned about cubes and cuboids. The results showed that while

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

some students still made errors in identifying the elements and properties of cubes and cuboids, all were able to draw the nets, though limited in variety. Some students knew the formulas for surface area and volume, but many still struggled when applying these to real-life problems.

Teaching Experiment of Cycle 2

In teaching experiment cycle 2, every deficiency in HLT was corrected to obtain optimal results. HLT 5 was an improvement of the lesson plan from HLT 1 with a focus on incorporating the use of the internet to help students gather information for designing cubes and cuboids building projects, which was not included in the initial HLT. Based on the sketches, the result of the completed projects are depicted in Figure 4.



(a) (b) (c)

Figure 4: Students' cubes and cuboids building project in cycle 2

The results of projects in Cycle 2 showed a significant improvement in students' understanding of building with cubes and cuboids. Students further appeared to understand project purpose more effectively, and nearly all projects featured structures composed of cubes and cuboids. The improvement focused on extending the time allocated for students to explore, analyze, and build an understanding of cubes and cuboids sub-topic. In HLT 2, the time was set at 3 sessions of 40 minutes each for a meeting. Observations during the implementation of HLT 2 further showed that the duration was insufficient for students to fully grasp all concepts related to cubes and cuboids. HLT 6 increased the time to 5 sessions of 40 minutes each, spread over 2 meetings. HLT 7 there were no significant changes. All tasks are in accordance with the student's train of thought and no changes were made in HLT 8 from the implementation of HLT 4, as learning process met the research objectives. Students were allowed to present and explain the results of projects and showcase the understanding of the concept of cubes and cuboids. In HLT 8, students also practiced solving problems as an application of the understanding gained. The implementation results of

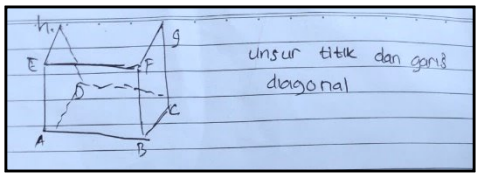
This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

HLT 8 showed that 80% of students were able to explain projects relevant to the concept of cubes and cuboids, improving problem-solving skills.

Improving Understanding of Students' Epistemological Learning Obstacles

Figure 5 shows an example of students' epistemological learning obstacles, namely limited context in understanding the properties of cubes and cuboids (EO-2) in Table 1**.**

Question: Mention what elements are in cubes and cuboids!
Answers (student 5):



Translate in English:

vertices and diagonal lines or face diagonal

Figure 5: Student answers on epistemological learning obstacle

Figure 5 shows students' answers before learning by using project-based HLT. The researcher tried to ask questions related to the elements of cubes and cuboids. The students' answers showed that they did not know the elements. The correct answers were vertex, edge, face, face diagonal, space diagonal and diagonal plane, but student 5 only mentioned two of the supposed elements. This didactic situation shows the limited context of students in understanding each part of the cubes and cuboids.

To overcome the existing obstacles through project-based HLT, students make direct observations of the results of the miniature cube and cuboid building project. Students are asked to show each element in the cube and cuboid building. For example, students will take the cubes from the project product in Figure 2(a), then show and mention each element in the cube. This activity provides students with a direct learning experience, so that they easily remember and understand the concepts on the elements of cubes and cuboids. Through direct experience with projects, students are involved in problem solving and provide opportunities for students to construct their own understanding (Pablos et al., 2017). They can work together with a team for better understanding(Gay, 2022b) . After the learning was conducted, students showed improvement on the limitations of previous understanding. Students can answer all elements in cubes and cuboids correctly. The didactical anticipation given is to give awards by saying ‘great’ to the correct group and provide scaffolding to students who are still mistaken by asking questions through pictures or project products directly.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Improving Understanding of Students' Ontogenical Learning Obstacles

One of the most common ontogenical learning obstacles is low academic awareness and interest in learning (OO-5) and operational learning obstacles (OO-3) as shown in Table 2. This is illustrated by the results of the researcher's interview with the teacher of the school mathematics lesson.

R (researcher)	:	*"In your opinion, what learning barriers have the most influence on students' success rate in learning math. Especially the material of cubes and cuboids?".*
T (teacher)	:	*"Based on my years of teaching experience, I noticed that many students were not enthusiastic when the math lesson started. They arrive late, ignore the lesson time and do not focus during the lesson. Of course this affects any math material".*
R (researcher)	:	*"Besides students' low interest in math, do you find other obstacles?".*
T (teacher)	:	*"Yes, I do. Among the issues are students' low operational skills in arithmetic. When learning about the surface area or volume of cubes and cuboids, students are expected to understand multiplication and division, but many still make mistakes."*

The teacher's statement from the interview shows that students have low interest in mathematics, and many still make mistakes in basic mathematical calculations. This didactic situation indicates the presence of ontogenical learning obstacles. The teacher's statement was also proven when the researcher made observations at the beginning of learning. This illustrates the importance of applying learning methods that can attract students' interest and make the process of learning mathematics more enjoyable.

To overcome these obstacles, a project-based HLT was designed through a series of learning activities that involved making miniature building projects from cubes and cuboids. The learning started with a reflection activity, where students were asked to mention the types of buildings around them that contained cube or cuboid shapes. After that, they are asked to create miniature buildings that they like, according to the predetermined learning criteria. The products of this project are then used as learning media to understand various concepts in cube and block materials, such as the elements of a building, nets, surface area, and volume.

This series of activities gives students the space to be creative and imaginative. They can also work together in teams to produce the best product. This approach is a special attraction for students, so they are more motivated to learn the cube and block material with enthusiasm. The results showed significant positive changes in students' attitudes towards learning mathematics. They arrived on time when the lesson started and were more focused during the learning process. One student even said: "*I really like learning like this. This project makes us excited to learn maths. Can it be done all the time*?". Another student expressed a similar opinion: *"Learning maths through projects like this is much more fun than just looking at the blackboard and books in class. I am always*

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

excited when the lesson starts". The students' statements reflect their increased academic awareness and interest in learning mathematics, especially in cubes and cuboids.

Improving Understanding of Students' Didactical Learning Obstacles

One of the didactical obstacles found in learning cubes and cuboids is the limited use of media and technology (DO-3, see Table 3). The results of observations and interviews show that teachers rarely use media or technology in explaining the material, and tend to teach traditionally. This obstacle disrupts the process of forming students' understanding, especially in imagining the shape of spatial shapes and developing their spatial imagination.

To overcome this, learning through project-based HLT is designed so that students gain a more concrete and visual learning experience. One of the activities is using unit cubes, which are arranged into larger cubes and cuboids. This activity helps students understand the concept of volume and surface area directly. In addition, at the application stage, students are given the opportunity to create digital models of cubes and blocks using GeoGebra software. The combination of using concrete media and technology is designed to improve students' spatial ability in understanding the concept of building space. The learning results showed a significant increase in student scores. This finding is in line with previous research which states that GeoGebra is able to improve students' spatial abilities and understanding of cube and cuboid materials (Gravemeijer & Eerde, 2009b; Miles et al., 2014b).

Retrospective Analysis

The retrospective analysis stage was used to evaluate the overall HLT based on the results of the teaching experiments in cycles 1 and 2. These results illustrate the impact of HLT implementation in overcoming students' learning obstacles on the topics of cube and cuboid. Retrospective analysis involves a careful review of the data and a reflection on the process of the teaching experiment in order to develop an explanatory model of what induced the changes observed in the learning ecologies (Gravemeijer & Eerde, 2009b). According to the research objectives, the development of HLT aimed to address students' learning obstacles regarding the topic of cubes and cuboids. Learning obstacle was considered resolved when 80% of students no longer experienced it. The implementation of HLT in Cycles 1 and 2 showed that 85.7% of students' learning obstacles were effectively overcome. The improvement of students' learning obstacles on the topic of cubes and cuboids was presented in Table 4.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Type of LO	Pre-Class	Implementation of HLT		Pasca Class
		Cycle 1	Cycle 2	
Epistemological Learning Obstacles (EO)				
EO-1	√	X	X	X
EO-2	√	√	X	X
EO-3	√	√	X	X
EO-5	√	√	X	X
EO-6	√	√	X	X
EO-7	√	√	X	X
EO-8	√	√	X	X
EO-9	√	X	X	X
EO-10	√	X	X	X
Ontogenic Learning Obstacles (OO)				
OO-1	√	X	√	√
OO-2	√	X	X	X
OO-3	√	√	√	√
OO-4	√	√	X	X
OO-5	√	X	X	X
Didactic Learning Obstacles (DO)				
DO-1	√	X	X	X
DO-2	√	√	√	√
DO-3	√	X	X	X
DO-4	√	X	X	X
DO-5	√	X	X	X
DO-6	√	X	X	X

Table 4. Improvement of Learning Obstacles

Project-based HLT for learning cubes and cuboids prompted changes in students' cognitive, affective, and psychomotor skills. Cognitive improvements were particularly evident in overcoming EO. However, some issues such as OO-1, OO-3, and DO-2 were not fully resolved. Some students still misinterpreted problem statements, leading to incorrect problem-solving stages. Errors in basic calculations also persisted, as some students remained confused about unit conversions for both surface area and volume. Students were still not accustomed to solving non-routine problems, suggesting a need for changes in teaching methods by the classroom teachers. Significant changes in students' learning obstacles were observed from the pre-class stage to Cycle 1, and from Cycle 1 to 2. These changes showed that the designed HLT gradually helped students overcome learning obstacles.

Although this study quantitatively demonstrated a high level of success in overcoming learning barriers in understanding the geometry concepts of cubes and blocks, the qualitative data from the interviews revealed deeper insights into students' experiences in constructing their knowledge. One student gave his opinion after implementing the learning, “At first I didn't know how the surface area or volume formula came about, but after learning with this miniature building project,

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

I understand better and can remember it better". This reflects the changes in students' reasoning generated through hands-on activities in acquiring new knowledge. Another student said, "I am very enthusiastic about learning with projects like this, I can exchange understanding with my friends and it helps me overcome what I don't understand". This statement shows the development of students' communication and problem-solving skills throughout the project.

During the learning process, the teacher gave feedback to each student's response to the task that was not in accordance with the expectations in the HLT by adjusting the didactical anticipation that had been prepared in cycle 1. This didactical anticipation was improved in cycle 2 to anticipate obstacles that might reappear. Improvements in feedback were made to encourage deeper exploration from students, by providing scaffolding such as: "What do you think?", "Let's try it again in another way", and "Try discussing it with your project team". These adjustments emphasize the dynamic relationship between plan and reflection in PBL that is continuously improved to overcome obstacles experienced by students.

DISCUSSION

The reflection and research stages motivate students to recall previous knowledge about cubes and cuboids. Reflection helps students learn from past experiences, transforming surface learning into deep learning. This process is crucial in PBL, as reflection is considered an essential component (Duryan, 2023; Sense, 2007; Söderlund et al., 2008). When students are asked to develop projects including cubes and cuboids, students interpret their existing understanding and transform it into the knowledge required for project. Previously acquired knowledge and experience cannot be fully used in projects without reflection. This reflection is reinforced by research activities that help students gather information related to project. In PBL, students participate in authentic research, using and constructing knowledge similarly to scientists (Novak & Krajcik, 2019). In HLT1, students initially relied on school textbooks and group discussions for research, which limited the ability to imagine buildings shaped such as cubes and cuboids. An improvement over HLT1 for HLT5 allowed students to search for information online, helping to better design the building sketches.

Observations from the reflection and research stages show that students critically analyze cubes and cuboids concepts and identify relationships between the concepts to develop projects and integrate new knowledge. Through this process, students construct an understanding of cubes and cuboids independently. The implementation results showed that these stages significantly reduced students' learning obstacles. Project activities during the reflection and research stages motivate students to explore various concepts such as shapes, elements, properties, nets, and skeletons, addressing epistemological learning obstacles (EO1-EO5). PBL also stimulates curiosity, engage-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

ment, and motivation, as students take responsibility for learning while addressing real-world obstacles (Chiang & Lee, 2016; Hsu et al., 2024; Whelan et al., 2022).

In the discovery stage, students begin to explore the concepts of cubes and cuboids through guided questions leading to the conclusion of the concepts independently. As students create building projects such as houses, hotels, and hospitals, mathematical concepts are not only learned but also relate the relevance of the concepts to daily life. This stage allows students to identify the shapes, elements, nets, and skeletons of cubes and cuboids in projects. The questions on students' worksheets further deepen the understanding. The discovery stage had a significant impact on overcoming learning obstacles related to the materials, including epistemological (EO-1 to EO-10), ontogenic (OO-1, OO-4, OO-5), and didactic learning obstacles (DO-1, DO-4, DO-5). Observations from HLT2 and HLT6 show a gradual improvement in students' understanding of these concepts.

During the application stage, students successfully applied their understanding by developing and analyzing cubes and cuboids using GeoGebra application. Technology provides immediate feedback, allowing students to self-correct and adjust more effectively (Bright et al., 2024). GeoGebra offers a visually rich and interactive learning experience, making math more interesting and engaging (Cai et al., 2020). Students can visualize cubes and cuboids from various perspectives and observe how cube nets transform into solid shapes and vice versa. The retrospective analysis showed that students' understanding of materials improved significantly. Learning obstacles related to low spatial abilities (DO-3) and low motivation (OO-5) are effectively addressed using GeoGebra. This correlates with previous research showing that GeoGebra enhances students' spatial abilities and understanding of cubes and cuboids (Kunrade et al., 2023; Ramdaniah, 2020). Additionally, GeoGebra significantly increases student learning motivation in the classroom (Hidayat et al., 2023), as students eagerly collaborate to develop and analyze cubes and cuboids. Last, in the communication stage, students show significant improvement in cognitive, affective, and psychomotor skills. Each group effectively presents project results, clearly explaining the concepts related to the materials. Students strive to showcase the best outcomes of projects and actively engage in discussions during question-and-answer sessions, which helps validate the understanding. These activities build decision-making skills, communication skills, and evaluation skills (Munna & Kalam, 2021).

The retrospective analysis shows a positive impact of project-based HLT on students' understanding. This improvement is evident in students' performance on practice questions towards the end of learning and in the interviews with the students. These results correlate with previous research by (Astuti & Wijaya, 2021) showing that project-based HLT aids in understanding mathematical concepts. The research also shows that students successfully overcome the 3 types of learning obstacles including epistemological, ontogenic, and didactic. These learning obstacles correspond to the 3 domains of learning outcomes as outlined: cognitive, affective, and psychomotor (Bloom et al., 1956).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

In the cognitive domain, students show improved understanding of concepts such as identifying elements and properties of cubes and cuboids, correctly applying formulas, overcoming the learning obstacles in visualizing 3-dimensional space, understanding the concept of volume in cubes and cuboids, and accurately using formulas. In the affective domain, students' academic awareness and interest in learning mathematics increase significantly. This enhancement in cognitive and affective domains parallels the improvement in students' psychomotor domain, particularly in spatial ability and learning independence, as showcased in projects. Generally, project-based HLT design supports meaningful geometry learning and effectively addresses learning obstacles related to cubes and cuboids.

During the implementation of project-based HLT, adjustments to the initial design were necessary, particularly concerning the time allocated for tasks. The original HLT design included 17 tasks with HLT2 containing 8 tasks completed in 1 meeting. However, during the retrospective analysis of Cycle 1, it became clear that more time was needed to ensure optimal construction of students' understanding. HLT5 is further extended to 2 meetings because students need more time to explore and develop the understanding of cubes and cuboids sub-topic. The freedom of time that teachers must give is an important concern in implementing PBL (Taylor et al., 2024). These results provide information to educational actors to anticipate more time when conducting PBL. An effort should be made to accommodate students' opportunities to construct knowledge more optimally.

CONCLUSIONS

This research concludes that project-based HLT successfully overcame learning obstacles identified in the topic of cubes and cuboids. The analysis found that HLT had a positive impact on students' understanding of the topic. This conclusion was supported by the retrospective analysis of students' practice question results and interview data. Generally, the design of learning activities using project-based HLT proved effective in supporting meaningful geometry learning and overcoming obstacles related to cubes and cuboids.

Future research could explore the application of this project-based HLT methodology to other geometric topics such as prisms, pyramids, or coordinate geometry to assess its broader applicability. Additionally, testing its effectiveness across different educational levels or learning environments—such as rural schools, inclusive classrooms, or digital/blended learning settings—would provide valuable insights. Learning obstacles arose in implementing project-based HLT, particularly concerning time allocation. Project-based learning required more time compared to traditional methods. Therefore, teachers should carefully adjust the time allocated to match the number of tasks. Addressing the learning obstacle should be a priority for those planning to implement PBL activity designs.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

These findings also can make a significant contribution to teacher training programs, particularly in developing teachers' skills in designing more structured, contextual, and student-centered learning experiences. By understanding how HLT can be used to anticipate and respond to students' thinking processes, teachers can design instructional strategies that are more adaptive and reflective of students' learning needs. In curriculum development, this approach can be utilized to design PBL sequences that are more exploratory and meaningful, while also promoting the integration of 21st-century skills such as collaboration, critical thinking, creative thinking, and problem-solving. A curriculum that accommodates HLT allows for flexibility in selecting activities that align with students' stages of thinking and the targeted learning outcomes.

ACKNOWLEDGEMENTS

This study was possible thanks to all the contributions of the researchers and supervisors in Universitas Pendidikan Indonesia, and the collaborating schools and mathematics teachers.

REFERENCES

[1] Ahamad, S. N. S. H., Li, H.-C., Shahrill, M., & Prahmana, R. C. I. (2017). Implementation of problem-based learning in geometry lessons. *Journal of Physics: Conference Series*, 943(1), 012008–0120018. https://doi.org/10.1088/1742-6596/943/1/012008

[2] Akker, J. V. D., Gravemeijer, K., McKenney, S., & Nieveen, N. (2006). Introducing educational design research. *Educational Design Research*, 3–7. https://doi.org/10.4324/9780203088364-9

[3] Astuti, W., & Wijaya, A. (2021). Learning trajectory berbasis proyek pada materi definisi him-punan. *Jurnal Riset Pendidikan Matematika*, 7(2), 254–266. https://doi.org/10.21831/jrpm.v7i2.16483

[4] Baker, C. N., Peele, H., Daniels, M. L., Saybe, M. C., Whalen, K., Overstreet, S., & Orleans, T.-I. S. L. C. T. N. (2021). The Experience of COVID-19 and Its Impact on Teachers' Mental Health, Coping, and Teaching. *School Psychology Review*, 50(4), 491–504. https://doi.org/https://doi.org/10.1080/2372966X.2020.1855473\

[5] Bardsley., & Ellen, M. (2006). Pre-Kindergarten Teachers' and Understanding of Hypothetical Learning Trajectories in Mathematics Education. In *Department of Learning and Instruction*. University of Utrecht.

[6] Baroody, A. J., Clements, D. H., & Sarama, J. (2022). Lessons Learned from 10 Experiments That Tested the Efficacy and Assumptions of Hypothetical Learning Trajectories. *Education Sciences*, 12(3). https://doi.org/10.3390/educsci12030195

[7] Bloom, B. S., Engelhart, M. D., Furst, E. J., Hill, W. H., & Krathwohl, D. R. (1956). Taxanomy of Educational Objectives The Classification of Educational Goals Handbook 1 Cognitive Domain. *Taxonomy of Educational Objectives*, 62–197.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[8] Bright, A., Welcome, N. B., & Arthur, Y. D. (2024). The effect of using technology in teaching and learning mathematics on student's mathematics performance: The mediation effect of stu-dents' mathematics interest. *Journal of Mathematics and Science Teacher*, 4(2), em059. https://doi.org/10.29333/mathsciteacher/14309

[9] Brousseau, G. (2002a). Theory of didactical situations in mathematics. In N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield (Eds.), *Didactique des mathematiques*, 1970-1990 (p. 228). Kluwer Academic Publishers.

[10] Brousseau, G. (2002b). Theory of didactical situations in mathematics. In N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield (Eds.), *Didactique des mathematiques*, 1970-1990 (p. 228). Kluwer Academic Publishers.

[11] Cai, S., Liu, E., Shen, Y., Liu, C., Li, S., & Shen, Y. (2020). Probability learning in mathematics using augmented reality: impact on student's learning gains and attitudes. *Interactive Learning Environments*, 28(5), 560–573. https://doi.org/10.1080/10494820.2019.1696839

[12] Capraro, R., Capraro, M., & Morgan, J. (2013). *STEM Project-Based Learning: An Integrated Science, Technology, Engineering, and Mathematics (STEM) Approach*. https://doi.org/10.1007/978-94-6209-143-6

[13] Cesaria, A., & Herman, T. (2019). Learning obstacles in geometry. *Journal of Engineering Science and Technology*, 14(3), 1271–1280.

[14] Chiang, C. L., & Lee, H. (2016). The Effect of Project-Based Learning on Learning Motivation and Problem-Solving Ability of Vocational High School Students. *International Journal of Information and Education Technology*, 6(9), 709–712. https://doi.org/10.7763/ijiet.2016.v6.779

[15] Chin, W. (2014). The effects of project-based learning in high school geometry. In *Doctoral Dissertation* (Issue August). University of Hawai'i at Manoa.

[16] Dimla, R. B. (2018). Probing Students' Levels of Geometric Thinking in Geometry and Their En-acted Example Space Function. *Journal of Education in Black Sea Region*, 4(1), 155–163. https://doi.org/10.31578/jebs.v4i1.162

[17] Dorn, E., Hancock, B., Sarakatsannis, J., & Viruleg, E. (2020, December 8). COVID-19 and learning loss: Disparities grow and students need help. *McKinsey & Company*., 12.

[18] Duryan, M. (2023). Reflective Practices and Learning in Construction Organisations via Profes-sional Communities of Practice. In *Construction Project Organising* (pp. 149–164). https://doi.org/https://doi.org/10.1002/9781119813798.ch10

[19] Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathemat-ics. *Educational Studies in Mathematics*, 61(1–2), 103–131. https://doi.org/10.1007/s10649-006-0400-z

[20] Farida, J. A., Siswono, T. Y. E., & Setianingsih, R. (2023). The Emergence and Form of Meta-cognitive Regulation: Case Study of More and Less Successful Outcome Groups in Solving Geometry Problems Collaboratively. *Mathematics Teaching Research Journal*, 15(1), 25–43.

[21] Fauzi, I., & Suryadi, D. (2020). Learning Obstacle the Addition and Subtraction of Fraction in Grade 5 Elementary Schools. *MUDARRISA: Jurnal Kajian Pendidikan Islam*,

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

12(1), 51–68. https://doi.org/10.18326/mdr.v12i1.51-68

[22] Fisher, D., Kusumah, Y. S., & Dahlan, J. A. (2020). Project-based learning in mathematics: A lit-eratur review. *Journal of Physics: Conference Series*, 1657(1). https://doi.org/10.1088/1742-6596/1657/1/012032

[23] Gal, H., & Linchevski, L. (2010). To see or not to see: Analyzing difficulties in geometry from the perspective of visual perception. *Educational Studies in Mathematics*, 74, 163–183. https://doi.org/10.1007/s10649-010-9232-y

[24] Gay, E. (2022a). *Project Based Learning in the Mathematics Classroom*. Assumption University.

[25] Gay, E. (2022b). *Project Based Learning in the Mathematics Classroom* [Thesis, Assumption University]. https://digitalcommons.assumption.edu/honorstheses

[26] Gravemeijer, K., & Eerde, V. D. (2009a). Design research as a means for building a knowledge base for teachers and teaching in mathematics education. *Elementary School Journal*, 109(5), 510–524. https://doi.org/10.1086/596999

[27] Gravemeijer, Koeno., & Eerde, D. Van. (2009b). Design research as a means for building a knowledge base for teachers and teaching in mathematics education. *Elementary School Journal*, 109(5), 510–524. https://doi.org/10.1086/596999

[28] Haji, S., Yensy, N. A., & Apryani, D. (2023). The Effectiveness of Problem-Based Learning Model on Geometry Theory. *Online Conference of Education Research International (OCERI 2023)*, 281–287. https://doi.org/10.2991/978-2-38476-108-1_27

[29] Henriksen, D., Gruber, N., & Woo, L. (2022). Looking to the future from the present: A call for well-being and mindfulness in education. *Journal of Technology and Teacher Education*, 30(2), 145–154.

[30] Herawati, H., & Sidik, G. S. (2023). Problem-Based Learning To Enhance Pupils' Conceptual Understanding in Geometry. *International Journal of Advance Research in Mathematics Education*, 1(1), 26–35.

[31] Hidayat, R., Kamarazan, N. A., Nasir, N., & Ayub, A. F. M. (2023). The Effect of GeoGebra Software on Achievement and Engagement Among Secondary School Students. *Malaysian Journal of Mathematical Sciences,* 17(4), 611–627. https://doi.org/10.47836/MJMS.17.4.06

[32] Hoffmann, M., & Németh, L. (2021). Is it a cube? Common visual perception of cuboid drawings. *Education Sciences*, 11(10), 1–17. https://doi.org/10.3390/educsci11100577

[33] Hsu, C. C., Su, C. S., Su, K. I., & Su, C. I. (2024). The Impact of Project-Based Learning on the Development of Statistical and Scientific Skills: A Study with Chilean University Students from the Faculty of Health Sciences. *Mathematics Teaching Research Journal*, 16(1), 5–32.

[34] Hudson, B., Henderson, S., & Hudson, A. (2015). Developing mathematical thinking in the primary classroom: liberating students and teachers as learners of mathematics. *Journal of Curriculum Studies*, 47(3), 374–398. https://doi.org/10.1080/00220272.2014.979233

[35] Kedikli, D., & Katranci, Y. (2022). Inquiry Learning Skills in Geometry : a scale development. *Acta Didactica Napocensia,* 15(1), 61–73. https://doi.org/10.24193/adn.15.1.6

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[36] Kuncoro, K. S., Juandi, D., Hidayat, R., & ... (2023a). Prospective Analysis on the Topic of Simi-larity: Designing a Hypothetical Learning Trajectory to Overcome Learning Obstacles. *Journal of Didactic …*, 1(1), 47–63.

[37] Kuncoro, K. S., Juandi, D., Hidayat, R., & ... (2023b). Prospective Analysis on the Topic of Simi-larity: Designing a Hypothetical Learning Trajectory to Overcome Learning Obstacles. *Journal of Didactic …*, 1(1), 47–63. https://doi.org/https://doi.org/10.17509/jds.v1i1.59051

[38] Kunrade, H. T., Sulangi, V. R., & Kaunang, D. F. (2023). Pengembangan Video Pembelajaran Kubus dan Balok dengan Model Project Based Learning (PjBL) Berbantuan Geogebra. *JTMT: Journal Tadris Matematika*, 4(1), 103–122. https://doi.org/10.47435/jtmt.v4i1.1869

[39] Laboy-Rush, D. (2011a). Integrated STEM education through project-based learning. *Learning.Com*, 12(4).

[40] Laboy-Rush, D. (2011b). Integrated STEM education through project-based learning. *Learning.Com*, 12(4).

[41] Lainufar, Mailizar, M., & Johar, R. (2021). Exploring the potential use of GeoGebra augmented reality in a project-based learning environment: The case of geometry. *Journal of Physics: Conference Series,* 1882(1). https://doi.org/10.1088/1742-6596/1882/1/012045

[42] Miles, Matthew. B., Huberman, Michael., & Saldana, Johnny. (2014a). *Qualitative Data Analysis: A Methods Sourcebook* (Third Edit). SAGE Publications, Inc.

[43] Miles, Matthew. B., Huberman, Michael., & Saldana, Johnny. (2014b). *Qualitative Data Analysis: A Methods Sourcebook* (Third Edit). SAGE Publications, Inc.

[44] MoEC. (2016). *Permendikbud no 022 tentang komponen perangkat pembelajaran [Regulation of the Indonesian MoEC Number 022 about component of learning instruments]*.

[45] Moru, E. K. (2009). Epistemological obstacles in coming to understand the limit of a function at undergraduate level: a case from the national university of lesotho. *International Journal of Science and Mathematics Education*, 7(3), 431–454. https://doi.org/10.1007/s10763-008-9143-x

[46] Munna, A. S., & Kalam, M. A. (2021). Impact of Active Learning Strategy on the Student En-gagement. *GNOSI: An Interdisciplinary Journal of Human Theory and Praxis*, 4(2), 96–114.

[47] Nazarovich, S. T., & Kurudirek, A. (2024). An Overview of Existing Problems in Teaching the Science ”Fundamentals of Geometry”. *International Journal of Social Sciences & Education-al Studies*, 11(2), 91–106. https://doi.org/10.23918/ijsses.v11i2p91

[48] NCTM. (2000). *National Council of Teacher of Mathematics, Principles and Standards for School Mathematics*. National Council of Teacher of Mathematics.

[49] Noreen, R., & Rana, A. M. K. (2019). Activity-Based Teaching versus Traditional Method of Teaching in Mathematics at Elementary Level. *Bulletin of Education and Research*, 41(2), 145–159.

[50] Novak, A. M., & Krajcik, J. S. (2019). A Case Study of Project-Based Learning of Middle School Students Exploring Water Quality. In *The Wiley Handbook of Problem-Based Learning* (pp. 551–572). https://doi.org/https://doi.org/10.1002/9781119173243.ch24

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[51] Nurhikmayati, I., Jatisunda, M. G., & Ratnawulan, N. (2022). The Practice of Reflection Based on Didactical Design Research: An Analysis of the Geometry Transformation Material. JTAM (*Jurnal Teori Dan Aplikasi Matematika*), 6(3), 565. https://doi.org/10.31764/jtam.v6i3.8441

[52] Pablos, V. B. G., Pozo, M. M. D., & Repiso, A. G. V. M. (2017). Project-based learning (PBL) through the incorporation of digital technologies: An evaluation based on the experience of serving teachers. *Computers in Human Behavior*, 68, 501. https://doi.org/10.1016/J.CHB.2016.11.056

[53] Plomp, T. (2006). Educational Design Research. In T. Plomp & N. Nieveen (Eds.), *Introducing Educational Design Research. Van Den, J Gravemeijer, K McKenney, S Nieveen, N* (p. 163). Netzodruk, Enschede.

[54] Priskila, P., Jamilah, J., & Oktaviana, D. (2023). Analisis Learning Obstacle Siswa Smp Pada Ma-teri Volume Kubus Dan Balok. *Journal of Comprehensive Science (JCS)*, 2(6), 1656–1663. https://doi.org/10.59188/jcs.v2i6.390

[55] Ramadhanti, F. T., & Juandi, D. (2022). Problem-based learning assisted by GeoGebra and Cabri 3D for understanding of geometrical concepts: A systematic review and meta-analysis. *International Conference on Mathematics and Science Education* (ICMScE 2021), 070034. https://doi.org/10.1063/5.0102475

[56] Ramdaniah, Mrs. (2020). Meningkatkan Pemahaman Konsep Volume Kubus Dan Balok Melalui Pembuatan Bangun Ruang 3D Geogebra. *JIRA: Jurnal Inovasi Dan Riset Akademik*, 1(4), 309–318. https://doi.org/10.47387/jira.v1i4.55

[57] Ratmokohadi, H. (2022). The Effectiveness of the Inquiry Learning Method to Improve the Ability to Identify Geometry in Mathematics Subjects. *Social, Humanities, and Educational Studies (SHES)*, 614–621.

[58] Rellensmann, J., Schukajlow, S., & Leopold, C. (2017). Make a drawing. Effects of strategic knowledge, drawing accuracy, and type of drawing on students' mathematical modelling per-formance. *Educational Studies in Mathematics*, 95(1), 53–78. https://doi.org/10.1007/s10649-016-9736-1

[59] Saadati, F., & Celis, S. (2023). Student Motivation in Learning Mathematics in Technical and Vo-cational Higher Education: Development of an Instrument. *International Journal of Education in Mathematics, Science and Technology*, 11(1), 156–178. https://doi.org/10.46328/ijemst.2194

[60] Safitri, G., & Dasari, D. (2022). Student's obstacles in learning volume of cube and cuboid. *Math Didactic: Jurnal Pendidikan Matematika*, 8(2), 112–122. https://doi.org/10.33654/math.v8i2.1844

[61] Sari, N., Saragih, S., & Elvis Napitupulu, E. (2024). Developing a Hypothetical Learning Trajectory with Problem-Based Learning and a Learning Medium for Middle School. *Educational Administration: Theory and Practice,* 30(1), 32–50. https://doi.org/10.52152/kuey.v30i1.714

[62] Sense, A. J. (2007). Structuring the project environment for learning. *International Journal of Project Management*, 25(4), 405–412. https://doi.org/https://doi.org/10.1016/j.ijproman.2007.01.013

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[63] Simon, M. A. (1995a). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. *Journal for Research in Mathematics Education*, 26(2), 114. https://doi.org/10.2307/749205

[64] Simon, M. A. (1995b). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. *Journal for Research in Mathematics Education*, 26(2), 114. https://doi.org/10.2307/749205

[65] Snyders, M. (1995). Geometry in senior secondary class. *Proceedings of the 4th Amesa Conf*, 5–13.

[66] Söderlund, J., Vaagaasar, A. L., & Andersen, E. S. (2008). Relating, reflecting and routinizing: Developing project competence in cooperation with others. *International Journal of Project Management*, 26(5), 517–526. https://doi.org/https://doi.org/10.1016/j.ijproman.2008.06.002

[67] Surya, F. Y., Pra, F., Dewi, U., & Julie, H. (2021). Hypothetical Learning Trajectory (HLT) to Build Understanding of Mathematics Education Students about What is and How to Apply Problem Based Learning (PBL) to Learn Mathematics. *International Conference on Mathematics and Learning Research Surakarta*, 18–25.

[68] Taylor, J. C., Allen, L. M., Van, J., & Moohr, M. (2024). The Effects of Project-Based Learning on Student Behavior and Teacher Burnout in an Emotional/Behavioral Support Classroom. *Journal of Emotional and Behavioral Disorders,* 32(2), 81–94. https://doi.org/10.1177/10634266241235933

[69] Triwahyuningtyas, D., & Suastika, I. K. (2021). Electronic module of geometry based on inquiry. *IOP Conference Series: Materials Science and Engineering*, 032086–032087. https://doi.org/10.1088/1757-899X/1098/3/032086

[70] Waluya, S. B., Sukestiyarno, Y. L., & Kharisudin, I. (2022). A Systematic Review on Geometric Thinking: A Review Research Between 2017-2021. *European Journal of Educational Research*, 11(3), 1535–1552. https://doi.org/10.12973/eu-jer.11.3.1535

[71] Whelan, A., Thomas, U., & Leat, D. (2022). *Project Based Learning: Developing Curiosity, Creativity and Skills for Life.* https://doi.org/10.13140/RG.2.2.21174.16969

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL
WINTER 2025
Vol 17 no 6

Discovering The Power of Realistic Mathematics Education: A Hypothetical Learning Trajectory for Teaching Social Arithmetic

Afifah Zafirah[1], Ahmad Fauzan[1*], Yerizon[1], Randika Irwa Risky[1], Fardatil Aini Agusti[2]

[1]Universitas Negeri Padang, Indonesia,

[2] Universitas Pendidikan Indonesia, Indonesia

afifahzafirah@student.unp.ac.id, ahmadfauzan@fmipa.unp.ac.id*, yerizon@fmipa.unp.ac.id,

randikairwa29@gmail.com, fardatilaini@gmail.com

Abstract: The limited use of contextualized mathematical problem design in social arithmetic, particularly with concepts of profit and loss, restricts student opportunities to develop the mathematical skills needed to understand and solve real-world financial problems. This study aimed to analyze students' thought processes using a hypothetical learning trajectory (HLT). To learn social arithmetic concepts, the HLT consisted of three components: learning objectives, mathematical tasks, and learning process hypotheses. Employing the design research methodology, this study was conducted in three phases: initial design, experimental teaching, and retrospective analysis. Three seventh-grade students from a junior high school in Indonesia were selected according to their mathematical levels, determined by low, medium, and high scores. Various methods were used to collect data, including in-depth interviews, observations, video recordings, and analysis of student work during the implementation of the intervention. To ensure students' understanding of social arithmetic concepts, the teacher provided guidance during the mathematical learning process; the students were encouraged to act and express their opinions. The results of this study indicated that the learning process, which was sequentially arranged based on the activities included in the HLT, encouraged students to think mathematically. They could construct knowledge and improve their understanding of social arithmetic concepts.

Keywords: hypothetical learning trajectory, realistic mathematics education, social arithmetic

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

INTRODUCTION

Social arithmetic is a branch of mathematics that directly connects with students' daily lives. Through social arithmetic, students learn practical skills, such as setting buying and selling prices, and calculating profits and losses, which are applicable in real-world scenarios (Laurens et al., 2017). Through studying mathematics, students not only acquire mathematical skills but also develop cognitive abilities that support strategic thinking and problem-solving, which are essential in analyzing and making decisions in everyday situations. Their daily interactions with storekeepers provide relevant experiences which make the concepts being taught familiar (Lerman, 2020). However, when social arithmetic appears as word problems in a mathematics class, students may find these problems challenging to interpret and solve. This gap between daily experiences and formal mathematical tasks highlights the need for teaching methods that connect practical knowledge with academic mathematical problem-solving (Dila & Zanthy, 2020).

Reports of the Programme for International Student Assessment show that in Indonesia, student abilities to solve and interpret contextual problems are still relatively low (OECD, 2018). Several studies have indicated that students experience conceptual and verbal difficulties in solving contextual problems (Haji et al., 2018), especially within the social arithmetic concept (Lestari et al., 2019). Students have conceptual difficulties understanding questions and transforming them into formulas (Khasanah & Sutama, 2015). Polya (1957) posited that students need to be able to use the formulas they have learned in solving problems. Difficulties in understanding social arithmetic concepts are additionally reflected in the students' struggle with prerequisite aspects. For example, students fail to write down what is known from the question and what is asked, have difficulty converting informal information into mathematical models, and lack mastery of relevant concepts. Hence, students find it difficult to determine which formula to use (Khasanah & Sutama, 2015; Fitri et al., 2019). Verbal challenges often manifest in students' misinterpretation and misunderstanding of questions, difficulty identifying question intent, and inability to rephrase the questions in their own words. Students frequently experience boredom with lengthy questions, struggle with interpreting their meaning, and tend to focus on memorization rather than conceptual understanding (Fitri et al., 2019). Additionally, students rarely complete problem-solving exercises, which contributes to frequent calculation errors. This is consistent with White's (2010) opinion that "the child identified an appropriate operation or sequence of operations but did not know the procedures necessary to perform these operations accurately." Errors in solving word problems in arithmetic are common, often resulting from student inability to correctly follow the procedures needed to execute operations (Dila & Zanthy, 2020; Zakiah, 2017).

To address the difficulties in solving social arithmetic problems, the teacher may provide learning content and context related to students' daily activities (Lestari et al., 2009; Khasanah & Sutama, 2015; Fitri et al., 2019). This will introduce a greater degree of efficiency and fun into teaching the material. Mathematics learning should be designed so that students are provided with contin-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

ued enjoyment and do not feel uncomfortable or forced. Students need to have a willingness to learn. Therefore, the employed learning approach must encourage the desire to learn (Laurens et al., 2017). A learning approach that can address the problems posed by traditional and abstract mathematics learning is realistic mathematics education (RME) (Bray & Tangney, 2016). Freudenthal's idea (1971) that mathematics is part of human life constitutes the roots of RME. This approach to learning allows students to reinvent mathematics by processing real-life situations into mathematical knowledge (Freudenthal, 2002).

Learning trajectory is fundamental in implementing RME across various subjects. It comprises a sequence of activities and tasks designed to foster students' understanding and help them achieve specific instructional goals (Simon, 1995). Gravemeijer (2004) argued that neither teachers nor researchers can rely on a fixed teaching sequence, as teachers must continually adapt to students' progress and thought processes. As such, a learning trajectory begins with a hypothesis and is therefore termed a hypothetical learning trajectory (HLT) (Simon, 1995; Gravemeijer (2004); Gravemeijer & Cobb, 2013).

This study aims to investigate students' thinking within an HLT framework applied to social arithmetic. The research focuses on examining students' mathematical reasoning, as well as the steps and decisions made by the teacher during the implementation of student-centered activities. The study addresses the following research questions:

1. How can a learning trajectory enhance students' strategies and thought processes in learning social arithmetic?
2. How does the learning trajectory affect students' independence from teacher assistance, their adaptability in strategy use, and their ability to communicate mathematically while learning social arithmetic?

LITERATURE REVIEW

RME is a learning theory derived from the ideas of Freudenthal and his colleagues at the Freudenthal Institute in the Netherlands (Sumirattana et al., 2017; Gravemeijer & Cobb, 2006; Van den Heuvel-Panhuizen, 2000). The RME theory emerged with the reform of the mechanistic approach in mathematics learning that had previously been used in the Netherlands (Freudenthal, 2000). Freudenthal (2002) emphasized that mathematics should be connected to reality, reflect human activities, and be closely tied to students' experiences and societal needs. In this context, social arithmetic is vital as it bridges the gap between abstract mathematical concepts and real-world situations, such as calculating profit, loss, and percentages. The emphasis on making things imaginable by the mind means that RME does not always use real-life problems but can employ contextual problems related to social arithmetic that students can relate to and visualize. Well-chosen

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

contextual problems in social arithmetic offer opportunities for students to develop informal and contextual solution strategies that can support the construction of mathematical concepts. Mathematics education is organized as a guided rediscovery process, allowing students to experience mathematics similarly to how it was invented (Sumirattana et al., 2017; Gravemeijer, 2004; Gravemeijer & Cobb, 2006).

According to the theory of RME, the real world is a source or starting point for the development of mathematical concepts (Sumirattana et al., 2017). The term realistic in RME comes from the Dutch term "zich realiseren," which means "to imagine." This term emphasizes the importance of problems that can be imagined by students, particularly in the context of social arithmetic. The relevance of RME in teaching social arithmetic is supported by studies indicating that contextual problems, such as those involving household finances or budgeting, facilitate a deeper understanding of mathematical processes (Cárcamo Bahamonde et al., 2017). The RME approach fosters an environment where students can engage with problems that reflect their daily lives, enhancing both their motivation and their ability to apply mathematical reasoning in practical contexts. This underlies the naming of RME (Freudenthal, 2002).

In designing instruction under RME principles, Gravemeijer (2004) proposed three core ideas: guided reinvention, didactical phenomenology, and emergent models. In the guided reinvention principle, students should have the opportunity to experience a process similar to how mathematics was invented. To accomplish this, a developer or instructional designer must design a route to allow students to discover mathematics on their own, particularly in social arithmetic scenarios, such as calculating discounts or interest rates. To implement the didactical phenomenology principle, the developer must provide students with contextual problems derived from phenomena that are real and meaningful to them (Gravemeijer & Cobb, 2013). As a result of the real phenomena, students will have the opportunity to observe the process of horizontal and vertical mathematization, especially in the context of social arithmetic. The emergent model's principle allows students to develop their own models when solving these contextual problems, beginning with informal models and gradually moving toward formal mathematical representations. The following characteristics must be present when the RME approach is used for teaching and learning: contextual problems (use context) and models (use models, bridged by vertical instruments); student contributions; interactivity; and intertwinement (Gravemeijer & Cobb, 2006).

HLTs provide a structured pathway for students to build mathematical understanding, aligning with the principles of RME. Simon (1995) defined HLTs as a set of assumptions about students' mathematical thinking, which guide the design of learning tasks to foster progressively sophisticated reasoning (Cárcamo Bahamonde et al., 2017; Akdeniz & Argün, 2021; Callejo et al., 2022). HLTs are particularly relevant in the context of social arithmetic, in which teachers hypothesize about the sequence of students' understanding as they engage with financial concepts and develop instructional experiences that support this progression (Callejo et al., 2022; Ivars et al., 2017; Wilson et al., 2014). An HLT includes three components: learning objectives, mathematical tasks (cognitive processes), and a hypothetical learning process that encompasses assumptions about

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

student understanding and reasoning that will develop during the learning process. For example, in social arithmetic, the learning trajectory may involve students starting with basic financial literacy concepts and gradually progressing to more complex calculations involving expenditure, income, profit, loss, and the method for calculating percentages (Daro et al., 2011). Teacher feedback is crucial in helping students achieve their learning goals and in ensuring that the learning trajectory remains aligned with their evolving understanding. In mathematics learning, two or more interrelated and interacting trajectories are aimed at achieving goals: (1) the teacher uses a hypothesized set of instructional experiences and tasks to "generate mental processes or actions" that develop or progress in the desired direction; and (2) students "think and learn... in a particular mathematical domain" through "developmental level progression," which must lead to the desired goal if the choice of instructional experience is to be successful (Clements & Sarama, 2004; Simon & Tzur, 2004; Dickinson et al., 2020).

An HLT focuses on the process by which students rediscover mathematical concepts. This process is divided into several levels of models that appear during learning (Gravemeijer, 2020). The models are the manners in which students perform every activity observed in learning. In the context of social arithmetic, four levels of activity are involved in the teacher's design of an HLT, namely situational, referential, general, and formal levels (Gravemeijer & van Eerde, 2009). The situational level refers to situational knowledge and strategies related to financial contexts, such as budgeting. The referential level demonstrates the financial situations described in the problems students encounter. The general level focuses on predominant strategies for solving mathematical problems related to finance, and the formal level requires reasoning with conventional symbolization to support mathematical activities in social arithmetic. Students go through these four levels of activity to solve financial problems with an informal approach and construct formal knowledge in stages (Andrews-Larson et al., 2017). Hans Freudenthal (2002) emphasized that in the RME approach, every HLT series must be connected to realistic mathematics. Realistic problems in social arithmetic can be integrated into contextual questions that can be divided into several different learning activities within HLTs (Dickinson et al., 2020; Van den Heuvel-Panhuizen & Drijvers, 2014).

Treffers (1987) distinguished two types of mathematization: horizontal and vertical. In the context of social arithmetic, horizontal mathematization involves students using mathematics to transform realistic financial problem situations into mathematical situations in the form of mathematical models, and vertical mathematization entails students working within the realm of symbolic mathematics through a process of model reorganization until problem solutions are found (Treffers, 2012; Deniz & Uygur-Kabael, 2017; Aniswita, 2024). Horizontal mathematization is related to the generalizing process: students identify regularities and relationships through visualization and schematization to develop mathematical concepts in the context of financial literacy (De Lange, 1987). By contrast, vertical mathematization formalizes these concepts, building upon the mathematical models obtained from horizontal mathematization to develop a more advanced mathematical understanding. There is no strict separation between the two processes, and they often occur in tandem, allowing students to gradually develop their understanding of both mathematical

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

concepts and their applications in social arithmetic (Deniz & Uygur-Kabael, 2017; Fauzan et al., 2018; Yilmaz, 2020).

METHODS

This study used a design research model introduced by Gravemeijer and Cobb (2013). The model has three stages: preparation for the experiment, experimenting in the classroom, and retrospective analysis. This activity began with a thought experiment (thinking about the learning trajectory that students would go through), followed by reflecting on the results of the trial experiment and further by the next thought experiment, as illustrated in Figure 1.

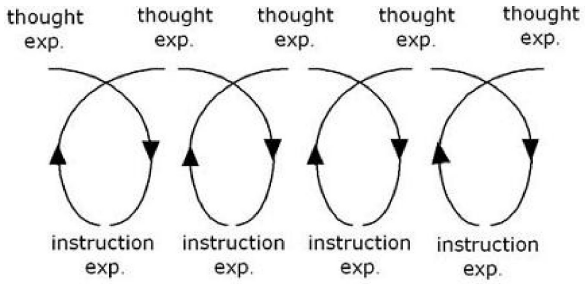


Figure 1: Gravemeijer & Cobb learning experiment (2013)

To prepare for the experiment, we established the end points of the instructions. The objectives of our social arithmetic classes were to help students rediscover concepts such as profit, loss, and the method for calculating percentages. Given that the students were already familiar with activities such as calculating sums of money and buying or selling items, we used these activities as a starting point for the lesson. Based on these starting and end points, we designed an HLT that consisted of three main activities involving solving contextual problems. This would facilitate students' horizontal and vertical mathematization while enhancing their conceptual thinking. A prototype HLT was developed (see Table 1).

Learning Activities	Social Arithmetic Concepts	Informal to Formal Mathematics Process
In the first activity, students solved a problem relating to buying goods and then selling them.	Capital (expenditures) and results (income)	Contextual understanding: Students familiarize themselves with real-world contexts by role-playing as sellers and buyers, which allows them to recognize how goods are bought and sold. Initial calculations: Students begin by calculating and

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Learning Activities	Social Arithmetic Concepts	Informal to Formal Mathematics Process
		organizing the costs of the items purchased. They estimate prices and consider which items might sell well, building a foundation for expenses and budgeting. Defining income and expenditures: They record the costs and revenue from sales, differentiating between money spent (capital) and money earned (income).
In the second activity, students solved a different problem relating to buying and selling goods.	Profit and loss	Calculating profit and loss: They compare total income with initial expenditure to determine if they gained (profit) or lost (loss) money. Introduction to profit and loss terms: The terms "profit," "loss," "selling price," and "cost price" are introduced. Students learn to define these terms in relation to their activities.
In the third activity, the students created financial reports.	Selling price, purchase price, and the percentages of profit and loss	Formulating financial statements: Students consolidate their earnings and expenses in the form of a simple financial report, transitioning from informal calculations to structured reporting. Using symbols and equations: Students start using mathematical symbols and equations to represent the selling price (HJ), purchase price (HB), and percentage profit ($U\% = \frac{U}{HB} \times 100$) or percentage loss ($R\% = \frac{R}{HB} \times 100$) Formal calculation of profit and loss: They use formulas to determine exact values for profit and loss percentages, applying these concepts to real-life data from their role-play. Reflection and analysis: Students analyze the cooperative's financial health based on their reports, which reinforces the importance of accurate calculations and leads to a formal understanding of profit, loss, and financial management.

Table 1: Learning trajectory of social arithmetic concepts

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

In addition, we designed the instructor's role in supporting students' mathematical thinking development through a sequence of tasks. Three seventh graders from junior high school in Indonesia participated in the experimental phase in small groups. We selected students based on their ability levels by grouping them into three categories: high, medium, and low. This grouping was based on the exam scores obtained from the school, as well as recommendations from their teachers. Additionally, the selection also considered gender differences to observe cognitive variations among the students. Therefore, two female students and one male student were chosen as participants in the study. This experiment aimed to observe and gain an understanding of how learning trajectories worked in shaping students' thinking from presumptive responses. In this experiment, one researcher acted as a teacher and another as an observer of the learning. Research data were collected through the triangulation technique. In the implementation process, all classroom activities, such as actions and conversations between students and teachers, were voice recorded, and students' answer sheets were collected and documented. The researchers' findings and impressions of the learning were discussed and documented immediately after the learning as additional supporting data to clarify the learning. Besides developing an HLT, the researchers conducted observations and analyses of the impact of the HLT on students' confidence when using their own strategies to solve context-related problems. Following this line of activities, student reasoning skills were examined in a pilot program.

A retrospective analysis was conducted to collect and identify student-generated data that could shape student understanding of social arithmetic. This analysis was formulated based on several questions: what learning activities were performed to generate conjectures from student responses in the HLT? How did this work? Why did it work or not work? To obtain answers to these questions, all the collected experimental data were comprehensively and critically analyzed. We reviewed the students' answer sheets according to the answer key. Next, we transcribed the interviews conducted during the learning process. This aimed to garner an understanding and explanation of the "what," "how," and "why" of the role of HLT in shaping student cognitive development. To answer the first question ("what"), we compared students' answers and responses generated during learning with the predicted student responses contained in each set of HLTs. After that, the information was examined to answer the second question ("how"). Finally, to answer the third question ("why"), we presented or discussed the findings and related them to relevant theories to gain a better understanding of the phenomenon.

RESULTS

The results of this study are presented here with a focus on the four components of an HLT, namely learning objectives regarding conceptual understanding, sequence of tasks (contextual questions), evolution/development of student's conceptual understanding, and the instructor's role in support-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

ing the development of students' mathematical thinking throughout the task sequence. The developed HLT includes three learning objectives: (1) discovering the concept of capital (expenditure) and results (income); (2) finding the concept of profit, loss, and break-even; and (3) conceptualizing the percentage of profit and loss from informal knowledge to formal knowledge.

Discovering the Concepts of Capital, Income, Profit, Loss, and Break-Even

The purpose of the first activity was to have the students discover the concepts of profit, loss, and break-even. This activity provided opportunities for students to analyze what a school cooperative officer experiences in their sales activities. A school cooperative officer is typically a student or staff member responsible for managing and overseeing the activities of a school cooperative, which is a student-run business or organization that provides goods or services to the school community. The contextual problem posed through this activity was as follows:

Bu Ani is a cooperative officer at school. Every day, buying and selling activities occur with school residents. Bu Ani sells various types of fried foods, including bakwan, stuffed tofu, and rissoles. To cook this food, Bu Ani needs to buy some ingredients. The ingredients needed by Bu Ani include 2 liters of oil, which is priced at Rp 30,000, vegetables, priced at Rp 40,000, 3 kg of flour, priced at Rp 36,000, and 15 eggs, priced at Rp 25,000. Bu Ani also needs to buy plastic packaging for Rp 10,000 and pay a cook's wage at Rp 50,000. Bu Ani will spend some money to buy these items.

In this activity, students were asked four questions: The first task was (a) *Help Bu Ani calculate the capital needed to buy all these items!* To this, students provided various answers. In the following section, a student's answer is described.

Student with Medium Ability (MA)

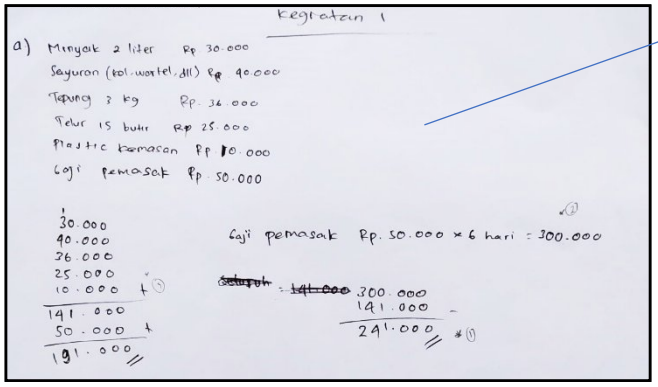
Only one student successfully provided the correct answer immediately. The other two students could only answer after the teacher asked several questions to prompt them to think about the correct answers. The researcher directed students to discover the concept of capital by asking contextual questions and directed them back to the expected flow. In this guided reinvention process, students had the opportunity to construct their own mathematical knowledge. The following is an excerpt of the teacher's questions and the student's answers:

Teacher : Why is it called capital? What are expenses?
Student MA : All items purchased, totaling Rp 241,000.
Teacher : Look again, have these items been calculated correctly?

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Student MA : Hmmm, the salary wage of a cook should still be Rp 50,000.
Teacher : Why did you get Rp 300,000?
Student MA : Because I thought all capital was for one week. It turns out that all capital was for one day.
Teacher : How much capital should Bu Ani have?
Students MA: Rp 191,000, sir.

In this dialogue, the teacher's probing questions revealed the student's thought process and understanding of social arithmetic concepts. Student MA initially showed a basic understanding of calculations by stating total expenses but struggled to distinguish between daily and weekly capital. The teacher's prompts encouraged Student MA to reevaluate their calculations, demonstrating a developing independence from teacher assistance. This interaction highlights the student's flexibility in strategy use, as they adjusted their reasoning upon realizing their misunderstanding of capital. Overall, the dialogue reflects Student MA's ability to communicate mathematically, although their errors indicate areas for further development in understanding and applying mathematical concepts related to capital and expenses. The subsequent answer can be observed in Figure 2.



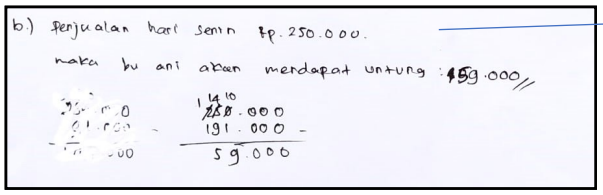
Activity 1

a) Oil, 2 liter : Rp 30,000
Vegetables (cabbage, carrots, etc.): Rp. 40,000
Flour, 3 kg: Rp 32,000
Eggs, 15 pieces: Rp 25,000
Plastic bags: Rp 10,000
Cook's salary: Rp 50,000
Cook's salary calculation:
Total for 6 days: Rp. 50,000 × 6 = Rp 300,000
Total capital calculation: Rp 191,000

Figure 2: Student MA's answer to Question (a) in Activity 1

Based on the answers of the three students (see appendix), it is concluded that Activity 1 part (a) could lead students to discover the concept of capital (expenses). This laid a foundation for solving the following problem, which was Question (b) *If Bu Ani succeeds in selling out her fried foods and earns Rp 250,000 on Monday, what will she experience? State your opinion!* Figure 3 shows the answer from one of the students.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.



Activity 1

a) Monday's sales: Rp 250,000.
Therefore, Mrs. Ani will get a profit of Rp 59,000.

Figure 3: Student MA's answer to Question (b) in Activity 1

The students' answers indicated that only one student had difficulty answering Question (b). After clarification and re-inquiring regarding Bu Ani's capital and income, the student MA explained the concepts of income and profit. The student MA was asked the following questions to explore their understanding:

Teacher : How did Bu Ani make a profit?
Student MA: Because the principal capital is Rp 191,000. Then, for example, she has fried foods, they sell well, and she earns a profit.
Teacher : Then how much should Bu Ani earn to experience a profit?
Student MA: Because the initial capital is Rp 191,000 and the income is more than that at Rp 250,000, then the profit is Rp 59,000.

In this dialogue, Student MA showed an emerging thought process in calculating profit. Student MA identified the initial capital of Rp 191,000 and recognized the successful sale of fried foods. When prompted by the teacher, the student accurately calculated the profit (Rp 59,000) by subtracting the initial capital from the income (Rp 250,000). This demonstrated Student MA's growing independence from teacher assistance and ability to apply mathematical concepts in real-world contexts. Additionally, Student MA effectively communicated their reasoning, reflecting flexibility in their strategy use, which aligned with the research question on student thought processes.

Next, students worked on Activity 1 part (c), with the following question: *If it rains heavily on Wednesday, and Bu Ani earns a total income of Rp 150,000 from selling her fried foods, what will Bu Ani experience on that day? Explain your opinion!* Figure 4 shows the answer from Student MA.

a) Therefore, Mrs. Ani earns a profit of Rp. 59,000, or a loss of Rp 100,000.
Loss calculation:
- Income: Rp 191,000
- Total expenses: Rp 150,000
- Loss: Rp 41,000

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

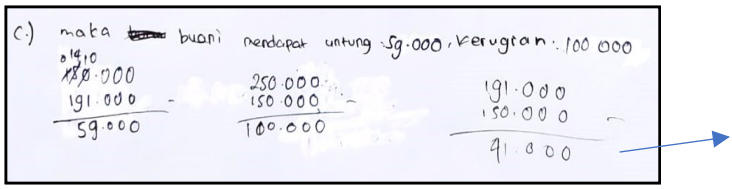


Figure 4: Student MA's answer to Question (c) in Activity 1

Teacher : What happens to Bu Ani on Wednesday?
Student (MA): Bu Ani does not sell as much as on Monday.
Teacher : (An income of) Rp 240,000 (means she) does not sell as much as on Monday, how about that?
Student (MA): A loss of Rp 100,000.
Teacher : Look at it again. How can it be called a loss? What is the benchmark we use to say that we incur a loss or earn a gain today?
Student (MA): Previous sales.
Teacher : Look again at the previous answer. Is the previous sales the correct answer?
Student (MA): Hmmm, capital, sir.
Teacher : Our benchmark is always capital, right? Not previous sales. So, what will happen to Bu Ani then?
Student (MA): It is a loss about Rp 41,000, sir.
Teacher : Why does Bu Ani suffer a loss?
Student (MA): Because the sales are less than the capital.

There was little doubt in Student MA's answer when the researcher attempted to clarify how the student obtained the answer. Student MA answered correctly, but they were unsure of what loss means. Upon direction from the researcher, Student MA gained an accurate understanding of the concept of loss. This exchange demonstrated Student MA's flexibility in adapting strategies when provided guidance as they shift from an initial, incorrect assumption to a correct understanding. The dialogue highlights Student MA's emerging independence and ability to communicate mathematically, as they corrected their reasoning and articulated that a loss occurs when sales are lower than the capital, aligning with the research focus on independence, strategy flexibility, and mathematical communication. As stated by Fauzan et al. (2018) and Möhring et al. (2021), a teacher's anticipation can help students process their knowledge well. Based on the answers of the three students (see appendix), it is concluded that Activity 1 part (c) could direct students to find the concept of loss.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Next, students worked on Activity 1 part (d), with the following question: *If on Saturday Bu Ani manages to sell fried foods with an income of Rp 191,000, what will Bu Ani experience on that day? State your opinion!* Figure 5 is the answer from the student MA.

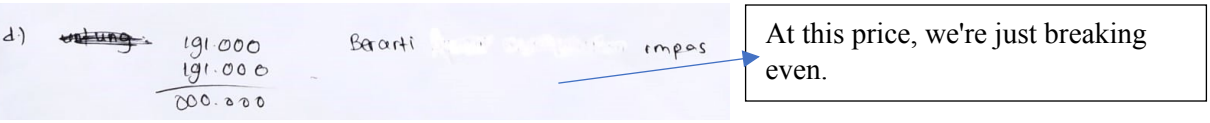


Figure 5: Student MA's answer to Question (d) in Activity 1

Teacher : What does Bu Ani experience?
Student (MA): Hmm, break-even, sir.
Teacher : Why does Bu Ani experience a return on investment or break-even?
Student (MA): Because the sales that day are valued at Rp 191,000, and the capital is also Rp 191,000.

The excerpt above shows that the student could answer correctly that Bu Ani experiences a break-even (return on investment). In this dialogue, the teacher clarified Student MA's answer regarding why they stated break-even. Then, the student explained that break-even occurs when the sales on that day match the capital spent. This shows the student's independence in thinking, as they were able to explain their reasoning without further prompting from the teacher. This exchange illustrates how the learning trajectory supports the development of students' thought processes, enhancing their ability to communicate mathematical concepts accurately.

A second activity with a more complex problem (consisting of contextual questions) was performed to retest students' understanding. This activity allowed students to determine what Bu Ani experiences as a school cooperative officer when she sells various gifts on National Teacher's Day. The contextual problem posed in this activity was as follows:

Besides food and stationery, Bu Ani intends to also sell gifts to commemorate National Teacher's Day, including flowers, chocolates, and brownies. As such, Bu Ani goes shopping for these items at once; the details are as follows:

15 stalks of red roses, bought for Rp 120,000;

10 stalks of white roses, bought for Rp 90,000;

25 pieces of chocolate, bought for Rp 100,000;

25 pieces of brownies, bought for Rp 50,000;

Plastic wrapping and gift tape, bought for Rp 30,000.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

With these items, it turns out that 25 ready-to-sell gift parcels can be produced, which are sold for Rp 18,000 per unit.

In this activity, students were presented with the first question (Question (a)), which asked, *Help Bu Ani determine what she will experience if 15 parcels are sold. Explain your answer!* The students exhibited unique characteristics in their responses, which were often unsystematic and appeared as rough scribbles. The researcher analyzed the students' thought processes by examining their answers and inquiring about their reasoning. Studies suggest that a teacher's readiness to ask probing questions can significantly enhance students' cognitive development (Sari et al., 2024; Gravemeijer & Cobb, 2006; Polya, 1957). The following questions and examples offer guidance for teachers to support students in processing information effectively (Paredes et al., 2020; Andrews-Larson et al., 2017).

Student with Low Ability (LA)

Student LA had misconceptions about the concept of loss. They mistakenly believed that subtracting the number of parcels sold from the total number of parcels available for sale would result in a loss (as shown in Figure 6). The teacher posed the following anticipatory questions to guide the students through a process of mathematization:

Teacher : Why does Bu Ani suffer a loss?
Student LA : Because she produces 25 parcels, but only 15 are sold.
Teacher : Look again! Is it right to consider that Bu suffers a loss by counting the number of parcels sold?
Student LA : Yes, sir.
Teacher : Let me give you an example, if you buy apples for Rp 2,000 per apple and you have three apples at hand, how much money did you spend on buying the apples?
Student LA : Rp 2,000 multiplied by 3, Rp 6,000, sir.
Teacher : Then you sold (the apples) for Rp 4,000 per apple. It turned out that you only sold two of apples and you ate the rest. What did you experience?
Student LA : It is a loss, sir.
Teacher : Look again! How much money did you spend earlier?
Student LA : Rp 6,000, sir.
Teacher : How much did you earn from your sales?
Student LA : Rp 8,000, sir.
Teacher : What did you experience?

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Student LA : Earning a profit, sir.
Teacher : You made a profit, even though you didn't sell all of the apples. So, what should we pay attention to in answering the questions?
Student LA : Sales and capital.

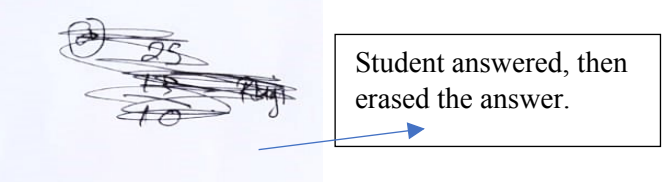


Figure 6: Initial Student LA's answers to Question (a) in Activity 2

In this dialogue, the teacher helped Student LA understand the concepts of profit and loss through a practical example. Initially, Student LA thought loss was determined by the number of items sold. However, guided by the teacher's questions, they realized that profit was based on total costs and revenues, reflecting a deeper understanding of financial concepts. Based on Figure 7, Student LA miscalculated the capital. Therefore, several questions were asked to direct Student LA to the correct answer, as shown in the excerpt below.

Teacher : Look again. Was the capital correctly calculated?
Student LA : Rp 290,000 (is) correct, sir.
Teacher : Be more careful. What is the result again?
Student LA : Rp 390,000, sir.

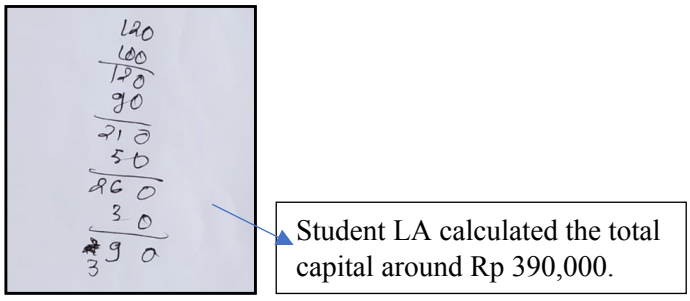


Figure 7: Student LA's answer to Question (a) in Activity 2

Although Student LA correctly identified the necessary capital, their answer lacked systematic organization, and they miscalculated the capital (as shown in Figure 7). Consequently, several questions were posed to guide Student LA to the correct answer. Student LA was then asked to contemplate the answer to the question: *What she (Bu Ani) would experience if 15 parcels were sold? Explain your answer!* Figure 8 shows a continuation of the student's answer.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

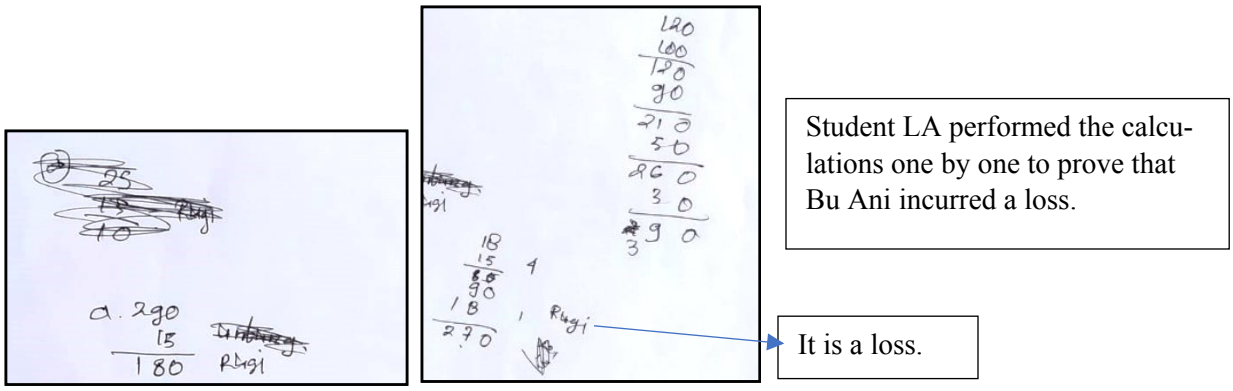


Figure 8: Student LA's final answers to Question (a) in Activity 2

Based on their answer shown in Figure 8, Student LA concluded that Bu Ani would experience a loss. However, in arriving at this conclusion, Student LA made a mistake: they did not compare the capital and income, but the capital and number of parcels sold. Additionally, Student LA still used the incorrect amount of capital, namely Rp 290,000. Therefore, to direct the student to identify the concept of loss, the following anticipatory questions were asked:

Teacher : Bu Ani only sells 15 parcels. How much will Bu Ani earn?
Student LA : Hmmm.
Teacher : Pay attention to the problem. How much does one parcel cost?
Student LA : Rp 18,000, sir.
Teacher : Then, how about 15 parcels?
Student LA : Hmm, Rp 60,000, sir.
Teacher : Look again. Is Rp 60,000 correct?
Student LA : Yes, sir. Rp 18,000 multiplied by 15. Oh my God, it is Rp 270,000, sir.
Teacher : That's right. What happens to Bu Ani then?
Student LA : It is a loss, sir.
Teacher : Why does Bu Ani suffer a loss?
Student LA : The capital is more significant, sir. Rp 390,000, sir.

Despite some mistakes in the calculation, with the help of anticipatory questions, the teacher managed to direct Student LA to understand the concept of loss. After being asked anticipatory questions, the student could understand the concept of loss by comparing capital and income. Although students successfully discovered the concept of loss with the teacher's guidance through anticipatory questions, they also significantly contributed to their learning process, allowing them to achieve the expected goals. This means that all student thinking processes (construction and pro-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

duction) could function adequately (Araújo & Lima, 2020; Amala & Ekawati, 2020; Jupri et al., 2021).

Next, students worked on Activity 2 part (b) with the question: *What is the minimum number of parcels that must be sold so that Bu Ani does not lose money?* This question was asked so that students could better understand the concepts of profit and loss, which are necessary for solving more complicated problems. Figure 9 depicts some answers from Student LA.

