



# Rethinking Instruction of the Tangent Line of a Circle

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*Abstract: The concept of a tangent line to a circle is a geometrically and algebraically important mathematics concept. However, finding the tangent line to a circle is not easy to find using ordinary methods. This article introduces how we get the equation of the tangent line of a circle as  $x_1x + y_1y = r^2$  using “Implicit Differentiation”, as well as the concept between a tangent line and a circle, and demonstrate how to teach this method to our students.*

## Introduction

A tangent line of a circle is a line that intersects with the circle at exactly one point and forms a 90-degree angle with the radius of the circle at that point. It is of great interest to mathematicians but also to scientists in general. Therefore, the concept is extensively used in our everyday lives, such as satellite orbits around the earth, a belt linking two rotating pulleys, and Ferris wheels at an amusement park.

Unfortunately, finding the equation of a line tangent to a circle using standard procedures such as quotient rule, product rule, and other methods is a very challenging and strenuous concept because a circle is not a function (Harshbarger & Yocco 2007). Instead, Implicit Differentiation can be used to find the tangent line of non-function objects (Hass, Weir, and Thomas, 2008). However, this method is not easy to apply. Are there any other methods to find the tangent line of a circle without using Implicit Differentiation?

There are alternative approaches for finding the slope of a tangent line. Authors such as Cartwright (2009) who demonstrated that the exact trans conductance of a field effect transistor can be derived without calculus, Xu (2014) who wrote elegant proofs for known mathematics without the use of calculus, and Rabin (2008) who determined that a tangent line equation could be determined by an algebraic calculation. In addition, Carroll (2009) introduced another algebraic way to find the equation of a tangent line of quadratic and cubic functions. These approaches imply there is a way to find the tangent equation to a circle without using implicit differentiation.

Both authors explored the commonly taught methods: Implicit Differentiation and the geometric concept of “the relation between the radius and tangent lines” to see whether there was anything in common between the two methods. After exploring tangents using implicit differentiation and the geometric approaches, there was an interesting finding that both methods demonstrated. They give the equation of the tangent line which passes through  $(x_1, y_1)$  of a circle  $x^2 + y^2 = r^2$  to be  $x_1x + y_1y = r^2$  after converting from slope-intercept form

to standard equation form. This article introduces how to get the equation of the tangent line of a circle as  $x_1x + y_1y = r^2$  using Implicit Differentiation as well as the geometric approach, and it will demonstrate how to instruct students on this approach.

In this article, we will describe and analyze not only a method of finding the tangent line to a circle without the use of calculus but also how students positively related to this method.

**Developing the idea that the equation of tangent line is  $x_1x + y_1y = r^2$**

*Using Implicit Differentiation*

If an equation of a circle is given as:  $x^2 + y^2 = r^2$ , and a tangent line of a circle passes through  $(x_1, y_1)$ , the slope of the tangent line can be found using implicit differentiation:

$$\frac{d}{dx} [x^2 + y^2 = r^2] \rightarrow 2x + 2y \frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

Based on this result, the slope of tangent a circle would be  $-\frac{x_1}{y_1}$  at the point  $(x_1, y_1)$ . Using the slope from above, the equation of the tangent line passing through  $(x_1, y_1)$  would be:

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

$$\rightarrow y = -\frac{x_1}{y_1}x + \frac{x_1 \cdot x_1}{y_1} + \frac{y_1 \cdot y_1}{y_1}$$

$$\rightarrow y = -\frac{x_1 \cdot x}{y_1} + \frac{x_1^2 + y_1^2}{y_1}$$

It can be converted to the standard formula:

$$y \cdot y_1 = \left[ -\frac{x_1 \cdot x}{y_1} + \frac{x_1^2 + y_1^2}{y_1} \right] \cdot y_1$$

$$\rightarrow y \cdot y_1 = -x_1 \cdot x + x_1^2 + y_1^2$$

Add  $x_1 \cdot x$  to both sides we get:  $x_1x + y_1y = x_1^2 + y_1^2$ . From the given equation of the circle:  $y_1^2 + x_1^2 = r^2$ , the equation of the tangent line of a circle passing through  $(x_1, y_1)$  is  $x_1x + y_1y = r^2$

