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Getting through Calculus without using the Trigonometric functions Cosecant, Secant, and Cotangent

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Abstract: We demonstrate how a student could approach problems containing Cosecant, Secant, and Cotangent using only Sine, Cosine and Arctangent.

Introduction

Trigonometric functions are widely used in most branches of mathematics as well as in solving real-world problems. When using trigonometric functions, it is often of value to change a trigonometric expression from one form to an equivalent form by the use of identities. Mathopenref states "Of the six possible trigonometric functions, secant, cotangent, and cosecant, are rarely used. In fact, most calculators have no button for them, and software function libraries do not include them. They can be easily replaced with derivations of the more common three: sin, cos and tan (2009) ". Axler (2013) states "Many books place too much emphasis on secant, cosecant and cotangent. You will rarely need to know anything about these functions beyond their definitions. Whenever you encounter one of the functions, simply replace it by its definition in terms of cosine, sine and tangent and use your knowledge of those familiar functions. By concentrating on cosine, sine and tangent rather than all six trigonometric functions, you will attain a better understanding with less clutter in your mind". We demonstrate how a student could approach problems containing sec x , csc x , cot x, $\sec^{-1}$ x, $\csc^{-1}$ x and $\cot^{-1}$ x using sin x, cos x and $\tan^{-1}$x . All the students would need to know are the trigonometric identities and how to use them. For example, if they have csc x they would change it to $\frac{1}{\sin(x)}$. This is how a group of our best students in our calculus I and II classes (engineering students) actually did the problems on their tests. These students changed sec x , csc x and cot x into expressions using sin x and cos x then proceeded to do the problems. We start this paper by discussing students" actual answers to our test questions. We next discuss the usual way of evaluating the integral of sec x and show the answer in terms of sin x and cos x. We include a method of partial fractions that also uses only sin x and cos x After this we show how to find the derivatives of $\sec^{-1}$ x , $\csc^{-1}$ x and $\cot^{-1}$ x using only sine and cosine. It seems the real purpose of $\sec^{-1}$ x is only for the integral $\int \frac{dx}{x\sqrt{x^2-1}}$. We will show four more answers to this integral, two using arctan ($\tan^{-1}\left(\sqrt{x^2-1}\right)+c$, $2\tan^{-1}(x+\sqrt{x^2-1})$ +c) , one using arcsin ($-\sin^{-1}\left(\frac{1}{x}\right)$ +c) and the other arccos ($\cos^{-1}\left(\frac{1}{x}\right)+c$) . We then reference some of the previous literature on the different techniques of integration that use sine and cosine only. Finally we discuss the only two places where students still use cosecant in the real world, they are offset bends that electricians use and radar.

We taught our calculus I and II classes the traditional way using sec(x), csc(x), and cot(x). If a student did not have a TI-89 or TI-Nspire CAS graphing calculator, we loaned the students the TI-89 for the semester. Students were allowed to use the calculator on all the exams. We had a group of our best students in calculus I and II (all engineering students) who changed every problem that had sec x, csc x, cot x into a problem in terms of sin x and cos x, and they were able to do calculus I and II without sec x, csc x and cot x. We show how some of our students used sec x, csc x, cot x to do the problems and then show how our students who did not using sec x, csc x, cot x did the problems.

We gave the following problem on a test: find $\frac{dy}{dx}$ for y=$\frac{\sec(x)}{\cot(6x)}$. We were expecting the students to get $\frac{sec(x)\tan(x)\cot(6x)+6csc^2(6x)\sec(x)}{cot^2(6x)}$.

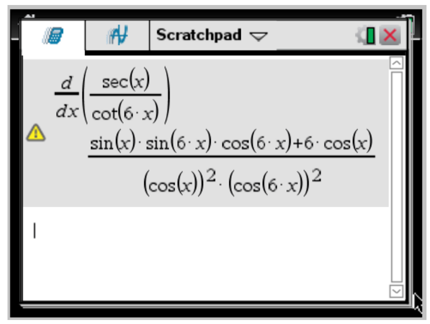
The majority of the students did get this result on the test. There was a different group of students (all were engineering students) who changed the problem to $y=\frac{\sin(6x)}{\cos(x)\cos(6x)}$

These students got the answer

$$\frac{6\cos(6x)\cos(x)\cos(6x)-(-(\sin(x)\cos(6x)-6\sin(6x)\cos(x))\sin(6x)}{cos^2(x)cos^2(6x)}.$$

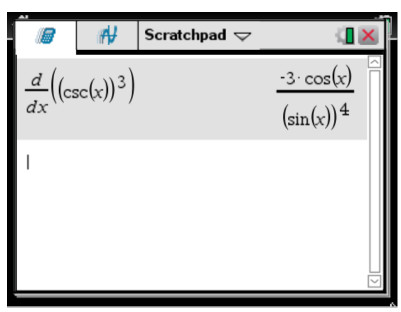
Some of these students simplified their answer to: $\frac{\sin(x)\sin(6x)\cos(6x)+6\cos(x)}{cos^2(x)cos^2(6x)}$.

