"Least Squares Estimation" instructional design based upon APOS theory: Laying Mathematical Representation and Transformation Bridge

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Abstract: *The teaching design of "least squares estimation" under the framework of APOS theory is discussed in this paper. The main points of instructional design is to create the sequence of that several mathematical representations and transformations in the four teaching phases of operation, process, object and schema. Different mathematical representations can not only connect four stages of teaching, but also achieve the mathematical transformation of the final statistical problems from the "uncertainty" to a linear function of " certainty" change, so that students really understand the idea of least squares, and the formation of Schemata of the idea of linear regression, between " algebraic expressions" and " geometric meanings"; between different " algebraic expressions".*

Keywords: APOS theory; least squares estimation; teaching design; mathematical representation and transformation.

The question is raised

The content standard set by the Ministry of Education in the "General High School Mathematics Curriculum Standard (Experiment)" (hereinafter referred to as "Mathematical Curriculum Standards") published by the Ministry of Education in April 2003 is: students learn

a) to estimate the overall population and its characteristics of thought through learning random sampling, sample estimation overall, linear regression and other basic methods;
b) through systematic experience of the entire process of data collection and processing, they learn statistical thinking and deterministic thinking differences**.**

So, what is statistical thinking? Dan Zhang believes that statistical educators and statisticians depict the students' statistical thinking in two aspects: (1) the statistical process and the statistical method level, which is reflected in the following aspects: "to collect information through data collection and analysis and to understand the randomness through data, and (2) to use statistics to explain phenomena and solve practical problems. This level is reflected in "using data to solve problems (including data awareness)" (DanZhang, 2010). Ningzhong Shi believes that the

statistical process should include two core features: extracting information through data analysis; and understanding randomness through data. The randomness of data means that for the same thing, the data collected each time may be different; as long as there is enough data, we can find out the law (Ningzhong Shi et al., 2005). Dan Zhang summary randomness of data means: First, uncertainty, and second, the regularity of sufficient data (Dan Zhang, 2010).

From the above perspectives of DanZhang and NingzhongShi, one of the important characteristics of the statistical problem is the randomness of the data. This "randomness" is the manifestation of "uncertainty." This "uncertainty" is reflected in the relationship between the two sets of data for bivariate data in life such as height and length; age and blood lipids, but this relationship cannot be determined by a definite functional relationship. The essence of a functional relationship is to determine the relationship between independent variables and dependent variables is determined, such as if x_1=1, y_1=f(x_1)= 1, x_2 = 2, y_2= f(x_2) =4, x_3=3, y_3= f(x_3) = 9,then the correspondence between the domain ($1 \le i \le 3, i \in Z_+$) and the values in the range can be functionally determined as a quadratic relationship that provides an element of certainty to the original randomness. The standard of mathematics curriculum believes that the teaching of high school statistics aims to improve students' ability of data analysis. The manifestation of data analysis ability is whether one grasps the ideas of random sampling, sample estimation population, linear regression, and independence test and regression analysis method.

Students can not only understand the uncertainty of statistical data while learning linear regression, but also know that the law of implication can be found out from the uncertainty of data. The essence of this law is the "certainty" of the functional relationship. Therefore, in the teaching of linear regression, the following questions need to be solved: What is the "uncertainty" characteristic of the data? How to find out the law that it contains? After finding out this law how to explain this law and use this law?

The content of the study

Textbook referenced in this text is "Mathematics 3 (Required)" (hereinafter referred to as "compulsory 3"), an experimental textbook of ordinary high school curriculum standard published

by Beijing Normal University Press, which is matched with "Mathematics Curriculum Standard". The topic of "Linear regression" is not designed as a single section in the first chapter of "Compulsory 3", but consists of Section 7 "Relevance" and Section 8 "Least Squares Estimation". The content requirements of the "Mathematics Curriculum Standards" on the "variable relevance" is:

(1) "by collecting real data of two variables associated with the scatter plot, and the use of scatter plot intuitive understanding of the correlation between variables' relationship ". The students, in the process of collecting data and drawing a scatter diagram can realize that the relationship between these two variables is not a "certain relationship", because it cannot be described by the functional relationship; one can only say that the two variables are related.

