Exploring the Part-Whole Subconstruct in Fractions Learning: Insights from a Design Research Study

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Abstract: The part-whole relationship is sometimes overlooked in fractions learning due to its perceived limitations. However, prior studies have shown that students often struggle to master the part-whole subconstruct and encounter difficulties. Despite these challenges, the part-whole subconstruct plays a critical role in supporting students' subsequent fractions learning. This study aimed to explore how students understand and reason about the part-whole subconstruct across its key components. Using a design research approach, a teaching experiment was conducted with 38 fourth-grade students. Students' written works, transcripts from the teaching experiment recordings, and observational notes were retrospectively analyzed to examine the learning trajectory. Several key findings emerged from the teaching experiment, each corresponding to a component of the part-whole subconstruct. We highlight that (1) learning the part-whole subconstruct involves progressively building upon students' prior fractions concepts across its key components and (2) working with incomplete situations helped students refine their understanding of what constitutes a valid fractional representation. These findings offer valuable insights for improving instruction on the part-whole subconstruct and addressing challenges associated with it. The study concludes with recommendations for future research based on the findings.

Keywords: Design Principle, Design Research, Fraction, Part-Whole, Teaching Experiment

INTRODUCTION

Prior research suggests that the usefulness of the part-whole subconstruct is limited. For example, it interferes with understanding of improper fractions. The part-whole subconstruct is described as a situation in which a whole is divided into equal parts, with a fraction representing how many of these parts are being considered (Charalambous & Pitta-Pantazi, 2006). Thus, given that a fraction is a part of a whole, the numerator of a fraction must be smaller than or equal to the denominator. The part-whole subconstruct relies on mental actions of partitioning and disembedding and misses out the mental action of iterating (Wilkins & Norton, 2018). Also, the part-whole subconstruct involves equal parts (one slice out of four and one t-shirt out of four t-shirts) and equal wholes (1/4 of a pie is the same as 2/8 of a pie); without a proper understanding students might answer that 1/4 of a family-sized pizza and 1/4 of a regular-sized pizza are the same (Pedersen & Bjerre, 2021). The part-whole subconstruct is fundamental in developing understanding of the other four subconstructs (measure, ratio, quotient, and operator) (Charalambous & Pitta-Pantazi, 2006), which also plays a role in students' subsequent understanding, such as equivalent fractions (Pedersen & Bjerre, 2021). Thus, the part-whole subconstruct of fractions is also fundamental to overall students' conceptual understanding of fractions.

Previous research indicated that by the end of the fifth grade, students have yet to attain mastery of the part-whole subconstruct of fractions and encounter difficulties in associating congruency with parts of the whole. Their common errors include: 1) failing to associate fractions with equal divisions of a whole; 2) linking fractions solely to congruent segments of the whole; and 3) identifying the portion of the whole that is represented (Hodnik Čadež and Manfreda Kolar, 2018). It is important to pay attention to these errors for excellent student understanding of the part-whole subconstruct.

Most prior studies on the part-whole subconstruct have primarily focused on equal partitioning (Cramer & Wyberg, 2009; Torres-Peña et al., 2024) or general partitioning (Purnomo et al., 2021). However, these components alone do not fully capture the complexity involved in understanding the part-whole subconstruct. There are several other components in the part-whole subconstruct that need to be accounted as well. These aspects are often overlooked in the literature, which tends to emphasize other subconstructs, like measure, ratio, quotient, or operator (Purnomo et al., 2021).

While some studies have questioned the effectiveness of the part-whole subconstruct to introduce fractions (Purnomo et al., 2021; Simon et al., 2018; Watanabe, 2006; Wilkins & Norton, 2018), it is important to acknowledge that subconstructs of fractions are interrelated (Kieren, 1976) and mutually supportive (Charalambous & Pitta-Pantazi, 2006). Each subconstruct contributes in a distinct and essential way to building a comprehensive understanding of fractions. For example, although the equal partitioning, unitizing, and reunitizing components of the part-whole subconstruct may be addressed through the partitioning and iterating components of the measure subconstruct (Purnomo et al., 2021), the inclusion component of the part-whole subconstruct supports students in distinguishing between fractions as part-whole relationships and as ratios, as in understanding the difference between $\frac{2}{5}$ and $\frac{2}{3}$ (Charalambous & Pitta-Pantazi, 2006). Moreover, development of fraction subconstruct in isolation does not guarantee in understanding of the other concepts (Brousseau et al., 2004).

