

Engaging elementary school students in mathematical reasoning using investigations: Example of a Bachet strategy game

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Abstract

Strategy games are known not only as genuine tools for amusement activities, but also as efficient teaching tools used to stimulate mathematical thinking in regular classrooms and to support construction of new knowledge in situations in which students become fully engaged in meaningful learning activities with enthusiasm, curiosity and excitement. Our ongoing study aims to design and implement teaching and learning tasks based on Bachet's strategy game. In this paper we discuss the design principles and examples of students' reasoning.

Games, investigations, reasoning, engagement, elementary school mathematics

Introduction

How does one enrich students' mathematical experiences while enhancing their understanding of the complex and abstract nature of mathematics? How can the teacher engage students in building conjectures, conducting in-depth investigations, and making generalizations? How does one do this based on students' natural curiosity and their desire to learn and succeed in mathematics?

Strategy games are known not only as genuine tools for amusement activities, but also as efficient teaching tools used to stimulate mathematical thinking in regular classrooms and to support construction of new knowledge in situations in which students are fully engaged in meaningful learning activities with enthusiasm, curiosity, and excitement. Those moments of joy can also promote deeper mathematical reasoning and investigation (Cañellas, 2008).

In this article, we analyse an example of such a mathematical game that may help to evoke patterns of the culture of reasoning and proof defined by the NCTM (2000) Standards as enabling all students to 'make and investigate mathematical conjectures; develop and evaluate mathematical arguments and proofs; select and use various types of reasoning and methods of proof' (<http://standards.nctm.org/document/chapter3/reas.htm>).

Context of the study: A Bachet game

The name of the game we used in our pilot study is attributed to the French mathematician Claude Gaspar Bachet de Méziriac (1581-1638). In 1612, he published a book featuring recreational mathematical problems entitled *Problèmes plaisants et délectables qui se font par les nombres* (in this paper, we refer to its later edition, Bachet, 1884, published nearly 250 years after his death) which contains several games and riddles that later became famous and whose variations are often used today in mathematical clubs, contests and competitions, as well as in other forms of mathematical challenge and entertainment. Among them we can find the problem 'Goat, cabbage and wolf', various card games, number games (guess a number), 'false coins' games, and many others (Bachet, 1884). As a recreational problem, the game was formulated by

Bachet in French (problem number XXII from this book in the fifth edition) as follows (*English translation*):

There are two players, who take turns naming numbers that are smaller than an initial predefined number, and add to that number at each turn. The first player to reach the "destination number" is declared the winner.

This problem belongs to the 'Nim-family' of mathematical games. Piggott and Sholten (2006) consider *Nim* games as a potentially challenging and enjoyable mathematical activity that can be a resource for the development of strategic planning and reasoning as well as concepts of analogy and, through this, generalizing. Although the real origins of the game remain unknown, Delahae (2009) refers to it as being played back in ancient China; its appearance in Europe is noted in the 16th century. The name itself can be attributed to the German word 'nimm' (take) or the graphical mirror-reflection of the English word 'WIN'. Programming an algorithm of those games has become a routine exercise in computer science courses, and there are many computer programs simulating them (Delahae, 2009). These algorithms are taught at the intermediate stage of learning, as compared to more complex algorithms for other well-known strategy games such as checkers, chess, and 'go' (Delahae, 2009).

There are many versions of Bachet's game in the literature. For example, Li (2003:1) formulates it in a general way:

Initially, there are n tokens on the table, whereby $n > 0$. Two persons take turns to remove at least 1 and at most k tokens each time from the table. The last person who can remove a token wins the game. For what values of n will the first person have a winning strategy? For what values of n will the second person have a winning strategy?

The game is also used in its 'backwards' version formulated by Engel (1998: 362):
Initially, there are n tokens on the table. The set of legal moves is the set $M = \{1, 2, 3, \dots, k\}$. The winner is the one to take the last token. Find the losing positions.

