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Assessment of the depth of knowledge (DoK) acquired during the Aha!Moment insight.

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Abstract

The inquiry into the nature of Aha!Moment based upon the theory of bisociation have been initiated since the publication of Czarnocha et al (2016) where the coordination between remedial mathematics classroom practice and theory of creativity of Aha!Moment of Koestler by Vrunda Prabhu (2016) was described. The results of the inquiry have been presented in Baker et al (2016); Czarnocha (2014), Czarnocha and Baker (2015); the processes of classroom facilitation and DoK assessment of Aha!Moment are being investigated presently. The proposed presentation we will sketch the DoK qualitative assessment approach with the help of Triad of Piaget and Garcia (1989) as the theory of schema development, reinforced by the Gestalt concept of restructuring (Weisberg, 1996). We will provide examples of such assessment with newly collected Aha!Moments.

Introduction. This Teaching-Research report informs about the pilot teaching experiment (TE) undertaken at Hostos CC with the aim to investigate facilitation and assessment of Aha!Moments in mathematics classroom. The design of the teaching experiment followed the suggestions of the Bisociation theory of Aha! Moment (Czarnocha et al, 2016). It was supplemented by the results of the analysis of several descriptions of Aha! Moments in professional literature (Czarnocha, Baker 2016). The analysis indicated presence of two distinct methods of facilitation: complex problems in the collaborative setting and scaffolding student understanding through one-on-one dialogues between the student and the teacher (or a knowledgeable peer).

The pilot teaching experiment was conducted in two remedial classes at Hostos CC, Mat 15 (integrated arithmetic/algebra) and Math 150 of Intermediate algebra. Important feature of the pilot was the participation of student-researchers, who as more experienced peers worked with students during and after the class. It is interesting to note that the majority of facilitated Aha!Moments during TE were obtained by student-researchers. The successful method of facilitation tuned out to be weekly/ biweekly assignments of moderately advanced problems assigned for every student in the classroom to be solved at home. Correct solutions were awarded of special credit, which later counted in highly in the final grade of the student.

Koestler defines Aha! Moment, Eureka Experience or bisociation as ”*the spontaneous leap of insight, which connects two or more unconnected matrices of experience, frames of reference*”(p.45). The terms matrix and code are defined broadly and used by Koestler with a great amount of flexibility. He writes, “*I use the term matrix to denote any ability, habit, or skill, any pattern of ordered behaviour governed by a code or fixed rules*” (p. 38). It follows that the term matrix can be applied to all coherent, logical or rule-based thought processes employed by an individual learning mathematics:
“*The matrix is the pattern before you, representing the ensemble of permissible moves. The code which governs the matrix…is the fixed invariable factor in a skill or habit, the matrix its variable aspect. The two words do not refer to different entities; they refer to different aspects of the same activity.* (Koestler, 1964, p. 40)”.
Central for Koestler’s theory is the notion of hidden analogy, which gets revealed during the insight. *”The discoveries of science, of works of art are explorations – more, are explosions of a hidden likeness*” (p.200).
In his opinion, and that of several other investigators such as Vygotsky (2004), the source of the discovery of hidden analogy lies in human imagination, “*it’s created by the imagination’; and once an analogy was created it’s of course for everyone to see.*” Thus to establish the adequacy of Koestler’s theory for a particular Aha! Moment it’s important to find the two or more not connected matrices which underlined the insight together with the discovery of the hidden analogy, which connects them.
PG Triad, Restructuring and Bloom-Webb DoK. The Piaget-Garcia Triad (Piaget and Garcia 1989) is a mechanism of thinking whose engine is the reflective abstraction. Its central structure are three developmental levels, Intra-, Inter-, Trans-; while the engine of development is the process called by Piaget, reflective abstraction. Baker (2016) had shown that bisociation is the basis for reflective abstraction. In the initial Intra- stage students experience actions or operations as isolated phenomena. They have difficulty coordinating these actions hence have a limited range in which they can successfully apply them; at this stage creativity is very difficult and tends to be brought out only through dialogue with instructor or peers. In the second, Inter- stage, coordination between actions is observed. The solver projects, according to Piaget and Garcia (1989) a lower order schema into the problem situation and through coordination with the problem information, problem goal and other actions, he or she constructs a more general schema to solve the problem. In this stage the different steps required to solve the problem are coordinated through bisociation (Koestler) of the concepts from their existing schema

