

Mathematical Curriculum, Mathematical Competencies and Critical Thinking

Dieter Schott

dieter.schott@hs-wismar.de

Hochschule Wismar, University of applied sciences technology, business and design
Wismar, Germany

ABSTRACT

A different view is given to mathematical curriculum by stressing competencies in mathematical education. Critical thinking is a very general competency which is an inherent part of mathematical and scientific thinking but which is also useful to analyze statements and events in everyday life. So mathematical education can not only contribute to problem solving in science and engineering or to an intellectual personality but also to making processes in society more transparent and to exposing dubious arguments and interests.

The paper presents some examples showing the interplay of mathematical curriculum, mathematical competencies, critical thinking and everyday life.

INTRODUCTION

In Germany, and also in other countries worldwide, the relation between mathematical knowledge and mathematical competencies within the educational process is controversially discussed. Indeed, there is a close connection between both concepts. Knowledge should create competencies and competences reflect on knowledge. The new trend of stressing competencies at least gives the opportunity to investigate this relation deeply, to make it conscious for the public and to draw fruitful consequences for mathematical education. But, some new prophets overemphasize competencies in such a way that bloodless and spongy categories (with empty or poor content) are left at the end of the educational process.

The mathematical curriculum considers content which is necessary to solve important problems and to understand other important parts of mathematics and their interplay. At the engineering faculty of Hochschule Wismar this curriculum is based on *Linear Algebra* (vector and matrix

concept), *Analysis* (calculus and differential equations), *Numerical Mathematics* (approximate methods, algorithmic thinking, error estimation, time and storage effort) and *Stochastics* (probability concept, distributions, conclusions from random samples). Critical thinking is trained based on logical thinking, on precision, on error estimation and on completeness of arguments. Besides, *paradoxes* and *counterexamples* in Mathematics are especially suitable to train such competences as critical thinking. Often they occur in the educational process as challenges of thinking. Paradoxes can play a very useful role by producing fruitful discussions, provoking deeper thinking about the subject, clarifying new concepts and giving better insight to the theory. So they can help removing potential conflicts between intuition, theory and reality [1], [7].

Finally, modern trends in education as *modelling* and *simulation*, use of *software* and *internet* sources as well as project work in teams are integrated to solve also more complex scientific or applied problems.

Often certain formulas measuring or relating quantities of interest are given in the public. Some of these formulas can be analysed by using basic mathematical means whether they are of more or less practical value, which modelling assumptions are hidden behind and which features are neglected by the model. Sometimes some specialists try to award a discipline more importance by creating useless mathematical formulas. This is called *pseudo-mathematisation*. Further, some interesting news containing mathematically looking data are presented in the media. Mathematically educated people as engineers should be able to decide if these data are seriously used or if they serve more or less ideological and personal interests, respectively. This is often not clear at the first view and needs some investigation.

The topic is already outlined in paper [6] with some other priorities. The paper [5] gives some insight into the interaction between mathematics and engineering concerning the development of competencies. The calculation of support forces in engineering mechanics corresponds to the solution of systems of linear equations. Support forces turn out to be statically determinate if and only if there is a unique solution of the corresponding linear system. Hence, the experience of engineers is backed by a clear mathematical criterion supplying a deeper understanding. Mathematics can play a similar part also in other disciplines and in everyday life.

CURRICULUM AND COMPETENCE

In earlier times the curriculum was the most important educational base. The contents and its volume were fixed for different study courses and different levels. Later a catalogue of learning aims was added. At the end of a curriculum section the learners should be able to solve fixed problems. Nowadays the competence model is the most favorite one. It has become a vogue word. It describes abilities and skills which are needed for a qualification, e.g. as Bachelor or Master in a certain engineering subject. There is more freedom for lecturers to fix contents and methods for acquiring of prescribed competencies. Besides, the ability for further study and the solution of everyday problems are included. Hence, the courses are also application oriented. Besides, there are methods to measure the quality development in learning processes. But, the definition of competencies is often rather vague and open for different interpretations. Finally, also the international discussion is rather broad. So the concept is used in most cases intuitively. Essential general mathematical competencies are:

- Use of *knowledge*: understanding mathematical theory, knowing important facts, linking between disciplines;
- Correct use of *technical elements*: mathematical language, logical reasoning, rearranging or transforming of terms, use of means as tables, formulas, computer software and media;
- *Problem solving*: applying and transferring solution methods, applying heuristic methods, generalizing, creating new connections and concepts;
- Use of *Methodology*: algorithmic, numerical, analytical and stochastic thinking, geometrical imagination;
- *Mathematical modelling*: creating suitable models, interpreting and validating results;
- *Critical thinking*: checking correctness and completeness of results;
- *Communication skills*: team working, networking.