In our paper, we use a simplified version of this latter form, with rules that can be adapted to students of all ages and abilities, to potentially engage them in meaningful

mathematical investigations. In so doing, we help them apply genuine mathematical thinking, making new conjectures, discussing, proving or disproving them and building new inquiries, thus differentiating the level of challenge, creating new games and then posing and solving new mathematical problems. Our ongoing study is divided into two steps, or objectives:

- (1) Develop a teaching-learning scenario integrating mathematical investigation of the game and validate it with elementary school students and teachers (professional development and initial training);
- (2) Develop research tools that enable studying patterns of mathematical thinking and reasoning emerging from students' investigations, experimenting with them in a classroom, and testing them as a case study.

In the following sections we briefly present our theoretical view, a scenario experimented on with one group of students and preliminary findings from the students' questionnaires.

Theoretical background: Educational potential of investigations with strategy games

Alro and Skovsmose (2004) stress the view of learning as an action which creates possibilities for investigations. Yerushalmy, Chazan and Gordon (1990), Leikin (2004), and Applebaum and Samovol (2002) emphasized the importance of instructional inquiry activities to develop more creative ways of learning, thus nurturing interest in a particular area and continuous motivation to learn more about it. In addition, students' willingness to choose problems to investigate, to design methods to explore them, and then attempt to generate the solutions can be facilitated. While designing tasks with the Bachet game, we draw on the works of Peressini and Knuth (2000) and Sheffield (2003) who suggest using rich tasks that promote open-ended investigations, allow several different approaches, are non-standard, focus on higher-order abilities, ensure interaction of practical and

theoretical thinking as an important element in the process of transition to more complex and abstract mathematical ideas like algebra.

Use of mathematical games in teaching and learning is described by a number of authors as favorable for fostering discussion between students (Olfield, 1991), enabling them to discover new mathematical concepts and develop thinking abilities ((Baek et al., 2008; Bragg, 2007). As an educational activity, each game contains an intriguing element that motivates children to play and explore rules and outcomes. Related mathematical content is uncovered in the process of playing and eventually posing a challenge of finding a better, (winning strategy. In this process of meaningful mathematical investigation, students can be further guided towards building models, testing, discussing, promoting hypotheses, posing new problems and refining them.

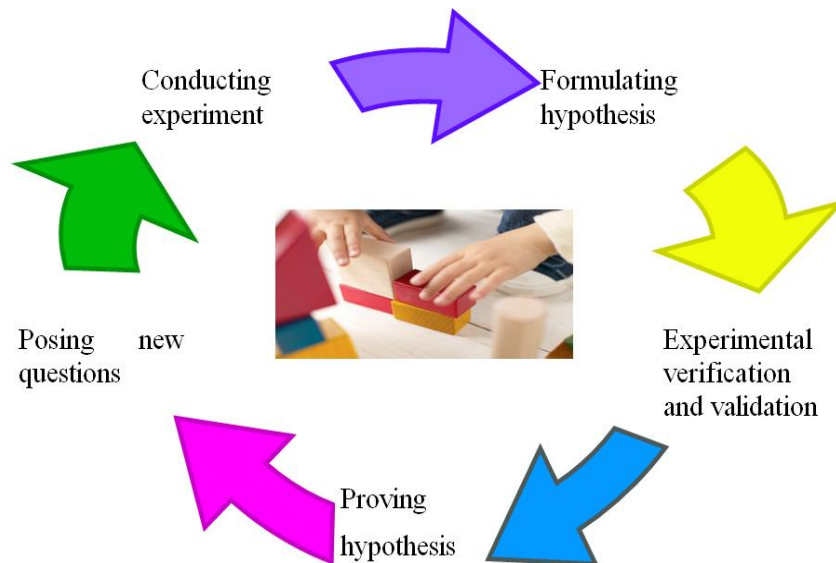
In their study of teachers' behavior that promotes mathematical reasoning, Diezmann, Watters, and English (2002) noticed that 'young children's reasoning can be enhanced or inhibited by teachers' behavior through their discourse, the type of support they provide for their class, and how they implement mathematical tasks'.