with other relevant schema and the problem situation yet these linked steps are not fully synthesized (Koestler) until the final Trans-stage of the Piaget-Garcia Triad.
The concept of "restructuring" (Weisberg, 1996) brought forward by Gestalt theorists will be helpful in the assessment of the Trans- stage of the Triad. Restructuring takes place, in the context of problem solving, as the solver is going through an initial preparation phase that does not result in a successful solution strategy, a discontinuity in which the original strategy is disregarded, and a moment of insight in an illumination phase (Aha moment). The strategy that led to the insight approached the problem through a different analytical process - the result of restructuring.

Assessment of the DoK became widely discussed in the professional literature in connection with the Common Core design which called for a thorough assessment of progress in student learning. One of the more popular measures has been Webb's Depth of Knowledge rubric which is the third stage in the development of Bloom taxonomy whose first version appeared in 1956. The revision of that taxonomy took place in 2001 by Krathwohl (2001) and in 2002, Norman Webb created his DoK (Webb, 2002), which addressed the requirements of new curriculum, objectives, standards and assessments. There is an unmistakable tendency in this process of transforming original noun based approach to the taxonomy into the verb based in the revised version, and finally in Webb's design we have the expansion of terms to suggest very particular aspects of thinking attributed to each stage. For example, where Blooms taxonomy classifies the stage of Application as *the use of learned materials in new and concrete situations*, revised Bloom taxonomy calls it the Applying stage, while Webb's taxonomy calls it the stage of Skills and Concepts, describes it as "*Engages mental processes beyond habitual response using information or conceptual knowledge*" and provides large collection of verbs, which do that job.
While possibly helpful to teachers in terms of assessing the state of knowledge of their students, it has important flaws as the method of DoK assessment, especially in the context of Aha!Moment creativity and insight:

1. Investigation of the depth of knowledge reached during the insight has to show the degree of progress of understanding, and that means to show the difference in understanding before and after the insight. Positioning of the DoK on a particular level of discourse doesn't help in this process so that the assessment of the progress is imprecise. Moreover, it doesn't give any information about the process that helped in the insight. Such information is helpful to the teacher in developing her/his teaching craft by addressing particular limitation or epistemological obstacles encountered by students.
2. The static character of the division into the stages doesn't allow to analyze student thinking which went through several stages of development as a consequence of the insight.

3. We are suggesting that the assessment of the depth of knowledge of Aha! Moments has two dimensions, one of the Bloom-like characterization of depth, another of intensity counted in the number of steps of progress in understanding that occurred.

We classified mild bisociation as that one that involves only one mathematical step or analogy, this includes the process in which discovering a hidden analogy involves employing elementary schemes that are intuitive or self evident to the solver and thus seen as relevant but in which the path to solution is not entirely clear as included in the range of mild; the normal level of bisociation is the building up process in which several of the elementary schemes are coordinated to form a functional whole; by a strong bisociation we classified as that one which has at least two steps and/or two cycles in progress of understanding.

Analysis of collected Aha! Moments.