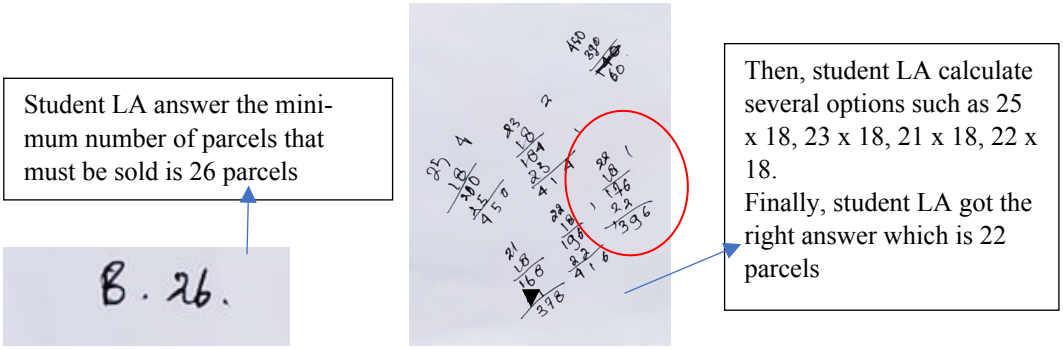


Figure 9: Student LA's answers to Question (b) in Activity 2

Based on these answers shown in Figure 9, the student LA initially answered that Bu Ani would not lose money if she sold 26 parcels. Since the student needed to understand the concepts of profit and loss through Question (b), the following anticipatory guiding questions were posed:

Teacher : Take a look. How many parcels does Bu Ani have?
Student LA : There are 25 parcels.
Teacher : What is the minimum number (that must be sold) so that Bu Ani doesn't suffer a loss?
Student LA : 26, sir.
Teacher : Why 26?
Student LA : The capital amounts to Rp 390,000. If (the parcels) are sold (in that number, the income) will be Rp 468,000. This way Bu Ani will earn a profit.
Teacher : Take a look. What is asked in the question?
Student LA : The minimum number of parcels (that must be sold) so that Bu Ani will not suffer a loss.
Teacher : What is meant by minimum?
Student LA : The least number.
Teacher : If Bu Ani has 23 parcels, what will she experience?
Student LA : It is a loss, sir.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Teacher : Why does Bu Ani suffer a loss? Take a look. If 25 parcels are worth Rp 450,000. What will 23 parcels be worth?
Student LA : Rp 414,000, sir.
Teacher : Then, what will Bu Ani experience?
Student LA : Earning a profit, sir.
Teacher : How can we know if Bu Ani suffers a loss? For example, if there are 24 packages or 21 gifts, can we know the profit or loss?
Student LA : I don't know, sir. You have to try one at a time, sir.
Teacher : After doing so, how many parcels should Bu Ani sell in order not to experience a loss did you find?
Student LA : 22 parcels, sir.
Teacher : Why does it have to be 22 parcels?
Student LA : Rp 396,000 (is the worth of) 22 parcels than the capital (it is higher than the capital), which is Rp 390,000.

In this dialogue, Student LA showed a developing understanding of profit and loss concepts in social arithmetic. Initially, they confidently stated that 26 parcels must be sold to avoid a loss, indicating a basic grasp of the problem. However, they struggled to explain the outcome when asked about selling 23 parcels, revealing gaps in their understanding. This exchange highlights the importance of teacher guidance in promoting student independence and flexibility in strategy use. Although Student LA ultimately correctly concluded the number of parcels needed to avoid a loss, the dialogue underscores the need for ongoing support to enhance their reasoning skills.

With directions and anticipatory questions, the student could finally discover the concepts of profit, loss, and break-even. However, the student could only do so by multiplying a certain number of parcels by the unit price of the parcels, and if the answer was wrong, they would repeat the process with a different number of parcels until they found the correct answer. They did not have a faster method to find an answer to Question (b). An alternative way to answer this is to divide the capital by the unit price of the parcel to determine the number of parcels that Bu Ani must sell to not lose money. Nevertheless, students were able to answer the questions by constructing their own steps and thoughts. In a realistic approach to mathematics, students may take a variety of pathways to solve a problem according to their own thinking abilities. Therefore, there is no wrong answer because the most important thing in this approach is the ability of students to construct nonformal knowledge into formal knowledge, which will allow them to solve contextual problems (Treffers, 2012; Aniswita et al., 2024; Zubainur et al., 2020). Based on the answers of the three

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

students (see appendix), it is concluded that Activity 2 part (b) could direct students to find the concept of profit.

Then, students worked on Activity 2 part (c) with the question: *What is the maximum profit that can be earned*? In answering this question, students encountered some difficulties. First, a student answered that the maximum profit was Rp 400,000, as shown in Figure 10.

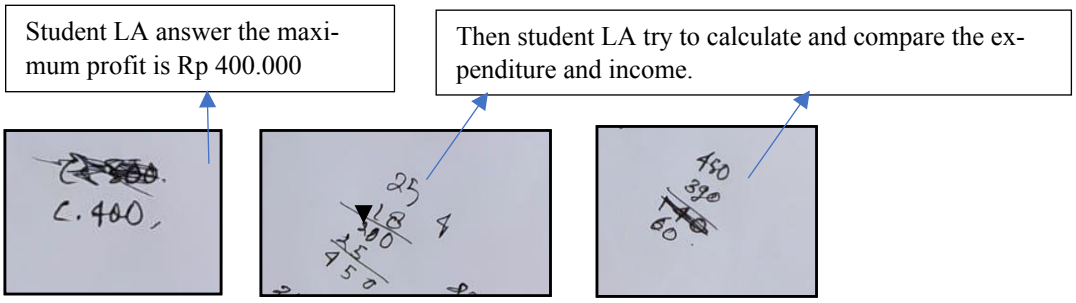


Figure 10: Student LA's answers to Question (c) in Activity 2

The teacher guided the student to the correct answer to Question (c) by presenting the following questions:

Teacher : How can Bu Ani get the maximum profit? What is meant by maximum?
Student LA : (It is) the most. All (items should be) sold.
Teacher : If all items should be sold, how many of them are there? What's the profit?
Student LA : There are 25 of them, sir. So, (the profit is) Rp 450,000.
Teacher : Is it correct to call Rp 450,000 a profit?
Student LA : Hmmm, (It is the value of the) sales, sir.
Teacher : If you want to make a profit, how do you do it? Try to remember activity (b) earlier. What was it? Profit or loss?
Student LA : Profit.
Teacher : Why was it called profit?
Student LA : Because the sales were more significant than the capital.
Teacher : How much profit does Bu Ani earn in Question (b)?
Student LA : The profit is Rp 6,000, sir.
Teacher : What is the maximum profit for sales? Take another look. How many parcels does Bu Ani sell?
Student LA : There are 25 of them, sir. So, it is Rp 450,000.
Teacher : Then, how much is Bu Ani's capital?
Student LA : Rp 390,000, sir.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Teacher : What can you conclude from this?
Student LA : Bu Ani earns a profit of Rp 60,000, sir.

The dialogue illustrates Student LA's cognitive process as they engaged with the concept of profit in a practical context. Initially, the student confused sales with profit, indicating a need for deeper understanding and independence from teacher assistance. However, as the conversation progressed, Student LA demonstrated improved flexibility in strategy use and mathematical communication, ultimately arriving at the correct conclusion about Bu Ani's profit; this reflects their developing cognitive skills in social arithmetic. However, Student LA did not write answers systematically using symbols and visual interpretations. This implies that although Student LA understood the concepts, their ability to achieve formal mathematical communication remained inadequate. The teacher should emphasize from the outset the importance of using symbols and clear interpretations when writing answers, especially for students with lower skill levels. Nevertheless, we confirmed that Student LA achieved a good understanding of the concepts of profit and loss, as demonstrated through the interview.

After being directed through anticipatory questions, students could find the concepts of profit, loss, and break-even. Students could also discover the concepts of maximum and minimum profit. Activities 1 and 2 involved RME characteristics. The use of context in problems made it easy for the students to imagine the problems. According to Freudenthal (2002), *realistic* in the RME approach means "easy to imagine," not only in direct relation to the real world but instead to the use of problems that students can imagine in the learning process.

Conceptualizing Percentages of Profit and Loss

The purpose of the third activity was for the students to discover the concept of the percentages of profit and loss and transform it from informal to formal knowledge. In this activity, students determined the percentage of profit earned by Bu Ani as a school cooperative officer. The contextual problem posed in this activity is as follows:

Bu Ani makes a school cooperative sales report every week. The report consists of the number of items sold, the percentage of profit, and the percentage of loss. Every week, Bu Ani buys goods with a total purchase of Rp 2,000,000 on average. Bu Ani wants to find the percentages of profit and loss that she experienced in the first week of June.

(a) If this week's total sales were worth Rp 2,600,000, what percentage of profit did the school cooperative earn?

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

(b) Based on the answer to Question (a), if Bu Ani made a profit with a purchase price HB, what is the profit percentage (P%)?
(c) Based on the answer to Question (b), if Bu Ani incurred a loss with a purchase price HB, what is the loss percentage (L%)?

In this activity, students were asked the first question in part (a): *If this week's total sales were worth Rp 2,600,000, what percentage of profit did the school cooperative earn?* Students were expected to find it easier to solve this problem because they already gained an understanding of the concepts of profit and loss through the previous activities. They answered correctly right away, as shown in appendix.

Student with High Ability (HA)

As shown in Figure 11, Student HA could easily find the profit percentage. Therefore, the teacher attempted to clarify the student's reasoning and learn about the student's thought processes. This was useful so that the process of mathematization performed by the student could be explored more deeply. Several questions below were asked to elicit the student's reasons for the answer.

Teacher : How much profit did Bu Ani get?
Student HA : Rp 600,000, sir.
Teacher : How could Bu Ani get this profit?
Student HA : Bu Ani's sales proceeds were Rp 2,600,000, and her capital was Rp 2,000,000. If it (the capital) is deducted (from the proceeds), she earned (a profit of) Rp 600,000.
Teacher : How to find the profit percentage?
Student HA : The profit is divided by the capital, sir, and then multiplied by 100%.

In this dialogue, Student HA demonstrated a clear understanding of the concepts related to profit calculation and percentage, indicating a strong mathematical thought process. Student HA accurately calculated the profit by deducting the capital from the sales proceeds, showcasing their ability to apply relevant mathematical operations independently and reflecting their level of independence from teacher assistance. Additionally, Student HA showed flexibility in strategy use by moving from a straightforward profit calculation to the concept of profit percentage, which requires a more advanced understanding of ratios and percentages. This ability to communicate mathematically is evident in the student's clear explanations, which not only provided the answers but also articulated the underlying processes. Overall, this dialogue highlights how the learning

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

trajectory has positively influenced the student's independence, strategy flexibility, and communication skills in social arithmetic.

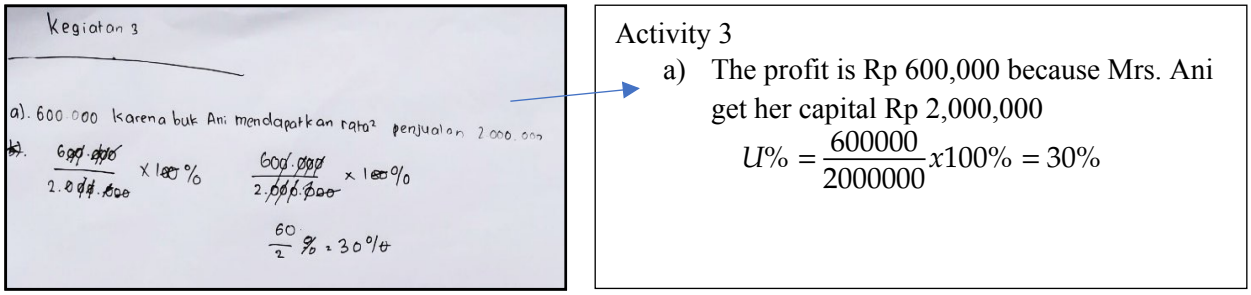


Figure 11: Student HA's answers to Question (a) in Activity 3

It also can be seen from the answers to Question (a) that Student HA could find the concept of profit percentage. Therefore, it is concluded that Activity 3 part (a) could lead students to discover the concept of profit percentages. Furthermore, Student HA was directed to establish formal knowledge in Questions (b) and (c). They were given anticipatory questions such as, "Is dividing the purchase price by the profit the correct way? What should we compare?" and the question, "What is known in the problem? Try to relate it back to the previous answer!" Through these questions, student HA could establish formal knowledge from the informal concepts found in Question (a), as shown in Figure 12.

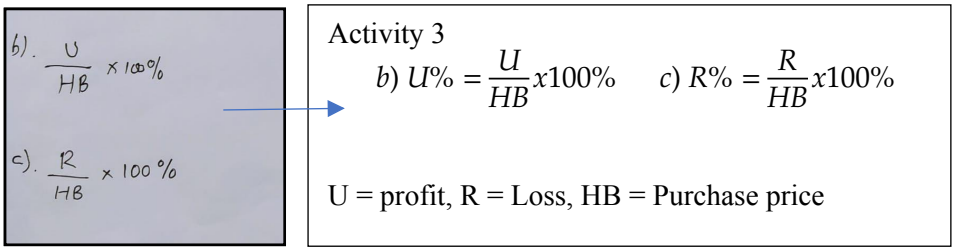


Figure 12: Student HA's answer to Questions (b) and (c) in Activity 3

DISCUSSION

Fostering Students' Thought Processes, Independence Strategies, and Mathematical Skills Through a Learning Trajectory

The obtained results suggest that students with high, medium, and low abilities had different strategies and thought processes (ways of thinking) in solving problems in each activity. The way of

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

thinking had already been established when the students started reading the assigned social arithmetic contextual problems; gradually, the students identified the problems. They began to develop models, formulas, or patterns for solving the problems, which led them to think formally. The answers provided in this article to questions in Activity 1 through Activity 3 show that Student HA could apply their mathematical abilities and explain the reasons for their answers well. Student MA applied a strategy that was not much different from that applied by the student with HA in Activity 1. However, Student MA took more time to answer the questions than Student HA. Student MA needed guidance from the teacher in completing procedures and applying concepts, facts, and arguments. By contrast, Student LA applied a strategy that was different from those applied by the other two students. The LA student tended to answer without communicating the answer mathematically. Student LA could understand the concepts of profit and loss well, but they needed guidance from the teacher to communicate these concepts. Overall, the activities provided here allowed students to use their own strategies in solving the various contextual problems given. This aligns with Aniswita's research (2024) which indicates that students with low abilities tend to require longer to solve problems and often respond with words and scribbles. Meanwhile, students with medium and high abilities tend to draw conclusions more quickly and use symbols.

De Lange (1987) referred to this as a "student's free production" or "self-developed or emergent model" (Gravemeijer, 2020). Students were provided with the freedom to solve contextual problems so that they could build their confidence in learning mathematics (Fauzan et al., 2018). The results also illustrate that RME can be a reference for teachers to construct students' problem-solving skills through the context of everyday problems. As explained by Yilmaz (2020), teachers must be trained in designing contextual problems based on RME. During the learning process, teachers are required to develop their creativity and critical thinking. The teacher's creativity will encourage them to develop HLTs with the RME approach, which encompasses cognitive domain, authenticity, and openness (Paredes et al., 2020). In testing an HLT, teachers must avoid directly supplying the answers to students. They must provide questions for each student's answer that diverges from the learning objectives. This was also the case in the implementation of the HLT developed in this study, as can be seen from the various questions asked to clarify some student answers (Möhring et al., 2021; Zubainur et al., 2020).

This learning trajectory helped students explore their ability to think mathematically by solving contextual problems about profit, loss, break-even, and percentages (Amala & Ekawati, 2020; Fauzan et al., 2018). The mathematization process occurred in every learning activity performed by the students (Widada et al., 2020; Fauzan et al., 2022). The first activity involved students acting as cooperative officers, who bought various items to be sold. Next, the students took turns acting as sellers and engaged in buying and selling transactions with other students in the school cooperative cafeteria. In the third activity, students played the role of the head of the school cooperative, preparing financial reports to evaluate the profit and loss experienced by the school

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

cooperative. From the implementation of the social arithmetic learning trajectory, it was found that the LA student had limitations in developing their mathematical communication skills. This can be seen from all figures depicting answers to questions in Activity 2. The student was unique in their way of thinking, and they needed more intensive guidance from the teacher (in the form of probing questions) to solve problems. This indicates that the learning trajectory helped the student with low ability identify how to solve problems and discover social arithmetic concepts through in-depth interviews. However, the learning trajectory was not perfect for improving the communication and critical thinking skills of the student with low ability. Observations revealed that the teacher did not remind the students to write answers systematically using symbols and visual interpretation and did not tell them how to draw conclusions. The teacher only posed questions to encourage the students to find the concept of social arithmetic on their own.

Nirawati (2021) introduced a way of thinking about HLT testing with students using the RME approach. The way of thinking can take the form of various visual and symbolic interpretations, how students explain the approaches or strategies that they use in problem-solving, and how they draw conclusions about a phenomenon. The more varied the students' thinking patterns or ways of thinking, the more developed the students' mathematical abilities. This is evident from the answers of the students with high and medium abilities who were able to develop their problem-solving skills and mathematical communication skills during the learning process. Some errors were still found when the students were solving contextual problems. These errors included counting errors and errors in applying procedures. Nevertheless, these students managed to construct their own ways of thinking to achieve the learning objectives of the HLT.

The implementation of a learning trajectory, especially through HLT within the RME framework, plays a significant role in promoting student independence from teacher assistance. By guiding students to explore and interpret contexts autonomously, they develop a more self-reliant approach to problem-solving. This independence fosters adaptability in strategy use, as students are encouraged to experiment with and select the strategies that best suit their understanding of a problem (Sahinkaya et al., 2024; Jupri et al., 2021; Fauzan et al., 2022). Consequently, this exploration of diverse strategies enhances their ability to communicate mathematical ideas clearly and effectively. When students are encouraged to articulate their way of thinking, both in explaining their strategies and in drawing conclusions, they reinforce their understanding of mathematical concepts and improve their mathematical communication skills (Nirawati et al., 2021). In social arithmetic learning, such autonomy and adaptability allow students to connect mathematical ideas to real-world contexts; this deepens their conceptual understanding and equips them with essential skills for communicating their solutions and reasoning comprehensively. Moreover, it is essential for both teachers and students to engage more thoroughly in checking activities to solidify student understanding of the mathematics learning process, ensuring that misconceptions are addressed

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

and that knowledge is effectively consolidated (Nusantara et al., 2023; Sampoerno & Meiliasari, 2019).

The activities in this research were designed to involve horizontal and vertical mathematization. As stated by Van den Heuvel-Panhuizen and Drijvers (2014), Jupri et al. (2021), and Widada et al. (2020), the mathematical process forms the basis of the HLT design that implements four levels of activity: situational, referential, general, and formal. The HLT implementation applied several RME key principles, namely guided rediscovery, didactic phenomenology, and self-developed models. In addition, the learning trajectory met the characteristics of RME, namely (1) use of context, (2) use of models for progressive mathematization, (3) use of students' construction results, (4) interactivity, and (5) linkage (Astuti, 2019; Gravemeijer, 2020). In the learning process, interactions between student and teacher and between student and student, such as negotiations, explanations, justifications, agreements, questions, and reflections, are used to achieve formal mathematical knowledge derived from informal mathematical knowledge based on the students' discoveries. In a realistic approach to mathematics, this is called interactivity. By designing an HLT, the teacher can develop a teaching and learning process that promotes students' creative thinking skills and ways of thinking and facilitates students' ability to construct new knowledge as a means of improving their understanding of mathematical concepts (Wewe & Juliawan, 2019; Fessakis et al., 2019; Laurens et al., 2017).

Practical Implications for The Teaching and Learning of Social Arithmetic

Teachers can use this learning trajectory as a reference for teaching social arithmetic and other mathematical topics in a more creative way by integrating cultural contexts and technology. The simplest initial step that teachers can take is to design a learning route that begins with clearly defined learning objectives. Subsequently, they can develop learning tasks based on contextual problems. In this regard, the use of context in RME has great potential, especially when the implementation is directly related to students' daily lives or to the cultural practices found in society. Sari et al. (2024) discussed the application of ethnomathematics in the batik-making tradition, specifically through the use of South Sumatera's Songket motif designs, to teach mathematical concepts such as symmetry, translation, and geometric transformation. Therefore, teachers can adopt a contextualizing problem to local culture approach by replacing generic examples with culturally relevant scenarios, such as traditional market transactions, local festivals, or community-based saving practices. In relation to ethnomathematics research, Nurjanah et al. (2021) suggested that the traditional Marosok livestock trading system could be used to teach social arithmetic by applying the ethnomathematical practices involved in buffalo trading within Minangkabau culture.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The use of ethnomathematical contexts presents a new experience for students in learning and teaching arithmetic through the RME approach. This implementation can enhance students' critical thinking and problem-solving abilities by facilitating the transfer of knowledge from school to real-life situations (Laurens et al., 2017; Nirawati et al., 2021). In teaching RME, teachers are also expected to be creative in preparing anticipatory questions based on various possible student responses. This aims to guide and deepen students' thinking during the problem-solving process. This approach encourages a process of reinvention and strengthens connections to broader mathematical concepts, aligning with the core principles of RME, which include: presenting problems through realistic contexts, identifying key mathematical elements, facilitating meaningful social interactions and guided teacher interventions to refine students' models, promoting reinvention throughout the problem-solving process, and emphasizing the interconnectedness of mathematical ideas (Van den Heuvel-Panhuizen, 2000).

Teachers can also combine the use of technology with the RME approach in teaching social arithmetic. Various studies have shown that technology has the potential to enhance student motivation, increase student engagement in learning, and create culturally relevant learning experiences (Zaranis, 2013; Zaranis & Exarchakos, 2018; Zhengtao & Hidayat, 2025; Bray & Tangney, 2016). Adaptive learning that utilizes emerging technologies has become an alternative solution for teachers in delivering social arithmetic lessons. For example, Zaranis and Exarchakos (2018) reported positive outcomes in teaching mathematical concepts to low-achieving students using a combination of RME and information and communication technologies (ICTs). Through this integration, students are able to investigate, solve problems, and understand concepts using various steps and tools provided by technology, particularly in the context of geometry (Zaranis, 2013).

Similarly, Zhengtao and Hidayat (2025) demonstrated that augmented reality (AR) technology can increase student interest and self-efficacy, thereby enhancing their mathematical abilities. Social arithmetic learning can empower students by incorporating cultural contexts, such as the traditional Marosok buying and selling technique or culturally contextualized word problems, with the aim of improving mathematical communication and problem-solving skills (Sahinkaya et al., 2024). Social arithmetic can also be taught through technological integration—such as AR or other ICTs-based software—to bridge cultural gaps by making abstract mathematical concepts more relatable. For instance, teachers can use real-world math applications that are connected to students' cultural backgrounds. For future research, scholars may explore the combination of Marosok techniques with AR for students with disabilities to support their numeracy development or investigate economic patterns and architectural designs from various cultures through educational software, including educational games. When thoughtfully integrated, mathematics teaching and learning can become more meaningful and engaging, offering students valuable and relevant experiences for everyday life.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

CONCLUSION

In general, the designed learning trajectory has helped students discover the concepts of profit, loss, break-even, and percentage formulas. The activities in the HLT provided students with experiences that built their conceptual understanding and mathematical thinking skills through solving problems. For instance, in activities simulating buying and selling scenarios, students independently developed strategies such as comparing selling prices to purchase prices to determine profit or loss. The teacher's role in the learning process constituted one of the main factors guiding the rediscovery of social arithmetic concepts, especially for students who faced difficulties and required more time to understand mathematical ideas. Through strategic questioning and scaffolding, teachers guided students toward rediscovering social arithmetic concepts rather than directly providing solutions, fostering deeper cognitive engagement.

Furthermore, the HLT effectively enriched students' strategies and thought processes in learning social arithmetic by creating an environment where they could engage more deeply with the material. As students navigated through the learning trajectory, those with intermediate and advanced abilities demonstrated good independence, strategies, and mathematical communication skills in solving problems. However, students with lower abilities required more intensive guidance from the teacher to achieve formal knowledge. In this study, students were intentionally provided with the flexibility to answer, but it was found that additional guidance was still necessary to enhance the mathematical communication skills of students with lower abilities. Overall, the HLT with a sequential RME approach gradually encouraged students to think mathematically, construct knowledge, and improve their understanding of social arithmetic material. It illustrated the relationship between students' cognitive processes and their independence, strategic flexibility, and mathematical communication skills within the framework of a learning trajectory in their educational experiences.

LIMITATIONS

The limitation of this study is its use of a small research sample. Nevertheless, the in-depth interviews conducted in this study were able to identify students' thought processes while they were discovering social arithmetic concepts through buying and selling activities. We suggest that future researchers conduct more in-depth research with a larger sample and expand the learning trajectory related to problem-solving and mathematical communication abilities within the context of technology and local culture.

ACKNOWLEDGMENTS

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

We would like to thank the Ministry of Education, Culture, Research, and Technology for the generous grant under the scheme of the Master to Doctoral Education Scholarship Program for Excellent Scholars (PMDSU). Any opinions, research findings, conclusions, and recommendations expressed in this work belong to the authors and do not reflect the views of the Ministry of Education, Culture, Research, and Technology. The Decree Number is 1350/UN35.15/LT/2024 and Agreement/Contract Number is 011/E5/PG.02.00/PL.PMDSU/2024.

REFERENCES

[1] Akdeniz, D. G., & Argün, Z. (2021). Learning trajectory of a student with learning disabilities for the concept of length: a teaching experiment. *The Journal of Mathematical Behavior*, 64, 100915. https://doi.org/10.1016/j.jmathb.2021.100915

[2] Amala, M. A., & Ekawati, R. (2020). The profile of horizontal and vertical mathematization process of Junior High School Student with high mathematical ability in solving contextual problem of fraction. *Jurnal Riset Pendidikan dan Inovasi Pembelajaran Matematika (JRPIPM)*, 3(2), 52-60. https://doi.org/10.26740/jrpipm.v3n2.p52-60

[3] Andrews-Larson, C., Wawro, M., & Zandieh, M. (2017). A hypothetical learning trajectory for conceptualizing matrices as linear transformations. *International Journal of Mathematical Education in Science and Technology*, 48(6), 809-829. https://doi.org/10.1080/0020739X.2016.1276225

[4] Aniswita, Fauzan, A, Armiati. (2024). How Students Understand the Area under a Curve: A Hypothetical Learning Trajectory. *Mathematics Teaching Research Journal*, 16(3), 80-100. https://mtrj.commons.gc.cuny.edu/volume-16-n-3/

[5] Araújo, J. D. L., & Lima, F. H. D. (2020). The mathematization process as object-oriented actions of a modelling activity system. *Bolema: Boletim de Educação Matemática*, 34, 847-868. https://doi.org/10.1590/1980-4415v34n68a01

[6] Astuti, N. R. (2019). The effect of RME on mathematics learning outcomes viewed mathematic communication skills. *International Journal of Educational Research Review*, *5*(1), 43-53. https://doi.org/10.24331/ijere.650864

[7] Bray, A., & Tangney, B. (2016). Enhancing student engagement through the affordances of mobile technology: a 21st century learning perspective on Realistic Mathematics Education. *Mathematics Education Research Journal*, 28, 173-197. https://doi.org/10.1007/s13394-015-0158-7

[8] Callejo, M. L., Pérez-Tyteca, P., Moreno, M., & Sánchez-Matamoros, G. (2022). The use of a length and measurement HLT by pre-service kindergarten teachers' to notice children's mathematical thinking. *International Journal of Science and Mathematics Education*, 20(3), 597-617. https://doi.org/10.1007/s10763-021-10163-4

[9] Cárcamo Bahamonde, A. D., Fortuny Aymemí, J. M., & Gómez i Urgellés, J. V. (2017). Mathematical modelling and the learning trajectory: tools to support the teaching of linear algebra. *International journal of mathematical education in science and technology*, 48(3), 338-352. https://doi.org/10.1080/0020739X.2016.1241436

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[10] Clements, D. and Sarama, J. (2004). *Learning trajectories in mathematics education*. Mathematical Thinking and Learning, Routledge.
[11] Daro, P., Mosher, F. A., & Corcoran, T. B. (2011). *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction*. CPRE Research Report. https://repository.upenn.edu/handle/20.500.14332/8415
[12] De Lange, J. (1987). *Mathematics, Insight and Meaning*. University Utrecht: OW & OC.
[13] Deniz, Ö. & Uygur-Kabael, T. (2017). Students' mathematization process of the concept of slope within the realistic mathematics education. *Hacettepe University Journal of Education*, 32(1): 123-142. https://doi.org/10.16986/HUJE.2016018796
[14] Dickinson, P., Eade, F., Gough, S., Hough, S., & Solomon, Y. (2020). Intervening with realistic mathematics education in england and the cayman islands—the challenge of clashing educational Ideologies. *International reflections on the Netherlands didactics of mathematics*, 341.
[15] Dila, O. R., & Zanthy, L. S. (2020). Identifikasi kesulitan siswa dalam menyelesaikan soal aritmatika social, *Teorema: Teori dan Riset Matematika*, 5(1), 17-26. http://dx.doi.org/10.25157/teorema.v5i1.3036
[16] Fauzan, A., Andita, C. D., Rada, G., Zafirah, A., & bin Abdullah, A. H. (2022). Developing RME-Based Learning Trajectory for Teaching Addition to A Dyscalculia Student in Elementary School. *Jurnal Didaktik Matematika*, *9*(1), 39-58. https://doi.org/10.24815/jdm.v9i1.25340
[17] Fauzan, A., Armiati, A., & Ceria, C. (2018). A learning trajectory for teaching social arithmetic using RME approach. In *IOP Conference Series: Materials Science and Engineering* (Vol. 335, No. 1, p. 012121). IOP Publishing. https://doi.org/10.1088/1757-899X/335/1/012121
[18] Fessakis, G., Karta, P., & Kozas, K. (2018). The math trail as a learning activity model for m-learning enhanced realistic mathematics education: A case study in primary education. In *Teaching and Learning in a Digital World: Proceedings of the 20th International Conference on Interactive Collaborative Learning–Volume 1* (pp. 323-332). Springer International Publishing. https://doi.org/10.1007/978-3-319-73210-7_39
[19] Fitri, N. M. A., Adhirakasiwi, A., & Utami, M. (2019). Analisis kesulitan siswa smp dalam menyelesaikan soal cerita aritmatika social, *Prosiding Sesiomadika*, 1(1a). https://journal.unsika.ac.id/sesiomadika/article/view/2153
[20] Freudenthal, H. (1971). *Geometry Between The Devil and The Deep Sea*. Educational Studies in Mathematics, 3, 413-435.
[21] Freudenthal, H. (2002). *Didactical Phenomenology of Mathematical Structures*. Kluwer Academic Publishers.
[22] Gravemeijer, K. (2004). *Developing Realistic Mathematics Education*. Utrecht: Freudenthal Institute.
[23] Gravemeijer, K. (2020). Emergent Modeling: an RME Design Heuristic Elaborated in a Series of Examples. *Educational Designer*, 4(13), 1-31.
[24] Gravemeijer, K. and Cobb, P. (2013). *Educational Design Research Part A: an Introductioned T Plomp and N Nieveen*. Enschede: SLO, p 72-11.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[25] Gravemeijer, K., & Cobb, P. (2006). *Design research from a learning design perspective*. In Educational design research (pp. 29-63). Routledge.

[26] Gravemeijer, K., & van Eerde, D. (2009). Design research as a means for building a knowledge base for teachers and teaching in mathematics education. *The Elementary School Journal*, 109(5), 510-524. https://doi.org/10.1086/596999

[27] Haji, S., Yumiati, Y., & Zamzaili, Z. (2018). Analisis kesulitan siswa dalam menyesaikan soal-soal PISA (Programme for International Student Assessment) di SMP kota bengkulu, *Jurnal Pendidikan Matematika Raflesia*, 3(2), 177-183. https://doi.org/10.33369/jpmr.v3i2.7522

[28] Ivars, P., Fernández, C., & Llinares, S. (2017). Pre-service teachers' uses of a learning trajectory to notice students' fractional reasoning. In *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 25-32). Singapore: PME.

[29] Jupri, A., Usdiyana, D., & Sispiyati, R. (2021). Teaching and Learning Process for Mathematization Activities: The Case of Solving Maximum and Minimum Problems. *Journal of Research and Advances in Mathematics Education*, 6(2), 100-110. https://eric.ed.gov/?id=EJ1301169

[30] Khasanah, U., & Sutama, M. P. (2015). *Kesulitan Menyelesaikan Soal Cerita Matematika Pada Siswa Smp Negeri 1 Colomadu Tahun Pelajaran 2014/2015* (Doctoral dissertation, Universitas Muhammadiyah Surakarta). https://eprints.ums.ac.id/32806/

[31] Laurens, T., Batlolona, F. A., Batlolona, J. R., & Leasa, M. (2017) How does realistic mathematics education (RME) improve students' mathematics cognitive achievement? *Eurasia Journal of Mathematics, Science and Technology Education*, 14(2), 569-578. https://doi.org/10.12973/ejmste/76959

[32] Lerman, S. (Ed.). (2020). *Encyclopedia of mathematics education*. Cham: Springer International Publishing. https://doi.org/10.1007/978-3-030-15789-0_300003

[33] Lestari, D. T., Senjayawati, E., & Rohaeti, E. E. (2019). Analisis kesulitan belajar siswa smp kelas viii dalam menyelesaikan soal aritmatika di tinjau dari kemampuan komunikasi matematis, *Journal on Education*, 1(2), 440-444. https://doi.org/10.31004/joe.v1i2.85

[34] Möhring, W., Ribner, A. D., Segerer, R., Libertus, M. E., Kahl, T., Troesch, L. M., & Grob, A. (2021). Developmental trajectories of children's spatial skills: Influencing variables and associations with later mathematical thinking. *Learning and Instruction*, 75, 101515. https://doi.org/10.1016/j.learninstruc.2021.101515

[35] Nirawati, R., Fatimah, S., & Juandi, D. (2021). Realistic Mathematics Learning on Students' Ways of Thinking. *Mathematics Teaching Research Journal*, 13(4), 112-130. https://eric.ed.gov/?id=EJ1332358

[36] Nurjanah, N., Mardia, I., & Turmudi, T. (2021). Ethnomathematics study of Minangkabau tribe: Formulation of mathematical representation in the Marosok traditional trading. *Ethnography and Education*, 16(4), 437-456. https://doi.org/10.1080/17457823.2021.1952636

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[37] Nusantara, T., As' ari, A. R., & Susanto, H. (2023). Exploring Students' Work in Solving Mathematics Problem through Problem-Solving Phases. *Mathematics Teaching Research Journal*, 15(3), 190-220. https://eric.ed.gov/?id=EJ1408184

[38] OECD, 2018. *Programme for International Students Assessment (PISA)-Results from PISA Country Note: Indonesia*. (OECD Publishing, Paris, 2019a). https://www.oecd.org/pisa/publications/PISA2018_CN_IDN.pdf.

[39] Paredes, S., Cáceres, M. J., Diego-Mantecón, J. M., Blanco, T. F., & Chamoso, J. M. (2020). Creating realistic mathematics tasks involving authenticity, cognitive domains, and openness characteristics: A study with pre-service teachers. *Sustainability*, 12(22), 9656. https://doi.org/10.3390/su12229656

[40] Polya, G. (1957). *How to Solve It. A New Aspect of Mathematical Method*. 2nd Edition. Princeton University Press, Princeton.

[41] Sahinkaya, N., Özcan, Z. Ç., & Obalar, S. (2024). Visualizing Math Word Problems: Impact on First-Grade Students' Problem-Solving Performance. *Mathematics Teaching Research Journal*, 16(3), 146-163.

[42] Sampoerno, P. D., & Meiliasari, M. (2019, December). Analysis of the mathematical learning materials with the characteristics of realistic mathematics education in the design research pre-service teachers' theses in Indonesia. In *Journal of Physics: Conference Series* (Vol. 1402, No. 7, p. 077105). IOP Publishing. https://doi.org/10.1088/1742-6596/1402/7/077105

[43] Sari, A., Putri, R. I. I., & Zulkardi, R. C. I. P. (2024). Ethnomathematics in Indonesian Woven Fabric: The Promising Context in Learning Geometry. *Mathematics Teaching Research Journal*, 16(5), 157-185. https://mtrj.commons.gc.cuny.edu/volume-16-n-5/

[44] Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for research in mathematics education*, 26(2), 114-145. https://doi.org/10.5951/jresematheduc.26.2.0114

[45] Simon, M. A., & Tzur, R. (2004). Explicating the Role of Mathematical Tasks in Conceptual Learning: An Elaboration of the Hypothetical Learning Trajectory. *Mathematical Thinking and Learning*, 6(2), 91–104. https://doi.org/10.1207/s15327833mtl0602_2

[46] Sumirattana, S., Makanong, A., & Thipkong, S. (2017). Using realistic mathematics education and the DAPIC problem-solving process to enhance secondary school students' mathematical literacy. *Kasetsart Journal of Social Sciences*, 38(3), 307-315. https://doi.org/10.1016/j.kjss.2016.06.001

[47] Treffers, A. (1987). *Three dimensions. A model of goal and theory description in mathematics education-the wiskobas project*. Springer, Netherland.

[48] Treffers, A. (2012). *Three dimensions: A model of goal and theory description in mathematics instruction—The Wiskobas Project* (Vol. 3). Springer Science & Business Media.

[49] Van den Heuvel-Panhuizen, M. (2000). *Mathematics education in the Netherlands: A guided tour*. Freudenthal Institute CD-rom for ICME9, 1-32. Retrieved from http://www.fi.uu.nl/en/ rme/TOURdefpref.pdf.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[50] Van den Heuvel-Panhuizen, M. and Drijvers, P. (2014). *Realistic Mathematics Education*. In Encyclopedia of Mathematics Education. https://doi.org/10.1007/978-3-030-15789-0_170

[51] Wewe, M., & Juliawan, I. W. (2019). Developing mathematical devices with characteristics realistic mathematics education. *Al-Jabar: Jurnal Pendidikan Matematika*, 10(1), 1-10. https://dx.doi.org/10.24042/ajpm.v10i1.3884

[52] White, A. L. (2010). Numeracy, literacy and newman's error analysis, *Journal of Science and Mathematics Education in Southeast Asia*, 33(2), 129-148. https://eric.ed.gov/?id=EJ970194

[53] Widada, W., Herawaty, D., Beka, Y., Sari, R. M., Riyani, R., & Nugroho, K. U. Z. (2020, February). The mathematization process of students to understand the concept of vectors through learning realistic mathematics and ethnomathematics. In *Journal of Physics: Conference Series* (Vol. 1470, No. 1, p. 012071). IOP Publishing. https://doi.org/10.1088/1742-6596/1470/1/012071

[54] Wilson, P. H., Sztajn, P., Edgington, C., & Confrey, J. (2014). Teachers' use of their mathematical knowledge for teaching in learning a mathematics learning trajectory. *Journal of Mathematics Teacher Education*, 17, 149-175. https://doi.org/10.1007/s10857-013-9256-1

[55] Yilmaz, R. (2020). Prospective Mathematics Teachers' Cognitive Competencies on Realistic Mathematics Education. *Journal on Mathematics Education*, 11(1), 17-44. https://eric.ed.gov/?id=EJ1237688

[56] Zakiah, N. E. (2017). Pembelajaran dengan pendekatan kontekstual berbasis gaya kognitif untuk meningkatkan self-awareness siswa, *Teorema: Teori dan Riset Matematika*, 2(1), 11-20. http://dx.doi.org/10.25157/teorema.v2i1.704

[57] Zaranis, N. (2013, July). The use of Information and Communication Technologies in the first grade of primary school for teaching rectangles based in Realistic Mathematics Education. In *IISA 2013* (pp. 1-6). IEEE. https://doi.org/10.1109/IISA.2013.6623694

[58] Zaranis, N., & Exarchakos, G. M. (2018). The Use of ICT and the realistic mathematics education for understanding simple and advanced stereometry shapes among university students. *Research on e-Learning and ICT in Education: Technological, Pedagogical and Instructional Perspectives*, 135-152. https://doi.org/10.1007/978-3-319-95059-4_8

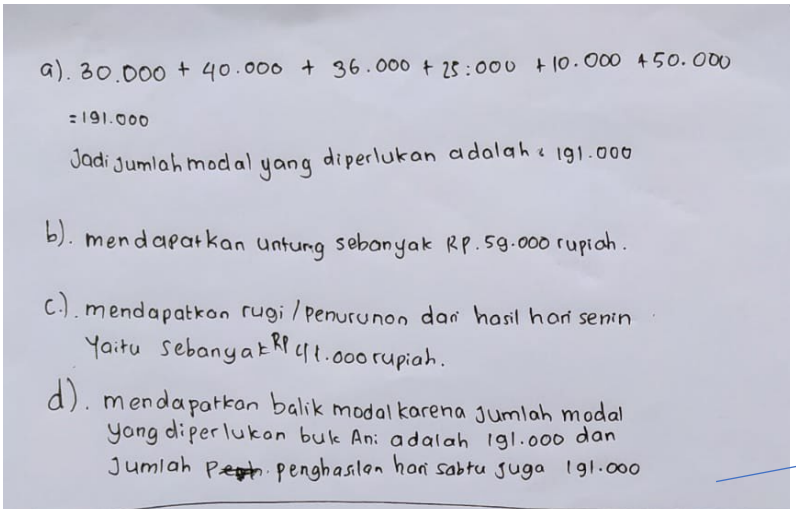
[59] Zhengtao, Z., & Hidayat, R. (2025). The Effects of Augmented Reality (AR) Toward Achievement on The Graphs and Geometry Topic Among Third-Grade Students. *Mathematics Teaching Research Journal*, 17(1), 82-98. https://mtrj.commons.gc.cuny.edu/

[60] Zubainur, C. M., Johar, R., Hayati, R., & Ikhsan, M. (2020). Teachers' understanding about the characteristics of realistic mathematics education. *Journal of Education and Learning (EduLearn)*, 14(3), 456-462. https://doi.org/10.11591/edulearn.v14i3.8458

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

APPENDIX

Appendix 1. Student HA's answer to question in HLT



a). 30,000 + 40,000 + 36,000 + 25,000 + 10,000 + 450,000 = 191,000
So, the total capital needed is 191,000.

b). Obtained a profit of 59,000 rupiah.

c). Incurred a loss/decline from the result on Monday, which was 11,000 rupiah.

d). Break-even because the total capital needed by Mrs. Ani is 191,000, and the total income on Saturday is also 191,000.

Figure 1. Student HA's answer to question in activity 1

Activity 1 part A & B

Teacher : How did you get a profit in part B? Why is there a profit?
Student HA : Because the main capital is Rp 191,000. Then, if, for example, the fried snacks sell out of them, we get a profit.
Teacher : How much is the profit? From where?
Student HA : Because the initial capital is Rp 191,000, then the profit is Rp 59,000 from the total calculation. The total sales are Rp 250,000.

Activity 1 part C

Teacher : What about part C?
Student HA : To determine a loss because sales went down.
Teacher : Why was there a loss?
Student HA : Because Mrs. Ani's sales weren't as high as on Monday.
Teacher : If sales were Rp 240,000, which is less than Monday, what happens?
Student HA : There's a loss of Rp 10,000.
Teacher : Why is there a loss? What's the benchmark for determining profit or loss when selling? Do we compare it to previous sales or something else?
Student HA : We compare it to previous sales.
Teacher : What if Mrs. Ani's sales today were Rp 200,000?
Student HA : It's a loss, because it's less than previous sales.
Teacher : How much preparation cost does Mrs. Ani need?

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Student HA : Rp 191,000, Teacher.
Teacher : If sales are Rp 200,000, is it profit or loss?
Student HA : Profit.
Teacher : So what should be our benchmark: previous sales or something else?
Student HA : The capital.
Teacher : in part C, why is there a loss?
Student HA : Because sales were lower than the previous day.
Teacher : What's the comparison? The previous day, right? So the profit is not Rp 41,000?
Student HA : Compared to the capital, Teacher.

Activity 1 part D

Teacher : Then for part d, breaking even means reaching the point where there's no profit or loss, right? Why is that?
Student HA : Because the sales that day were Rp 191,000, and our capital was also Rp 191,000.
Teacher : Okay, that's correct.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Appendix 2. Student LA's answer to question in Activity 1

Activity 1 part A

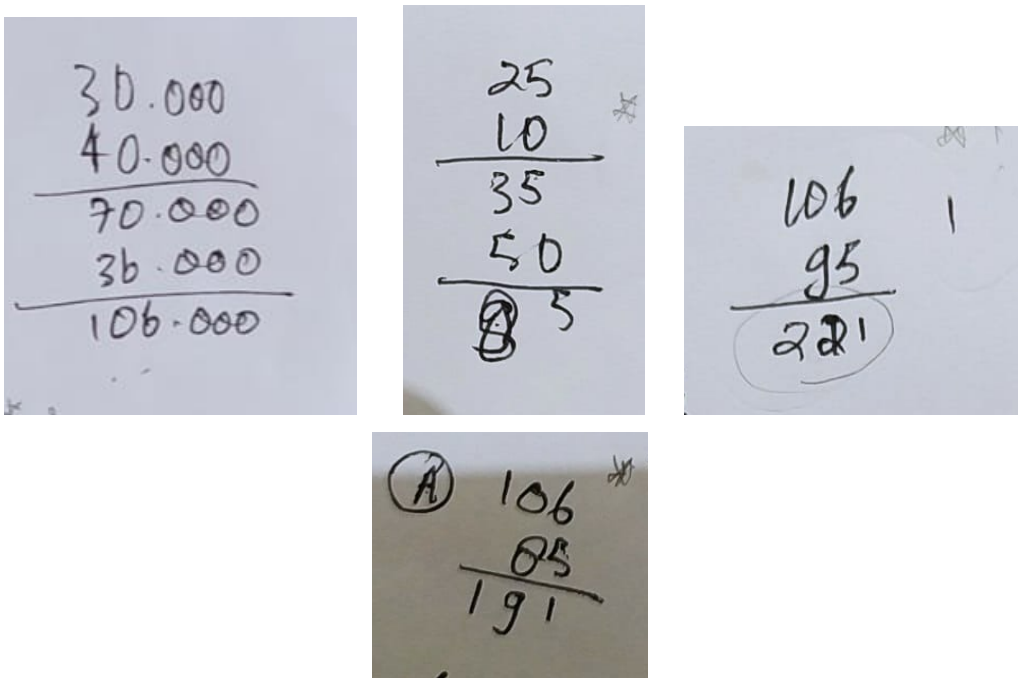


Figure 2. Student LA's answer (a) to question in activity 1

Student LA : What does this mean, sir?
Teacher : Has the number (a) been completed? What is the result?
Student LA : Rp 221,000, sir.
Teacher : Hmmm, Rp 221,000, okay. Now, where does the Rp 95,000 come from?
Student LA : This is Rp 25,000 plus Rp 10,000, so Rp 35,000, then add Rp 50,000, making it Rp 95,000.
Teacher : Hmm, what's 35 plus 50? Hmm, what is 3 plus 5?
Student LA : Hmm, 3 plus 5 is 9, ehh, 8.
Teacher : Yes, 8, okay. Try correcting it and just cross it out there. Try to fix your answer at the beginning.
Student LA : So, what's the answer?
Student LA : Six plus five, hmm, eleven; one plus eight, hmm, nine, so the answer is 191,000

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Activity 1 part B

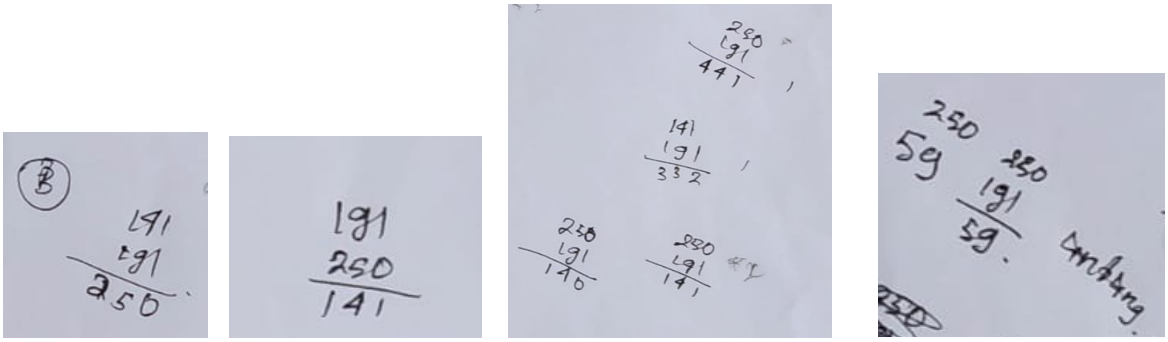


Figure 3. Student LA's answer to question (b) in activity 1

Teacher : How about the answer for (b)?

Student LA : It's done, sir. Here, Rp 191,000 plus Rp 141,000 equals Rp 250,000.

Teacher : Rp 141,000, is it correct?

Student LA : Yes, sir.

Teacher : It seems like there's still an error. Check if it was added correctly.

Student LA : Eh, it should be subtracted, sir. So it's 140,000, sir.

Teacher : Look carefully, why does 5 minus 9 give you 4?

Student LA : Ehhh. It's because I borrowed using the borrowing system, so 250,000 minus 191,000. The difference is 59,000, sir.

Teacher : How did you know that's a profit? What did you understand?

Student LA : Hmm, sales.

Teacher : Yes, so what does it mean?

Student LA: If the sales are higher than expenditure, it's a profit, sir.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Activity 1 part C ***Activity 1 part D***

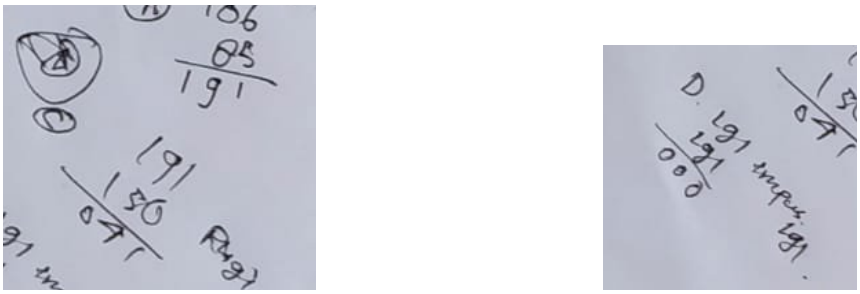
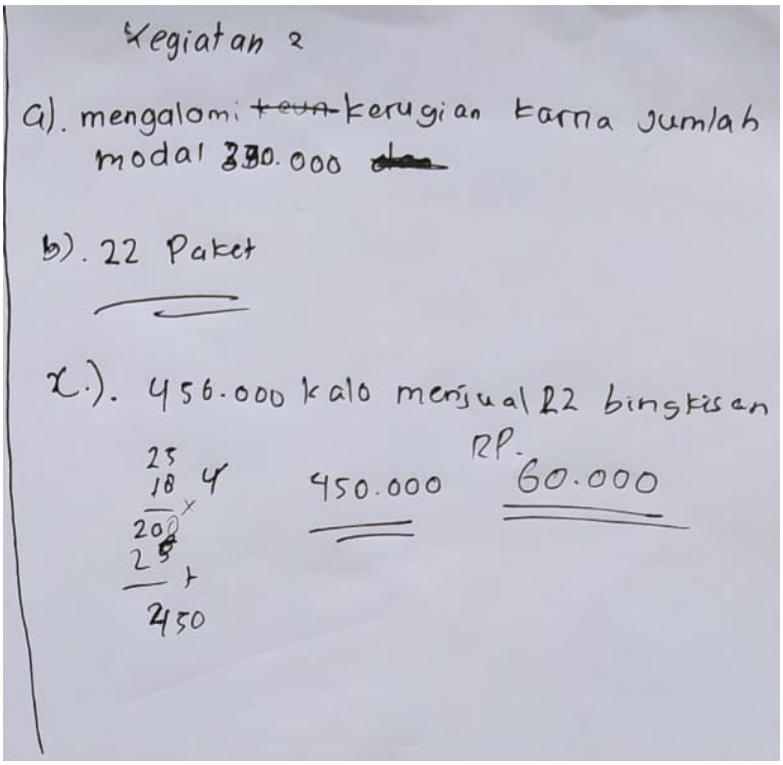


Figure 4. Student LA's answer to question (c) and (d) in activity 1

Teacher : Now, for (c). How much did Bu Ani earn from sales?
Student LA : Rp 150,000, sir.
Teacher : What was the cost price earlier?
Student LA : Rp 191,000, sir.
Teacher : So, what does that mean for Bu Ani now?
Student LA : A loss, sir.
Teacher : Why is it a loss?
Student LA : Hmmm, because the cost is higher than the sales, ehh, the revenue.
Teacher : So, it's a loss. That's what you need to understand, Dimas. How much is the loss?
Student LA : Hmmmm.
Teacher : So, what's the difference?
Student LA : Rp 41,000, sir.
Teacher : How much is the sales for (d)?
Student LA : Rp 191,000, sir.
Teacher : What was the cost price?
Student LA : Rp 191,000, sir.
Teacher : So, which is greater?
Student LA : They're the same, sir.
Teacher : If they're the same, what does that mean?
Student LA : Hmmm.
Teacher : Is it a loss, a profit, or what?
Student LA : Break-even, sir.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Appendix 3. Student HA's answer to question in Activity 2



Activity 2

a) Experiencing a loss because the total capital is Rp 250,000.

b) 22 Packages

c) Rp 456,000 if selling 22 packages.

Calculation:

- Total = Rp 450,000

- Profit = Rp 60,000

Figure 5. Student HA's answer to question in activity 2

Activity 2 part A

Teacher : Why did Mrs. Ani experience a loss?
Student HA : Because the income earned is less than the capital.

Activity 2 part B

Teacher : What's the minimum number of packages needed so that Mrs. Ani does not incur a loss?
Student HA : 22 packages.
Teacher : Why is it 22?
Student HA : Because 22 times 18 equals 456,000.
Teacher : How did you figure that out?
Student HA : multiplying by 18.
Teacher : What did you multiply by 18?
Student HA : 22 times 18, Sir.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Teacher : What if it's 21? Would it be a profit or a loss?
Student HA : A loss.
Teacher : If she sells 21, how much money would that be?
Student HA : 378,000, still below 390,000.
Teacher : And if 22 is multiplied by 18, what do you get?
Student HA : 456,000, Sir.
Teacher : Hmm, is that correct? It seems there's an error in the multiplication.
Student HA : 396,000 for 22 packages.

Activity 2 part C

Teacher : What's the maximum profit Mrs. Ani can make?
Student HA : Rp 450,000, Sir.
Teacher : When is the maximum profit achieved? What does "maximum" mean?
Student HA : It means the most, Sir.
Teacher : So, how many packages need to be sold for the maximum profit?
Student HA : More than 22, around 29, Sir.
Teacher : Why is it 29?
Student HA : Because it's above the capital, Sir.
Teacher : What is Mrs. Ani selling to make a profit?
Student HA : Gift packages, Sir.
Teacher : How much is each package?
Student HA : Rp 18,000 each.
Teacher How many packages are prepared?
Student HA : 25, Sir.
Teacher : To achieve maximum profit, does everything have to be sold?
Student HA : Yes, everything has to be sold, Sir.
Teacher : What's the total if everything is sold?
Student HA : Rp 18,000 times 25, Sir.
Teacher : That's the sales revenue, not the profit. So, how do you calculate the profit?
Student HA : By subtracting the capital from the sales revenue, Sir.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Appendix 4. Student MA's answer to question in Activity 2

Activity 2 part A

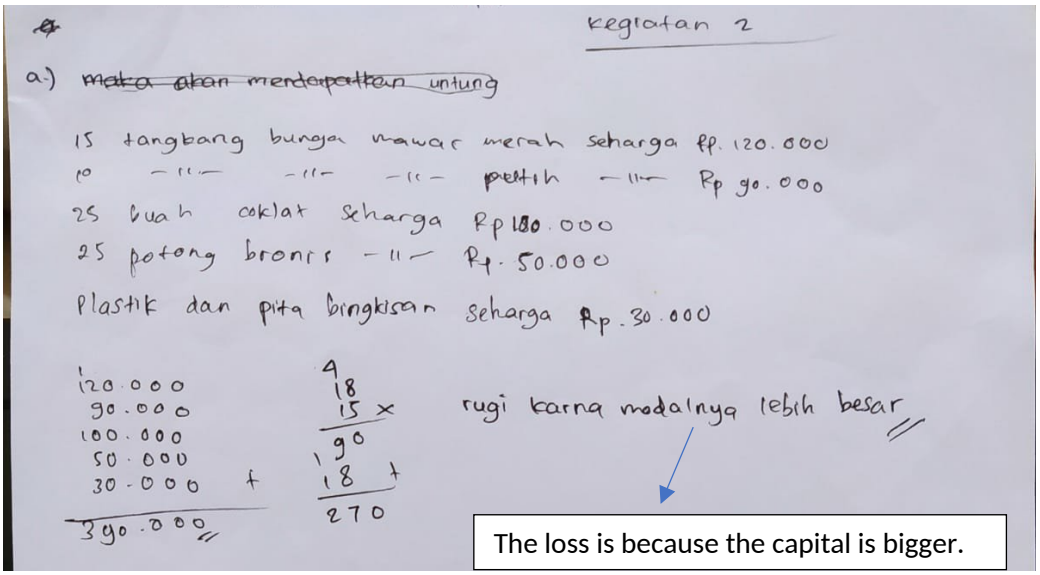


Figure 6. Student MA's answer to question (a) in activity 2

Teacher : If there are 15 packages, how much money do they get?

Student MA : If it's 25 packages, it's Rp18,000.

Teacher : Now only 15 packages are sold. How much money does Bu Ani get?

Student MA : Hmm...

Teacher : How much is one package?

Student MA : Rp 18,000 multiplied by 15 packages... hmm... Rp 270,000.

Teacher : Hmm, what's the cost price?

Student MA : the correct answer is Rp 390,000.

Teacher : What about the sales?

Student MA : Hmm, Rp 270,000 pak.

Teacher : What does it mean?

Student MA : It is a loss, sir.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Teacher : Why is it a loss?

Student MA : Because the cost is higher than capital.

Activity 2 part B

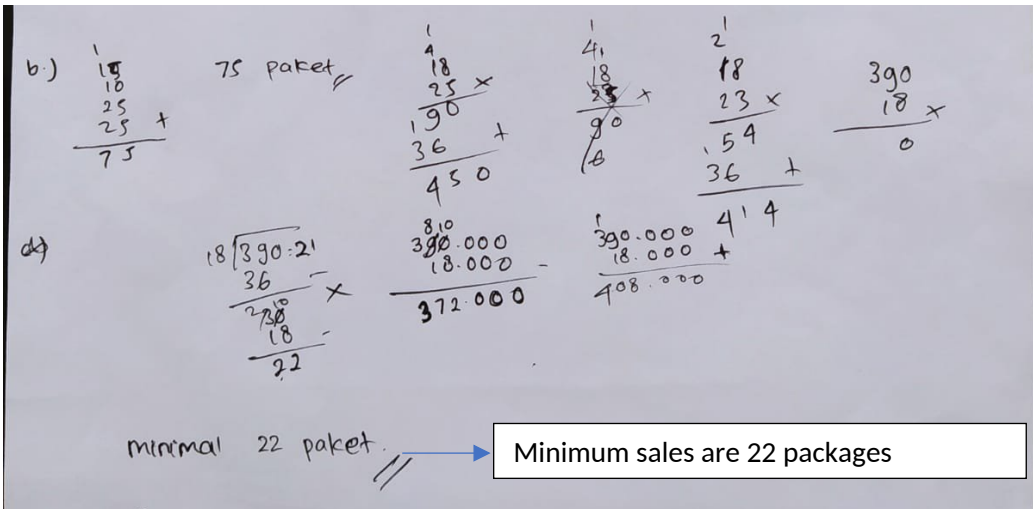


Figure 6. Student MA's answer to question (b) in activity 2

Teacher : To know the total, how do we calculate it?
Student MA : Make it larger than this (total of package).
Teacher : For example, how many?
Student MA : it is 25, sir.
Teacher : If it's 25, is there a loss or not?
Student MA : (Hesitant)
Teacher : How about 23?
Student MA : (Hesitant)
Teacher : Is it a loss or a profit?
Student MA : Profit, Pak.
Teacher : What if it's break-even, is it a loss or not?
Student MA : Not a loss, sir.
Teacher : How do we know for sure?
Student MA : Multiply, sir.
Teacher : What do we multiply?
Student MA : The cost, sir, Rp 390,000.
Teacher : Multiply by what?
Student MA : Rp 18,000, sir.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Teacher : If we multiply, it will be very large. How do we handle that?
Student MA : Divide, Pak.
Teacher : The result?
Student MA : 21.1, Pak.
Teacher : So, what's the minimum to avoid a loss?
Student MA : Round it up to 22, Pak.
Teacher : What if it's 22?
Student MA : Profit, Pak. So, the answer to avoid a loss is 22 packages.

Activity 2 part C

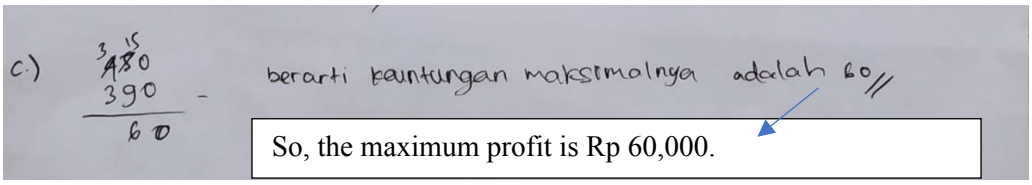


Figure 7. Student MA's answer to question (c) in activity 2

Teacher : If everything is sold out, what is it called?
Student MA : The profit is maximized, Pak.
Teacher : How many packages did Bu Ani make in total?
Student MA : 25, Pak.
Teacher : Price per package?
Student MA : Rp 18,000.
Teacher : So, if 25 packages are sold, what's the total sales?
Student MA : Rp 450,000, Pak.
Teacher : Is that profit or sales?
Student MA : Sales, Pak.
Teacher : What's the cost price?
Student MA : Rp 390,000.
Teacher : So, what's the profit?
Student MA : Rp 60,000, Pak.
Teacher : Correct. Please write the explanation.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Appendix 5. Student MA's question in activity 3

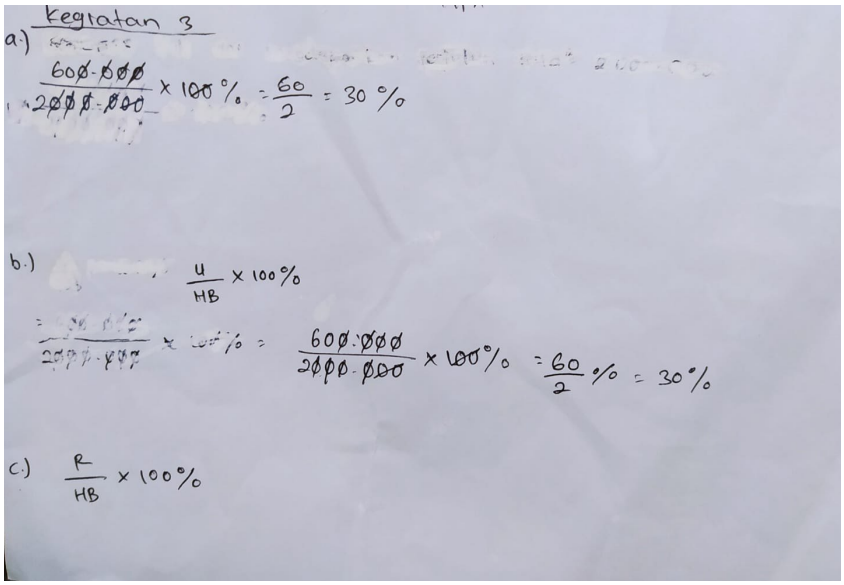


Figure 7. Student MA's answer to question in activity 3

Teacher : Where did this come from? Why is the answer Rp 600,000 divided by 2,000,000?
Student MA : Yes, because the average sales are Rp 2,000,000.
Teacher : Is that really the sales?
Student MA : Hmm...
Teacher : Then, what's this Rp 600,000?
Student MA : Hmm... that's the profit.
Teacher : How do you know that's the profit?
Student MA : The sales are Rp 2,600,000.
Teacher : So, what is this Rp 600,000?
Student MA : Hmm... just remove the 2's, Pak. You get Rp 600,000. It can't be bigger than Rp 2,000,000.
Teacher : Why can't it be?
Student MA : Hmm... this Rp 2,000,000 is the capital.
Teacher : What is the sales?
Student MA : Rp 2,600,000, Pak.
Teacher : So, how much is the profit?
Student MA : Hmm... 600,000, Pak.

Appendix 6. Student LA's question in activity 3

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

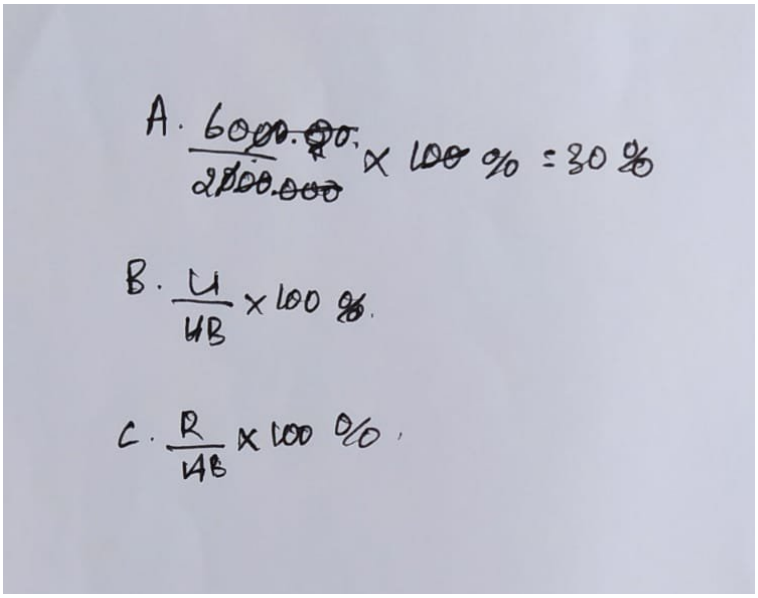


Figure 7. Student LA's answer to question in activity 3

Teacher : Now, if we want to know the profit percentage, what should we compare?
Student LA : Hmm...
Teacher : What terms have we discussed so far?
Student LA : Sales, capital, loss, profit, break-even.
Teacher: So, how do we know if an item has a profit or loss?
Student LA : Sales.
Teacher : What else?
Student LA : Compared to the capital.
Teacher : How can there be a loss?
Student LA : If the capital is higher.
Teacher : How can there be a profit?
Student LA : If the capital is lower.
Teacher : That's the basic idea behind finding the profit percentage. Try to find it, okay?

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The Application of Problem-Based Learning in Improving Junior High School Students' Mathematical Communication Skills

Asri Ode Samura[1], In Hi Abdullah[2], Muhammad Daut Siagian[3], Habibi Ratu Perwira Negara[4]

[1]Institut Agama Islam Negeri Ternate, Ternate, Indonesia

[2]Universitas Khairun Ternate, Ternate, Indonesia

[3]Universitas Singaperbangsa Karawang, Karawang, Indonesia

[4]Universitas Islam Negeri Mataram, Mataram, Indonesia

asriodesamura@iain-ternate.ac.id

Abstract: Mathematical communication is a core competency in mathematics education, yet many students continue to experience difficulties in expressing mathematical ideas coherently. This study investigates the effectiveness of problem-based learning (PBL) in enhancing students' mathematical communication skills in junior high school mathematics. A quasi-experimental design with a nonequivalent control group was employed, involving 78 students from a public junior high school in Indonesia. Students' mathematical communication skills were measured using essay-based tests administered as pretests and posttests. The data were analyzed using nonparametric statistical techniques to examine differences in learning outcomes between groups. The results indicate that students who received instruction through problem-based learning demonstrated significantly higher gains in mathematical communication skills compared to those taught using conventional instructional methods. These findings highlight the pedagogical value of problem-based learning in fostering students' ability to articulate mathematical communication and suggest its potential as an effective instructional approach in secondary mathematics education.

Keywords: Problem-Based Learning, Mathematical Communication Skills

INTRODUCTION

One of the skills that students need to possess is mathematical communication. Through reasoning, deliberation, and decision-making, this skill enables pupils to comprehend mathematics (Wardono et al., 2020; Sercenia et al., 2023). Additionally, communication skills can help students illustrate mathematical concepts in a variety of ways. Thus, the main goal of learning mathematics should be developing mathematical communication skills in order to encourage students' pas-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

sion and critical thinking as well as to communicate a variety of concepts (Kamid et al., 2020; Triana et al., 2019; Sa'adah et al., 2023).

Presenting mathematical concepts orally, in writing, and using graphs, pictures, and other visual aids is part of mathematical communication skills. Students' mathematical knowledge can be strengthened by mathematical communication, both written and verbal (Kusumah et al., 2020; Ikhsan et al., 2020). Students can exchange ideas through a mathematical communication process. Therefore, it is critical to effectively incorporate mathematical communication into the classroom and guide students in expressing and recording their thoughts, questions, and solutions. According to several accounts, mathematical communication is expressing mathematical concepts and symbols verbally, in writing, in pictures, or in diagrams (Çelik Demirci & Baki, 2023; Balyan et al., 2022; Smith et al., 2023).

In mathematical communication, one must ask questions, formulate arguments, and provide answers (Pourdavood et al., 2020). Given the enormous variation in intelligence across students, brain function plays a significant influence in determining how well individuals are able to think and communicate. In order to increase student engagement and yield more meaningful learning, teachers should create a fun learning environment and assess students' critical thinking skills according to van Dijk & Lane (2020) and Öztaş (2023).

In order for pupils to become proficient communicators in mathematics, teachers must encourage their curiosity. Teachers can give students the chance to communicate their thoughts by utilizing the learning opportunities that are offered (Fonseca, 2021; Chasanah et al., 2020). If a teacher consistently employs a learning model or strategy, learning can be successful. A learning platform that can assist teachers in teaching content such that pupils grasp it fast is called a learning model or approach. In order to facilitate students' understanding of the content being taught quickly, the problem-based learning approach is used. Problem-based learning is thought to improve students' ability to communicate mathematical ideas to others (Ahdhianto et al., 2020; Kurniasih et al., 2022; Greenstein & Fernández, 2023; Iwuanyanwu, 2021).

Problem-based learning is a learning approach that provides students with real-world situations to help them learn how to communicate mathematical ideas and solve problems, as well as help them retain the knowledge and concepts taught. When problem-based learning is used, students must be given problems (Darhim et al., 2020; Treepob et al., 2023). Problem-based learning involves the following steps: (1) the teacher poses a problem in front of the class; (2) students have conversations in small groups; (3) students carry out an independent study of problems that need to be solved; and (4) students review the problems that have been solved. Students work in home groups to solve problems, share knowledge, and learn from each other; (5) students present the solutions they found; and (6) teachers help students complete assessments related to all learning activities (Tadjer et al., 2022; Ukobizaba et al., 2021; Magaji, 2021). Specifically, the problem-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

based learning approach calls for the long-term development of students' mathematical communication skills, even while it theoretically facilitates the development of those skills based on applied designs. Through conversations and the presentation of issues that may be used with the problem-based learning paradigm, mathematical communication skills can be fostered (Dinç et al., 2022; Palupi et al., 2020; Cresswell-Yeager, 2021).

Creating new mathematical problems or questions by reformulating existing ones is known as posing mathematical problems (Lee, 2021; Putra et al., 2020). The process of creating a math problem consists of the following steps: (1) choosing a starting point; (2) quality recording (registering property); (3) What-if-not-ing (aka asking "what if not?"); (4) raising questions or concerns; and (5) evaluating problems and finalizing them after analysis. In addition to fostering students' mathematical thinking, mathematical problem-solving offers opportunities for critical interpretation and analysis of reality (Muhtarom et al., 2020; Passarella, 2021; Papadopoulos et al., 2022; Piñeiro et al., 2021).