By designing and implementing investigative tasks with the Bachet game, we aimed to model such kinds of behavior and make explicit the elements of mathematical reasoning that emerge under these conditions. Since the study is on-going research, at this stage we discuss preliminary observations from the field notes and the students' questionnaires.

Methods of inquiry: Design and validation of the teaching and learning scenario

Related to our first and second objective, in order to study the emergence of mathematical reasoning by investigation of the Bachet game, we designed a teaching and learning scenario that has been implemented with several groups of pre-service and in-service teachers and elementary school students (Grades 2-5) in Israel and Canada (New Brunswick). To construct a learning activity with Bachet's game we used the following

teaching - learning cycle of mathematical investigation suggested by Applebaum and Samovol (2002).



In this scheme, we see how students gradually become engaged in mathematical investigations through the game: posing questions, conducting experiments, formulating hypotheses, verifying and validating, proving, and formulating new questions for further investigations, thus launching a new learning cycle.

Initial steps

Each session starts with an explanation of the rules of a simplified version (compared to the one in Bachet's book). The game starts with placing 15 tokens on the table. Then, two players (or groups of players) make their moves, at each one's turn, according to the following rules: The first player (or team) begins by removing one, two or three tokens, the second player (or team) removes one, two or three tokens; then the game continues with each player (or team) taking turns. The player who picks up the last token left on the table is the loser of the game. Usually, we play a couple of times with the whole class to make sure participants understand the rules, but not to give any advice or prompting on how to play. In all settings and with all audiences (students and teachers), the rules of the

game were understood by every participant after 2-3 rounds of such collective play. While trying out the game by playing it in dyads, participants were asked to make a number of initial observations which they then shared with their colleagues and peers in an all-group discussion. Here are some examples of the comments they shared: ‘I just copy all the moves my partner (opponent) makes and often win’; ‘Each time I play I use a different strategy’; ‘I see that each time I leave my ‘partner’ (opponent) with 5 tokens I win’; ‘I always take 2 tokens off and win’.

We can see that whereas some of the observations took the form of reflection on the results of the game (who won – ‘I always won’), others showed a focus on strategy (how to play better in order to win – ‘I always took two at each turn and won most of the games’). Since our task as facilitators of the activity was simply taking notes from the participants' discourse and writing them on the board, we were able to notice several things. For example, we observed some spontaneous questioning initiated by participants (e.g., ‘what number of tokens should I have in order to win?’); argumentation (e.g., ‘no, you are not right by saying that the person who starts always wins. I started twice, and in both cases I lost.’); or even providing proof (e.g., ‘I noticed that in the case of 6 tokens, I could always take 1 and leave my opponent with 5 tokens, so he would lose every time.’). The general observation at this stage (regardless of the age and type of audience) was that the most plausible conjectures are related to a situation in which few tokens are left (3 – 4 – 5 – 6), so the end of the game becomes ‘calculable’. After this initial discussion, students return to playing in dyads, but this time they are explicitly asked to verify the list of observations or conjectures written on the board and, eventually, others they may wish to add to this list. For mathematical reasoning to occur, the process of verifying (and eventually, proving or disproving initial observations/ conjectures) is very important. A good understanding of the ‘5-case’ (player left with five tokens will lose in any possible case) is an example of such an opportunity that emerges in all groups we led. While construction of the ‘proof’ seems to be possible by students who Sheffield (1999) would

call ‘mathematically promising’, presenting it and validating it with the rest of the class is an essential step for all participants – i.e., the ‘new’ knowledge is constructed, debated, shared, and validated at the highest level of motivation to learn the ‘winning strategy’.

Proving and disproving conjectures: Fostering exploration and questioning

Hence, by working in dyads, and discussing their findings, students become engaged in more active exploration which would implicitly involve some conjecturing and validation. The first attempts in proving (disproving) emerge spontaneously. Some students may explain why **5 tokens left** by their partner ensure them a win.