Fir Tree Aha! Moment, a **strong** bisociation is specially important because it shows that an Aha! Moment **does not have to occur** in the context of problem solving. Here it took place as further refinement of the solution, which was obtained before the insight took place through the standard process of abstraction and generalization of a pattern. Once the solver abstracted the relationship $n(n-1)$ as representing the value of squares at stage 'n' she coordinated this with the problem situation to gain a further insight. She realized that the fir tree will contain N squares if there is a 'n' such that $n(n-1) = N$ and at this point reflection upon solution activity to find such an 'n' leads her to synthesize understanding of how to solve this with her understanding of factoring trinomials. Again this abstraction creates a new code or scheme in which she reduces the problem to whether one can factor the trinomial $n^2 + n - N = 0$. Thus she has engaged in constructive generalization i.e. the creation of a new scheme with a 'viewable' artifact as an end product. This constructive generalization is founded upon her work building up the previous scheme of finding 'n' such that $n(n-1) = N$ and thus supports Hershkowitz et al (2001) claim that structural abstraction frequently results during the building up process. In this situation the 'building up' process consists of the solver repeating variations perceived as related but distinct cases of solutions until they abstract or synthesize the common code in both Hershkowitz et al (2001) and here the solver records variations in solution attempts in a table form, while in Simon (2014) the student abstracts the results of solving several different related problem all presented as unrelated and without the use of tables. However, we argue that in each case there was a 'building up' phase as the student worked on examples they considered separate but related and then a bisociative synthesis of the codes or structural abstraction that unified these examples and propelled the solver into the second and third stage of the Piaget triad.

In terms of Bloom taxonomy, we have here the synthesis level of the pattern generalization with the concept of a quadratic equation and the role of factorization in solving the equation. The extended thinking of Webb's Depth of knowledge suggests "complex reasoning", however no planning through extended period of time was necessary as the Aha!Moment of bisociation is instantaneous, though solving the original problem can be taken as the required extended time.

Square root domain a **strong** bisociation.
The dialog goes through the Application stage facilitated by the instructor; student jumping over the Analysis stage to land in the Synthesis as an abstraction and generalization. Then again Application of the new understanding and second Synthesis. So, the Aha! Moment is again taking place, twice through iteration in between levels of the Bloom Taxonomy.
The bisociation takes place in line 10, when student finds hidden analogies in the concrete examples discussed earlier during the dialogue, which are abstracted and generalized. A similar process takes place in lines 11-14.

The student makes an incomplete association from the matrix of finding the domain of a function such as f(x) = $\sqrt{x}$ to this problem situation which involves a transformation of the previous situation (the argument within the square root, x+3)
In lines (6) and (8) the instructor employs concrete counter examples to provide a *perturbation,* or a *catalyst*, for cognitive conflict and change. "...perturbation is one of the conditions that set the stage for cognitive change" (Von Glasersfeld, 1989a, p. 127).
In lines (**6**) – (**9**) the student reflects upon the results of the solution activity. Through the comparison of the results (records) they abstract a pattern, — "the learners' mental comparisons of the records allows for recognition of patterns" (Simon et al., 2004).

Evidence of the abstraction of the principle that the transformation of the argument x+3 beneath the square root shifts the domain 3 to the left is given in line (10). Thus, this realization can be seen as reflective abstraction in the sense of Simon et al. (2004). In that the solution activity of substitution was projected into and coordinated with the solver's knowledge of the domain of the square root function it is also an example of reflective abstraction according to Piaget and Garcia (1989). In that solution activity through substitution with the students matrix or schema of evaluating x − a is coordinated or integrated by the student with the student's understanding (matrix) of the domain of the square root function their realization in line (10) can be viewed as bisociative in nature.

In lines (**11**) and (**12**), the perturbation brought about by the teacher's questions, leads the student to enter the second stage of the Piaget and Garcia's Triad. The student understood that the previously learned matrix or domain concept of radical functions, with proper modifications, extended to this example. They student was then able to reflect upon this pattern and abstract a general structural relationship in line (**14**). Thus, the student's understanding of finding the domain of a square root has undergone constructively generalization to accommodate transformation providing evidence of the structural understanding noted by Sfard (1991) and the third stage of the Triad. (Piaget and Garcia, 1989)