*Using the geometric approach: the relation between tangent lines and the radius of a circle*

Given the equation of a circle:  $x^2 + y^2 = r^2$ , and the tangent line of a circle passes through  $(x_1, y_1)$ , the slope of the radius between the center and the point where the tangent line passes is:  $m_{radius} = \frac{y_1 - 0}{x_1 - 0} = \frac{y_1}{x_1}$ . According to Jurgensen R., Brown R., & Jurgensen J. (2009), the

tangent line is the perpendicular line to the radius, the slope would be:  $m_{tan} = -\frac{x_1}{y_1}$ .

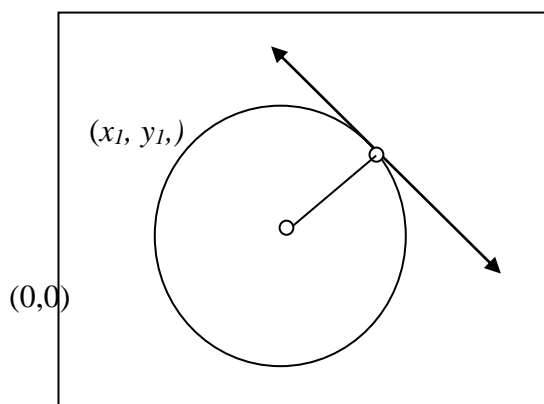


Figure 1: Description of a tangent line on a circle.

After we find the slope, the equation of the tangent line is:

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

$$\rightarrow y = -\frac{x_1}{y_1}x + \frac{x_1 \cdot x_1}{y_1} + \frac{y_1 \cdot y_1}{y_1}$$

$$\rightarrow y = -\frac{x_1 \cdot x}{y_1} + \frac{x_1^2 + y_1^2}{y_1}$$

Convert it to the standard form:  $y \cdot y_1 = \left[ -\frac{x_1 \cdot x}{y_1} + \frac{x_1^2 + y_1^2}{y_1} \right] \cdot y_1$

$$\rightarrow y \cdot y_1 = -x_1 \cdot x + x_1^2 + y_1^2$$

Add  $x_1 \cdot x$  to both sides:  $x_1x + y_1y = x_1^2 + y_1^2$ .

From the given equation of the circle we get:  $y_1^2 + x_1^2 = r^2$ .

The equation of the tangent line of a circle passing through  $(x_1, y_1)$  is  $x_1x + y_1y = r^2$ . Since the equation of a tangent of a circle is defined as  $x_1x + y_1y = r^2$ , problems can be solved easily.

For example: Find the equation of the tangent line of a circle passes through  $(-3, 4)$  if the given circle is  $x^2 + y^2 = 25$ .

Based on the proofs, the equation of the tangent line of the circle would be  $x_1x + y_1y = r^2$ , so  $-3x + 4y = 25$  is the equation of the tangent line of the given question.

The standard form of a linear equation can be used to find a tangent line of a circle can be shown with GeoGebra. Figure 2 shows the tangent line, the normal line, and the point of tangency. The figure also shows the equations of the two lines. You can ask your students following questions: What do you notice about the two equations? Can we make any conjectures about the angle they form? What do you notice about the equation of the tangent line and the point of tangency?

The equation of tangent line to a circle can be derived many different ways but the picture does show that it will be  $x_1x + y_1y = r^2$ .

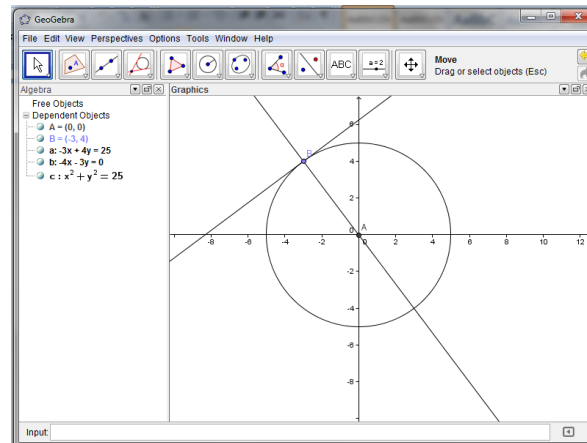


Figure 2: Example of the tangent line equation using GeoGebra

Point A represents the center of the circle, and the point B represents the point on the circle that the tangent line intersects. As shown in Figure 2, the tangent line equation is  $-3x + 4y = 25$ .