The next round of classroom discussion may lead to more formal proofs, by looking at all possible combinations with 5 tokens:

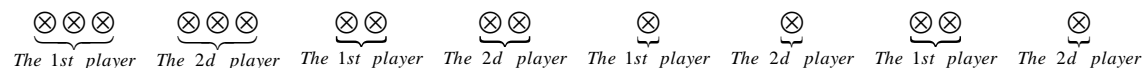
If you remove 1 token, I will remove 3, and so you are left with one and lose.

If you remove 2 tokens, I will remove 2, and so you are left with one and lose.

If you remove 3 tokens, I will remove 1, and so you are left with one and lose.

There are no other options, so you will lose in all cases.

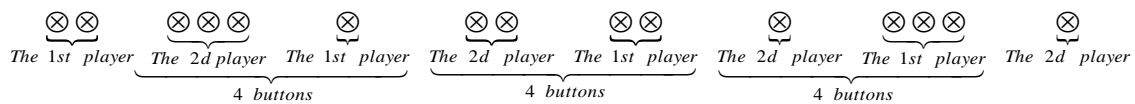
It is also important to have at least one conjecture being disproved like ‘repeating the same move as my partner causes me to win’. The next example illustrates the case for which the conjecture does not hold:



The second player repeated the same move as his partner and lost the game. A follow-up discussion can lead students to the understanding that one single case is sufficient to disprove the statement and eventually to a deeper comprehension of the logic behind proving and disproving conjectures.

By all means, it is important to make students realize that from 6 – 7 – 8 left token configurations, the player who starts can win by generating a ‘**5 tokens left**’ case. This moment of realization is important in order to boost further investigation that would lead students to discover other winning numbers – ‘**9 tokens left**’ and ‘**13 tokens left**’ and eventually to prove that in the initial situation, the player who starts first can remove 2 tokens (to reach the ‘**13 tokens left**’ position) and let the second player choose whatever tokens he/she wants and then complete it to ‘4’ tokens (second player gets 1 – first player gets 3; second player gets 2 – first player gets 2; or second player gets 3 – first player gets 1) and bring it to ‘**13 – 9 – 5 – 1 tokens left**’ pattern.

The next diagram presents an example of possible moves:



Studying the ‘4’ pattern here is an important step in moving towards reasoning and generalizations in a purer mathematical way.

It is very important to guide the students to the comprehension of the idea that the ‘4’ pattern does not appear randomly, but has to be obtained as a sum of 3 (the maximum number of tokens that can be taken) and 1 (the minimum number of tokens that can be taken). An explanation for this can be the following: we need a constant total number of tokens at each move to maintain control over the game. In this game, 4 is the unique total of tokens at each move that gives the first player a full control over the game's development and its (winning in *all* cases) outcome. Other numbers, for example, 5 cannot guarantee the same outcome. The reason is: if your partner takes only one token, you will not be able to complete the sum up to 5. Number 3 does not work for the same reason : if the first player takes 3 tokens, the second player has no way to complete to 3.

Prompting Further Investigations

Once the game with 15 tokens is well mastered by students it is important to bring them to further questioning about the rules. For example, students or the teacher can suggest changing the number of tokens, and seeing what happens if the number of tokens is 20 (or any other number larger than 15). It is plausible to anticipate that students will try to apply the same (or a similar) strategy and eventually come up with the solution (by going from the end to the beginning). For further generalization of the findings it might be interesting to ask students they think what would happen with 200 tokens; thus encouraging them to move towards more abstract mathematical exploration and eventually helping them to discover mathematical formula to model the game.

It might also be interesting to ask students to make their own rules and investigate strategies for newly created games or challenge their classmates to investigate. Here are some innovative ideas that may emerge:

- (a) The players may choose 1, 2, 3 or 4 tokens at each turn,
- (b) A player who gets the last token is the winner.