Physics Aha! Moment , a **normal** bisociation takes place exactly from the Analysis to Synthesis level of understanding: ”*Opening the right tringles on a piece of paper I was able to see the problem differently. When I opened the right triangles I noticed that they were the same and since we were dealing with variables and not actual numbers I could apply the given variables…I determined that the blue triangle and red triangle were equal when dealing with variables, as shown below.”* The description of the circumstances which accompanied the Aha!Moment here conform with the Analysis level of Bloom described as “Break(ing) down material into component parts so that its organizational structure maybe understood” followed by the Synthesis described as ” Put(ing) parts together to form a given whole” and/or characterization of creativity (revised Bloom taxonomy) by “Put(ing) elements together to form a coherent or functional whole….”

Strategic thinking of Webb’s DoK stage is reflected in reasoning, differentiating and development of a plan (*opening triangles and noticing they are the same*) while the Extended Thinking stage is reflected by the student in terms of connecting different triangles through similarity, critique of the situation as well as explaining phenomena in terms of new concepts.

We see here the fundamental problem of these taxonomies where assigning a specified thinking skills disregards thinking at different levels which led to the assigned one. We have here a standard example showing that what’s important in the assessment of DoK is the extent of the gap, which was dealt with during the insight. Here, the Aha! Moment or bisociation was within the transition between Analysis and Synthesis level.

Aha! Moment to factorize $2x^2 + 3x + \square$.

Normal to strong bisociation in the context of trial and error method, suggested restructuring of the problem to integrate algebraic and visual. So, we have here first Application of the coordination, the difficulty with Application was the force that propelled me to algebraic generalization as Synthesis. Again Aha! Moment took place in between the levels of taxonomy. The hidden analogy leading to Aha! Moment was between the concrete examples and the general form of factorization monomials.

The student solves the problem with finding first the polynomial $2x^2 + 3x + 1 = (2x + 1)(x + 1)$ then after a period of time finds another solution one with a negative factor: $2x^2 + 3x - 2 = (2x - 1)(x + 2)$.

A this point the student appears to be using trial and error substituting values a & b into the expression $(2x + a)(x + b)$ and asking if the result adds up to 3 when the binomials are multiplied i.e. using a multiplication technique to verify if the substituted values are correct. After the point where she/he chooses a negative integer the student realizes that there is always the same pattern to this multiplication that is the code can be written as: 2a + b. Thus the student has translated or abstracted the code she/he is using during this trial and error process into an algebraic equation 2a + b = 3. This certainly qualifies as reflective abstraction using the definition of Simon et al (2004) as reflection as based upon goal directed solution activity during which the solver has mental records of the result such activity. The solver's mental comparison of these records allows them to see a pattern and the abstraction of the activity-effect relationship is the beginning of concept formation a "coordination of conceptions." (p319)

In our view Simon's understanding of reflective abstraction moves closer to Koestler understanding of bisociation when he states that, "*knowledge of the logical necessity of a particular pattern or relationship is generated through reflective abstraction. By anticipation of the logical necessity…As a result of reflective abstraction the student learns not only that the relationship exists but why the relationship is necessary*" Simon (2010, p.365). For Simon understanding logical necessity implies the solver not only notices the pattern but further understands the reason for the pattern i.e. the equivalence of the mental records she has accumulated and can thus anticipate her actions and their effect within this problem type. "…she came to see the equivalence of particular problems and can explain the logical necessity of their relationship." (Simon et al, 2010, p.99) For Simon the solver noticed the pattern and then abstracted the pattern as she became aware of the logical necessity of multiply a by 2 and adding this to b to obtain 3 at which point (certainly with the help of the formula) she can anticipate the results of her action more efficiently in particular without participating in the process of binomial multiplication

In the terminology of Koestler one could say she synthesized the codes she was reflecting upon in the pattern of her experiences with her object level understanding of variables to create an artifact i.e. the algebraic expression 2a+b = 3. Once again this like the earlier fir tree problem can be seen as the synthesis of codes during a 'building up' in the abstraction takes previously related but separate solutions into a unified scheme and is thus structural abstraction within the Piaget Triad.