These competencies are also developed in our mathematical curricula at Hochschule Wismar. The nomination of competencies allows drawing links between subjects and competencies. Unfortunately, for all our efforts, many students can only solve problems mechanically following given algorithms. The introduction of new teaching methods can not eliminate this phenomenon, although some experts in didactics insist to claim that.

PSEUDO-MATHEMATICS

There are different roots for applying mathematics in a wrong way:

- Mysticism, giving subjects a curious mathematical meaning;
- Naivety, missing competence in the sphere of application;
- Use of mathematical models without checking the model assumptions;
- Mathematical modelling in a sphere not yet prepared for (*Quételet*: Social Physics [4], *Fechner*: Psycho-Physics [3]);
- Trickery, upgrading a subject by linking to mathematics;
- Rationalism (Mathematics is behind all things);
- Propaganda (ideology, interests), using mathematics to influence persons in a certain direction.

Nevertheless the book [4] of *Quételet* (1869) is considered as a starting point and milestone for the quantitative analysis of human and social qualities. According to the book [3] the work of *Fechner* is also well estimated as the beginning of a new era in psychology. Although the legitimacy of mathematics in natural sciences is recognised, the applications of mathematics in social sciences and economy are often discussed controversial. The complexity in society is very high and of another quality as in nature. The models are often either too simple or uncritically transferred from other fields. The consequences of pseudo-mathematics are double-edged. A lot of people believe at the omnipotence of mathematical formulas and models and feel helpless because mathematical power seems to dictate the world. Other people mistake mathematics as an elite discipline with no or only little value for the general public.

MODELING

Often simple models are used in everyday life. If the applications are successful some people do not think at Mathematics at all and some others believe that Mathematics regulates the world. In any case, they take the success for granted. But what happens, if the result turns out to be wrong. Then people often blame Mathematics for the disaster. They do not realize the true causes for misleading results:

- The model is too simple for the given facts.
- The model is given a wrong meaning.
- The range of application is not observed. Prerequisites of the model are ignored.
- The model is used in a new context (without proving the legitimacy).

In everyday life the problems are not uniquely given. It is necessary to see problems and to formulate them adequately in a mathematical model. There are a lot of possibilities to create such models. The art is to find a sufficiently simple one which is good enough for successful applications.

Modelling in science or engineering shows that there is a gap between models and reality. Models can replace parts of reality under certain conditions. Models show then a similar behaviour as real systems. Often models have only a restricted range of application which has to be known and observed to get reasonable results. Models which hold under some conditions have to be replaced by more general models if these conditions are violated. Sometimes these more general models are already known; sometimes they still have to be found. E.g., the simple linear differential equation

$$x'(t) = c \cdot x(t) \quad (1a)$$

of first order describes exponential growth of a population or a process. Indeed such a model is realistic if there are enough resources for a completely free development (bacteria, certain time intervals of growth processes). The British scientist *Robert Malthus* assumed such a model for the human population and predicted a catastrophe on earth (1798). In the meantime it is known that limited resources slow down the

growth of a population. *Verhulst* proposed a new model with a self-limitation for large populations (bounded carrying capacity, 1838):

$$x'(t) = c \cdot x - d \cdot x^2 \quad (c > 0, d > 0) \quad (1b)$$

The negative second term on the right side of the so-called logistic differential equation reduces the growth to a certain limit. This equation is also of first order, but nonlinear. For $d=0$ the old model (1a) arises. Since the logistic model was successfully applied to a lot of growth processes in nature and economy some people believed that there is a logistic principle in the world. But a lot of other models with restricted growths were found considering also the age process in a population or other additional parameters of influence. A free harmonic oscillator is described by the linear homogenous differential equation