To make the investigation even deeper and eventually more interesting for students with higher mathematical abilities, the following questions may be helpful:

- (1) Could you suggest a case in which the player who is first to take tokens will lose the game, when the number of tokens is more than 20 and students can take 1,2,3,4 or 5 tokens on each turn? (Both players know the winning-strategy as well and the player that gets the last token loses the game)
- (2) Two players Dan and Sam know the winning-strategy of this game. Dan is the first player and he should decide how many tokens: 28, 29 or 30 will be on the table. Sam is the second player and he should decide about the maximal number of tokens that may be taken in each turn: 6, 7 or 8 (you can take 1-6, 1-7 or 1-8 tokens). Who will win this game?

- (3) Two players Kate and Carole know the winning-strategy of this game. Kate has the right to decide if she wants to be a first player. As first player she can state how many tokens will be on the table: 43,45,47,49 or 51. Then Carole as the second player will choose the maximal number of tokens to be taken in each turn: 7,8,9 or 10. If Kate decides to be the first player, who will win this game?

Finally, for some very advanced students, investigations can be enriched by moving to any number of tokens:

Investigation 1

There are N ($N > 20$) tokens placed on the table. The game is for two players. The players may take 1, 2 or 3 tokens. The player who is left with the last token loses the game. How can we know who will win this game?

Explanation: To win this game you have to take the $(N-1)$ -th token. How can we guarantee taking the $(N-1)$ -th token? We have to take the $(N-5)$ -th token! And then we'll take the $(N-9)$ -th token, and so on... Now if we get 1, 2 or 3 tokens at the end of the process, the first player can win the game by using the strategy we described above. If we get 0, then the first player will lose the game.

First step to generalization

First of all we have to reduce the number of all tokens by one: $(N-1)$. Then we'll divide $(N-1)$ by 4 (remember: $4=3+1$). If we get a remainder (1, 2 or 3) then the result of the game depends only on the first player; if he uses the win strategy (completing to 4), he will win this game. If we do not get a remainder, then the second player can win this game using the strategy we revealed above.

Solving the next two questions may help students gain a deeper understanding of Investigation 1.

Question1: There are 47 tokens on the table and two players that are playing this game.

As in previous cases, they can take, by turns, 1, 2, 3,... k tokens ($5 < k < 20$). It is known

that the first player is going to lose the game (both players know the winning strategy).
Find the possible k value.

Question 2: There are 62 tokens on the table and two players that are playing this game. As in previous cases, they can take, by turns 1, 2, 3, ..., k tokens ($5 < k < 20$). It is known that the second player is going to lose the game (both players know the winning strategy). Find the possible k value.

Investigation 2

There are N tokens placed on the table. The game is for two players. The players may take 1, 2, 3, ..., or k ($2 < k < \frac{N}{2}$) tokens. The player who is left with the last token loses the game. What is the winning strategy?

Explanation: First of all, we need (as in Investigation 1) to reduce the number of all tokens by one: $(N-1)$. Then we'll divide $(N-1)$ by $(k+1)$. If we get a remainder (1, 2, 3... or k) then the result of the game depends only on the first player, and he can win this game. If we does not get a remainder then the second player can win this game by using the already well-known strategy. We can write:

$$\frac{N-1}{k+1} = \begin{cases} \text{whole number} \rightarrow \text{the second player is the winner} \\ \text{some remainder} \rightarrow \text{the first player is the winner} \end{cases}$$

In order to win a player needs to take the number of tokens that completes to $k+1$ the number of tokens that had been taken by the other player.

CASE STUDY with grade 5 students and their teacher: Classroom setting for the activity and research tools employed

In order to meet our second objective and thus gain more insight into students' 'laboratory of thought' during the investigation of the game, at the second stage of our study we developed a questionnaire asking students to share their ideas and perceptions. Our first data come from an experiment conducted with two groups of 5th grade students (a total of 88 students, 10-11 years old) in an Israeli elementary school. The activity was organized at the end of the school year.