(2) "to experience the use of different estimation methods describing the two variables linearly related; to experience the process of knowing the idea of the least squares method, when linear regression equation can be established according to the linear regression coefficient equation." The linear regression equation is the "law" found from the correlation between these two variables. However, the essence of this equation is a linear function. Since it is a functional relation, the relation between variables is then "definite". So why is the relationship between the two variables that are not "definite" finally "determined"? The idea of the least-squares method embodies the process of "determining" the relationship.

The purpose of this paper is to explore the teaching of least-squares estimation using APOS theory and reveal how least-squares transform "uncertainty" into "certainty."

Literature review

The same mathematical concept often shows the characteristics of duality: it is both a process operation and an object structure; it is both an operation behavior and a structural relationship; both dynamic and static characteristics (Weiping Zhang, 2014). As the concept of process is related to APOS theory (Tall, Gray, 2001; Tall, 2006), APOS theory can be used to guide the learning and teaching of statistical knowledge.

The theory of APOS was first put forward by Ed Dubinsky of Georgia State University on the theory of Reflecting Abstraction. Shiqi Li thinks that to reflect on the abstract is to do some practical activities, then take a step back and review one's own practice activities to consider one's activities in the position to be considered. Then the activity becomes the object of thinking and the related abstraction is completed. Activity experience only provides the basis for the concept of organization, and the activity itself does not provide a concept (Shiqi Li, 1996).

APOS is the acronym for the four English words Action, Process, Object, and Schemas respectively. According to the theory, learners can obtain and establish the connection between knowledge based on reflection and construction after learning the mathematics through several stages of operation, process and object (LijuanCao , 2012). ShiqiLi think mathematics learning is an empirical activity. He believes that the recognition of mathematical objects must first have the basic behavior of mathematical cognition, namely "operational computing", with the "operational computing" experience after the realization of mathematical objects may be further manipulated (Shiqi Li, 1996).

Therefore, "action stage" refers to the "operation" the student experiences. This "operation " is not only "operation" in the narrow sense but also all mathematical activities in a broad sense, including conjecture, memory, calculation, reasoning, observation graphics (Weiping Zhang, 2014). This "operational computing" phase is the "activity phase," where students experience these maths and experience the relationship between the intuitive background and concepts of mathematics (Xuefen Gao, 2013). The "process stage" is the process of thinking about "activity", experiencing the internalized process of compressing the mind, describing the activity in the mind, and abstracting the concept-specific nature (Xuefen Gao, 2013).

When "activity" repeats and leads to constant reflection on the part of learners it establishes an inner mental structure that can perform similar activities without external stimuli (Weiping Zhang, 2014). At this stage, learners form programs that do not need to be manipulated in their minds and can reverse the program or combine it with other programs to form procedures, steps and abstraction of the peculiar nature of the concept (Lijuan Cao, 2012). After the learner has experienced the essence of the concept through the "abstraction" of the previous stage, the

formalized definition and symbol of the concept are given as a concrete object. This stage is the "stage of the object" (Xuefen Gao, 2013). After "process" is treated as a whole and can be transformed, "process" condenses into "objects" (Weiping Zhang, 2014).

Learners establish a connection with other concepts, rules, graphics, etc. The stage in which the mental schemas form in the mind is the "schemata" stage (Xuefen Gao, 2013). At this stage learners build a coordinated knowledge network that remembers the problem context associated with the concept (Weiping Zhang, 2014).

As can be seen from the above, the four phases of APOS can interpenetrate each other, and the boundaries of distinction are not necessarily clear, but gradually transiting. For example, repetition of activities can lead learners to reflect on "activities" and then to transit to the "process stage", that is, some "activities" belong to the "operational phase" while others are “Process stage ". The criteria for this division are: whether the sequence of activities can lead to reflection on the activity or not; entering the "object stage" after formalizing the definition and the sign of the essence of the concept recognized in the "process stage". After formal definition and symbol of the concept is recognized in the process stage, it enters the" object stage ". During the development of understanding complex concepts," process stage "and" object stage " will be often repeated.