Theoretical Framework

The theoretical framework is structured around the lens of a deeper examination of the conceptual understanding required to develop students' understanding of the part-whole subconstruct to guide the present study.

Understanding the Part-Whole Subconstruct

Charalambous and Pitta-Pantazi (2006) have outlined several ideas or concepts that students need to develop to master the part-whole subconstruct of fractions. First, they proposed that students must comprehend that the parts into which the whole is partitioned must be of equal size, and they should be capable of partitioning a continuous area and a discrete set into equal parts, as well as determining whether the whole has been partitioned in equal parts. This statement highlights that equal partitioning is an important component of the part-whole subconstruct. However, the subconstruct entails more than simply partitioning the whole into equal parts. Other aspects must be taken into account, like recognizing different types of wholes and comparing parts in relation to the whole. Next, they talked about the idea of inclusion or embeddedness and they provided an illustration of this component: unless pupils develop this understanding, they might erroneously count some parts twice—once for the numerator and again for the denominator. For example, when asked to identify 2/3, students are likely to choose a representation illustrating 2/5. They emphasized the importance of understanding that the parts are to be considered as integral components of the whole. Furthermore, they also underscored that the part-whole relationship should remain consistent despite changes in the representation of the parts. Finally, they concluded the ideas of mastering the part-whole subconstruct with the component of unitizing and reunitizing—a component that allows students to fully understand the part-whole subconstruct. This statement emphasizes the necessity of reconstructing the whole based on its parts. The first four points in Table 1 summarizes these ideas.

In addition to Charalambous and Pitta-Pantazi's (2006) ideas on mastering the part-whole subconstruct, we also see the potential to support students' understanding through the problem-solving component. Pramudiani et al. (2022) found that the ambiguity regarding what constitutes the whole can be used to foster students' understanding of the part-whole relationship. They proposed a task designed to stimulate students' reasoning about the part-whole relationship in an incomplete situation, posing the question: "Would students understand a fraction as a part of a whole when one part is missing?". We see the potential of this problem to further support students' understanding of the part-whole relationship by encouraging them to recognize the relationship even in less straightforward contexts. The last point in Table 1 summarizes this idea.

No	Components	Description	Source
1	Equal partitioning of the whole	Students should understand that the parts into which the whole is partitioned must be of equal size.	Charalambous and Pitta-Pantazi, 2006.
2		Student should be able to partition a continuous area or a discrete set into equal parts and discern whether the whole has been partitioned in equal parts.	Charalambous and Pitta-Pantazi, 2006.
3	Inclusion	Students should develop the idea of inclusion or embeddedness, according to which parts are considered as components of the whole.	Charalambous and Pitta-Pantazi, 2006.
4	Unitizing and reunitizing	Students develop unitizing and reunitizing abilities, which allow them to reconstruct the whole based on its parts.	Charalambous and Pitta-Pantazi, 2006.
5	Part-whole relationship in incomplete situations	Students' understanding of fractions should be well addressed with challenging problems on the part-whole relationship, in this case in incomplete situations.	Pramudiani et al., 2022.

Table 1: The Framework of the Part-Whole Subconstruct

Hence, this study differs from prior studies in the aspect of building a strong foundation of the part-whole subconstruct by integrating a series of questions that contain important part-whole components. This study contributes to both the practical design of instructional tasks and the theoretical development of learning trajectories in fractions. Therefore, the main research question addressed in this study is "How do students understand and reason about the key components of the part-whole subconstruct in fractions?".

In order to help students understand and develop an ability to reason about the key components of the part-whole subconstruct, we have formulated the following specific sub-research questions:

R1. How are previous fractional concepts used to understand new ones?
R2. How incomplete problem situations can expand insights into fractions learning, particularly in terms of the part-whole subconstruct?