According to national external exams (Ministry of Education) the school is in the top 10% of all Israeli schools. The groups of students participating in our activity had the same mathematics teacher from the 3rd grade and up till the time of the experiment. The average marks in the two groups were: 80 and 83. According to the teacher, there were no students with behavioral problems in these groups.

Regarding the mathematical background of our participants, the teacher informed us that until the time of the experiment, while following the regular curriculum, students had learnt the following topics: integers and the 4 arithmetic operations with them, and addition and subtraction of fractions. During the school year, the teacher had frequently integrated problems oriented towards developing mathematical thinking in her lessons. In addition to the regular curriculum, mathematically promising students received one lesson per week, over the course of 3 months, which dealt with solving non-standard problems.

According to the teacher, mathematical investigation of a strategy game framed within a scenario similar to the one we developed with Bachet's game was relatively new to our participants. The teacher said that sometimes she opened her lessons with a math game or puzzle. But it was usually of rather a small scope and a short activity compared to the one we suggested to the students. According to the teacher's observation, the end of the school year was not the ideal time for this kind of activity since the children's motivation for any kind of intellectual work was rather low; however, most of the students participated and they seemed to have fun. Some of the students were challenged by the game and activated deep thinking that led them to discover some form of strategy. Others

just "played the game" for game's sake and also enjoyed it. As suggested in our scenario, students had to explore the game on their own with minimum guidance from their teacher.

Analysis of Questionnaires

While playing in dyads, students made several observations and produced protocols of the game rounds. Although, only one student in each dyad filled-in the questionnaire (as we mentioned above, in total we collected 44 questionnaires). The first part of the questionnaire asked about students' perceptions of mathematics, of learning mathematics, of lessons in general, and playing the game in particular. The following table summarizes data for each item (including mean and standard deviation based on the 4-point Likert scale: ranging from 1- completely disagree up to 4 – completely agree).

Item	Mean	Standard deviation
I like to solve new math problems.	2.89	.97
I solve Math problems easily.	2.84	.78
Math is important to me and useful in everyday life.	3.74	.57
My motivation to learn Math increased after playing this game.	2.87	.94
I would recommend learning through games to my friends.	3.3	.73
When playing this game I learnt new ways of problem solving.	3.17	.94
I enjoyed playing this game.	3.68	.66
I'm active during Math lessons.	3.23	.63
My success in Math depends only on my effort.	3.68	.75
I always prepare my homework in Math.	3.38	.77
I like to solve Math tasks.	2.87	.90
I have anxiety during Math tests.	1.91	1.04
I'm interested in Math.	3.26	.79
I'm successful in Math.	3.15	.83
I enjoy Math lessons.	3.23	.70

As we learn from the data, almost all students seem to agree that math is important to them and useful in everyday life ($M=3.74$). Furthermore, they believe that their success in math depends only on their own effort ($M=3.68$). The majority of students are interested in math ($M=3.26$), active in math lessons ($M=3.23$) and enjoy them ($M=3.23$). They always do their homework ($M=3.38$), do not have anxiety during math tests ($M=1.91$), and seem to be quite successful in math ($M=3.15$). Regarding the question regarding their experience with problem solving, the opinions seem to be more diverse, although many students agree that they like to solve math tasks (2.87), including new problems ($M=2.89$) and also claim to solve problems easily ($M=2.84$).

Regarding students' experience with the game, they almost unanimously confirm having enjoyed the game ($M=3.68$); and many generally affirm having learnt new ways to solve math problems ($M=3.17$). Although their opinions on the role of the activity in increasing their motivation to learn math ($M=2.87$) are more diverse, the majority would recommend it to their friends ($M=3.3$).

The second part of the questionnaire was related to reflection about the game and to reporting students' observations and discoveries. More specifically, there were questions related to the initial observations while playing in dyads (conjecturing). Students were asked to note whether the winner had played well and why.

Another set of questions was based on conjectures made, and their investigation was recorded in the form of a table in which children note the number of the move, number of tokens taken by the 1st player, number of tokens taken by the 2nd player, and total of tokens taken at each move. Examples of two recorded games are given [in Hebrew] in the Figure 1 below.