Statistical knowledge has process characteristics, which can be coupled with the APOS theory and procedural concepts so teaching statistics can utilize APOS theory.

The research process

The following are four stages of APOS theory to analyze and design "least squares estimation" teaching:

(1) Activity stage. At this stage, one of the activities of teachers and students is to review the knowledge of the last lesson. As students learned in Section 7, "Relevance," students already know that some of the bivariate data in their life have a linear correlation, reflected in the image that the scatter points are not in a straight line, but appear to be near a straight line. At this stage, one of the activities of teachers and students is reviewing the knowledge of the last lesson. The examples of activity stage are questions to be solved by "least square estimation": "what kind of line can best

characterize the scatter plot", and "How to find this line?" Therefore, at this stage, the second part of the activities of teachers and students is to observe the graphics, and guess the conclusion. These two teacher-student activities provide the foundation for the next phase of the organization's concept. Observing the graph, the conjecture conclusion itself does not provide yet a concept, and therefore these two activities are included in the activity phase.

(2) Process stage. Here we seek the connections between different aspects of the concept. According to the previous literature review, we can see that the teaching activities in the activity stage and the process stage may be similar. Then one criterion for judging whether the teaching activities are in the process is whether the activity provoked students' reflection. For the first question, students may have different answers, teachers can guide students to think about the discussion and find out that if the scatter points are all near the straight line. If not all the points fall on the straight line, the more suitable straight line should satisfy the diagram. “The distance from the point to the straight line is all 'close' enough so that there is a need to express the "point-to-line distance". Since the straight line is indefinite, the coefficients in the line equation $y = bx + a$ are to be determined. In fact, the process of determining the coefficient and value is the solution to the second problem.

When the distance of the scatter A(x_i, y_i) to the straight line is expressed as follows: $d = \frac{|bx_i - y_i + a|}{\sqrt{b^2 + 1}}$, we can see that the minimum is the minimum. Then the next question is: If the suspected minimum d= $\frac{|bx_i - y_i + a|}{\sqrt{b^2 + 1}}$ is $|bx_i - y_i + a|$ really the smallest? If so, then thinking back to the question also leads to: if the suspected $\frac{|bx_i - y_i + a|}{\sqrt{b^2 + 1}}$ is minimum whether the distance $|bx_i - y_i + a|$ is the minimum, and therefore the smallest distance? The conclusion drawn here is a very crucial step.

(3) Object stage. According to the previous literature review, when the learner has experienced the essence of the concept through the "abstraction" of the previous stage, the formalized definition of the concept and the symbol given to it become a concrete object. This

stage is the "object stage" (Xuefen Gao, 2013). The key question to understand the Least Squares method of thinking is why you use "deviation" to mean "point-to-line distance." Therefore, "deviation" is the "object" of understanding, and students' understanding of this concept cannot appear out of nowhere. The next (and very easy to overlook) question is: in case $\frac{|bx_i - y_i + a|}{\sqrt{b^2+1}}$ is min, whether $|bx_i - y_i + a|$ is also minimum? If so, then the next (and very easy to overlook) problem is: What is the geometric meaning of $|bx_i - y_i + a|$?

In other words, if $|bx_i - y_i + a| = |y_i - (bx_i + a)|$ from the images (see below (1)) can be seen, $|y_i - (bx_i + a)|$ that is the distance of AB. So, the sum of the distances of the scatter points to the straight line is the minimum. The sum of the vertical distance AB (vertical segment AB is also called deviation) is the smallest possible. In terms of the geometric meaning to this formula, then the expression expressed by this formula is the shortest distance from point to line.