Numerous pertinent studies, such as Domu et al. (2023), have provided evidence for the aforementioned claim. They found that problem-based learning in reverse online classes enhances students' statistical literacy skills in remote learning. Positive feedback was also observed from students working on entrepreneurship-related statistics projects. This is evident from the opinions of students in this study who expressed a high level of satisfaction with their entrepreneurship-related projects. Additionally, the study's findings support the notion that incorporating students into distance learning is crucial.

According to Susanti et al. (2023), their study developed a problem-based learning model implemented in a probability theory course, which consists of several instructional stages. These stages include introducing students to a given problem, organizing a structured learning schedule, facilitating individual and group investigations, guiding students in developing and presenting their solutions, and conducting analysis and evaluation of the problem-solving process. The findings indicate that both lecturers and students perceive problem-based learning as a practical and effective instructional approach. In the context of probability theory courses, the application of problem-based learning supports students in improving problem-solving abilities and fostering critical thinking skills. Therefore, the integration of problem-based learning is considered a viable strategy for optimizing learning outcomes in probability theory instruction.

According to Rohid et al. (2019), one in three pupils are able to communicate mathematical concepts through words, notation, and symbols as well as comprehend, analyze, evaluate, and respond to mathematical concepts. According to this research, students still need to improve their mathematics communication skills. Math teachers should take note of this research so they can use creative and innovative learning activities to help students develop their mathematical communication skills in addition to teaching mathematics.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Despite numerous studies emphasizing the significance of mathematical communication skills and the role of problem-based learning (PBL) in enhancing these skills (e.g., Wardono et al., 2020; Sercenia et al., 2023; Ahdhianto et al., 2020; Kurniasih et al., 2022), there remains a need for more comprehensive research investigating the specific impact of PBL on junior high school students' mathematical communication abilities. Existing studies have predominantly focused on theoretical frameworks and applied designs, emphasizing the long-term development of these skills (Dinç et al., 2022; Palupi et al., 2020; Cresswell-Yeager, 2021). However, there is limited empirical evidence demonstrating whether PBL directly enhances students' ability to communicate mathematical concepts effectively, particularly in the context of junior high school education.

Furthermore, although studies such as Rohid et al. (2019) have indicated that students continue to encounter challenges in mathematical communication, there is a lack of detailed exploration into how PBL can be tailored to address these challenges. Specifically, the effectiveness of PBL in fostering skills such as oral and written mathematical expression, the utilization of visual aids, and collaborative problem-solving among junior high school students remains underexplored. Additionally, while some research has examined PBL in higher education or specific subjects like probability theory (Susanti et al., 2023), its application and outcomes in junior high school mathematics classrooms require further investigation.

LITERATURE REVIEW

Mathematical Communication

Students' mathematical communication skills can be developed through the education process at school. This happens because one of the elements of mathematics is the science of logic, which develops students' thinking abilities. So, mathematics is vital in developing mathematical communication skills (Ya-Amphan et al., 2024; Rajagukguk et al., 2022).

Mathematical communication is a method of expressing and interpreting mathematical ideas orally or in writing, pictures, tables, diagrams, formulas, or demonstrations. Studying mathematics without being balanced with mathematical communication ensures the learning process will not run well. Mathematical communication can help students develop their personality and ability to solve problems in learning mathematics, convey ideas correctly, and make it easier for students to convey ideas to other people so that the information is easy to understand (Uyen et al., 2021; Kusumah et al., 2020).

Greenes and Schulman (in Ata Baran & Kabael, 2021) explain that mathematical communication is (1) the ability to express mathematical ideas through speech, writing, demonstration, and depict them visually in different types, (2) the ability to understand, interpret, and assess ideas presented

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

in writing, orally, or in visual form, and (3) the ability to construct, interpret and connect various representations of ideas and their connections.

According to the National Council of Teachers of Mathematics, mathematical communication skills refer to the ability to organize mathematical thinking, communicate mathematical ideas logically and clearly to others, analyze and evaluate the mathematical thinking and strategies of others, and use mathematical language to express ideas. The ability to articulate mathematical ideas during problem solving and to communicate the processes and outcomes of problem solving also constitutes essential skills that foster higher-order mathematical thinking, including logical, analytical, systematic, critical, creative, and productive thinking (Trgalová & Tabach, 2023; Estabrooks & McArdle, 2022)

Mathematical communication skills can be classified into two parts, namely written and unwritten or verbal mathematical communication skills. Oral mathematical communication skills are mathematical communication built verbally in the form of speaking, listening, discussing, and exchanging opinions without being proven in writing. Meanwhile, written mathematical communication skills are students' mathematical communication skills using vocabulary, structures, and mathematical notation in the form of reasoning connections, as well as in problems presented in the form of graphs, pictures, tables, equations, or writing (Smith et al., 2023; Pourdavood et al., 2020).

The mathematical communication skills analyzed in this study are specifically focused on written communication. This is based on the objectives and indicators studied, which are to describe mathematical communication skills in writing. The indicators of written mathematical communication skills utilized in this study encompass Grammatical Ability, Sociolinguistic Ability, and Strategic Ability.

Grammatical Ability comprises two primary aspects. Firstly, the appropriate use of mathematical symbols or notations. This involves students' ability to present problems using tables, figures, mathematical models, or symbols. Students are expected to articulate statements or problems in mathematical language utilizing appropriate models, symbols, or visuals. In problem-based learning, students have been trained to solve problems using models and subsequently express their solutions in the form of mathematical representations or symbols. Secondly, the formulation of mathematical definitions. Students are expected to define terms in mathematics with clarity. This definition serves as a foundation for introducing concepts, elucidating mathematical topics, or discussing linear equations.

Sociolinguistic Ability also comprises two aspects. Firstly, the expression of problems in mathematical language. This entails students' ability to construct mathematical models and conclude problems presented. Secondly, the mathematical solution of problems. This includes students'

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

proficiency in providing reasons for their solutions, explaining the methods employed to solve problems, and concluding appropriately.

Furthermore, Strategic Ability encompasses students' ability to extract information from given problems. Students must identify the available information in the question and determine whether it is sufficient to solve the problem. Additionally, this aspect includes students' ability to describe the strategies employed to solve contextual problems. Students must be able to articulate the systematic steps involved in solving problems with appropriate and correct strategies. In problem-based learning, students are trained to collaborate with peers and teachers, fostering their accustomedness to express their ideas or thoughts and optimizing the teaching and learning process.

Problem-Based Learning

The method of addressing problems through science is the main focus of a set of learning exercises known as problem-based learning. Problem-based learning is characterized by three key elements. First, problem-based learning is an activity-based learning approach, which means that students must complete multiple tasks in order to utilize it. Problem-based learning involves students actively thinking, communicating, searching and processing facts, and ultimately drawing conclusions. It does not assume that students will just listen, take notes, and then memorize the course information. Secondly, the goal of learning activities is to solve issues. Problem-based learning is a crucial term in education. This implies that in order for learning to occur, difficulties must exist. Third, a scientific thinking technique is used for solving problems. A scientific method of thinking involves both inductive and deductive reasoning. This methodical, empirical thought process is employed. Systematic refers to the steps that scientific thought takes, whereas empirical refers to the use of accurate data and facts as the foundation for problem-solving (Katsara, 2023; Liu et al., 2022; Rogers et al., 2021).

According to Duch (in Kelly et al., 2022), problem-based learning is an instructional approach that promotes students' ability to learn and collaborate in groups to find answers to issues in the real world. Before beginning to learn a subject, pupils' curiosity is piqued by problem simulations. Problem-based learning equips students with the critical and analytical thinking skills necessary to locate and make effective use of learning materials.

Problem-based learning is an educational approach that develops students' thinking, problem-solving, social, and autonomous learning skills as well as their ability to construct or acquire new knowledge by using authentic, real-world problems that are left open-ended for student solution (Dakabesi & Luoise, 2019; Tadjer et al., 2022). This type of learning is distinct from traditional forms of learning, which only use real-world situations as a means of applying the knowledge that have been acquired at the end of the learning process. Genuine issues were chosen according to

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

how well they will help students develop key skills (Sancar-Tokmak & Dogusoy, 2023; Ernawati et al., 2022).

Problem-based learning is characterized by several characteristics, according to Barrows (in Buus & Pedersen, 2021): (1) student-centered learning; (2) learning occurs in small groups; (3) the teacher acts as a facilitator or guide; (4) the problems presented in the learning environment are arranged with a particular form and focus and function as a learning stimulus; (5) new information is obtained through independent learning (self-directed learning); and (6) problems serve as a means to develop clinical problem-solving abilities.

The following describes the characteristics of problem-based learning based on Barrows' explanation:

(1) Learning starts from problems.
(2) The problems should be relevant to students' daily lives.
(3) Learning must be organized based on problems rather than scientific disciplines.
(4) Students should be responsible for forming and implementing their learning process.
(5) Small groups should be used.
(6) Students must demonstrate their learning through performances and products.

This problem-based learning approach is based on students addressing real-world problems, but they also need to gain experience with unforeseen circumstances. Problem-based learning must so accomplish a number of particular goals (Noordzij & Wijnia, 2020). The following are the goals of putting this program into practice in order to raise the caliber of students:

(1) To enhance pupils' ability to choose and make decisions using critical thinking.
(2) Teach problem-solving techniques in a methodical, responsible manner, and make plans for successful outcomes.
(3) Problem-based learning aids kids in accurately comprehending the function of adults in society.
(4) Students are encouraged to develop into responsible, self-sufficient adults.

Problem-based learning must be implemented correctly and in accordance with the guidelines in order to meet its intended purpose. It is known that certain unique syntaxes or rules were purposefully developed in order to facilitate its ideal implementation in this regard. Figure 1 is the syntax for problem-based learning (Balac & Ozogul, 2024).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

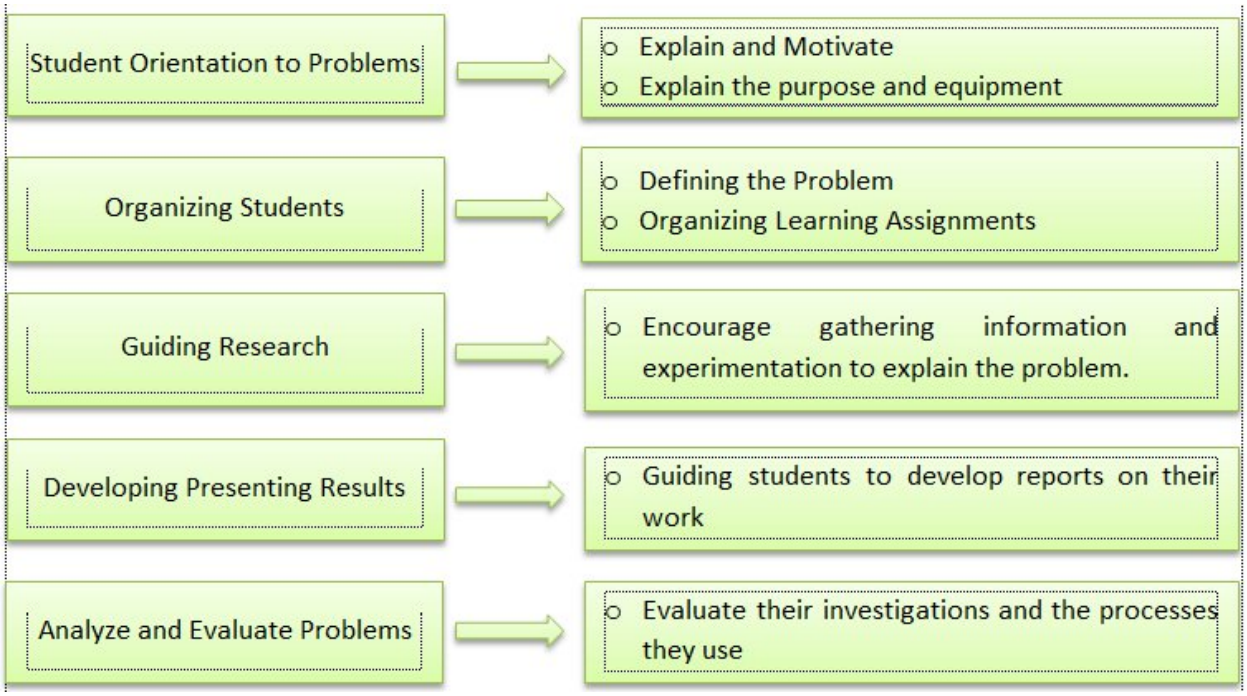


Figure 1: Problem-Based Learning Syntax

Based on the aforementioned explanation, the implementation of Problem-Based Learning (PBL) is designed to assist students in understanding the concept of linear equations. The design of the PBL model is structured not only to enable students to learn about formulas and graphs but also to effectively communicate their mathematical ideas.

Through the classroom arrangement using the PBL model, students not only learn about linear equations theoretically but also develop essential mathematical communication skills. They learn to articulate their ideas, listen to others' perspectives, and collaborate within groups. In this way, PBL creates a dynamic and interactive learning environment where students can refine their communication skills while comprehending complex mathematical concepts.

The primary objective of this study is to address these gaps by investigating the following:

(1) The extent to which problem-based learning influences junior high school students' mathematical communication skills.
(2) Does problem-based learning enhance the mathematics communication skills of junior high school students.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

By conducting this research, this study seeks to provide actionable insights for educators to design innovative and effective learning strategies that enhance students' mathematical communication abilities.

METHODS

Design

This study uses a quantitative approach and is problem-based. The purpose of this study was to evaluate how problem-based learning and traditional learning environments are handled. As a result, this study employs quasi-experiments. Nonequivalent control group design was selected as the study design. The experimental and control groups in this design were not chosen at random. The researchers did not select the research subjects at random; instead, they accepted the students' circumstances exactly as they were, making no modifications to the other pupils in the class. Both the experimental and control groups are contrasted in this research design. While the control group did not receive treatment, the experimental class did.

To measure their mathematical communication skills, students in the experimental class were given treatment using problem-based learning. For students in the control class, they were given regular learning, as shown in Figure 2.

O_1	X	O_2
O_1		O_2

Figure 2: Nonequivalent Control-Group Design. Key: O1 = Pretest; O2 = Posttest; X = Problem-based learning.

Research Sample

The sample used in the research included 78 students at State Middle School 4, Ternate City, Indonesia, of VIII grade. The number of students was grouped based on study groups from the school without particular criteria. The school arranged the learning schedule and supporting administration. Therefore, researchers did not randomize individual students but accepted the subjects as they were in each class. Determination of the experimental and control classes was based on the researcher's direct appointment. The number of students in the experimental class was 39 people, and the control class was 39 people.

Research Instrument

The instruments utilized in this study encompass training materials, non-test instruments, and test instruments. Mathematical communication skills assessments are administered before and after

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

the acquisition of knowledge for measuring the progression of students' abilities. The initial assessment (pre-test) is administered prior to the commencement of learning, while the final assessment (post-test) is administered upon the culmination of the learning process. This assessment comprises descriptive questions that encompass three primary domains: Grammatical Ability, Sociolinguistic Ability, and Strategic Ability. Grammatical Ability encompasses the accurate utilization of mathematical symbols or notations and the formulation of a lucid definition of a mathematical term. Sociolinguistic Ability involves students' proficiency in expressing mathematical concepts in language and subsequently solving them mathematically. Furthermore, Strategic Ability assesses students' aptitude for systematically recording information contained within a problem and describing the strategies employed in solving the problem.

One instance of a pre-test and post-test question utilized in this study is as follows: In the post-test, students were tasked with determining the equation of a straight line that passes through the point $(4, 5)$ and is perpendicular to the line $y = -2x + 4$. Concurrently, in the pre-test phase, students were tasked with determining the equation of a straight line that passes through two specified points: $(1, -2)$ and $(3, 4)$. The comprehensive mathematical communication skills assessment instrument is available in the appendix.

Prior to its utilization in the study, the test instrument underwent a rigorous trial and validation process. The trial was conducted with classes that had studied the material in alignment with the content of the test instrument. Item analysis was conducted to assess the quality of the questions, ensuring their appropriateness based on difficulty, discriminatory power, validity, and reliability. The difficulty level was analyzed to strike a balance between easy, medium, and challenging questions. Discriminating power was employed to measure the extent to which the questions differentiated students with varying abilities. The validity of the instrument was evaluated to ascertain its alignment with the mathematical communication skills under investigation. Additionally, its reliability was assessed to ensure consistency in measurement outcomes across diverse situations. Based on the trial results, the test instrument employed in this study demonstrated high validity and reliability standards, rendering it an accurate and reliable data collection tool.

Data Collection Technique

Data collection was conducted through the administration of tests and documentation of the instructional treatment implemented in each group. Each group participated in a total of six meetings. The first meeting was allocated for the administration of the pre-test, while the final meeting was used to administer the post-test. The remaining four meetings were dedicated to the instructional treatment, with each treatment session conducted over a duration of 120 minutes per meeting. In the experimental class, treatment was provided through problem-based learning, as illustrated in the following steps:

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Step 1: Student Orientation to the Problem

The researcher begins by introducing an engaging problem: "Creating a City Map Using Straight Lines." Students are divided into small groups and tasked with designing a city map using linear equations. They must determine the locations of various places, such as schools, parks, and shopping centers, and illustrate the relationships between these locations using straight lines.

Step 2: Organizing Students

Each group begins exploring the concept of linear equations. They learn about the general form of a linear equation, $y = mx + b$, where m represents the slope and b is the y-intercept. Students investigate how straight lines are applied in everyday life, such as in architecture and design.

Step 3: Guiding Research

During this process, students engage in active discussions. They exchange ideas on how to determine the appropriate slope to connect various locations on their map. These discussions encourage them to use precise mathematical language and articulate their reasoning to their peers.

Step 4: Developing and Presenting Results

After completing their maps, each group presents their work to the class. They explain how they used linear equations to design the map, as well as the rationale behind their choice of locations and slopes. This presentation provides students with an opportunity to practice mathematical communication skills, such as explaining concepts, listening to feedback, and responding to questions.

Step 5: Analyzing and Evaluating Results

Following the presentations, the teacher guides students in reflecting on their experiences. Students are asked to discuss what they learned about linear equations and how they communicated with their peers during the project. This reflection helps students recognize the importance of communication in understanding and applying mathematical concepts.

The data presented in this study were collected through a mathematical communication skill test administered to students, consisting of four questions. This test was conducted in two stages: an initial test administered before the treatment and a final test administered after the treatment for each group. Each test session was scheduled for 60 minutes, during which students were required to work independently on the problems without discussing or interacting with each other to ensure

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

the validity of the data obtained. Additionally, data was collected from the results of assignments given to students during the learning process. Documentation of these assignments serves as an additional source to evaluate students' mathematical communication skills in greater depth.

Data Analysis Technique

Based on the collected data, an analysis was conducted to evaluate students' mathematical communication skills in the context of implementing problem-based learning and conventional learning models. The data analysis process was carried out in several key stages. First, a comprehensive data collection was performed, followed by a data quality assessment to ensure that the data met the required standards. If any missing data were identified, data extraction procedures were conducted to complete the necessary information. Subsequently, prerequisite tests were carried out to examine the normality and homogeneity of the data, ensuring that it met the assumptions required for inferential statistical analysis.

Descriptive statistical analysis was applied by calculating pretest, posttest, and gain scores, including statistical measures such as mean, variance, and standard deviation. This analysis aimed to provide a general overview of the data distribution without making inferences about a broader population.

For inferential analysis, the Wilcoxon test and independent sample t-test were employed to assess the impact of the learning models. The Wilcoxon test was used to measure the improvement in mathematical communication skills within each class, while the independent sample t-test was utilized to compare the differences in improvement between the problem-based learning group and the conventional learning group.

Furthermore, the enhancement of students' mathematical communication abilities was evaluated employing the Normalized Gain metric (Hake, 1998), which is presented in Table 1.

Normalized Gain value Score	Interpretation
$g \geq 0{,}70$	High
$0{,}30 \leq g \leq 0{,}70$	Medium
$g < 0{,}30$	Low

Table 1: Normalized Gain Value Score Criteria (Hake, 1998)

The effectiveness of the learning process is also evaluated by determining the percentage of scores that fall within the categories outlined in Table 2.

Percent (%)	Interpretation
< 40	Ineffective
40 - 55	Less effective

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

56 - 75	Effective enough
> 76	Effective

Table 2: Category Interpretation of Gain value Effectiveness (Hake, 1998)

Thus, this analysis provides a comprehensive understanding of the effectiveness of problem-based learning in enhancing students' mathematical communication skills.

RESULTS

The discussion and results of this study are relevant to the goals of the study. The objectives of this study are based on current issues: (1) To determine the impact of problem-based learning on junior high school students' mathematical communication skills and (2) To determine the increase in junior high school students' mathematical communication skills after implementing problem-based learning. Data were assessed based on how students who participated in learning with the problem-based learning model and students who learned conventionally were influenced and improved in their mathematical communication skills. With the use of the SPSS 20 software program, descriptive and inferential statistics were used for data analysis. To describe the abilities of both student groups before and after treatment, as well as their improvement, a descriptive statistical analysis was conducted. Concurrently, inferential statistical analysis was carried out to determine the impact and variations on students' development of mathematical communication skills.

The quantitative data collected for the study were processed in accordance with the need to facilitate discussions about the issues brought up by the investigation. Table 3 displays data on students' mathematical communication skills based on their learning.

Statistics	n	Problem-Based Learning			Conventional Learning		
		Pretest	Posttest	Gain Value	Pretest	Postest	Gain Value
Minimum	39	8	60	.30	5	60	.42
Maximum		50	95	.95	35	85	.78
Mean		21,35	79,72	.7332	21,14	70.18	.6265
Standard Deviation		11,51	9,52	.14162	6,02	7.126	.08287
Ideal Maximum Score: 100							

Table 3: Description of Mathematical Communication Ability Test Data Based on Learning

Before the implementation of the Problem-Based Learning model (see Table 3), students exhibited varying levels of mathematical communication skills. The average pre-test score of 21.35 indicates that many students were at a low level of understanding. The wide range of scores (from 8 to 50) and the high standard deviation (11.51) suggest significant differences in students' initial abilities. Some students may have a stronger background in mathematics, while others may face

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

difficulties. After the application of the Problem-Based Learning model, there was a significant improvement in students' mathematical communication skills. The average post-test score reached 79.72, indicating that students, overall, made substantial progress. The narrower range of post-test scores (60 to 95) and the lower standard deviation (9.52) indicate that students' learning outcomes became more consistent. This suggests that the Problem-Based Learning model successfully enhanced students' understanding and reduced the gap in their abilities.

In the class that utilized conventional learning model, students had a slightly lower average pretest score (21.14) compared to the Problem-Based Learning model. The narrower range of scores (10 to 35) and the lower standard deviation (6.02) indicate that students in this class had more homogeneous abilities. Nevertheless, many students still remained at a low level of understanding. After the implementation of conventional teaching, students showed improvement in their mathematical communication skills, with an average post-test score of 70.18. Although there was an increase, the range of post-test scores (60 to 85) and the lower average compared to the Problem-Based Learning class indicate that the impact of conventional learning was not as strong as that of Problem-Based Learning. The lower standard deviation (7.126) suggests that students' learning outcomes remained consistent, but there was no significant leap in their understanding.

Based on the results of this descriptive analysis, it can be concluded that the Problem-Based Learning class demonstrated a greater improvement in students' mathematical communication skills compared to the conventional learning class. The Problem-Based Learning model not only increased the average post-test score but also reduced the variability in students' learning outcomes, indicating that more students achieved a better understanding. The conventional learning class showed a smaller increase, although students still demonstrated progress. However, the results were not as varied as those in the Problem-Based Learning class, suggesting that conventional learning model may not be sufficient to foster deeper understanding. To strengthen this hypothesis, inferential statistical tests were subsequently conducted.

1. The Effect of Problem-Based Learning on Students' Mathematical Communication Skills

In this section, we will test the influence of problem-based learning on students' mathematical communication skills. Before inferential statistical tests, prerequisite tests were carried out: data normality tests and equality of variance (homogeneous) tests. Data from the normality test results for the experimental and conventional classes are presented below, listed in Table 4.

Learning	Test	Shapiro-Wilk			Conclusion
		Statistics	df	*Sig.*	
Problem-Based	Pretest	0.863	39	0.001	Not Normal
	Posttest	0.996	39	0.279	Normal
	N-gain	0.943	39	0.076	Normal
	Pretest	0.971	39	0.404	Normal

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Conventional	Posttest	0.943	39	0.048	Not Normal
	N-gain	0.965	39	0.261	Normal

Table 4: Summary of Normality Test for Mathematical Communication Ability Data

Pay attention to Table 4. The data obtained from the pretest and posttest results of both modes of learning, namely problem-based learning and conventional learning, shows that the data are not generally distributed because, from the results of the Shapiro-Wilk test, there are several significant values from the pretest and posttest in problem-based learning and conventional learning. Conventional is less than alpha (0.05). Based on the testing rules for statistical tests, testing can be carried out using nonparametric statistical tests; in this case, it can be done using the Wilcoxon test. The research hypotheses formulated are as follows:

H_0: Problem-based learning does not affect students' mathematical communication skills.

H_1: Problem-based learning influences students' mathematical communication skills.

The following is a summary of the results of nonparametric statistical tests (Wilcoxon), which can be used in decision-making when implementing problem-based learning. A summary of the test results are presented in Table 5.

	PostTets PBL - PreTest PBL	PostTest Conventional - PreTest Conventional
Z	-5.444[b]	-5.481[b]
Asymp. Sig. (2-tailed)	.000	.000

a. Wilcoxon Signed Ranks Test
b. Based on negative ranks.

Table 5: Summary of Wilcoxon Test of Students' Mathematical Communication Ability

The results of the Wilcoxon test for the problem-based learning class indicated a very small p-value (e.g., $p < 0.05$), suggesting that the difference between the pre-test and post-test scores is statistically significant. This implies that the implementation of the problem-based learning model effectively enhances students' mathematical communication skills. The Wilcoxon test results for the conventional class also revealed a significant p-value (e.g., $p < 0.05$), indicating that there is a significant difference between the pre-test and post-test scores. However, the smaller improvement compared to the problem-based learning class suggests that while the conventional method is effective, its impact is not as pronounced as that of the problem-based learning model.

The analysis results demonstrate that the Problem-Based Learning model is more effective in enhancing students' mathematical communication skills compared to conventional learning model. The significant increase in post-test scores in the PBL class indicates that students are better able to understand and communicate mathematical concepts after participating in this learning approach. Although conventional learning also showed significant improvement, the results were not as robust as those observed in the PBL class. This suggests that the PBL method may be more capable of fostering student engagement and deeper understanding of the material.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The results of calculations using the Wilcoxon tests show that the application of problem-based learning has a significant influence on mathematical communication skills, as shown in the results of student work after the application of the problem-based learning model in the following Figure 2.

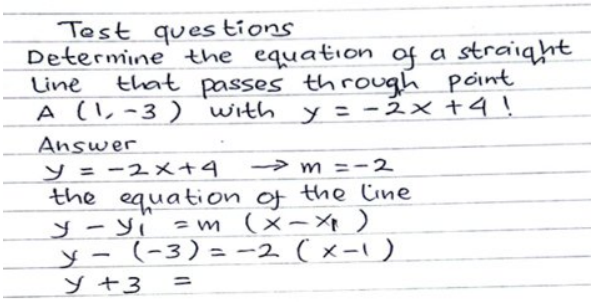


Figure 2: Example of student pretest results

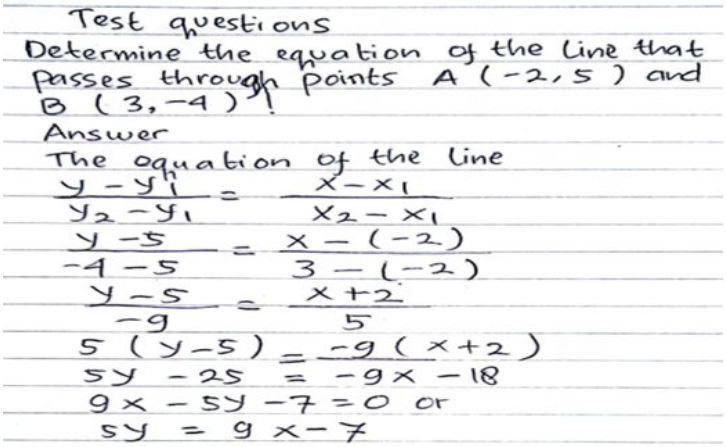


Figure 3: Example of student posttest results

After implementing problem-based learning, (1) students can communicate the flow of thought or strategy used clearly and coherently so that other students can follow their thinking process; (2) students can discuss, assess, and provide additional explanations or clarifications to mathematical ideas in discussions; and (3) Students can write mathematical answers or solutions with logical, coherent steps that the reader can understand. This explanation is shown in Figure 3. Figure 2 shows the results of student work before the implementation of problem-based learning. Based on the results, students cannot write mathematical solutions using logical steps, so they have not reached a correct solution set. By looking at Figure 3, it can be said that students can communicate their understanding of determining the equation of a straight line.

2. Increasing students' mathematical communication skills by implementing problem-based learning

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Based on the prerequisite test results for the normality test, the results show that the data were normally distributed. So, the test can be used with parametric statistics, namely the t-test. Testing the improvement in students' mathematical communication skills can be done using the Normalized Gain test. The results of the independent t-test are presented below in Table 6.

		Levene's Test for Equality of Variances		t-test for Equality of Means		
		F	Sig.	t	df	Sig. (2-tailed)
N-gain	Equal variances assumed	6.300	.014	4.211	76	.000
	Equal variances not assumed			4.211	60.861	.000

Table 6: Comparison of N-gain values in the two classes

The results of the independent t-test (assuming equal variances) revealed a significant difference ($p<0.05$) between the two groups of students taught with distinct learning models. This analysis primarily focused on the N-gain score of students' mathematical communication skills, which quantifies the enhancement in their abilities following the implementation of the learning model.

Students taught with the problem-based learning model exhibited a notable increase in their mathematical communication skills. The average N-gain for the problem-based learning model group (0.7332) was significantly higher than that of the conventional group (0.6229). This finding underscores the effectiveness of the problem-based learning model, which emphasizes active and collaborative learning, in fostering students' engagement and involvement in the learning process. Not only do students acquire mathematical concepts, but they also practice communicating and explaining their understanding to peers.

In contrast, students taught with the conventional learning model also demonstrated an improvement, albeit with a lower N-gain compared to the problem-based learning model group. While progress was made, the more passive conventional learning model may not provide sufficient opportunities for students to effectively practice mathematical communication. Students in this group tend to prioritize mastering the material without substantial interaction, which can hinder their ability to communicate effectively.

The gain value, which assesses the effectiveness of enhancing learning outcomes, indicates that the problem-based learning class has a value of 0.7332, placing it within the high improvement category. This result demonstrates a substantial increase of 73.32% of its potential, suggesting the efficacy of implementing the problem-based learning model. Conversely, the conventional class has a gain value of 0.6265 (categorized as moderate improvement), indicating an increase of approximately 62.65% of its maximum potential. This suggests the effectiveness of implementing the conventional model.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Overall, these findings demonstrate that the application of the Problem-Based Learning model exerts a greater impact on enhancing students' mathematical communication skills compared to the conventional learning model. The presence of problem-based challenges in the learning process encourages students in the problem-based learning class to engage actively in thinking, discussing, and developing more effective problem-solving strategies. While conventional learning also yields significant improvements, it falls short of the problem-based learning model in optimizing students' mathematical communication skills.

DISCUSSION

The findings of this study demonstrate that students who engaged in problem-based learning (PBL) exhibited significantly higher gains in their mathematical communication abilities compared to those who were taught using conventional methods. This aligns with the theoretical framework of PBL, which emphasizes active, collaborative, and contextual learning as a means to enhance both conceptual understanding and communication skills (Abate et al., 2022; Martin, 2022; Alenezi, 2023). The success of PBL in fostering mathematical communication can be attributed to its structured approach, which requires students to engage in group discussions, articulate their reasoning, and collaboratively solve real-world problems. This process not only deepens their understanding of mathematical concepts but also provides ample opportunities to practice and refine their communication skills.

The results of this study are consistent with prior research that highlights the effectiveness of PBL in improving students' ability to communicate mathematical ideas. For instance, Bailey (2022), Yong et al. (2020), and Ilhan and Akin (2022) argued that traditional, teacher-centered methods often fail to cultivate students' passion for communicating mathematical concepts, as these approaches typically prioritize rote memorization over active engagement. In contrast, PBL creates a dynamic learning environment where students are encouraged to explain their thought processes, justify their solutions, and respond to feedback from peers and teachers. This active participation is critical for developing the ability to communicate mathematical ideas in a clear, accurate, and logical manner.

Furthermore, the contextual nature of PBL, as highlighted by Nugraheni and Marsigit (2021) and Nurmasari et al. (2024), plays a pivotal role in enhancing students' motivation and engagement. By connecting mathematical concepts to real-world problems, PBL makes learning more relevant and meaningful for students. This relevance not only increases their enthusiasm for learning but also provides a practical framework for applying mathematical communication skills. For example, in this study, students were tasked with designing a city map using linear equations, a problem that required them to interpret, analyze, and communicate their solutions in a real-world context. This approach aligns with the findings of Das (2020) and Susanti et al. (2023), who emphas-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

ize that PBL's five key components—orienting students, organizing learning, guiding research, developing and presenting results, and analyzing outcomes—collectively contribute to the development of higher-order thinking and communication skills.

The data presented in Table 3, 5 and 6 further underscore the superiority of PBL over traditional methods in enhancing mathematical communication skills. While both approaches showed some improvement, the difference in average scores between the two groups highlights the unique advantages of PBL. Specifically, students in the PBL group demonstrated a greater ability to interpret mathematical results, explain their reasoning, and use mathematical language effectively. These findings are consistent with the observations of Sakir and Kim (2020), who noted that PBL encourages students to articulate their ideas and collaborate on problem-solving tasks, thereby fostering both conceptual understanding and communication proficiency.

However, it is important to acknowledge that students in traditional learning environments still showed moderate improvement in their communication skills, albeit to a lesser extent. This suggests that while conventional methods may provide some foundation for mathematical communication, they are insufficient for developing the depth and fluency required for effective communication. As noted by Novikasari (2020), Sevinc and Lesh (2022), and Luritawaty et al. (2024), traditional approaches often fail to provide students with opportunities to engage in meaningful dialogue or present their work, which are critical components of communication skill development.

CONCLUSIONS

In conclusion, this study provides compelling evidence that problem-based learning is a highly effective approach for enhancing students' mathematical communication skills. By integrating real-world problems, collaborative learning, and structured reflection, PBL creates an environment where students can actively engage with mathematical concepts and practice communicating their ideas. These findings not only support the existing literature on PBL but also highlight the importance of adopting innovative teaching methods to address the challenges of mathematical communication in the classroom. Future research could explore the long-term impact of PBL on students' communication skills and its applicability across different cultural and educational contexts.

By applying learning models in the classroom, we can ignite a transformative process where students discover something new or make various updates that are formed in a product, idea, design, and so on. The problem-based learning model, in particular, harnesses the real world as a context that students use to learn, thereby enhancing their mathematical communication skills. This approach holds immense potential to revolutionize the way we teach and learn mathematics.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The implementation of mathematics learning should shift its focus from the teacher as the main source of knowledge to the students as active participants in their learning journey. Learning should be a process where students are empowered to create meaningful learning experiences. Fun learning, undoubtedly, is the first step to achieving quality learning outcomes. However, the true value of learning lies in its meaningfulness, which can be achieved when students are given the opportunity to experience and apply what they are learning (Ndiung & Menggo, 2024).

Before problem-based learning was implemented, most students had difficulty communicating mathematics. Problem-based learning helps students communicate abstract mathematics. The abstractness of mathematics and problem-based learning is a solution to recognize, understand, appreciate, and practice mathematics in communication (Ningsih et al., 2023; Aguilar & Telese, 2018).

Mathematical communication skills are reflected through several key indicators. Students should be able to: (1) express a situation or problem using language, symbols, ideas, or mathematical models, such as pictures, diagrams, graphs, or mathematical expressions; (2) explain mathematical ideas, situations, and relationships using everyday language; (3) listen to, discuss, and write about mathematical concepts; (4) interpret mathematical representations; and (5) re-express mathematical descriptions in their own words (Bron & Prudente, 2024).

Problem-based learning can improve students' mathematical communication skills in the high category. Meanwhile, conventional learning can also improve mathematical communication skills in the moderate category. Both lessons were quite effective in enhancing mathematical communication skills.

ACKNOWLEDGEMENTS

The author would like to thank all parties who have helped in this research. Especially the Head of State Junior High School 4, Ternate City, Indonesia, who has allowed researchers to collect data.

REFERENCES

[1] Abate, A., Atnafu, M., & Michael, K. (2022). Visualization and problem-based learning ap-proaches and students' attitude toward learning mathematics. *Pedagogical Research*, 7(2), em0119. https://doi.org/10.29333/pr/11725

[2] Aguilar, J., & Telese, J. A. (2018). Mathematics understanding of elementary pre-service

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

teachers: The analysis of their procedural-fluency, conceptual-understanding, and problem-solving strategies. *Mathematics Teaching-Research Journal*, 10(3–4), 24–37.

[3] Ahdhianto, E., Marsigit, M., Haryanto, H., & Santi, N. N. (2020). The effect of metacognitive-based contextual learning model on fifth-grade students' problem-solving and mathematical communication skills. *European Journal of Educational Research*, *9*(2), 753–764. https://doi.org/10.12973/eu-jer.9.2.753

[4] Alenezi, A. (2023). Using project-based learning through the madrasati platform for mathematics teaching in secondary schools. *International Journal of Information and Communication Technology Education*, *19*(1), 1–15. https://doi.org/10.4018/IJICTE.332372

[5] Ata Baran, A., & Kabael, T. (2021). An investigation of eighth grade students' mathematical communication competency and affective characteristics. *Journal of Educational Research*, *114*(4), 367–380. https://doi.org/10.1080/00220671.2021.1948382

[6] Bailey, J. (2022). Learning to teach mathematics through problem solving. *New Zealand Journal of Educational Studies*, *57*(2), 407–423. https://doi.org/10.1007/s40841-022-00249-0

[7] Balac, V., & Ozogul, G. (2024). Design and implementation of a problem-based learning module in a clinical radiography course to foster image critique skills: An evaluative case study. *Interdisciplinary Journal of Problem-Based Learning*, *17*(1). https://doi.org/10.14434/ijpbl.v17i1.35150

[8] Balyan, R., Arner, T., Taylor, K., Shin, J., Banawan, M., Leite, W. L., & McNamara, D. S. (2022). Modeling one-on-one online tutoring discourse using an accountable talk framework. *Proceedings of the 15th International Conference on Educational Data Mining, 2022*. https://doi.org/10.5281/zenodo.6852936

[9] Bron, J. F., & Prudente, M. S. (2024). Examining the effect of problem-based learning approach on learners` mathematical creativity: A meta-analysis. *International Journal of Research in Education and Science*, *10*(3), 653–668. https://doi.org/10.46328/ijres.3456

[10] Buus, K., & Pedersen, L. M. (2021). How differences in motivation and identification shape four types of student experiences with problem-based learning. *Journal of Problem Based Learning in Higher Education*, *9*(2), 22–45. https://doi.org/10.5278/ojs.jpblhe.v9i2.6620

[11] Çelik Demirci, S., & Baki, A. (2023). Characterizing mathematical discourse according to teacher and student interactions: The core of mathematical discourse. *Journal of Pedagogical Research*, *7*(4), 144–164. https://doi.org/10.33902/JPR.202321852

[12] Chasanah, C., Riyadi, & Usodo, B. (2020). The effectiveness of learning models on written mathematical communication skills viewed from students' cognitive styles. *European Journal of Educational Research*, *9*(3), 979–994. https://doi.org/10.12973/EU-JER.9.3.979

[13] Cresswell-Yeager, T. (2021). Forming, storming, norming, and performing: Using a semester-long problem-based learning project to apply small-group communication principles. *Communication Teacher*, *35*(2), 155–165. https://doi.org/10.1080/17404622.2020.1842476

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[14] Dakabesi, D., & Luoise, I. S. Y. (2019). The effectiveness of problem-based learning model to increase the students' critical thinking skills. *Journal of Education and Learning (EduLearn)*, *13*(4), 543–549. https://doi.org/10.11591/edulearn.v13i4.12940

[15] Darhim, D., Prabawanto, S., & Susilo, B. E. (2020). The effect of problem-based learning and mathematical problem posing in improving student's critical thinking skills. *International Journal of Instruction*, *13*(4), 103–116. https://doi.org/10.29333/iji.2020.1347a

[16] Das, K. (2020). Realistic mathematics & vygotsky's theories in mathematics education. *Shanlax International Journal of Education*, *9*(1), 104–108. https://doi.org/10.34293/education.v9i1.3346

[17] Dinç, E., Sezgin-Memnun, D., Lee, E., & Aydın, B. (2022). Predicting non-routine mathematical problem-solving anxiety of ninth graders. *Proceedings of the 19th International Conference on Cognition and Exploratory Learning in the Digital Age, CELDA 2022*, Celda, 169–176. https://doi.org/10.33965/celda2022_202207l022

[18] Domu, I., Pinontoan, K. F., & Mangelep, N. O. (2023). Problem-based learning in the online flipped classroom: Its impact on statistical literacy skills. *Journal of Education and E-Learning Research*, *10*(2), 336–343. https://doi.org/10.20448/jeelr.v10i2.4635

[19] Ernawati, M. D. W., Rusdi, M., Asrial, A., Muhaimin, M., Wulandari, M., & Maryani, S. (2022). Analysis of problem based learning in the scaffolding design: Students' creative-thinking skills. *Cypriot Journal of Educational Sciences*, *17*(7), 2333–2348. https://doi.org/10.18844/cjes.v17i7.7551

[20] Estabrooks, C., & McArdle, D. (2022). The design and implementation of a course in mathematical research and communication. *Primus*, *32*(7), 785–797. https://doi.org/10.1080/10511970.2021.1931996

[21] Fonseca, K. (2021). Self-reported mathematical problem-solving skills of future mathematics teachers. *South African Journal of Childhood Education*, *11*(1), 1–8. https://doi.org/10.4102/sajce.v11i1.1011

[22] Ford, J. (2018). Blending team-based learning with standards-based grading in a calculus classroom. *Mathematics Teaching-Research Journal*, *10*(3-4), 52–70.

[23] Greenstein, S., & Fernández, E. (2023). Learning mathematics with mathematical objects: Cases of teacher-made mathematical manipulatives. *Contemporary Issues in Technology and Teacher Education*, *23*(1), 103–115. https://www.learntechlib.org/primary/p/222417/

[24] Hake, R. R. (1998). Interactive-engagement vs traditional methods: A six-thousand student survey of mechanics test data for introductory physics courses. *American Journal of Physics*, *66*(1), 64–74. https://files.eric.ed.gov/fulltext/ED441679.pdf

[25] Ikhsan, F., Pramudya, I., & Subanti, S. (2020). An analysis of mathematical communication skills. *International Online Journal of Education and Teaching*, *7*(4), 1300–1307. https://files.eric.ed.gov/fulltext/EJ1271035.pdf

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[26] Ilhan, A., & Akin, M. F. (2022). Analysis of contextual problem solutions, mathematical sentences, and misconceptions of pre-service mathematics teachers. *International Electronic Journal of Mathematics Education*, *17*(1), em0666. https://doi.org/10.29333/iejme/11470

[27] Iwuanyanwu, P. N. (2021). Contemporary problems of teaching and learning in mathematics education. *Mathematics Teaching-Research Journal*, *13*(2), 23–35. https://files.eric.ed.gov/fulltext/EJ1383115.pdf

[28] Kamid, K., Rusdi, M., Fitaloka, O., Basuki, F. R., & Anwar, K. (2020). Mathematical communication skills based on cognitive styles and gender. *International Journal of Evaluation and Research in Education*, *9*(4), 847–856. https://doi.org/10.11591/ijere.v9i4.20497

[29] Katsara, O. (2023). Reconceptualizing pedagogy within the context of an internationalized problem-based learning approach. *Journal of Comparative & International Higher Education*, *15*(1), 111–130. https://doi.org/10.32674/jcihe.v15i1.4864

[30] Kelly, D., Conway, C., Harney, S., & McKeague, H. (2022). Temporary transitions to online problem-based learning: Advice for tutors and learners. *Journal of Problem Based Learning in Higher Education*, *10*(1), 1–19. https://doi.org/10.54337/ojs.jpblhe.v10i1.6362

[31] Kurniasih, A. W., Purwanto, P., Hidayanto, E., & Subanji, S. (2022). Teachers' skills for attending, interpreting, and responding to students' mathematical creative thinking. *Mathematics Teaching-Research Journal*, *14*(2), 157–185. https://files.eric.ed.gov/fulltext/EJ1350654.pdf

[32] Kusumah, Y. S., Kustiawati, D., & Herman, T. (2020). The effect of geogebra in three-dimensional geometry learning on students' mathematical communication ability. *International Journal of Instruction*, *13*(2), 895–908. https://doi.org/10.29333/iji.2020.13260a

[33] Lee, S. Y. (2021). Research status of mathematical problem posing in mathematics education journals. *International Journal of Science and Mathematics Education*, *19*(8), 1677–1693. https://doi.org/10.1007/s10763-020-10128-z

[34] Liu, M., Li, C., & Pan, Z. (2022). Creating an interactive dashboard to support middle school teachers' implementation of a technology-supported problem-based learning program. *International Journal of Designs for Learning*, *13*(1), 1–18. https://doi.org/10.14434/ijdl.v13i1.31243

[35] Luritawaty, I. P., Herman, T., & Prabawanto, S. (2024). A case study on students' critical thinking in online learning: Epistemological obstacle in proof, generalization, alternative answer, and problem solving. *Mathematics Teaching-Research Journal*, *15*(6), 74–93. https://files.eric.ed.gov/fulltext/EJ1417844.pdf

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[36] Magaji, A. (2021). Promoting problem-solving skills among secondary science students through problem based learning. *International Journal of Instruction*, *14*(4), 549–566. https://doi.org/10.29333/iji.2021.14432a

[37] Martin, D. (2022). The impact of problem-based learning on pre-service teachers' mathematics pedagogical content knowledge. *Australian Journal of Teacher Education*, *47*(4), 55–73. https://doi.org/10.14221/ajte.2022v47n4.4

[38] Muhtarom, M., Shodiqin, A., & Astriani, N. (2020). Exploring senior high school student's abilities in mathematical problem posing. *Journal of Research and Advances in Mathematics Education*, *5*(1), 69–79. https://doi.org/10.23917/jramathedu.v5i1.9818

[39] Ndiung, S., & Menggo, S. (2024). Problem-based learning analysis in strengthening college students' interpersonal communication skill. *Journal of Education and Learning*, *18*(4), 1350–1361. https://doi.org/10.11591/edulearn.v18i4.21219

[40] Ningsih, W., Prayitno, B. A., & Santosa, S. (2023). The effectiveness of environment-oriented e-books based on problem-based learning for problem-solving skills. *JPBI (Jurnal Pendidikan Biologi Indonesia)*, *9*(3), 511–520. https://doi.org/10.22219/jpbi.v9i3.25603

[41] Noordzij, G., & Wijnia, L. (2020). The role of perceived quality of problems in the association between achievement goals and motivation in problem-based learning. *Interdisciplinary Journal of Problem-Based Learning*, *14*(1), 1–18. https://doi.org/10.14434/ijpbl.v14i1.28593

[42] Novikasari, I. (2020). Pre-service teacher's mathematical knowledge for teaching in problem-based learning. *Journal of Research and Advances in Mathematics Education*, *5*(2), 160–174. https://doi.org/10.23917/jramathedu.v5i2.10556

[43] Nugraheni, L. P., & Marsigit, M. (2021). Realistic mathematics education: An approach to improve problem solving ability in primary school. *Journal of Education and Learning (EduLearn)*, *15*(4), 511–518. https://doi.org/10.11591/edulearn.v15i4.19354

[44] Nurmasari, L., Budiyono, B., Nurkamto, J., & Ramli, M. (2024). Realistic mathematics engineering for improving elementary school students' mathematical literacy. *Journal on Mathematics Education*, *15*(1), 1–26. https://doi.org/10.22342/jme.v15i1.pp1-26

[45] Öztaş, E. T. (2023). Investigation of the learning outcomes of the 2018 middle school mathematics curriculum in terms of mathematical communication skills 3. *International Journal of Curriculum and Instructional Studies*, *13*(1), 224–248. www.ijocis.com

[46] Palupi, B. S., Subiyantoro, S., Rukayah, R., & Triyanto, T. (2020). The effectiveness of guided inquiry learning (GIL) and problem-based learning (PBL) for explanatory writing skill. *International Journal of Instruction*, *13*(1), 713–730. https://doi.org/10.29333/iji.2020.13146a

[47] Papadopoulos, I., Patsiala, N., Baumanns, L., & Rott, B. (2022). Multiple approaches to problem posing: Theoretical considerations regarding its definition, conceptualisation, and

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

implementation. *Center for Educational Policy Studies Journal*, *12*(1), 13–34. https://doi.org/10.26529/cepsj.878

[48] Passarella, S. (2021). Mathematics teachers' inclusion of modelling and problem posing in their mathematics lessons: An exploratory questionnaire. *European Journal of Science and Mathematics Education*, *9*(2), 43–56. https://doi.org/10.30935/SCIMATH/10773

[49] Piñeiro, J. L., Castro-Rodríguez, E., & Castro, E. (2021). Mathematical problem-solving in two teachers' knowledge models: A critical analysis. *Mathematics Teaching-Research Journal*, *13*(1), 71–93. https://files.eric.ed.gov/fulltext/EJ1385079.pdf

[50] Pourdavood, B. R., McCarthy, K., & McCafferty, T. (2020). The impact of mental computation on children's mathematical communication, problem solving, reasoning, and algebraic thinking. *Athens Journal of Education*, *7*(3), 241–254. https://doi.org/10.30958/aje.7-3-1

[51] Putra, H. D., Herman, T., & Sumarmo, U. (2020). The impact of scientific approach and what-if-not strategy utilization towards student's mathematical problem posing ability. *International Journal of Instruction*, *13*(1), 669–684. https://doi.org/10.29333/iji.2020.13143a

[52] Rajagukguk, W., Bina, N. S., & Samosir, K. (2022). The effect of prior knowledge, emotional intelligence and motivation on mathematical communication. *Education Quarterly Reviews*, *5*(4), 146–159. https://doi.org/10.31014/aior.1993.05.04.581

[53] Rogers, H. L., Hitt, S. J., & Allan, D. G. (2021). Problem-based learning in institutional and curricular design at the new model institute for technology and engineering. *Journal of Problem Based Learning in Higher Education*, *9*(2), 131–141. https://doi.org/10.5278/ojs.jpblhe.v9i2.6814

[54] Rohid, N., Suryaman, S., & Rusmawati, R. D. (2019). Students' mathematical communication skills in solving mathematics problems: A case in Indonesian context. *Anatolian Journal of Education*, *4*(2), 19–30. https://files.eric.ed.gov/fulltext/EJ1244446.pdf

[55] Sa'adah, N., Faizah, S., Sa'dijah, C., Khabibah, S., & Kurniati, D. (2023). Students' mathematical thinking process in algebraic verification based on crystalline concept. *Mathematics Teaching-Research Journal*, *15*(1), 90–107. https://files.eric.ed.gov/fulltext/EJ1391469.pdf

[56] Sakir, N. A. I., & Kim, J. G. (2020). Enhancing students' learning activity and outcomes via implementation of problem-based learning. *Eurasia Journal of Mathematics, Science and Technology Education*, *16*(12), em1925. https://doi.org/10.29333/ejmste/9344

[57] Sancar-Tokmak, H., & Dogusoy, B. (2023). Novices' instructional design problem-solving processes: Second life as a problem-based learning environment. *Interactive Learning Environments*, 31(1), 562–575. https://doi.org/10.1080/10494820.2020.1799025

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[58] Sercenia, J. C., Ibañez, E. D., & Pentang, J. T. (2023). Thinking beyond thinking: Junior high school students' metacognitive awareness and conceptual understanding of integers. *Mathematics Teaching-Research Journal*, *15*(1), 4–24. https://files.eric.ed.gov/fulltext/EJ1391473.pdf

[59] Sevinc, S., & Lesh, R. (2022). Preservice mathematics teachers' conceptions of mathematically rich and contextually realistic problems. *Journal of Mathematics Teacher Education*, *25*(6), 667–695. https://doi.org/10.1007/s10857-021-09512-5

[60] Smith, K. H., Silva, C., Weinburgh, M. H., Jones, N. S., & Riggs, B. (2023). Change in emergent multilingual learners' mathematical communication: Attending to language use and needs. *Electronic Journal for Research in Science & Mathematics Education*, *27*(2), 81–98. https://files.eric.ed.gov/fulltext/EJ1395971.pdf

[61] Susanti, M., Suyanto, S., Jailani, J., & Retnawati, H. (2023). Problem-based learning for improving problem-solving and critical thinking skills: A case on probability theory course. *Journal of Education and Learning (EduLearn)*, *17*(4), 507–525. https://doi.org/10.11591/edulearn.v17i4.20866

[62] Tadjer, H., Lafifi, Y., Seridi-Bouchelaghem, H., & Gülseçen, S. (2022). Improving soft skills based on students' traces in problem-based learning environments. *Interactive Learning Environments*, *30*(10), 1879–1896. https://doi.org/10.1080/10494820.2020.1753215

[63] Treepob, H., Hemtasin, C., & Thongsuk, T. (2023). Development of scientific problem-solving skills in grade 9 students by applying problem-based learning. *International Education Studies*, *16*(4), 29–36. https://doi.org/10.5539/ies.v16n4p29

[64] Trgalová, J., & Tabach, M. (2023). Affordances of virtual learning environments to support mathematics teaching. In *Digital Experiences in Mathematics Education* (Vol. 9, Issue 3). Springer International Publishing. https://doi.org/10.1007/s40751-023-00127-4

[65] Triana, M., Zubainur, C. M., & Bahrun, B. (2019). Students' mathematical communication ability through the brain-based learning approach using autograph. *Journal of Research and Advances in Mathematics Education*, *4*(1), 1–10. https://doi.org/10.23917/jramathedu.v4i1.6972

[66] Ukobizaba, F., Nizeyimana, G., & Mukuka, A. (2021). Assessment strategies for enhancing students' mathematical problem-solving skills: A review of literature. *Eurasia Journal of Mathematics, Science and Technology Education*, *17*(3), 1–10. https://doi.org/10.29333/ejmste/9728

[67] Uyen, B. P., Tong, D. H., & Tram, N. T. B. (2021). Developing mathematical communication skills for students in grade 8 in teaching congruent triangle topics. *European Journal of Educational Research*, *10*(3), 1287–1302. https://doi.org/10.12973/eu-jer.10.3.1287

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[68] Van Dijk, W., & Lane, H. B. (2020). The brain and the US education system: Perpetuation of neuromyths. *Exceptionality*, *28*(1), 16–29. https://doi.org/10.1080/09362835.2018.1480954

[69] Wardono, W., Rochmad, R., Uswatun, K., & Mariani, S. (2020). Comparison between generative learning and discovery learning in improving written mathematical communication ability. *International Journal of Instruction*, *13*(3), 729–744. https://doi.org/10.29333/iji.2020.13349a

[70] Ya-Amphan, D., Thinwiangthong, S., & Sythong, P. (2024). Comparative study of means of mathematical communication in Japan, Laos, and Thailand. *Journal on Mathematics Education*, *15*(1), 99–114. https://doi.org/10.22342/jme.v15i1.pp99-114

[71] Yong, S. T., Karjanto, N., Gates, P., Chan, T. Y. A., & Khin, T. M. (2020). Let us rethink how to teach mathematics using gaming principles. *International Journal of Mathematical Education in Science and Technology*, *52*(8), 1175–1194. https://doi.org/10.1080/0020739X.2020.1744754

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

APPENDIX

Pretest Questions

Please ensure that you answer these questions correctly:

1. Determine the equation of the line that passes through the point (3, 4) and has a gradient of 2.
2. Find the equation of the line that passes through the points (1, -2) and (3, 4).
3. Determine the gradients of the parallel and perpendicular lines to the given line $y = \frac{1}{2}x + 4$
4. Determine the equation of the line that is perpendicular to the given line $y = 4x + 1$. The line passes through the point (2, -3).

Posttest Questions

Please ensure that you answer these questions correctly:

1. Determine the equation of the line that passes through the point (5, -3) and has a gradient of $-\frac{3}{4}$.
2. Find the equation of the line that passes through the points (2, 3) and (6, -1).
3. Determine the gradients of the parallel and perpendicular lines to the given line $y = \frac{5}{3}x + 7$.
4. Determine the equation of the line that is perpendicular to the given line $y = -2x + 3$. The line passes through the point (4, 5).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL 223
WINTER 2025
Vol 17 no 6

Well-Stated Problem: What is the Level of Problem-Posing Performance of Pre-Service Mathematics Teachers?

John Rey Oficiar[1], Jerwin Reonisto[2], Anthony Bactol[3]

[1]Aurora State College of Technology, Baler, Aurora, Philippines

[2]San Jose City National High School – Senior High School, Nueva Ecija, Philippines

[3]Core Gateway College Inc., Nueva Ecija, Philippines

oficiarjohnreyg@gmail.com

Abstract: The Philippine mathematics education system aims to develop learners with strong critical thinking and problem-solving skills. However, students often struggle with problem-solving due to its complexity and ambiguity of tasks designed by educators. The use of problem-posing activities was seen as a new pedagogical approach in mathematics education. This study examined the problem-posing performance of preservice mathematics teachers. Quantitatively, it described their scores using the "Free Problem-Posing Performance Assessment", moreover, this study explored the strategies and difficulties in problem-posing performances. Results showed a satisfactory performance in Statistics and Probability, Number Sense, and Algebra, but a fair performance in Geometry and Measurement—mainly due to poor alignment between visual representations and mathematical goals. Common issues included ambiguous wording, grammatical errors, misleading contexts, and lack of authenticity. Socio-demographically, sex did not significantly affect performance, but academic year level indicates that higher-year students performs better than lower years. The recurring deficiencies in mathematics performance highlight the urgent need for curriculum enhancement; effective mathematics education must go beyond content delivery by fostering awareness of how learners think and approach problems. Promoting self-awareness among educators help them better understand diverse student approaches to both problem-solving and problem-posing, ultimately leading to more meaningful and effective instruction.

Keywords: Mathematics Education, Performance task, Pedagogical approach, Pre-service Education, Problem-posing

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

INTRODUCTION

Mathematics education emphasizes the importance of the problem-solving ability of the learners. Critical thinking and problem-solving skills were highlighted in the twin goals of the Philippine K-12 Mathematics Framework (Department of Education, 2016). The development of these competencies is the top priority of the education system gearing towards the advancement of mathematics education. However, despite the efforts to enhance the mathematical abilities of the learners through these concepts, learners still have a relatively low performance in the international mathematics assessment like Programme for International Student Assessment (PISA) where the performance of the Philippines in mathematics did not improve from 2018 to 2022 (Acido & Cabales, 2024). One possible reason is the complexity of the nature of problem-solving, combined with the ambiguity of the problem-solving activities developed by the educators.

To address this, new core competencies must be developed for mathematics teachers, which include learning to devise problem-solving activities that involve a heuristic approach and target the mathematical thinking, critical thinking, and creativity of the learners. One way to uphold this competency is through designing a well-structured problem through problem-posing. The line "A well-stated problem is half-solved" emphasizes the importance of clearly stated problem structure. In general, problem-posing in mathematics is defined as the ability of an individual to construct an effective mathematical problem.

Mathematical competency was recognized as an essential aspect of mathematics and mathematics education as it not only enriched the mathematical investigation, but also opened a promising approach to exciting and enriching activities for the development of mathematical thinking (Peng et al., 2022) In mathematics learning, teachers should present mathematics problems that can encourage the students to learn and demonstrate their critical and creative thinking (Masriyah et al., 2018). Problem-posing activities are characterized by the active learning approach where learners have control over their thinking processes, ultimately leading to the application of their prior learning experience that develops their mathematical knowledge (Wakhata et al., 2023).

Since it is a common practice to expose learners to an active learning environment, teachers can emphasize the role of problem-posing in classroom activities. In a common mathematics classroom, learners answer a mathematics problem from the book or where student and material engagement is rarely seen. While problem-posing activities deal with process-oriented and outcome-oriented skills, learners incorporate their interests and curiosity which leads to posing a problem that suits their critical thinking and their creative mind to come up with a solvable problem that is contextualized through their learning experiences (Baumanns & Rott, 2024; Divrik, 2023). This learning approach could target the mathematical interest of the learners, which can enhance the content knowledge in an engaging way.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The importance of problem-posing is that is a skill that emphasizes the competency of a learner to creatively construct a mathematical problem. Interestingly, previous studies emphasized that problem-posing activities are a method that can be used to incorporate into a teaching and learning process (Divrik, 2023; Li et al, 2020; Polat & Özkaya, 2023). Problem-posing can be an essential aspect of mathematics teachings and learning; it can be used as an assessment strategy to monitor the learner's capability in mathematical content and mathematical thinking (Cai et al., 2020). Moreover, teachers can encourage innovative thinking through creative thinking process.

Problem-Posing as Pedagogical Approach

By emphasizing the pedagogical approach of problem-posing, educators can bring a new contextualized environment to the mathematics classroom, not just by solely determining students' ability to solve problems but by employing the strategy of making mathematical problems. This could unlock a goal in teaching learning through gaining insights into every learner's approach coupled with their understanding and their competency not just about their knowledge of mathematical content, but also their perspective on a certain task through employing their learning experiences (Leung, 2012).

Given that the key persons in developing the teaching and learning process are the teachers, they must possess adequate knowledge and understanding not just of the content, but also an immersive idea of the pedagogical approach (Masriyah et al., 2018). In a teacher education program, preservice mathematics teachers were exposed to different mathematical contents: abstract, symbolic, and even interactive approaches in mathematics. They were taught the essential competencies of a mathematics teacher. Additionally, this ability is needed to be able to explore and evaluate the extent to which learners understand the material being taught. In addition, the teacher's ability to pose problems can assist students in reducing their dependence on textbooks and help students become more involved in learning activities.

However, principles and strategies in teaching mathematics remain intact on the traditional approaches due to the limitation of equitable materials on how to teach mathematics and the understanding of how a young learner thinks during a complex task like mathematical problem-solving. This may be linked to the absence of exposure to a real classroom setting scenario, which can be addressed through gaining insights on what a learner can do by posing their mathematical task. It can serve as a measure of their understanding and their level of competency.

Previous studies indicate that teachers themselves had difficulty engaging in problem-posing activities because of the unfamiliarity of these concepts (Kozakli Ulger et al., 2022; Ozgen, 2019; Vale & Barbosa, 2024). Moreover, the output quality produced by the educators in their content and pedagogical approach to problem-posing is not always valid, which leads to a deficiency in

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

the quality of teaching and learning (Koichu & Kontorovich, 2013). Ironically, mathematics educators are the ones who must demonstrate a strong understanding of the given concept. Hence, if we could produce mathematics teachers that possess an equitable knowledge in varied approaches of problem-posing performance, it could significantly improve the mathematics education in the Philippines.

Mathematics teachers play a critical role in the development of strong new approaches in mathematical pedagogy. Hence, the main goal of this study is to assess if these future mathematics teachers could properly construct effective mathematical problem solving through their problem-posing strategies. Introducing problem-posing activities in mathematics classrooms could revolutionize the conventional approaches of problem-solving, thus it is a crucial to produce highly competent teachers in this concept to properly guide and evaluate the learners.

This study determines the current performance of the preservice mathematics teachers in problem-posing activities through evaluating their approaches and their content knowledge in mathematics. Moreover, determining the socio-affective domains in terms of gender roles, sex, and the developmental factor in terms of academic year level could significantly improve the targeted intervention in problem-posing performance. Taking these concerns into account could improve the relevance of the study.

Specifically, this paper answers three main questions:

(1) What is the competency level of the preservice mathematics teachers in problem-posing?
(2) What are the strategies and approaches used by the preservice mathematics teachers in posing a mathematical problem?
(3) Do problem-posing performance of the pre-service mathematics teachers differ across sex, and academic year level?

Addressing these concepts could inform curriculum experts and educators through understanding the underlying principles and concepts that might affect the learners in these competencies. Determining their competency in this matter will inform the educators and the teacher education programs to devise and design a strategy where the use of varied approaches in improving critical thinking, creativity, and problem-solving skills could be maximized. This could then eventually lead to the advancement of teaching and learning in mathematics education that might attain the twin goal in the mathematics framework.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

METHOD

Research Design

This study utilizes a quantitative approach to determine the problem-posing competency level of the respondents; a descriptive design was used by quantifying the responses and interpreting it into descriptive categories. To properly evaluate and interpret the output made by the respondents, each performance was assessed to generate more in-depth analysis by examining and evaluating the common mistakes, strengths, and differentiated approaches performed by the respondents in the problem-posing activity. For the inferential part, mean comparisons on determining the differentiated performances of the respondents were utilized.

Sample and Data Collection

The research sample consisted of 75 pre-service mathematics teachers, who are currently enrolled in the Teacher Education Program–Bachelor of Secondary Education major in Mathematics. These participants were selected using a stratified sampling procedure with proportional allocation. This method allows the researchers to obtain a diverse research sample that represent every group in the population of interest, accounting for sex and academic year level.

The study was conducted during the first semester of the Academic Year 2024–2025, aligning with the period when the participants were taking the course "Principles and Strategies in Teaching Mathematics." This course explores various instructional strategies and methodologies essential for effective mathematics teaching. Its focus on pedagogical approaches made it an ideal context for investigating the participants' abilities in problem-posing.

To ensure meaningful data collection, the researchers introduced the participants to a series of structured problem-solving activities, incorporating both routine and non-routine mathematical problems. These activities were designed to expose pre-service teachers to different problem types, fostering an understanding of how mathematical concepts can be constructed and represented from various perspectives. The exposure aimed to cultivate their ability to generate cognitively demanding mathematical problems which are crucial skill for educators in teaching and assessment. Data collection took place after this instructional intervention, where students were tasked with crafting their own mathematics problems. These outputs were evaluated to determine the depth of conceptual understanding, creativity, and clarity of presentation.

The underlying assumption was that exposure to high-level cognitive tasks and diverse problem types would enhance their capacity to pose meaningful and engaging problem. By situating the activity within the framework of coursework and instructional development, this study emphasizes the potential of problem-posing as both a pedagogical strategy and an assessment tool. It highlights how intentional instructional design can foster a deeper understanding of mathematical thinking, preparing future educators to deliver richer learning experiences in their future classrooms.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Ethical Considerations

The study obtained an ethical clearance approval from the institution. The research complies with all relevant legal standards and regulations governing the rights and welfare of research participants. Participation in the study was entirely voluntary, and participants could withdraw at any stage without facing any negative consequences. The voluntary nature of participation was emphasized both in the informed consent process and during the study. Participants were informed about the study's purpose and procedures to maintain transparency throughout their involvement, ensuring strict confidentiality of the generated data; any published results do not include personally identifiable information to protect the privacy of participants.

Instrumentation

This research utilized a "Free Problem-Posing Performance Assessment." This is a researcher-made performance test that aligns with several research studies about the assessment of problem-posing performances and is validated by a group of experts in mathematics education, including a person holding a doctorate degree in Mathematics education and an in-service mathematics teacher from high school level.

This instrument comprises several tasks in which participants will generate or pose two problems based on the indicated task or given concept on each specific item. Free problem-posing activities can be the most effective tool in problem-generation as problem-posers can develop learners' mathematical thinking. Situations in free problem-posing activities encourage authentic assessments that target daily life scenarios that learners can relate to. Use of their prior knowledge will give them an advantage in solving and approaching the problem. This instrument consists of three main concepts, aligned to the content areas of K-12 Curriculum in the Philippines: Measurement and Geometry, Number Sense and Algebra, and Statistics and Probability. Sample tasks were provided as follows:

Sample item: Pose a problem that involves the length of measurements (mm, cm, and m) in it.

Sample item: Pose a problem involving a division of fractions.

Sample item: Pose a problem that deals about the probability of an event.

To objectively measure the performance of the participants in the problem-posing activity, a modified analytical rubric developed by Polat and Özkaya (2023) was utilized (Table 1). Given the qualitative nature of problem-posing tasks, a standardized and objective evaluation tool is necessary to ensure the reliability, validity, and fairness of the assessment process.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Criterion	Indicators	Score
Problem text criterion	The problem text is not clear and understandable.	0
	Problem text is partially understandable.	1
	The problem text is understandable, it is not clear what is asked.	2
	The problem text is clear, concise and understandable.	3
Compatibility with mathematical principles	The problem does not conform to the principles of mathematics	0
	The problem partially complies with the principles of mathematics (such as the use of misconceptions).	1
	The problem does not comply either mathematical or conceptual error, but the data used is not compatible with daily life	2
	The problem complies with mathematical principles and real life.	3
Type and structure of the problem	It is a simple exercise type.	0
	It is of the exercise type.	1
	It is of the simple word problem type.	2
	It is of word problem type.	3
Solvability of the problem	The data and information in the problem are not sufficient to solve the problem.	0
	Although the given in the problem is sufficient, it cannot be solved because it is too complex.	1
	The problem can be solved, but the data is either incorrect or missing.	2
	The problem is solvable because the data and information in the problem are complete and appropriate.	3

Table 1: Adapted Scoring Rubric developed by Polat and Özkaya (2023)

Each criterion captures a different but interdependent aspect of problem quality, ranging from language and context clarity to mathematical correctness and real-world relevance. This multi-dimensional assessment is necessary because problem-posing is a higher-order cognitive task that demands both pedagogical insight and mathematical competence.

This rubric quantifies qualitative responses, turning open-ended outputs into measurable data that can be analyzed statistically. This rubric will yield a total of 12 points in each problem, 24 points in each competency, and will generate a total of 72 points. To determine the performance of the participants in each competency, and the over-all problem-posing performance, a qualitative classification was used (Table 2).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Analysis of Data

This study used various methods on interpreting the obtained data. All are aligned to the stated objectives. To determine the level of the problem-posing performance, mean and standard deviation were utilized. The mean was transmuted into qualitative description given below (Table 2). To answer if there are significant differences based on sex and academic year level, an independent sample's t-test was utilized, along with a One-way Analysis of Variance, followed by a univariate analysis (Post-hoc) using Bonferroni analysis for a proper evaluation of the existing mean differences.

The use of descriptive and inferential statistical methods in this study is justified by the nature of the research objectives and the type of data collected. This study aims to determine the level of problem-posing performance among pre-service mathematics teachers; using the mean and standard deviation provides a clear summary of overall trends in their performance. The inclusion of a transmuted qualitative scale further enhances the interpretability of the scores in an educational setting.

