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MINUTES FROM MATH MEETINGS WITH AN UNDERGRADUATE STUDENT

Malgorzata Aneta Marciniak

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Abstract: *The notes contain minutes from meetings with an undergraduate student who is preparing to join a graduate program in mathematics*.

Some ideas

I had some ideas to work with John:

-Harmonics on a torus: since torus is a product of two circles can we solve and visualize the harmonics on a torus? What does it mean from the perspective of physics?

- What is the meaning of the linear combinations of spherical harmonics? What is the meaning of elliptic harmonics? How about other quadric surfaces? Saddle surface?

- From today's dream: What is the meaning of the equidistant points (with distance in 3D) on the surfaces representing the spherical harmonics?

Few days ago, John officially became my student. I am not sure what it exactly means to him and to me but I have a strong commitment to take care of him. Today we discussed what we should study.

I gave John an example of my research groups and mentioned that my students have been working on particular problems since the summer and now they have material for presentation. One group just recently presented their results. I asked John if he would have something to present and he admitted that he does not. I asked whether he would like to work on problems and present at conferences and meetings and he agreed. Thus, we began to search.

John said that he wants to work on problems that include physics, complex analysis, and computer-generated geometry. It seems that we like the same thing.

More ideas

John wanted to study four books (listed in the References) and purchased hard copies. A long time ago I decided to go through my life lightly and avoided purchasing anything heavy, thus, equipped in digital copies of the same books downloaded from the library, I was waiting for John at the lobby of the Math Department.

We both began our first meeting by arriving late because of an unreliable subway service. A colleague of mine was studying at the lobby making an appearance that being present there is a fashionable thing. While waiting for John I was entertained by a small talk. Later, during the meeting with John, my colleague was listening carefully to the conversation and I was wondering whether he was curious how to handle a meeting with a student. To make it easier to him I decided to record parts of the conversation.

That's how we began
Me: How do you want to read these books? Do you have a plan?
J: First I should master the concept of an ideal.
Me: Excellent! Which book are you going to use first?
J: (Pointing towards [1]) This one gives the definition of an ideal in Chapter 1 so maybe here.

John apparently did his homework and looked through the books before bringing them to the meeting. The first chapter of [1] began with the definition of a ring and went straight to the definition of a homomorphism between rings.

Me: What is the motivation for a concept of an ideal? Why does [1] define rings and their homomorphisms before defining ideals? Is there a relationship between the order of the concepts and the meaning? (I wish someone pointed it to me when I was a student, long before I had to write my own articles)
J: Yes. Ideals are the kernels of homomorphisms between rings.

Why do we care about the kernels? Maybe because taking a quotient of a ring R_1 by a kernel of a homomorphism $\phi: R_1 \to R_2$ leads to a one-to-one homomorphism of rings $R_1/ker\phi \to R_2$.

Me: Great! Let us write the definition of an ideal on the board and prove that a kernel is an ideal.
Thus, I wrote:
Definition 1: A subset I of a ring R is an ideal if the following conditions are satisfied:
-If $x, y \in I$ then $x + y \in I$
- If $x \in I$ and $r \in R$ then $rx \in I$
John proved on the board that a kernel is an ideal. He did not prove that an ideal is always a kernel and did not question this concept but we will certainly return to this topic later.
Then he had a question:
J: Is this definition of an ideal valid only in commutative algebra? What is the difference for noncommutative rings?
Me: Let us examine the conditions in Definition 1 The first one: `` If $x, y \in I$ then $x + y \in I$

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" has nothing to do with commutativity of the ring. But the second one `` If $x \in I$ and $r \in R$ then $rx \in I$ " is a property of a left ideal. A right ideal would be described as follows: `` If $x \in I$ and $r \in R$ then $xr \in I$." The difference is in multiplying x by r either from the left or the right side.

Curiosity led John to the question that was not answered in any of the commutative algebra books.
Me: Shall we write few examples of ideals on the board? It would be helpful to provide the rings R_1, R_2 and the homomorphism $\phi: R_1 \to R_2$ such that $ker\phi = I$.
J: Could you give one example?
Me: Yes, of course. Here it is.
What I wrote on the board:
Example 1:
Let $R_1 = k[x]$ be a ring of polynomials over a field k. We can assume for simplicity that k is $\mathbb{C}$ or $\mathbb{R}$. An ideal I can be defined as follows
$I = \{f(x) \in R_1 : f(0) = 0\}$
We could either prove that this set is an ideal in R_1 by checking the conditions from Definition 1 or show a homomorphism ϕ such that $ker\phi = I$.
John proved on the board that I indeed fulfills the assumptions from Definition 1 but I followed the other path. For a ring R_2 choose k and a homomorphism $\phi: k[x] \to k$ is defined as follows:
$\phi\big(f(x)\big) = f(0)$
where $f(x)$ is a polynomial over k. Then the set of those polynomials $f(x)$ such that $f(0) = 0$ is exactly the kernel of the homomorphism ϕ.
It was John's turn to present his examples. This is what he wrote on the board:
$I_1 = \{r \in \mathbb{Z}, x \in \mathbb{R}: x^r\}$
and another example
$I_2 = \left\{a, b, c, d \in \mathbb{Z}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\}$

Since I did not understand I asked John to write down the elements of the ideal. So, he wrote:
$I_1 = \{x, x^2, x^3, \dots\}$
and
$I_2 = \left\{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, \dots\right\}$

Me: Did you check the assumptions of an ideal?
J: No. I see that I_1 does not fulfill neither condition.
Me: How can we fix that?
J: For condition 1 we can add the monomials: $x + x^2, x + x^2 + x^3, \dots$
Me: How about $x + x^3$ or $x + x^4$?

J: Ah, yes, I have to think about it.
Me: How about I_2? What ring is it in?
J: The ring of all matrices.
Me: Of any size? With what terms?
J: No. Just2×2. I see now. What I wrote is a ring and now I have to find an ideal within this ring. I have to think about it.

The time was up and we finished the meeting. Jonathan's homework consisted of finding three examples of ideals I indicating the rings and their homomorphisms ϕ such that $ker\phi = I$.
Before we parted we discussed briefly whether the set of all ideals of a ring has a structure? Certainly, some ideals are subset of other ideals but is there something more than that? This question seemed to strike John so we added it to the homework.

John's Examples
John send me an email with two examples

$$R_1 = \cos x \sum_{i=0}^{\infty} a_i x^i$$

with the ideal

$I_1 = \left\{n \in \mathbb{Z}: \frac{\pi(n+1)}{2}\right\}$ and

$$R_2 = \sin x \sum_{i=0}^{\infty} a_i x^i$$

with the ideal

$$I_2 = \left\{n \in \mathbb{Z}: \frac{\pi n}{2}\right\}$$

which convinced me that we should fist master the concept of a ring before we can approach the concept of an ideal. Notation needs some revisions as well.

[1] Atiyah, MacDonald, *Introduction to Commutative Algebra*
[2] Cox, Little, O'Shea, *Ideals, Varieties, and Algorithms*
[3] Eisenbud *Commutative Algebra with a view Towards Algebraic Geometry*
[4] Kunz *Introduction to Commutative Algebra and Algebraic Geometry.*