### What if the center is not the origin?

If the center is not the origin, then the equation of a circle is defined as  $(x-h)^2 + (y-k)^2 = r^2$ . What is the equation of tangent line for a circle with center at  $(h, k)$  and radius  $r$ ? The tangent line of a circle  $x_1x + y_1y = r^2$  can be written as  $(x_1-0)(x-0) + (y_1-0)(y-0) = r^2$ . Based on this, the tangent line of a circle with the form  $(x-h)^2 + (y-k)^2 = r^2$  is  $(x_1-h)(x-h) + (y_1-k)(y-k) = r^2$ .

Here is the proof:

The circle with center  $(h, k)$  is moved from  $(0, 0)$ ,  $(x_1, y_1)$  is moved  $x_1$  to  $x_1 - h$ , and  $y_1$  to  $y_1 - k$ . The slope of tangent line becomes:  $m = -\frac{(x_1 - h)}{(y_1 - k)}$ . Then, the equation of a tangent

line is:

$$\begin{aligned}(y-k)-(y_1-k) &= m[(x-h)-(x_1-h)] \\ \Rightarrow (y-k) &= -\frac{(x_1-h)}{(y_1-k)}[(x-h)-(x_1-h)] + (y_1-k) \\ \Rightarrow (y-k)(y_1-k) &= -(x_1-h)[(x-h)-(x_1-h)] + (y_1-k)^2 \\ \Rightarrow (y-k)(y_1-k) &= -(x_1-h)(x-h) + (x_1-h)^2 + (y_1-k)^2 \\ \Rightarrow (x_1-h)(x-h) + (y-k)(y_1-k) &= (x_1-h)^2 + (y_1-k)^2\end{aligned}$$

Since  $(x_1 - h)^2 + (y_1 - k)^2$  is  $r^2$ , the equation of tangent line is  $(x_1 - h)(x - h) + (y - k)(y_1 - k) = r^2$ .

For example: given the circle equation  $(x - 3)^2 + y^2 = 25$ , find the equation of the tangent line of the circle passing through  $(6, 4)$ . According to this proof, the equation would be  $(6 - 3)(x - 3) + 4y = 25$ . After simplifying, the equation would be

$$\begin{aligned}(6 - 3)(x - 3) + 4y &= 25 \\ \rightarrow 3x - 9 + 4y &= 25 \\ \rightarrow 3x + 4y - 9 &= 25 \\ \rightarrow 3x + 4y &= 34\end{aligned}$$

Therefore, the equation of the tangent line is:  $3x + 4y = 34$ .

Students can check this answer using the GeoGebra. First of all, ask students to create the circle with the center  $(3, 0)$  and a radius of 5. After that, create the tangent line passing through  $(6, 4)$ . Figure 3 shows this example.

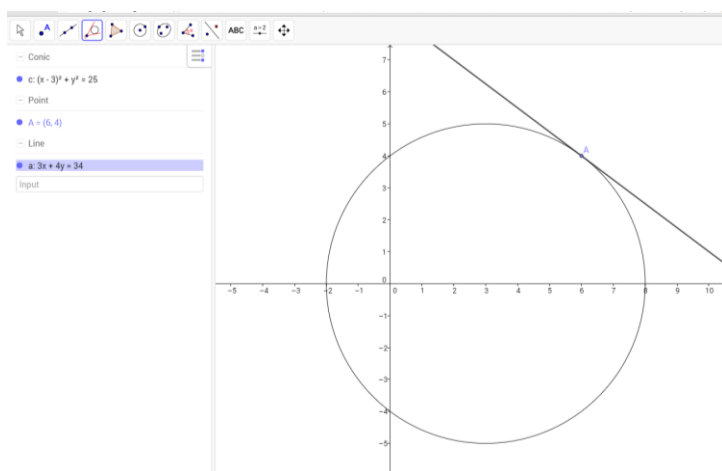


Figure 3: Example of the tangent line using Geo Gebra

Figure 3 shows that the tangent line equation of the circle  $(x-3)^2 + y^2 = 25$  passing through  $(6, 4)$  is  $3x+4y=34$ .