Description	For each Competency	For each task	For Overall Performance
	Scoring	Scoring	Scoring
Poor	1.00 – 6.75	1.00 – 3.75	1.00 – 18.75
Fair	6.76 – 12.50	3.76 – 6.50	18.76 – 36.50
Satisfactory	12.51 – 18.25	6.51 – 9.25	36.51 – 54.25
Very Satisfactory	18.26 – 24.00	9.26 – 12.00	54.26 – 72.00

Table 2: Standard criteria for the level of Problem-posing Performance

To evaluate the respondents' demonstrated performances in problem-posing activity, the researchers investigated participants' responses and thoroughly analyzed the patterns, common mistakes, strengths, and weaknesses of their provided outcomes. Aside from quantitatively describing the differences in the performances of problem-posing activities, the researchers explored the answers of each response and compared it across the demographic profile of the respondents. The qualitative analysis of the participants' responses was integrated to gain deeper insight into their problem-posing abilities. This approach allowed for access beyond numerical scores by identifying common patterns, strengths, and weaknesses in how the participants construct mathematical problems. Exploring individual responses across demographic profiles supports a more comprehensive understanding of how different factors may influence performance.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

RESULTS

Table 3 presents the breakdown of the scores on problem-posing performance of the respondents on different tasks, and the overall score on each competency in mathematics curriculum.

Area of Competency		Mean	SD	Level
Geometry and Measurement	Task 1	6.19	3.348	Fair
	Task 2	6.73	4.323	Satisfactory
	Total	12.20	6.877	Fair
Statistics and Probability	Task 1	7.07	3.306	Satisfactory
	Task 2	6.84	2.927	Satisfactory
	Total	13.91	5.199	Satisfactory
Number Sense and Algebra	Task 1	5.53	3.310	Fair
	Task 2	7.65	3.100	Satisfactory
	Total	13.19	5.399	Satisfactory
Overall Problem-Posing Performance		39.29	14.050	Satisfactory

For each task: Poor (1.00–3.75), Fair (3.76–6.50), Satisfactory (6.51– 9.25), Very satisfactory (9.26–12.00)
For total of Competency Area: Poor (1.00 – 6.75), Fair (6.76–12.50), Satisfactory (12.51-18.25), Very satisfactory (18.26–24.00)
For Overall Problem-posing performance: Poor (1.00–18.75), Fair (18.76–36.50), Satisfactory (36.51-54.25), Very satisfactory (54.26 –72.00)

Table 3: Problem-Posing Performance of pre-service mathematics teachers

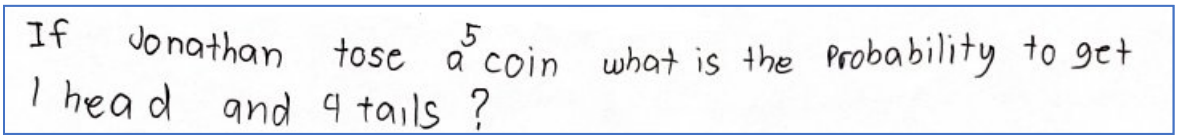
The demonstrated performance of the respondents in the problem-posing activity was properly analyzed using descriptive statistics. The mean and standard deviation were obtained and interpreted in a qualitative description reflecting poor to very satisfactory performance. Findings revealed that the overall problem-posing performance of the respondents is at satisfactory level (μ=32.29, SD=14.50) and shows a high variability of the results. When examining the combined scores in Task 1 and 2 in each competency area, Statistics and Probability and Number sense and Algebra is at Satisfactory level (μ=13.91, SD=5.199, and μ=13.19, SD=5.399, respectively). However, when it comes to Geometry and Measurement, respondents demonstrate a lack of competency as the results reveal that respondents demonstrated a Fair level of performance (μ=12.20, SD=6.877). Although the performance of the respondents in the Number sense and Algebra is at satisfactory level, respondents demonstrated a relatively low score in task 1. A consistent finding of fair performance of the respondents in task 1 in Geometry and Measurement imply a lack of demonstrated competency in figures and measurements.

Moreover, the overall performance of the preservice mathematics teachers is at satisfactory level, indicating that these future mathematics teachers possess the necessary skill in problem generation. However, the mastery in this concept was not achieved, as they may still lack the creativity and innovative approaches in problem-posing.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Qualitative Analysis of the Demonstrated Performance in Problem-Posing Activity

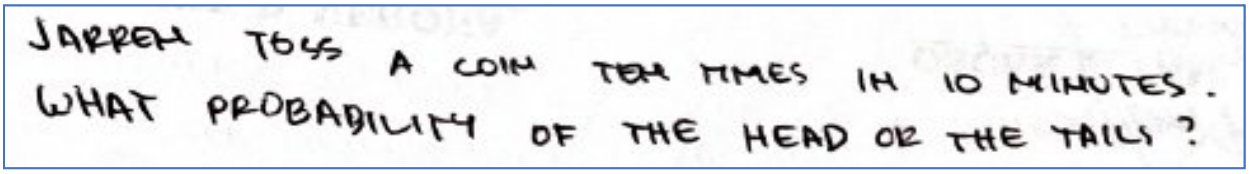
To properly evaluate the problem-posing performance of the preservice mathematics teachers, this study explored and assessed each response and noted all the observed deficiencies, lack of indicated domains, and notable approaches used in the problem-posing.



*"If Jonathan **tose** a 5 coin what is the probability to get 1 head and 4 tails?"*

Figure 1: Statistics and Probability in Task 1 of 1st year student

Several students' posed problems in the context of tossing a coin and rolling a die, which is used in teaching and learning in mathematics classrooms. Most of the examples used by an educator during the discussion of the probability of an event is the use of these examples as well. Because of the common nature of the concept, these examples are readily available in the textbooks used by the teachers. However, common errors were observed including grammatical errors and misspelled words, which significantly affects the presentation of the problem as it affects the clarity of thought. In the case of Figure 1, the word "toss" was misspelled to "tose," and the clarity of the thought was out of the context that the respondent wants to present it. In the researcher's understanding, the problem pertains to *"If Jonathan tosses five (5) pieces of coin, what is the probability of getting one (1) head and four (4) tails as an outcome?"* However, this problem still lacks depth and an authenticity to support learners in properly cultivating their critical and logical thinking. A similar problem presented by one 2nd year student (Figure 2).

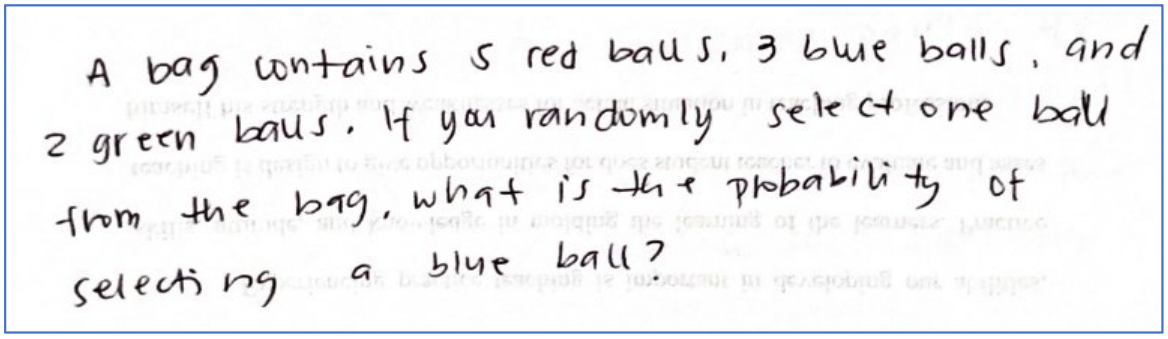


"Jarren toss a coin ten times in 10 minutes. What probability of the or the tail?"

Figure 2: Problem-posed of 2nd year student in Statistics and Probability Task 1

The problem was phrased clearly and presented an insufficient story to tell the context of the problem. Similar to the approach used in Figure 1, this student used a name to enhance the relevance of the problem. A slightly different approach made by the 3rd year student. The 3rd year student used a commonly posed problem regarding the probability of getting colored balls (object) in a pool of selections. This type of problem is commonly used in a mathematics classroom; thus, students tend to imitate this approach.

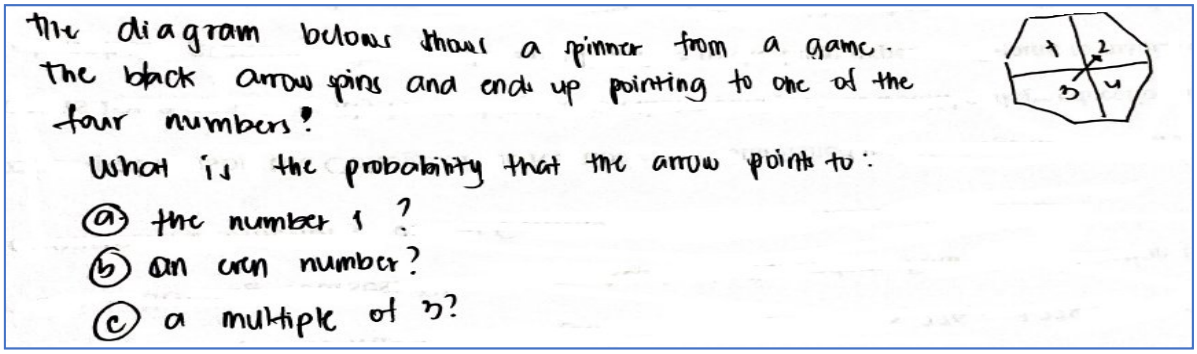
This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.



"A bag contains 5 red balls, 3 blue balls, and 2 green balls. If you randomly select one ball from the bag, what is the probability of selecting a blue ball?"

Figure 3: Problem-Posed in task 1 by 3rd year student

Although the presentation of data is complete, this problem (Figure 3) does not cater a complex analysis of problem-solving approaches. The structure of the problem mostly targets a simple exercise problem, but it does not conform to the mathematical principles that relates to the applicability in real life scenarios that learners could relate their experiences and prior knowledge to further enhance their understanding of the problem.



"The diagram below shows a spinner from a game. The black arrow spins and each up pointing to one of the four numbers. What is the probability that the arrow points to:

a. the number 1? b. an even number? c. a multiple of 3?"

Figure 4: Problem-posed in task 1 by 4th year student

The approach used in Figure 4 seems promising as the illustration adds to the clarity of the problem. The learners could properly visualize the context, which leads to better comprehension. Although the words in the problem presentation are partially erroneous and the given numbers limit

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

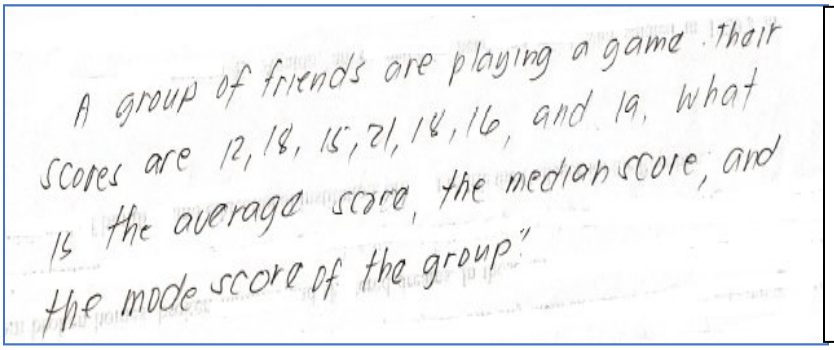
the complexity of the problem, this problem may be used as an activity; however, further enhancement is needed.

The problem lacks context. They assume students understand the mechanics of the spinner and equal probability, but do not explicitly state that each outcome is equally likely. For improvement, the problem posed must clarify the assumptions by adding a statement specifying that the spinner is divided into equal sections and that each number has an equal chance of being landed on. Consistent phrasing for each question (a to c) could also enhance the problem. Furthermore, the main goal in problem-posing is to encourage deeper thinking; hence, generating contextualized problems are crucial. For example, we can add a follow-up question here: *"If the spinner is spun 100 times, how many times would you expect the arrow to point to an even number? Justify your reasoning."* By doing this, we can promote good reasoning and thinking processes.

Task 2: Statistics and Probability

Most of the student's output in task 2 in statistics and probability does not vary, as the conceptual understanding revolves on simple statements of finding the mean, median, and mode.

Figure 5 shows a little variation of the problem. The respondent simply added details to tell a story that serves as context and then proceeded to mathematical content. However, this problem shows potential ambiguity in what to do. The problem asks for multiple measures of central tendency, but it might be clearer if it specifies the steps expected, especially for learners new to these concepts. Also, to aid learners, it is a good strategy to give learners a prompt on what to do in the mathematical problem. This problem can include the definition of mathematical terms (mean, median, and mode); while these are common terms, some learners might need clarification or definitions.

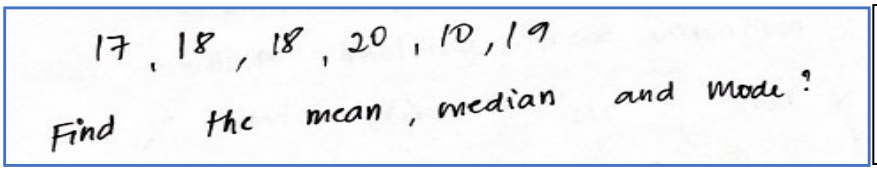


"A group of friends are playing a game, their scores are 12, 18, 15, 21, 18, 16, and 19. What is the average score, the median score, and the mode score of the group"

Figure 5. Variation of presentation of mean, median, and mode

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

However, the main issue arises in Figure 6. The problem-posed was simply presenting a set of numbers and proceeding to ask the value of the mean, median, and mode.



"17, 18, 18, 20, 10, 19 Find the mean, median and mode?"

Figure 6: Common observation on the problem-posed in mean, median, and mode

Although, the structure of the problem (Figure 6) is valid for the exercise type as it is complete and solvable, this problem clearly shows an absence of authenticity of the problem and the clarity of instruction. The data values were listed but not explicitly labeled as a data set which leads to ambiguity of the problem. While the question asks for the "mean, median, and mode," it does not define these terms or guide the reader on how to calculate them. For some learners, this could cause confusion.

Geometry and Measurement

The overall mean in the Measurement and Geometry fell under the "fair" category, implying that the performance of the respondents is below the expected threshold of the competency that must be possessed by a teacher. Through further analysis and evaluation, respondents displayed insufficient contextual understanding of measurement; they pose a different approach. However, the content and the context of the problem does not properly execute.

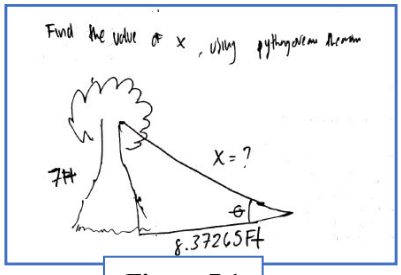


Figure 7.1

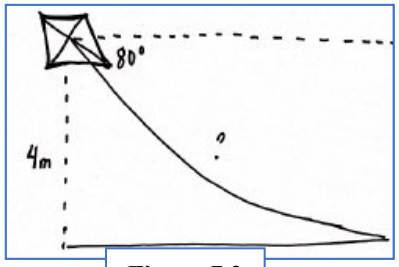


Figure 7.2

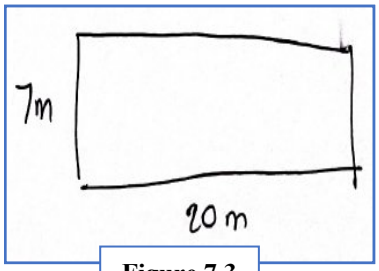


Figure 7.3

Figure 7: Examples of Inaccurate and Misaligned Student-Generated Illustrations in Geometry and Measurement Problem-Posing Task by 3 respondents.

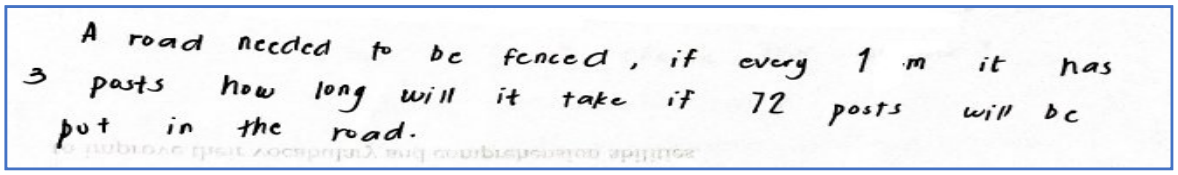
This performance shows an insufficient grasp of knowledge in the proper presentation of a problem. Although the use of figures for mathematical notations added to the value of clarity, the over reliance on figures and illustrations is irrational as it does not provide the context of the problem, resulting in confusion and an incomprehensible structure of the mathematical problem. If the learners encountered this problem in an activity and exercises, this would bring puzzlement and may hinder the learner's development in mathematical content.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Furthermore, Figure 7.1 is explicitly asking about the value of x using Pythagorean theorem and the indicated values. That drawing is a right triangle, although it does not present it that way. Another is the misleading label; if it is a right triangle, a small square on the intersection between two sides can be added to indicate that one angle measures 90°. Furthermore, unnecessary inclusion of theta (θ) leads to another form of question which adds confusion to the aim of the problem. Moreover, the figure is inaccurate. The Pythagorean theorem pertains to the sides of the right, and the problem is asking for the hypothenuse of the right triangle. Solving for the value of x, $x = \sqrt{((7)^2 + (8.37265)^2)} \approx 10.913$, which is the longest side; however, the scale in the drawing does not visually reflect this relationship.

The same issues were observed in Figure 7.2, although it uses another approach by introducing an exterior angle and only one given side. To solve this problem, the concept of trigonometric functions is now involved. In this case, it is crucial that the drawing clearly shows that it has one right angle. By assuming that the intersection between two sides forms right angle, the problem is now solvable using the trigonometric concept $sin\ (\theta) = \frac{opposite}{hypothenuse}$.

Looking at the last Figure 7.3, the problem lacks clarity of the mathematical concept it is asking. Is the figure a rectangle? If so, is it pertaining on calculating the area or the perimeter of the rectangle? This figure clearly lacks the clarity of problem presentation; hence it is not a valid approach in problem-posing activities.



"A road needed to be fenced, if every 1 m it has 3 post how long will it take if 72 posts will be put in the road."

Figure 8: An increase performance in problem-posing skills

There are respondents who demonstrate a skill in problem-posing that can conceptualize a problem (Figure 8) that targets the critical-thinking skills of the learners. This problem could capture the learner's interest as they can relate to the problem as the context is encountered daily. However, despite of the edge of the problem, it still presents an ambiguity and confusion about the context was clearly seen. A further question may arise in the mind of the learner, "Why would you fence the road? Does that mean you are blocking it?" In problem-posing, the clarity and preciseness of the words are crucial as it will lead to an increase comprehension. Moreover, the mathematical content of the given problem is not precise; the phrase, "how long will it take," implies a time calculation, but there is no mention of the speed or rate at which posts are placed. The in-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

tent might be to calculate the total length of the road, but the wording is misleading. If the goal is to calculate the road length based on the number of posts, it is important to clarify whether posts are placed at the ends of the road or at regular intervals in between.

The findings clearly suggest the need for structured learning opportunities that refine the pre-service teachers' ability to integrate visual representations and contextual elements in problem construction. The demonstrated performance in Measurement and Geometry reveals that while learners can use figures to support mathematical problems, they often fail to ensure alignment between illustrations and the problem's mathematical goals. Therefore, the intervention must emphasize conceptual alignment, such as accurately representing geometric relationships (e.g., using right-angle indicators, consistent side lengths, and logical visual scaling). These errors highlight a gap not only in content knowledge but in pedagogical reasoning. Future educators must be trained to anticipate student confusion and eliminate sources of ambiguity in problem design.

Moreover, the performance in these tasks suggests that learners are not fully internalizing the communicative function of problem-posing—that is, the ability to clearly convey a mathematical challenge through coherent language and visual structure.

Number Sense and Algebra

The expectation towards this area of competency is slightly higher because mathematics majors usually enjoy the context of manipulating numbers. However, the results suggest that although learners are good in solving mathematical algebraic problems, respondents demonstrate a "fair" level of competency in the problem-posing activity in Task 1, which involves fraction. The common problem that was observed is the presentation of the division of two fractional numbers.

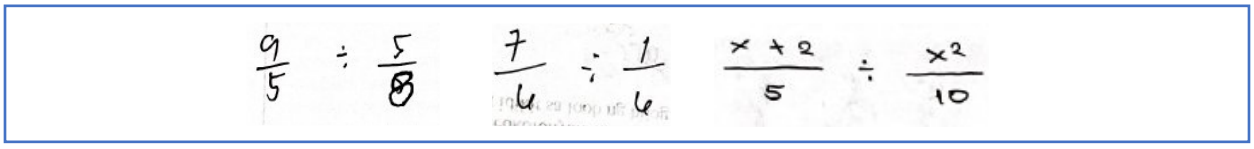
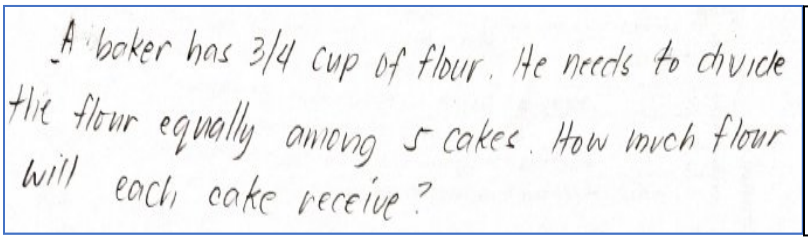


Figure 9: Missteps in Posing Fraction Division Problems in Algebraic Contexts

This approach is inappropriate (Figure 9). This kind of problem does not conform to the principle of problem-posing activities; it does not offer any context in problem-text criterion which creates a vagueness of the problem, and the main question of the problem is not explicitly known. These problems were usually used in the process demonstration in a mathematics discussion, in which teachers assumed the role of explaining the context of the process of division of fractions. This entails that this type of problem is a tool for understanding the mathematical content. However, when it comes to the approach of problem-posing, this problem presentation is not valid. Aside from non-appearance of words that discuss the problem, it does not conform to the solvability of the problem as the question and task for this type of problem-posed is unclear and ambiguous. This

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

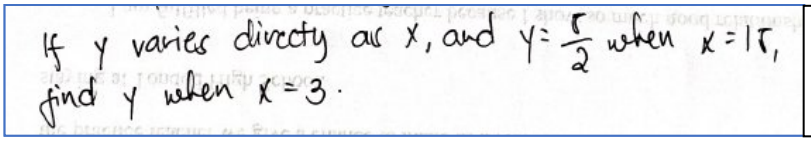
observation is common to lower year level students. The scenario is different from higher years, as they started to introduce concepts of verbal problems.



"A baker has ¾ cup of flour. He needs to divide the flour equally among 5 cakes. How much flour will each cake receive?"

Figure 10: Problem-posed by higher year student.

Figure 10's simple presentation of words added merit to the problem. It not only enhanced the presentation, but is also provided valuable information that could aid the learner's mind to comprehend the mathematical context. Aside from explicitly asking the learners to divide a certain number, incorporating a story enhances the solvability of the problem. Despite this increased performance, the innovative approach is still lacking. Due to the deficiency of depth in problem-presentation, the principle of reality was not obtained, and more importantly, the lapses of contextualize approach in high school learners is concerning.



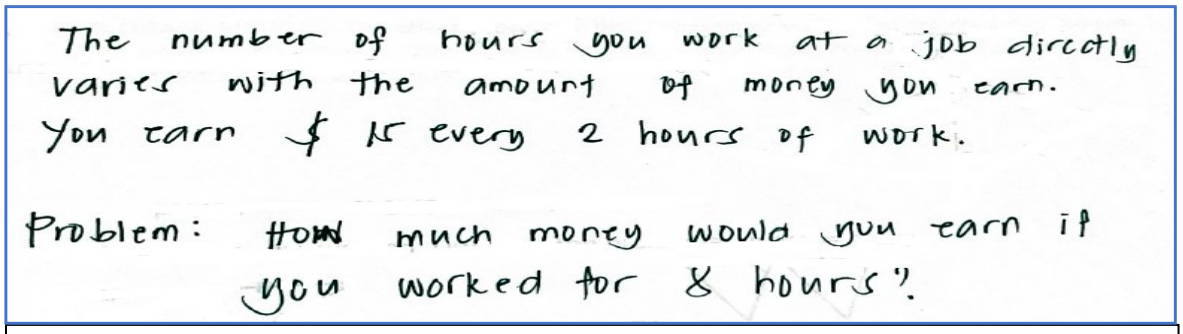
If y varies directly as x, and y =$\frac{5}{2}$ when x=15, find y when x=3.

Figure 11: Task 2, Problem-posed in Direct variation

A typical textbook problem appears in Figure 11 which shows a common problem observed in the respondents from 1st year to 4th year. This clearly implies that learners are textbook driven and are taking this pattern throughout their learning process. However, when we properly analyze the problem, the given proportion is incorrect. For y to vary directly as x, the ratio between y and x must remain the same for all values of x and y. Given y=kx, the constant of proportionality k is equal to y/x. If we compute for the given, $y = \frac{5}{2}$, and $x = 15$, we have $k = \frac{\left(\frac{5}{2}\right)}{15} = \frac{1}{6}$. Using this as value of k, the relationship becomes $y = \frac{1}{6}x$. However, when we substitute $x = 3$, the resulting value of y now is $\frac{1}{2}$. The problem fails to clarify that this computation aligns with the relationship described. Instead, it leaves room for misinterpretation, as some might expect that the given information forms a basis for another type of variation.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

This issue in given presentation is very crucial in mathematical content, as this might lead to an inaccurate result. This approach may hinder the learner's development. Aside from sticking to a conventional approach, this mathematical concept remains abstract in the mind of the learners. Hence, it is imperative to construct a mathematical problem that builds a context with correct content.



The number of hours you work at a job directly varies with the amount of money you earn. You earn $15 every 2 hours of work.

Problem: How much money would you earn if you worked for 8 hours?

Figure 12: Other form of direct variation in word problem

The mathematical problem (Figure 12) is straightforward, avoiding unnecessary complexity. This ensures students can focus on the mathematical relationship without being distracted by unrelated details. Moreover, the problem uses real-world context (working hours and earnings), making it relatable and engaging for students. Realistic scenarios help learners see the relevance of mathematics in daily life. Most importantly, it explicitly addresses the concept of direct variation, where one variable changes proportionally with another. This helps students understand the relationship between variables in practical settings. By eliminating unnecessary complexity, students can focus on the relationships between variables while appreciating the relevance of mathematics in daily life. This approach fosters engagement, critical thinking, and the practical application of mathematical skills.

Demographic Profile	Grouping Variable	Mean	SD	t, F	df	p-value
Sex	Male	39.75	12.295	3.301	73	0.810
	Female	38.95	15.361			
Year Level	1st year	27.07	16.020	8.791	3, 71	0.000
	2nd year	49.00	11.888			
	3rd year	37.44	8.031			
	4th year	41.65	12.397			

Table 4: Mean Comparison of the problem-posing performance across demographic profiles

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

It is crucial to emphasize the gender differences in the complex tasks of mathematical concepts as both genders may have different perceptions on the problem-posing approaches. Findings from independent samples t-test indicates that the mean differences of the gender in problem-posing performance is consistent, implying that the gender roles have nothing to do with the problem-posing approaches ($p>0.05$).

Furthermore, One-way Analysis of Variance (ANOVA) was performed to determine if there is a significant difference on the mean scores to the overall problem-posing performance when grouped according to year level. Results indicates that there is a significant difference on the academic year levels on the problem-posing performance ($F = 8.791, p < 0.01$). Moreover, the post-hoc analysis for these significant differences were explored. Performance in problem-posing activity revealed that the 2nd to 4th year students are better when compared to 1st year students; however, although the performance of 2nd year students is better when compared to 1st year students, this increased performance is not enough when compared to 3rd year and 4th year students. There is no significant difference between 3rd year and 4th year students.

This suggests that instructional strategies should focus on skill development rather than gender-based differences. This reinforces the idea that all students, regardless of gender, can become effective mathematical thinkers through equitable learning opportunities. Moreover, the increased performance of problem-posing performance in academic year level underscores the developmental nature of problem-posing skills, which is likely enhanced through increased exposure and experience in mathematics. These findings support the need for scaffolded and progressive learning experiences, ensuring students build the necessary cognitive and creative capacities over time.

DISCUSSION

Presenting a clear and coherent storyline allows readers to understand different scenarios. Without this clarity, an incomprehensible scene can lead to serious confusion. The same context applies to problem-posing; a good mathematical problem not only relies on the content presentation, but on properly merging the mathematical content to a good story that could capture the learner's interest and make them think critically (Yu & Yun, 2015). The more learners are engaged to material, their motivation and disposition to learn helps them to become successful in mathematics learning (Roche et al., 2021). This implies that giving learners' good material is the best initiative to expose learners to a comprehensive learning process.

However, the demonstrated performances by pre-service mathematics teachers unveil a call for an innovative approach in strengthening creativity and logical thinking. It is imperative to produce teachers that are empowered with knowledge in problem generation as it significantly relates to shaping learners' mathematical learning (Leung, 2013). However, an issue in technicalities, especially grammatical errors and unclear problems due to the poor problem presentation is widely observed. It is concerning, because of the objective nature of mathematics in which every context or even a simple word might significantly affect the meaning and approach of the mathematical

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

problem (Baumanns & Rott, 2023). Thus, the presentation of the mathematical problem must be systematical and not cognitively demanding to avoid learner's confusion on the approach in the problem (Erath et al., 2018). Hence, it is important to deliberately develop the English language proficiency of the mathematics teachers as it might significantly improve the problem-posing performance of the learners leading to enhanced mathematics learning.

Moreover, the contextual understanding in mathematical problem-posing is limited. Adding names in the generated problem does not guarantee that the learners will relate to it. Also, the over reliance on textbook mathematical problems does not provide an innovation for teaching and learning (Kozakli Ulger et al., 2022). Posing nonproblem statements were also observed even though that the task is clear, highlighting the inattentiveness to details (Cai et al., 2015). The main concerns in problem generation is making the context relatable for the learners and ensuring the clarity of the mathematical content is properly presented (Ornek & Soylu, 2021). Koichu and Kontorovich (2013) emphasized that problem-posing activity is only valid if it aligns to the criterion of an authentic mathematical activity. Thus, teachers must be aware of their learning environment to be able to design a mathematical problem that address these concerns.

Preservice teachers lack reflection on problem-posing. They posed problems that were not contextually appropriate for the developing learners; this was observed through the ambiguity of the context of the problems. One possible reason, indicated by Koichu (2019), is the limited exposure in managing a real classroom environment. This may explain the possible advantage of the posed problem of the higher-year level students as they have already had classroom interactions that support their ideas in problem generation (Ozgen, 2019). Similarly, the use of illustrations as an assistive guide targets the visual learning approach that allows learners to comprehend mathematical concepts with clear representations, helping them break the mathematical content into more manageable chunks of knowledge (Jacobs, 2005). Involving figures in mathematical problems offers a holistic approach to facilitate the learner's mind through visualizing problem scenarios which opens an opportunity to deliver good mathematics learning by helping the learners overcome difficulties in understanding the abstract presentation of mathematical problems (Maula et al., 2024; Pehkonen, et al., 2016). Hence, the use of drawings and mathematical representations must be presented accurately.

Furthermore, providing clear instruction is important. It serves as a guide, helping students understand the expectations and processes involved in creating meaningful and coherent mathematical problems. Clarity reduces confusion and ensures that students focus on the task's objectives rather than struggling with ambiguity. Well-structured instructions provide a framework that encourages creativity while maintaining alignment with the desired learning outcomes. By explicitly outlining criteria such as the mathematical concepts to be included, the level of complexity, and the context of the problems, educators can support students in developing their problem-posing skills systematically (Zubieta & Lapinid, 2024). This may imply that designing knowledge from a perspective of the learner could significantly improve the mathematics learning outcome (Parks, 2020).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The development of the competency of the pre-service mathematics teachers is very important because their competency determines their skills and approach in mathematics teaching, which is directly related to learners' development. Leung (2013) emphasized that problems posed by a teacher can shape the mathematical learning of the students; thus, these deficiencies and lapses on problem-posing reflects the current performance of the pre-service mathematics that calls for an innovation to improve. This aligned with Cai et al. (2015) that found that teachers are prolific problem-posers when the set of information necessary for problem generation is provided; however they struggle when it comes to crafting a valid problem without the given prompts. Additionally, problem-posing must involve several steps: comprehension, translation, and finally editing and selecting to ensure that the problem-posed is valid and contextualized.

Problem-posing as a mathematical pedagogical approach opens an opportunity for a more exciting knowledge delivery. The development of these concepts signifies the attainment of mathematical literacy, as it allows individuals to identify mathematical issues within real-life contexts and translate them into solvable problems that require critical thinking and a deep understanding of mathematical concepts. It's not just about solving problems, but also about formulating problems effectively (Kozakli Ulger et al., 2022; Ozgen, 2019).

CONCLUSIONS

The insufficient performance of the Philippines in international assessments highlights the prevailing issues in mathematics education. This calls for a strategic shift toward the development of innovative pedagogical approaches. Among these, problem-solving and mathematical modeling present promising avenues for enhancing learners' cognitive and problem-solving abilities. However, the integration of problem-posing activities adds a critical dimension. It provides varied pathways for learning, where the learners' ability to generate problems serves as a reflection of their conceptual understanding, creativity, and mastery of mathematical content.

Similar to problem-solving, a contextualized approach is vital in problem-posing. Learning materials must be designed with learners' perspectives in mind that they can relate to, understand, and engage with meaningfully. The complexity of the task should align with their cognitive level. Although assessing learners' thinking through their posed problems may seem abstract, a careful evaluation of their thought processes and problem structures can offer a comprehensive view of their learning outcomes. Thus, there is a need to prepare future professionals who are equipped to explore and assess problem-posing effectively.

This study examined the competencies of pre-service mathematics teachers in problem-posing, revealing that their current level of performance does not meet the desired standard. Their outputs, when analyzed, expose gaps in both content and structure. Common issues include grammatical errors, incoherent problem construction, and a lack of contextual relevance—all of which could have been addressed at earlier educational stages. This is particularly concerning, as these individuals are training to become future educators. Additionally, inaccuracies in mathematical givens

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

and illogical figure presentations signal a deeper issue: inadequate mastery of core mathematical concepts.

The recurring deficiency in problem-posing knowledge and skills indicates a pressing need for curriculum improvement. To create meaningful learning opportunities, mathematics education must evolve. Strengthening the problem-posing abilities of pre-service and in-service teachers is essential. Beyond content and pedagogy, developing an understanding of how learners think and approach mathematical problems can significantly enhance educational outcomes. This begins with self-awareness; understanding one's own learning process can provide insight into how students might approach problem-solving and problem-posing in diverse ways.

This study highlights the variety of strategies used by pre-service mathematics teachers, revealing both strengths and areas needing development. Teacher education programs must respond by offering methodologies that target these deficiencies. Contextualizing problem-posing as a key component of mathematical literacy may further enhance understanding and application of the concept.

To support this, a conceptual framework (Figure 13) was proposed, illustrating a developmental flow from problem-solving to problem-posing. The framework posits that engaging learners in rich, non-routine problem-solving activities lays the cognitive foundation for effective problem generation. This developmental sequence ensures that learners acquire strategic, logical, and structural competencies necessary for creating meaningful mathematical problems.

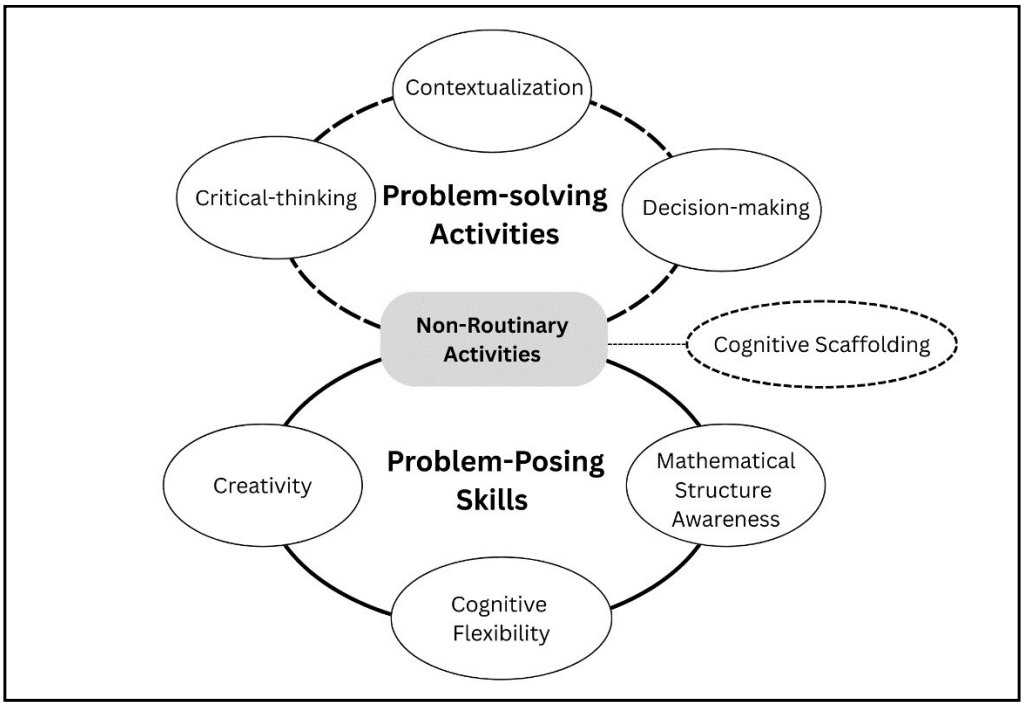


Figure 13: Linking Problem-Solving to Problem-Posing in Math Education

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The proposed framework draws upon foundational educational theories—Vygotsky's Zone of Proximal Development (ZPD) and Bruner's concept of scaffolding—to illustrate how learners can gradually develop problem-posing skills through guided experiences. In mathematics education, this means learners benefit from structured problem-solving tasks that are slightly beyond their independent capabilities but achievable with guidance (Geteregechi, 2025; Santos et al., 2024).

Bruner's scaffolding theory complements this by highlighting the importance of temporary instructional support that can be gradually removed as learners internalize concepts and strategies. Applying this framework, students begin with non-routine problem-solving tasks that require contextualization, critical thinking, and decision-making, which emphasizes cognitive processes that strengthen mathematical understanding (van Velzen, 2016). As learners engage with these tasks, they build the foundational skills needed to transition into problem-posing, which demands creativity, cognitive flexibility, and an awareness of mathematical structures.

Thus, teacher education programs must embed explicit instruction on problem-posing strategies. This includes, (1) Guiding how to derive problems from given contexts, solutions, or mathematical principles; (2) Guiding learners through refining the structure, clarity, and relevance of problems; and (3) Embedding reflective exercises that deepen their understanding of the pedagogical value of problem generation.

In classroom practice, problem-posing empowers students to take ownership of learning, promotes mathematical discourse, and enables teachers to identify misconceptions and tailor instruction accordingly. Therefore, strengthening the problem-posing competencies of pre-service mathematics teachers is essential to developing more learner-cantered, reflective, and inquiry-based mathematics classrooms.

ACKNOWLEDGEMENTS

The authors would like to thank the Core Gateway College Inc. for the institutional support they provided.

REFERENCES

[1] Acido, Josephine & Caballes, Dennis. (2024). Assessing educational progress: A comparative analysis of PISA results (2018 vs. 2022) and HDI correlation in the Philippines. *World Journal of Advanced Research and Reviews,* 462-474. http://dx.doi.org/10.30574/wjarr.2024.21.1.0020

[2] Baumanns, L., & Rott, B. (2024). Problem-posing tasks and their influence on pre-service teachers'creative problem-posing performance and self-efficacy. *Journal of Mathematical Behavior, 73,* Article 101130. https://doi.org/10.1016/j.jmathb.2024.101130

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[3] Cai, J., Hwang, S., Jiang, C., & Silber, S. (2015). Problem-Posing Research in Mathematics Education: Some Answered and Unanswered Questions. *In F.M. Singer et al. (eds.), Mathematical Problem Posing, Research in Mathematics Education.* https://doi.org/10.1007/978-1-4614-6258-3_1

[4] Department of Education. (2016). K to 12 curriculum guide: Mathematics (Grade 1 to grade 10). https://bit.ly/3IwrXLE

[5] Divrik, R. (2023). Effects of teaching mathematics supported by problem-posing strategies on problem posing skills. *International Journal of Modern Education Studies, 7*(2), 371-408. https://doi.org/10.51383/ijonmes.2023.308

[6] Erath, K., Prediger, S., & Quasthoff, U. (2018). Discourse competence as important part of academic language proficiency in mathematics classrooms: the case of explaining to learn and learning to explain. *Educational Studies in Mathematics, 99,* 161-179. https://doi.org/10.1007/s10649-018-9830-7

[7] Geteregechi, J. (2025). An examination of undergraduate students' problem-posing and its interaction with problem-solving processes. *International Electronic Journal of Mathematics Education, 20*(1), em0809. https://doi.org/10.29333/iejme/15738

[8] Jacobs, K. L. (2005). Investigation of interactive online visual tools for the learning of mathematics. *International Journal of Mathematical Education in Science and Technology, 36*(7), 761–768. https://doi.org/10.1080/00207390500271149

[9] Koichu, B., Kontorovich, I. (2013). Dissecting success stories on mathematical problem posing: a case of the Billiard Task. *Educational Studies in Mathematics, 83,* 71–86. https://doi.org/10.1007/s10649-012-9431-9

[10] Koichu, B. (2019). Problem posing in the context of teaching for advanced problem solving. *International Journal of Educational Research.* https://doi.org/10.1016/j.ijer.2019.05.001

[11] Kozakli Ulger, T., Bozkurt, I., & Altun, M. (2022). Analyzing in-service teachers' process of mathematical literacy problem posing. *International Electronic Journal of Mathematics Education, 17*(3), em0687. https://doi.org/10.29333/iejme/11985

[12] Leung, S. (2012). Teachers implementing mathematical problem posing in the classroom: challenges and strategies. *Educational Studies in Mathematics, 83,* 103–116. http://dx.doi.org/10.1007/s10649-012-9436-4

[13] Li, X., Song, N., Hwang, S., & Cai, J. (2020). Learning to teach mathematics through problem posing: teachers' beliefs and performance on problem posing. *Educational Studies in Mathematics.* https://doi.org/10.1007/s10649-020-09981-0

[14] Masriyah, M., Kurniasari, I., & Palupi, E. L. (2018). Characteristics of pre-service teachers' performance in problem posing. *The 6th South East Asia Design Research International Conference (6th SEA-DR IC), Journal of Physics: Conf. Series 1088.* https://doi.org/10.1088/1742-6596/1088/1/012115

[15] Maula, I., Nusantara, T., Sulandra, I. M., & Sisworo (2024). Mathematical Problem Posing in Differentiated Learning. *Mathematics Teaching Research Journal, 16*(6), 144-167.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[16] Parks, A. N. (2020). Centering children in mathematics education classroom research. *American Educational Research Journal, 57*(4). https://doi.org/10.3102/0002831219873853

[17] Pehkonen, E., Ahtee, M., Laine, A. (2016). Pupils' Drawings as a Research Tool in Mathematical Problem- Solving Lessons. In: Felmer, P., Pehkonen, E., Kilpatrick, J. (eds) Posing and Solving Mathematical Problems. *Research in Mathematics Education,. Springer,* Cham, pp. 167-188. https://doi.org/10.1007/978-3-319-28023-3_11

[18] Peng, A., Li, M., Lin, L., Cao, L., & Cai, J. (2022). Problem Posing and Its Relationship with Teaching Experience of Elementary School Mathematics Teachers from Ethnic Minority Area in Southwest China. *EURASIA Journal of Mathematics, Science and Technology Education, 18*(2). https://doi.org/10.29333/ejmste/11534

[19] Örnek, T., & Soylu, Y. (2021). A model design to be used in teaching problem posing to develop problem- posing skills. *Thinking Skills and Creativity, 41.* https://doi.org/10.1016/j.tsc.2021.100905

[20] Ozgen, K. (2019). Problem-Posing Skills for Mathematical Literacy: The Sample of Teachers and Pre-Service Teachers. *Eurasian Journal of Educational Research, 84.* https://doi.org/10.14689/ejer.2019.84.9

[21] Polat, H., & Özkaya, M. (2023). The effect of problem posing-based active learning activities on problem- solving and posing performance: The case of fractions. *Journal of Pedagogical Research, 7*(1). https://doi.org/10.33902/JPR.202317880

[22] Roche, A., Gervasoni, A., & Kalogeropoulos, P. (2023). Factors that promote interest and engagement in learning mathematics for low-achieving primary students across three learning settings. *Mathematics Education Research Journal, 35,* 525-556. https://doi.org/10.1007/s13394-021-00402-w

[23] Santos, R., Santiago, A., & Cruz, C. (2024). Problem Posing and Problem Solving in Primary School: Opportunities for the Development of Different Literacies. *Education Sciences*, 14(1), 97. https://doi.org/10.3390/educsci14010097

[24] Vale, I., & Barbosa, A. (2024). Exploring the creative potential of mathematical tasks in teacher education. *International Electronic Journal of Mathematics Education, 19*(4), em0790. https://doi.org/10.29333/iejme/15075'

[25] Van Velzen, J. H. (2016). Evaluating the suitability of mathematical thinking problems for senior high-school students by including mathematical sense making and global planning. *The Curriculum Journal.* https://doi.org/10.1080/09585176.2016.1174140.

[26] Wakhata, R., Balimuttajjo, S., & Mutarutinya, V. (2023). Building on students' prior mathematical thinking: exploring students' reasoning interpretation of preconceptions in learning mathematics. *Mathematics teaching research journal, 15*(1), 127-151.

[27] Yu, E., & Yun, J. (2015). A research on the real state of story-telling mathematics class of middle school. *Communications of Mathematical Education, 29*(3), 441-463. http://dx.doi.org/10.7468/jksmee.2015.29.3.441

[28] Zubieta, V. C., & Lapinid, M. R. (2024). Analysis of grade 7 learners´ problem posing skills and difficulties in a free problem posing situation. *Mathematics Teaching Research Journal, 16*(6), 120-143.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

APPENDIX
Free Problem-Posing Performance Assessment

Problem-posing activity in Statistics and Probability

Task 1: Pose a problem that deals about the probability of an event.

Task 2: Pose a problem about measures of central tendency.

Problem-posing activity in Geometry and Measurement

Task 1: Pose a problem that involves the length of measurements (mm, cm, and m) in it.

Task 2: Pose a problem pertaining to the distance of points in a cartesian plane.

Problem-posing activity in Number-sense and Algebra

Task 1: Pose a problem involving a division of fractions.

Task 2: Pose a problem where it shows a direct variation.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Technology-Based Worksheets to Develop Students' Mathematical Argumentation Skills

Lia Budi Tristanti[1], Syarifatul Maf'ulah[1], M.Bahrul Subkhi[2]

[1]Department of Mathematics Education, Universitas PGRI Jombang, East Java, Indonesia

[2]Department of Information Technology, Universitas PGRI Jombang, East Java, Indonesia

btlia@rocketmail.com, syarifatul.m@gmail.com, bahruls27@gmail.com

Abstract: This study examines the need for and effectiveness of technology-based worksheets in improving students' mathematical argumentation skills. The worksheets were developed based on the Toulmin argumentation model, including mathematical problems, data, claims, warrants, qualifiers, rebuttals, and backing to guide students in constructing logical mathematical arguments. A pretest–posttest design with two unequal experimental groups was used. The population consisted of 200 prospective mathematics teachers in East Java, Indonesia, and 80 students were randomly selected and divided into two groups. Experimental Group 1 used digital-based worksheets, while Experimental Group 2 used printed worksheets. Data were collected through questionnaires and tests. Initial findings showed that although lecturers allowed students to express opinions, debate, and challenge peers' ideas, the worksheets commonly used did not promote structured mathematical argumentation or integrate technology. Therefore, technology-based worksheets were needed. The results showed that the average post-test score of the digital worksheet group (82.88) was significantly higher than that of the printed worksheet group (69.22). Statistical testing ($p < 0.05$) confirmed a significant difference between the two groups. It can be concluded that digital-based worksheets have a positive and significant effect on students' mathematical argumentation abilities.

Keywords: mathematical argumentation skills, printed worksheets, technology-based worksheets

INTRODUCTION

One challenge encountered when teaching mathematics at the university level is the insufficient proficiency among students in constructing mathematical arguments, particularly when tackling argumentation problems. Within the realm of mathematical argumentation, certain students occasionally neglect to employ deductive reasoning (Inglis et al., 2007; Tristanti, 2019a; Tristanti et al., 2016). However, deductive reasoning stands as the sole valid form of argumentation, as its

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

premises derive from verified definitions, theorems, and/or facts (Harel, 2001; Lodder, 2004; Tall, 2004).

The inadequate proficiency of students in mathematical argumentation stems from their insufficient grasp of the significance of argumentation. Additionally, they lack proper training in executing effective arguments (Harel & Sowder, 1998). Enhancing their argumentation skills necessitates a transition from non-deductive to deductive argumentation. Hence, it is crucial for the teaching objectives to be clearly defined to progressively enhance students' competency in mathematical argumentation, ultimately revealing formal evidence (Harel, 2001). Teachers should guide students in constructing and evaluating mathematical arguments (Ball et al., 2008).

Consideration of the use of worksheets to improve mathematical argumentation skills is important. This is in accordance with the suggestions given by Pruitt (2015) and NGSS (2013), which emphasize that the teacher's role is not only limited to guiding students in understanding concepts and skills, but also facilitating students' abilities in conducting investigations, building models, arguing, and achieving certainty in the context of science and mathematics. Learning approaches that integrate collaborative components and software technology are seen as effective strategies for improving students' argumentation skills (Fukawa-Connelly & Silverman, 2015). Kartika et al. (2023) emphasize the importance of structured teaching activity design. Integrating technology into worksheets can enhance interaction and collaboration, as well as help students better understand and apply mathematical concepts.

One effort to optimize learning activities is to empower students through the use of worksheets (Mutia & Prasetyo, 2018). Considering the teacher's role as a facilitator, worksheets enable students to become more active in learning. Students can use worksheets as a guide in understanding concepts and problem-solving, developing skills, and providing opportunities for students to demonstrate their knowledge (Karsli & Şahin, 2009; Utami et al., 2016). The advancement of technology today encourages researchers to utilize it in creating worksheets. In this study, worksheets are designed in such a way with the help of technology to enhance students' mathematical argumentation abilities. One of the goals of combining worksheets with technology is to motivate students in learning, as the appearance of the worksheets becomes more engaging and less monotonous.

Well-designed Student Worksheets (LKS) can promote active engagement and reflective thinking in students during mathematics learning (Suryawati et al., 2020). Along with the increasing use of technology in education, several studies have shown that integrating digital tools in LKS can enrich the learning experience and facilitate the development of higher-order thinking skills, including mathematical argumentation (Drijvers, 2013; Zbiek & Hollebrands, 2008). However, most of these studies focus on the general effectiveness of technology without specifically developing or evaluating teaching tools designed directly to build students' argumentation skills. Therefore, this

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

study aims to describe the needs and outcomes of implementing technology-based worksheets specifically designed to improve students' mathematical argumentation skills as a contribution to the development of evidence-based learning design

The components in the technology-based worksheet refer to the Toulmin scheme. The Toulmin scheme is an argumentation model developed by philosopher Stephen Toulmin to analyze and describe the structure of arguments in detail (Toulmin, 2003). This scheme consists of six main components consisting of data, claim, warrant, backing, disclaimer and qualifier. Data is evidence or reasons that support a claim. Data provides the factual basis for claims. A claim is a statement that you want to prove or support in an argument. This is the main conclusion drawn from the argument. Warrants are a logical bridge that connects data with claims. The Warrant explains why the data is relevant to support the claim. Backing is additional support that strengthens the warrant. Backing provides further justification and is often based on deeper principles or data. A disclaimer is an acknowledgment of a potential objection or condition that could weaken a claim. Rebuttals help make arguments more realistic and credible by considering exceptions. Qualifiers are words or phrases that indicate the extent to which a claim applies or introduces uncertainty. Qualifiers help set claim limits.

Previous studies have shown that technology integration in mathematics education can enhance students' engagement, conceptual understanding, and higher-order thinking skills (Drijvers et al., 2010; Zbiek et al., 2007). Specifically, digital tools such as dynamic geometry software, online collaborative platforms, and interactive simulations have been found to support the development of mathematical reasoning and discourse (Hoyles & Noss, 2016; Kieran et al., 2015). However, much of the existing literature focuses on general reasoning skills or problem-solving abilities, with limited emphasis on structured mathematical argumentation as a distinct competency. Tristanti & Nusantara (2021), Tristanti et al. (2024), and Tristanti & Nusantara (2023) have explored students' mathematical argumentation from both content-based and semiotic perspectives, identifying challenges and variations in students' abilities to construct valid arguments across different topics and learning models. Nevertheless, their studies primarily focus on face-to-face instruction and do not address the integration of digital tools in facilitating argumentation. Furthermore, while some research has explored the use of digital worksheets or scaffolding tasks (e.g., Niess et al., 2013), these often emphasize content transmission rather than engaging students in constructing and critiquing mathematical arguments. In contrast, this study offers a more targeted approach by designing technology-based worksheets explicitly aimed at promoting students' ability to construct logical, evidence-based arguments in mathematics. By aligning the worksheet design with Toulmin's model of argumentation and integrating interactive features that prompt justification and reflection, this study addresses a gap in the literature and contributes a novel instructional tool that blends pedagogical theory with practical classroom technology.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Research related to learning to develop mathematical argumentation skills includes applying the infusion learning model (L. B. Tristanti & Nusantara, 2021), applying PBL and CIRC (L. B. Tristanti & Nusantara, 2022a), developing student argumentation through small group collaboration using the dynamic geometry software (DGS) software environment (Fukawa-Connelly & Silverman, 2015), and developing a web-based multimedia whiteboard system to assist students in learning mathematical problem solving where students are asked to criticize solutions and reply to other people's arguments (Hwang et al., 2006). It appears that no previous research has used technology-based worksheets to develop students' mathematical argumentation abilities. Therefore, this research was carried out with the aim of:

A. Describing the need for technology-based worksheets to develop students' mathematical argumentation abilities.
B. Describing the results of applying technology-based worksheets to develop students' mathematical argumentation abilities

LITERATURE REVIEW

Mathematical Arguments Worksheet

Worksheets or student worksheets are defined as sheets containing guidance for students to perform tasks or activities related to learning (Yaden, 2017). Worksheets can help improve students' understanding of subject matter, enhance learning motivation, support active student engagement in the learning process, assist students in organizing information, and enhance critical thinking skills (Basuki & Wijaya, 2018; Kaymakci, 2012). According to Abd-El Khalick et al. (2010) and Susantini et al. (2016), worksheets are part of teaching materials that can be used to develop students' thinking skills, pose and answer questions, make connections between concepts, and assess students' learning progress.

The application of worksheets in mathematics education has a positive impact on students' learning outcomes (Nurhidayah et al., 2020). Worksheets help improve conceptual understanding, critical thinking skills, and students' ability to solve mathematical problems (Novitasari et al., 2022). Therefore, it is recommended that teachers use worksheets as part of their mathematics teaching strategies to enhance students' learning outcomes

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

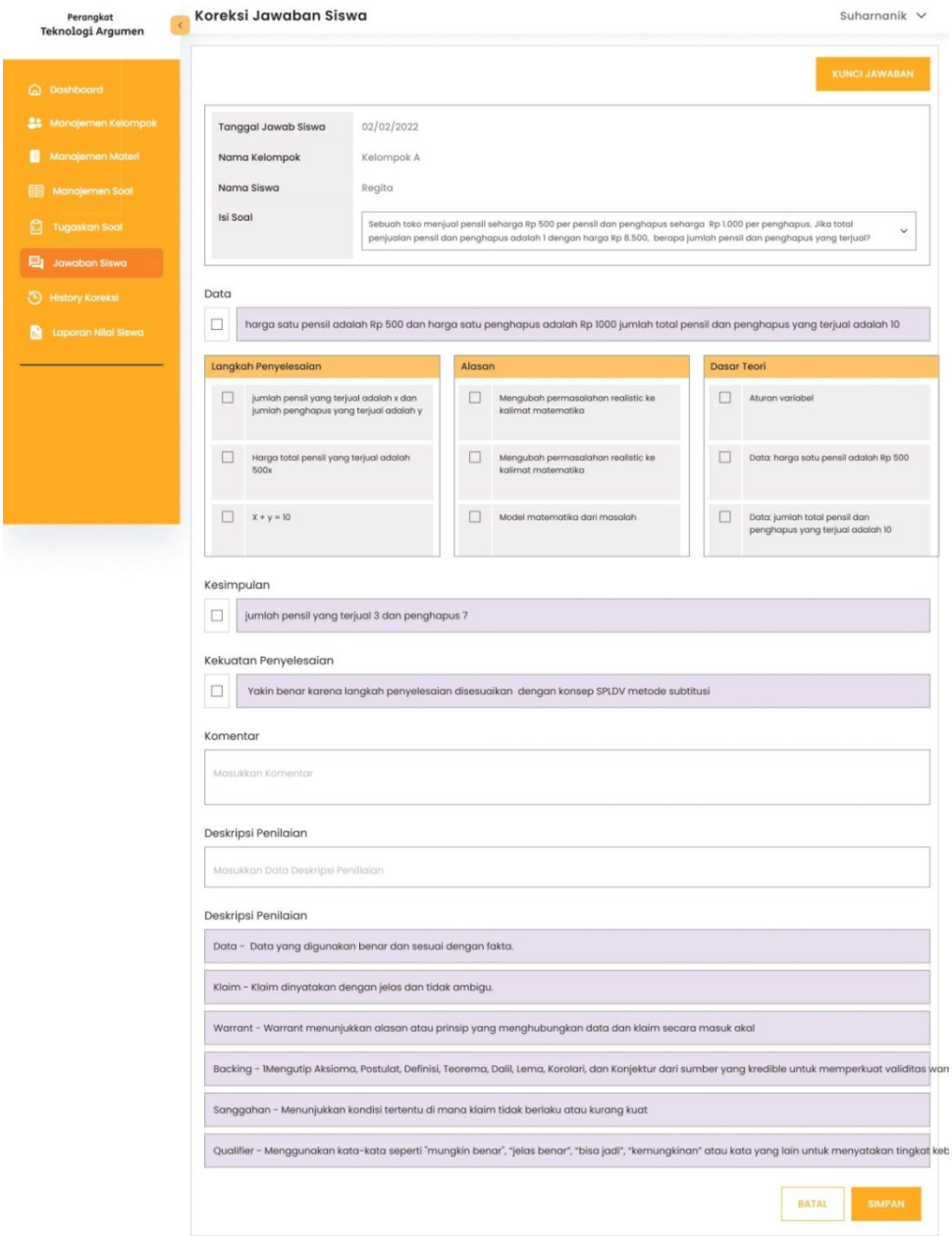


Figure 1. Mathematical Argument Worksheet Framework

Mathematical Argument Worksheet is a worksheet used to help students develop and organize mathematical arguments systematically and logically. This worksheet contains a series of tasks or questions designed to lead students through the process of critical thinking and mathematical problem solving, with a focus on forming valid and coherent arguments. The components in the math-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

ematical argumentation worksheet refer to the components of Toulmin's argument scheme (Toulmin, 2003), which include:

(1) Mathematical problem or statement, which presents a mathematical problem to be solved or a question to be answered.
(2) Data, which requires students to provide a factual basis for proving claims.
(3) Claim, which requires students to convey the statement they want to prove or support in an argument.
(4) Warrants, which require students to create a logical bridge/completion steps that connect the data with the claim, and explain why the data is relevant to support the claim.
(5) Backing, which requires students to provide additional support that strengthens the warrant.
(6) Rebuttal, which requires students to consider exceptions makes the argument more realistic and credible.
(7) Qualifiers, which require students to express words or phrases that indicate the extent to which a claim applies or introduces uncertainty.
(8) Description of mathematical argumentation abilities, in which the lecturer conveys mathematical argumentation skills based on filling in data, claims, warrants, qualifiers, rebuttals and backing in student worksheets.

Figure 1 shows the framework of the Mathematical Argument Worksheet. In this research, mathematical argument worksheets are packaged in digital-based worksheets and printable worksheets. Digital-based worksheets can be accessed via a link https://argumen.oldmapps.com/.

Mathematical Argumentation Skills

Mathematical argumentation is defined as a process that involves making claims, using evidence to support those claims, and presenting and evaluating arguments in a mathematical context (Wells, 2014). Mathematical argumentation ability refers to a person's ability to use logic and mathematical evidence in explaining, defending, and convincing others about the truth of a mathematical statement or concept (Tristanti & Nusantara, 2022b). Mastering mathematical argumentation is a valuable skill in mathematics education, which not only enhances academic understanding but also prepares students for challenges beyond the academic environment (Sriraman & Umland, 2020). Table 1 describes the elements of student competence in mathematical argumentation.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Component	Indicators
Data	1. The data used must be correct and in accordance with the facts. 2. Data must be directly related to the claim being made. 3. All data necessary to support the claim must be included. 4. Data comes from reliable and valid sources. 5. Data is presented clearly and easy to understand. 6. The data logically supports the claims made without any contradiction.
Claim	1. Claims are stated clearly and unambiguously. 2. Claims must be specific and focused on the topic discussed. 3. Claims can be proven or supported with strong evidence and reasons. 4. Claims are relevant to the context of the argument and discussion topic. 5. Claims are directly related to the data and evidence presented 6. Claims are based on logical and rational thinking.
Warrants	1. Warrant shows reasons or principles that make a reasonable connection between data and claims 2. The explanation in the warrant must be acceptable to the audience and based on logic or strong evidence 3. There is a specific explanation of the data that supports the claim, not just an assumption. 4. Warrants are based on general principles, theories, or verifiable evidence
Backing	1. Quote axioms, postulates, definitions, theorems, propositions, lemmas, correlations and conjectures from credible sources to strengthen the validity of the warrant 2. Include widely accepted theories or principles to strengthen the basis of the warrant. 3. Provide concrete examples or illustrations that support the warrant.
Rebuttal	1. Indicates certain conditions where the claim is not valid or lacks strength 2. Present data or evidence to weaken the claim
Qualifiers	Using words such as "probably true", "obviously true", "could be", "likely" or other words to express the level of truth of the claim

Table 1. Competence In Mathematical Argumentation

METHOD

Research Design

This study utilized a quantitative experimental method to determine the impact of learning with technology-based worksheets on students' mathematical argumentation skills. The research employed a pretest-posttest, nonequivalent multiple group design. The research design of this study is illustrated in Figure 2.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Group	Beginning	Treatment	End
Experiment 1	Y_1	T_a	Y_2
Experiment 2	Y_1	T_b	Y_2

Figure 2. Research Design

Y1 is the pretest of students' mathematical argumentation skills, Ta is the treatment with digital-based worksheets, Tb is the treatment with printable worksheets, Y2 is the post-test of students' mathematical argumentation skills.

The treatment is considered to have a significant effect on students' mathematical argumentation skills if a significant difference in mathematical argumentation skills is found between experimental group 1 and experimental group 2. The learning process with digital-based worksheets was conducted for experimental group 1 over three sessions, with the post-test administered in the fourth session. Each session lasted 100 minutes. For experimental group 2, students received the same material as experimental group 1. However, they did not use digital-based worksheets as their treatment; instead, they used printable worksheets. Both groups were then given the same validated post-test, which was a mathematical argumentation test.

Population and Sample

The population of this study consisted of 200 prospective math teacher students at a tertiary institution in East Java, Indonesia, all approximately 20 years old. The sample for this study included 80 prospective math teacher students, divided into an experimental group 1 (40 students) and an experimental group 2 (40 students). The experimental group 1 comprised 15 males and 25 females, while the experimental group 2 comprised 16 males and 24 females.. This selection was based on preliminary test results provided to all classes, ensuring that both groups had equivalent initial abilities. This is also supported by the results of interviews with teachers in both groups, which indicated that both groups have equal mathematical ability and activity (homogeneous). In terms of gender, both groups have a comparable number of male students, with 15 male students in Experiment group 1 and 16 male students in Experiment group 2. Both groups also have a comparable number of female students, with 25 female students in Experiment group 1 and 24 female students in Experiment group 2. This indicates that the researcher implemented strict control over external variables that could affect the study's results.

Worksheet Design

In this study, the worksheets used consist of two formats: digital format and printed format. The digital format worksheets are designed using interactive learning software that allows students to interact directly with the content. This format includes multimedia elements, such as explanatory videos, interactive images, and quizzes that can be completed online. Meanwhile, the printed

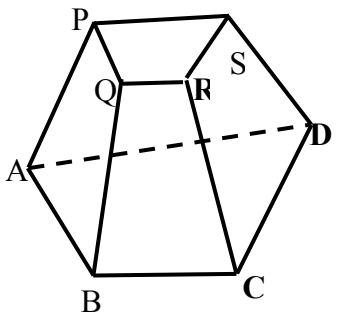
This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

worksheets are the physical version of the same material, which includes text, static images, and clear instructions for students to complete on paper.

The content of both formats is identical in terms of topics and learning objectives. Each worksheet is designed to cover the same concepts and achieve the desired learning outcomes. However, the digital format allows for a more dynamic and engaging presentation of content. Several adaptations are made to ensure that the digital worksheets are more effective and interactive. These include: the addition of interactive elements, such as quizzes and tasks that can be completed online; the integration of multimedia, such as videos or animations, that support concept understanding; and adjustments to instructions to ensure that students understand how to use the interactive features available in the digital format. Figure 3 illustrates a task used in the worksheet for spatial geometry content.

Task 1 **Information:**

The following image represents a pyramid with its top cut off



Introductory Questions:

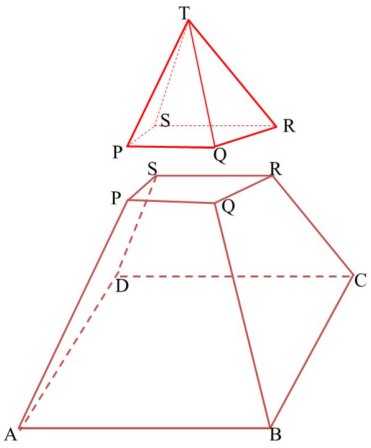
1. Identify the characteristics of image!
2. Name and define the image according to the characteristics you have described!

Problem 1:

Prove that the bases of a frustum of a pyramid are similar!

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Task 2



Prove that $V_{Frustum} = \frac{1}{3}\, t\, \left(A + B + \sqrt{AB}\right)$.

Here t is the height of the frustum, A is the area of the base of the frustum, and B is the area of the top of the frustum!

Figure 3. Argumentation Ability Task

The spatial geometry task have the potential to be used in learning that develops students' mathematical argumentation skills, as these tasks require students to prove a formula or theorem, which involves logical reasoning steps and mathematical argumentation.

The spatial geometry task is important in students' learning as it involves fundamental geometric concepts, such as volume and surface area, which are essential for understanding three-dimensional structures. Proving the volume formula of a frustum deepens students' understanding of spatial geometry, which is a crucial foundation in advanced mathematics. Furthermore, the proof process enhances students' skills in deductive reasoning and formal logic, which are essential abilities for higher-level mathematics learning.

This task also challenges students to think about the relationships between various geometric elements, helping them develop analytical and problem-solving skills. The task supports creative mathematical thinking by asking students not only to understand the formula but also to prove and explain its origins. Creativity emerges when students seek different ways to approach the proof, such as using visual methods, algebra, or a combination of both. Creative thinking can be expected when students try to connect this task with the concept of the volume of a pyramid or use substitution techniques from simpler problems. Creativity may also arise in how students organize their arguments or create illustrations that aid in the proof. Thus, this task provides space for exploration and the discovery of new solutions, supporting the development of creative thinking skills in mathematics.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The tasks designed in the worksheets were implemented through small-group learning activities consisting of 3-4 students per group. Students were asked to collaboratively discuss and solve the problems, particularly those that required constructing mathematical arguments. The teacher acted as a facilitator, monitoring group discussions, providing limited guidance when necessary, and occasionally posing prompting questions to encourage the development of arguments. After working in groups, selected students were invited to present their group's solutions to the class. This was followed by whole class discussions involving responses from both peers and the teacher, thereby fostering a learning environment that supports idea exchange and the strengthening of mathematical argumentation.

Instrument

This instrument consists of a questionnaire and a test. Two types of questionnaires were used: the digital worksheet needs questionnaire and the digital worksheet assessment questionnaire. The digital worksheet needs questionnaire was filled out by students before they participated in learning using the digital worksheet. The digital worksheet assessment questionnaire was filled out by students after they participated in learning using the digital worksheet. Both questionnaires were provided on Google Forms. The assessment results from the respondents were analyzed by calculating the average score obtained for each component of the digital worksheet design, then converted into qualitative statements according to Table 2.

Score Interval	Decision
4.20 - 5.00	Digital worksheet design is very appropriate
3.40 - 4.19	Digital worksheet design is appropriate
2.60 - 3.39	Digital worksheet design is somewhat appropriate
1.80 - 2.59	Digital worksheet design is inappropriate
1.00 - 1.79	Digital worksheet design is very inappropriate

Table 2. Score Conversion

This test was a mathematical argumentation test used to collect data about students' mathematical argumentation skills. The data on students' mathematical argumentation competence were analyzed quantitatively by assigning scores to each of the elements. The scoring guidelines were defined as follows: a score of 2 was assigned if students correctly identified the elements, a score of 1 if students incorrectly identified the elements, and a score of 0 if students did not identify any elements.

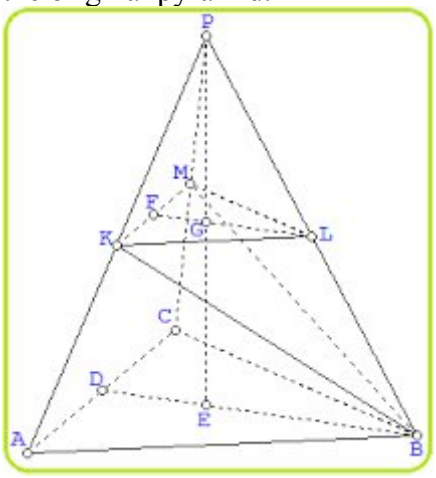
This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICAL ARGUMENTATION TEST

Instructions:

a. Fill in your Name, Class, and Phone Number in the provided columns!
b. Read the instructions carefully first!
c. Show all your ideas and solutions!
d. If there are any incorrect answers, cross them out without erasing!

1. **Investigate the truth of this statement!**
   In every simple triangular pyramid, the area of the top face and the base area of the pyramid are proportional to the square of the height of the upper pyramid with the height of the original pyramid.



2. Based on the answer to question 1, create a mathematical argumentation scheme!

Figure 4. Mathematical Argumentation Test

The test was in a descriptive format and included two evidentiary problems. Before administering it to the students, the test was validated by experts, specifically two mathematics education specialists and one mathematician. The purpose of this validation was to ensure the test met content and construct validation criteria. These criteria were assessed in three components: the appropriateness of the material content, the construction of the material, and the language suitability. If there were any revisions, the instrument was revised according to the suggestions provided by the validators. Once the validators declared the instrument fit for use, it was administered to the students to assess their mathematical argumentation skills. Figure 4 below is a mathematical argumentation test.

Essentially, both groups received the same learning process, including the teaching strategies, the teacher, the material taught, the content of the worksheets, and the duration of the learning time.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The only difference lies in the type of worksheet used. Experiment group 1 utilized digital-based worksheets, while Experiment group 2 used printable worksheets.

Research Hypothesis

The treatment is considered to have a significant effect on students' mathematical argumentation skills if a significant difference in mathematical argumentation skills is found between experimental group 1 and experimental group 2. The research hypotheses are as follows:

H_0: There is no difference in mathematical argumentation skills between experimental group 1 and experimental group 2.
H_1: There is a difference in mathematical argumentation skills between experimental group 1 and experimental group 2.