METHOD

Research Procedure

Design research seeks to develop theories with a focus on specific learning processes (Bakker, 2018). It aims to establish a trajectory for students to transition from their existing knowledge to the desired learning objectives (Putri et al., 2021). Therefore, this study adopted design research as its research methodology to answer the research question by enabling the students to understand fractions as parts of a whole through a series of activities such as equal partitioning, unitizing, and solving problems involving incomplete partitions. The design research methodology consists of three phases: 1) preparation for experiment, 2) teaching experiment, and 3) retrospective analysis (Bakker, 2018).

Participants and Context of the Study

This study was conducted in a public elementary school in a suburb area in Palembang, Indonesia. The participants of the teaching experiment were 38 fourth-grade students in a whole class setting with heterogeneity in their mathematical performance, as informed by the teacher. The selection of the class was influenced by the teacher's willingness to participate in the study. The teaching experiment involved two lessons over two weeks. The classroom setting was equipped with basic teaching tools, such as a whiteboard and visual aids, supplemented with hands-on materials—printed pictures of objects, such as *martabak*, Indonesian stuffed sweet pancake—to engage students through concrete representations of fractions. This context was chosen for reasons of its alignment with the study's aim of integrating students' informal knowledge into the teaching of fractions as parts of a whole.

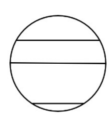
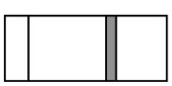
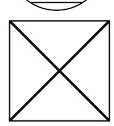
Task Design

The task employed in this study was designed based on the components of the part-whole subconstruct, as reviewed in the literature (see Table 1). The first component is the equal partitioning of the whole. Charalambous and Pitta-Pantazi (2006) proposed that students must comprehend that the parts into which the whole is partitioned must be of equal size and they should be capable of partitioning a continuous area and a discrete set into equal parts as well as determining whether the whole has been partitioned in equal parts. In the teaching experiment conducted by Yoshida and Sawano (2002), the importance of equal partitioning and equal whole was emphasized for understanding fractions. Their findings showed that students who received an instruction focused on these two aspects outperformed the control group in solving non-routine fractions tasks. Simon (2006) further explained that in equal partitioning, students need to understand that the fraction produced represents a new unit whose size is relative to the original unit (whole). Therefore, we present questions 1, 3, and 5 to address the first component (see Table 2). These questions address students' understanding of equal partitioning, adapted from the study by Pantziara and Phillipou (2011). Question 1 focuses on students' ability to determine whether the whole has been partitioned equally. Question 3 assesses students' ability to reunitize the whole to produce equal partitions. Finally, question 5 requires students to partition a discrete set into equal parts.

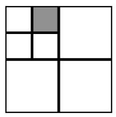
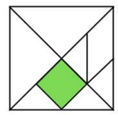
The next component is inclusion. This component helps students develop the idea of inclusion or embeddedness (Charalambous & Pitta-Pantazi, 2006). Lamon (2020) further suggested using discrete units of various types, including units composed of more than one object, since children often struggle with units that consist of multiple objects. Incorporating this component is essential for advancing students' part-whole understanding—particularly in recognizing and identifying the correct whole in more complex representations. Therefore, we adapted a task from Lamon (2020) that addresses students' understanding of inclusion for question 4.

The component of unitizing and reunitizing refers to the ability to reconstruct the whole based on its part (Charalambous and Pitta-Pantazi, 2006). Bruce et al. (2023) defined a unit fraction as the representation of the smallest segment of a fraction. Unit fractions can be combined or iterated to generate larger quantities, or, conversely, the quantities can be deconstructed to generate the underlying unit fractions. They further explained that engaging with unit fractions helps pupils internalize the meaning of fractions as quantities. To engage students with this component, we came up with 6 questions. According to Simon et al. (2018), this type of task facilitates the process of building students' understanding of fractions based on composite unit concepts. This enables students to coordinate their composite unit concepts with reunitizing to construct a unit system for measurement. We believe that this kind of task would be beneficial for students' fractions understanding.