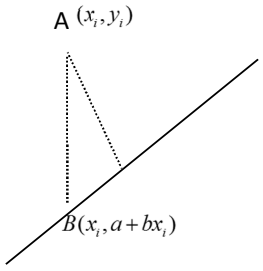


图 (1)

FIG. 1

The next problem is that the sum of the accumulated deviations involves the sum of the absolute values, and the square of the sum of the deviations can be summed up in order to remove the sign of the absolute

value that is advantageous for the additive deviations without any positive or negative cancellation. Then the final determination of the coefficient and value is to consider the square of the square of the deviation as a quadratic function and to use the knowledge of the quadratic

function to solve it. Derivation formula, "mathematics curriculum standards" and "compulsory 3" are not required, but recommended for interested students to try derivation.

(4) Schematic stage. Through the first three stages of learning, students can form a psychological schemata around the core issues are: point-to-line distance can be described as "deviations", the shortest deviation is the point-to-line distance is the shortest. According to the previous analysis, this straight line can satisfy the fact that the distance between the existing scatter point and this straight line is "minimum", and the essence of "minimum distance" is " Minimum error ", so if the linear error based on the existing data to obtain the smallest, then according to this line to make the prediction error will be the smallest. In addition, with least-squares estimation, students develop a larger mental schemata that knows how to use a straight line to fit a scatterplot and find an "optimal" line that makes the line and scatter.