Analysis of Data

The research data, comprising the post-test scores of students, underwent statistical analysis. A t-test was conducted to assess the significance of the treatment, specifically the impact of learning with digital-based worksheets. The treatment is deemed significant if there is a notable difference in post-test scores between experimental group 1 and experimental group 2.

The criterion for supporting H_0 states, "H_0 is supported if the Sig. value ≥ 0.05," indicating no significant difference exists in post-test scores between experimental group 1 and experimental group 2. Absence of a significant difference suggests the treatment lacks a substantial effect on the experimental group. Here, the treatment involves implementing

Before conducting the t-test analysis, it is necessary to perform tests for normality and homogeneity. The normality test assesses whether the data distribution follows a normal distribution, where a distribution is considered normal if the asymptotic significance value is > 0.05. The homogeneity test examines the similarity between the experimental-1 and experimental-2 groups, specifically comparing their variances. Comparison between the groups is valid only if their variances are similar. Homogeneity is confirmed if the Sig. value > 0.05, indicating no significant difference in variance at the 5% significance level.

When the data is not normally distributed or when the assumption of homogeneity of variances is not met, the data is analyzed using the Mann-Whitney test. The results of the Mann-Whitney test are confirmed with the statistical significance value from the Z test in the Mann-Whitney test.

RESULTS

Preliminary research results

The preliminary research results were based on the analysis of the digital worksheet needs, filled out by students through Google Forms as shown in Table 3.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

No	Statement	Percentage
1	Lecturers use worksheets in learning	20
2	The worksheets used by lecturers are fun and make students actively learn	80
3	Lecturers use worksheets that are appropriate to the learning material	90
4	The worksheet encourages students to construct mathematical arguments based on the Toulmin scheme	15
5	Lecturers utilize technology in using worksheets	10
6	Lecturers use worksheets to introduce technology	10
7	Lecturers use worksheets to guide students to develop mathematical argumentation skills	45
8	Lecturers use worksheets to allow students to evaluate observations	85
9	Lecturers use worksheets to enable students to create problem models and predictions	85
10	Lecturers use worksheets to make it easier for students to design steps to compose mathematical arguments	60
11	Lecturers use worksheets to facilitate students in finding answers	95
12	Teachers use worksheets to integrate technology and mathematics in the learning process, making it easier for students to understand concepts	45
13	The lecturer gives students the opportunity to refute their friends' statements	100
14	Lecturers provide opportunities for students to express their opinions	100
15	Lecturers give students the opportunity to debate with their friends	100

Table 3. Needs Analysis Results for Digital Worksheets

Based on Table 3, four important points are evident. First, not all lecturers use worksheets. This is in line with statement 1, which indicates that only 20% of lecturers use worksheets in their teaching. Second, the worksheets used by the lecturers are enjoyable and encourage active learning among students, are aligned with the learning materials, enable students to evaluate observations, help students create problem models and predictions, facilitate students in designing steps to construct mathematical arguments, and assist students in finding answers. Third, lecturers provide opportunities for students to challenge their peers' statements, express opinions, and engage in debates with their peers. Fourth, the worksheets used by lecturers have not yet encouraged students to construct mathematical arguments based on the Toulmin model, have not utilized technology in the use of worksheets, and have not guided students in developing their mathematical argumentation skills. As a result, students' ability to provide answers and reasons for solving problems remains weak. This is also supported by research from Tristanti Nusantara (insert citation year), which indicates that, overall, the argumentation skills demonstrated by students, particularly in refutation skills, are still in the poor category.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Many students have not yet mastered the mathematical concepts they are studying. A strong grasp of these concepts is essential for making counterarguments, as it allows students to provide alternative answers (Dinantika et al., 2020). Tristanti (2019) indicates that students tend to focus on empowering claims without utilizing all elements of argumentation skills. This leads to a weak understanding of the concepts of floating and submerged materials. Ideally, a comprehensive empowerment of argumentation skills would enhance students' conceptual understanding. The findings of this research serve as a reference for making improvements in future studies to achieve positive changes by empowering argumentation skills, thereby directly enhancing students' conceptual understanding. Therefore, technology-based worksheets are needed to develop students' mathematical argumentation skills.

The design of the worksheets refers to the components of the Toulmin model. The worksheets are available in both digital and printable formats. The structure and components of the digital and printable worksheets are the same; the difference lies in their usage. Digital worksheets are used on electronic devices such as computers or smartphones. Users can fill out and save their answers directly on the device, and the worksheets can be used for online learning. Printable worksheets, on the other hand, are printed on paper and used manually, requiring writing tools to complete them.

Mathematical Argumentation Ability Results

After the mathematics lessons using worksheets, students in experiment group 1 and experiment group 2 were given a posttest to measure their achievement in mathematical argumentation ability. The posttest data were analyzed for normality, homogeneity, and t-tests. The following are the results of the normality test, homogeneity test, and t-test on the students' mathematical argumentation ability data.

Test of Normality

Based on the data processing results of the posttest scores for experiment group 1 and experiment group 2, with the number of participants being $n_1 = 40$, $n_2 = 40$ respectively, the mean scores obtained were a $\bar{x}_1 = 82.8750$, $\bar{x}_2 = 69.2188$, with standard deviations of $s_1 = 10.10828$, $s_2 = 8.66485$. To determine the effect of digital worksheets on students' mathematical argumentation abilities, several statistical tests need to be performed. First, a normality test was conducted using SPSS software, which produced the Shapiro-Wilk normality test output for the posttest scores, as shown in Table 4 below.

Group	n	Mean	Std. Deviation	Shapiro-Wilk Statistic	Shapiro-Wilk Sig.
Experiment-1	40	82.8750	10.10828	0.945	0.069
Experiment-2	40	69.2188	8.66485	0.927	0.031

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Table 4. Shapiro-Wilk Normality Test Output For The Posttest Scores

The results indicate that the posttest scores for both experiment groups were tested for normality using the Shapiro-Wilk test. For experiment group 1, the Shapiro-Wilk statistic is 0.945 with a significance value of 0.069, indicating that the data is normally distributed. For experiment group 2, the Shapiro-Wilk statistic is 0.927 with a significance value of 0.031, indicating that the data is not normally distributed.

Homogeneity of Variance Test

Since the posttest scores for one of the groups are not normally distributed, the next step is to perform a homogeneity of variance test using Levene's test for the posttest scores. Table 5 below shows the results of Levene's test for the homogeneity of variance for the posttest scores.

Levene Statistic	df1	df2	Sig.
2.135	1	78	0.148

Table 5. Results of Levene's Homogeneity of Variance Test for Posttest Scores

The results of Levene's test indicate that the significance value (Sig.) is 0.148. Since this value is greater than 0.05, it suggests that the variances of the posttest scores for both groups are homogeneous, meaning there is no significant difference in the variances of the two groups. Given these results, further analysis can be conducted using appropriate statistical methods that do not assume normal distribution, such as the Mann-Whitney U test, to compare the posttest scores of the two groups.

In the experiment-1 group, the posttest scores do not follow a normal distribution. Therefore, the Mann-Whitney test was used to compare the means of the experiment-1 and experiment-2 groups. The results of the data analysis using SPSS software, including the R_1 and R_2 values, are presented in Table 6 below.

	Group	N	Mean Rank	Sum of Ranks
Posttest Value	Exprerimental_1	40	43.89	1404.50
	Exprerimental_2	40	21.11	675.50
	Total	80		

Table 6. R_1 dan R_2 Values from the Mann-Whitney U Test

Based on the data analysis results in Table 6, the values obtained are $R_1 = 1404.50$ and $R_2 = 675.50$. The Experimental-1 group has a higher posttest score based on the average rank (43.89) compared to the Experimental-2 group (21.11). This may indicate that the Experimental-1 group tends to perform better in the posttest compared to the Experimental-2 group; however, this result is confirmed by examining the statistical significance value from the Mann-Whitney test. The next step

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

involves using SPSS software to perform the Mann-Whitney-Wilcoxon test, as shown in Table 7 below.

	Posttest Value
Mann-Whitney-Wilcoxon	147.500
Wilcoxon W	675.500
Z	4.902
Asymp. Sig. (2-tailed)	0.000

Table 7. Mann-Whitney-Wilcoxon Test

Based on the output in Table 7, the calculated Z value is 147.500 and the significance (Sig) value is 0.000. Since Sig < 0.05, the null hypothesis (H_0) stating $\mu_1 = \mu_2$ is rejected. It can be concluded that the average post-test scores for the experiment-1 and experiment-2 groups are not the same. Therefore, it can be concluded that there is an effect of using digital worksheet-assisted mathematics learning on students' mathematical argumentation skills

Digital Worksheet Design Assessment Results

The assessment of the digital worksheet design was conducted to evaluate the suitability of the digital worksheet and the design of learning activities aimed at developing students' argumentation skills. This assessment process ensures that the digital worksheet is appropriate and effective for use in learning activities designed to enhance students' argumentation abilities. The results of the digital worksheet design assessment are presented in Table 8.

Based on the assessment results in Table 8, all evaluated components received scores ranging from 4.20 to 5.00. According to Table 2, the digital worksheet design is deemed very appropriate and can be used to develop students' mathematical argumentation skills. Figure 5 represents the results of students' mathematical arguments through digital worksheets.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

No	Digital-based worksheet design	Score
1	Suitability of digital-based worksheets	
	a. Suitability of activities on digital-based worksheets with Toulmin scheme components	4.30
	b. Images, writing, symbols and animations presented are appropriate to the context of the material	4.45
2	Learning Activities	
	a. Through digital-based experimental activities and worksheets, students are able to train in collecting and presenting data correctly and in accordance with the facts	4.60
	b. Activities on digital-based worksheets can train students in making claims that are relevant to the context of arguments and discussion topics	4.65
	c. Activities on digital-based worksheets can train students in making warrants that show reasons or principles that connect data and claims in a reasonable way	4.50
	d. Activities on digital-based worksheets can train students in citing axioms, postulates, definitions, theorems, postulates, lemmas, correlations and conjectures from credible sources as backing to strengthen the validity of the warrant.	4.40
	e. Activities on digital-based worksheets can train students to indicate certain conditions where a claim does not apply or is not strong enough	4.30
	f. Activities on digital-based worksheets can train students to use words such as "probably true", "obviously true", "could be", "possible" or other words to express the level of truth of a claim.	4.50
	g. Activities on digital-based worksheets can educate students' skills in expressing objections to something that is different from what they understand	4.35

Table 8. Digital Worksheet Design Assessment Results

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Data

Limas terpancung adalah bentuk tiga dimensi yang terbentuk dari limas yang dipotong sejajar dengan alasnya.

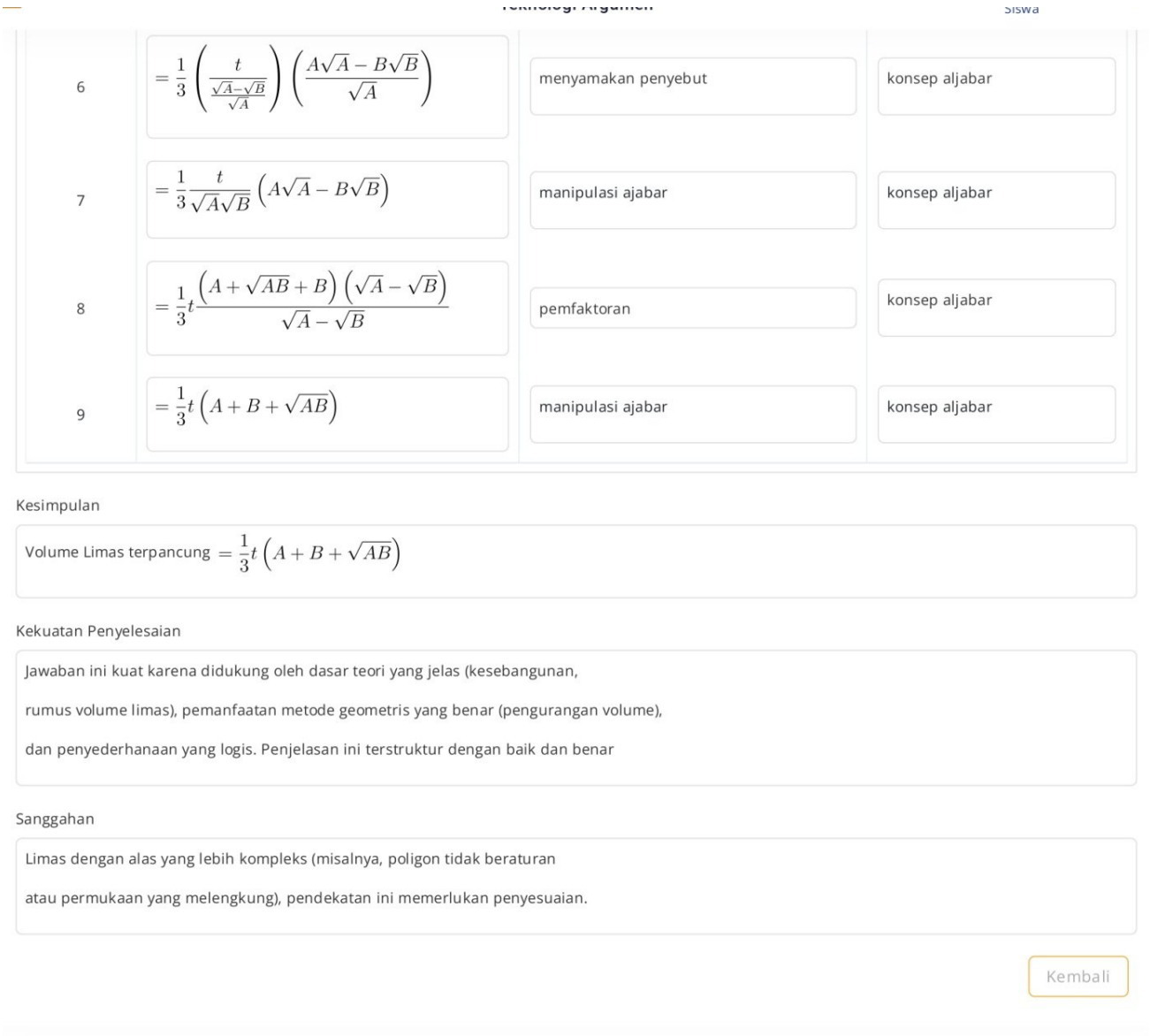
Volume dari limas penuh dapat dihitung dengan rumus umum: $\frac{1}{3} \times t \times \left(A + B + \sqrt{AB}\right)$

Jawaban Langkah

No	Langkah Penyelesaian	Alasan	Dasar Teori
1	$t_2 = \frac{\sqrt{B}}{\sqrt{A}} \times t_1$	Limas penuh sebangun dengan limas kecil, maka rasio tinggi antara dua limas yang berhubungan dengan luas alasnya	Kesebangunan
2	$t_1 = \frac{t}{1 - \frac{\sqrt{B}}{\sqrt{A}}}$	hubungan antara tinggi t1 dari limas penuh, dan tinggi t dari limas terpancung, serta luas alas besar A, dan luas alas kecil B	Definisi limas terpancung dan kesebangunan
3	Volume Limas Terpancung = Volume Limas Penuh - Volume Limas Kecil	Limas terpancung terbentuk dari limas yang dipotong sejajar dengan alasnya.	Definisi Limas terpancung
4	Volume Limas Terpancung $= \frac{1}{3} \times A \times t_1 - \frac{1}{3} \times B \times t_2$	subtitusi rumus volume limas penuh dan volume limas kecil	Rumus volume limas
5	$= \frac{1}{3} \frac{t}{1 - \frac{\sqrt{B}}{\sqrt{A}}} \left(\frac{A\sqrt{A} - B\sqrt{B}}{\sqrt{A}}\right)$	suntitusi langkah 1 dan langkah 2	konsep aljabar

(a)

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.



6	$= \frac{1}{3}\left(\frac{t}{\frac{\sqrt{A}-\sqrt{B}}{\sqrt{A}}}\right)\left(\frac{A\sqrt{A}-B\sqrt{B}}{\sqrt{A}}\right)$	menyamakan penyebut	konsep aljabar
7	$= \frac{1}{3}\frac{t}{\sqrt{A}\sqrt{B}}\left(A\sqrt{A}-B\sqrt{B}\right)$	manipulasi ajabar	konsep aljabar
8	$= \frac{1}{3}t\frac{\left(A+\sqrt{AB}+B\right)\left(\sqrt{A}-\sqrt{B}\right)}{\sqrt{A}-\sqrt{B}}$	pemfaktoran	konsep aljabar
9	$= \frac{1}{3}t\left(A+B+\sqrt{AB}\right)$	manipulasi ajabar	konsep aljabar

Kesimpulan

Volume Limas terpancung $= \frac{1}{3}t\left(A+B+\sqrt{AB}\right)$

Kekuatan Penyelesaian

Jawaban ini kuat karena didukung oleh dasar teori yang jelas (kesebangunan,
rumus volume limas), pemanfaatan metode geometris yang benar (pengurangan volume),
dan penyederhanaan yang logis. Penjelasan ini terstruktur dengan baik dan benar

Sanggahan

Limas dengan alas yang lebih kompleks (misalnya, poligon tidak beraturan
atau permukaan yang melengkung), pendekatan ini memerlukan penyesuaian.

Kembali

(b)

Figure 5. The results of students' mathematical arguments through digital worksheets

Figure 6 represents the results of students' mathematical arguments through printable worksheets.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

1. Berdasarkan masalah, tulis data yang diketehui! (Data merupakan rangkaian fakta-fakta yang digunakan untuk membuktikan)

Volume Limas terpancung = Volume Limas penuh — Volume Limas kecil

2. Tulis klaim! (Klaim merupakan pernyataan yang akan dibuktikan)

Volume Limas terpancung = $\frac{1}{3} \cdot t \left(A + B + \sqrt{AB}\right)$

3. Tulis langkah penyelesaian, alasan dan dasar teori

Langkah Penyelesaian	Alasan	Dasar Teori
$t_2 = \frac{\sqrt{B}}{\sqrt{A}} \cdot t_1$	t_2 = tinggi limas kecil t_1 = tinggi limas penuh	Kesebangunan
$t_1 = t : \left(1 - \frac{\sqrt{B}}{\sqrt{A}}\right)$		Kesebangunan
$V_{Limas\ terpancung} = V_{L.penuh} - V_{L.kecil}$		konsep limas terpancung
$= \frac{1}{3} \cdot A \cdot t_1 - \frac{1}{3} \cdot B \cdot t_2$		Volume Limas
$= \frac{1}{3} \frac{t}{1 - \frac{\sqrt{A}}{\sqrt{B}}} \left(A - \frac{B\sqrt{B}}{\sqrt{A}}\right)$		Langkah 1 & 2
$= \frac{1}{3} \frac{t}{\frac{\sqrt{B} - \sqrt{A}}{\sqrt{B}}} \left(\frac{A\sqrt{A} - B\sqrt{B}}{\sqrt{A}}\right)$	menyamakan penyebut	
$= \frac{1}{3} \frac{t\sqrt{A}}{\sqrt{A} - \sqrt{B}} \left(\frac{A\sqrt{A} - B\sqrt{B}}{\sqrt{A}}\right)$	menyamakan penyebut	
$\frac{1}{3} t \frac{(A + \sqrt{A}\sqrt{B} + B)(\sqrt{A} - \sqrt{B})}{\sqrt{A} - \sqrt{B}}$		Pemfaktoran
$\frac{1}{3} t \left(A + B + \sqrt{AB}\right)$		

4. Tulis Kesimpulan!

Volume Limas terpancung = $\frac{1}{3} \cdot t \left(A + B + \sqrt{AB}\right)$

5. Bagaimana dengan tingkat kekuatan (*qualifier*) dari kesimpulan?

pasti benar karena didasarkan pada langkah logis dan konsep kesebangunan

6. Tulis sanggahan dari kesimpulan?

tidak ada.

Figure 6. The results of Students' Mathematical Arguments through Printable Worksheets

Based on the results of students' answers in Figures 5 and 6, it appears that students fill in all argument components on digital worksheets. However, students do not fill in all argument compo-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

nents on Printable Worksheets, where students do not fill in all the reasons and theoretical bases used when compiling mathematical arguments as in Figure 6.

During the lesson on the volume of a frustum of a cone, formulated by the equation $V_{Frustum} = \frac{1}{3}\,t\,\left(A + B + \sqrt{AB}\right)$, students showed various responses to the tasks in the digital worksheets. Some students actively discussed in their groups to construct logical arguments, linking the frustum's height *t*, the areas of the large base *A*, the small base *B*, and the square root of the product of both base areas $\sqrt{AB}$. Meanwhile, other students required prompting from the teacher to clarify the relationship between the elements of the formula and how to apply them in calculating the volume of the frustum of a cone. For example, in one task, there was a group that initially presented an argument based on simple geometric calculations, such as estimating the volume without considering the deeper relationship between the base areas and the height of the frustum. After receiving a prompting question from the teacher, such as "How do you relate the values of *A* and *B* with the frustum's height *t* in the volume formula?", the group began to develop a deeper understanding and referred to basic concepts of geometry and calculus to ensure that the formula $V_{Frustum} = \frac{1}{3}\,t\,\left(A + B + \sqrt{AB}\right)$ could be applied correctly. The teacher provided selective feedback to maintain students' independent thinking, such as suggesting they review their calculations or compare arguments between groups regarding the appropriate way to calculate the base areas and height. During the class discussion, students commented on each other's group presentations, both in the form of agreement and rebuttal, supported by mathematical reasoning. This not only strengthened their understanding of the frustum volume formula but also created opportunities for deeper development of mathematical argumentation. Thus, the use of digital worksheets in this learning process not only promotes individual cognitive activity but also enhances social interactions that support the development of mathematical thinking in an authentic learning context. Students are not only learning how to calculate volume but also developing the ability to argue and discuss mathematical concepts more critically and in greater depth.

DISCUSSION

The findings of this study indicate that mathematics learning assisted by digital worksheets has a positive impact on students' mathematical argumentation skills. This is consistent with previous research that emphasizes digital-based learning to enhance cognitive abilities and academic performance (Shapiro, 2006). Engaging students in activities that integrate authentic projects and online learning can promote active learning, critical thinking, and collaborative problem-solving, which are essential skills for success in the 21st-century workforce (Hasnunidah et al., 2023). Learning models that implement argumentation have a significant impact on improving students' argumentation abilities (Nussbaum, 2008). Additionally, the use of e-worksheets makes learning

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

more engaging and encourages students to be more active during the learning process (Kanselaar et al., 2002)

The mathematical argumentation skills of students in the experiment-1 group show a higher average compared to the experiment-2 group (mean scores of $\bar{x}_1 = 82.8750$, $\bar{x}_2 = 69.2188$, respectively). This indicates that students in the experiment-1 group find the interactive electronic worksheets more engaging and easier to understand. With electronic-based worksheets, students not only learn theory but also develop relevant practical skills (Hafsah et al., 2024). Creating clear, accurate, and simple teaching materials to ensure student comprehension is crucial, and this is a primary need in the development of e-worksheets (Prastowo, 2019; Sadjati, 2012). The use of e-worksheets facilitates teachers in presenting teaching materials and assignments, and it allows students to complete tasks online anytime and anywhere, significantly impacting students' cognitive learning outcomes (Kim & Hannafin, 2011; Mispa et al., 2022).

The results of this study indicate that digital-based worksheet-assisted mathematics learning has an effect on students' mathematical argumentation skills. Therefore, teachers must adapt to digital tools for teaching, communication, evaluation, and problem-solving in the context of education. Joshi et al. (2021) suggest that teachers should possess digital pedagogy skills. In the era of digitalization, these digital pedagogy skills help teachers maximize the use of technology to support the teaching and learning process. Nursyahidah and Albab (2021) also argue that mathematics teachers can integrate cultural contexts into mathematics learning and enhance students' understanding of mathematical concepts through the use of educational technology to make learning more effective. Overall, this research highlights the importance of integrating digital tools into mathematics learning to enhance students' mathematical argumentation skills. With appropriate adaptation to technology, teachers can not only enrich students' learning experiences but also create a more inclusive and responsive learning environment that meets students' needs in this digital era

Conversely, the limitation of the experiment-2 group, which uses printable worksheets for mathematics learning, lies in the static and non-interactive nature of printable worksheets. This restricts students' ability to interact dynamically with the material, thereby limiting their mathematical argumentation skills. Students learn more effectively when they can interact with the material through various forms of representation and manipulation (Van de Walle et al., 2003). Printable worksheets cannot provide immediate feedback to students, whereas Hattie & Timperley (2007) emphasize the importance of immediate feedback for improving students' understanding and skills. According to Prabjandee (2023), one of the main weaknesses of printable worksheets is the lack of interactivity, which can lead to students losing motivation and engagement in the learning process.

In the use of digital tools for learning, there are several potential biases that need to be considered, such as accessibility issues and varying levels of digital literacy. Not all students have equal access to digital devices and internet connections. Students from economically disadvantaged back-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

grounds struggle to access the digital tools and resources needed, which can result in inequality in learning experiences. Additionally, students with special needs face challenges in using digital tools if they are not designed with accessibility in mind, such as the lack of support for assistive devices.

Students also have varying levels of digital literacy. Some students are very proficient in using digital tools, while others struggle and lack confidence in utilizing the technology. This can affect how they interact with learning materials and contribute to group activities. Variations in digital literacy can also impact students' ability to search for, evaluate, and use information obtained online, which is important in the learning context.

Research findings indicate that mathematics learning supported by digital worksheets (LKS digital) has a positive impact on students' mathematical argumentation skills. This result aligns with Piaget's constructivist theory, which emphasizes the importance of active student engagement in constructing knowledge through meaningful learning experiences (Piaget, 1970). Digital worksheets enable students to explore mathematical concepts interactively and provide them with opportunities to reflect on and independently develop logical arguments.

Furthermore, these findings are consistent with Vygotsky's theory of the zone of proximal development (ZPD), in which social interaction through discussion and collaboration can foster the development of higher-order thinking skills, including argumentation (Vygotsky, 1978). In the context of digital worksheets, students can work in groups or pairs to solve problems, explain their reasoning for choosing certain strategies, and critique each other’s arguments processes that support the development of strong mathematical argumentation.

The instructional implication of these findings is the crucial role of teachers in designing digital worksheets that not only present practice problems but also facilitate open-ended questions, problem-solving scenarios, and tasks that encourage students to express reasoning, provide justification, and defend their opinions logically. This is supported by findings from (Tristanti et al., 2024), which demonstrate that digital tools based on problem-solving tasks can significantly enhance students’ mathematical communication and argumentation skills. Thus, digital worksheets have the potential to serve as strategic tools in creating a mathematics learning environment that is more reflective, interactive, and conceptually grounded, rather than merely procedural.

The worksheets used in this study were designed to develop students' mathematical argumentation skills in the topic of frustum volume. However, these worksheets can also be applied to other mathematical topics for the same purpose. The components within these worksheets are general and flexible, making them easily adaptable by other educators or researchers for different topics or teaching contexts. Therefore, although this study focuses on a specific topic, its findings can contribute to the broader development of mathematics education.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

CONCLUSIONS

The research results show that not all lecturers use worksheets in their teaching. However, during the learning process, lecturers always provide opportunities for students to challenge their peers' statements, express opinions, and engage in debates. When lecturers do use worksheets, these worksheets do not encourage students to construct mathematical arguments based on Toulmin's schema, do not leverage technology, and do not guide students in developing their mathematical argumentation skills. Therefore, technology-based worksheets are needed to enhance students' mathematical argumentation skills, adhering to Toulmin's schema.

The average post-test score for the experiment-1 group is 82.8750, while for the experiment-2 group it is 69.2188. With a calculated Z value of 147.500 and a significance (Sig) value of 0.000, which is less than 0.05, the null hypothesis (H_0) stating that the means of the two groups are equal ($\mu_1 = \mu_2$) is rejected. Thus, it can be concluded that mathematics learning using digital worksheets has a positive effect on students' mathematical argumentation skills.

The assessment of the digital worksheet design was conducted to evaluate its suitability and the effectiveness of the learning activity design in developing students' argumentation skills. This assessment aims to ensure that the digital worksheets are indeed appropriate and effective for use in learning aimed at enhancing students' argumentation skills. The assessment results show that all evaluated components received scores ranging from 4.20 to 5.00. In other words, the digital worksheet design is deemed very appropriate and effective for developing students' mathematical argumentation skills.

Although the findings of this study indicate an improvement in students' mathematical argumentation skills through the use of digital worksheets, we acknowledge several limitations that warrant consideration. First, the successful implementation of this approach largely depends on adequate access to technology and students' levels of digital literacy, which may vary across educational institutions. Second, it is possible that part of the observed improvement in student performance was influenced by the novelty effect of the digital format, suggesting the need for long-term investigation to assess the consistency of outcomes. Therefore, further research is recommended to explore the long-term impact and effectiveness of this approach across diverse contexts with varying levels of technological access and digital readiness.

ACKNOWLEDGEMENTS

The authors would like to thank the Directorate of Research, Technology, and Community Service, Directorate General of Higher Education, Research, and Technology, Ministry of Education, Culture, Research, and Technology. In accordance with the 2024 Fiscal Year Research Contract, Contract Number SP DIPA-023.17.1.690523/2024, 1st revision dated February 4, 2024.

REFERENCES

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[1] Abd-El Khalick, F., BouJaoude, S., Duschl, R., Lederman, N. G., Anderman, L. H., & Midgley, C. (2010). Effectiveness of genetics student worksheet to improve creative thinking skills of teacher candidate students Eficacia de la hoja de trabajo para mejorar las habilidades de pensamiento creativo de los estudiantes de genética. *Educational Leadership*, *68*(1), 34–37. https://www.researchgate.net/publication/310254538_Effectiveness_of_genetics_student_worksheet_to_improve_creative_thinking_skills_of_teacher_candidate_students

[2] Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge For Teaching: What Makes It Special? *Journal Of Teacher Education*, *59*((5)), 389–407. https://doi.org/10.1177/0022487108324554

[3] Basuki, W. A., & Wijaya, A. (2018). The development of student worksheet based on realistic mathematics education. *Journal of Physics: Conference Series (Vol. 1097, No. 1, P. 012112). IOP Publishing*. https://iopscience.iop.org/article/10.1088/1742-6596/1097/1/012112/pdf

[4] Dinantika, H. K., Viyanti, & Suyatna, A. (2020). Design of Student Worksheet to Accommodate Learning Style and Initial Knowledge and Reduce The Differences of Scientific Reasoning and Argumentation Performance. *Proceedings of the 4th International Conference on Learning Innovation and Quality Education*, 1–7. https://dl.acm.org/doi/pdf/10.1145/3452144.3453752

[5] Drijvers, P. (2013). Digital technology in mathematics education: Why it works (or doesn't). *In Selected Regular Lectures from the 12th International Congress on Mathematical Education*, 135–151. https://dspace.library.uu.nl/bitstream/handle/1874/322431/978-3-319-17187-6_8.pdf?sequence=2

[6] Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics*, *75*, 213–234. https://link.springer.com/content/pdf/10.1007/s10649-010-9254-5.pdf

[7] Fukawa-Connelly, T., & Silverman, J. (2015). The Development of Mathematical Argumentation in an Unmoderated, Asynchronous Multi-User Dynamic Geometry Environment. *Contemporary Issues in Technology and Teacher Education*, *15*(4), 445–488. https://www.learntechlib.org/d/150824/

[8] Hafsah, D. S., Aznam, N., & Rohaeti, E. (2024). The Development of Electronic Student Worksheets Based on Problem-Based Learning and Its Impact on Motivation and Problem-Solving Skills. *Jurnal Penelitian Pendidikan IPA*, *10*(9), 6984–6992. https://jppipa.unram.ac.id/index.php/jppipa/article/view/8873

[9] Harel, G. (2001). The Development of Mathematical Induction as a Proof Scheme : A Model for DNR-Based Instruction. *Learning and Teaching Number Theory, Journal of Mathematical Behavior*, 185–212. https://mathweb.ucsd.edu/~harel/The Development of Mathematical Induction as a Proof Scheme - A Model for DNR-Based Instruction.pdf

[10] Harel, G., & Sowder, L. (1998). Students' Proof Schemes: Results from Exploratory Studies. In *CBMS Research in Collegiate Mathematics Education. III* (Vol. 7, pp. 234–283). https://mathweb.ucsd.edu/~harel/Students' Proof Schemes.pdf

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[11] Hasnunidah, N., Rosidin, U., Maulina, D., & Ismi, R. (2023). Students' Argumentation Skillstowards Using Biology e-Worksheetbased on Project-Argumentative Learning Model. *Eurasian Journal of Educational Research*, *103*(2), 341–361. http://repository.lppm.unila.ac.id/52431/1/1194-Article Text-2468-1-10-20230821.pdf

[12] Hattie, J., & Timperley, H. (2007). The Power of Feedback. *Review of Educational Research*, *77*(1), 81–112. https://blog.intuyuconsulting.com.au/wp-content/uploads/2012/03/Power-of-Feedback-Article.pdf

[13] Hoyles, C., & Noss, R. (2016). Mathematics and digital technology: challenges and examples from design research. *In: Proceedings of the 13th International Congress on Mathematical Education. International Commission on Mathematical Instruction: Hamburg, Germany*, 1–8. https://discovery.ucl.ac.uk/id/eprint/1476986/1/TSG36Finalv2CHRN[1].pdf

[14] Hwang, W. Y., Chen, N. S., & Hsu, R. L. (2006). Development And Evaluation Of Multimedia Whiteboard System For Improving Mathematical Problem Solving. *Computers & Education*, *46*(2), 105–121. https://doi.org/10.1016/j.compedu.2004.05.005

[15] Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics*, *66*(1), 3–21. https://doi.org/10.1007/s10649-006-9059-8

[16] Joshi, D. R., Neupane, U., & Joshi, P. R. (2021). Synthesis Review of Digital Frameworks and DEPSWALIC Digital Competency Framework for Teachers from Basic to University Level. *Mathematics Teaching Research Journal*, *13*(2), 108–136. https://files.eric.ed.gov/fulltext/EJ1383146.pdf

[17] Kanselaar, G., Erkens, G., Prangsma, M. E., & Jaspers, J. G. M. (2002). Computer support for collaborative and argumentative writing. *Rapport ISOR*, *2*, 1–214. https://www.kanselaar.net/wetenschap/files/cosarboek.pdf

[18] Karsli, F., & Şahin, Ç. (2009). Developing worksheet based on science process skills: Factors affecting solubility. *In Asia-Pacific Forum on Science Learning and Teaching*, *10*(1), 1–12. https://www.eduhk.hk/apfslt/download/v10_issue1_files/sahin.pdf

[19] Kartika, H., Budiarto, M. T., Fuad, Y., Clark, L. J., & Jeonghyeon, K. (2023). Comparative Analysis of Students' Argumentation Patterns in the Context of Algebraic Problems. *Mathematics Teaching Research Journal*, *15*(5), 105–128. https://files.eric.ed.gov/fulltext/EJ1412239.pdf

[20] Kaymakci, S. (2012). A Review of Studies on Worksheets in Turkey. *Online Submission. Retrieved from ERIC Database (ED529088)*. https://files.eric.ed.gov/fulltext/ED530699.pdf

[21] Kieran, C., Doorman, M., & Ohtani, M. (2015). Frameworks and principles for task design. *Task Design in Mathematics Education: An ICMI Study*, *22*(19-81). https://doi.org/10.1007/978-3-319-09629-2_13

[22] Kim, M. C., & Hannafin, M. J. (2011). Scaffolding problem solving in technology-enhanced learning environments (TELEs): Bridging research and theory with practice. *Computers & Education*, *56*(2), 403–417. https://doi.org/10.1016/j.compedu.2010.08.024

[23] Lodder, A. R. (2004). Law, Logic, Rhetoric: A Procedural Model Of Legal Argumenta-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

tion. *Logic, Epistemology, and the Unity of Science*, 569–588. https://papers.ssrn.com/sol3/Delivery.cfm?abstractid=1312456

[24] Mispa, R., Putra, A. P., & Zaini, M. (2022). Penggunaan E-LKPD berbasis live worksheet pada konsep protista terhadap hasil belajar peserta didik kelas X SMAN 7 Banjarmasin. *Jurnal Pendidikan Indonesia*, *3*(1), 1–12. https://media.neliti.com/media/publications/424625-none-a9684bff.pdf

[25] Mutia, N. B., & Prasetyo, Z. K. (2018). The effectiveness of students' worksheet based on multiple representations to increase creative thinking skills. *Journal of Education and Learning (EduLearn)*, *12*(4), 631–637. http://edulearn.intelektual.org/index.php/EduLearn/article/view/8487

[26] NGSS, L. S. (2013). *Next Generation Science Standards: For States, by States*. The National Academies Press.

[27] Niess, M. L., Ronau, R. N., Shafer, K. G., Driskell, S. O., Harper, S. R., Johnston, C., & Kersaint, G. (2013). Mathematics teacher TPACK standards and development model. *Contemporary Issues in Technology and Teacher Education*, *13*(1), 38–70. https://citejournal.org/volume-9/issue-1-09/mathematics/mathematics-teacher-tpack-standards-and-development-model/

[28] Novitasari, M., Narimo, S., Fajri, D. N., & Raisia, A. (2022). Critical Thinking Skills Through Literacy and Numeration Oriented Mathematics Student Worksheet. *Jurnal Basicedu*, *6*(4), 5775–5784. https://jbasic.org/index.php/basicedu/article/view/3173

[29] Nurhidayah, N., Ninawati, M., & Soro, S. (2020). The influence of worksheet implementation towards students mathematics learning outcomes. *Indonesian Journal of Elementary Teachers Education*, *1*(1). https://journal.uniku.ac.id/index.php/ijete/article/view/2874

[30] Nursyahidah, F., & Albab, I. U. (2021). Learning Design on Surface Area and Volume of Cylinder Using Indonesian Ethno-Mathematics of Traditional Cookie Maker Assisted by GeoGebra. *Mathematics Teaching Research Journal*, *13*(4), 79–98. https://files.eric.ed.gov/fulltext/EJ1332350.pdf

[31] Nussbaum, E. M. (2008). Using argumentation vee diagrams (AVDs) for promoting argument-counterargument integration in reflective writing. *Journal of Educational Psychology*, *100*(3), 549–565. https://psycnet.apa.org/record/2008-10939-005

[32] Piaget, J. (1970). *Science of Education and the Psychology of the Child*. New York: Viking Press.

[33] Prabjandee, D. (2023). A Review of the Website Liveworksheets.com. *Computer Assisted Language Learning*, *24*(1), 269–279. https://old.callej.org/journal/24-1/Le-Prabjandee2023.pdf

[34] Prastowo, A. (2019). *Panduan kreatif membuat bahan ajar inovatif menciptakan metode pembelajaran yang menarik dan menyenangkan*. Diva Press. https://ecampus-fip.umj.ac.id/repo/handle/123456789/5218

[35] Pruitt, S. L. (2015). The Next Generation Science Standards. *Science and Children*, *52*(9), 7. https://files.eric.ed.gov/fulltext/EJ1153875.pdf

[36] Sadjati, I. M. (2012). *Pengembangan Bahan Ajar*. Universitas Terbuka. http://reposi-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

tory.ut.ac.id/4157

[37] Shapiro, N. S. (2006). Learning communities: Reforming undergraduate education. *The Journal of Higher Education*, *77*(3), 550–553. https://doi.org/https://doi.org/10.1080/00221546.2006.11778940

[38] Sriraman, B., & Umland, K. (2020). Argumentation in mathematics education. In *Encyclopedia of mathematics education* (pp. 63–66). https://link.springer.com/rwe/10.1007/978-94-007-4978-8_11

[39] Suryawati, E., Suzanti, F., Zulfarina, Z., Putriana, A. R., & Febrianti, L. (2020). The implementation of local environmental problem-based learning student worksheets to strengthen environmental literacy. *Jurnal Pendidikan IPA Indonesia*, *9*(2), 169–178. https://journal.unnes.ac.id/nju/jpii/article/view/22892

[40] Susantini, E., Isnawati., & Lisdiana, L. (2016). Effectiveness of genetics student worksheet to improve creative thinking skills of teacher candidate students Eficacia de la hoja de trabajo para mejorar las habilidades de pensamiento creativo de los estudiantes de genética. *Research in Science & Technological Education*, *30*(1), 65–80. https://www.researchgate.net/publication/310254538_Effectiveness_of_genetics_student_worksheet_to_improve_creative_thinking_skills_of_teacher_candidate_students

[41] Tall, D. (2004). Introducing Three Worlds of Mathematics. *For the Learning of Mathematics*, *23*(3), 29–33. https://www.semanticscholar.org/paper/Introducing-Three-Worlds-of-Mathematics-Tall/1a91ed4cb77347755cc37344d6eb3ecf94b69f41

[42] Toulmin, S. (2003). *The uses of argument*. Cambridge University Press. https://doi.org/10.2307/2183556

[43] Tristanti, L. B. (2019). The Process Of Thinking By Prospective Teachers Of Mathematics In Making Arguments. *Journal of Education and Learning (EduLearn)*, *13*(1), 17~24. https://doi.org/http://dx.doi.org/10.11591/edulearn.v13i1.6853

[44] Tristanti, L. B., & Nusantara, T. (2021). Improving students' mathematical argumentation skill through infusion learning strategy. *Journal of Physics: Conference Series*, *1783*(1), 012103. https://doi.org/10.1088/1742-6596/1783/1/012103

[45] Tristanti, L. B., & Nusantara, T. (2022a). The advantage and impact of CIRC-typed and problem-based cooperative learning models on students' mathematical argument. *2nd International Conference on Education and Technology (ICETECH 2021)*, 172–178. https://www.atlantis-press.com/proceedings/icetech-21/125968176

[46] Tristanti, L. B., & Nusantara, T. (2022b). The influence of infusion learning strategy on students' mathematical argumentation skill. *International Journal of Instruction*, *15*(2), 277–292. https://e-iji.net/ats/index.php/pub/article/view/384

[47] Tristanti, L. B., Nusantara, T., & Maf'ulah, S. (2024a). Development of digital tools to improve mathematical argumentation skills. *Journal of Hunan University Natural Sciences*, *51*(10). https://jonuns.com/index.php/journal/article/view/1662

[48] Tristanti, L. B., Nusantara, T., & Maf'ulah, S. (2024b). Implementation of inquiry learning model in collaboration with pbl to improve student understanding in number theory course. *JTAM (Jurnal Teori Dan Aplikasi Matematika)*, *8*(1), 150–163. https://journal.ummat.ac.id/index.php/jtam/article/view/17421

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[49] Tristanti, L. B., Sutawidjaja, A., As'ari, A. R., & Muksar, M. (2016). The construction of deductive warrant derived from inductive warrant in preservice-teacher mathematical argumentations. *Educational Research and Reviews*, *11*(17), 1696–1708. https://doi.org/https://doi.org/10.5897/ERR2016.2872

[50] Tristanti, L., & Nusantara, T. (2023). The Effectiveness of Infusion Learning Model in Linear Algebra Course. *Education Research International*, *2023*(9004072), 1–10. https://doi.org/https://doi.org/10.1155/2023/9004072

[51] Utami, W. S., Ruja, I. N., & Utaya, S. (2016). The Effectiveness of Geography Student Worksheet to Develop Learning Experiences for High School Students. *Journal of Education and Learning*, *5*(3), 315–321. https://files.eric.ed.gov/fulltext/EJ1111407.pdf

[52] Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2003). *Elementary and Middle School Mathematics: Teaching Developmentally*. Pearson Education. https://api.pageplace.de/preview/DT0400.9781292331447_A39573276/preview-9781292331447_A39573276.pdf

[53] Vygotsky, L. S. (1978). *Mind in Society: The Development of Higher Psychological Processes*. Cambridge, MA: Harvard University Press. https://w.pauldowling.me/rtf/2021.1/readings/LSVygotsky_1978_MindinSocietyDevelopmentofHigherPsycholo.pdf

[54] Wells, J. (2014). Developing argumentation in mathematics: The role of evidence and context. In *Retrieved from https://core.ac.uk/download/pdf/43064325.pdf*. https://core.ac.uk/download/pdf/43359810.pdf

[55] Yaden, Z. (2017). A Development of students' worksheet based on contextual teaching and learning. *International Journal of Learning, Teaching and Educational Research*, *16*(6), 64–79. https://www.ijlter.org/index.php/ijlter/article/view/951

[56] Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education: A perspective of constructs. In *Second handbook of research on mathematics teaching and learning. 2* (pp. 1169–1207). https://www.scirp.org/journal/paperinformation?paperid=103439

[57] Zbiek, R. M., & Hollebrands, K. (2008). A research-informed view of the process of incorporating mathematics technology into classroom practice by in-service and prospective teachers. *Research on Technology and the Teaching and Learning of Mathematics*, *1*, 287–344. https://www.emerald.com/books/edited-volume/19473/chapter-abstract/104103027/A-Research-Informed-View-of-the-Process-of?redirectedFrom=fulltext

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Boosting Junior High School Students' Mathematical Proficiency Utilizing the Integration of AR in the Classroom

Wanda Nugroho Yanuarto[1], Elfis Suanto[2], Ira Hapsari[1], Masanori Fukui[3]

[1]Universitas Muhammadiyah Purwokerto, Indonesia

[2]Universitas Riau, Indonesia

[3]Mie University, Japan

wandanugrohoyanuarto@ump.ac.id

Abstract: Several attempts have been made to enhance education through innovative technologies like Augmented Reality (AR). Examining the effects of these settings or technologies on various student populations is essential. The primary goals of this research are to determine how open students are to using AR applications for educational purposes and how well it helps junior high school students understand geometry. Seventy-seven students from the junior high school in Purwokerto City, Indonesia, which is located on the edge of both urban and rural areas, were selected for this study. We select 34 participants for the experimental group and 32 for the control group. The experimental group used custom-built AR applications to help students better understand geometry concepts, and the control group stuck with more conventional instruction. The findings of the experiment proved that students' grasp of mathematical ideas can be improved with the use of AR learning applications. Results from the survey and the free-form questions further support the students' positive views on applications. Research into the effects of AR on students' more subjective traits, such as learning anxiety, and on a wider range of app users may be undertaken in the future.

Keywords: Augmented Reality, experimental study, high-school students, mathematical proficiency

INTRODUCTION

A multitude of teachers are intrigued by virtual learning due to the unique potential they provide for students. Pramuditya et al. (2022) assessed various educational applications of 3D technology and discovered that students may acquire knowledge more effectively in well-structured virtual classrooms with reduced cognitive strain compared to traditional classrooms. Research by Fendi et al. (2021) indicates that surgeons who consistently engaged with simulation applications for three hours enhanced their efficiency and diminished their error rates during surgeries, in contrast

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

to those who did not utilize such applications. Numerous educational institutions have been utilizing third-person virtual worlds, such as Second Life, for online instruction. Opportunities for higher education may become increasingly available to a broader population. Its assistance for low-income students can substantially reduce the "Digital Divide" (Salim et al., 2020).

The unique opportunity that virtual learning gives to students has sparked the imagination of many teachers. Students in well-designed virtual classrooms could learn more with less mental effort than in more traditional classrooms, according to a study of several educational applications of 3D technology (Pramuditya et al., 2022).

AR is occasionally perceived only as an advanced kind of virtual reality. The three fundamental features of this method in computer science are three-dimensional registration, real-time interaction, and the superimposition of virtual components onto the actual learning (Suzanna & Gaol, 2021). AR incorporates both digital and physical elements. Users may interface with digital, three-dimensional items in these actual spaces. In an ideal scenario, an AR classroom would help students see 3-D concepts that are otherwise impossible to envision, give solid form to abstract ideas, and better understand concepts with which they are struggling. AR has the potential to be used in conjunction with different learning styles (Abdullah et al., 2020).

However, it's still a good idea to find out if all educational audiences get anything from using contemporary technology. There has been a lot of effort to provide richer information to students by employing modern learning settings or technologies. However, research on the effects of these settings or technologies on students with varied qualities is necessary (Richardo et al., 2023). There is some research on AR in the classroom, but not much that especially targets students from various backgrounds. In this project, we designed a collection of AR learning applications for a Chromebook to aid students in mastering the mathematical underpinnings of geometry. Despite its base in reality, the knowledge is abstract, making it difficult to visualize. This study intends to assess the amount of receptivity to employing AR applications as a learning tool and the efficiency of such applications in increasing junior high school students' grasp of geometry.

LITERATURE REVIEW

Constructivism Theory

Constructivist theory emphasises the process of discovering knowledge, ideas, and concepts rather than the outcomes of individual student responses. According to constructivist learning theory, learning is a process that allows pupils to build their knowledge. Knowledge that already exists cannot be communicated, except through the process of imparting knowledge between teachers and students (Vygotsky, 1999). Constructivist learning is also based on the concept that pupils go through a process that is similar to construction. The process of construction is based on learners' experience (Belbase, 2014). In constructivist learning, the teacher serves as a facilitator, allowing students to discover their potential. Thus, pupils become the focal point or play an active role in

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

growing the knowledge they possess. This is consistent with the notion of differentiated learning, which recognises all of students' particular abilities and places them as learning subjects. Students are encouraged to be creative while submitting ideas or thoughts they have. This is done so that students can absorb learning information using their preferred learning style, making it easier to understand (Gold, 1999).

Lev Vygotsky is one of the founders of constructivist learning theory. Lev is an educator who views learning as a social process (Abida & Muhammad, 2012). Students' cognitive and language development occurs in a social context (Belbase, 2014). More than half a century ago, Russian psychologist Lev recognised this fundamental aspect of the child's cognition. Vygotsky's idea attracted increased attention near the end of the twentieth century. Vygotsky believes that every individual develops within a social framework. All intellectual growth, including meaning, memory, thinking, perception, and awareness, shifts from the interpersonal to the intrapersonal domain. The mechanism that underpins high-level mental effort is a replication of social interaction (Gold, 1999). According to Vygotsky, all higher-order cognitive function in humans begins with each individual's social interactions within a specific cultural environment. Cognition refers to the internalisation of social interaction. Vygotsky's theory of social cognition emphasises the need for a new social foundation for comprehending the educational process (Dangel, 2017).

Information Technology for Geometry

Geometry should be highlighted in junior high school courses. The Minister of Education introduces geometry topics to Indonesian high school students as independent sections in the curriculum (Abidin & Herman, 2023). A unit on geometry is also required in the mathematics curriculum of numerous Asian countries' secondary schools. Efforts to strengthen students' reasoning and statistical knowledge have been the focus of various recent examinations in mathematics curriculum in high schools. From a psychological and educational standpoint, Pujiastuti et al. (2020) explored students' knowledge of geometry. Students commonly create subjective and erroneous first ideas of geometry. Students' subjective opinions may contradict the formal assumptions of geometry. Important principles, such as independent occurrences, are comprehended by a considerably smaller percentage of high school students than they were on the National Assessment of Educational Progress. The results showed that the students did not fare well in geometry (Kihwele & Mkomwa, 2023).

Widyasari and Mastura (2020), many students should not rely on their beliefs about geometry but rather on more traditional understandings of the idea. The significance of students' conceptual knowledge and the use of appropriate technology in well-planned learning activities was underlined in Mohammed et al.'s (2022) systematic evaluation of geometry education, which focused on the field as a whole. The study's authors found that this could help students improve their reasoning and comprehension abilities. Sadly, most research on the use of technological tools in geometry lessons has been undertaken in universities rather than high schools. When it came to gathering, analysing, visualising, computing, transforming, and checking data, high school math

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

students relied on graphing calculators (Marrahi & Belda-Medina, 2023). Excel was a typical tool for high school geometry teachers to employ. When compared to the typical group, Excel performed better on the unit tests. Lai and Cheong (2022) used an Excel add-on to help high school teachers and students grasp data modelling. In conclusion, the dynamic statistical tool helped students to arrange enormous data sets and obtain valuable findings effectively (Pramuditya et al., 2022).

Three computer-assisted simulations analyzed using SPSS and Microsoft Excel helped students better grasp sample distributions and associated disciplines (Fernández-Enríquez & Delgado-Martín, 2020; Talan et al., 2022). These simulations were incorporated in some studies that took advantage of relevant technology resources in academic contexts. Preservice teachers can also help students polish their prediction and inference abilities by using simulation geometry tools like Assembler Edu (Fernández & Delgado, 2020). In addition, Elsayed and Al-Najrani (2021) compared the usefulness of numerous technology aids. Technology had a minor effect on boosting students' statistical achievement, according to his examination of 46 studies. Simulation was the most influential of these technologies. In their review of computer-assisted geometry training, Umam and Dwi (2024) synthesised the results of twenty-five distinct empirical studies. The findings reveal that the two most beneficial types of applications are expert systems and drill-and-practice programs. Embedded assessment, students' educational background, and the length of time they spend in class can all change their statistical accomplishments (Somby & Mølstad, 2022).

The Role of AR in the Classroom

Using AR to develop technology could be advantageous for high school students since it provides a more participatory learning experience compared to standard virtual classrooms. Research on AR's usefulness in the classroom has yielded largely positive outcomes. It may act as an excellent teaching tool in elementary school (Ahmad & Junaini, 2020). The possibilities of AR in the classroom have recently drawn the attention of various schools and advocacy groups for better education. The most widely acknowledged advantage of AR was its capacity to lessen students' cognitive load while simultaneously boosting their motivation, understanding, and engagement (Nugraha, 2023).

Students learning attitudes, performance, motivation, and critical thinking abilities could be improved by utilizing an AR system (Guntur et al., 2019). Mathematical attentiveness is better when students employ visual assistance (Suzanna & Gaol, 2021). The researchers opted to implement AR in solid geometry because of its three-dimensional features. A method that was built to help high school students understand solid geometry through AR has been evaluated and proven to be effective (Widyasari & Mastura, 2020). Also, after implementing AR learning tools, low-achieving students' attitudes improved. The open-source, AR-built 3D virtual classroom has enabled geometry and function education and research, analyzing its application in junior high school mathematics use specifically.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

On the other hand, studies have also shown that AR can make abstract concepts easier to understand by presenting visual representations of them. Somby and Mølstad (2022) constructed an AR book and examined its usefulness in helping students learn electromagnetism. Narrative results revealed that AR books might benefit in comprehending complicated spatial concepts, as evidenced by the gap in the average proportion of accurate replies given by the two groups of participants. Students may find it simpler to absorb otherwise difficult course material with the usage of AR learning tools (Ahmad & Junaini, 2020).

To study the effect of AR on conceptual understanding, researchers employed a psychological model of AR learning. This paradigm rests on three pillars to characterize the impact of AR on the educational process: physical, cognitive, and environmental. On a cognitive level, AR may help students understand metaphors. According to Nasrudin et al. (2021), this procedure is facilitated by AR scaffolding. Some teachers may choose to employ animated videos generated using Adobe Flash to replicate data presentation as technology progresses. Regardless, some students could conclude the work isn't authentic. Maybe if students cared about the outcome of the experiment, they could finish gathering data much more rapidly. So, we built a course to teach students the basics of geometry, including sample space and more. The objective of this study was to present junior high school students with an introduction to geometry and to suggest an alternative way for them to experience data collecting through AR.

METHOD

Participants

Seventy students from Purwokerto City, Indonesia, reside at the intersection of urban and rural areas. Participants' ages varied from thirteen to fifteen. All students enrolled in the same course possessed a robust understanding of geometry before the classroom learning. The class was partitioned into four equal segments for the group projects. Consequently, two separate groups were formed from the previously mentioned. Both groups of students performed nearly identically on the preliminary examinations. Ultimately, the experimental group was comprised of 34 students, whereas the control group consisted of 32 students. We ensured that all students were informed of the voluntary nature of their participation. Furthermore, we guaranteed that their replies and responses would stay confidential and would not influence their mathematics marks.

Research Design

Figure 1 depicts the instructional and learning framework for the geometry unit for both groups. While the control group continued to use more traditional education methods, the experimental group utilised custom-built AR applications to explore geometry concepts further, individually or in small groups. Each of the three courses ran for a full six weeks.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

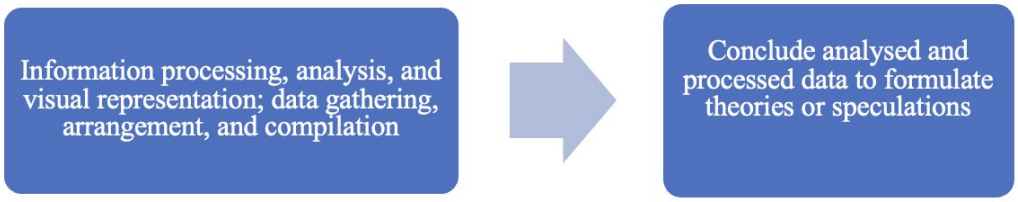


Figure 1. A Review of Mathematical Material

Figure 2 shows the general layout of all the classes. We started class with a "Conclude the current activity", which allowed students to work on a brief assignment on the board as soon as they entered the room. During this time, we also encouraged group discussion. Students in both groups were introduced to the day's topic during the fifteen minutes that followed. The format of this section of the class was somewhere between a lecture and a group project. During a fifteen-minute session, students explored geometry using a Chrome book equipped with AR applications.

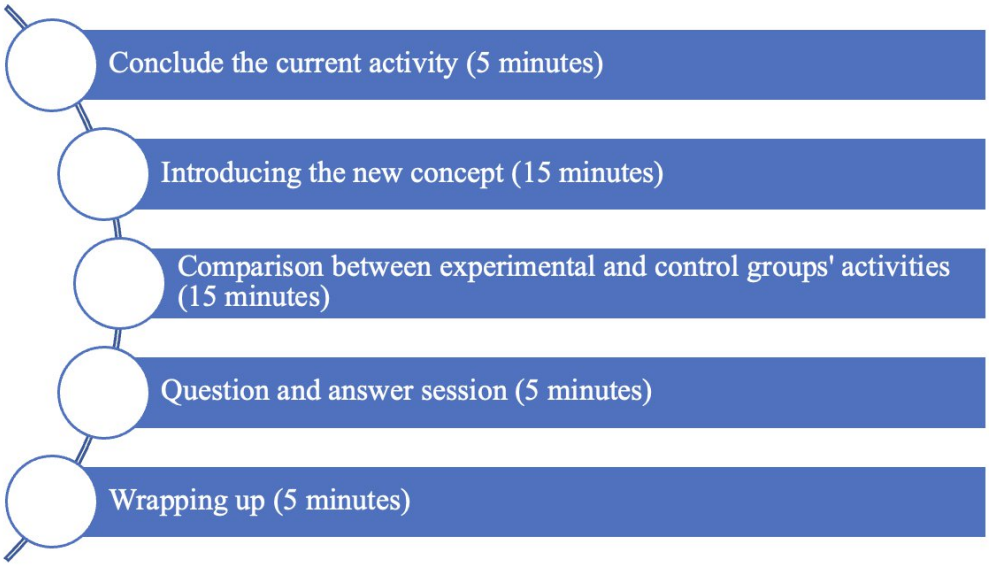


Figure 2. A Classroom Exercise

Students could collaborate with the teacher while keeping track of experimental results (Figure 3). At the end of the experiment, students had five minutes to discuss and share their findings. A guided recap took up the final five minutes of the lesson. This lesson plan would be used for all three subjects.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

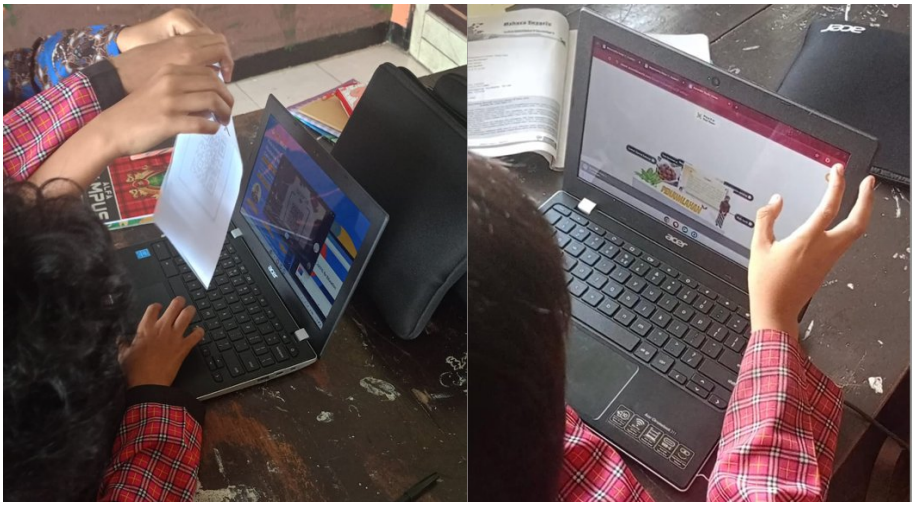


Figure 3. The Student Activity of the Experimental Group

Implementing AR in Experimental Group Design

The initial application was driven by the "Data Presentation through Diagram" instruction in Figure 1. Only the two of them were able to engage in the game. To activate the applications, a student positioned the AR card they had constructed in front of the camera. Upon the vehicle's appearance, the camera triggered. The outcome was visually represented on the screen. The subsequent student emulated the action, and the game advanced to the second round. Students had the option to either re-roll the dice in this round or terminate the game if they judged their numbers sufficient for winning. Unless a student elected to cancel the game, it would progress to the future round. However, the game would terminate quickly for the student to obtain seven points or more. This application enables students to encounter geometry in real-world situations. Upon completion of the program, individuals should possess a firm understanding of when to persist and when to cease.

The second application was specifically built for the "Analysed and Processed Data" lesson represented in Figure 2. This program gives sample spaces of equally probable events to students. The application requires students to collaborate in pairs, each employing a single mobile device. Each page depicted a different, equally geometric occurrence. Upon a student presenting a piece of paper to the camera, the whole sample area would be displayed on the screen. Students had to work with their groups on the sample space of the stated geometric event and thereafter employ their Chrome book to evaluate their findings. The examined and processed data would comprise a multiple sample space of two probability occurrences if students could concurrently show both papers in front of the camera.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

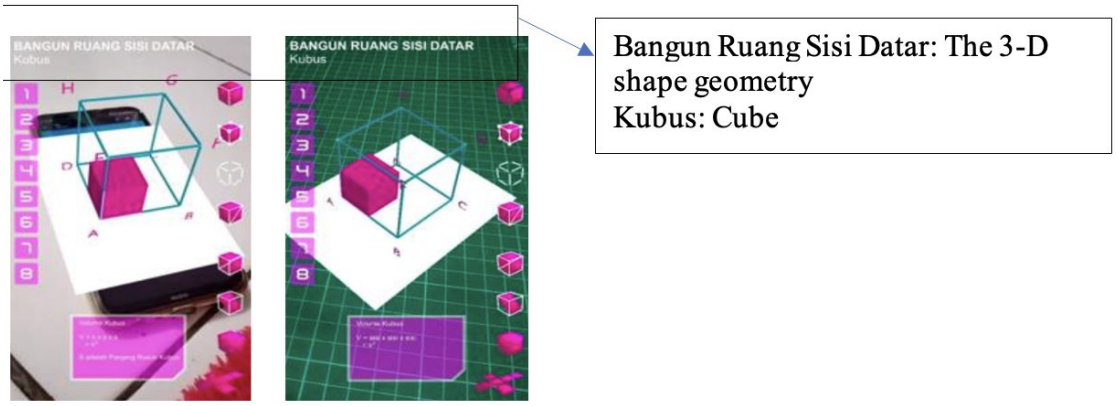
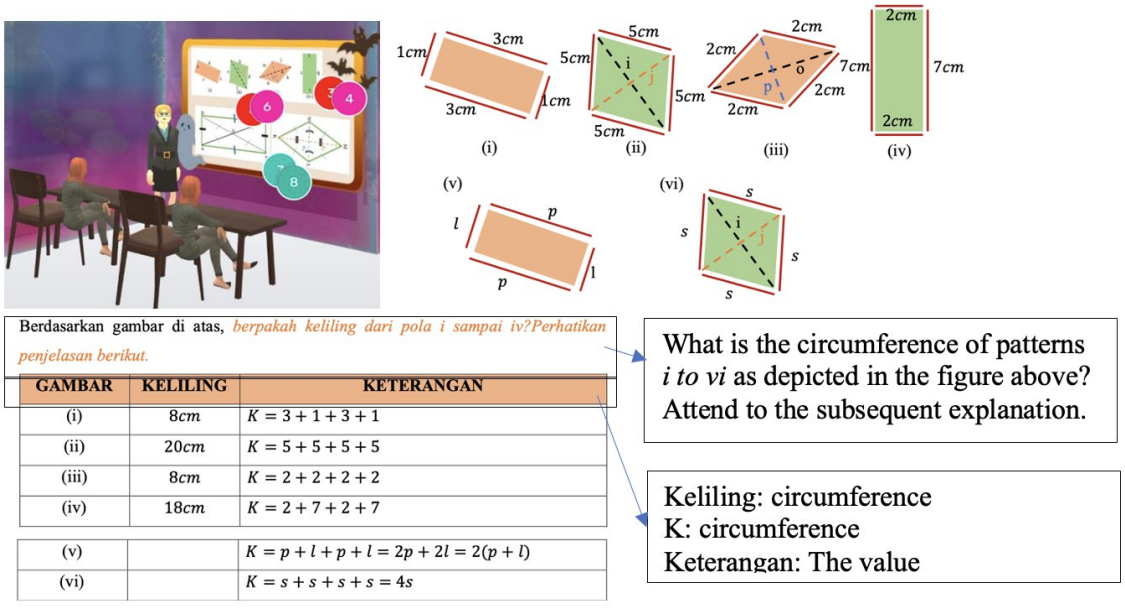


Figure 4. The Sample of AR Applications in the Experimental Group

Students were introduced to three AR programs as part of their coursework as shown in figure 4. Using the physical device layer to connect with an Android phone's camera allows for shooting pixel-perfect photos. The developing layer sends an application development kit to the application layer, allowing the application layer to call the interface methods. An interface for engaging with various AR features is given via the function module layer. Simultaneously, the applications count the data points on the diagram and provide the findings on the upper left of the screen. Figure 5 demonstrates that the bottom left corner of the screen displays the head-side geometry 2-D shape in real time.



GAMBAR	KELILING	KETERANGAN
(i)	8cm	$K = 3 + 1 + 3 + 1$
(ii)	20cm	$K = 5 + 5 + 5 + 5$
(iii)	8cm	$K = 2 + 2 + 2 + 2$
(iv)	18cm	$K = 2 + 7 + 2 + 7$
(v)		$K = p + l + p + l = 2p + 2l = 2(p + l)$
(vi)		$K = s + s + s + s = 4s$

Figure 5. Presenting AR Data Visualisations

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The third application might be valuable for students since it can do away with the obligation to collect data and construct curve graphs by hand in class. This app dives into numerous AR applications in the classroom, not to transmit knowledge but to aid students in their topic research, which could result in additional class time being spent in many circumstances (Marrahi & Belda, 2023). The procedure of sending a QR code to the server allows teachers to examine all of their students' data (Figure 3). Additionally, the program keeps past data in a local database, which users can retrieve when they exit the app. Teachers recommended the feature for its capability to help students comprehend the connection between two geometry categories using classroom data and for its impact on teachers' ability to explain the lesson's theme. These results could help students visualize how the real chance approaches the theoretical chance of 0.5.

Instruments and Theories for Experimentation

Experimental research methodologies were employed in this study. We investigated the impact of AR on students' understanding and perspective through the use of questionnaires, tests, open-ended questions, and in-depth interviews. The working hypothesis is as follows:

H_1: Students who utilised AR to augment their learning may have achieved better results than those in the control group.

H_2: The students were enthusiastic about using the AR program in their studies.

Research Instruments

Pre- and Post-Test

Our objective was to assess the students' grasp of 2D geometry using the standard test that was administered to all local schools in Purwokerto, Indonesia, by the local Education Department. If students could perform better on the local tests, the consequences of this study would be more practical and meaningful. Therefore, rather than creating a new test, we relied on the current one. On the other hand, we replaced several technical terms in the practice test with more common ones so that students who haven't studied them yet could understand them. On both tests, students would have to solve sixteen geometry problems. All of the exams had a maximum score of 100. The samples of the pre- and post-test are impressive below.

Given the measure of angle A is, 33^0*. find the sum of the measures of the complementary angle, supplementary angle, and vertical angle for angle A.*

a. 257^0 *b.* 237^0 *c.* 279^0 *d.* 227^0 *e. NOTA*

The perimeter of a rectangle is 28 in. if its length is 9 in. Find its area.
a. 6 sq. in. b. 45 sq. in. c. 90 sq. in. d. 2.5 sq. in. e. NOTA

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Survey on the Use of AR Applications

This questionnaire is derived from the original survey instrument created by Chao and Chang (2018). Thamrongrat (2021) revised the survey to assess the degree of acceptance of AR applications. The survey for this study comprises three sections: cognitive processes, perceptions of cognitive access, and the notion of contentment. This poll aimed to assess student sentiments and satisfaction with AR applications for Chromebook devices. This survey was conducted retrospectively. No participants from the control group completed the survey; it was only administered to the experimental group. The results can be deemed highly reliable since the questionnaire exhibits a Cronbach's Alpha value of 0.92, with the three constructs demonstrating values of 0.91, 0.97, and 0.93, respectively. The findings indicate that the questionnaire demonstrates reliability.

Semi-Interview Questions

In addition to the post-test, certain students were asked to answer many open-ended questions alongside the quantitative research instruments. Researchers questioned students involved in the study in person after class. The post-test for the experimental group had five open-ended questions.

1. What aspect of the lesson did you find most enjoyable?
2. In what ways do you believe these AR features would enhance this unit?
3. What are the limitations of the unit's AR applications?
4. What advantages does the utilisation of a tablet and AR offer over traditional instructional methods in a mathematics lesson?
5. What advantages does the conventional approach to teaching and learning mathematics (without the use of Chromebook and AR) offer to students?

The questions improved the questionnaire aimed at comprehending students' viewpoints on the utilisation of AR in the classroom. Subsequently, after the class, four students were randomly selected to engage in in-person interviews concerning their perceptions and emotions about AR teachings. The inquiries encompassed numerous topics, including:

1. In what ways does the utilisation of AR in the classroom differ from traditional methodologies?
2. What mathematical information did you gain from the initial mobile AR geometry application (Experience the Data Presentation)?
3. What mathematical insights did you gain from the second mobile AR geometry application (Data Analyse)?

This study sought to compare conventional mathematics training with instruction incorporating learning activities using AR technologies. Therefore, the initial question was selected accordingly. Questions two, three, and four are essential, as students' grades are significant in determining the study's effectiveness.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

RESULTS

Learning Gains Analysis in Experimental and Control Groups – RQ1

The control group achieved a score of 32 out of a maximum of 36 points, while the experimental group attained a score of 36 points both before and following the intervention. The maximum attainable score on the test is 100. Table 1 presents the essential descriptive statistics of the primary group prior to and following the test. Subsequently, paired t-tests were employed to compare the pre- and post-test scores of both the control and experimental groups. The results are presented in Table 2. Table 2 indicates that students in both the experimental group ($t = 11.41$, $p < 0.01$) and the control group ($t = 4.24$, $p < 0.01$) performed better on the post-test compared to the pre-test.