In the terminology of reflective abstraction as described by Piaget, the solver has projected her solution activity –trial and error involving substitution and binomial multiplication scheme into the problem situation and coordinated it with the goal of finding values of c and in this process engaged in constructive generalization. This constructive generalization certainly involved coordinating or synthesizing the code or logical reasoning underlying the pattern with her object level understanding of variables. In this case the end product of constructive generalization is an artifact –the algebraic equation that she employs to solve the problem

The Elephant **, a normal** bisociation.

The elephant Aha! Moment takes place from below the Bloom's first level to the second level of understanding. The hidden analogy is between the logic/geometric approach and perceptual matrix. The student B. has difficulty with the concept of the unknown in the context of a linear equation. The bisociation takes place when his peer, student P. realizes he has to reach beyond mathematics into the different perceptual matrix to be able to explain the concept to B. What makes a normal bisociation is the length and variety of the process that led to the single bisociation. This problem is noteworthy because it demonstrates a teaching research bisociation as there are two realizations occurring one by the learner and the other by his peer-tutor. The one student-learner finally makes the abstraction required to consider an unknown as an object with the analogy of the elephant as being some unknown quantity the other realization is by his peer tutor who realizes several times that the analogy he has presented is not being understood by the other student and thus he changes his chosen analogy from line to box to window and finally to an elephant, thus demonstrating natural teaching skills.

Calculus problem
This is **mild** bisociation. The solver goes through a clear period of uncertainty that nicely fits Norton's (2008,2009) description of abduction that is to say the solver recognizes a previous scheme that is related and conjectures that it will help solve the problem but is uncertain over the results as there are clear difference between his previous mental records or scheme and the current situation. "I was looking at the limit, and said to myself why not apply the same rule for the fraction when we have the radical in the denominator". Consequently, the hidden analogy is here between algebraic expression positioned in the numerator and denominator. Thus, the solver conjectures his existing scheme is relevant and coordinates this with the problem information to produce a new scheme. This process is referred to by Norton (2008) as type of accommodation called "generalizing assimilation" in which existing schemes are applied intact to new situations with only slight if any modification the accommodation-change being that their domain of application is extended.

Composition Problem represents second **mild** bisociation, or Aha! Moment, whose hidden analogy is the meaning of the exponent in two different mathematical expressions, the variable x and the monomial (2x+1) both raised to the second power. Where exactly to place the exponent was the problem for the student, whose origins were understood by the student at the mild moment of insight.

Observations and Conclusions.

Observations:

1. Aha! moments are taking place in-between the Bloom-like levels of thinking.
2. In many of them the level of analysis is missing.
3. What I do not see in Bloom-derivatives is the process of transition between them.

What has to be done to move from one level to another is not assessed.

Purely heuristically the collection of Aha! Moments has arranged itself along the intensity (or depth) gradation axis of mild, normal and strong. We classified mild bisociation as that one that involves only one mathematical step or analogy, this includes the process in which discovering a hidden analogy involves employing elementary schemes that are intuitive or self-evident to the solver and thus seen as relevant but in which the path to solution is not entirely clear as included in the range of mild; the normal level of bisociation is the building up process in which several of the elementary schemes are coordinated to form a functional whole; by a strong bisociation we classified as that one which has at least two steps and/or two cycles. The second or structural abstraction often occurs during the building up process in which a single solution process built is repeated in what is understood by the solver as separate but related solution activity. Often the results recorded in a table until a synthesis of codes allows the solver to understand the logical necessity behind their solution activity and thus to engage in a structural abstraction.

That means that the assessment of the depth of knowledge of Aha! Moments has two dimensions, one of the Bloom-like characterization of depth, another of intensity counted in the number of steps of progress in understanding that occurred.

Appendix

Pdf Collection of Aha!Moments attached

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