$$x''(t) + b \cdot x'(t) + c \cdot x(t) = 0 \quad (b \geq 0, c > 0) \quad (2a)$$

of second order. The solutions $x(t)$ are undamped ($b=0$) or damped ($b>0$) harmonic oscillations. Often the pendulum is also considered as a harmonic oscillator. But the model equation for the angular displacement $x=x(t)$ is

$$x''(t) + b \cdot x(t) + c \cdot \sin x(t) = 0 \quad (b \geq 0, c > 0) \quad (2b)$$

in this case. Hence, this is a nonlinear differential equation of second order. The solutions $x(t)$ are quite different from the previous ones except for small angular displacements x where $\sin x \approx x$.

FRACTIONS

In physics or electrical engineering the simple law of *Ohm* is well-known:

$$R [\Omega] = \frac{U}{I} \left[\frac{\text{V}}{\text{A}} \right]. \quad (3)$$

This law connects three different magnitudes, namely voltage U , current intensity I and current resistance R in an electrical DC circuit. It states that the resistance is directly proportional to the voltage (for constant current intensity) and indirectly proportional to the current intensity (for constant voltage). The law can be checked by experiments. For instance, the equation $U = R \cdot I$ shows that in a circuit with fixed resistance R the measurement points (I_k, U_k) have to lie on a straight line in the I - U plane. Considering small errors in modelling and measurement this is the case at least approximately. A more sophisticated analysis shows that the resistance depends also on the temperature T . So a complex model should describe the influence of T .

In mathematics the sine of angle $\alpha < 90^\circ$ in a rectangular triangle is the ratio of the side (length) a opposite α and the hypotenuse (length) c :

$$\sin \alpha = \frac{a}{c}. \quad (4)$$

This relation is a definition of sine function or at least a consequence of the definition. It can be experimentally checked by comparing the ratio of side lengths in a plotted rectangular triangle and the sine of the corresponding angle calculated by a computer. There will be an approximate equality. But, this is no mathematical proof. It only shows that the model of Euclidian geometry is adequate in the case of plotted triangles. But fractions

$$z = \frac{x}{y} \quad (5)$$

are also used in other fields expressing the fact or the conviction that the magnitude z increases if the magnitude x increases and the magnitude y decreases. The fraction (5) is indeed the simplest model with these consequences. But the same qualitative effects can be obtained by more complex functions $f(x, y, \dots)$ of two or more variables. How to decide which of these functions f is the most realistic one. If there are experimental data of the involved magnitudes, an optimal function f can be found in a defined model class (least squares method).

There are a lot of attempts to use fractions or more general formulas in the humanities. The Russian poet *Anton Chekov* proposed a fraction rule to measure the *value* V of man, where the power (mind, true reputation and other personal properties) determines the nominator and the meaning about himself the denominator of the fraction:

$$V = f(P, S) = \frac{P}{S} \left[\frac{\text{Power}}{\text{Self-Confidence}} \right]. \quad (6)$$

According to this formula the value increases if the power increases (at constant self-confidence) or if the self-confidence decreases (at constant power). Perhaps people believe that these statements are true or at least reasonable. But, P and S are vaguely determined. Besides, other variables should be considered, too. Perhaps it is useless to give a formula for that at all. But, if people can handle mathematical fractions, the formula suggests a social message, namely that modest people with great power are of excellent moral value. By the way, the message depends also on the spirit of the era. Surprisingly such primitive formulas are presented up today. *Marion Wolf*, a German journalist born in 1950, proclaimed: “The greatness of a person is reckoned by his ability in proportion to his modesty”.

BODY MASS INDEX

The book [8] presenting some mathematical topics to a broad audience was the reason to investigate the value of body mass index (BMI). This index was already introduced in 1870 by *Adolphe Quételet*, a Belgian mathematician and statistician. He measured the weight (mass) m and the height l of 5738 Scottish soldiers and found out that the numbers

$$I = \frac{m}{l^2} \left[\frac{\text{kg}}{\text{m}^2} \right] \quad (7)$$

gave a good impression of the soldiers body constitution if the following simple scale was used:

- I is less than 20: *thin*;
- I is between 20 and 25: *normal*;
- I is between 25 and 30: *thick*;
- I is over 30: *fat*.