Experimental Test	Group	N	Mean
Pre-test	“Experiment” category	34	82.34
	"Control" category	32	81.90
Post-test	“Experiment” category	34	84.66
	“Control” category	32	82.53

Table 1. The Results of Pre-and Post Test

Group		Mean	S.D.	t
The range for Pre-and Post-test	“Experiment” category	8.21	4.62	11.41*
	“Control” category	3.35	5.72	4.24*

Table 2. The t-Test Value for Pre- and Post-Test of The Experimental Design

		Pre-Experiment		Post-Experiment		Univariate ANCOVA			
Group	N	Mean	S.D.	Mean	S.D.	Adjusted Mean	Standard Error	F	eta^2
Experimental Group	34	81.44	12.52	84.66	12.31	86.52	.72	30.81*	.25
Control Group	32	80.50	11.56	82.52	11.56	83.56	.68		

Table 3. Results from the Experimental Design's Pre- and Post-Test ANCOVA Value

To determine the impact of AR, we combined the post-test results of the two groups and performed an analysis of covariance (ANCOVA). Table 3 shows the performance of the independent variables and covariances before and after AR treatment is defined as the student's performance on the pre- and post-tests. The experimental group displayed significantly higher post-test scores compared to the control group, as indicated by the analysis of covariance (ANCOVA) results ($F = 30.81$, $p < 0.01$, $\eta^2 = 0.25$). Lastly, it should be noted that the utilisation of AR learning tools considerably enhanced the students' ability to retain course material. Furthermore, the difference

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

between the two groups becomes even more pronounced after the classes, with the experimental group achieving better results than the control group. The results support hypothesis 1.

Statistical Analysis of the AR-Based Questionnaire - RQ2

All 36 students in the experimental group completed the "Satisfaction" questionnaire, and the descriptive statistics are presented in Table 4. An average score of 4.05 indicates that the predominant response was, "Mathematical learning with AR is enjoyable for me," suggesting that students may significantly benefit from the application of AR principles in areas beyond mathematics. The statement, "Geometry is a captivating discipline, and I believe it is intrinsically linked to the essence of these applications," which received a high average score, supports this perspective. In the context of mathematics education, the elevated mean score for the statement, "These AR-based applications can help me discover new problems and questions," indicates that students gain to identify new topic explanations before using this AR-based application. AR serves as an excellent resource for math classes as it enhances problem-solving instruction. The average score for all claims exceeds 3.00, indicating that students are generally satisfied with these AR applications.

The Item	Mean	S.D.
AR learning applications are significantly more engaging than the monotonous traditional classroom environment of the past.	3.85	.89
These AR applications enable me to explore new difficulties and enquiries.	3.67	.89
I can visualise mathematical concepts like geometry by using AR applications.	3.68	.91
I enjoy learning maths via AR.	4.08	.85
I enjoy using AR to learn.	3.52	.96
If AR applications could improve the teaching of other disciplines as well, such as physics and chemistry, that would be fantastic.	3.84	.93
I envision a future where I can employ AR programs to assist me with my mathematics assignments.	3.81	1.05
I wholeheartedly support the integration of AR in educational settings.	3.73	1.02
I am fascinated with AR educational applications.	3.81	.96
I believe the intriguing discipline of geometry is directly connected to the core of these applications.	3.56	.89
AR technology for education enables me to learn both independently and collaboratively.	3.56	.90
The appearance of these applications is pleasant and authentic.	3.74	.91
The colours employed in the applications are appropriately calibrated; they achieve the intended effect without being excessively conspicuous.	3.78	1.02

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

In my view, learning geometry and geometry using AR is crucial.	3.58	.97

Table 4. The Mean Form of the Questionnaire's "Content" Construct

The descriptive geometry of the items that measured the "Cognitive Processes" may be found in Table 5.

The Item	Mean	S.D.
AR, in my view, makes course materials more thorough and easier to understand.	3.78	.92
I contend that AR will transform mathematics.	3.89	.89
Utilising AR enables me to thoroughly comprehend fundamental subjects that were previously obscure to me.	3.68	.91
Employing AR allows me to contemplate and experiment without limitations, considerably improving my problem-solving talents.	4.02	.94

Table 5. The Questionnaire's "Cognitive Processes" in Geometry Context

A maximum score of 4.02 for the statement " Employing AR allows me to contemplate and experiment without limitations, considerably improving my problem-solving talents " suggests that students' study and comprehension abilities are positively influenced by these applications. This result demonstrates that AR has a beneficial effect on learning (Cabero et al., 2021).

The Item	Mean	S.D.
Operating AR applications is not difficult.	4.07	.92
Learning to utilise AR has required no effort on my part.	4.06	.94
The materials and procedures employed in the educational exercise are clear and comprehensible.	4.04	.91
I swiftly determine how to employ AR.	4.01	.89

Table 6. "Cognitive Access Learning" Mean Geometry in the Questionnaire

Items with high scores on the "Cognitive Access Learning" scale are described statistically in Table 6. All four claims in this table show that the AR applications are easy for junior high students to use, with a mean score over 4.00. Students' enthusiastic attitudes and readiness to utilise AR in the classroom are demonstrated by the outcomes of three components, lending credence to hypothesis 2.

Semi-Structured Interviews Overview

There were 34 students in the experimental group, and each of them answered the five free-form questions honestly and openly. Students were asked to identify the most interesting component of the unit in the initial open-ended question. All students in the class utilised one of three AR programs to engage in the “Do Now” exercises. The AR applications were the most favoured

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

aspect of the course, as reported by all students, except for one who outright declined to participate. Students exhibited the greatest interest and preference for the third AR application among the three options.

The second and third open-ended questions required students to evaluate the advantages and disadvantages of integrating AR into the classroom. A poll revealed that 82% of students saw AR as an engaging and stimulating educational tool, 65% deemed it an efficient instrument for teaching and learning, and 56% considered it beneficial in enhancing their motivation to study. Forty percent of teachers believed that certain students were daydreaming or otherwise disengaged in class due to their fixation on their phones.

In the fourth and fifth open-ended questions, we prompted students to evaluate the advantages and disadvantages of conventional mathematics instruction. Among the questioned students, 37% said that traditional teaching methods enhanced their examination performance. Given that the predominant emphasis of mainland Chinese education is on test preparation, certain students may have perceived that more conventional pedagogical methods would enhance their examination performance. Simultaneously, all the students perceived traditional mathematics classes as tedious and excessively rigid.

Students View AR Applications

For this study, we randomly surveyed four students from the experimental group regarding their perceptions and sentiments toward AR instruction. A student was asked, "What is your perception of the distinction between traditional mathematics courses and the use of AR technology in contemporary instruction?" His response was as follows:

> "AR in the classroom encourages student participation, rather than sitting back and observing a teacher do a task. This, in turn, helps students retain more abstract mathematical concepts because they spend less time on data recording and curve graphing."

One additional student contributed:

> “The applications improved the readability of the study's findings by automatically drawing an empirical geometry graph. Using the AR app, students were able to get the needed graph in a fraction of the time it takes using traditional methods.”

The students also felt that the real-world applications of sample spaces improved their understanding of the theoretical concept behind them. One student who participated in an interview regarding the mathematical insights gained from AR geometry applications for ChromeBook noted that the initial app, "Experience the likelihood," was significant in helping him understand geometry in its abstract form. As a result, he gained a deeper appreciation for the relationship between theoretical and practical geometry. Applying the abstract concept of sample spaces, he

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

provided new and impressive examples. The researchers spoke with the class's original maths teacher. As part of her reply, she stated:

> "Immediate data collection from all students and the graphical representation of empirical geometry on the instructional screen enhances the results, making them more impressive and comprehensible for all students, thereby fostering greater enthusiasm and curiosity regarding the class materials in AR compared to traditional mathematics classes."

DISCUSSION

The findings indicate that AR can effectively inspire students, aid in the transition between physical and virtual realms, and enhance the manipulation of tangible objects via a tactile interface metaphor. All of these advantages align with the need to instruct mathematics, particularly in mathematical modeling, where students want to understand the application of abstract concepts to real-world scenarios. According to Cabero et al. (2021), mathematical modeling is a sub-competence that helps students understand real-world problems; this program enables students to match 3D objects with their suitable real-world scenarios.

Salim et al. (2020) discovered that students exhibited increased motivation to utilise AR applications for constructing mathematical models derived from real-world instances. This is an element of the educational environment that inspires students to engage in learning within the classroom. The purpose of each task strengthens these two phases of education. Moreover, Elsayed and Al-Najrani (2021), discover that students may utilise the app's suggestion feature to implement previously acquired mathematical principles when encountering difficulties with certain issues.

An analysis of the performance of the two groups on pre- and post-lesson assessments indicated no statistically significant difference. According to ANCOVA statistics, students in the AR group who utilised the app for geometry comprehension outperformed those in the control group following the classes. Previous research arrived at the same conclusion (Elsayed & Al-Najrani, 2021; Talan et al., 2022). The quantitative analysis indicated that students acquired significantly more knowledge. Conversely, the findings of the qualitative study, encompassing students' responses to the open-ended survey questions, indicated that the implementation of AR in the classroom significantly improved students' engagement and excitement for mathematics. According to Somby and Mølstad (2022), the AR applications received unanimous acclaim from the majority of users. The poll results were corroborated by positive feedback from participants, who said that AR might enhance their motivation, learning efficiency, and create more engaging experiences.

During the face-to-face interview, one student stated that AR enables him to actively investigate mathematical relationships that showed how AR has the potential to increase classroom participation. Numerous interviewers from the student body emphasised the necessity of

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

comprehensively grasping sample space and other abstract notions. From a constructivist perspective, students may utilise AR to enhance their statistical understanding (Nasrudin et al., 2021). The mathematics instructor asserts that the student's mathematical understanding significantly improved following the implementation of AR technology. The AR technology enabled students to view animated cartoon characters in three dimensions. The integration of this feature into the otherwise unremarkable AR exercise was well-received by high school students.

Moreover, the results of this study indicate that AR classes influence students' proficiency in geometry. Students who learn geometry via AR exhibit more positive attitudes than those who learn through conventional techniques. The overwhelming majority of respondents demonstrate a keen curiosity and enthusiasm for understanding AR information (Schutera et al., 2021). Despite this, students exhibited a comparatively low level of confidence in the beginning phase due to their unfamiliarity with the concept of AR. Ultimately, they found the learning experience to be entertaining and interactive, which motivated them to further explore geometry. AR-enhanced students' comprehension of the subject matter and heightened their passion for the educational experience within the classroom. The results align with the research of Su et al. (2022), as well as Del and Méndez (2021), which demonstrated that AR aids students in comprehension, enhances learning proficiency, fosters a positive attitude, and increases enjoyment in the educational experience.

In a 2021 study, Moreno et al. (2021) discovered that AR technology serves as an alternative educational tool that enhances students' cognitive by necessitating independent research on their subject matter. Students have the opportunity to explore mathematical concepts instead of receiving rote arithmetic instruction during the AR learning process (Hamzah et al., 2021). In education, the application of any technological tool is not solely confined to novelty and invention but is primarily determined by its efficacy in engaging students, as well as in fostering a conducive learning environment (Schmalstieg et al., 2020). Understanding their students' cognition of geometry will enable educators to recognise those with adverse attitudes and provide appropriate interventions (Vakaliuk et al., 2020).

However, the concept of "active learning" involves both explicit and implicit actions, such as students engaged in critical thinking about the material being taught. Examples of explicit behaviours include asking questions or making requests about the material. The findings suggested that the members of the full-AR group participated in more active learning; nevertheless, this does not imply that all of the students were not involved in the learning process. Despite the fact that they did not participate verbally, the students in the Basic-AR group gave the impression that they were paying attention in class. The cognitive load evaluations and targeted enquiries that are carried out before and after examinations are two alternative approaches that have the potential to be useful in the study that will be undertaken in the future to analyse the cognitive processes of students. Last but not least, we acknowledge that the activity has an impact on the activities that students engage in to investigate. When students are exposed to a wide variety of learning topics

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

and when they take part in less prescriptive tutoring activities, they may exhibit a variety of inquiry patterns.

LIMITATIONS

This study has multiple drawbacks. The precise impact of the complexity variations between the two scenarios on students' learning and engagement remains indeterminate owing to the study's limited sample size. The circumstances regarding the quantity, nature, and anchoring of AR representations varied. Future studies must examine more controlled variations to enhance understanding of the differences in AR application designs for these specific factors. The limited sample size precluded the statistical detection of conditional differences, rendering the results and discussion largely speculative and indicating potential avenues for further research. The instructional exercise may have experienced a low volume of questions due to insufficient question prompts. A bigger sample size engaged in an open-ended tutoring activity for an extended period should yield more detectable statistical impacts.

CONCLUSION

In this study, the impacts of AR learning applications on the comprehension of geometry among students in junior high school are investigated. Students were able to better learn and grasp difficult ideas related to geometry with the use of two AR applications that were developed as a supplement to classroom inquiry activities. Rather than merely displaying visual items, our AR solution incorporates them into the overall experience of exploration. This is the primary distinction between our approach and others. The findings of the experiments revealed that AR learning applications have the potential to improve students' comprehension of mathematical topics. The positive perceptions that the students have of applications are supported by the findings of the attitude survey as well as the free-form questions. Additionally, the researchers had a conversation with the mathematics teacher at the junior high school. The effects of AR on learning outcomes and attitudes may be investigated in the future, but it is also possible that research may investigate how AR influences students' more subjective features, such as learning anxiety. Different teaching approaches to AR could be contrasted.

Moreover, qualitative studies of students' references to AR visuals just utilise data from enquiries that explicitly pertain to the visuals. Nevertheless, students posed numerous enquiries unrelated to the visuals, and this data could provide a clearer understanding of the comparative dynamics among the groups. The statistical implications and reproducibility of descriptive findings should be examined in future studies utilising larger datasets. Moreover, our sample exhibited a pronounced bias towards females, with women constituting around two-thirds of the participants in each condition. Although there seems to be no variation in gender disparity among situations, it remains uncertain whether this influenced the outcomes, as the current study did not do a gender

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

analysis. Subsequent research may ascertain whether the current findings apply to larger populations.

REFERENCES

[1] Abdullah, A., Yahaya, M. F., & Mat Isa, N. (2020). The Impact of Nearpod Interactive Learning Platform in Quality Accounting Education for Sustainable Development. *SSRN Electronic Journal*, October, 4–6. https://doi.org/10.2139/ssrn.3563907

[2] Abida, K., & Muhammad, A. (2012). Constructivist Vs Traditional : Effective instructional approach in teacher education. *International Journal of Humanities and Social Science*, 2(5), 170–177. https://doi.org/10.3389/fonc.2014.00249

[3] Abidin, Z., & Herman, T. (2023). The Computational Thinking in Elementary School in the Indonesia New Curriculum: A Teacher's Perspective. *Sekolah Dasar: Kajian Teori Dan Praktik Pendidikan*, *32*(02), 178–185. https://doi.org/10.17977/um009v32i-22023p178-185

[4] Ahmad, N. I. N., & Junaini, S. N. (2020). Augmented Reality for Learning Mathematics: A Systematic Literature Review. *International Journal of Emerging Technologies in Learning*, 15(16), 106–122. https://doi.org/10.3991/ijet.v15i16.14961

[5] Belbase, S. (2014). Radical versus social constructivism: An epistemological-pedagogical dilemma. *International Journal of Contemporary Educational Research*, 1(2), 98–112. http://cepa.info/2616

[6] Cabero-Almenara, J., Barroso-Osuna, J., & Martinez-Roig, R. (2021). Mixed, augmented and virtual, reality applied to the teaching of mathematics for architects. *Applied Sciences*, 11(15), 1–16. https://doi.org/10.3390/app11157125

[7] Chao, W.-H., & Chang, R.-C. (2018). Using Augmented Reality to Enhance and Engage Students in Learning Mathematics. *Advances in Social Sciences Research Journal*, 5(12), 455–464. https://doi.org/10.14738/assrj.512.5900

[8] Dangel, J. R. (2017). An Analysis of Research on Constructivist Teacher Education. *Journal of Curriculum and Teaching*, 17(2), 87–113. https://doi.org/10.37119/ojs2011.v17i2.85

[9] Del Cerro Velázquez, F., & Méndez, G. M. (2021). Application in augmented reality for learning mathematical functions: A study for the development of spatial intelligence in secondary education students. *Mathematics*, 9(4), 1–19. https://doi.org/10.3390/math9040369

[10] Elsayed, S. A., & Al-Najrani, H. I. (2021). Effectiveness of the Augmented Reality on Improving the Visual Thinking in Mathematics and Academic Motivation for Middle School Students. E*urasia Journal of Mathematics, Science and Technology Education*, 17(8), 1–16. https://doi.org/10.29333/ejmste/11069

[11] Fendi, R. D., Suyatna, A., & Abdurrahman, A. (2021). Augmented Reality-Based Student Worksheet to Stimulate Students' Critical Thinking Skills. *Indonesian Journal of Science*

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

and Mathematics Education, 4(2), 118–133. https://doi.org/10.24042/ijsme.v4i2.9017

[12] Fernández-Enríquez, R., & Delgado-Martín, L. (2020). Augmented reality as a didactic resource for teaching mathematics. *Applied Sciences*, 10(7), 1–19. https://doi.org/10.3390/app10072560

[13] Gold, B. (1999). Social Constructivism as a Philosophy of Mathematics. *The American Mathematical Monthly*, 106(4), 373–380. https://doi.org/10.2307/2589577

[14] Guntur, M. I. S., Setyaningrum, W., Retnawati, H., Marsigit, M., Saragih, N. A., & Noordin, M. K. bin. (2019). Developing augmented reality in mathematics learning: The challenges and strategies. *Jurnal Riset Pendidikan Matematika*, 6(2), 211–221. https://doi.org/10.21831/jrpm.v6i2.28454

[15] Hamzah, M. L., Ambiyar, Rizal, F., Simatupang, W., Irfan, D., & Refdinal. (2021). Development of Augmented Reality Application for Learning Computer Network Device. *International Journal of Interactive Mobile Technologies*, 15(12), 47–64. https://doi.org/10.3991/ijim.v15i12.21993

[16] Kihwele, J. E., & Mkomwa, J. (2023). Promoting students' interest and achievement in mathematics through "King and Queen of Mathematics" initiative. *Journal of Research in Innovative Teaching and Learning*, 16(1), 115–133. https://doi.org/10.1108/JRIT-12-2021-0083

[17] Lai, J. W., & Cheong, K. H. (2022). Adoption of Virtual and Augmented Reality for Mathematics Education: A Scoping Review. *Digital Object Identifier*, 10(3), 13693–13703. https://doi.org/10.1109/ACCESS.2022.3145991

[18] Marrahi Gomez, V., & Belda-Medina, J. (2023). The Integration of Augmented Reality (AR) in Education. *Advances in Social Sciences Research Journal*, 9(12), 475–487. https://doi.org/10.14738/assrj.912.13689

[19] Mohammed Jabar, J., Hidayat, R., Fakhrul Hazim Rohizan, M., Rosdin, A., Salim, N., & Aisyah Norazhar, S. (2022). Augmented Reality Learning in Mathematics Education: A Systematic Literature Review. *Journal of Higher Education Theory and Practice*, 22(15), 183–202. https://doi.org/10.33423/jhetp.v22i15.5570

[20] Moreno, L. A. H., Solórzano, J. G. L., Morales, M. T. T., Villegas, O. O. V., & Sánchez, V. G. C. (2021). Effects of using mobile augmented reality for simple interest computation in a financial mathematics course. *PeerJ Computer Science*, 7(8), 1–33. https://doi.org/10.7717/peerj-cs.618

[21] Nasrudin, N. H., Azfar, M. H., Hatta, M., Azmi, A. Z., & Junid, R. A. (2021). Mobile Application: Learning Basic Mathematics Operation using Augmented Reality. *Mathematical Sciences and Informatics Journal*, 2(1), 9–20. http://www.mijuitmjournal.com

[22] Nugraha, I. (2023). Application development of augmented reality for elementary English teaching. *Indonesian EFL Journal (IEFLJ)*, 9(1), 19–24. https://doi.org/10.25134/ieflj.v9i1.6423

[23] Pramuditya, S. A., Pitriayana, S., Subroto, T., & Wafiqoh, R. (2022). Implementation of augmented reality-assisted learning media on three-dimensional shapes. *Jurnal Elemen*, 8(2), 480–493. https://doi.org/10.29408/jel.v8i2.5238

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[24] Pujiastuti, H., Haryadi, R., & Arifin, M. (2020). The development of Augmented Reality-based learning media to improve students' ability to understand mathematics concept. *Unnes Journal of Mathematics Education*, 9(2), 92–101. https://doi.org/10.15294/ujme.v9i2.39340

[25] Richardo, R., Wijaya, A., Rochmadi, T., Abdullah, A. A., Nurkhamid, Astuti, A. W., & Hidayah, K. N. (2023). Ethnomathematics Augmented Reality: Android-Based Learning Multimedia to Improve Creative Thinking Skills on Geometry. *International Journal of Information and Education Technology*, 13(4), 731–737. https://doi.org/10.18178/ijiet.2023.13.4.1860

[26] Salim, S., Fitrah, D., & Jainuddin. (2020). Augmented Reality-based Mathematics Worksheet for Online Learning During Covid-19 Pandemic. *Indonesian Journal of Educational Studies (IJES)*, 24(1), 81–90. https://doi.org/ 10.26858/ijes.v23i2.18934

[27] Santosa, E. B., Hadi, B., Subiyantoro, S., Irmade, O., & Sukmawati, F. (2021). Augmented Reality for Student Learning Media: A Bibliometric Analysis and Visualization. *Indonesian Journal of Instructional Media and Model*, 3(2), 47–55. https://doi.org/10.32585/ijimm.v3i2.2007

[28] Schmalstieg, D., Kaufmann, H., Reitmayr, G., & Ledermann, F. (2020). Mathematics and Geometry Education with Collaborative Augmented Reality. *Computers & Graphics* 27(3), 339-345. https://doi.org/ 10.1016/S0097-8493(03)00028-1

[29] Schutera, S., Schnierle, M., Wu, M., Pertzel, T., Seybold, J., Bauer, P., Teutscher, D., Raedle, M., Heß-Mohr, N., Röck, S., & Krause, M. J. (2021). On the potential of augmented reality for mathematics teaching with the application cleARmaths. *Education Sciences*, 11(8), 1–18. https://doi.org/10.3390/educsci11080368

[30] Somby, H. M., & Mølstad, C. N. (2022). Augmented reality in mathematics: Enhancing pupils' everyday school lives. *Digital Culture & Education*, 14(3), 87–104. https://www.digitalcultureandeducation.com/volume-14-3

[31] Su, Y.-S., Cheng, H.-W., & Lai, C.-F. (2022). Study of Virtual Reality Immersive Technology Enhanced Mathematics Geometry Learning. *Frontiers in Psychology*, 13(February), 1–8. https://doi.org/10.3389/fpsyg.2022.760418

[32] Suzanna, S., & Gaol, F. L. (2021). Immersive Learning by Implementing Augmented Reality: Now and The Future. *Journal of Computer Science and Visual Communication Design*, 6(1), 22–28. https://journal.unusida.ac.id/index.php/jik/article/view/402

[33] Talan, T., Yilmaz, Z. A., & Batdi, V. (2022). The Effects of Augmented Reality Applications on Secondary Students' Academic Achievement in Science Course. *Journal of Education in Science Environment and Health*, 8, 333–347. https://doi.org/10.55549/jeseh.1193695

[34] Thamrongrat, P. (2021). The Effectiveness of Augmented Reality with Gamification in Learning 3D. *Journal for Research in Mathematics Education*, 3(April), 11–24. https://doi.org/10.25392/leicester.data.16593008

[35] Umam, K., & Dwi Susandi, A. (2024). An Exploration of Students' Anxiety in Mathematics Online Learning using Augmented Reality at Junior High School: A Rasch Model Construct. *Pegem Journal of Education and Instruction*, 14(1), 186–196.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

https://doi.org/10.47 750/pegegog.14.01.21
[36] Vakaliuk, T. A., Shevchuk, L. D., & Shevchuk, B. V. (2020). Possibilities of using AR and VR technologies in teaching mathematics to high school students. *Universal Journal of Educational Research*, 8(11B), 6280–6288. https://doi.org/10.13189/ujer.2020.082267
[37] Vygotsky, L. S. (1999). Vygotsky 's Sociocultural Theory. *Development of Higher Psychological Processes*, 3(4), 269–279. https://doi.org/10.1007/978-94-007-4065-5
[38] Widyasari, N., & Mastura, L. I. (2020). Improving Geometry Thinking Ability through Augmented Reality Based Learning Media. *Eduma: Mathematics Education Learning and Teaching*, 9(1), 80–85. https://doi.org/10.24235/eduma.v8i2.5304

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Joint Research and Design of Educational Adaptation Tools in Mathematics for Foreign Students in the LMS Moodle

Veronika I. Belousova, Alla A. Knysh, Kseniia S. Potorochina

Ural Federal University, Russia

v.i.belousova@urfu.ru, knysh.alla84@gmail.com, ksen83@mail.ru

Abstract: The number of foreign students in the universities of Russia is increasing every year. Traditionally, one of the most difficult subjects for 1-2 years of study is mathematics. For foreign students who have difficulties with the perception of Russian speech, problems with different levels of basic mathematical training often become an insurmountable difficulty. Therefore, the design of educational adaptation tools that help students learn mathematical disciplines becomes an integral part of the teacher's methodological work. The purpose of the article is to consider the most effective tools for adaptation in joint work with foreign students. During the studying and summer practicum, 10 international students participated in the investigation and development of adaptive instructional materials. Subsequently, 42 international students evaluated the outcomes of this work. As a result, a set of educational adaptation tools was proposed, along with techniques for organizing interaction between instructors and non-native language learners within a blended learning model. The online course, built in the LMS Moodle, allows not only the use of developed tools for foreign students, but also organization of joint work with them on the improvement of these tools. The article presents the list of the most effective educational adaptation tools in students' opinion.

Keywords: academic adaptation, foreign students, bilingual students, educational adaptation tools, bilingual approach, studying mathematics, blended learning.

INTRODUCTION

Currently, the number of foreign students in Russian universities is constantly increasing. In the Institute of Radioelectronics and Information Technologies-RTF of the Ural Federal University, which is located in the Yekaterinburg city, Russia, over the past 6 years (2018-2024), admission to the bachelor's degree of foreign students included 57, 83, and 137 in 2018-2020, 194 students in 2022-23, and 203 students in 2023-24. To date, the total number of international students of the university exceeds 4,600. The geography of the foreign contingent is diverse: China, African countries (Sudan, Egypt, Morocco), neighboring countries (Kazakhstan, Uzbekistan, Tajikistan,

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Ukraine), Iraq, Syrian Arab Republic, Turkish Republic, Yemen, South American countries (Colombia, Ecuador, Peru), and more.

On the one hand, the expansion of the number of foreign students has a positive effect on the status of the university, while on the other hand, it requires solving issues of social and educational adaptation of students (Arkatova, Danakin & Shavyrina, 2015). We consider the educational adaptation as the socio-pedagogical process of interaction of students with the changed conditions of the educational and social environment in order to coordinate their individual characteristics and capabilities at the linguistic, interpersonal, and didactic levels. The purpose of the article is to search and select the most effective, according to the opinion of students, tools for the educational adaptation of foreign students when they study mathematical disciplines.

In the process of teaching mathematics, foreign students and teachers face a lot of problems, primarily related to the language barrier. The level of students' proficiency in the Russian language is insufficient for learning at the pace that the university curriculum assumes. The problem connected with the lack of readiness to study mathematical disciplines is also associated with the discrepancy between the level of basic training of most foreign students and the level and training program of Russian students (Kochetova, Mumryaeva & Egorchenko, 2018; Pykhtina, 2017). This fact is confirmed by statistical studies on this issue and feedback from teachers at the university.

The application of a bilingual approach to teaching foreign students, the experience of which is described, for example, by colleagues of the Financial University under the Government of the Russian Federation (Moscow) (Ganina, Dubinina & Stepanyan, 2019), seems successful if students at a high level speak one of the languages (most often, English). In practice, we are dealing with a mixed audience of native speakers and equally low proficiency in Russian and English. Therefore, the selection of educational literature in the student's native language to accompany the Russian-language course is reasonable to be carried out by linguistic groups. This raises the question of how to organize linguistic groups. The so-called random option is ineffective due to spontaneity and is possible only if students actively communicate outside of classes. This approach is not suitable for students who have difficulties in communication. It remains an option to form linguistic groups through the guidance of a teacher with constant student-lecturer interaction. How to organize such communication taking into account the opinions of students belonging the same linguistic group but studying in different academic groups? The article suggests a variant of using a blended learning model on the Moodle platform.

The idea to create a multilingual online support course is also considered in papers (Snegurenko et al 2019; Sosnovsky et al, 2012) where the Math-Bridge platform is offered as a learning environment. This platform supports 7 languages: English, German, French, Spanish, Finnish, Dutch and Hungarian and has a wide functionality for studying various math sections with interactive tasks, illustrative examples, and theoretical material. However, the Math-Bridge platform may be inconvenient for students with Arabic, Chinese, and other native languages who study Russian as a foreign language. Moreover, LMS Moodle has become more widespread in Russia than Math-Bridge.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Czarnocha and Prabhu (2016) explored the influence of learning mathematics on the development of language. Their research demonstrates an influence of mathematical thinking on the development of descriptive writing. In the Elementary Algebra course, the authors propose using the new instructional strategies for mastering algebra to learn English. These strategies include verbalization, explication, and analysis of algebraic procedures. For example, students are asked to write a symbolic statement in English. Such an approach could be also effective on the supporting course of Math in the universities in which the goal of studying math is connected with the studying a second language.

In addition to the linguistic problem, researchers (Kochetova, Mumryaeva & Egorchenko, 2018; Osipova & Tereshchenko, 2018; Pykhtina, 2017) unanimously focus on organizing a mandatory propaedeutic course in mathematics to equalize the level of mathematical training of students. In the process of revising the basic sections of the school mathematics course, future students learn mathematical terminology in Russian and at the same time fill in the existing gaps in knowledge of the discipline.

Unfortunately, training in a propaedeutic course does not exclude the problems of further perception of new terminology by students. Therefore, the task arises to design the structure of the educational course and determine the requirements for its content, including propaedeutic or adaptive material for each topic and class in higher mathematics. The task, including the development of educational tools for students' adaptation to the course of higher mathematics, was solved in the interaction of teachers with foreign students during the semester and their summer practice.

METHODS

There is a lot of research devoted to the question of academic adaptation for foreign students and revealing the diversity of their key problems. Readiness for adaptation is being studied. This term characterizes a wide range of adaptation components: cognitive, informational, evaluative, motivational, dispositional, emotional, volitional, operational, communicative, and sociopsychological (Arkatova, Danakin, & Shavyrina, 2015). The phenomenon of sociocultural adaptation helps explain learners' lack of readiness to perceive educational information, which may be influenced by personal anxiety, situational anxiety, aspects of culture shock, and language anxiety (Valieva, Ivanova, & Dashkina, 2020). Ways to reduce anxiety are associated with supporting students not only in learning and are described in terms of psychological and socio-cultural assistance or adjustment (Marinenko, 2021).

In contrast to the surveys aimed at studying the problems and structure of adaptation of foreign students, various approaches and recommendations to the organization of the adaptation process, this paper focused on the question: "What means of adaptation to mathematical courses do foreign students consider effective in practice?" The study emphasizes the fact that students are partially involved in the creation of adaptation tools; they fill in multilingual dictionaries and texts and evaluate the materials received.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The article synthesizes both the practical experience of teachers working with international students at Ural Federal University (Yekaterinburg) and relevant scholarly literature addressing the issues discussed. It also presents the results of analyses conducted jointly with students on methods and approaches to teaching higher mathematics, along with findings from a survey of international students.

Nineteen foreign students, who were undergoing summer practice at the Department of Mathematics, participated in the joint discussion about the structure of the adaptation course in mathematics and search for adaptive teaching tools. The students were asked to listen to a traditional lecture (it was a lecture on the parametric equation of an ellipse), read the text of the lecture, and then solve a typical problem on the same topic with an explanation of the solution in Russian. The task imitated typical actions in the educational process. After completing the task, the students made a list of problems they experienced when working with the task. Analyzing the difficulties they encountered in the process of learning mathematics, students and teachers identified a range of tools for adapting to the educational process.

To evaluate the results of the study, namely the practical significance and effectiveness of the tools received, an experiment was conducted with 42 foreign 1st year students who agreed to take part in the study. During the term, they were offered to learn more about the set of tools for adaptation within the training course on the Moodle platform and use them in the learning process. Teachers monitored the completion of tasks on the platform and gave recommendations on working with the course. The students participating in the experiment study in a mixed environment with Russian-speaking students from different teachers. The final control for all students is carried out in the form of independent automated testing. Therefore, in order to get feedback from foreign students on how useful and effective the adaptation tools offered to them were, a survey using a Google form was conducted. For each adaptation tool in the questionnaire, students were asked whether the tool was used in the learning process, to rate its effectiveness on a scale from 0 to 10, and to provide recommendations or comments regarding its use.

RESULTS

The most convenient form of work with foreign students is blended learning with the support of full-time training with e-learning in LMS Moodle. Researchers note the importance of interactive teaching methods in working with a foreign contingent using electronic resources (Kochetova, Mumryaeva & Egorchenko, 2018). The Moodle educational platform attracts teachers with no payment for use, open source code, customization of the service to individual needs, support for courses in the popular SCORM format, and the ability to integrate the platform with other services (Handayanto, Supandi & Ariyanto, 2018; Surjono, 2014).

In matters of training organization, the advantage of e-educational content lies in its flexibility in the presentation of information, ease of filling, changes, and speed of transmission in accordance with the needs of students. The Moodle platform allows you to design an electronic course that

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

includes various means to solve the problems of foreign students' adaptation: lecture presentations, texts of theoretical and practical orientation, tests, surveys, and game elements. In addition, Moodle allows you to organize the course while taking individual learning styles into consideration.

When preparing educational content, special attention should be paid to the ways in which information is presented. Based on the results of an experimental comparative analysis examining the quality of students' performance on different types of tasks, Yakovleva (2020) concluded that a cognitive-visual approach is advantageous for teaching foreign students. Using generalizing, systematizing, and visualizing methods to present information helps reduce the cognitive load for foreigners working with the text. By considering individual learning styles—whether visual, auditory, or kinesthetic—the presentation methods can be adapted, such as employing global or sequential approaches, with the latter being more suitable for an adaptation course.

The principles of presenting instructional material include three-level adaptation, information visualization, invariance of mathematical competencies across languages of instruction, repetition of mathematical terminology, as well as individualization and motivational strategies (Weinstein, Esin, & Zykova, 2022). The literature identifies three main types of adaptation materials based on their placement within the course:

- Introductory adaptation, which can be considered in relation to the entire course and each of its topics. This is a description of the competencies necessary for the assimilation of the material, motivational material, and the formulation of problems that will be solved in the lecture/topic/course.
- Ongoing adaptation of the studied material, which involves the use of specially structured materials adapted to the presentation considering the principles listed above. Adaptation also involves the use of additional elements of the electronic course (e.g., videos, game elements, tests, surveys, etc.) to simplify the perception of the material by students.
- Assessment and corrective adaptation, which includes the adaptation of control and measuring materials, requirements for verification activities, and means of correcting the results obtained.

The main issue of the study was to find and analyze techniques and means of adapting lecture and practical material. During the analysis of literature, communication with teachers and foreign students, the following options and means of work were considered and proposed:

1. Revision of the lecture (practice) listened to: Involves attending the scheduled lecture and taking notes on the materials posted in the electronic course.
2. Preliminary work with the lecture text: The student reviews the online course materials or prints the lecture text, then listens to the lecture while summarizing only the main points in a concise format (e.g., diagrams or figures) and highlighting important or unclear information.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

3. Notes-commenting: Students work with pre-downloaded or printed notes containing gaps (e.g., empty columns) for annotation. Since international students cannot take notes in Russian at the pace of the lecture, teachers should prepare printed materials or practice tasks with gaps for comments. For both options, using an electronic tablet is recommended to avoid printing notes for each lesson.
4. Fragments of a lecture or practice exercises with word/phrase selection: Allows students to practice Russian grammar, expand vocabulary, and reinforce the topic studied.
5. Step-by-step problem-solving plan: Language barriers in understanding symbolic mathematical texts are ad-dressed by providing students with ready-made instructions. For example, the classroom problem "Build a second-order curve in a rectangular coordinate system given in polar coordinates" can be adapted as follows:
    a. Substitute expressions for x and y using formulas for converting from polar to rectangular coordinates.
    b. Complete the square in expressions with one variable.
    c. Transform the equation to canonical form and determine the type of second-order curve.
    d. Construct the second-order curve and specify its key parameters (e.g., center or vertex, eccentricity, lengths of semi-axes, asymptotes, or directrix).
6. Brief reference material for the lesson: Summarizes concepts and formulas to be used during the lesson and can serve as a reference for individual tasks.
7. Test simulator based on a completed lecture or practice: Assesses understanding of key concepts and formulas. Tests can be bilingual or include task texts in multiple languages. Task enrichment in different languages is developed gradually in collaboration with foreign students.
8. Multilingual glossary: Can be thematic (per section) or general (course-wide). Thematic glossaries may include audio components, such as short videos demonstrating correct pronunciation of key terms.
9. Feedback survey (online course): Evaluates difficulties affecting comprehension on a scale of 0–10, including language barriers, gaps in mathematical knowledge, complexity of the material, self-organization, tutor interaction, and other social or personal factors (Rusticus, Pashootan, & Mah, 2023)."

In addition to the development of adaptation tools, it is important to take into account the organizational nuances of their implementation in the educational process. Normally, the solution of issues that are incomprehensible to the student on the educational material is carried out at weekly consultations with the teacher. But for the implementation of organizational, linguistic support to a foreign student, the most convenient way is to assign an intragroup tutor. This is a student who helps a foreign classmate to navigate the online course (e.g., what elements of the course they need to view before the lesson, which to prepare for work in the classroom, which to use to consolidate the topic, etc.) to deal with the teacher's setting of educational tasks. The designation of the course elements can be accompanied by prefixes PRE, IN, AFTER–before, during and after class, respectively. The tutor may be changed as the topic changes or remain a tutor throughout the

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

semester. Successful tutoring is an additional factor in providing a student with an increased scholarship, benefits, etc.

The following is an example of the structure of an educational module of a mathematical course with adaptation tools for international students:

Pre-Tasks

1. Before listening to the full-time lecture, read the glossary of the lecture topic. Write down the topic in your notebook and write down the key terms from the glossary in Russian.
2. Study the purposes of the lecture and the requirements for its development (they are listed in the training module of the course). Continue the phrases: "To understand the lecture, I need to know / be able to...", "After the lecture, I will know....".

In-Tasks

1. During a lecture (or reading a presentation for a lecture), pay attention to the phrases highlighted in blue in the presentation slides. These are the key points in understanding the material. A presentation with missing "blue phrases" is attached to the task. Write down these phrases from memory and test yourself.
2. According to the plan with step-by-step recommendations, describe the solution to the problem.
3. Write comments on the given study text. Use the questions like what problem is being solved, what methods, mathematical statements, transformations are used.
4. Describe the steps of action in this solution of the problem. Decipher the notation.

After-Tasks

1. After listening to the full-time lecture, make a short mind map indicating the main elements of the lecture in your native and Russian languages. Mind map is a common international name for a diagram used to structure information. The main concepts, formulas, ideas are arranged radially from the central concept outward.
2. In the lecture presentation add comments in your native and Russian languages: which points in this part of the lecture are the most significant; describe what this part of the lecture is (theorem, introduction of the concept, example); if there are questions about this part of the lecture, unclear points – write about them.
3. Take a test to check the understanding of the material you have studied. The emphasis in the test is on the assimilation of terms, theorems, statements of typical tasks, methods of solving problems and reasoning.

The proposed set of basic tasks is repeated in each learning module, which ensures that students develop a strategy for studying the material, providing the ability to identify the main elements of the course and assimilate them.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

To assess the proposed educational adaptation tools and methods of organizing the teaching of mathematics, questionnaire were given to 42 foreign students. The chart below (Table 1) shows the results of the questionnaire by groups assessing the practical significance of the several tools: 0-4 points - low; 5-7 points - average; 8-10 points - high.

For assessment, the following training tools were proposed:

1. Tasks for the translation of mathematical text into the native language and Russian.
2. Dictionary of mathematical terms in different languages.
3. Tests to consolidate the material in different languages.
4. Preparatory work with the material of the lecture/ practice before the lesson according to the schedule (familiarization with the terms, problems of the lesson).
5. Audio reference material with pronunciation of the basic concepts of lectures and practice in Russian language.
6. Forum for discussing problems when studying the material.

Tool's number	Low (%)	Average (%)	High (%)	Positive effect (%)
1	19.05	19.05	61.90	80.95
2	28.57	19.05	52.38	71.43
3	19.05	28.57	52.38	80.95
4	19.05	38.10	42.86	80.95
5	23.81	14.29	61.90	76.19
6	14.29	28.57	57.14	85.71

Table 1. Distribution of the number of students by groups assessing the level of importance of adaptation tools (compiled by the authors).

The result of the survey confirmed that the proposed educational adaptation tools are relevant and effective, since most students (Table 1) find practical significance in the use of additional means for mastering educational material in a non-native language.

In addition to the proposed means, students indicated the desire to increase the hours of practice of more problems accompanied by a simplified explanation of their solution.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

DISCUSSION

As the study showed, 80% of the experiment participants on average indicated a positive result in the application of the proposed adaptation tools to the educational process. All the experiment participants successfully passed the exam test: with a maximum of 100 points, the marks are in the range of 55-100 points. Given the varying levels of Russian proficiency and, consequently, different rates of material assimilation, while all students had the same time to complete the test, a direct correlation between exam scores and knowledge level is questionable. Therefore, investigating this relationship was not the focus of the study. The primary goal was to gather student feedback on the use of adaptation tools, which facilitated self-organization in learning a foreign language, mastery of core course elements, structuring reasoning in a non-native language, and earning points in ongoing assessments that they might not have obtained in class due to language difficulties.

The tasks repeated in the training modules create a comfortable environment for students to understand the structure of the course and the requirements of the teacher, as well as form a minimum set of necessary knowledge and skills for mastering the material.

The speed of perception during the study of the material by Russian-speaking and foreign students in a mixed group is very different. Therefore, the tasks from the PRE-TASK series, such as preparing the perception of the material in a foreign language, are especially important. Not understanding the teacher during face-to-face lecture and practice causes foreign students to quickly lose interest in learning and stop attending classes. Therefore, we consider the use of adaptation tools at the introductory, current, and assessment-corrective stages of adaptation in a convenient online format on educational platforms as a necessary condition for working with foreign students.

Since, even in the presence of additional educational elements, the main factor of success is motivation to learn, the involvement of foreign students in the educational process is a priority for the teacher. Joint research with foreign students of the most convenient and effective academic adaptation tools contributes to a more conscious and active construction of educational content and its use. Therefore, the study of the opinions of foreign students, wishes, and suggestions helps university teachers to identify ways to overcome the problems of interaction with their multilingual audience.

Feedback in the educational process is one of the most important tools for assessing its effectiveness. For example, information from students could be used to correct the presentation of course material according to their learning styles (Surjono, 2014). In the case of foreign students, conducting joint surveys and implementing proposals based on the results of the study creates positive motivation from the foreign audience, which needs special attention and assistance.

Based on the research data obtained, it is possible to design an online course to accompany the teaching of mathematics to foreign students. Taking into account the individual level of mathe-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

matical training and the student's knowledge of the Russian language, in addition to the main elements of the online course, it is possible to build additional elements of the course together with students and evaluate their effectiveness within one language group.

CONCLUSION

The joint analysis of adaptation tools in mathematics with foreign students made it possible to conclude that the design of electronic educational content should be aimed at helping the student in elementary educational activities that do not cause problems for a Russian-speaking student.

Based on the results of a survey of foreign students, a list of the most effective and useful adaptation tools was identified:

- Notes-commenting
- Step-by-step problem-solving plan
- Test simulator
- Multilingual glossary

In the case of foreign students, the stage of preparation for the lecture, which is absent for Russian-speaking students, can help the perception of the lecture material and, therefore, should become mandatory, at least in the form of familiarization with the terminology (pronunciation and spelling of new terms) and the problems of the lecture and practice.

The selection of educational literature and the compilation of multilingual educational content should be carried out in a dialogue, jointly filling the student's individual library. Translation of educational texts into the native language by a student, compilation of a glossary of mathematical terms in Russian and native languages, and other tasks not only create a complex of individual adaptive learning tools, but also contribute to increasing awareness and motivation of students in a foreign language environment

LIMITATIONS

Interpreting the results of the study, it should be noted that the stage of selecting effective adaptation tools together with foreign students took 1.5 years and the study continues. The number of students participating in the experiment is expanding, and new students fully confirm the results stated above. The presence of students who chose a low level of significance for some adaptation tools indicates that they have subjective preferences related to their cognitive style, mastery of self-organization techniques for learning, etc. Analysis of students' choices may be a further topic of research.

REFERENCES

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[1] Arkatova, O., Danakin, N., Shavyrina, I. (2015). Enhancing Adaptability of Foreign Students. *Mediterranean Journal of Social Sciences* 6(6S7). DOI 10.5901/mjss.2015.v6n6s7p276.

[2] Czarnocha, B., Prabhu, V. (2016). Algebra/English as a Second Language (ESL) Teaching Experiment. In book: *The Creative Enterprise of Mathematics Teaching Research*, 455-473.

[3] Ganina, E., Dubinina, G., Stepanyan, I. (2019). Cross-Cultural Analysis of Bilingualism in Preparinginternational Students for Training Professionally Oriented Disciplines. *Humanities and Social Sciences. Bulletin of the Financial University* 9, 142-147. DOI 10.26794/2226-7867-2019-9-4-142-147.

[4] Handayanto, A., Supandi, S., Ariyanto, L. (2018). Teaching using moodle in mathematics education. *Journal of Physics: Conference Series* 1013, 012128. DOI 10.1088/1742-6596/1013/1/012128.

[5] Kochetova, I. V., Mumryaeva, S. M., Egorchenko, I. V. (2018). Features of the system of additional mathematical education of foreign students at the university. *Pedagogical education in Russia* 8, 189-196.

[6] Marinenko, O. (2021). Assistance and support provided to international students by university teachers. *International Journal of Research in Education and Science* (IJRES) 7(3), 580-592. DOI 10.46328/ijres.1551

[7] Osipova, S.I., Tereshchenko, Yu.A. (2018). Organization of training of foreign students on the example of the propedeutics of their mathematical education. *Pedagogical Journal* 8(6A), 59-68.

[8] Pykhtina, N.A. (2017). Features of foreign students pre-university mathematical training. *RUDN Journal of Informatization in Education* 14(1), (pages 59-68). DOI 10.22363/2312-8631-2017-14-1-59-68

[9] Rusticus, S.A., Pashootan, T., Mah, A. (2023). What are the key elements of a positive learning environment? Perspectives from students and faculty. *Learning Environ Res*. 26, 161–175. DOI 10.1007/s10984-022-09410-4

[10] Snegurenko, A.P., Sosnovsky, S.A., Novikova, S.V., Yakhina, R.R., Valitova, N.L., Kremleva, E.Sh. (2019). Using E-Learning Tools to Enhance Students-Mathematicians' Competences in the Context of International Academic Mobility Programmes. *Integration of Education* 23(1), 8-22. DOI 10.15507/1991-9468.094.023.201901.008-022

[11] Sosnovsky, S., Dietrich, M., Andres, E., Goguadze, G., Winterstein, S. (2012). MathBridge: Adaptive Platform for Multilingual Mathematics Courses. *Conference: Proceedings of the 7th European conference on Technology Enhanced Learning*, 7563, 495-500. DOI 10.1007/978-3-642-33263-0_49.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[12] Surjono, H. (2014). The Evaluation of a Moodle Based Adaptive e-Learning System. *International Journal of Information and Education Technology* 4, 89-92. DOI 10.7763/IJIET.2014.V4.375

[13] Valieva, F. I., Ivanova, E. A., Dashkina, A. I. (2020). Socio-Cultural Adaptation of Students in a Foreign Language Environment: Factor Analysis. In E. Tareva, & T. N. Bokova (Eds.), *Dialogue of Cultures - Culture of Dialogue: from Conflicting to Understanding* 95, 82-91. DOI 10.15405/epsbs.2020.11.03.10

[14] Weinstein, Y.I., Esin, R.V., Zykova, T.V. (2022). E-learning of mathematical disciplines in the university. *Sib. feder. Un-t*, Krasnoyarsk, 164.

[15] Yakovleva, E.V. (2020). Teaching mathematics to foreign students at the University based on a cognitive-visual approach. *Herald of Vyatka State University* 1(135), 84-93. DOI 10.25730/VSU.7606.20.010

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Jambi *Batik* Pattern: Developing PISA-Like Mathematics Problems

Tria Gustiningsi*, Feri Tiona Pasaribu, Yelli Ramalisa

Universitas Jambi, Indonesia

triagustiningsi@unja.ac.id*, feri.tiona@unja.ac.id, yelli.ramalisa@unja.ac.id

Abstract: This study aims to develop valid and practical PISA-like mathematics problems contextualized within Jambi batik patterns. A design research methodology in the form of development studies was chosen for this study, which consists of preliminary, prototyping, and assessment phases. The prototyping phase included self-evaluation, expert review, one-on-one evaluation, small group evaluation, and field testing. The participants were 30 eighth-grade junior high school students in Jambi City. Data were collected through walk-throughs, tests, and questionnaires. The study produced valid PISA-like mathematics problems within the context of Jambi batik patterns. They were valid in terms of content, aligning with the PISA framework: construct, as they conformed to the eighth-grade curriculum; language, by adhering to the PUEBI (General Guidelines for Indonesian Spelling); and avoiding ambiguity. The problems were also deemed practical, as they were usable, understandable, and engaging for students. The findings of this study were that the questions produced had a potential effect, in enhancing students' mathematical literacy skills.

Keywords: PISA-like mathematics problems; Jambi *batik* patterns; design research

INTRODUCTION

One of the essential skills that all students must have is mathematical literacy. It refers to an individual's ability to apply mathematical reasoning to formulate, employ, and interpret mathematical concepts to solve problems related to real-world contexts or everyday life (OECD, 2018b). Individuals with strong mathematical literacy skills can apply their mathematical knowledge to solve everyday problems (Gustiningsi et al., 2023; Kemendikbud, 2017; Stacey & Turner, 2015). In addition, mathematical literacy is an important focus of the Programme for International Student Assessment (PISA), which measures the students' capacity to apply mathematical concepts to solve problems. Based on the results of PISA in 2022, Indonesia is ranked 70th out of 81 countries. Indonesia's average mathematical literacy score is 366, compared to the average score achieved by OECD countries, which is 472 (OECD, 2023). This indicates a low level of mathematical literacy. Similarly, previous studies have shown that Indonesian students have not been able to formulate questions from the problems given (Gustiningsi, Indra Putri, et al., 2024; Gustiningsi, Putri, et al., 2024a), have difficulty interpreting information in the form of images in the

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

problems given (Gustiningsi & Somakim, 2021), and can only solve basic-level questions (Gustiningsi, 2015; Gustiningsi & Somakim, 2021; Pratiwi, 2019; Stacey, 2011; Widjaja, 2011). According to preliminary research conducted at a school in Jambi City, data was obtained that students were unable to solve PISA questions properly, had difficulty constructing geometric shapes, failed to write arguments in each answer, and struggled to interpret the meaning of the questions.

One contributing factor to students' low mathematical literacy skills is the lack of experience in solving problems relevant to real-life context (Lisnani et al., 2025; Putri & Zulkardi, 2020). Based on the interviews with teachers, the questions given to students during learning are routine. This is also related to the lack of availability of contextual questions in mathematics textbooks(Gustiningsi, Putri, Zulkardi, & Hapizah, 2022; Gustiningsi & Somakim, 2021; Wijaya et al., 2015a). Wahyuni et al. (2023) analyzed the content of mathematical literacy questions in the Merdeka Curriculum mathematics textbook and found that 69% of the questions were included in levels 1, 2, and 3. In addition, the content also does not include all content in mathematical literacy according to the PISA framework (Wijaya et al., 2015b).

One approach to addressing low mathematical literacy is the development of learning resources in the form of PISA-like mathematics problems in everyday life contexts or situations (Gustiningsi, Putri, Zulkardi, Sari, et al., 2022; Zulkardi & Kohar, 2018). Galen & Eerde (2019) stated that providing contextual problems can stimulate students' engagement and learning. Incorporating real-life situations into the development of PISA-like mathematics problems can increase the vocabulary of questions that can be used by teachers in learning and support students' mathematical literacy skills (Gustiningsi et al., 2023; Gustiningsi, Putri, Zulkardi, Sari, et al., 2022). A particularly relevant and accessible context for students is local culture (Deda & Maifa, 2021; Muslimahayati, 2020).

Internationally, it is also recognized that there is a need for integration of local culture in mathematics education (Suherman & Vidákovich, 2024). Several studies have demonstrated integration of local culture in mathematics education, such as the South African curriculum (Madusise, 2015), Malaysian culture elements in permutations and combinations (Ismail & Ismail, 2010), cultural contexts in the South Pacific and Northern Brazil (François & Fantinato, 2018), as well as Nias culture (Ramadhani et al., 2025) and Javanese batik motifs (Hamidah et al. 2024) in Indonesia. These studies highlight that culture can be a medium that connects with mathematics so that students can understand mathematics learning. Therefore, incorporating local culture into mathematics education is essential.

Several studies have developed PISA-like mathematics problems using diverse cultural contexts, such as Minang Kabau (Isnaniah & Imamuddin, 2022), North Sumatran snack (Pangaribuan et al., 2023), Bangka Belitung tourism (Dasaprawira & Aspriyani, 2020), Lampung (Putra et al., 2016), and Yogyakarta tourism (Rokhima et al., 2023). Jambi, a province that is rich in culture and has unique tourism, traditional clothing, batik, and traditional houses, can be used as a context for PISA questions. Therefore, it is possible to develop PISA questions with the Jambi context to

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

increase the vocabulary of PISA-like mathematics problems. The uniqueness that is used as the context in the PISA questions to be developed is the Jambi batik pattern.

Several studies in Indonesia have explored the use of *batik* patterns in mathematics education. Sofliatin et al., (2024) showed that Pekalongan *batik* motifs can serve as geometry teaching materials in elementary schools after being validated by experts and receiving positive student responses. Meanwhile, Susanti & Budiarto (2020) found that Jonegoroan *batik* motifs demonstrate mathematical literacy concepts such as symmetry and transformation. These works highlight the pedagogical value of indigenous cultural resources in mathematics instruction. However, most have not explicitly aligned with international frameworks such as PISA. In addition, few have focused on the unique characteristics of Jambi batik as a cultural context. This study addresses these gaps by designing and evaluating PISA-like mathematics problems based on the Jambi batik tradition.

Although previous studies have developed PISA-like items using real-world contexts, limited attention has been given to incorporating local cultural elements, such as traditional *batik* patterns, in contextualizing mathematical problems. Most existing studies have not specifically addressed how such cultural integration supports the development of mathematical literacy aligned with PISA. This study addresses this gap by integrating culturally responsive pedagogy with PISA-based problem design.

This study is based on the theory of mathematical literacy as defined by the OECD (2018b), which emphasizes the ability to formulate, employ, and interpret mathematics in real-world contexts. Additionally, the development of items is informed by the construct of culturally responsive pedagogy (Gay, 2002), which promotes the use of instructional materials to enhance student engagement and understanding.

Based on the study's objective, the following research questions are proposed:

1) How can PISA-like mathematics problems be developed using Jambi batik patterns?
2) Are the developed PISA-like mathematics problems valid and practical according to expert valuations and student feedback?

METHOD

Types and Stages of Research

This study employs a design-research approach in the form of development studies (Bakker, 2019). This study aims to produce valid and practical PISA-like mathematical problems. The research was conducted through preliminary, prototyping, and assessment phase (Van den Akker et al., 2007). The stages are outlined as follows:

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

1) *Preliminary Phase:* The researcher analyzed the curriculum, PISA framework, Jambi batik motifs, and existing PISA problems tested by the OECD. The researcher also gave a pre-test to students to determine their initial knowledge before using the questions developed. Then, the researcher compiled the problems to be developed based on the results of the analysis that had been carried out.

2) *Prototyping Phase:* Prototyping was carried out by following the formative evaluation flow consisting of (a) self-evaluation stage, (b) expert review, (c) one-to-one, (d) small-group, and (e) field test, as in Figure 1.

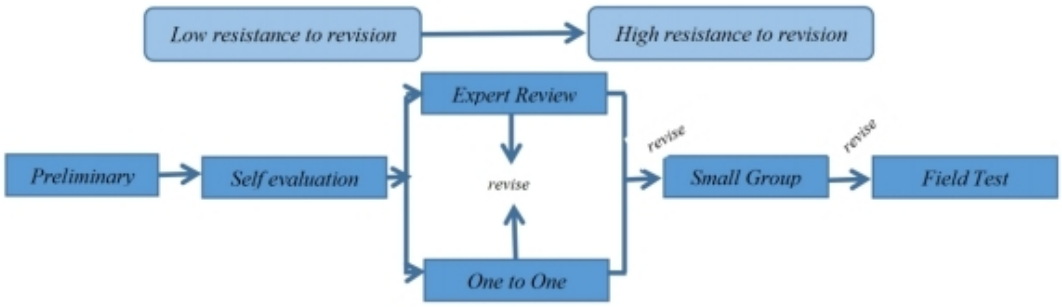


Figure 1. Formative evaluation flow (Tessmer, 1993; Zulkardi, 2002)

Based on Figure 1, the stages carried out consist of the following: (1) Self-evaluation stage, (2) Expert-review and one-to-one evaluation, (3) Small-group evaluation, and (4) Field-test stage. During the self-evaluation stage, the researcher drafted the PISA-like questions and conducted a self-assessment, resulting in prototype I which will be tested in the next stage. The Expert-review and one-to-one evaluation was carried out in parallel; the researcher gave prototype I to two experts for validation. The expert in this study has expertise in the field of mathematics education and is also a PISA researcher. They assessed the product from the aspects of content, structure (construct), and language validity. Simultaneously, the prototype I was tested at the one-to-one evaluation with two students. Comments and suggestions from the expert review and students during one-to-one evaluation were used as material to revise prototype I. The revised draft is called prototype II which was tested at the next stage. During Small-group evaluation, prototype II was tested on six students. Students worked on the PISA questions that were developed in groups. At this stage, the practicality of the developed product is seen. Students' comments and suggestions are used as material to revise prototype II. The revised result is called prototype III. Finally, during the Field-test stage, prototype III was tested on 30 students at a junior high school in Jambi City. The students who were the subjects of the one-to-one evaluation, small group evaluation, and field test stages were different students.

3) *Assessment Phase:* In the assessment phase, students' answer sheets at the field test stage were analyzed to evaluate the potential effects of the developed questions. The assessment

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

focused on the mathematical process that occurs in solving PISA questions which consists of the ability to formulate, employ, and interpret and evaluate.

Data Collection Technique

Data in this study were collected through walk-throughs, questionnaires, and tests. Walk-throughs were conducted to validate the developed prototype in terms of content, construct, and language at the expert-review stage by providing the developed prototype and assessment instruments to experts. Feedback obtained from experts were used as material to improve the developed prototype.

Open and closed questionnaires were given to students during the one-to-one, small-group, and field-test stages. Closed questionnaires were used to see the practicality assessment scores of the developed prototype but were supplemented with open questionnaires to obtain students' suggestions or comments on the practicality of the developed prototype.

Tests were used to assess the usability of the developed prototypes and to measure students' mathematical literacy skills.

Data Analysis Techniques

Walk-through and questionnaire data were analyzed qualitatively, while test data were analyzed using mathematical processes in the PISA framework, as outlined in Table 1(Gustiningsi, Putri, Zulkardi, & Hapizah, 2022).

Mathematical Process	Description
Formula	Express the problem in mathematical terms to find a solution.
Employ	Applying mathematical concepts, procedures, facts, and tools to obtain solutions to mathematical problems.
Interpret and evaluate	Evaluating a mathematical solution or outcome and interpreting it within the context of the problem or challenge.

Table 1. Mathematical Process

The developed problems are considered valid if experts confirm that the questions are valid in terms of content, construct, and language, and said to be practical if students can easily use the problems (Nieveen et al., 2006; Van den Akker et al., 2007). The valid and practical criteria are presented in Table 2.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

No.	Criteria	Description
1.	Valid	
	Content	The problem align with the PISA framework consisting of mathematical processes (formulate, employ, and interpret), real world contexts (personal, occupational, societal, and science), content domains (quantity, change and relationship, space and shape, uncertainty and data) (OECD, 2018b), the Jambi cultural context, and relevant basic competencies in the school curriculum.
	Construct	The questions align to the indicators of mathematical literacy as seen from the mathematical process, presented systematically, and the level of difficulty is calibrated to the character of the students.
	Language	The questions use Indonesian language according to the rules (PUEBI), feature clear and concise language, avoid ambiguity, local cultural terms are explained through definitions or visuals and apply mathematical terms accurately and contextually.
2.	Practical	The questions are easy for students to understand, appropriate for the available class time, supported by accessible resources or tools for problem solving.

Table 2. The valid and practical criteria

RESULTS & DISCUSSION

This study developed PISA-like mathematics problems using the context of Jambi batik motifs. The research was conducted through preliminary and prototyping stages.

Preliminary Phase

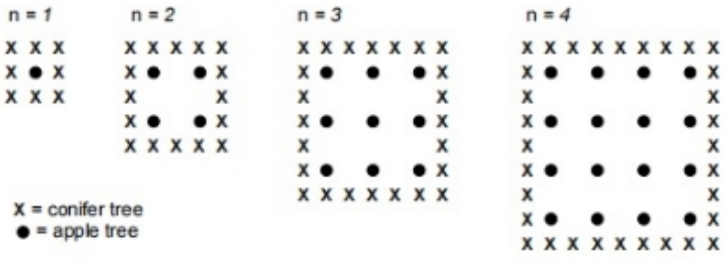
In this phase, analysis of the PISA framework resulted in the selected content being “change and relationship” content domain, while the context used was the work context of making Jambi batik production. The researcher also referenced PISA questions that inspired the development of the questions, as shown in Figure 2.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

M136: Apples

A farmer plants apple trees in a square pattern. In order to protect the apple trees against the wind he plants conifer trees all around the orchard.

Here you see a diagram of this situation where you can see the pattern of apple trees and conifer trees for any number (n) of rows of apple trees:



Question 1: APPLES

M136Q01- 01 02 11 12 21 99

Complete the table:

n	*Number of apple trees*	*Number of conifer trees*
1	1	8
2	4	
3		
4		
5		

(a)

Question 2: APPLES

M136Q02- 00 11 12 13 14 15 99

There are two formulae you can use to calculate the number of apple trees and the number of conifer trees for the pattern described above:

Number of apple trees = n^2

Number of conifer trees = $8n$

where n is the number of rows of apple trees.

There is a value of n for which the number of apple trees equals the number of conifer trees. Find the value of n and show your method of calculating this.

Question 3: APPLES

M136Q03- 01 02 11 21 99

Suppose the farmer wants to make a much larger orchard with many rows of trees. As the farmer makes the orchard bigger, which will increase more quickly: the number of apple trees or the number of conifer trees? Explain how you found your answer.

(b)

Figure 2. PISA items (OECD, 2006)

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Figure 2 presents a PISA question tested by the OECD in 2006. The researcher selected two patterns of Jambi *batik* as a suitable context for the question. Based on inspiration from PISA items, PISA-like mathematics problems were developed as in Figure 3 and Figure 4.

Jambi *Batik*



Jambi batik motif
(source: personal documentation)

The figure shows one of the Jambi batik motifs made on a cloth measuring 30 cm x 20 cm. A description of the motif is available in the table.

Figure	Description Motif
	Swan facing left
	Swan facing right
	Flower
	Broken durian

(a)

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The following pattern is formed in the batik motif in the figure.

n = 1 n = 2



a) Complete the following table:

n	Number of broken durian images	Number of flower images
1	1	4
2	2	8
3	…	…
4	…	…
5	…	…

Explain your answer.

b) If the batik motif in figure is made on a cloth measuring 100 cm x 100 cm. How many swans are facing right? Explain your answer.

(b)

Figure 3. Developed PISA questions with the first Jambi *Batik* Pattern

Figure 3 asks students to complete a table provided by analyzing the first Jambi *batik* patterns, followed by a question on the number of swan motifs if the cloth is expanded. Within the same context, questions were also posed with other Jambi *batik* pattern, as shown in Figure 4.

Jambi's *angso duo* batik motif is one of the famous batik motifs in Jambi city. The tailor makes several pieces of cloth using the *angso duo* batik motif as follows:

n = 1 n = 2 n = 3



Information:

 : flower motif

 : swan motif

(Source: batik-tulis.com)

a. Based on the information provided, please complete the tables below.

n	Number of swan motifs	Number of flower motifs
1	1	4
2	2	….
3	….	….
5	….	….
10	….	….

b. Based on the previous question, on which piece of cloth there are 22 swan motifs and 44 flower motifs? Please explain your opinion.

Figure 4. The other PISA question which was developed with the second Jambi *Batik* pattern

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Questions prepared during the preliminary phase are then continued into the next prototyping phase.

Prototyping Phase

The prototyping phase uses formative evaluation, consisting of self-evaluation, expert review, one-to-one evaluation, small group evaluation, and field testing.

a) *Self-evaluation*: In formative evaluation, the first stage of prototyping is self-evaluation. The researcher composes the questions and evaluates the content, construct, and language suitability. The questions prepared during the self-evaluation stage are referred to as prototype I, which is then advanced to the next prototyping stage.

b) *Expert-review:* At the expert review stage, prototype I was validated by experts in mathematics education and PISA researchers. The validators provided suggestions and comments on the questions, as presented in Table 3. These inputs were used by the researcher to revise the questions.

Topic	Validators	Comments and Suggestions
Question of the first Jambi *Batik* patterns (Figure 3)	Validator 1	• Better if the question is changed, "if the cloth is expanded to a size of 100 cm x 100 cm, is the number of swan motifs facing right and left the same?" • Improve the level of the question • Is it better to find the formula for the nth term? Add the sentence "What is your strategy, show the mathematical calculation"
	Validator 2	What is the purpose of the students looking for a batik motif with a flower image? Each question should be able to be added to their purpose in completing the question.
Question of the second Jambi *Batik* patterns (Figure 4)	Validator 1	• Question *b* should be adjusted again to the original PISA question. • The sentence "based on previous questions" does not match the PISA questions in general. • Add one more question that matches the apples unit in the original PISA question.
	Validator 2	• Questions that are not included in the high-level category should be changed to become questions that require high-level thinking skills. • In the question section, "complete the tables below" the word table should only be written once.

Table 3. Comment and Suggestion of Expert

c) *One-to-one Evaluation:* In the one-to-one evaluation, the questions were tested on three students. They were asked to complete the questions and respond to a questionnaire to as-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

sess the usability of the items. Table 4 shows the findings of comments or responses in the one-to-one evaluation.

Topic	The finding
Question of the first Jambi *Batik* patterns (Figure 3)	Students did not explain how to find the nth term in question *a*, indicating the need to revise the wording of the question.
Question of the second Jambi *Batik* patterns (Figure 4)	Students were unable to answer the second question (question *b*).

Table 4. Findings during one-to-one evaluation

From the expert review comments and the findings of the one-to-one evaluation, revisions were made available in Table 5.

Topic	Revision Decision
Question of the first Jambi *Batik* patterns (Figure 3)	• Change the question sentence to "If the fabric is expanded to a size of 100 cm x 100 cm. Is the number of swan motifs facing right and left the same? Show your strategy. • Adding the sentence "To be able to create batik with a larger cloth size, we must know the pattern and number of motifs formed, one of which is as in the picture, there are durian and flower motifs in different numbers." • Adding the sentence “What is your strategy? Show the mathematical calculations.”
Question of the second Jambi *Batik* patterns (Figure 4)	• Adjusting question *b* according to the second question of the apples unit in the PISA questions that have been tested by the OECD. • Delete the sentence "based on previous questions." • Adding one more question (the third question) according to the apples unit in the PISA questions. • Changing the question so that the question falls into the high-level question category, namely by asking students to draw a batik motif according to the requested cut. • Correction of the words “the tables” to “the table”

Table 5. Revision Decision of Prototype I

Based on the revision decision, the questions were revised to be as shown in Figure 5 and Figure 6.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The following pattern is formed in the batik motif in the figure.

n = 1 n = 2



To be able to create batik with a larger cloth size, we must know the pattern and number of motifs formed, one of which is as in the picture, there are durian pecah and flower motifs with different numbers.

a) Complete the following table:

n	Number of broken durian images	Number of flower images
1	1	4
2	2	8
3	…	…
4	…	…
5	…	…

What is your strategy to fill the table above? Show your mathematical calculations.

b) If the cloth is expanded to a size of 100 cm x 100 cm, is the number of swan motifs facing right and left the same?

Addition of sentences according to expert suggestions.

Improvement of questions according to expert suggestions.

Figure 5. Prototype II with the first Jambi *Batik* patterns question

The question with the context of the second Jambi batik pattern, which is presented in Figure 6.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Jambi's *angso duo* batik motif is one of the famous batik motifs in Jambi city. The tailor makes several pieces of cloth using the *angso duo* batik motif as follows:

n = 1 n = 2 n = 3



Information:

 : flower motif

 : swan motif

Source: https://batik-tulis.com/blog/batik-jambi/#google_vignette

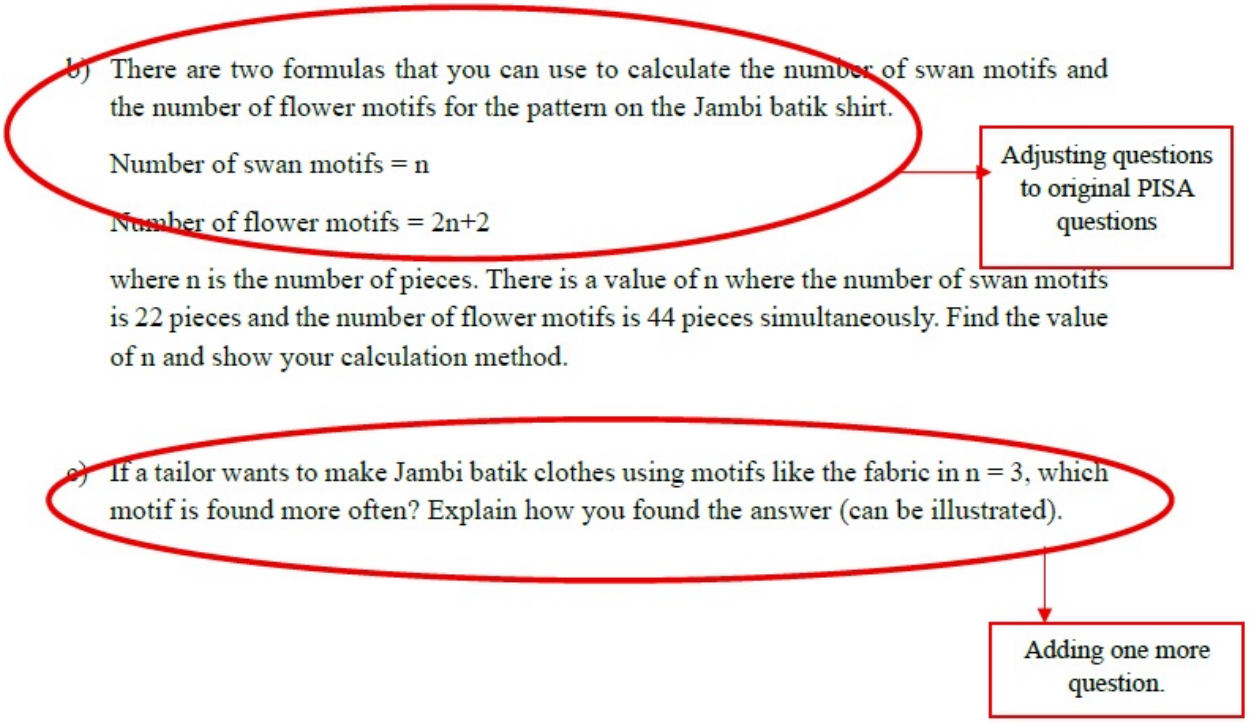
a) Based on the information provided, draw a batik motif according to the cuts requested in the table below.

n	Batik pattern
4	
5	
6	
8	

Changing the question to ask students to draw a batik motif.