Number of tokens taken by both players	Number of tokens taken by player 2	Number of tokens taken by player 1	Number of each move
2	1	1	1
5	3	2	2
3	2	1	3
4	1	3	4
	1		5
			6
			7
			8

Number of tokens taken by both players	Number of tokens taken by player 2	Number of tokens taken by player 1	Number of each move
4	1	3	1
3	1	2	2
4	1	3	3
4	1	3	4
			5
			6
			7
			8

Figure 1. Example of protocols of two students: The right column shows the number of each move, then (from right to left) we have: number of tokens taken by player 1, number of tokens taken by player 2, and (left column) number of tokens taken by both players at each move.

From our analysis of the questionnaires, we learned that in 14 of the 44 dyads (31.82%), the children were able to find that leaving the partner with 5 tokens is a winning strategy (this is actually the first step in discovering such a strategy).

In 37 of the 44 dyads (84.09%), students who got the 10th token won the game. It is clear that it could not be coincidental and students directed themselves to getting the 10th token. In 14 out of 37 cases (37.84%) students that got the 10th token (or 5th token, if counted backwards), also won the game and described the winning strategy. In 23 out of 37 cases (62.16%) students that got the 10th token (5th token, if counted backwards), and won the game were not able to draw conclusions or describe the winning strategy. A number of metacognitive comments made by students about their way of thinking and reasoning seem to indicate their appreciation of the task as thought provoking. The following table (Table 1) presents categories that emerged from students' responses and corresponding quotations.

Table 1. Categories emerging from students' observations

Category	Examples of students' responses
Promoting thinking	'I liked the fact that I needed to think a lot in this game.'
Combining joy and learning	I liked this game because I enjoyed playing and studying at the same time; 'when we enjoy it we learn in an effective way.'
Need to search for the winning	'I had to look for ways to win using different

strategy	strategies.’
New mathematics learning opportunities for students	‘I learnt to think with logic and to find patterns.’
Making discoveries and investigating them	‘I make sure that by the end of the game I will have 5 tokens left, and then it does not matter how many he [my opponent] gets, I’ll win anyway’; ‘It worked exactly as I thought.’
Opportunity to ask new questions	‘Will the number of tokens influence the game? Will the number of players influence the game?’

Overall, we found that students perceive the game as an enjoyable learning experience that makes them think in an attempt to find a winning strategy, as they need to apply logic and look for patterns, an ability particularly well-demonstrated by mathematically promising students (Sheffield, 1999). Few students go beyond these first steps of reasoning by conducting deeper investigations and asking new questions; this is a culture of mathematical proving yet to be developed in the classroom (Bieda, 2010).

We also asked students to predict and explain whether 8 rows in the form (which is equivalent to the maximum number of moves) would be sufficient to complete any round of the game. By asking this question, we solicited some reasoning based on the total number of tokens (15) and the number of players involved at each move (2). Almost all students gave an affirmative answer, but only 12 of the 44 dyads (27.28%) were able to explain their answer in a more or less plausible way. Table 2 represents five different patterns found in students’ responses.

Table 2. Students’ reasoning about the structure of the game

Students’ statements	Our remarks
Even if each player gets one token, 7 rows will be sufficient to complete the round of the game	The student seems to understand the structure of the game and condition of looking for a maximum number of possible steps. She probably anticipates the situation in which the last remaining token would not require a move – so to her, the game stops after 7 moves (rows).
8 is a little bit more than we need.	This student did not provide the details of her definition of ‘a little bit’ but may also get a sense of 7 rows needed and then 1 token would be left over.

8 rows are enough because we have 15 tokens and only 2 players.	In this effort of a plausible explanation, the student seems to omit the explanation that this situation would happen if every player took 1- the minimal number of tokens at each move.
If each player takes only one token we'll need only 7 and a half rows.	Interesting explanation with the student interpreting the record of her last move as "a half row" – question of making a distinction between the structure of the table (row) and its content (the need to fill in a complete row to count it as a 'row').
The answer is "yes", because in both games we played we needed only 3 rows.	Incorrect attempt to generalize based on an insufficient number of examples; no reasoning is provided.