Figure 1 is a screenshot from the TI-Nspire CAS graphing calculator.



Another example we had on a test was as follows: find $\frac{dy}{dx}$ for y= $\csc^3(x)$. We were expecting the answer $-3\csc^3(x)\cot(x)$. These same students who changed the previous problem to use only sine and cosine changed the problem to $y=\frac{1}{sin^3(x)}=\sin^{-3}(x)$. They then had $\frac{dy}{dx}=-3\sin^{-4}(x)\cos(x)=-\frac{3\cos(x)}{sin^4(x)}$.

Figure 2 is a screenshot from the TI-Nspire CAS graphing calculator.



On yet another test, we gave the problem: $\lim_{x\to 0}\frac{\csc(x)}{1+\cot(x)}$ We were expecting students would do the problem in the following way: $\lim_{x\to 0}\frac{\csc(x)}{1+\cot(x)}=\lim_{x\to 0}\frac{-\cot(x)s\csc(x)}{-csc^2(x)}=\lim_{x\to 0}\frac{\cot(x)}{\csc(x)}$. Most of the students would continue using L'Hospital's rule until they got lost if they did not realize that $\frac{\cot(x)}{\csc(x)}=\cos(x)$.

These same students who changed the previous problem to use only sine and cosine did the following: $\lim_{x\to 0}\frac{\csc(x)}{1+\cot(x)} = \lim_{x\to 0}\frac{-\frac{\cos(x)}{sin^2(x)}}{\frac{-1}{sin^2(x)}} = \lim_{x\to 0}\cos(x)$ =1.

One problem we discussed in our classes was $\int \sec(x)dx$. The traditional answer is: ln[sec(x)+tan(x)] . We then asked our students to put $\int \sec(x)dx$ directly into the calculator. The result they got was $\ln\left(\frac{-|\cos(x)|}{\sin(x)-1}\right)$.

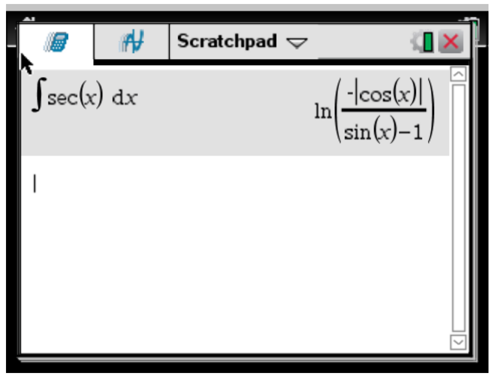


Figure 3 is a screenshot from the TI-Nspire CAS graphing calculator

They asked if this was correct and we told them to prove that | sec(x)+tan(x) | = $\left|\left(\frac{\cos(x)}{\sin(x)-1}\right)\right|$ using only the basic identities. We now explain how to get the answer $\ln\left(\frac{-|\cos(x)|}{\sin(x)-1}\right)$ using just sin(x) and cos (x).

$$\int \sec(x)\,dx = \int \frac{1}{\cos(x)}dx$$

$$=\int \frac{\frac{-1}{\sin(x)-1}}{\frac{-\cos(x)}{\sin(x)-1}}\text{dx}$$

$$=\ln\left(\frac{-\cos(x)}{\sin(x)-1}\right) + c \quad (\text{let } u = \frac{-\cos(x)}{\sin(x)-1} \text{ and use } \int\frac{du}{u} = \ln(u)\ ).$$