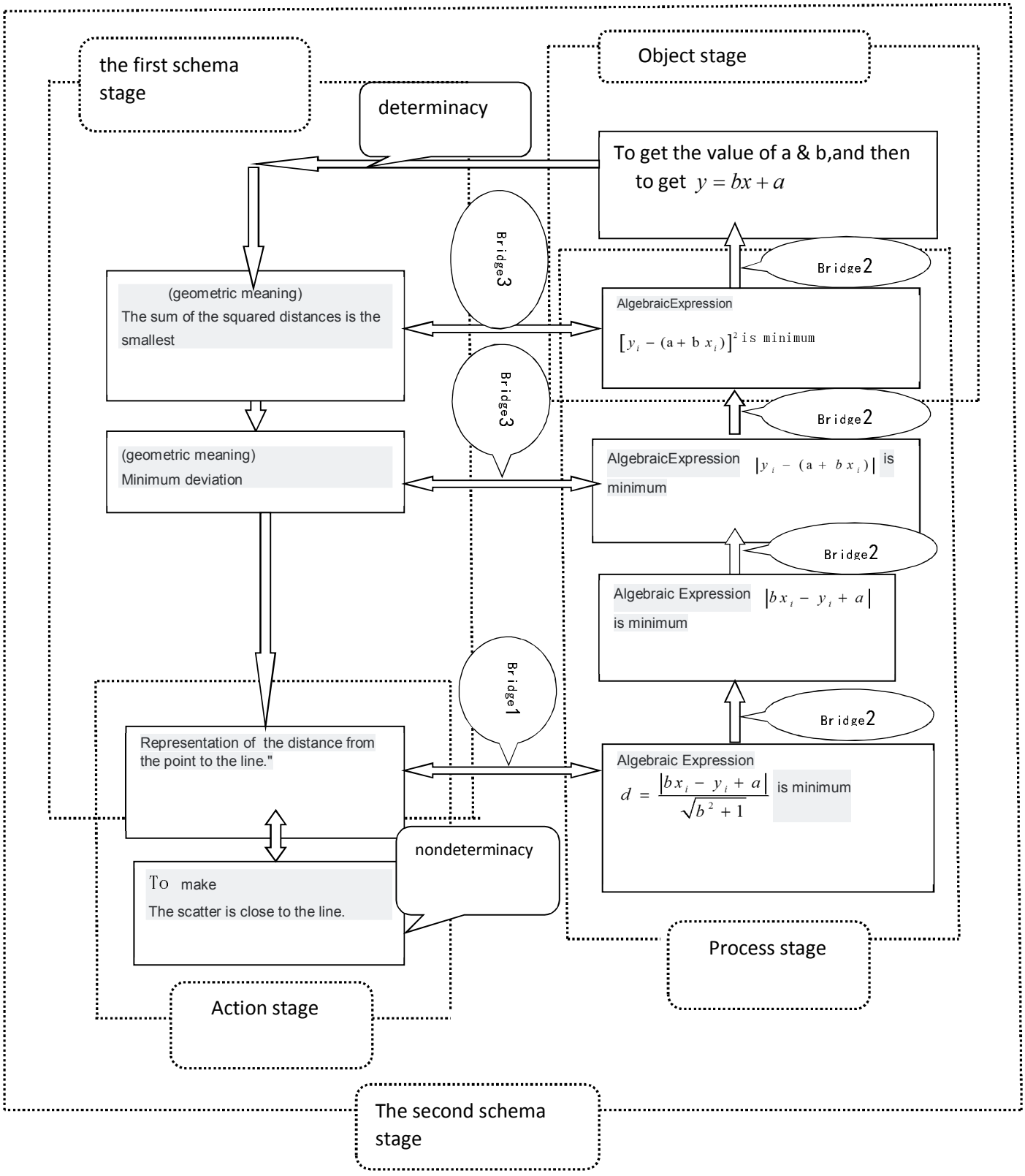
The conclusion of the research

From the above analysis point of view, to enable students to establish the link from "uncertainty" to "certainty", the key is to set up several bridges of mathematical representation and transformation, so that students gradually establish experience through activities, processes; After a period, the final form of linear regression method becomes clear. These bridges are (see Figure (2)).

Bridge One is between the "problem or the target to be solved" and the "algebraic expression," and to approximate the scatter to a straight line, describe the mathematical concept of "proximity" as "distance", and is "point-to-line distance," so the process of mathematical representation from "problem or goal to be solved" to "point-to-line distance".

Bridge two is between different "algebraic expressions", the process has two mathematical transformations,

The first math transformation is from $\frac{|bx_i - y_i + a|}{\sqrt{b^2+1}}$ to $|bx_i - y_i + a|$, the difference between the two algebraic expressions is whether to consider the denominator. It is a complicated process;



The second mathematical transformation is from $|bx_i - y_i + a|$ to $(bx_i - y_i + a)^2$.

The process, to find the absolute value of the problem cannot be a linear operation.

Bridge three is in the "algebraic expression" and "geometric meaning" between the mathematical characterization process $|bx_i - y_i + a|$ expressed as $|y_i - (bx_i + a)|$. After that, it can be understood as "dispersion", which is the process of interpreting the algebraic interpretation into a geometric meaning. In the learning process of least square estimation, students can form two schema structures. The first schema structure is between "point-to-line distance" and "dispersion" and "square of dispersion", but this structure must be in the operating stage, the process stage, so that the object stage can be established, or else students cannot understand why the expression of scattered points and the distance from the straight line to represent "square of dispersion".

The second schema connection is from the whole process of thinking between "uncertainty" to "certainness". After forming this schema structure, students can understand the nature of the linear regression method and can use this method to reason and predict the data. Because this part of the score is not high, not a college entrance examination hot spots, many teachers are generally doing simple teaching, teaching most teachers will not let students experience "with" deviation "to characterize the scattered point to the 'distance'. They think that as long as students know the formula for calculating the sum of the coefficients in the regression equation, they will imitate the training (YangchunXie, 2016). The result of this teaching is that students do not really understand the essence of the least-squares method.

Shiqi Li think mathematics object is a kind of thinking object, the mathematical content can be divided into two types of processes and concepts. The establishment of mathematical concepts and methods is, in essence, a process of reflection and construction on the basis of experience, on the basis of experience, and lacking in operational procedures, so that the process of this activity and operation must be personally experienced by the students, (Shiqi Li,1996).

"Least Squares Estimation" realizes the transformation between the "uncertainty" (or "randomness") of statistical problems and the "certainty" of functional relationships. The establishment of this method of thinking requires that after experiencing the four Therefore, when teaching, teachers need to build a bridge of understanding for students through mathematical representation and transformation and care for the integrity of the learning process. In particular, teachers should pay attention to the teaching of "process stage" Stage of mathematical representation and thinking about the purpose of mathematical transformation, the lack of this stage will lead to student thinking "chain" fracture, not only will affect the formation of mathematical characterization and transformation of students, but also further affect their knowledge and experience will eventually Affect the real formation of the schema structure.

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