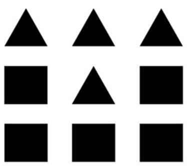
In terms of the problem-solving in fractions as parts of the whole subconstruct, Pramudiani et al. (2022) found that comprehending the relationship of fractions as parts of the whole should be well addressed. They designed tasks to stimulate students' reasoning regarding the part-whole relationship. Therefore, we adapted their task of the part-whole relationship in incomplete situations to address students' understanding of fractions as parts of the whole in question 2 by encouraging students to recognize the part-whole relationship in less clear situations. Table 2 encompasses all the questions utilized in this study.

No.	Question
1	Which of the following figures represents the fraction $\frac{1}{4}$? a.  c.  b.  d. 
2	Write the fraction that is represented by the cheese-topped *martabak.*

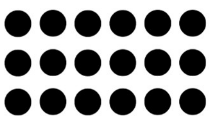
3 Write the fraction that is represented by of the shaded area.

a.  b. 

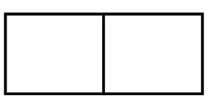
4 What fraction is represented by the triangle?



5 Circle $\frac{2}{3}$ of these objects.



6 a. The fraction of the following figure represents the fraction $\frac{2}{3}$. Determine what the figure would look like if the fraction is $\frac{3}{3}$.



b. The fraction of the following figure represents the fraction is $\frac{2}{3}$. Determine what the figure would look like if the fraction is $\frac{4}{3}$.

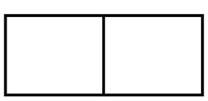


Table 2: The Questions on the Part-Whole Subconstruct

Conjectures of Students' Thinking

It was conjectured that for question 1 students could choose figure a. We conjectured for question 2 that students might answer $\frac{1}{7}$ because they had not yet been exposed to the part-whole relationship in incomplete situations. For question 3a, students could draw another four small squares in each big square to discern whether the whole had been partitioned in equal parts, then give $\frac{1}{16}$ as the answer. Meanwhile, students might find question 3b difficult since the already-partitioned parts are not congruent. Question 3b assesses students' ability to combine certain processes and exhibit signs of conceptual understanding to a certain extent (Pantziara & Philipou, 2012). For question 4, we conjectured that students would understand that when they were asked to determine the fractions, they would be able to identify the whole being referred to. For question 5, we conjectured that students could circle 12 objects. Lastly, for question 6, we expected that some students would come up with the idea of finding the unit fraction and then re-unitizing it to determine the desired fraction.

Data Sources and Analytical Procedure

The data sources for this study comprised students' written works, recordings of the teaching experiment, and observational notes from whole-class conversations. The data analysis technique used involved a retrospective analysis by comparing the hypothetical learning trajectory with the actual learning to determine its alignment with the learning objective. The retrospective analysis entailed: 1) examining students' works, 2) transcribing classroom discussions and scrutinizing the observational notes, 3) describing students' understanding and reasoning during whole-class discussions recorded in the observational notes, and 4) evaluating the match of learning hypotheses and students' actual learning. The selection of students' answers or responses for analysis, described in the Results and Discussion section, was done by purposive sampling. Specifically, examples were chosen to represent the range of students' understanding. This approach allowed the study to capture diverse reasoning patterns, difficulties encountered during the learning process, and how to tackle the difficulties. While all the 38 students participated in the teaching experiment, selected students' works and dialogues were highlighted to illustrate key aspects of learning related to each component of the part-whole subconstruct.

RESULTS AND DISCUSSION

Equal Partitioning of the Whole

For question 1, it was not hard for students to circle figure a. However, there were some students who circled figure b as well. What one student knew, for example, was that to determine the fraction, all he had to do was count the shaded parts and the unshaded parts. Below is an excerpt of the teacher and the student (S1) (see Figure 1).

S1: I think figure b is also (the answer).

T: And why is that?

S1: Because there is one shaded part

T: And then?

S1: Hmm... I don’t know. It’s usually like that.