Chen and Fulford (2004) solve the integral of $\int \sec\theta d\theta$, by first replacing $sec\theta\ with \frac{1}{cos\theta}$ and then use partial fraction as follows:

$$\int \sec\theta d\theta = \int \frac{1}{cos\theta}d\theta.$$

$$=\int \frac{cos\theta}{cos^2\theta}d\theta.$$

$$=\int \frac{cos\theta}{1-sin^2\theta}d\theta$$. Using the substitution $y = sin\theta, dy = cos\theta d\theta$ yields

$$=\int \frac{1}{1-y^2}dy = \frac{1}{2}\int\left[\frac{1}{y+1} - \frac{1}{y-1}\right]dy$$

$$=\frac{1}{2}[ln|y+1|] - ln|y-1| + C$$, then, we have

$$=\frac{1}{2}ln\left|\frac{y+1}{y-1}\right| + C$$

$$=\frac{1}{2}ln\left|\frac{sin\theta+1}{sin\theta-1}\right| + C.$$

We leave it to the reader to verify that $\left(\frac{-|\cos(x)|}{\sin(x)-1}\right) = \sqrt{\frac{sin\theta+1}{sin\theta-1}}$. This would be an interesting problem to put on an exam, probably as extra credit.

Similarly, we can integrate $\int \csc\theta d\theta$ using a similar technique of partial fractions as the integration of $\int \sec\theta d\theta$ above. However, Weierstrass' half-angle substitution is a useful technique to integrate $\int \csc\theta d\theta$. Steward (1995) states, "Karl Weierstrass (1815-1897) noticed that the substitution $t = tan\frac{x}{2}$ will convert any rational function of $sinx$ and $\cos x$ into an ordinary rational function (p. 465)."

$\int csc\theta d\theta = \int \frac{1}{sin\theta} d\theta$. Let $tan\frac{\theta}{2} = t, \theta = 2tan^{-1}t\, , and\ d\theta = \frac{2}{1+t^2}dt$. By substitution,

$$= \int \frac{1}{\frac{2t}{1+t^2}} * \frac{2}{1+t^2} dt$$

$$= \int \frac{dt}{t} = ln|t| + C$$

$$= ln\left|tan\frac{\theta}{2}\right| + C.$$

(The authors leave it to the reader to prove the identity $\frac{2tan\frac{\theta}{2}}{1+\left(tan\frac{\theta}{2}\right)^2}$ =sinθ, hint use sin θ =sin$\left(\frac{\theta}{2}+\frac{\theta}{2}\right)$ or use tan$\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)}$)

For integrals of $\int sec^m\theta d\theta$ with higher powers of m, we can apply the same technique of partial fractions. However, the algebra involved can be lengthy at best.

For the case of m=3, we can first rewrite $sec^3\theta = sec\theta sec^2\theta$ and perform integration by parts. However, by using the reciprocal function we can achieve the same answer and thus avoid the use of the secant as follows:

$$\int sec^3\theta d\theta = \int \frac{1}{cos^3\theta} d\theta$$

$$= \int \frac{cos\theta}{cos^4\theta} d\theta$$

$$= \int \frac{cos\theta}{[1-sin^2\theta]^2} d\theta$$ Let $y = sin\theta, dy = cos\theta d\theta$, thus

$$= \int \frac{1}{[1-y^2]^2} dy$$

$$= \int \left[\frac{1}{2}\left(\frac{1}{y-1} - \frac{1}{y+1}\right)\right]^2 dy$$

$=\frac{1}{4}\int\left[\frac{1}{(y-1)^2}-\frac{1}{y-1}+\frac{1}{y+1}+\frac{1}{(y+1)^2}\right]$ After integrating the integral above and making the appropriate substitutions, we have:

$$=\frac{1}{4}\left[-\frac{1}{y-1}-\frac{1}{y+1}+ln|y+1|-ln|y-1|\right]+C$$
$$=\frac{1}{4}\left[-\frac{2}{y^2-1}+ln\left|\frac{y+1}{y-1}\right|\right]+C$$
$$=\frac{1}{2}\frac{y}{1-y^2}+\frac{1}{4}ln\left|\frac{y+1}{y-1}\right|+C$$
$$=\frac{sin\theta}{2cos^2\theta}+\frac{1}{4}ln\left|\frac{sin\theta+1}{sin\theta-1}\right|+C$$

For a more lengthy coverage of the integrals of the type $\int csc^m\theta\, sec^n\theta d\theta$, Chen and. Fulford use $\int\frac{1}{sin^m\theta cos^n\theta}d\theta$. (c.f. Chen & Fulford ,2004)

The authors will now show how to find the derivative of $sec^{-1}(x)$, $csc^{-1}(x)$ and $cot^{-1}(x)$, using only sine and cosine. We proceed in the following way:

$y=sec^{-1}(x)=cos^{-1}\left(\frac{1}{x}\right)$

$\frac{1}{x}=\cos(y)$ then

$x=\frac{1}{\cos(y)}$. We next differentiate to get

$$1=\frac{\sin(y)}{cos^2(y)}\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{cos^2(y)}{\sin(y)}$$
$$=\frac{cos^2(y)}{\sqrt{sin^2(y)}}$$
$$=\frac{cos^2(y)}{\sqrt{1-cos^2(y)}}$$
$$=\frac{\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}}$$
$$=\frac{1}{x\sqrt{x^2-1}}.$$

To find the derivative of $csc^{-1}(x)$ using only sine and cosine we proceed in the following way:

$y=csc^{-1}(x)=sin^{-1}\left(\frac{1}{x}\right)$

$\frac{1}{x}=\sin(y)$ then

$x=\frac{1}{\sin(y)}$. We next differentiate to get

$$1=\frac{-\cos(y)}{sin^2(y)}\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{-sin^2(y)}{\cos(y)}$$

$$=\frac{-sin^2(y)}{\sqrt{cos^2(y)}}$$
$$=\frac{-sin^2(y)}{\sqrt{1-sin^2(y)}}$$
$$=\frac{-\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}}$$
$$=-\frac{1}{x\sqrt{x^2-1}}.$$

To find the derivative of $\cot^{-1}(x)$ using only sine and cosine we proceed in the following way:

$y = \cot^{-1}(x) = tan^{-1}\left(\frac{1}{x}\right)$

$\frac{1}{x} = \tan(y)$ then

$x=\frac{1}{\tan(y)}$.

$x=\frac{\cos(y)}{\sin(y)}$. We next differentiate to get

$$1=\frac{-\sin(y)\sin(y)-\cos(y)\cos(y)}{sin^2(y)}\frac{dy}{dx}$$

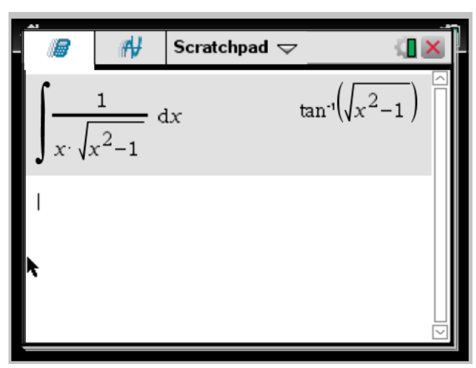
$$1=\left(-1-\frac{cos^2(y)}{sin^2(y)}\right)\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{-1}{1+\frac{cos^2(y)}{sin^2(y)}}$$
$$=\frac{-1}{x^2+1}.$$

We now present an integral with five answers. The integral is: $\int\frac{dx}{x\sqrt{x^2-1}}$.

Here are the different possible answers

- $\sec^{-1}(x)$ +c (1)
- $cos^{-1}\left(\frac{1}{x}\right)+c$ (2)
- $\tan^{-1}(\sqrt{x^2-1})$ +c (3)
- $-\ \sin^{-1}\left(\frac{1}{x}\right)$ +c (4)
- $2\tan^{-1}(x+\sqrt{x^2-1})$ +c (5)

Figure 4 is a screenshot from the TI-Nspire CAS graphing calculator.



Formula (1) is in any standard calculus book, formula (2) the authors derived earlier in this paper (when we showed how to find the derivative of $cos^{-1}\left(\frac{1}{x}\right)$), formula (3) is from the calculator, formula (4) we discovered when trying to derive (5), (5) is from Fulling (2005).

For formula (3) we use $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\frac{x}{\sqrt{x^2-1}}}{1+x^2-1} dx$

$$= \int \frac{\frac{x}{\sqrt{x^2-1}}}{1+\left(\sqrt{x^2-1}\right)^2} dx$$

$=\tan^{-1}\left(\sqrt{x^2-1}\right)$ +c (let u $=\sqrt{x^2-1}$ and use $\int \frac{du}{1+u^2} =$ $\tan^{-1}(u)$).

For formula (4) we use $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{dx}{x\sqrt{x^2(1-\frac{1}{x^2})}}$

$$= \int \frac{\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}} dx$$

$$= \int \frac{-\left(-\frac{1}{x^2}\right)}{\sqrt{1-\frac{1}{x^2}}} dx$$

$= - \sin^{-1}\left(\frac{1}{x}\right) + c$ (let u $=\frac{1}{x}$, then du= $-\frac{1}{x^2}$ dx and use $\int \frac{-du}{\sqrt{1-u^2}}$).