(a)

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.



(b)

Figure 6. Prototype II with the second Jambi *Batik* patterns question

The revised result is called Prototype II. The revised questions (prototype II) were then tested at the small group stage.

d) *Small-group Evaluation:* At the small group stage, the questions were tested on six students. They were given time to complete the questions, followed by a questionnaire to identify difficulties related to pictures, tables, or sentences in the questions. Based on their answers, students were able to complete the questions successfully without difficulties; therefore, no revisions were made. The questions were then referred to as prototype III, as presented in Figures 7 and Figure 8.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Jambi *Batik*



Jambi batik motif
(source: personal documentation)

The figure shows one of the Jambi batik motifs made on a cloth measuring 30 cm x 20 cm. A description of the motif is available in the table.

Figure	Description Motif
	Swan facing left
	Swan facing right
	Flower
	Broken durian

The following pattern is formed in the batik motif in the figure.

n = 1



n = 2



To be able to create batik with a larger cloth size, we must know the pattern and number of motifs formed, one of which is as in the picture, there are durian pecah and flower motifs with different numbers.

a) Complete the following table:

n	Number of broken durian images	Number of flower images
1	1	4
2	2	8
3	...	...
4	...	...
5	...	...

What is your strategy to fill the table above? Show your mathematical calculations.

b) If the cloth is expanded to a size of 100 cm x 100 cm, is the number of swan motifs facing right and left the same?

Figure 7. Prototype III with the question of the first Jambi *batik* pattern

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Jambi's *angso duo* batik motif is one of the famous batik motifs in Jambi city. The tailor makes several pieces of cloth using the *angso duo* batik motif as follows:

n = 1 n = 2 n = 3



Information:

 : flower motif

 : swan motif

Source: https://batik-tulis.com/blog/batik-jambi/#google_vignette

a) Based on the information provided, draw a batik motif according to the cuts requested in the table below.

n	Batik pattern
4	
5	
6	
8	

(a)

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

b) There are two formulas that you can use to calculate the number of swan motifs and the number of flower motifs for the pattern on the Jambi batik shirt.

Number of swan motifs = n

Number of flower motifs = 2n+2

where n is the number of pieces. There is a value of n where the number of swan motifs is 22 pieces and the number of flower motifs is 44 pieces simultaneously. Find the value of n and show your calculation method.

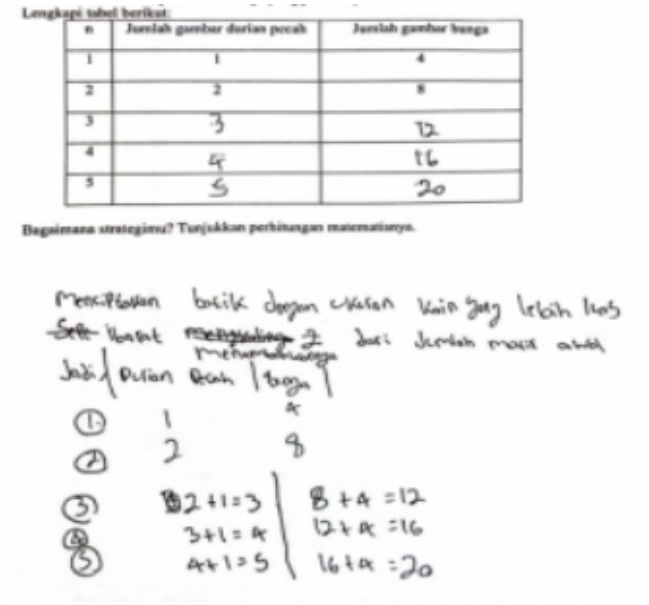
c) If a tailor wants to make Jambi batik clothes using motifs like the fabric in n = 3, which motif is found more often? Explain how you found the answer (can be illustrated).

(b)

Figure 8. Prototype III with the question of second Jambi *batik* pattern

The revised questions (prototype III) were then tested at the field test stage.

e) *Field Test:* In the field test stage, the questions were administered on 30 students. Their responses were analyzed to determine the potential effects of the developed PISA-like mathematics problems. The students' answers from this stage are presented in Figure 9.



Translation:

Creating batik on a larger cloth is analogous to adding to the initial number of motifs.

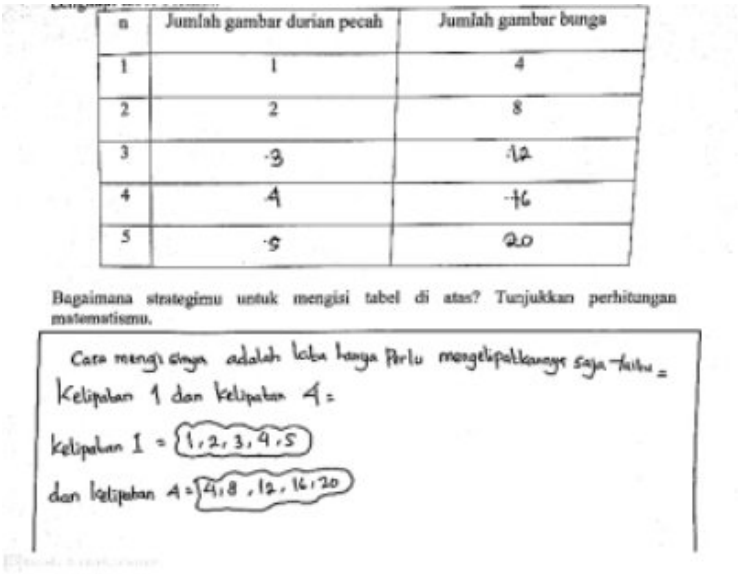
So,

No.	Broken durian	Flower
1	1	4
2	2	8
3	2+1	8+4 = 12
4	3+1	12+4 = 16
5	4+1	16+4 = 20

Figure 9. Student's answer to the question *a* of the first Jambi *batik* pattern question

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

In Figure 9, students identified a pattern for determining the number of broken durian motifs using the formula (n-1) + 1, and for determining the number of flowers, which increases by 4 with each successive n. Figure 9 shows that students can formulate problems mathematically, relate the number of broken durian motifs to flower motifs using certain patterns, apply appropriate mathematical concepts and calculations, and interpret these patterns to solve the problems to solve problems (Gustiningsi & Somakim, 2021; OECD, 2018b; Rawani et al., 2019; Utari & Gustiningsi, 2021). Additional student responses are shown in Figure 10.



n	Jumlah gambar durian pecah	Jumlah gambar bunga
1	1	4
2	2	8
3	3	12
4	4	16
5	5	20

Bagaimana strategimu untuk mengisi tabel di atas? Tunjukkan perhitungan matematismu.

Cara mengisinya adalah kita hanya perlu mengelipatkangan saja-saja =
Kelipatan 1 dan kelipatan 4 =
kelipatan 1 = {1, 2, 3, 4, 5}
dan kelipatan 4 = {4, 8, 12, 16, 20}

Translation:

It can be completed by folding based on multiples of 1 and 4.

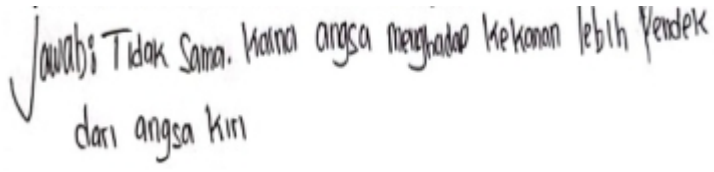
multiples of 1 = 1, 2, 3, 4, 5

multiples of 4 = 4, 8, 12, 16, 20

Figure 10. Another student's answer

Figure 10 shows that students identified the number of durian burst motifs as a multiple of 1 and the number of flower motifs as a multiple of 4. The responses in Figure 10 show that students can find a pattern between the two motif types. These responses demonstrate the potential of the developed questions to foster students' mathematical literacy skills, as reflected in the mathematical processes of formulating, employing, and interpreting (Gustiningsi et al., 2024; Putri & Zulkardi, 2020, 2019; Zulkardi & Putri, 2019).

In addition, the student's answer of the question *b* of the first Jambi *batik* pattern question is presented in Figure 11.



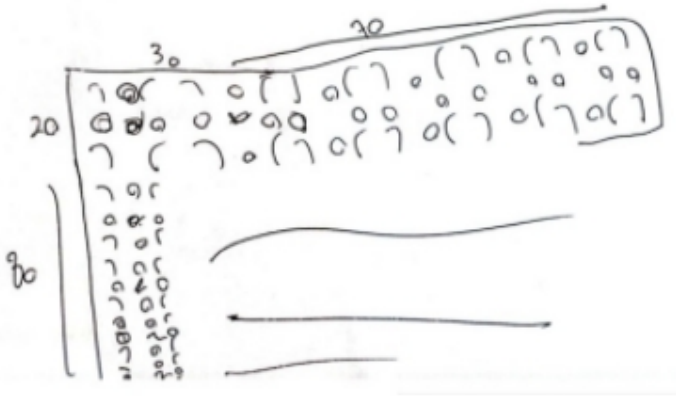
Jawab: Tidak Sama. Karna angsa menghadap kekanan lebih pendek dari angsa kiri

Translation:

Different, because the goose facing right is shorter than the goose facing left.

(a)

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.



(b)

Figure 11. Student's answer to the question *b* of the first Jambi *batik* pattern question

In Figure 11, the student draws a batik pattern measuring 100 cm x 100 cm. It shows that the student can determine the repetition of the pattern when the cloth is expanded to that size. The student can analyze the batik patterns and interpret whether the number of goose images facing right and left is equal based on the pattern formed. This evidence supports the attainment of the targeted mathematical processes (Gustiningsi, Putri, Zulkardi, & Hapizah, 2022; Gustiningsi & Somakim, 2021; OECD, 2018b).

Then the students' answers to questions of the second Jambi *batik* pattern are presented in Figures 12, 13, and 14.

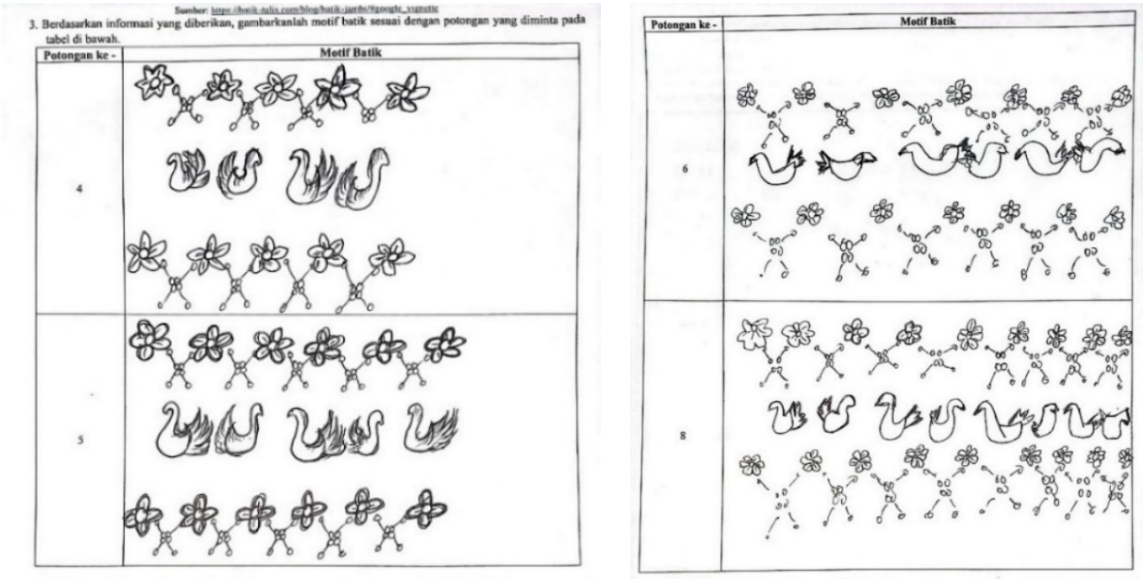
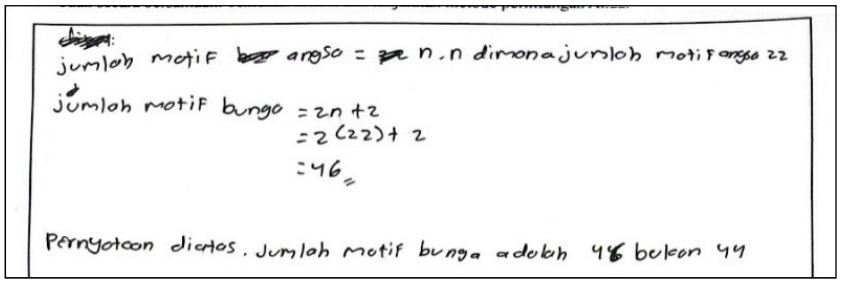


Figure 12. Student's answer to question *a* of the second Jambi *batik* pattern question

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Based on Figure 12, students demonstrate mathematical literacy skills. In the formulating aspect, students can recognize that batik patterns consist of several repeating elements, and can identify the basic structure of the pattern, both in terms of shape and direction of the motif. In terms of employing, students demonstrate the skill of placing motifs in an organized arrangement, both horizontally and vertically, and maintaining a consistent repetition sequence, which reflects an understanding of the mathematical structure of the pattern. Meanwhile, in the interpreting and evaluating aspect, students can connect patterns between pieces (for example from the 4th to the 5th or from the 6th to the 8th), and continue the pattern with appropriate adjustments in size and direction, which shows that they can interpret mathematical structures in a visual context and evaluate the suitability of the results to the previous pattern. Then, the students' answers to question *b* on the second Jambi batik motif are shown in Figure 13.



Translation:
The number of swan motifs = n, where the total number of swan motifs is 22.
The number of flower motifs = 2n + 2
= 2(22) + 2
= 46
Based on the calculation above, the number of flower motifs is 46, not 44.

Figure 13. Student's answer to question *b* of the second Jambi *batik* pattern question

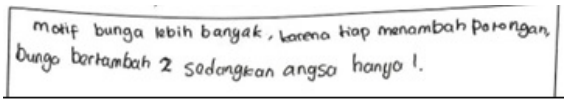
Based on Figure 13, students demonstrate mathematical literacy skills that include indicators of applying, interpreting, and evaluating. In the applying aspect, students are able to use the given mathematical model, namely substituting the value n=22 into the equation for the number of flower motifs and obtaining a result of 46. Furthermore, in the interpreting and evaluating aspect, students are able to check the results of their calculations and criticize the discrepancy between the results and the information in the question, which states that the number of flower motifs should be 44. Students can also correctly conclude that the results obtained (46) do not match the conditions of the question, and correctly explain that the number of flower motifs they calculated is 46, not 44. This shows that student has good evaluative skills regarding the suitability of the mathematical model and the context of the question. The students' answers to question b on the second Jambi batik motif are shown in Figure 14.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.



Translation:
The flower motif appears more often because, in the picture of the batik shirt above, there are 16 swans and 27 flowers.

(a)



Translation:
There are more flower motifs because for each additional pattern, the flowers increase by 2 while the swans only increase by 1."

(b)

Figure 14. Student's answer to the question *c* of the second Jambi *batik* pattern question

Based on Figure 14(a), the student demonstrates mathematical literacy skills that encompass the aspects of formulating, applying, interpreting, and evaluating. In the formulating aspect, the student is able to distinguish between the types of motifs used in the batik pattern, namely flower motifs and swan motifs, and identify the relevant elements from the given context. In the applying aspect, the student draws a batik shirt as part of the solution and adjusts the motifs in the illustration according to the given information, showing the ability to use visual representation as a mathematical strategy. For the interpreting and evaluating aspect, the student connects the question to the appropriate response, as seen in the reasoning provided at the beginning of the answer. The student states that the flower motif appears more frequently, which is supported by a logical explanation or visual evidence. This indicates that the student understands the context, applies the given information correctly, and evaluates the relationship between data in a meaningful way. Based on Figure 14(b), the student was able to identify relevant elements in the context of the problem, namely the swan motif and the flower motif, and recognize that both need to be counted and compared. In the applying aspect, students calculate the number of each motif based on the available images or information. Furthermore, in the interpreting and evaluating aspect, the student was able to compare the calculation results and draw the right conclusions about which motif is more, accompanied by logical arguments to support his answer. This shows that the student can use mathematical results meaningfully in the given context.

Students' answers indicate that the questions have the potential to enhance their ability to formulate, employ, and interpret mathematical problems. The potential is evident not only qualitatively through students' answers but also quantitatively from the pre-test and post-test scores presented in Table 6.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

No.	Initial Name	Score	
		Pre-test	Post-test
1.	FD	20	20
2.	PPR	10	37,5
3.	AH	35	65
4.	ACC	20	60
5.	DSA	10	27,5
6.	CDS	15	60
7.	FPO	10	15
8.	MAG	10	70
9.	MPA	5	12,5
10.	MSK	10	30
11.	ZAA	45	72,5
12.	DOF	10	57,5
13.	KA	20	75
14.	FDA	5	35
15	NDA	0	5
16.	RK	5	60
17.	MAGA	10	70
18.	EKA	15	60
19.	MHR	25	70
20.	RFZ	5	65
21.	MRA	45	70
22.	DA	35	67,5
23.	CPV	30	55
24	BDP	45	87,5
25	ZR	5	10
26.	OK	25	55
27.	ZPKR	25	17,5
28.	NHN	5	77,5
29.	VDA	20	62,5
30.	AAS	0	60
	Percentage	28,88%	51%

Table 6. Pre-test and Post-test score

Based on Table 6, it can be seen that there was an increase in the average student score from the pre-test of 28.88% to the post-test of 51%. This shows an increase of 22.12 percentage points after students used the developed questions. This increase reflects that the questions contributed positively to strengthening students' mathematical literacy skills.

Based on students' answers, none expressed confusion about the meaning of the question. This shows that the questions are usable and practically meaningful, aligning with Nieveen et al. (2006),

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

who stated that practicality is achieved when a designed product can be effectively used. In addition, the questions developed align with the criteria of a work context, namely the creation of Jambi *batik* patterns (OECD, 2018a).

The description above indicates that the questions developed are valid, as validated during the expert review and one-to-one stages. Their validity is supported by content, construct, and language considerations (Van den Akker et al., 2007). The questions were also tested for practically during the one-to-one stage and small group stages, demonstrating their usability. Moreover, the questions showed a potential effect, as reflected in students' responses during the field-testing stage. The findings reveal that the developed PISA-like mathematics problems have the potential to foster students' mathematical literacy abilities.

In this study, a test item is considered "PISA-like" if it aligns with the PISA framework outlined by the OECD, which includes the use of real-world contexts and the incorporation of mathematical processes—formulating, employing, and interpreting—in various contexts (OECD, 2018b). The questions developed incorporate local cultural contexts, especially Jambi batik patterns, within the occupational context. They also involve the three mathematical processes and align with the content of change and relationship.

The methodology used in this study—developing PISA-like mathematics problems based on the Jambi *batik* pattern, specifically the *angso duo* motif—demonstrates characteristics that can be adapted to other regions in Indonesia, particularly those with similar educational infrastructure and socio-cultural dynamics. For instance, batik motifs such as *Parangklitik*, *Parangbarong*, and *Sidomukti* in Yogyakarta (Prahmana & D'Ambrosio, 2020), Javanese *batik* patterns (Selamet, 2018), and other regional *batik* designs across Indonesia (Trixie, 2020) offer culturally rich alternatives that serve as authentic mathematical contexts. Beyond national application, this methodology holds global potential by incorporating local cultural elements such as traditional textiles, architecture, or crafts. However, successful contextualization requires careful consideration of cultural practices, linguistic diversity, and policy frameworks. Culturally responsive learning models have been implemented in countries such as Malaysia and the Philippines, where they have been shown to enhance student engagement and learning relevance (Idrus, 2014; Mercado, 2020). This approach can also be applied to other subjects, such as science and social studies, by incorporating local wisdom. Integrating this model into teacher training programs can enhance educators' capacity to design culturally meaningful, competency-based assessments and provide valuable insights for curriculum developers in aligning global standards with local identities.

Limitations and Future Research

Despite the promising results, several limitations must be acknowledged. First, the expert review involved a relatively small number of participants, which may limit the generalizability of the validity judgments. Although expert selection was purposive, it may introduce bias due to their

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

shared academic background and familiarity with the local context. Additionally, the study was conducted in a single region (Jambi), which may restrict the broader applicability of the findings to other cultural or educational settings in Indonesia or beyond. Future research should involve a more diverse group of experts from various regions and academic disciplines to reduce potential bias and enhance the credibility of the validation process. Expanding the study to different regional contexts would also strengthen external validity and enable comparative insights across cultural settings.

CONCLUSION

Based on the question development process, PISA-like mathematics problems can be developed by integrating contextual elements from Jambi *batik* patterns, particularly by utilizing the motifs and structures to construct problems that require mathematical reasoning, representation, and problem-solving in real-life contexts. PISA-like mathematics problems using the context of Jambi *Batik* patterns were produced and found to be qualitatively valid and practical. The question was declared valid in terms of content, as they aligned with the PISA framework and the work context of creating Jambi Batik Patterns. The question was also valid in terms of construct, as they corresponded to the grade VIII curriculum, and in terms of language, as they adhered to PUEBI *(General Guidelines for Indonesian Spelling)* and avoided ambiguity. The questions were considered practical, as they could be used by students, were clearly understood, and effectively elicited students' mathematical literacy skills. This study suggests that similar PISA-like problems should be developed using other batik motifs or cultural elements beyond batik.

ACKNOWLEDGEMENT

This article is part of a research project funded by a research grant from the University of Jambi with the letter number Chancellor's letter number 395/UN21.11/PT.01.05/SPK/2024.

REFERENCES

[1] Bakker, A. (2019). *Design Research in Education: A Practical Guide for Early Career Researhers*. Routledge. https://doi.org/https://doi.org/10.4324/9780203701010

[2] Dasaprawira, M. N., & Aspriyani, R. (2020). Pengembangan Soal Matematika Tipe PISA Menggunakan Konteks Puri Tri Agung. *EPSILON*, *2*(1). https://jurnal.stkippgribl.ac.id/index.php/epsilon/article/view/246

[3] Deda, Y. N., & Maifa, T. (2021). Efek Potensial Lembar Kerja Siswa Matematika Menggunakan Konteks Makanan Tradisional Timor pada Materi Perbandingan. *AKSIOMA: Jurnal Program Studi Pendidikan Matematika*, *10*(3). https://doi.org/10.24127/ajpm.v10i3.3214

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[4] François, K., & Fantinato, M. C. (2018). Local Mathematical Education: The implementation of local mathematical practices into the mathematics curriculum. *Philosophy of of Mathematics Education Journal*, *33*, 1–18.

[5] Galen, F. Van, & Eerde, D. van. (2019). Mathematical Investigations for Primary School. In *Utrecht: Utrecht University*. http://www.fisme.science.uu.nl/en/impome/

[6] Gay, G. (2002). Preparing for Culturally Responsive Teaching. *Journal of Teacher Education*, *53*(2). https://doi.org/10.1177/0022487102053002003

[7] Gustiningsi, T. (2015). Pengembangan Soal Matematika Model PISA Untuk Mengetahui Kemampuan Berpikir Kritis Matematis Siswa Kelas VII. *Jurnal Pendidikan Matematika RAFA*, *1*(1), 139–158. http://jurnal.radenfatah.ac.id/index.php/jpmrafa/article/view/1228

[8] Gustiningsi, T., Indra Putri, R. I., Zulkardi, & Hapizah. (2024). Supporting Students' Mathematical Literacy Skill Using Digital Tools. *AIP Conference Proceedings*, *3046*(1). https://doi.org/10.1063/5.0194695

[9] Gustiningsi, T., Putri, R. I. I., Zulkardi, & Hapizah. (2022). Secondary Mathematics Teachers' Ability in Solving PISA-Like Mathematics Problems. *AIP Conference Proceedings*, *2577*, G. https://doi.org/10.1063/5.0096217

[10] Gustiningsi, T., Putri, R. I. I., Zulkardi, & Hapizah. (2023). Developing a PISA-Like Mathematical Problem: Using Traditional Food Context. *International Journal of Education & Curriculum Application*, *6*(3), 324–337. https://doi.org/10.31764/ijeca.v6i3.20200

[11] Gustiningsi, T., Putri, R. I. I., Zulkardi, & Hapizah. (2024a). Learning Video for Supporting Mathematical Literacy: How to Design It? *AIP Conference Proceedings*, *3052*(1). https://doi.org/https://doi.org/10.1063/5.0201006

[12] Gustiningsi, T., Putri, R. I. I., Zulkardi, & Hapizah. (2024b). LEPscO: Mathematical Literacy Learning Environment for the Guru Penggerak Program. *Journal on Mathematics Education*, *15*(2), 661–682. https://doi.org/https://doi.org/10.22342/jme.v15i2.pp661-682

[13] Gustiningsi, T., Putri, R. I. I., Zulkardi, Sari, D. K., Marlina, L., Rawani, D., Sari, A., Azmi, Z. L., Septimiranti, D., & Lisnani. (2022). Designing Student Worksheet on Relation and Function Material for Mathematics Learning: Jumping Task. *Mathematics Teaching-Research Journal*, *14*(4). https://files.eric.ed.gov/fulltext/EJ1361679.pdf

[14] Gustiningsi, T., & Somakim. (2021). Pengembangan Soal Matematika Tipe PISA Level 5 Konteks Pribadi. *Jurnal Aksioma*, *10*(2), 915–926. https://ojs.fkip.ummetro.ac.id/index.php/matematika/article/view/3535/pdf

[15] Hamidah, Kusuma, J. W., & Auliana, S. (n.d.). *Development of Discovery-Based Etnobra (Ethnomathematics Geogebra) Geometry Learning Model to Improve Geometric Skills in Terms of Student Learning Styles and Domicile*. https://files.commons.gc.cuny.edu/wp-content/blogs.dir/34462/files/2024/07/Hamidah-53-16n3-2024-p25.pdf

[16] Idrus, F. (2014). Initiating Culturally Responsive Teaching for Identity Construction in the

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Malaysian Classrooms. *English Language Teaching*, *7*(4). https://doi.org/10.5539/elt.v7n4p53

[17] Ismail, M. R., & Ismail, H. (2010). Exploring Malay-Islamic ethnomathematics: Al-Khatib's combinatoric theory in Àlam Al-Hussab And Raudah Al-Hussab. *Procedia - Social and Behavioral Sciences*, *8*. https://doi.org/10.1016/j.sbspro.2010.12.102

[18] Isnaniah, I., & Imamuddin, M. (2022). Pengembangan Soal Literasi Matematika Konteks Budaya Minangkabau untuk Meningkatkan Literasi Matematika Siswa. *AKSIOMA: Jurnal Program Studi Pendidikan Matematika*, *11*(4). https://doi.org/10.24127/ajpm.v11i4.5985

[19] Kemendikbud. (2017). Materi Pendukung Literasi Numerasi. In *Kementerian Pendidikan dan Kebudayaan*. https://gln.kemdikbud.go.id/glnsite/wp-content/uploads/2017/10/cover-materipendukung-literasi-numerasi-gabung.pdf.

[20] Lisnani, Stevany, A. L., Tanujaya, B., & Widodo, S. A. (2025). Android-based Mathematics Learning Media on Geometry Using the Context of Tourist Destinations. *Mathematics Teaching-Research Journal*, *17*(2). https://s3.amazonaws.com/files.commons.gc.cuny.edu/wp-content/blogs.dir/34462/files/2025/05/3.Android-based-Mathematics-.pdf

[21] Madusise, S. (2015). Cultural villages as contexts for mediating culture and mathematics education in the South African curriculum. *Revista Latinoamericana de Etnomatemática*, *8*(2).

[22] Mercado, M. G. M. (2020). Delivering Culturally Responsive Curriculum : A Case Study of IP School in the Philippines. *Research Gate*, *1*(1).

[23] Muslimahayati, M. (2020). Pengembangan Soal Kemampuan Berpikir Kritis Berbasis Kearifan Lokal Sumatera Selatan Pada Materi Trigonometri. *AKSIOMA: Jurnal Program Studi Pendidikan Matematika*, *9*(1). https://doi.org/10.24127/ajpm.v9i1.2459

[24] Nieveen, N., Van den Akker, J., Gravemeijer, K., & McKenney, S. (2006). Educational Design Research. In *Educational Design Research*. Routledge. https://doi.org/https://doi.org/10.4324/9780203088364

[25] OECD. (2018a). PISA 2021 Mathematics Framework (Draft). In *OECD Publishing*.

[26] OECD. (2018b). *PISA 2022 Mathematics Framework*. https://www.oecd.org/pisa/publications/pisa-2021-assessment-and-analytical-framework.htm

[27] OECD. (2023). *PISA 2022 Results (Volume I)*. OECD. https://doi.org/10.1787/53f23881-en

[28] Pangaribuan, F., Simamora, R., & Pangaribuan, L. R. (2023). Pengembangan Soal Matematika Setara PISA pada Konteks Jajanan Daerah Sumatera Utara. *JIMPS: Jurnal Ilmiah Mahasiswa Pendidikan Sejarah*, *8*(4), 3791–3800. https://doi.org/10.24815/jimps.v8i4.26553

[29] Prahmana, R. C. I., & D'Ambrosio, U. (2020). Learning geometry and values from patterns: Ethnomathematics on the batik patterns of yogyakarta, indonesia. *Journal on Mathematics Education*, *11*(3). https://doi.org/10.22342/jme.11.3.12949.439-456

[30] Pratiwi, I. (2019). Efek Program PISA Terhadap Kurikulum di Indonesia. *Jurnal Pendidikan Dan Kebudayaan*, *4*(1), 51–71. https://jurnaldikbud.kemdikbud.go.id/index.php/jpnk/article/view/1157/482

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[31] Putra, Y. Y., Zulkardi, Z., & Hartono, Y. (2016). Pengembangan Soal Matematika Model PISA Level 4, 5, 6 Menggunakan Konteks Lampung. *Kreano, Jurnal Matematika Kreatif-Inovatif*. https://doi.org/10.15294/kreano.v7i1.4832

[32] Putri, R. I. I., & Zulkardi. (2020). Designing PISA-Like Mathematics Task using Asian Games Context. *Journal on Mathematics Education*, *11*(1), 135–144. https://doi.org/10.22342/jme.11.1.9786.135-144

[33] Putri, R. I. I., & Zulkardi, Z. (2019). Designing Jumping Task on Percent using PMRI and Collaborative Learning. *International Journal on Emerging Mathematics Education*, *3*(1). https://doi.org/10.12928/ijeme.v3i1.12208

[34] Ramadhani, R., Syahputra, E., & Simamora, E. (2025). Enhancing Students' Informal Statistical Reasoning Skills Through The Development of A Worksheet Rooted in the Nias Culture. *Mathematics Teaching-Research Journal*, *17*(2). https://s3.amazonaws.com/files.commons.gc.cuny.edu/wp-content/blogs.dir/34462/files/2025/05/5-Enhance-informal-reasoning-.pdf

[35] Rawani, D., Putri, R. I. I., & Hapizah. (2019). PISA-like mathematics problems: Using taekwondo context of Asian games. *Journal on Mathematics Education*, *10*(2), 277–288. https://doi.org/10.22342/jme.10.2.5243.277-288

[36] Rokhima, N., Pamungkas, D., & Prahmana, R. C. I. (2023). Pengembangan Soal Matematika Tipe PISA Menggunakan Konteks Tugu Yogyakarta. *Jurnal Pendidikan Dan Pembelajaran Matematika Indonesia*, *12*(1), 49–59. https://ejournal2.undiksha.ac.id/index.php/JPM/article/view/2110/1170

[37] Selamet, J. (2018). Indonesian batik translation: A case study. *International Journal of Visual Design*, *12*(3). https://doi.org/10.18848/2325-1581/CGP/v12i03/11-17

[38] Sofliatin, U. K., Fiantika, F. R., & Rusminati, S. H. (2024). *Exploring Pekalongan Batik Motifs as Geometry Learning Materials for Elementary School Students*. *10*(2). https://doi.org/10.30870/jpsd.v10i2.28337

[39] Stacey, K. (2011). The PISA view of mathematical literacy in Indonesia. *Journal on Mathematics Education*, *2*(2). https://doi.org/10.22342/jme.2.2.746.95-126

[40] Stacey, K., & Turner, R. (2015). Assessing mathematical literacy: The PISA experience. In *Assessing Mathematical Literacy: The PISA Experience*. https://doi.org/10.1007/978-3-319-10121-7

[41] Suherman, S., & Vidákovich, T. (2024). Relationship between ethnic identity, attitude, and mathematical creative thinking among secondary school students. *Thinking Skills and Creativity*, *51*. https://doi.org/10.1016/j.tsc.2023.101448

[42] Susanti, S. A., & Budiarto, M. T. (2020). Etnomatematika Batik Jonegoroan ditijau dari Aspek Literasi Matematis. *Media Pendidikan Matematika*, *8*(2). https://doi.org/10.33394/mpm.v8i2.3092

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[43] Tessmer, M. (1993). Planning and Conducting Formative Evaluations: Improving the Quality of Education and Training. In *Planning and Conducting Formative Evaluations*. Kogan Page. https://www.taylorfrancis.com/books/mono/10.4324/9780203061978/planning-conducting-formative-evaluations-tessmer-martin

[44] Trixie, A. A. (2020). Filosofi Motif Batik sebagai Identitas Bangsa Indonesia. *Folio*, *1*(1). https://journal.uc.ac.id/index.php/FOLIO/article/download/1380/1148

[45] Utari, R. S., & Gustiningsi, T. (2021). Developing of Higher Order Thinking Skill in Relation and Function to Support Student's Creative Thinking. *Jurnal Pendidikan Matematika*, *15*(1), 49–60. https://doi.org/https://doi.org/10.22342/jpm.15.1.12876.49-60

[46] Van den Akker, J., Bannan, B., Kelly, A. E., Nieveen, N., & Plomp, T. (2007). *An Introduction to Educational Design Research*. https://doi.org/https://ris.utwente.nl/ws/portalfiles/portal/14472302/Introduction_20to_20education_20design_20research.pdf

[47] Wahyuni, I., Firnanda, D. T. F., & Munawaroh, L. (2023). Muatan Soal Literasi Buku Matematika Kurikulum Merdeka Kelas X Berdasarkan PISA. *Jurnal Pendidikan Matematika Dan Matematika*, *7*(2), 199–210. https://doi.org/10.36526/tr.v%vi%i.2873

[48] Widjaja, W. (2011). Towards Mathematical Literacy in the 21st Century: Perspectives from Indonesia. *Southeast Asian Mathematics Education Journal*. https://doi.org/10.46517/seamej.v1i1.12

[49] Wijaya, A., van den Heuvel-Panhuizen, M., & Doorman, M. (2015a). Opportunity to Learn Context-Based Tasks Provided by Mathematics Textbooks. *Educational Studies in Mathematics*, *89*, 41–65. https://doi.org/10.1007/s10649-015-9595-1

[50] Wijaya, A., van den Heuvel-Panhuizen, M., & Doorman, M. (2015b). Teachers' Teaching Practices and Beliefs Regarding Context-Based Tasks and Their Relation with Students' Difficulties in Solving These Tasks. *Mathematics Education Research Journal*, *27*, 637–662. https://doi.org/10.1007/s13394-015-0157-8

[51] Zulkardi, & Putri, R. I. I. (2019). *New School Mathematics Curricula, PISA and PMRI in Indonesia*. https://doi.org/10.1007/978-981-13-6312-2_3

[52] Zulkardi, Z., & Kohar, A. W. (2018). Designing PISA-Like Mathematics Tasks in Indonesia: Experiences and Challenges. *Journal of Physics: Conference Series*, *947*(1). https://doi.org/10.1088/1742-6596/947/1/012015

[53] Zulkardi. (2002). Developing a Learning Environment on Realistic Mathematics Education for Indonesian Student Teacher [Thesis University of Twente]. In *Thesis University of Twente, Enschede. - With refs. - With summary in Ducth ISBN 90 365 18 45 8 Subject*. https://research.utwente.nl/en/publications/developing-a-learning-environment-on-realistic-mathematics-educat

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

Bridging the Gap: Understanding Students' Struggles with Algebraic and Graphical Representations of Functions

Santosh Pathak

Department of Mathematics, University of Utah Asia Campus, Incheon, South Korea
spathak2039@gmail.com

Abstract: Students often find it challenging to move between algebraic and graphical representations of functions, which can lead to a fragmented understanding and make problem-solving more difficult. Research and classroom observations suggest that putting too much emphasis on algebraic procedures may limit students' ability to think graphically and connect different representations. In this study, we looked at how well students could use both algebraic and graphical approaches to solve function-related problems. We also explored their preferences for different representations and how aware they were of alternative strategies. To do this, we gave a five-question survey to 300 Grade 12 STEM students in Nepal. The survey included a variety of function types and required students to reason both symbolically and visually. The results showed a strong preference for algebraic methods, while performance on graph-based tasks was significantly lower. Many students were not even aware that graphical strategies were an option and had trouble transferring their knowledge between the two forms. These findings point to a clear need for changes in teaching and curriculum design. A better balance between symbolic and visual reasoning—including more graph-based tasks, the use of visualization tools, and training to build metacognitive skills—could help students develop a more flexible and deeper understanding of mathematics.¶

Keywords: Functions and Graphs, Algebraic and Graphical Approach, Problem-Solving, Mathematics in Nepal

INTRODUCTION

Understanding functions through both algebraic and graphical representations is essential in high school mathematics. Functions are considered central to mathematics education and foundational for more advanced topics such as calculus and statistics (Dubinsky & Harel, 1992; Romberg et al., 2012). Algebraic forms support symbolic manipulation, while graphs offer intuitive insights into behavior such as *trends*, *intercepts,* and *slopes* (NCTM, 2000; Schoenfeld et al., 1993)

Despite their importance, many students enter higher education with a weak grasp of functions, often due to insufficient focus on graphical reasoning in earlier education (Durant & Garofalo, 1994). Even those who encounter functions in high school struggle to understand them deeply

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

(Leinhardt et al., 1990; Markovits et al., 1988). Students typically prefer algebraic manipulation over graphical interpretation (Hitt, 1998; Knuth, 2000a), a tendency linked to curricular and instructional practices that emphasize procedures over conceptual understanding (Booth & Newton, 2012).

Graphical literacy is essential across disciplines. Shuard and Neill (1977) highlight that students unable to interpret graphs are disadvantaged, especially since graphs serve roles ranging from data recording to predicting behavior. However, differing views between mathematics and science educators on the role of graphical understanding can lead to inconsistencies in instruction (Booth, 1981).

The ability to move between algebraic and graphical representations—known as representational fluency—is a critical skill for mathematical reasoning and problem-solving (Lesh et al., 1987). Yet research shows students often fail to connect these forms, resulting in fragmented understanding (Even, 1990; Leinhardt et al., 1990). This gap highlights the importance of *metacognitive awareness* and instructional approaches that emphasize multiple representations (Dubinsky & Harel, 1992; Shuard & Neill, 1977). Moreover, developing this fluency supports cognitive growth as students transition to abstract thinking (Piaget, 1972) and builds communication skills necessary for mathematical literacy (CCSSM, 2010). By helping students interpret functions across symbolic systems, educators can promote deeper, more adaptable mathematical understanding.

Theoretical Framework

This study is grounded in a representational perspective on mathematical understanding, which emphasizes the central role of external representations—such as algebraic symbols, graphs, diagrams, and verbal descriptions—in the learning and use of mathematics. Rather than perceiving mathematical *objects* as directly accessible or intuitive, this perspective suggests that such *objects* are constructed and understood through representational systems (Duval, 1999, 2017). Each *system*, or *register*, provides a distinct way of expressing mathematical ideas and contributes uniquely to conceptual development.

For instance, the function $f(x) = 2x + 3$ can be expressed algebraically or visualized as a straight line on a Cartesian plane. Effective learning involves not only the ability to perform procedures within a single *register* (e.g., solving $f(1) = 5$) but also the ability to translate between *registers*—such as moving from a symbolic expression to a graphical representation, or vice versa. This distinction between *treatment* (within-register operations) and *conversion* (*cross-register translations*) is crucial in understanding students' reasoning (Duval, 1999; Pavlopoulou, 1993).

Empirical studies show that students frequently face challenges in coordinating algebraic and graphical representations. Many students can carry out symbolic manipulations but struggle to interpret or generate corresponding graphs, often treating representations as disconnected entities rather than as linked expressions of the same concept (Lobato et al., 2012; Wilkie, 2024).

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

Particularly difficult is the reverse *conversion*—from graph to equation—which demands a conceptual grasp that goes beyond procedural skill (Arnal-Palacián et al., 2022; Lesh et al., 1987; Schliemann et al., 2021). This asymmetry in representational fluency suggests that some *conversions* are cognitively more demanding or less intuitive than others.

Students often rely heavily on procedural strategies like table-making or formula substitution, which may allow them to answer specific questions but hinder their ability to recognize underlying structures or make generalizations (Lobato et al., 2012; Wilkie, 2024). These difficulties become especially visible in applied contexts—such as motion problems—where students must relate changes over intervals (e.g., speed or acceleration) to features of a graph. However, rather than attending to rates of change, many focus on accumulated values (such as total distance), which can lead to a superficial understanding of function behavior. Similarly, students may fail to grasp how parameters in a quadratic function like $f(x) = ax^2 + bx + c$ influence the graph's shape and position, suggesting weak connections between symbolic manipulation and visual interpretation (Arnal-Palacián et al., 2022; Wilkie, 2024). Graphs are often viewed at a surface level, reproduced by memory or habit rather than interpreted meaningfully (Duval, 1999).

Relevance to this Study

The representational perspective described above is central to this study, as it helps explain the specific challenges students faced when working with algebraic and graphical representations of functions. Many students relied mostly on algebraic methods and either overlooked graphs or weren't sure how to use them effectively. Their difficulties in connecting equations to graphs—and in interpreting key features like *domain, intercepts*, and *turning points*—reflect well-known struggles with moving between different forms of representation (Lobato et al., 2012; Wilkie, 2024).

This framework also shaped the study's analysis. It guided the examination of how well students could work within a single *register* (e.g., solving an equation or reading a graph) and how effectively they could move between *registers*. Particular attention was given to the direction of these translations, such as whether students found it easier to graph an equation or to write an equation based on a graph. Finally, the study used this lens to assess students' development of representational fluency, a key aspect of flexible and meaningful mathematical thinking.

By highlighting where students struggle and why, this perspective provides useful insights for teaching—especially in helping students make stronger connections between algebra and graphs.

Objective

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

The objective of this study is to investigate students' ability to navigate between algebraic and graphical representations of functions, and to identify the *cognitive* and *instructional* barriers that hinder this transition. Specifically, the study aims to assess students' problem-solving strategies, determine their representational preferences, and evaluate their metacognitive awareness of alternative solution methods. By analyzing responses from 300 grade 12 STEM students in Nepal, the study seeks to uncover the extent to which students can flexibly apply both symbolic and visual reasoning when working with functions. The findings are intended to inform instructional practices and curricular reforms that promote a more balanced and conceptually grounded understanding of functions.

METHOD

This section outlines the participants, the design of the assessment instrument, and the approach taken to analyze student responses, with the aim of understanding students' use of algebraic and graphical representations in problem-solving.

Sample and Context

The study was conducted with 300 Grade 12 STEM students from various high schools in Kathmandu, Nepal. These students had completed instruction in graphing functions as part of their Grade 11 curriculum (*linear* and *quadratic* functions are also part of Grade 10 curriculum) and had some exposure to applying these concepts in calculus. The selection was made using a random sampling method to ensure a diverse and representative group across schools. In the *Nepali education system*, there is a strong emphasis on algebraic manipulation, and graphical reasoning often receives less instructional time. As a result, students were expected to be more comfortable with symbolic procedures than with visual or graphical approaches, making this group suitable for exploring the contrast in representational fluency.

Instrument

To assess students' ability to work with and move between algebraic and graphical representations, a paper-based questionnaire was developed. It contained five problems with a total of fifteen tasks, each involving different function types: *linear, quadratic, square root,* and *rational*. The tasks were designed to evaluate both procedural fluency and conceptual understanding through algebraic and graphical reasoning. Some questions offered multiple ways to solve the problem, inviting students to reflect on different approaches. The questionnaire was inspired by earlier studies on students' function understanding (Hitt, 1998; Knuth, 2000a; Van Dyke & White, 2004), and students were encouraged to show their reasoning and reflect on alternative strategies when appropriate.

Data Analysis

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

Student responses were analyzed using a *cognitive framework* that integrates both *constructivist* and *metacognitive* perspectives. From a *constructivist* viewpoint (Gale & Steffe,1995), we examined how students built upon their prior knowledge of algebra and graphing to approach unfamiliar problems. At the same time, a *metacognitive* lens (Hacker et al., 2009) allowed us to assess how students monitored and adapted their strategies—particularly their awareness of different solution methods and their flexibility in choosing between them. Each task was evaluated in terms of *metacognitive knowledge* (what students knew and how accurately they applied it) and *metacognitive regulation* (how they planned, monitored, or revised their approach). For example, in Question 1, *metacognitive knowledge* was reflected in students' understanding of the equation $2x + 3y - 6 = 0$ and their ability to apply algebraic and graphical methods. *Metacognitive regulation* was evident in how students chose their strategy, checked their work, and responded to part (c), which explicitly asked them to reflect on alternative approaches. Responses were coded as *correct*, *partially correct*, or *incorrect*, and attention was given to whether students demonstrated an awareness of multiple representations and switched between them appropriately. This combined framework supports a deeper understanding of students' problem-solving processes and provides insights into how instruction can better balance symbolic and visual reasoning for more effective mathematical learning.

RESULTS

Question 1

The graph of the equation $2x + 3y - 6 = 0$ is given below in Figure 1.

(a) What value of x gives $y = 2$?
(b) What value of y gives $x = 3$?
(c) Can you answer the questions in part (a) and (b) by a method different than the one you used? Explain your answer.

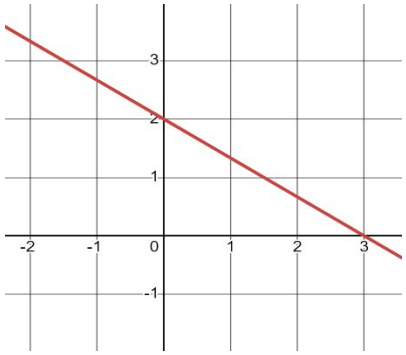


Figure 1: Graph of a straight line

The problem assesses whether students can interpret a linear equation both symbolically (through algebraic computation) and visually (by analyzing the graph in Figure 1). Students must realize that the same problem can be solved using either an algebraic approach or a graphical method. The third part of the question explicitly asks students to reflect on alternative methods.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

Out of 300 students, 290 participated students responded at least one part of the question. Of these, 195 students (about 67% of the respondents) chose to answer parts (a) and (b) algebraically, showing a clear preference for algebraic methods, even though the graphical method is often quicker and easier. Only 95 students, which is nearly 33% of the 290 students chose to answer it graphically. This preference for algebra over graphical solutions was consistent across both parts. In part (c), only 178 students responded to the question. Of the 195 who answered part (a) and (b) algebraically, only 83 responded to the question in part (c), and of these 83, only 60 students were aware of the alternative approach for part (a) and (b). Hence 155 (about 52%) students were aware of graphical and algebraic approach of solving the problem whereas 145 (about 48%) students either had no idea at all or no graphical approach of solving the problem. Some students expressed this lack of awareness of the graphical approach with statements like the following:

- "No, I cannot answer the question by a different method."
- "There is no other method to answer the question in part (a) and (b)."
- "Sorry, I have no clue."
- "It can't be explained, sorry."

This indicates that while students overwhelmingly began with algebraic methods, many of them only realized the possibility of a graphical solution when confronted with part (c). As one student noted, "The easiest way to do it is by putting $x = 0$ $y = 0$,"highlighting the strong reliance on algebraic techniques. The responses in part (c), particularly the statements like "I have no clue" and "There is no other method to answer the question," suggest a lack of *metacognitive regulation*. Many students were not consciously aware that graphical solutions were an option, indicating that they were unaware of alternative methods beyond the algebraic ones they were comfortable with. The bar graph below (Figure 2) summarizes the outcome of the students' responses to question 1.

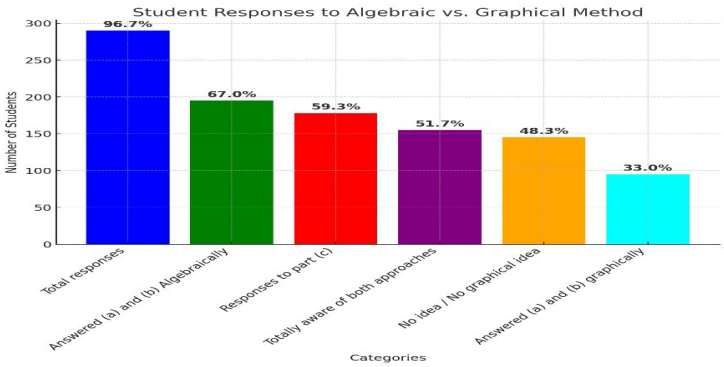


Figure 2. Students' Responses to Question 1.

Figure 3 below (from left to right) consists responses from three students in which first student demonstrates a clear understanding of the question and provides a correct answer. The second student initially applies the algebraic method but recognizes the alternative approach upon

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

reaching part (c). In contrast, the third student's response reveals a lack of awareness or knowledge of the alternative approach for solving the problem.

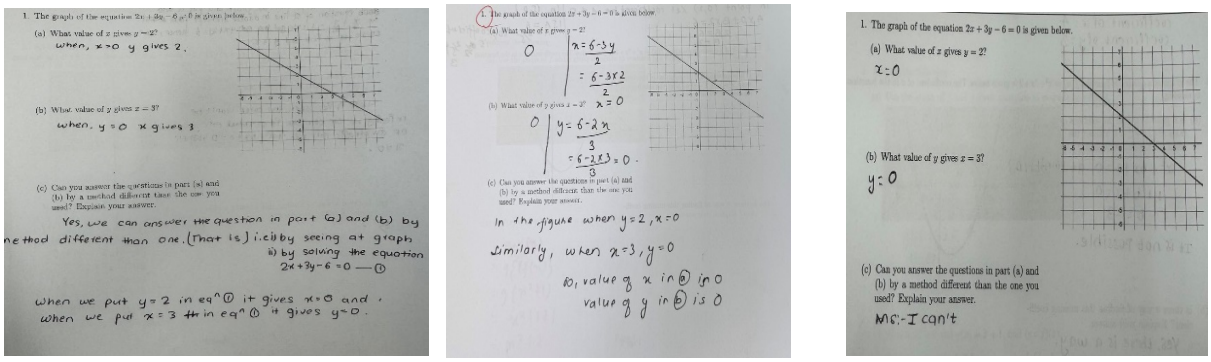


Figure 3. Sample Responses to Question 1.

Question 2

The graph of a linear function $?x + ?y = 6$ is given below (Figure 4).

(a) Can you find some obvious solution of the given equation without the missing coefficient? Explain your answer.
(b) If possible, find the missing coefficient of the equation. Explain your answer.

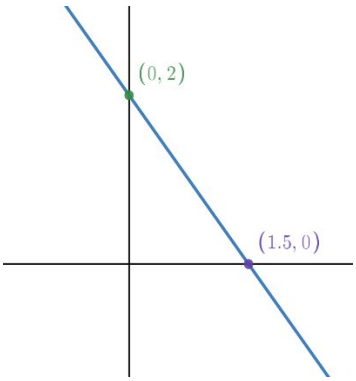


Figure 4. Graph of a straight line.

The purpose of this problem was to assess whether students can relate the graph to the equation (e.g., recognizing intercepts as direct solutions), whether they express a preference for graphical or algebraic methods and justify their choice, and if they demonstrate *metacognitive regulation* by identifying mistakes and adjusting their approach accordingly.

Out of 300 students, only 44 (14.67%) answered correctly, while 51 (17%) provided partially correct answer of part (a); 95 students (about 32%) gave either correct or partially correct answer. Of these 95 students, only 20 students (6.6%) demonstrated a clear understanding of the connection between the equation and its graph. The majority—205 students (68%)—showed little understanding of the question or its graphical representation. Many students were con-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

fused by the missing coefficients of x and y in the equation. Some identified the midpoint of the given line as the solution, while others suggested that a complete equation would allow them to check the points on the graph by substitution, though they failed to recognize that these points are solutions to the equation. In part (b), which requires an algebraic approach—use of *slope-point formula* of a straight line—to find the missing coefficients, about 55% students—significantly higher than in part (a)—answered either correctly or partially correctly, —reflecting their strong reliance on algebraic skills in problem-solving.

Many students believed that finding the "*obvious*" solutions required solving for the coefficients, with one student commenting, “Without solving, we can’t find the missing coefficients.” This reliance on algebra is further highlighted by a student who wrote, “We can’t find the obvious solution without the missing coefficients, as they determine the constant nature of the graph.” Interestingly, a few students first found the equation of the line using the *two-point formula* and then verified the given points on the graph, demonstrating their heavy reliance on algebraic skills.

Students showed more confidence with algebraic methods than with interpreting graphs. While their metacognitive skills helped with algebra, they struggled to reflect on and understand graphical representations, which caused confusion. This suggests a need for more training to encourage flexible problem-solving and improve graph interpretation. The bar graph below (Figure 5) summarizes the results for question 2.

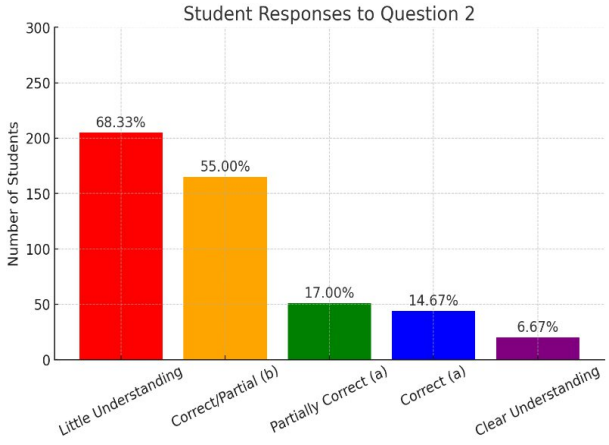


Figure 5: Students Responses to Question 2.

The following Figure 6 (from left to right) includes responses from three students. The first student demonstrates a clear understanding of *solution points*. The second student shows some confusion about the concept, while the third student completely misunderstands the idea of *solution points*.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

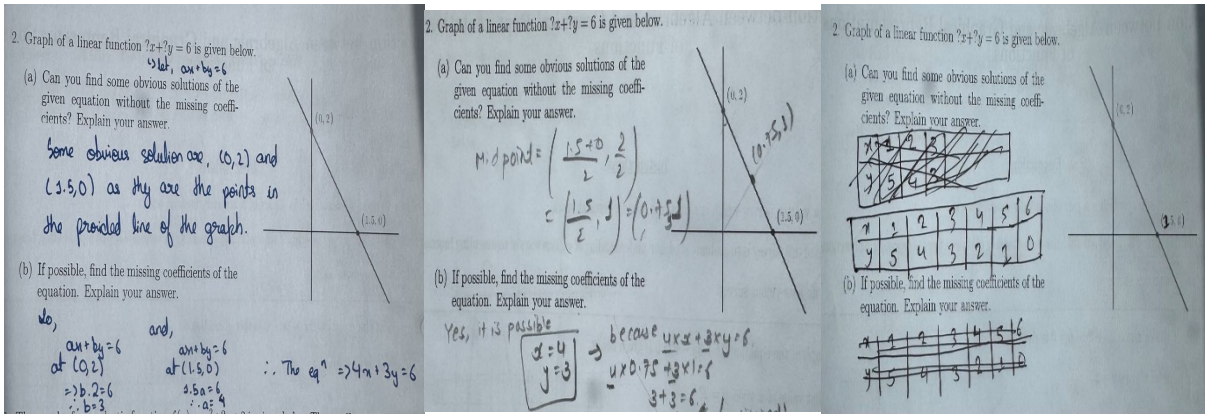


Figure 6. Sample Responses to Question 2.

Question 3

Refer to the graph below in Figure 7 to answer the questions.

a) $f(4) = g(-2)$
b) $f(4) = g(4)$
c) $f(-2) = g(-2)$
d) More information is necessary to say something definitive about $f(x)$ and g(x).

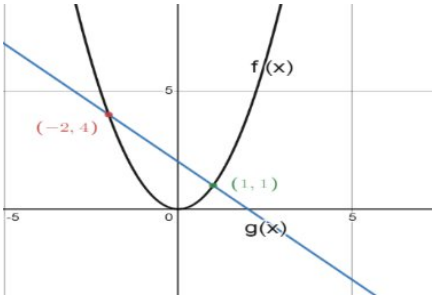


Figure 7. Graph of a straight line and parabola.

The question requires students to interpret the graph of two functions, $f(x)$ and $g(x)$ in Figure 7 and make connections between their values at specific points. According to the *constructivist* framework, students will build their understanding through interaction with the graph, recognizing how the two functions relate to one another. The focus would be on their ability to construct meaning from the graph and draw conclusions based on the points of intersection.

Out of the 300 students, only 31 (about 10%) answered the question correctly. A surprising 52% of students were completely unsure of what the question was asking, with one student

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

commenting, “I don’t know. Very bad question.” Additionally, 66 students (22%) selected choice (d), believing that there was insufficient information to answer the question. Around 12% chose answers (a) or (b). In total, 90% of students either selected an incorrect answer, gave an unclear response, or did not attempt the question at all. This outcome is particularly surprising, given that many students struggled to understand the *domain* and *solution points* of a function, even when a graphical representation was provided. Some students even attempted to find the algebraic form of the function from the graph to determine the correct answer. Despite the relatively low level of difficulty of the problem, the percentage of correct responses was alarmingly low. This result mirrors the trends observed in previous problems, where students showed a strong reluctance to use graphical representations and instead relied heavily on algebraic methods. The bar graph below (Figure 8) gives an outlook of the results of question 3.

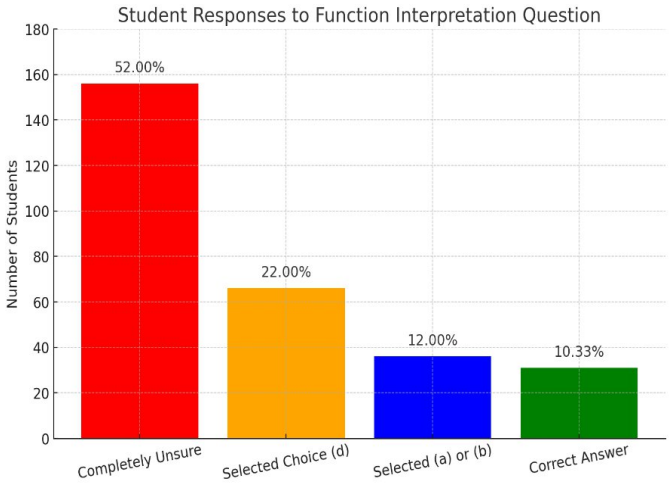


Figure 8: Students’ Responses to Question 3.

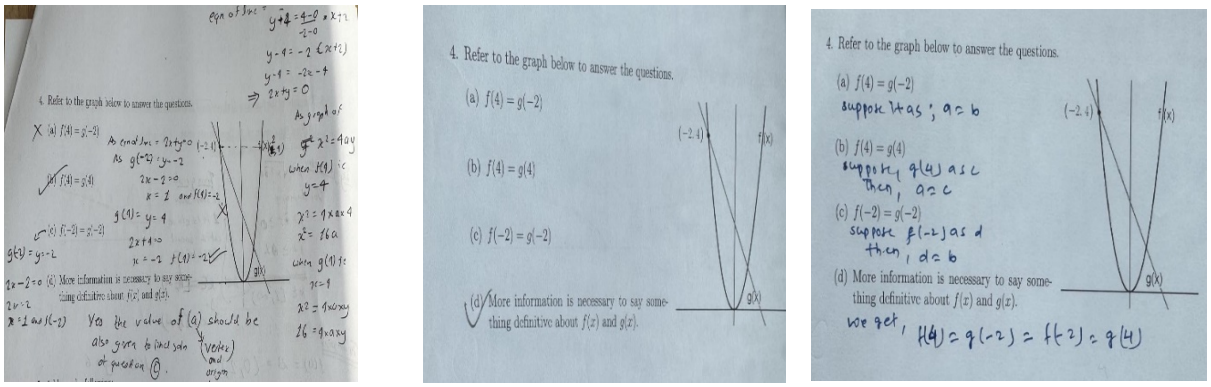


Figure 9. Sample responses to question 3.

Figure 9 (left to right) presents responses from three students, each displaying different approaches to the problem. The students either provide incorrect answers or attempt to solve the problem algebraically, despite a graphical representation of the function being provided. The

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

first and third responses involve algebraic methods, while the middle response is an incorrect answer. All three responses fail to demonstrate the necessary *metacognitive regulation* in problem-solving.

Question 4

The graph of the function $f(x) = \sqrt{(x-1)}$ is given alongside in Figure 10. Answer the following questions.

a) Find the domain of f. Demonstrate or explain how you found it.
b) Find $f(0), f(1), f(5), f(6)$.
c) Find the x values on the domain of f that correspond to the points A, B, C, D. Include details in your answer.

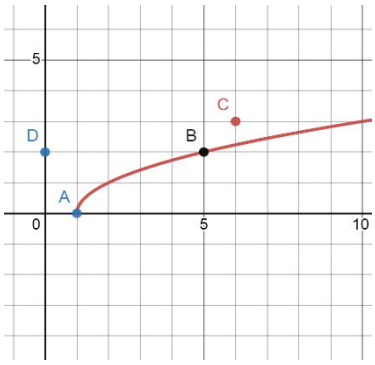


Figure 10. Graph of a square root function.

This task's problem analysis framework examines how students approach problem-solving, focusing on identifying key strategies and evaluating various methods. Students must apply their knowledge of domains, evaluate function values, and understand the connection between algebraic expressions and their graphs—essential concepts in high school mathematics (National Council of Teachers of Mathematics [NCTM], 2000; Common Core State Standards Initiative [CCSSI], 2010). These concepts are introduced in grade 10 and 11 in Nepal and must be carried forward to grade 12.

Based on students' responses in part (a), only 42 out of 300 students (14%) understood the domain and attempted the problem graphically. In contrast, 79 students (26%) understood the concept but approached the problem algebraically. However, 179 students (60%) had no clear understanding of how to approach the problem. This highlights a significant gap in students' understanding of fundamental concepts like domain, even when different function representations are provided.

The aim of the question in part (b) was to assess whether students recognize that $x = 0$ is excluded from the domain found in part (a) and to evaluate their willingness to use the graph (Figure 10) to find values at various points within the domain. The survey results show that only 22 students (about 7.5%) correctly identified that $f(0)$ is undefined, as $x = 0$ is not part of the domain. Meanwhile, three out of four students attempted the problem without understand-

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

ing this concept, simply plugging in x-values into the equation. Many of these students recognized that $f(0)$ resulted in an imaginary number and responded with terms like "undefined," "impossible," or "infinity." However, there was insufficient evidence to suggest they understood that $x = 0$ is not part of the domain, and thus $f(0)$ cannot be computed either algebraically or graphically. In contrast, more than half of the students—189 out of 300—had no idea how to approach the problem. They either did not attempt it or provided incorrect answers.

As seen in part (b), a similar pattern emerged in part (c) too. Only 16 students (about 5.5%) met the expectations of the problem and answered correctly. In contrast, 103 students (approximately 34%) answered the question without realizing that points C and D cannot correspond to any x-values, as they are not *solution points* of the equation. One student wrote, "By studying the graph, I found the values of x for points A, B, C, and D," despite not understanding that C and D are not on the graph. Meanwhile, 63% of students either had no idea how to approach the problem or gave incorrect answers. Below is an example of responses from three students.

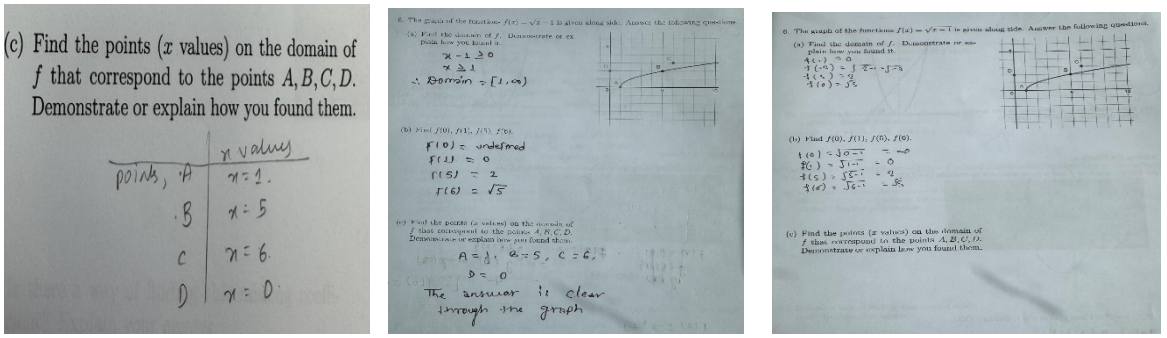


Figure 11. Sample Responses to Question 4.

Figure 11 (left to right) displays three responses to Question 5, illustrating that the students struggled to understand the domain of the function, despite the graphical representation provided. The second and third responses include the correct answer to part (a), showing sufficient evidence of using the algebraic equation rather than the graph.

In conclusion, there is a significant gap in students' understanding of the domain of the function and how to evaluate points on the graph. At the same time, a large portion of students failed to recognize key *domain restrictions* and the inability of certain points (C and D) to correspond to any x-values. Also, many students struggled to use the graph effectively, and there was widespread confusion around function evaluation and domain concepts. The bar graph below (Figure 12) summarizes the outcome of the responses to the question.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

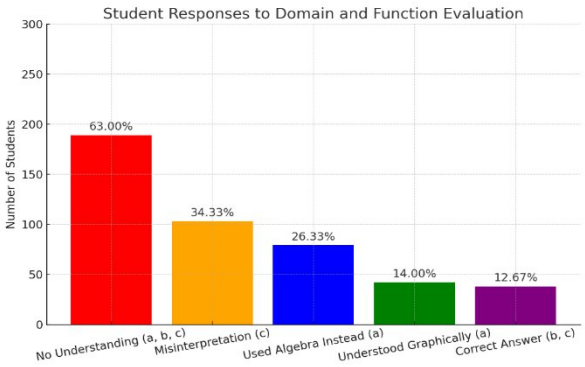


Figure 12. Students' Responses to Question 4.

Question 5

Answer the following questions. Try to include as much detail as possible in your answer.

a) Find the domain of the function $f(x) = \frac{x-2}{x+2}$.

b) Find the domain of the function whose graph is given below in Figure 13.

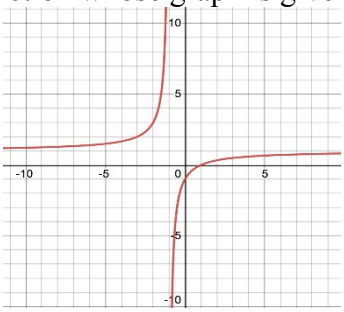


Figure 13. Graph of a Rational Function.

c) Which question (a) or (b) did you find easier to answer? Explain.

The objective of this task is to assess students' understanding of the domain of function, including *restrictions* caused by undefined values. The *cognitive demand* requires algebraic reasoning to determine where $f(x)$ is undefined, interpretating a graphical representation to determine *domain restriction*, and evaluates students' perceived difficulty in switching between algebraic and graphical methods.

The outcome of the survey indicates, only 75 students (25%) correctly identified the domain in part (a), and an unexpectedly low number—only 25 students (about 9%)—were able to identify the correct domain in part (b) where a graphical representation (Figure 13) of the function was provided. This low number is striking, especially given that 275 students struggled to approach the problem in part (b). In part (c), 27% of students felt comfortable finding the domain when presented with the algebraic form, while only about 10% found the graphical

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

method easier. Surprisingly, 62% of respondents did not select any method as a compatible approach for finding the domain, and only 2% felt equally comfortable with both methods.

These results suggest that the majority of students lack a clear understanding of the domain of a function, regardless of how the function is represented. A clear pattern emerged showing that more students were comfortable determining the domain when given the algebraic representation. Many students noted that part (a) was easier because the equation was directly provided. The following are a few student comments from part (c) that reflect this preference:

- "(a) is a bit easier than (b) because the direct function is given in this question. But in (b), the function should first be determined."
- "(a) is easier to answer since the function is straightforward, but (b) is more difficult since the function in the graph is not easy to identify."
- "(a) was easier because the graph is difficult to understand."

From these comments, it's clear that many students find it easier to determine the domain when the function is given algebraically. They expressed difficulty in part (b) because they first needed to determine the function from the graph before identifying its domain.

Additionally, some students commented on how the education system in Nepal places greater emphasis on algebraic and computational problem-solving methods, leaving students less comfortable with graphical approaches:

- "I found (a) easier because we have done this type of problem before using just one method. We are not as familiar with the graphical approach since our education system focuses more on algebraic methods."
- "(a) is easier because solving algebraically is easier, and we're not used to graphical assignments."
- "I found (a) easier because it was taught more in class, and the teacher focused more on part (a) than part (b)."

Students showed a strong preference for algebraic methods over graphical ones when determining the domain of a function. While 25% correctly answered part (a) algebraically, only 9% succeeded with the graphical approach in part (b). Many struggled with graphs due to limited exposure, as their education emphasized algebraic computation. This highlights the need for greater focus on graphical reasoning to improve students' ability to interpret functions across different representations. Figure 14 presents a bar graph summarizing the responses to Question 5

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

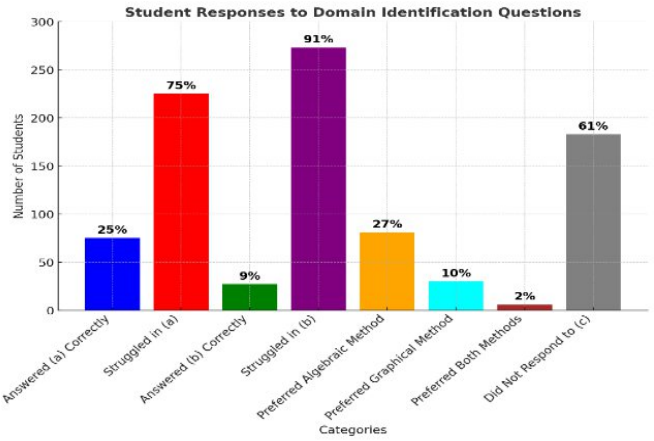


Figure 14. Students' Responses to Question 5.