Discussion and Conclusion

While recent mathematics curricula place great emphasis on the development of mathematical reasoning in all students, the optimal teaching and learning conditions to foster this reasoning is not yet known. Our ongoing study explores the potential of recreational mathematics, in general, and strategy games, such as Nim, in particular, which shed light on how to build learning and teaching scenarios which prompt investigations of patterns that emerge during the game. Issues of their implementation in the classroom, as well as studying their ultimate impact on students' learning were also examined. As a case study, we used a "Bachet's game" which we presented to teachers and students in a way in which clear and simple rules hold nearly infinite potential for students to make observations, conjecture, verify, explain, and ask new questions while playing. The entire process potentially leads to the development of high-level reasoning abilities even in young students in elementary school. The students participated in our study, in the context of a strategy game, in a similar way to that discussed in previous studies, e.g., Piggot and Sholten (2006), Diezmann et al. (2002), and Sheffield (1999), showed a high level of engagement and task commitment. The latter was accompanied with 'continual attempts to make sense of their actions and discourse, and challenges for their peers to do likewise' (Piggot and Sholten, 2006). Similar observations were also noted by Civil (2002) while investigating Nim-like games with her Grade 5 students by

involving students in the exploration of mathematical principles behind the game; she found students engaged, appearing to enjoy themselves, and persistent. At the same time, the author noticed that while (almost) all students showed initial enthusiasm about the game and curiosity in finding the winning strategy, few of them were actively involved in mathematical analysis (Civil, 2002). The same situation was also observed in our case study, with only a small number of questionnaires providing a clear explanation of the winning strategy and a justification for the number of moves needed, so the question of ‘bridging’ students’ overall excitement with meaningful mathematical reasoning is still an open question that requires more research and teaching practice. Moreover, in our study we had to comply with a short time-period given to the students to conduct investigations, which is a clear limitation of our study. While promoting investigations with a strategy game, which can also be considered to be an open-ended problem, teachers may face difficulty in explicitly suiting the task (open-ended) of investigating patterns to the school curriculum and learning outcomes. This issue was also brought up by a teacher who said that focusing on specific content and tasks suggested by curriculum places additional pressure on her, although she does see multiple benefits of deeper learning that can be enhanced by strategy games.

Looking back on our scenario may help to uncover potential pitfalls when guiding students through investigative tasks; some were already mentioned in earlier studies, like the one conducted by Henningsen and Stein (1997). In their study, which focused on geometry, students’ failure to engage in the intended high-level cognitive processes was attributed to a number of factors, such as lack of clarity and specificity of the task expectations which were not specific enough to guide students toward discovering the relevant mathematical properties; another example, the lack of prior knowledge needed to make effective comparisons and differentiations, was already mentioned as a factor that could potentially hinder students’ efforts to systematically record and generalize their findings (Henningsen and Stein, 1997).

The teachers' readiness to support students' deeper thinking through mindful and purposeful scaffolding is also important, as noted by McCosker and Diezmann (2009) who argue for pressing students to provide meaningful explanations, working from students' ideas, distinguishing positive encouragement and cognitive scaffolding, as well as providing students with unambiguous task instructions and clear expectations, while ensuring that the investigation remains open-ended in the form of teachers' strategies that enhance reasoning. Our other publication, addressed specifically to teachers, also illustrates the benefits of this approach (Applebaum and Freiman, 2014).

Overall, our preliminary results indicate promising paths in fostering mathematical reasoning in young children using strategy games that need to be studied in more depth. In the next stage of this research, we plan to continue working with teachers and collecting data in order to gain more insight into the impact of such activities on students' motivation and performance in learning mathematics. We also seek to explore how these activities may be used by more teachers in a more efficient way.

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