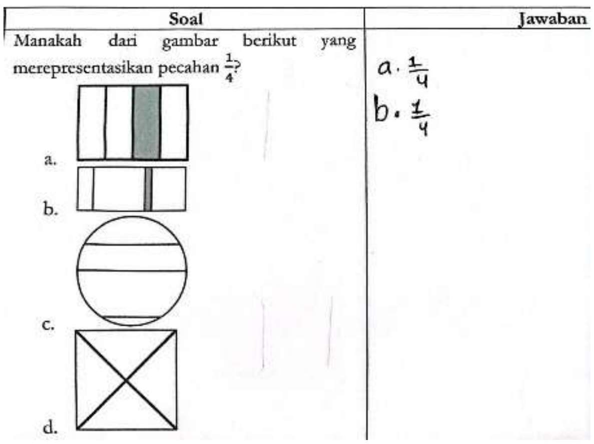


Figure 1: Students’ representation of $\frac{1}{4}$

The teacher then asked students to continue to task 3a. Unfortunately, the real process missed the conjecture. No students were able to repartition another three big squares into four smaller squares. Instead, they answered $\frac{1}{7}$. The teacher then returned to question 1 and asked the students what the difference is between figure a and b.

S2: The sizes (of the parts) are different.

T: And are the parts in task 3a different (in size)?

S2: yes, so the answer is not $\frac{1}{7}$. It’s $\frac{1}{4}$.

T: Where did you get $\frac{1}{4}$?

S2: One, two, three, four (counting the four smaller squares in the one big square).

T: The object is not only these (the four small squares) but all of this (the whole figure).

S2: (repartitioning the object like in figure 2)

T: Can you name the fraction of the shaded part now?

S2: It’s one-sixteenth.

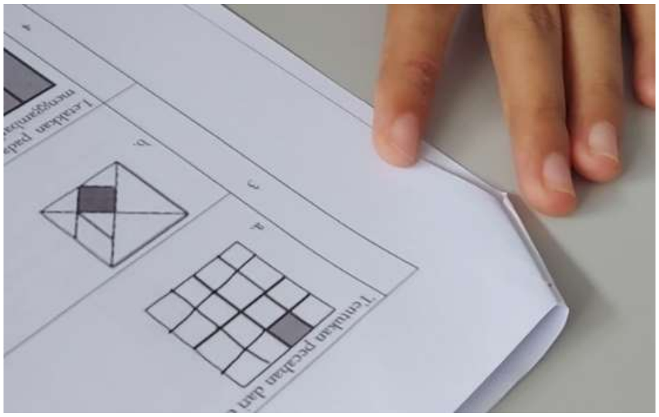


Figure 2: Student's repartitioning the figure in question 3a

Unlike task 3a, task 3b was quite difficult to be repartitioned. However, S2 understood that the repartition should be in the triangle (see the red circle in figure 2) even though it was still hard for her since she mistakenly drew a wrong triangle (see the green circle in figure 3). To support the student, the teacher gave her a hint to look at the shape she thought the partitions would be and ensured that all the partitions she made look exactly alike. On the other side, task c was quite easy for students to solve. They were all able to circle 12 circles objects to mark $\frac{2}{3}$ of all the objects.

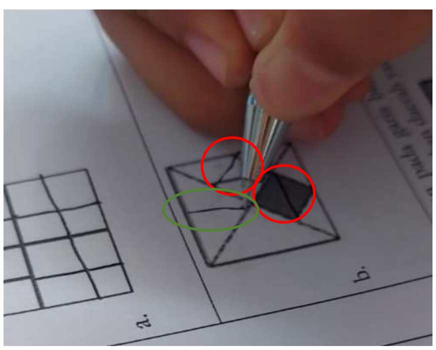


Figure 3: Student's attempt to repartition the figure in question 3b

The teaching experiment on equal partitioning tasks revealed insights into students' understanding related to fractions as parts of a whole. For question 1, while most students correctly identified figure a as representing $\frac{1}{4}$, some students demonstrated a procedural approach to fractions. S1 relied on counting shaded and unshaded parts without considering equal partitioning. Prior studies, such as one by Hodnik Čadež and Manfreda Kolar (2018), have highlighted that children often associate the partitioning of a whole with dividing it into congruent parts, which is a fundamental aspect of understanding fractions. However, our findings suggest that some students may not even recognize the necessity of congruent parts in determining fractions, reflecting a more profound gap in their conceptual understanding. This highlights a prevalent issue in fractions learning: a focus on part-whole relationships without verifying congruence.