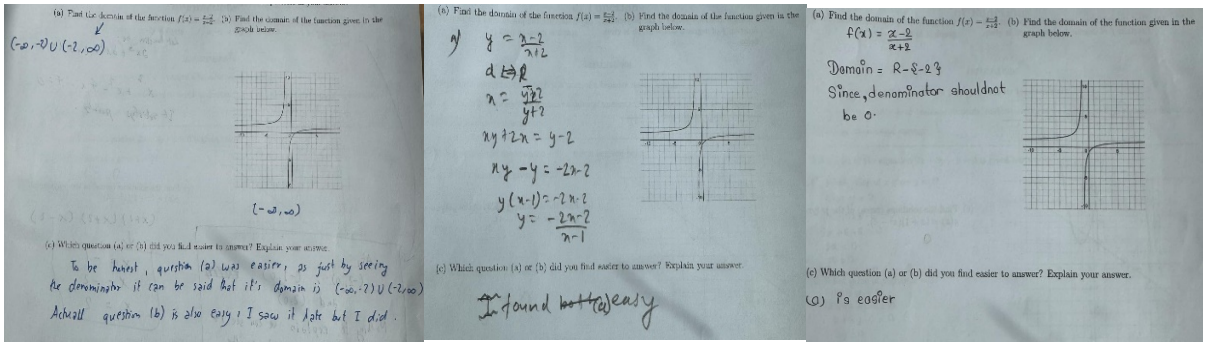


Figure 15. Sample Responses to Question 5.

The responses in Figure 15 (left to right) show that none of the three students answered part (b). They seem to believe that determining the domain is easier when an algebraic equation is provided, rather than when the information is presented graphically. In the second response, the student deviates from the question's intent by applying an algebraic method typically used to find the inverse of a function. Despite this, the student still finds part (a) easier than part (b).

DISCUSSION

The findings of this study reveal a consistent pattern in students' problem-solving approaches: a strong preference for algebraic methods and limited comfort with graphical representations. This tendency was evident across all five questions administered to grade 12 STEM students in Nepal. Even when graphical approaches were potentially more intuitive or efficient, students overwhelmingly defaulted to symbolic procedures.

In Question 1, although both methods were viable, 67% of students chose algebraic solutions, and nearly half were unaware of graphical alternatives. Responses such as "There is no other

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

method" or "I have no clue" suggest a lack of exposure to graph-based reasoning and highlight limited *metacognitive regulation*—students' ability to reflect on and evaluate their strategies (Hacker et al., 2009). These findings align with Duval's (1999, 2017) notion that students struggle to convert between *semiotic registers*, particularly from algebra to graph and vice versa.

The difficulty interpreting graphs was more pronounced in Questions 2 and 3. In Question 2, which asked students to identify "obvious" solutions from a graph, only 14.67% answered correctly. Many were confused by missing coefficients, indicating a rigid reliance on equation-based reasoning. In Question 3, where students compared values of two functions at specific points on a graph, only 10% responded correctly, while over half expressed uncertainty or frustration. These results are surprising given the relatively low cognitive demand of the tasks and suggest that even when visual information is clearly presented, many students default to symbolic thinking or abandon the task altogether.

Performance on Questions 4 and 5 reinforced this trend. When asked to determine the domain of a function, 25% answered correctly using algebraic expressions, but only 9% succeeded when a graph was provided. Many students attempted to reconstruct the function algebraically before answering, reinforcing their discomfort with visual reasoning. These behaviors are consistent with international findings (e.g., Knuth, 2000a; Knuth, 2000b; Van Dyke & White, 2004), but the lower success rates observed here suggest deeper instructional gaps in the Nepali context.

Moreover, many students appeared to have forgotten key concepts introduced in Grade 11, such as *domain restrictions, function evaluation,* and *solution points*. This supports concerns about knowledge decay and lack of review, as discussed in studies on learning retention (Bjork & Bjork, 2011; Brown et al., 2014; Rohrer & Pashler, 2007). While students' algebraic competence seemed relatively stronger, their graphical reasoning and ability to coordinate representations were notably underdeveloped.

Taken together, the results suggest that many students show a strong preference for symbolic methods and have difficulty using or interpreting graphs. Most were not aware that problems could be solved in different ways and did not reflect on which method might be more efficient or appropriate. This points to limited *metacognitive engagement*—students rarely questioned their own approaches or considered alternatives. These patterns highlight a need to better support students in making connections between representations and in thinking more deeply about their problem-solving strategies. They may also reflect broader challenges in teaching practices and curriculum design that deserve further attention.

CONCLUSIONS

The findings of this study point to several important takeaways for mathematics education, especially in countries like Nepal, where instruction and curriculum have long focused more on symbolic procedures than on helping students build conceptual understanding. If we want

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

students to develop flexible thinking and strong problem-solving skills, it's crucial to better balance algebraic and graphical reasoning in how we teach functions.

Balanced Curriculum Design

Students' strong preference for algebra—even when using a graph would have been easier—suggests that current textbooks and lessons in Nepal place a heavy emphasis on procedural fluency. A recent informal analysis of Grade 9–12 mathematics textbooks found that over 84% of the problems focus mainly on symbolic computation, with very few involving graphs. This imbalance limits students' opportunities to think visually and understand functions in multiple ways (Booth & Newton, 2012; Fey & Harel, 2004; Hiebert & Carpenter, 1992). To address this, the curriculum should incorporate more graph-based problems, not just equations; include tasks that encourage students to move between different representations (Duval, 1999); and provide opportunities for students to explain and compare different solution methods. Visual thinking should not be treated as an optional extra but should be integrated into regular lessons from the early grades onward.

Metacognitive Instruction

Many students did not seem to realize that problems could be solved in more than one way. Even when asked directly, they tended to stick with familiar methods and showed little reflection on their choices. This observation aligns with research indicating that students do not automatically develop metacognitive skills unless those skills are explicitly taught and practiced (Hacker et al., 2009; Mainali, 2021; Orhun, 2012; Schoenfeld, 1992; Tiew et al., 2023). Teachers can help build metacognition into regular lessons by asking students to explain why they chose a particular method, encouraging them to compare algebraic and graphical strategies, and using simple prompts such as "Could there be a faster way to do this?" Modeling their own thinking and decision-making during lessons is another powerful tool. In addition, teacher training should include Duval's (1999, 2017) framework on switching between representations, so teachers are better equipped to support students who struggle to connect different forms of mathematical thinking.

Use of Visualization Tools

Visualization tools like *Desmos* and *GeoGebra* can be very effective in helping students understand the behavior of functions, such as *intercepts*, *symmetry*, and overall trends (Artigue, 2009; Larkin & Simon, 1987). These tools make it easier for students to connect equations with their graphical representations and develop visual understanding. However, simply providing access to these tools is not enough; teachers need to design purposeful activities that encourage students to predict what a graph will look like before plotting it, explore how changing

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

parameters affect the graph's shape, and reflect on when a visual approach might be more appropriate than an algebraic method.

Assessment Reform

Currently, most mathematics tests reward students primarily for obtaining the correct answer through algebraic methods, which discourages the use and appreciation of visual reasoning. To address this, assessments should include problems that integrate both equations and graphs, require students to interpret graphs directly, and award credit for explaining their problem-solving strategies rather than just for arriving at the correct solution. Implementing these changes would send a clear message that understanding and reasoning are valued over mere computation (NCTM, 2000; Schoenfeld, 1992).

Support Retention

Many students struggled to recall or apply basic concepts like domain and function evaluation, despite having learned them in grades 10 and 11. This aligns with research showing that if topics are not revisited regularly, students tend to forget important material (Bjork & Bjork, 2011; Rohrer & Pashler, 2007). To address this issue, teachers can incorporate spaced practice and weave review activities into different units, design cumulative reviews that connect earlier ideas to new contexts, and begin the school year with short assessments to identify areas that need refreshing.

Local Challenges, Global Relevance

Although this study focused on Nepal, the issues are familiar to educators in many countries. Students often favor algebra over graphs because that's what they're used to (Knuth, 2000a; Van Dyke & White, 2004). What seems different here is just how rarely students successfully used graphs, which may be a result of fewer opportunities to practice or less exposure in class. Future research could look at how different teaching practices and educational systems affect students' ability to move between representations. Lessons from other countries might help inform what works best in different contexts.

In the end, these results show that improving students' understanding of functions isn't just about teaching more content—it's about teaching differently. We need to help students see that mathematics can be approached in more than one way, and that being flexible in how they think is just as important as getting the right answer. Fixing these issues will take coordinated changes to curriculum, instruction, assessment, and teacher support—but the payoff is students who are better prepared to think deeply and solve problems in a connected, meaningful way.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

REFERENCES

[1] Arnal-Palacián, M., Baeza, M. Á., & Claros-Mellado, J. (2022). Quadratic Functions Representation: Pencil and Paper vs GeoGebra. *International Journal for Technology in Mathematics Education*, *29*(3), 165-178.

[2] Artigue, M. (2009). Didactical design in mathematics education. *Nordic Studies in Mathematics Education, 14*(1), 7-18.

[3] Bjork, R. A., & Bjork, E. L. (2011). Making things hard on yourself, but in a good way: Creating desirable difficulties to enhance learning. In M. A. Gernsbacher et al. (Eds.), *Psychology and the real world: Essays illustrating fundamental contributions to society* (pp. 56-64). Worth Publishers.

[4] Booth, L. (1981). Graphs in Mathematics and Science. *Mathematics in School, 10*(4), 2–5

[5] Booth, J. L., & Newton, K. J. (2012). Conceptualizing algebraic reasoning: Developing the capacity for multiple representations. In L. R. Wixted & S. D. Zannetti (Eds.), *Teaching and learning mathematics through multiple representations* (pp. 181-206). Springer.

[6] Brown, P. C., Roediger, H. L., & McDaniel, M. A. (2014). *Make it stick: The science of successful learning.* Harvard University Press.

[7] Common Core State Standards Initiative. (2010). *Common Core state standards for mathematics*. Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

[8] Dubinsky, E., & Guershon Harel. (1992). *The Concept of function: aspects of epistemology and pedagogy.* Mathematical Association of America.

[9] Durant, K., & Garofalo, J. (1994). Teaching functions and graphing in pre-calculus mathematics. *Journal of Developmental Education, 18*(1), 18.

[10] Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. In F. Hitt & M. Santos (Eds.), *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 3–26). Cuernavaca, Morelos, México

[11] Duval, R. (2017). Understanding the mathematical way of thinking: The registers of semiotic representations. Springer International Publishing. https://doi.org/10.1007/978-3-319-56910-9.

[12] Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21*(6), 521–544. https://doi.org/10.1007/bf00315943

[13] Fey, J. T., & Harel, G. (2004). The roles of algebraic and graphical representations in understanding functions. In G. Harel & E. Dubinsky (Eds.), *Proceedings of the 12th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 109-114).

[14] Hacker, D. J., Dunlosky, J., & Graesser, A. C. (2009). *Metacognition in Educational Theory and Practice*Kotovsky (Eds.). In *Complex information processing: The impact of Herbert A. Simon* (pp. 285–318). Lawrence Erlbaum Associates, Inc.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

[15] Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). Macmillan Publishing.

[16] Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *The Journal of Mathematical Behavior, 17*(1), 123–134. https://doi.org/10.1016/s0732-3123(99)80064-9

[17] Knuth, E. J. (2000a). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education, 31*(4), 500. https://doi.org/10.2307/749655

[18] Knuth, E. J. (2000b). Understanding connections between equations and graphs. *The Mathematics Teacher*, *93*(1), 48-53.

[19] Lobato, J., Hohensee, C., Rhodehamel, B., & Diamond, J. (2012). Using student reasoning to inform the development of conceptual learning goals: The case of quadratic functions. *Mathematical Thinking and Learning*, *14*(2), 85-119.

[20] Larkin, J., & Simon, H. A. (1987). Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science, 11*(1), 65–99. https://doi.org/10.1111/j.1551-6708.1987.tb00863.x

[21] Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, Graphs, and Graphing: Tasks, Learning, and Teaching. *Review of Educational Research, 60*(1), 1. https://doi.org/10.2307/1170224

[22] Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33–40). Lawrence Erlbaum Associates.

[23] Markovits, Z., Eylon, Bat Sheva, & Bruckheimer, M. (1988). Difficulties students have with the function concept. In *The ideas of algebra, K12: 1988 yearbook* (pp. 43–60). The National Council of Teachers of Mathematics.

[24] Mainali, B. (2021). Representation in teaching and learning mathematics. *International Journal of Education in Mathematics, Science, and Technology (IJEMST), 9*(1), 1-21. https://doi.org/10.46328/ijemst.1111

[25] National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. NCTM.

[26] Orhun, Nevin. (2012). Graphical Understanding in Mathematics Education: Derivative Functions and Students' Difficulties. *Procedia - Social and Behavioral Sciences, 55*, 679–684. https://doi.org/10.1016/j.sbspro.2012.09.551

[27] Pavlopoulou, K. (1993). Un problème décisif pour l'apprentissage de l'algèbre linéaire : La coordination des registres de représentation. In R. Duval (Éd.), *Annales de didactique et de sciences cognitives*, 5, 67–93. IREM de Strasbourg.

[28] Piaget, J. (1972). *The psychology of the child*. Basic Books.

[27] Romberg, T. A., Fennema, E., & Carpenter, T. P. (2012). Towards a common research perspective. In *Integrating Research on the Graphical Representation of Functions* (pp. 1–9). Routledge.

[29] Rohrer, D., & Pashler, H. (2007). Increasing retention without increasing study time. *Current Directions in Psychological Science, 16*(4), 183-186.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

MATHEMATICS TEACHING RESEARCH JOURNAL

WINTER 2025
Vol 17 no 6

[30] Schliemann, A. D., Carraher, D. W., & Teixidor-i-Bigas, M. (2022). Teacher development structured around reasoning about functions. *International Journal of Science and Mathematics Education*, *20*(4), 793-816.

[31] Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). Macmillan Publishing.

[32] Schoenfeld, A. H., Smith, J. P., III, & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 4, pp. 55–175). Erlbaum.

[33] Shuard, H., & Neill, H. (1977). *From Graphs to Calculus (The mathematics curriculum: a critical review).* Blackie For the Schools Council.

[34] Steffe, L. P., & Gale, J. (Eds.). (1995). *Constructivism in education*. Lawrence Erlbaum Associates.

[35] Tiew, N., Teoh, S., Singh, P., Walida, S., & Surya, F. (2023). An analysis of students' understanding of algebraic concepts. *Jurnal Pendidikan Sains dan Matematik Malaysia, 13*, 86-95. https://doi.org/10.37134/jpsmm.vol13.2.8.2023

[36] Van Dyke, F., & White, A. (2004). Examining students' reluctance to use graphs. *The Mathematics Teacher*, *98*(2), 110-11.

[37] Wilkie, K. J. (2024). Coordinating visual and algebraic reasoning with quadratic functions. *Mathematics Education Research Journal*, *36*(1), 33-69.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for non-commercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Analyzing Mathematics 10 First Quarter Examination and Students' Test Taking Skill Through the Rasch Model: A Comprehensive Study

Jay A. Balasabas[1], Alvic A. Arnado[2]

[1]Bayugan National Comprehensive High School, Philippines

[2] Caraga State University, Philippines

jaybalasabas95@gmail.com

Abstract: The study aimed to facilitate a comprehensive understanding of how well the Grade 10 First Quarter Mathematics Examination aligns with students' abilities, allowing insights into item difficulty in the subject matter. This test was administered to 51 Grade 10 students from a regular class, 23 males and 28 females, at Bayugan National Comprehensive High School. The results show that two items are considered very easy for the students, while the other two are considered very difficult for their abilities. It also indicates that seventeen students were misfits (outliers), which could be interpreted as just guessing in answering the test items. At the same time, some of them could have misunderstood the items. Lastly, some of these students may have test anxiety or stress. The researcher listed some recommendations that the teacher may promote a positive attitude and belief to overcome test anxiety and build self-efficacy. The teacher could also equip students with test-taking strategies like eliminating incorrect answers and estimating answers. The teacher may review the individual response patterns of the students and look for inconsistencies or unexpected responses. Individual interviews with students are also encouraged to understand possible causes of underperformance.

Keywords: item characteristics, mathematics, person characteristics, Rasch model, test assessment

INTRODUCTION

Mathematics is connected with everyday problems and using imagination, intuition, and reasoning to find new ideas and solve puzzling problems (Khan, 2015). In the study conducted by Al Ali and Rehab (2020), Al Ali stated that, according to the National Center for Assessment in Higher Education, there is a significant weakness in the basic mathematical principles of most students. Many students fear and worry about mathematics and consider it complex, abstract, and uncreative

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

(Mutodi & Ngirande, 2014). According to Ngunjiri (2022), students need to know how they do as they learn. This is because knowledge of current understanding gives students a source of awareness of their achievement, which may motivate them to learn more. Thus, mathematics teachers must assess students' learning and give them immediate feedback (Anderson, 1993). Furthermore, assessments should be given to students as frequently as possible using written and oral tests, observation, and projects where possible with feedback given accordingly (Gore, 2000).

According to Isnaini et al. (2019), student learning assessment activities are among the essential tasks teachers must perform. In education, an assessment of student learning achievements is conducted to determine students' progress in the curriculum that has been taught. One effort to assess students is to give examinations. However, sometimes, giving questions that are too difficult or too easy makes it easier for lecturers to distinguish students' abilities. Therefore, an analysis of exam questions is needed in the hope that the exam results present students' abilities.

Assessment in education is a fundamental process of educational activities. With that, it is easier to know whether the progress of learning has been achieved. According to Sumintono (2017), instrument testing and determining students' abilities in educational assessments are fundamental. An analysis that can result in more precise measurements (produce an equal-interval scale) will determine the quality of the investigation results and the improvement of the educational process to help students learn. The Rasch model can help teachers assess and improve the quality of the analysis performed because it applies the appropriate basic principles of data processing. This is because the Rasch model addresses five objective measurement requirements.

Rasch model, according to Isaini et al. (2019), is a dichotomous scoring model that merely has two categories, namely the correct answer with a score of 1 and the incorrect answer with a score of 0. According to Al Ali and Shehab (2020), the Rasch model is a probabilistic unidimensional model that confirms that the more manageable the question, the more likely the student will respond correctly to it, and also, the more able the student, the more likely he will pass the question. In constructing tests using this model, one frequently discards those items that do not fit and that do not meet the assumptions of the Rasch model (Wright & Stone, 1979). Rasch analysis gives the reliability coefficient for each person and item (El-Korashy, 1995). A study by Kurniawan and Mardapi (2015) showed that the Rasch model provides complete information about test items, including their difficulty level.

A study by Ramadhani, Saragih, and Napitupulu (2022) demonstrated the practical use of the Rasch model in evaluating students' statistical reasoning abilities within real classroom contexts. Their research highlights how Rasch analysis provides detailed diagnostics on item difficulty and student performance, supporting its value in developing meaningful and adaptive instruction, a goal shared by the current study.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

In the Philippines, teachers and lawmakers are making ways to improve the educational system through continued assessments. Magno (2010) highlights the diverse applications of educational assessment, measurement, and evaluation, leading to a field shaped by numerous influences. The passage of the Philippine Psychology Act has further clarified the roles of psychologists and teachers in school-based assessment. Current research suggests that the field is poised for continued growth and development in the future. Filipino researchers utilized a modern test theory approach with various model parameters, while other social science studies employed a simpler one-parameter Rasch model. Within the Philippines' educational assessment landscape, professionals are becoming increasingly vital due to the growing emphasis on quality assurance in schools, especially concerning instruction and program effectiveness. This necessitates stronger partnerships between teachers and psychometricians to create and interpret assessments that effectively measure student learning.

In the context of Grade 10 mathematics examinations, the Rasch model helps to assess how well the test matches the students' levels of ability and how suitable each test item is. Although the Rasch model is widely used in educational research internationally, there are few studies that apply it in secondary schools in the Philippines, especially in regular classes where student performance varies significantly. This limits the understanding of how Rasch-based analysis can support teaching and assessment in typical classrooms. Therefore, this study aims to provide teachers with useful information on the level of difficulty of each test item and the performance of students, which can help improve instructional planning, support student needs, and guide future assessment development.

Research Questions

This study explored how the Rasch model can assess the extent to which a standardized mathematics test reflects student ability and how effectively each test item functions. Specifically, the study addressed the following research questions:

1. How well does the Grade 10 Mathematics First Quarter Examination align with students' abilities based on Rasch model analysis?
2. What item characteristics—such as difficulty level and fit statistics—emerge from the Rasch model, and how do they inform the quality of the test?
3. How do students' response patterns (e.g., misfitting or guessing behavior) reflect their test-taking skills and affect the interpretation of performance data?
4. What instructional strategies can be developed to address issues such as test anxiety, motivation, and response inconsistency identified through the Rasch analysis?
5. How can the Rasch model be used to improve assessment design and teaching practices in regular Grade 10 mathematics classrooms?

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

METHODS

This study was descriptive and diagnostic in nature. No instructional interventions were administered during the research period. The Rasch model was employed to evaluate the psychometric properties of an existing classroom assessment. The results are intended to inform future revisions to test design and instructional planning.

Participants

The participants of the study are 51 Grade 10 students, selected through purposive sampling, from a regular class with 23 males (45.10%) and 28 females (54.90%) at Bayugan National Comprehensive High School. The students had varied learning styles and levels of mastery of the subject matter.

Instrument

The first quarter mathematics examination for grade 10 students at Bayugan National Comprehensive High School was developed by the researcher and aligned to a Table of Specifications that an expert and highly competent professor validated. The examination consisted of 50 multiple-choice items and was composed of 60% low-level thinking skill questions (remembering and understanding), 30% moderate-level thinking skill questions (applying and analyzing), and 10% high-level thinking skill questions (evaluating and creating). These levels were based on the Revised Bloom's Taxonomy of Cognitive Objectives by Anderson and Krathwohl (2001).

Procedure

The researcher developed and administered a first-quarter mathematics examination to collect the data for this study. The researcher acquired permission from the school principal, students, and parents to ensure the willingness and participation of the participants.

Analysis

The data collected were tabulated and treated according to the problems presented in this study. Data was cleaned through descriptive statistics by examining the valid N and missing values. Responses were coded as 1 if the student answered the item correctly and 0 if incorrect. The data were analyzed on Jamovi version 2.3.28 using its free module on snowIRT on the Dichotomous Rasch Model.

Rasch analysis was employed to assess the test's structural validity, item fit, person fit, and difficulty distribution. The primary statistics examined included:

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

- Infit and outfit mean square values are used to evaluate how well individual items and students align with model expectations.
- Person reliability index, which reflects the consistency of student responses across varying item difficulties.
- Item difficulty measures, expressed in logits, indicate the cognitive demand required for each item.
- MADaQ3 statistic, used to assess the overall model fit through residual correlations.

A Wright Map was generated to visually compare student ability against item difficulty. Additionally, response patterns were reviewed to identify students exhibiting high outfit values (potential guessing behavior) or low values (possible anxiety or disengagement). Misfitting items and students were flagged for further instructional analysis.

RESULTS

Model Fit

Model Fit is a measure of how well the model fits the data. A good model fit is crucial because it means the model accurately represents the relationships between the variables. Person Reliability measures how reliably the model can estimate the person's ability. A higher person reliability indicates that the model is more reliable. MADaQ3 is a measure of model fit, with lower values indicating a better fit. The range of values a MADaQ3 can take on can be between 0 and infinity. The MADaQ3 is calculated by taking the mean of the absolute values of the centered Q_3 statistic. The Q_3 statistic measures how well the person's responses fit the model. P-value is the probability of obtaining a MADaQ3 as extreme or more extreme than the observed MADaQ3, assuming that the model is correct. A p-value less than 0.05 is generally considered statistically significant, meaning that the model fit is unlikely to have occurred by chance.

	Person Reliability	**MADaQ3**	**p**
scale	0.552	**0.136**	**0.001**

Note. MADaQ3= Mean of absolute values of centered Q_3 statistic with p-value obtained by Holm adjustment; H_0= the data fit the Rasch model.

Table 1. Model fit

Table 1 shows the results of a model fit for a person reliability model. The model fit is good, with a person reliability of 0.552, demonstrating that the students have consistently correctly answered questions and a MADaQ3 of 0.136. The MADaQ3 measures model fit, with lower values indicating a better fit. The p-value for the MADaQ3 is 0.001, which indicates that the model fit is statistically significant.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Item Characteristics

Item characteristics are a property of an item in a test or assessment that describes how well the item can distinguish between individuals with different ability levels or traits being measured. A complete summary of item characteristics is presented in Appendix A. The proportion column shows the proportion of people who answered each item correctly. The measured column shows the estimated measure for each item, which measures the item's difficulty. The S.E. Measure column shows the standard error of the measure. The infit and outfit columns show the infit and outfit mean square errors, which measure how well each item fits the model. Sumintono and Widhiarso (2015) guide assessing these items into four categories, namely (1) Measure value < -1 = very easy item, (2) Measure value -1 to 0 = easy item, (3) Measure value 0 to 1 = difficult item, and (4) Measure value > 1 = very difficult item.

The result reflects the characteristics of each item in the first quarter examination in Mathematics 10. Twelve items are considered very difficult. Subsequently, it shows in Appendix A that items 39, 28, and 11 are some of the items that are considered very difficult, with 2%, 4%, and 10%, respectively, of the number of students who got the answer correctly. Seven items are considered difficult, with 31% (0.31) to 43% (0.43) of the students getting the answer correctly. Eleven out of 50 items are considered easy questions with measure coefficients ranging from -0.04 to -0.91 and a proportion value of 0.51 to 0.71 (or 51% to 71% of the students got the answer correctly, respectively. Lastly, there are 20 items interpreted as very easy for the student's level of subject knowledge.

Boone et al. (2014) suggest the following criterion to check whether an item is fit (item fit) or not fit (outlier or misfit). This criterion is the OUTFIT value. If the OUTFIT value is greater than 0.5 and less than 1.5, then these items are considered misfits since they undermine the Rasch model’s assumptions of unidimensionality, local independence, and monotonicity. Hence, if the outfit value is greater than 1.5 and less than 0.5, the item is considerably unfit (outlier). The result shows that the items' outfit coefficient lies between the ranges 0.00 to 1.38, with only items 1 and 3 being considerably more outliers than any other items because of its coefficient of 0.00, which is out of the range of a fit item. With this, items 1 and 3 must be improved or changed. Items 1 and 3 on the test were too easy, as every student answered them correctly. These items covered common knowledge that was directly taught from the textbook and lecture notes, suggesting the teacher may have focused heavily on this material. Additionally, the questions may have lacked effective distractors, making the correct answers obvious. The teacher should carefully review the test questions, particularly items 1 and 3, to determine if they were too simple and direct, given that all students answered them correctly, to increase the difficulty.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Person Characteristics

According to Sumintono (2017), the Rasch model can measure an individual's ability more precisely. Based on the accuracy of the response given, the pattern will show individual tendency regarding how one performs solving test items.

The total score is the sum of the scores on all the items. It is a measure of the student's overall ability on the test. The measure is the student's ability estimate on the Rasch scale. It is a more precise measure of the student's ability than the total score because it considers the items' difficulty. The S.E. is the standard error of the measure. It is a measure of the uncertainty in the measure. A lower S.E. indicates a more precise measure. The infit measures how well the student's responses fit the Rasch model. A value close to 1 indicates that the student's responses fit the model well. A value above 1 indicates that the student's responses are more variable than expected, and a value below 1 indicates that the student's responses are less variable than expected. The misfit measures how well the student's responses fit the Rasch model when the student is removed from the analysis. In general, both infit and outfit values between 0.7 and 1.3 are acceptable. Values outside this range may indicate that the student is a misfit or an outlier.

The student participants in the results represent a wide range of abilities. The students with the lowest total scores are likely to struggle with the material, while those with the highest scores are likely to excel. The students with negative measurement values are likely to underperform on the assessment, while those with positive ones will likely overperform. In the result in Appendix B, it can be seen that student 28 has the lowest total score of 21, and students 8 and 13 have the highest total score of 37. Note that students 10, 7, and 30 have an outfit value greater than 1.3, with 1.47, 1.58, and 1.53, respectively, which could mean that these students may be just guessing when answering the test items. In addition, fourteen (14) students have an outfit value lower than 0.7, which could mean that students misunderstood the items and were not motivated to take the examination. These students may have been careless or speeding in answering the examination. They may not pay close attention to the test items, which leads to random or guessing. Apart from this, these misfit students may have experienced test anxiety or stress as students might second-guess themselves, which leads to guessing due to stress. Hence, the teacher may review the individual response patterns of the misfit students and look for inconsistency or unexpected responses. Interviewing the students individually may help the teacher understand why their students might have struggled with the exam.

Wright Map

The Wright map shows the difficulty of the items in a test, as well as the ability of the test-takers. The items are arranged in order of difficulty, from the most difficult at the top to the easiest at the bottom. The test-takers are arranged in order of ability, from the highest-ability test-takers at the top to the lowest at the bottom.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

To find out the distribution of items and the ability of the student to respond to items in general, the person-item map can be seen in Figure 1. The left side is the distribution of the students' abilities, while the right side is the distribution of items. From the map, the easiest items are items 1 and 3, which are in the lowest position, while the most difficult item is item 39. The map allows us to see which students are likely to have successfully answered which items. Generally, the test questions are lower (easier) than the student's ability. In other words, the questions could be more challenging for test takers who generally have high abilities.

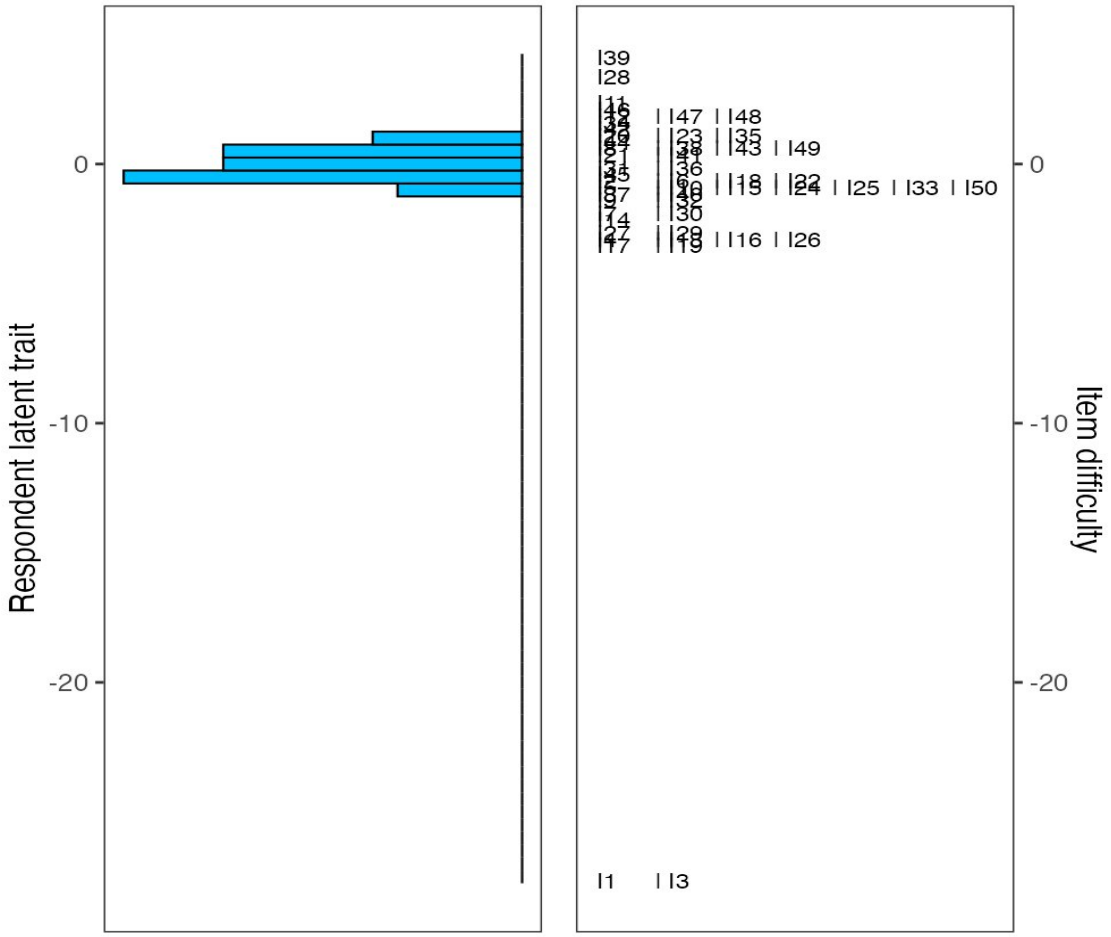


Figure 1. Wright map of students' latent trait and item difficulty

DISCUSSION

Based on the model fit results, the first quarter examination for Grade 10 in Mathematics to assess students' mathematics skills, gauge their understanding, identify areas for improvement, and

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

enhance overall academic achievement represents an instrument that can measure the Grade 10 students' mathematics skills. Additionally, the result accurately represents the relationships between the variables, and the model fit is statistically significant. Based on the item characteristics (Appendix A) and the Wright Map (Figure 1), items 1 and 3 are very easy, and all students got both items correctly. This would mean that all students understand this item or subject matter, which needs to be higher than their abilities. Note that items 1 and 3 were made with low-level thinking skills, such as specifically remembering the facts and definitions of the subject matter. Subsequently, the students understand item 1 in finding the next term given a series of terms under the learners' learning competency, which generates a pattern. Also, the students understand item 3, which is remembering in the Revised Bloom's taxonomy of cognitive level of Anderson and Krathwohl (2001), which asks about the formula for finding the nth term of an arithmetic sequence. Some reasons why all students may have answered items 1 and 3 correctly are if the question was very basic or directly repeated information from the textbook or class notes. This highlights a potential flaw in the question, as it does not effectively differentiate between students who understand the material and those who do not. Alternatively, it could be that if the teacher discussed the specific answer or concept during class or made it very clear what the expected response was, students might all answer correctly for that reason.

A possible reason is that both items are general knowledge, given that remembering level and their easy calculations or straightforward applications. Maybe the construction of the test item has obvious distractors and its alignment with the learning objectives. This could also be that the teacher has built a strong foundation of essential knowledge and skills to ensure students have the necessary tools to tackle such questions. Moreover, items 1 and 3 must be improved or changed because they considerably outfit or misfit the students' abilities and assess their mathematics skills. On the characteristics of the students, it can be seen in Appendix B that three (3) students are merely guessing when answering the examinations, which may be because they need more preparation. Lacking preparation due to insufficient understanding of the concepts of the subject matter or procrastination and lack of time management skills are reasons why these results were seen. However, it could also be attributed to unclear or ambiguous questions, or they feel rushed to complete the exam since it was given quickly. In addition, fourteen (14) students needed to have understood the items and were not motivated to take the examination. This could be because the students have mathematics anxiety. Difficulty in mathematics is related to cognitive components of social perception, which are highly varied among students. Anxiety in mathematics is related to the emotional component of social perception, which varies slightly among students (Al Ali, 2016). It could also be because they need more confidence and believe they are not good at math. Other factors could be the lack of support at home or school, or personal issues like stress, anxiety, and other problems that can also impact their motivation and ability to learn.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Contribution to the Broader Literature

Although Rasch model analysis has been utilized in various international studies, its application in evaluating classroom-level assessments within the Philippine secondary school setting remains underexplored. This study contributes to the wider body of literature by showcasing how item-level diagnostics and student response patterns derived from Rasch analysis can inform both assessment quality and teaching strategies in actual classroom environments. The emphasis on identifying misfit items, understanding student behavior such as guessing and disengagement, and aligning instruction with ability profiles presents a practical framework for educators in similar contexts. This localized approach provides evidence that supports the development of responsive assessment practices not only in mathematics but also in other subjects and grade levels where student diversity and emotional factors affect performance.

LIMITATIONS

This study has several drawbacks. First, the most evident limitation was the tiny data set, which reduced the validity of the inferences derived from the data. The data analysis used in this study was based on a sample of data gathered only in one section of Bayugan National Comprehensive High School with a limited number of respondents (N = 51), which impacted generalization to other groups. Second, the study is limited to the characteristics of the participants. The school is located in the heart of Bayugan City and is one of the biggest schools, hence, the school has more varied students. The study participants were students in a regular class whose level of understanding of the mathematics content varied from one another. They were slower than how science classes learn. It is important to note that some items in higher-level thinking are considered very easy for the students, with 72.55% of them answering it correctly. This could be because the students understand more about this subject matter. Additionally, female participants outnumbered male participants in this research. It is possible to study gender bias with this instrument. The test questionnaire was made for grade 10 students taking Mathematics 10 for the first quarter, hence, it cannot be inferred for other quarters and other groups. Future studies should use bigger random samples from other groups of students and look for evidence of different backgrounds.

CONCLUSION

In conclusion, the model fit is good, as evidenced by a personal reliability of 0.552 and a MADaQ3 value of 0.136. The MADaQ3 serves as an indicator of model fit, with lower values signifying a more favorable fit. The p-value associated with the MADaQ3 is 0.001, indicating statistical significance in the model fit. Analysis of the first quarter Mathematics 10 examination reveals that among the 50 items, twelve are classified as very difficult, seven as difficult, eleven as easy, and thirty as very easy. The Wright map illustrates that items 1 and 3 are the easiest, located at the bottom, while item 39 is the most challenging. Both items 1 and 3 are outliers or misfits and are suggested to be improved or changed. Overall, the test questions are relatively easier than the students' abilities, suggesting that the questions may be too simple for test takers with generally

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

high aptitudes. Thus, the items of the first-quarter mathematics examination for grade 10, developed by the researcher, are effective in assessing students' mathematics skills, especially in regular classes. The results of this research can be directly applied to classroom instruction in several impactful ways. Teachers can use item-level analysis to revise ineffective questions, aligning difficulty levels with students demonstrated proficiencies. The Wright Map provides diagnostic insights that allow teachers to differentiate instruction by grouping students according to ability, tailoring lesson complexity, and designing appropriate scaffolds. Additionally, identifying students with response behavior enables the implementation of targeted support strategies, such as confidence-building exercises, mindfulness practices, and individualized feedback. Since some students are merely guessing in answering the test items and with the given factors, the teacher may promote a positive attitude and belief to overcome test anxiety and build self-efficacy. Also, provide clear and well-designed items like using unambiguous language and applying appropriate time constraints to minimize confusion and reduce the likelihood of students needing to guess. The teacher could also equip students with test-taking strategies like eliminating incorrect answers, estimating answers, and checking their work to enhance their confidence and accuracy. Most importantly, the teacher may talk to students individually to understand and find ways to address test anxiety. The students should acknowledge that test anxiety is normal and can be managed through test-taking strategies. Introduce students to deep breathing exercises and mindfulness practices. The teacher may help students with positive affirmations to replace negative thoughts. During the test, the teacher may create a calm and supportive environment and minimize time pressure by avoiding frequent announcements about time limits, as this can increase anxiety.

Additionally, assessment should not only measure knowledge—it should inform teaching practice. Teachers are encouraged to use this data to create a more inclusive and responsive classroom environment by promoting a positive attitude, building self-efficacy, and teaching effective test-taking strategies like eliminating distractors and estimating answers. Emotional and behavioral factors, such as anxiety, must also be addressed through affirmations, calming techniques, and reducing time-related stress during exams. Post-assessment feedback must emphasize perseverance, effort, and growth mindsets to cultivate resilience and long-term academic success.

Beyond the immediate classroom context, the findings of this study have broader implications for mathematics education and assessment practice. The use of Rasch analysis offers a data-driven method that educators in similar grade levels or regions can adopt to improve test quality, evaluate student performance, and identify instructional gaps. The patterns observed—such as item misfit, varying difficulty levels, and inconsistent student responses—can inform curriculum planners and teachers in designing more balanced assessments that align with diverse learner abilities. Furthermore, addressing non-cognitive factors such as test anxiety and engagement through evidence-based strategies can enhance learning outcomes across various school settings. This study contributes to a growing body of educational research advocating for diagnostic assessment tools that support effective teaching and equitable learning environments in Philippine secondary education and beyond.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

REFERENCES

[1] Al Ali, R. M., & Shehab, R. T. (2020). Psychometric properties of social perception of mathematics: Rasch model analysis. *International Education Studies*, *13*(12), 102–111. https://doi.org/10.5539/ies.v13n12p102

[2] Anderson, A. (1993). Assessment: A means to empower children. In N. L. Webb & A. L. Oxford (Eds.), *Assessment in the mathematics classroom*. NCTM.

[3] Anderson, L. W., & Krathwohl, D. R. (Eds.). (2001). *A taxonomy for learning, teaching, and assessing: A revision of Bloom's taxonomy of educational objectives*. Longman.

[4] Bond, T. G., & Fox, C. M. (2015). *Applying the Rasch model: Fundamental measurement in the human sciences* (3rd ed.). Routledge.

[5] Clements, D. H., Sarama, J. H., & Liu, X. H. (2008). Development of a measure of early mathematics achievement using the Rasch model: The research-based early maths assessment. *Educational Psychology*, *28*(4), 457–482. https://doi.org/10.1080/01443410701777272

[6] El-Korashy, A. (1995). Applying the Rasch model to the selection of items for a mental ability test. *Educational and Psychological Measurement*, *55*(5), 753–763. https://doi.org/10.1177/0013164495055005006

[7] Isnaini, Utami, W. B., Susongko, P., & Lestiani, H. T. (2019). Estimation of college students' ability on real analysis course using Rasch model. *Research and Evaluation in Education*, *5*(2), 95–102. https://doi.org/10.21831/reid.v5i2.20924

[8] Khan, L. A. (2015). What is mathematics – an overview. *International Journal of Mathematics and Computational Science*, *1*(3), 98–101. https://doi.org/10.13140/RG.2.1.3626.8967

[9] Kurniawan, D. D., & Mardapi, D. (2015). Penyetaraan vertikal tes matematika SMP dengan teori respons butir model Rasch [Vertical equating of junior high school mathematics tests using the Rasch model item response theory]. *Jurnal Evaluasi Pendidikan*, *3*(1), 12–25. http://journal.student.uny.ac.id/ojs/index.php/jep/article/view/1221/1093

[10] Magno, C. (2010). A brief history of educational assessment in the Philippines. *Educational Measurement and Evaluation Review*, *1*, 140–149. https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1638490

[11] Mutodi, P., & Ngirande, H. (2014). The influence of students' perceptions on mathematics performance: A case of a selected high school in South Africa. *Mediterranean Journal of Social Sciences*, *5*(3), 431–438. http://doi.org/10.5901/mjss.2014.v5n3p431

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

[12] Ngunjiri, M. (2022). The role of assessment in mathematics classrooms: A review. *International Journal of Advanced Research*, *5*(1), 156–160. https://doi.org/10.37284/ijar.5.1.887

[13] Ramadhani, R., Saragih, S., & Napitupulu, E. (2022). Exploration of students' statistical reasoning ability in the context of ethnomathematics: A study of the Rasch model. *Mathematics Teaching-Research Journal*, *14*(1), 138–168. https://eric.ed.gov/?id=EJ1361651

[14] Sumintono, B. (2017). Rasch model measurements as tools in assessment for learning. *Advances in Social Science, Education and Humanities Research*, *173*, 57–61. https://doi.org/10.2991/icei-17.2018.11

[15] Sumintono, B., & Widhiarso, W. (2015). *Application of Rasch modeling on educational assessment*. Cimahi, Indonesia: Trim Komunikata Publishing House.

[16] Wright, B. D., & Stone, M. H. (1979). *Best test design: Rasch measurement*. Chicago, IL: MESA Press.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

APPENDIX

Appendix A

Item	Proportion	Measure	S.E.Measure	Infit	Outfit	Interpretation
I39	0.02	4.00	1.01	1.01	1.02	very difficult
I28	0.04	3.28	0.72	0.97	0.71	very difficult
I11	0.10	2.29	0.47	1.04	1.17	very difficult
I46	0.12	2.08	0.44	1.05	1.15	very difficult
I47	0.14	1.90	0.41	1.02	1.05	very difficult
I12	0.16	1.74	0.39	1.03	1.07	very difficult
I48	0.16	1.74	0.39	1.12	1.38	very difficult
I34	0.18	1.60	0.37	1.12	1.28	very difficult
I42	0.22	1.34	0.35	1.05	1.10	very difficult
I35	0.24	1.23	0.34	1.05	1.09	very difficult
I20	0.27	1.01	0.32	1.14	1.22	very difficult
I23	0.27	1.01	0.32	1.09	1.14	very difficult
I44	0.31	0.82	0.31	1.00	1.03	difficult
I38	0.35	0.63	0.30	1.09	1.10	difficult
I43	0.35	0.63	0.30	0.88	0.86	difficult
I49	0.35	0.63	0.30	0.98	0.99	difficult
I8	0.37	0.55	0.30	1.00	0.99	difficult
I21	0.43	0.29	0.29	0.99	0.99	difficult
I41	0.43	0.29	0.29	1.00	0.99	difficult
I31	0.51	-0.04	0.29	1.00	1.00	easy
I36	0.51	-0.04	0.29	0.99	0.99	easy
I45	0.57	-0.28	0.29	0.96	0.95	easy
I2	0.67	-0.72	0.30	1.00	0.99	easy
I6	0.67	-0.72	0.30	0.98	0.95	easy
I18	0.67	-0.72	0.30	0.91	0.88	easy
I22	0.67	-0.72	0.30	0.91	0.88	easy
I10	0.69	-0.81	0.31	0.92	0.89	easy
I5	0.71	-0.91	0.31	1.02	1.01	easy
I25	0.71	-0.91	0.31	0.98	0.97	easy
I33	0.71	-0.91	0.31	0.98	0.95	easy
I15	0.73	-1.00	0.32	0.94	0.91	very easy
I24	0.73	-1.00	0.32	0.96	0.92	very easy

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

I50	0.73	-1.00	0.32	0.99	0.96	very easy
I40	0.75	-1.11	0.33	0.94	0.89	very easy
I37	0.76	-1.22	0.34	0.91	0.84	very easy
I9	0.78	-1.33	0.35	0.96	0.91	very easy
I32	0.78	-1.33	0.35	1.04	1.08	very easy
I7	0.86	-1.89	0.41	1.03	1.06	very easy
I30	0.86	-1.89	0.41	1.00	1.02	very easy
I14	0.88	-2.07	0.44	1.07	1.23	very easy
I27	0.92	-2.53	0.52	0.99	0.89	very easy
I29	0.92	-2.53	0.52	1.02	1.03	very easy
I4	0.94	-2.84	0.60	1.00	0.94	very easy
I13	0.94	-2.84	0.60	1.01	0.96	very easy
I16	0.94	-2.84	0.60	0.98	0.82	very easy
I26	0.94	-2.84	0.60	1.00	0.93	very easy
I17	0.96	-3.27	0.72	1.00	0.91	very easy
I19	0.96	-3.27	0.72	1.01	1.02	very easy
I3	1.00	-27.52	123953.03	NaN	**0.00**	very easy
I1	1.00	-27.52	123953.03	NaN	**0.00**	very easy

Table 2. Item characteristics

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Appendix B

Student	Total score	Measure	SE	Infit	Outfit
28	21	-1.03	0.36	0.98	0.84
31	22	-0.90	0.36	1.14	1.30
44	22	-0.90	0.36	0.92	0.95
27	23	-0.78	0.36	1.10	1.24
48	23	-0.78	0.36	1.04	1.12
3	24	-0.65	0.36	1.05	1.16
33	24	-0.65	0.36	1.11	1.23
10	25	-0.52	0.36	1.45	**1.47**
21	25	-0.52	0.36	0.86	0.75
37	25	-0.52	0.36	1.23	1.28
14	26	-0.39	0.36	0.91	0.78
29	26	-0.39	0.36	1.20	1.12
36	26	-0.39	0.36	1.08	1.01
39	26	-0.39	0.36	1.26	1.30
49	26	-0.39	0.36	1.09	1.22
6	27	-0.26	0.36	0.90	1.00
7	27	-0.26	0.36	1.47	**1.58**
15	27	-0.26	0.36	0.76	0.77
19	27	-0.26	0.36	1.58	1.71
26	27	-0.26	0.36	1.26	1.11
50	27	-0.26	0.36	1.05	1.01
20	28	-0.13	0.36	0.86	0.83
34	28	-0.13	0.36	1.04	1.02
51	28	-0.13	0.36	1.18	1.13
2	29	0.00	0.36	0.58	**0.42**
5	29	0.00	0.36	0.94	0.73
23	29	0.00	0.36	0.78	**0.64**
25	29	0.00	0.36	1.12	0.89
30	29	0.00	0.36	1.32	**1.53**
35	29	0.00	0.36	1.18	1.17
1	30	0.13	0.37	0.74	**0.56**
11	30	0.13	0.37	0.69	**0.60**
40	30	0.13	0.37	1.13	0.95

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

4	31	0.26	0.37	0.91	1.01
16	31	0.26	0.37	1.03	1.05
42	31	0.26	0.37	0.92	0.90
9	32	0.40	0.37	0.54	**0.40**
12	32	0.40	0.37	1.22	1.10
24	32	0.40	0.37	0.87	**0.68**
45	32	0.40	0.37	0.92	**0.68**
46	32	0.40	0.37	0.92	0.83
47	32	0.40	0.37	0.77	**0.57**
18	34	0.68	0.38	0.68	**0.52**
41	34	0.68	0.38	0.49	**0.32**
43	34	0.68	0.38	1.12	0.99
17	35	0.82	0.38	0.66	0.70
22	35	0.82	0.38	0.58	**0.38**
32	36	0.97	0.39	0.85	0.92
38	36	0.97	0.39	0.81	**0.51**
8	37	1.12	0.40	0.58	**0.35**
13	37	1.12	0.40	0.75	**0.54**

Table 3. Person characteristics

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Appendix C

First Quarter Examination in Mathematics 10

1. Which pattern follows the sequence: 2, 4, 8, 16, __?
   A. 24 C. 64
   B. 32 D. 12

2. What is an arithmetic sequence?
   A. A sequence where each term is the product of the preceding term and a fixed number.
   B. A sequence where each term is the sum of the preceding term and a fixed number.
   C. A sequence where each term is a random number.
   D. A sequence where each term alternates between odd and even numbers.

3. What is the formula for the nth term of an arithmetic sequence?
   A. $a_n = a_1 + (n-1)d$ C. $a_n = a_1 \times n \times d$
   B. $a_n = a_1 \times d^{n-1}$ D. $a_n = a_1 + n \times d$

4. What defines a geometric sequence?
   A. A sequence where each term is the sum of the preceding term and a fixed number.
   B. A sequence where each term is a random number.
   C. A sequence where each term is the product of the preceding term and a fixed number.
   D. A sequence where each term alternates between odd and even numbers.

5. What distinguishes a geometric sequence from an arithmetic sequence?
   A. In a geometric sequence, each term is found by adding a constant value to the previous term, whereas in an arithmetic sequence, each term is found by multiplying a constant value to the previous term.
   B. In a geometric sequence, each term is found by multiplying a constant value to the previous term, whereas in an arithmetic sequence, each term is found by adding a constant value to the previous term.
   C. Geometric sequences are entirely random in their progression, while arithmetic sequences follow a predictable pattern.
   D. Geometric sequences involve both addition and multiplication operations, while arithmetic sequences only involve addition operations.

6. What is the formula to find the nth term (a_n) of a geometric sequence?
   A. $a_n = a_1 + (n-1)d$ C. $a_n = \frac{a_1 + a_n}{2}$
   B. $a_n = a_1 r^{n-1}$ D. $a_n = \frac{n}{2}(a_1 + a_n)$

7. Which of the following sequences is an arithmetic sequence?
   A. 3, 6, 9, 12, 14 C. 1, 3, 9, 27, 81
   B. 2, 4, 8, 16, 32 D. 5, 10, 15, 20, 25

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

8. Which method among the following is specifically used for dividing polynomials by linear factors or divisors?
   A. Synthetic division
   B. Long division
   C. Horner's method
   D. Newton's method

9. Which theorem states that if a polynomial *f*(x) is divided by x+c, the remainder will be *f*(-c)?
   A. Remainder theorem
   B. Factor theorem
   C. Rational root theorem
   D. Intermediate value theorem

10. Factor the polynomial $(x^2 - 4)$.
   A. $(x - 2)(x - 2)$
   B. $(x + 2)(x - 2)$
   C. $(x + 2)(x + 2)$
   D. $(x - 4)(x + 4)$

11. Identify the type of polynomial equation represented by the expression: $3x^4 - 2x^2 + 5x + 1 = 0$
   A. Quadratic equation
   B. Cubic equation
   C. Quartic equation
   D. Quintic equation

12. What is the degree of the polynomial equation represented by $4x^3 - 2x^2 + 5x - 1 = 0$
   A. Degree 1
   B. Degree 2
   C. Degree 3
   D. Degree 4

13. In the sequence 3, 6, 12, 24, __, which rule or pattern is being followed?
   A. Addition by 12
   B. Multiplication by 2
   C. Subtraction by 2
   D. Division by 3

14. If the first term in an arithmetic sequence is 7 and the common difference is 4, what is the fifth term in the sequence?
   A. 19
   B. 23
   C. 27
   D. 31

15. What is the formula to find the sum of the first *n* terms of an arithmetic sequence when the nth term in unknown?
   A. $S_n = \frac{n}{2}(a_1 + a_n)$
   B. $S_n = n\ x\ a_1\ x\ d$
   C. $S_n = \frac{n}{2}(2a_1 + (n-1)d)$
   D. $S_n = a_1 + (n-1)d$

16. In a geometric sequence where the first term (a_1) is 5 and the common ratio (r) is 3, what are the first four terms of the sequence?
   A. 5, 8, 13, 18
   B. 5, 10, 15, 20
   C. 5, 15, 30, 60
   D. 5, 15, 45, 135

17. Differentiate between an arithmetic sequence and a geometric sequence based on their defining characteristics.
   A) Arithmetic sequences have a common ratio, while geometric sequences have a common difference.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

B) Arithmetic sequences have a common difference, while geometric sequences have a common ratio.
C) Arithmetic sequences have an exponential growth, while geometric sequences have a linear growth.
D) Arithmetic sequences have a common ratio, while geometric sequences have an exponential growth.

18. What is the formula for the sum of the first n terms (S_n) of a finite geometric sequence when a_n is known?
    A. $S_n = \frac{n}{2}(a_1 + a_n)$
    B. $S_n = \frac{n}{2}(2a_1 + (n-1)d)$
    C. $S_n = \frac{a_1(1-r^n)}{1-r}$
    D. $S_n = a_1 r^n$

19. If the 7th term in an arithmetic sequence is 31 and the common difference is 5, what is the 15th term of this sequence?
    A. 51
    B. 65
    C. 61
    D. 71

20. When should synthetic division be used instead of long division for polynomial division?
    A. When the dividend is a higher-degree polynomial than the divisor.
    B. When the divisor is a higher-degree polynomial than the dividend.
    C. When the divisor is a linear polynomial.
    D. When the divisor is a quadratic polynomial.

21. Which theorem helps in determining the possible roots of a polynomial equation and the relationship between the roots and the factors of a polynomial?
    A. Remainder theorem
    B. Factor theorem
    C. Rational root theorem
    D. Intermediate value theorem

22. Factor the polynomial $x^2 + 5x + 6$.
    A. $(x+3)(x+2)$
    B. $(x+1)(x+6)$
    C. $(x-3)(x-2)$
    D. $(x-1)(x-6)$

23. Identify the degree of the polynomial equation: $6x^3 - 4x^2 + 2x - 1 = 0$
    A. Degree 1
    B. Degree 2
    C. Degree 3
    Degree 4

24. Solve the equation $2x^2 + 5x - 3 = 0$ using factorization.
    A. $x = 1/2, x = -3$
    B. $x = 1/2, x = 3$
    C. $x = -1/2, x = -3$
    D. $x = -1/2, x = 3$

25. Using a pattern or rule, extend the following sequence: 5, 10, 20, __.
    A. 25
    B. 35
    C. 40
    D. 50

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

26. In an arithmetic sequence, if the first term a_1 is 3 and the common difference d is 7, what would be the tenth term of this sequence?
    A. 66 C. 83
    B. 76 D. 97

27. In an arithmetic sequence, if the first term (a_1) is 7 and the common difference (d) is 4, what is the 15th term (a_{15}) and the sum of the first 15 terms in this sequence?
    A. $a_{15} = 63,\ S_{15} = 435$ C. $a_{15} = 63,\ S_{15} = 525$
    B. $a_{15} = 63,\ S_{15} = 465$ D. $a_{15} = 63,\ S_{15} = 635$

28. In a geometric sequence, if the 5th term is 81 and the 3rd term is 9, what is the common ratio of this sequence?
    A. 3 C. 27
    B. 9 D. 36

29. Identify the following sequence as either arithmetic or geometric: 2, 4, 8, 16, 32, ...
    A. Arithmetic Sequence C. Neither
    B. Geometric Sequence D. Both

30. In a geometric sequence where the first term (a_1) is 3 and the common ratio (r) is 2, what is the third term (a_3) of the sequence?
    A. 6 C. 12
    B. 9 D. 15

31. In a geometric sequence, if the first term (a_1) is 8 and the common ratio (r) is 3, what is the 6th term (a_6) and the sum of the first 5 terms (S_5)?
    A. a_6=1944, S_5=968 C. a_6=729, S_5=3280
    B. a_6=729, S_5=1092 D. a_6=1944, S_5=19683

32. Divide $3x^3 - 7x^2 + 5x - 9$ by $x - 2$ using the most suitable method. Which method should be employed for this division?
    A. Long division C. Horner's method
    B. Synthetic division D. Newton's method

33. Given a polynomial $f(x)=2x^3-5x^2+7x+3$. When divided by $(x-2)$, the remainder is 0. What can be concluded using this information?
    A. $f(2)=0$ C. $f(2)=3$
    B. $f(2)=7$ D. $f(2)=2$

34. Factor the following polynomial completely: $x^3 - x^2 - 6x$
    A. $x(x-3)(x+2)$ C. $x(x+3)(x+2)$
    B. $x(x+3)(x-2)$ D. $x(x-3)(x-2)$

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

35. Determine the values of k that satisfies the equation $2k^2 - 5k + 2 = 0$.
    A. $k = 2, k = \frac{1}{2}$
    B. $k = 1, k = 3$
    C. $k = 2, k = 1$
    D. $k = 3, k = 2$

36. Solve the equation $x^3 - 5x^2 + 8x - 4 = 0$ given that one root is $x = 1$. What is the quadratic equation formed by the remaining roots?
    A. $x^2 - 4x + 4 = 0$
    B. $x^2 - 6x + 8 = 0$
    C. $x^2 - 4x + 6 = 0$
    D. $x^2 - 6x + 10 = 0$

37. In an arithmetic sequence, if the 5th term is 23 and the 10th term is 43, what is the common difference of the sequence?
    A. 3
    B. 4
    C. 5
    D. 6

38. In an arithmetic sequence, if the 7th term is 32 and the 15th term is 72, what is the sum of the first 15 terms in this sequence?
    A. 445
    B. 555
    C. 635
    D. 685

39. If in a geometric sequence the first term (a_1) is 5 and the common ratio (r) is 1/3, what is the sum of the first 5 terms of this sequence?
    A. 325/81
    B. 495/81
    C. 545/81
    D. 605/81

40. In a sequence, the first term (a_1) is 10, and the second term (a_2) is 16. If this sequence is an arithmetic sequence, what is the 6th term (a_6)?
    A. 28
    B. 32
    C. 36
    D. 40

41. Factor the polynomial $x^2 + 7x + 12$.
    A. $(x + 3)(x + 4)$
    B. $(x + 2)(x + 5)$
    C. $(x - 3)(x - 4)$
    D. $(x + 6)(x + 2)$

42. Given the equation $x^3 - 4x^2 + 5x - 2 = 0$, what is the sum of the roots?
    A. 2
    B. 3
    C. 4
    D. 5

43. Consider the equation $3x^3 - 11x^2 + 8x = 0$. What conclusion can be drawn about the roots based on the coefficients?
    A. The equation has one real and two complex roots.
    B. The equation has real roots.
    C. the equation has all complex roots.
    D. The equation has one real and one complex conjugate pair of roots.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

44. In an geometric sequence, if the first term (a_1) is 3 and the common ratio (r) is 5, what is the 4th term (a_4) and the sum of the first 4 terms?
    A. $a_{15} = 375,\ S_{15} = 438$
    B. $a_{15} = 375,\ S_{15} = 468$
    C. $a_{15} = 375,\ S_{15} = 528$
    D. $a_{15} = 375,\ S_{15} = 638$

45. In an arithmetic sequence, the 10th term is 72, and the 15th term is 102. What is the 20th term of this sequence?
    A. 132
    B. 112
    C. 142
    D. 152

46. Factor the polynomial $2x^2 + 5x - 3$.
    A. $(2x - 1)(x + 3)$
    B. $(2x + 3)(x - 1)$
    C. $(2x - 3)(x + 1)$
    D. $(2x + 1)(x + 3)$

47. Solve the equation $2x^2 - 7x + 3 = 0$. Identify the nature of the roots.
    A. $x = \frac{1}{2}, x = 3$, Real and unequal
    B. $x = 1, x = 3$, Real and unequal
    C. $x = 3, x = -\frac{1}{2}$, Real and unequal
    D. $x = 3, x = 3$, Real and equal

48. Solve the equation $x^3 - 6x^2 + 11x - 6 = 0$. Determine the nature of the roots.
    A. Real and unequal
    B. Real and equal
    C. Two real and one complex
    D. One real and two complex

49. Create a geometric sequence where the first term (a_1) is 4 and the common ratio (r) is 3. Then, determine the 5th term (a_5) and the sum of the first 4 terms (S_4).
    A. a_5=324, S_4=160
    B. a_5=243, S_4=260
    C. a_5=243, S_4=160
    D. a_5=324, S_4=260

50. Create a polynomial expression that can be factored by difference of squares.
    A. $x^2 - 4$
    B. $x^2 - 5$
    C. $x^2 - 6$
    D. $x^2 - 7$

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

The Problem Corner



Ivan Retamoso, PhD, *The Problem Corner* Editor
Borough of Manhattan Community College
iretamoso@bmcc.cuny.edu

The Purpose of ***The Problem Corner*** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than ~~in~~ the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word~~,~~ using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to the posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word~~,~~ using math type or equation editor, and email your proposed problem and its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, problem solvers!

We are pleased to share that Problems 42 and 43 in *The Problem Corner* have received exceptional solutions, marked by both accuracy and originality. Highlighting these remarkable approaches aims to inspire others and foster a deeper appreciation for mathematical thinking worldwide.

Solutions to **problems** from the previous issue.

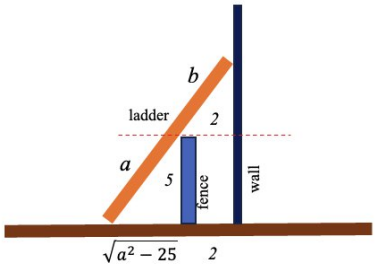
Problem 42
Proposed by Ivan Retamoso, BMCC, USA.

A 5-foot-tall fence stands parallel to a tall building, 2 feet away from it. Determine the length of the shortest ladder that can extend from the ground, over the fence, to touch the wall of the building.

First solution to problem 42

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (**CC BY-NC-SA 4.0**). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

By JUDEL V. PROTACIO, Capiz State University, Capiz, Philippines.



Solution. Let $c = a + b$ be the length of the ladder where a is the part of the ladder from its foot to the point that meets the fence while b is the remaining part as shown. By Pythagorean theorem, the distance of the foot of the ladder from the base of the fence is $\sqrt{a^2 - 25}$. By similarity of two right triangles,

$$\frac{a}{\sqrt{a^2-25}} = \frac{b}{2}$$

so that $b = \frac{2a}{\sqrt{a^2-25}}$. Since $c = a + b$, then

$$c = a + \frac{2a}{\sqrt{a^2-25}}.$$

We calculate $\frac{dc}{da}$.

$$\frac{dc}{da} = 1 + \frac{2\sqrt{a^2 25} - 2a(\frac{1}{2})(a^2-25)^{-\frac{1}{2}}(2a)}{a^2-25}$$

so that

$$\frac{dc}{da} = \frac{(a^2-25)^{3/2}-50}{(a^2-25)^{3/2}}$$

We set $\frac{dc}{da} = 0$ and solve for a to get $a \approx 6.21\ ft$. We then solve for b,

$$b = \frac{2a}{\sqrt{a^2-25}} = \frac{2(6.21)}{\sqrt{6.21^2-25}} \approx 3.37 ft$$

Therefore, $c = a + b = 9.58\ ft$.

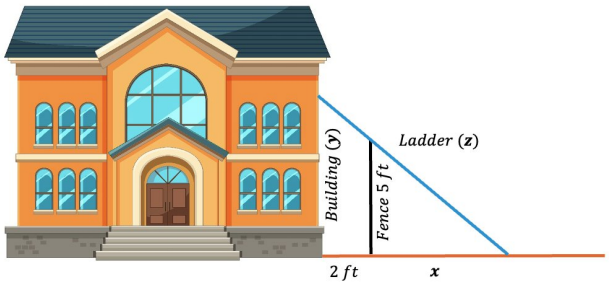
To verify that c is minimum, we compute $\frac{d^2c}{da^2}$. Since

$$\frac{d^2c}{da^2} = \frac{50}{(a^2-25)^3} > 0\ for\ a > 5,\ c\ is\ minimum.$$

Second solution to problem 42
Dr. Abdullah Kurudirek, Al-Naji University, Baghdad.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

Solution: First, let's begin our work by representing the given information on a diagram



First, using the Pythagorean theorem, we can write

$$z^2 = y^2 + (x+2)^2 \quad (1)$$

Next, using the similar triangles formed by the building, the ground, and the fence, we can write

$$\frac{5}{x} = \frac{y}{2+x} \quad (2)$$

From equation (2), we obtain $y = \frac{5(2+x)}{x}$, and after substituting this into equation (1), we obtain

$$z^2 = (\frac{5(2+x)}{x})^2 + (x+2)^2 \quad (3)$$

To find the minimum length, we differentiate z with respect to x and set the derivative to zero.

$$\frac{dz}{dx} = \frac{(x+2)(x^3-50)}{x^2\sqrt{x^4+4x^3+29x^2+100x+100}} = 0 \quad (4)$$

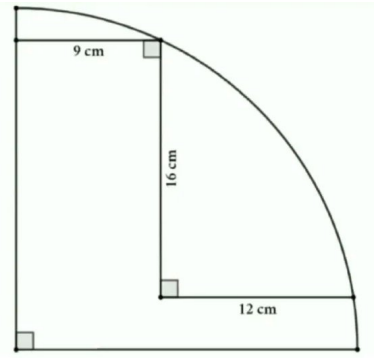
$x = -2$ (impossible) so $x = \sqrt[3]{50}$

The shortest ladder length (z) is approximately 9.58 ft.

Problem 43
Proposed by Ivan Retamoso, BMCC, USA.

In the figure below find the radius of the circle.

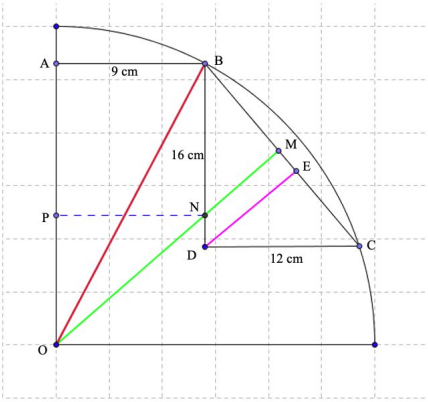
This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.



First Solution to problem 43

By JUDEL V. PROTACIO, Capiz State University, Capiz, Philippines.

In the figure, find the radius of the circle.



Solution.

1. Initially label the intersection points as A, B, C, D and O as the center of the quadrant as shown.
2. Draw $\overline{BC}$, the hypotenuse to ΔBCD so that by Pythagorean theorem, $BC = 20\ cm$. Draw the altitude $\overline{DE}$ to the hypotenuse $\overline{BC}$.
3. Locate the midpoint M of $\overline{BC}$. Draw $\overline{OM}$. By chord-radius relationship, $\overline{OM}$ is the perpendicular bisector of $\overline{BC}$ so that $BM = CM = 10\ cm$.
4. Label the intersection of $\overline{BD}$ and $\overline{OM}$ as point N.
5. With $\overline{DE}$ as altitude to the hypotenuse,

   $EC = \frac{36}{5}$ cm, so that $EC = \frac{64}{5}$ cm and $DE = \frac{48}{5}$ cm.
6. Likewise, $\Delta DEB \sim \Delta NMB$ so that $NB = \frac{25}{2}$ cm and $NM = \frac{15}{2}$ cm.

This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.

7. Draw $\overline{PN}$ parallel to $\overline{AB}$ so that $PN = 9\ cm$ and $AP = 12.5\ cm$. It follows that $\Delta NMB \sim \Delta OPN$ so that by similarity,

   $ON = \frac{45}{4}$ cm and $OP = \frac{27}{4}$. This gives $AO = \frac{77}{4}$ cm.
8. Finally, draw $\overline{OB}$, the radius of the circle, so that by Pythagorean theorem, $OB = \frac{85}{4}\ cm$.

Second Solution to problem 43

Dr. Abdullah Kurudirek, Al-Naji University, Baghdad.

Solution: In solving this problem, we can make use of a coordinate system. For this purpose, I placed our quarter circle as shown below with its center at $(0,0)$, and ~~let's~~ added auxiliary lines with dashed lines alongside the x- and y-axes to support our solution. From here, we can easily identify two points on the quarter circle, $A(9,h)$ and $B(21,h-16)$, and these points will satisfy the circle equation $x^2 + y^2 = r^2$.

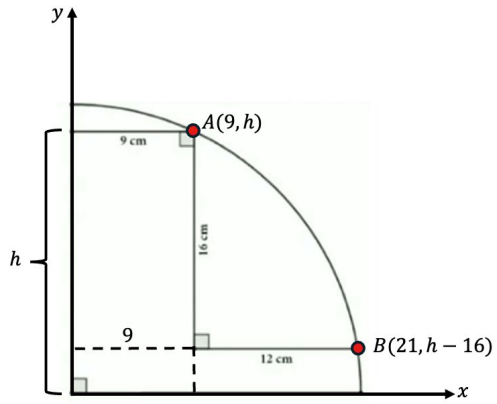
Now, when we write these points, we obtain the following two expressions:

$$9^2 + h^2 = r^2 \quad (1)$$

$$21^2 + (h-16)^2 = r^2 \quad (2)$$

From this, $9^2 + h^2 = 21^2 + (h-16)^2$ and we obtain $h = \frac{77}{4}$

If you substitute this value h into either of the above equations (1) or (2), you get radius $r = \frac{85}{4}$.



This content is covered by a Creative Commons license, Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0). This license allows re-users to distribute, remix, adapt, and build upon the material in any medium or format for noncommercial purposes only, and only so long as attribution is given to the creator. If you remix, adapt, or build upon the material, you must license the modified material under identical terms.