### Implementing the method

For the first exposure of this method in a classroom setting, the researchers concentrated on collecting students' opinions. Thus, this study did not include the comparison between regular classes and research classes. Students were selected based on volunteering. All participants had taken a Calculus course, so they were all knowledgeable of Implicit Differentiation and the concept of the tangent line to a circle.

At the beginning, each participant was given questions to find tangent line of a circle. They were asked to solve questions using any method. After that, the researchers asked them to generalize how to find the tangent of a circle. Most of the participants described how they solved the problems:

*"The equation of a circle is not a function, so we need to find the implicit differentiation in order to find the slope of a tangent line. After that, find the equation of a tangent line based on the given point"*

*"I found the slope of a line passing through the center and the given point. Because the slope the tangent line is the perpendicular to the line from the center, so using the perpendicular line concept, found the slope the tangent line. After that, I found the equation of the tangent line based on the given point."*

Soon after, we introduced the new algorithm, and discussed the method with them.

*"That's it!! Is that simple?"*

*"Oh My God!! It is really simple to find the tangent line of a circle"*

For deeper understanding, we also demonstrated the answer graphically.

*"Wow. It is truly simple and easy. From now on, I definitely remember this method"*

*"I love this method. Can we use it for other conic problems, such as ellipse and hyperbola as well?"*

The participants were interested in learning this new method to find the tangent lines of a circles, and also interested in finding other conic problems as well.

When each participant was given some review questions, they solved all the questions easily and accurately.

GeoGebra provided the visualization of the new method. The participants checked the tangent line of a circle based on GeoGebra demonstration, and also confirmed the distance between the center and a given point is always the radius “r”. However, it did not necessarily provide the conceptual understanding of the new method.

### Conclusion

The equation of a tangent line of a circle is an important concept for mathematics and science. The tangent line is usually described by the on geometric definition: the radius and the tangent line are always perpendicular, and implicit differentiation. Gordon (2009) showed detailed steps for finding the equation of tangent of a circle. However, the steps are so long and tedious. If students learn the concept of derivative, then the method would be easier. But, students still need to understand all the derivative properties, and they especially need to understand the concept of “Implicit Differentiation” since the equation of a circle is not a function.

If we consider the standard circle formula:  $x^2 + y^2 = r^2$ , then defining the tangent line equation of a circle is more understandable. As shown in this article, the equation of tangent which passes through  $(x_1, y_1)$  of a circle  $x^2 + y^2 = r^2$ , is  $x_1x + y_1y = r^2$ . Based on this fact, the equation of the tangent line  $x_1x + y_1y = r^2$  can be rewritten.  $(x_1 - 0)(x - 0) + (y_1 - 0)(y - 0) = r^2$ . It gives an idea that the equation of a circle is  $(x - h)^2 + (y - k)^2 = r^2$ , the tangent line which passes through  $(x_1, y_1)$ , can be defined:  $(x_1 - h)(x - h) + (y_1 - k)(y - k) = r^2$ .

Instructors can provide a dynamical view of tangent line of a circle using the GeoGebra software. GeoGebra generates the standard form equation to describe tangent.

Fortunately, an equation of a circle is similar to standard form, so both instructors and students can explore alternative approaches to find a tangent of a circle using the standard form. As a couple examples were shown in this article, students and instructors can check and find the equation of the tangent line of a circle easily using the formulas:  $x_1x + y_1y = r^2$  and  $(x_1 - h)(x - h) + (y_1 - k)(y - k) = r^2$ .

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