In question 3a, students faced challenges with repartitioning unequal parts into equal partitions. Contrary to the conjecture, no student initially attempted to subdivide the larger squares into smaller, congruent parts. This underscores a gap in students' reasoning, highlighting the importance of explicitly teaching strategies for creating equal partitions. Question 5, in contrast, was relatively straightforward, with all students successfully circling 12 out of 36 circles objects to represent $\frac{2}{3}$.

Inclusion

In Question 4, the whole is 9, as the figure contains 9 objects. Half of the students were able to answer $\frac{4}{9}$, but the other half gave $\frac{4}{5}$ as an answer, as in ratio. See the excerpt bellow between the teacher and a student who answered $\frac{4}{5}$ and the answer (figure 4).

T: Why did you answer $\frac{4}{5}$?

S3: Because (the question is) asking what fraction is represented by the triangle. There are 4 triangles and there are 5 squares.

T: That's why your answer is $\frac{4}{5}$?

S3: Yes, Ma'am.

T: What if…. The question is "What fraction is represented by the triangle compared to all of the objects?"

S3: $\frac{4}{5}$?

T: Are you sure?

S3: No… I guess it should be $\frac{4}{9}$, because you said compared to all of the objects.

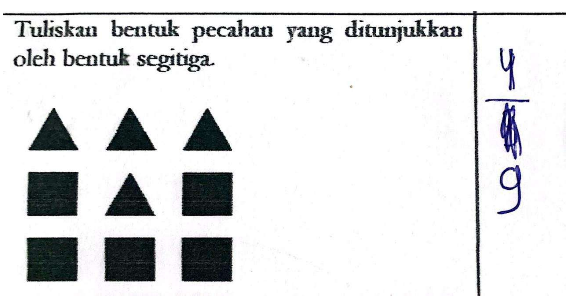


Figure 4: Student's changing his answer

S3's initial struggle aligns with Lamon's (2020) observation that children often have difficulties working with units composed of multiple objects. From another perspective, Möhring et al. (2016) explained that spatial proportional reasoning differs between part-whole and part-part reasoning. This means proportions can be represented as either part-whole relations or part-part relations. Their review also revealed that children's thinking about proportions and their approach to solving such problems changes with age and depends on how the problems are framed. To address this issue, it may be beneficial to articulate questions more explicitly and align them with students' current level of understanding, particularly by clearly defining what is considered the whole.

Unitizing and Reunitizing

For question 6, the idea is to partition the given area to obtain the unit fraction and then iterate it until the desired fraction is achieved. We expected that some students would be able to come up with the idea of finding the unit fraction and then reunitizing it to determine the desired fraction. However, no students were able to generate this idea without additional scaffolding by the teacher. Here is the excerpt of the teacher and one student.

T: Could you draw for me the representation of the fraction $\frac{2}{3}$?

S4: Yes, ma'am (see figure 5a).

T: Now, could you draw for me the representation of the asked fraction?

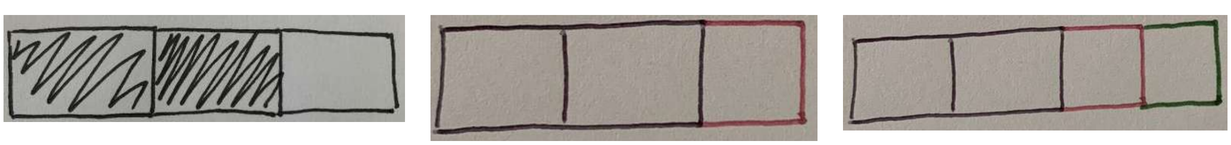
S4: $\frac{3}{3}$? (then she shaded one more unit).

T: Okay, now could you answer this question (question 6)?

S4: So… I have to draw another rectangle (see figure 5b).

T: How about question 6b?

S4: I think I should add one more rectangle to it (see figure 5c).



a b C

Figure 5: Student's unitizing and reunitizing

A similar case was observed in the teaching experiment conducted by Simon et al. (2018). They emphasized the importance of balancing students' exploration with sufficient support to facilitate breakthroughs in understanding. After receiving additional guidance, S4 iterated the unit fraction to generate the requested fraction. Through this process, she learned how fractions function both as parts of a whole and as measures, which will support her future learning of fractions (Simon et al., 2018).