Although this sample was not at all representative for the whole mankind, the BMI (7) is used with some variations up today. The simple message is that everyone should reach normal weight. Many websites of body care and nutrition industries offer calculators for BMI and advertise products and treatments for getting optimal BMI values. In Germany fat persons have no chance to start a carrier as civil servants. So the BMI is in some sense proclaimed as a measure for health and beauty, respectively. An analysis shows that the index (7) has a questionable value. Since the denominator represents a surface according to the measure, the BMI can be interpreted as average mass load per surface unit of the body. Is the risk of health really increasing

proportional to the mass and indirectly proportional to the squared height (body surface)? Which influence do ages, sex, build or distributions of muscles and fat tissue have? Statistical investigations can help to get a more specific insight. In medicine indeed the scale is modified for woman, children and people with amputated body parts. Another criticism is based on geometrical arguments. Assuming that the body is approximately a cuboid with average density ρ , width $a = \alpha \cdot l$, thickness $b = \beta \cdot l$ and height $h = l$, the BMI is transformed to

$$I = \frac{\rho \cdot a \cdot b \cdot h}{l^2} = \frac{\rho \cdot \alpha \cdot \beta \cdot l^3}{l^2} = \gamma \cdot l \quad (\gamma = \rho \cdot \alpha \cdot \beta). \quad (7a)$$

Hence, the BMI increases proportional with the body height for all people with constant body proportions (α, β constant) and constant density ρ . This is a strange consequence. By the way, supposing more realistic body forms, the consequence is similar. Considering the function

$$z = f(x, y) = \frac{x}{y^2} \quad (x > 0, y > 0) \quad (7b)$$

of two independent variables x, y it has the same structure as BMI in (7). The graph of this function is a surface in the three-dimensional space. Interpreting x as weight and y as height of persons, z is their BMI. Hence, the points (x, y, z) lie on this surface. Since weight and height are dependent in some way, certain surface regions are expected to remain free. Looking for features with constant BMI $z=c$ the corresponding level curves on the surface are parabolas with points $(c \cdot y^2, y, c)$. The BMI values for persons with constant proportions and density lie on a certain curve within the surface having the parameter representation $(\gamma \cdot l^3, l, \gamma \cdot l)$ (see Figure 1).

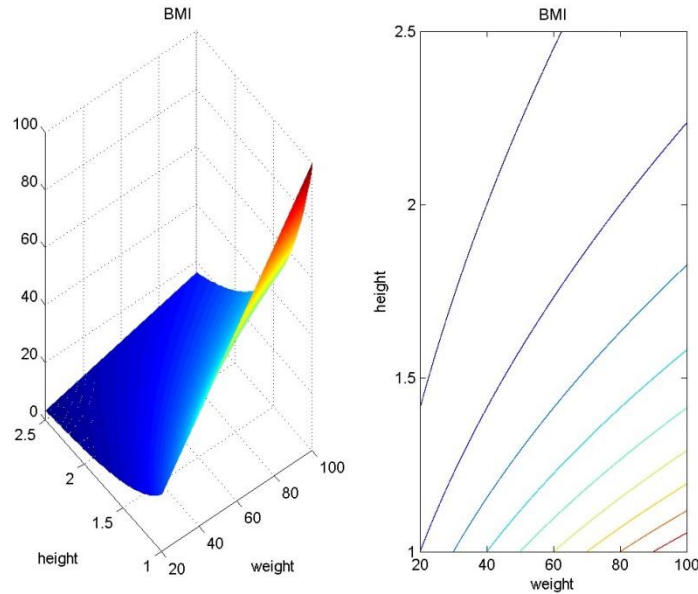


Figure 1: Surface and level curves of function (7b) in the domain $[20, 100] \times [1, 2.5]$.

PROBABILITIES

There are a lot of paradoxes in the field of probabilities, perhaps more than in any other scientific discipline [1], [7]. Some examples containing paradoxes are quite simple to understand, but the common sense struggles against accepting the mathematical results (e.g. Monty Hall paradox, Prisoners' paradox). Who is interested in the mentioned well-known paradoxes should check literature or internet. The crucial point is in both cases to have in mind the effect of conditional probabilities.