Part-Whole Relationship in Incomplete Situations

When students have a solid understanding of the part-whole relationship, Hodnik Čadež and Manfreda Kolar (2018) concluded, they must be able to identify parts of an equally divided but incomplete whole. Pramudiani et al. (2022) designed tasks to address this issue and found that incomplete situation problems have the potential to encourage mathematical reasoning and deepen the understanding of fractions. Their results also revealed that, when determining a fraction, students sometimes count only the physically visible parts and struggle to refer to the correct whole.

Our findings align with the conjecture for question 2: all students answered $\frac{1}{7}$. Based on the conversation between the teacher and the students, it is evident that they counted only the visible parts of the *martabak* as the whole. To challenge their reasoning, the teacher asked one question: "Would you be okay if you received this kind of *martabak* in this condition from the seller?" See the excerpt below.

T: Would you be okay if you received this kind of *martabak* in this condition from the seller?

S: No…

T: Why?

S5: Because it's incomplete, I would be mad if I pay full.

T: Then, how many parts are there if it's complete?

S5: 8? Because I think it's missing one slice.

T: Therefore, what is the fraction of that is represented by the cheese-topped *martabak*?

S5: $\frac{1}{8}$, ma'am.

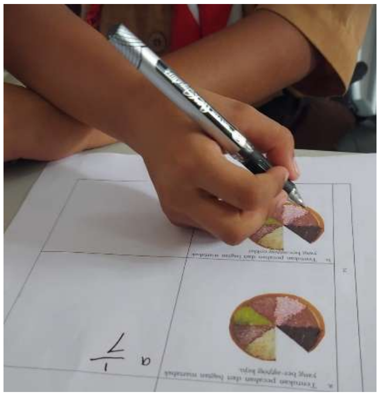


Figure 6: Student's attempt to referring the correct whole

Based on the findings of Pramudiani et al. (2022), students' ability to determine a fraction in an incomplete situation largely depends on their comprehension of fractions as parts of a whole. They suggest stimulating students' persistence in understanding fractions by providing appropriate aids. In this study, the aim was to help students recognize the incompleteness of the *martabak* slices and understand how it affects the fraction that is representeds it. Fosnot and Dolk (2002) observed that such challenges help children build a deeper understanding of fractions by encouraging them to reason through the relationships.

The result of the teaching experiment led to several key findings. The first relates to the component of equal partitioning of the whole. The most critical aspect is ensuring that students comprehend how to determine a fraction, specifically that the whole must be partitioned into equal-sized parts. Although prior studies have found that constructing a fraction is generally considered the easiest aspect of fractions learning (Jiang et al., 2020; Pantziara & Philippou, 2012), this was not observed in the present study. A lack of understanding in this component can hinder students' ability to perform fractional operations and leave them vulnerable to perceptual distractors, especially for those with an unstable understanding of fractions (Noordin et al., 2012). As observed, some students considered figure b in question 1 as a representation of $\frac{1}{4}$. They were also unable to repartition the figures in questions 3a and 3b to ensure the parts were of equal size. Therefore, students' understanding of constructing fractions based on equal partitioning should be carefully emphasized in instructional practices.

The next key finding relates to the component of inclusion. A discrete unit consisting of multiple objects may lead students to interpret fractions as ratios. This observation aligns with the findings of Pramudiani et al. (2022), who noted that students often equate comparing (to the whole) to ratio. Being explicit in such cases is essential, as how children perceive a problem depends on how the teacher presents it (Möhring et al., 2016).

The third key finding relates to the component of unitizing and reunitizing. It is worth noting that unitizing to generate the desired fraction may not come naturally to students. Providing fractional representations that they are familiar with can be beneficial. This also aligns with the finding of Pramudiani et al. (2022). In such cases, engaging with unitizing and reunitizing can help extend students' understanding of the measurement subconstruct of fractions (Simon et al., 2018).

The final key finding pertains to problem-solving within the part-whole relationship in incomplete situations. Previous research has emphasized the interrelation between students' ability to determine a fraction in an incomplete whole and their part-whole comprehension (Hodnik Čadež & Manfreda Kolar, 2018; Pramudiani et al., 2022). Such problems have the potential to stimulate students' reasoning in fractions learning, highlighting the interplay between conceptual understanding, persistence, and the use of aids to support their learning.