The *arithmetic average* is a statistical quantity. It is used to estimate the expectation value of a probability distribution from a random sample. Often people think that average characterizes the typical. So people orientate at average income in their country and compare it between different countries. The country is rich if the average income is high and poor if it is low. But the average income is not at all typical, if only some filthy rich people and a lot of very poor people live in the

country or if middle incomes are very rare. But nowadays the distributions are often not unimodal (as the Gaussian distribution). Figure 2 shows a bimodal distribution. The already mentioned scientist *Quételet* made even a study about *average human* which caused vehement debates. This concept is problematic, among other things, because the averages of single characteristics (as height and weight) do not correspond. Other people even claimed the average is the source of beauty.



Figure 2: Bimodal noisy distribution

It is claimed in the March edition 2009 of the American magazine “Wired” that a mathematical formula caused the Wall Street to collapse and led to the deep finance crisis in the whole world. The well-known Gaussian copula was applied by *David X. Li* to actuarial problems [2]. He used the formula

$$P(T_A < 1, T_B < 1) = \Phi_2(\Phi^{-1}(F_A(1)), \Phi^{-1}(F_B(1)), \gamma) \quad (8)$$

to estimate the risk of finance institutions for investments in correlated securities. The formula gives the probability P that enterprises A and B go bust simultaneously (F_A, F_B distribution functions of T_A, T_B , correspondingly; Φ, Φ_2 one- and two-dimensional standardized Gaussian normal distribution function, correspondingly, γ parameter). The formula can also be extended to more than two enterprises. *Li* used parallels to methods in life insurance in calculating the probability that married couples die in the same year. It was simple to apply this formula, and finance managers used it widely. But the formula contains a parameter, the correlation coefficient γ . *Li* assumed this coefficient to be constant and estimated it on the base of historical data. But at some stage this coefficient

started to increase rapidly, turning into a fast increasing function. The classical model failed under the new conditions. So-called finance experts used Mathematics without understanding the meaning of the underlying actual phenomena. Policy and greed played an important part that the formula (8) was not guilty in this case.

CONCLUSIONS

The essential findings concerning curriculum, competencies and critical thinking in mathematical education of engineering students are:

- It is especially important to know the basics (basic facts as well as basic techniques) because they are used in all mathematical disciplines and in practical applications again and again. Besides, these basics do hardly depend on the historical development. Poor knowledge of basics reduces the chance to get a satisfactory job.
- In some sense it is more important to learn the kind of thinking in mathematics (the methodology) than to learn the solution of certain time dependent problem classes.
- Modelling should be an essential part of the curriculum, since it is necessary to understand the part of mathematics in engineering and in practice. Already simple models show what can happen and why.
- A reasonable curriculum should supply not only problem solving competencies for certain disciplines but also for everyday life to make general processes more transparent. Hence mathematical competencies are important to contribute essentially to the welfare of our society.
- Mathematical educators should encourage young people to act not only emotional to societal developments but also rational using mathematical or logical arguments.

REFERENCES

1. Klymchuk, S., Kachapova, A., Schott, D., Sauerbier, G. and Kachapov, I., Using paradoxes and counterexamples in teaching probability: A parallel study. *Mathematics Teaching Research Journal Online*, 5, 1, 50-70 (2011).
2. Li, D., On Default Correlation: A copula function approach. *Journal of Fixed Income*, Vol.9, Iss.4, 43-54 (2000).
3. Meischner-Metge, A. (Hg.), *Gustav Theodor Fechner – Werk und Wirkung*. Leipziger Universitätsverlag, Leipzig 2010.
4. Quételet, A., Essai de Physique sociale; L’homme moyen. *Physique Sociale*, vol.2, Bruxelles (1869).
5. Schott, D., Mathematical components of engineering competency, *Proc. 10th UICEE Baltic Region Seminar on Engineering Education*, Szczecin (Poland), 47 - 50 (2006).
6. Schott, D., Mathematical curriculum, mathematical competencies and critical thinking. *Proc. 16th SEFI MWG Seminar*, Salamanca (Spain), Electronic Edition, Session 1, 1 - 6 (2012).
7. Székely, G., *Paradoxes in probability theory and in mathematical statistics*. Reidel 1986.
8. Ziegler, G., *Darf ich Zahlen? Geschichten aus der Mathematik*. Piper, München 2011.