However, there is a limitation to the current study. This study focused on advancing students' understanding and reasoning about the part-whole subconstruct only in incomplete situations. Further research on problems related to the part-whole relationship that can potentially expand insights into the part-whole subconstruct, such as those involving more complex scenarios of multiple wholes, could provide deeper insights into fractions learning, particularly within the part-whole subconstruct. Additionally, this study did not systematically examine students' mastery of each component of the part-whole subconstruct as a sequential learning process. Future research could focus on treating each component as a milestone to better understand the progression of students' fractional reasoning.

CONCLUSIONS

The study aimed to explore how students understand and reason about the key components of the part-whole subconstruct in fractions. To provide more comprehensive insights, we divided this aim into two sub-questions: the first focused on how students utilize previous fractional concepts to understand new ones, and the second focused on how incomplete problem situations can expand students' insights into the part-whole subconstruct. In the following, we reflect on how the results obtained contribute to answering these sub-questions.

This study found that students' understanding of the part-whole relationship—particularly the idea that a whole must be partitioned into equal-sized parts—played a crucial role in learning more advanced components of the part-whole subconstruct. However, some students overlooked whether the parts are actually equal, which shows the importance of emphasizing that fractions must come from equally sized parts of a whole.

Students' experiences with part-whole representations also supported their understanding of the inclusion component. For example, their ability to recognize a fraction as a part of a whole helped them begin to distinguish between situations requiring comparison to the whole (part-whole) and those requiring comparison between parts (ratio). This distinction was not always clear to students, especially when the whole is composed of multiple or composite units. Moreover, students' recall of earlier representations helped them make sense of unit fractions. In one case, a student identified two-thirds as a composite unit, and correctly extended it by drawing an additional rectangle to represent three-thirds and four-thirds. This demonstrated the use of previous understanding (i.e., identifying parts of a whole and interpreting them visually) to support learning about improper fractions and iteration. These findings suggest that learning the part-whole subconstruct involves building upon foundational concepts in a progressive manner. Prior knowledge acts as a stepping stone that helps students navigate increasingly complex components, including inclusion and unitizing, thereby strengthening their overall understanding of fractions.

This study also found that presenting students with incomplete problem situations—where certain parts of the whole are missing or not explicitly shown—encouraged them to think more deeply about the part-whole relationship. These situations required students to infer the missing parts and consider what the complete whole should look like, which prompted them to reason beyond surface-level representations.

By recognizing the incompleteness of a situation, students began to understand that a fraction cannot be determined solely based on the visible parts; it also depends on knowing the total number of equal parts that make up the whole. This awareness helped them refine their understanding of what constitutes a valid fraction representation. For instance, they learned that if a whole is divided into unequal parts or if some parts are missing, they need to think more carefully about what the fraction is actually represented. Therefore, incomplete situations can serve as a productive challenge that pushes students to revisit and solidify their foundational understanding of the part-whole relationship. They also support the development of critical reasoning skills, particularly in identifying the whole and verifying the basis of comparison in fraction tasks.

This study focused specifically on students' understanding and reasoning in incomplete problem situations related to the part-whole subconstruct. However, it did not explore other types of problem-solving contexts within the part-whole relationship. Future research is recommended to investigate more complex scenarios, such as tasks involving multiple wholes, to further expand our understanding of fractions learning within this subconstruct. Additionally, future studies may benefit from treating each component of the part-whole subconstruct as a milestone that students need to comprehend before progressing to the next. Experimental research comparing control and intervention groups could provide deeper insights into how students build a more robust and sequential understanding of each component.

ACKNOWLEDGMENTS

The authors thank SD Negeri 120 Palembang for granting permission to conduct this research there. This study was supported by the Ministry of Higher Education, Science, and Technology of the Republic of Indonesia through HIBAH PMDSU 2022-2024, under Main Contract No.

016/E5/PG.02.00/PL.PMDSU/2020 and Derivative Contract No. 0088.06/UN9/SB3.LP2M.PT/2024.

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