Fulling (2005) uses hyperbolic trigonometric functions to solve the $\int \frac{dx}{x\sqrt{x^2-1}}$, but we get this formula without the use of hyperbolic trigonometric functions. We do it in the following way

For formula (5) we use $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{1}{\sqrt{x^2-1}} \frac{2}{2x} dx$

$$= \int \frac{2(x+\sqrt{x^2-1})}{\sqrt{x^2-1}(2x)(x+\sqrt{x^2-1})} dx$$

$$= \int \frac{2(x+\sqrt{x^2-1})}{\sqrt{x^2-1}(2x^2+2x\sqrt{x^2-1})} dx$$

$$= \int \frac{2(x+\sqrt{x^2-1})}{\sqrt{x^2-1}(x^2+1+2x\sqrt{x^2-1}+x^2-1)} dx$$

$$= \int \frac{2(x+\sqrt{x^2-1})}{\sqrt{x^2-1}\left(1+\left(x+\sqrt{x^2-1}\right)^2\right)} dx$$

$$= \int \frac{\frac{2(x+\sqrt{x^2-1})}{\sqrt{x^2-1}}}{1+\left(x+\sqrt{x^2-1}\right)^2} dx$$

$$= 2\tan^{-1}\left(x+\sqrt{x^2-1}\right)+c$$

(Let $u = x+\sqrt{x^2-1}$, then $du = \frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}dx$ and use $\int \frac{2du}{1+u^2}$).
We now show that all five answers differ by a constant, and in fact three are equal.
For (2) we recall that $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$.
For (3) we see that $\sec^{-1}(x) - \tan^{-1}\left(\sqrt{x^2-1}\right) = 0$.
For (4) $\sec^{-1}(x) - \left(-\sin^{-1}\left(\frac{1}{x}\right)\right) = \sec^{-1}(x) + \csc^{-1}(x) = \frac{\pi}{2}$.
(5) Involves more work. We use $\tan\left(\frac{a}{2}\right) = \frac{\sin(a)}{1+\cos(a)}$, then $\cot\left(\frac{a}{2}\right) = \frac{1+\cos(a)}{\sin(a)}$, and let $\theta = \sec^{-1}(x)$.

$$\cot\left(\frac{\frac{\pi}{2}-\theta}{2}\right) = \frac{1+\cos(\frac{\pi}{2}-\theta)}{\sin(\frac{\pi}{2}-\theta)} = \frac{1+\frac{\sqrt{x^2-1}}{x}}{\frac{1}{x}} = x+\sqrt{x^2-1}, \text{ then}$$

$$\tan\left(\frac{\pi}{2}-\left(\frac{\frac{\pi}{2}-\theta}{2}\right)\right) = \cot\left(\frac{\frac{\pi}{2}-\theta}{2}\right) = x+\sqrt{x^2-1}. \text{ This leads to}$$

$$\frac{\pi}{4}+\frac{\theta}{2} = \tan^{-1}\left(x+\sqrt{x^2-1}\right). \text{ We conclude that}$$

$$\sec^{-1}(x) - 2\tan^{-1}\left(x+\sqrt{x^2-1}\right) = -\frac{\pi}{2}.$$

Several mathematical papers have been published to address the problem of solving trigonometric integrals by nontraditional methods, namely, using trigonometric substitution. In a paper recently published in *The College Mathematics Journal*, Fulling(2005) points out the need to bring to closure, in the traditional sense, the teaching of trigonometric integrals from trigonometric substitution to hyperbolic substitution. Fulling (2005) states "one might have expected that after a decade of calculus reform, the secant function and its inverse would have been de-emphasized to the point, along with its even less useful siblings, cosecant, cotangent, and their inverses(p. 381)." Furthermore, Fulling (2005) states that he hopes "to convince the reader that there is nothing that the secant and its inverse secant do in the traditional techniques of integration chapter that cannot be done better by the hyperbolic sine and cosine and their inverses. It is time for sec, csc, cot, $\sec^{-1}$, $\csc^{-1}$, $\cot^{-1}$ to be retired from our calculus syllabus(p. 382)". While Fulling (2005) uses hyperbolic substitution to solve trigonometric integrals, Velleman (2002) uses combinatorics identities involving binomial coefficients for integrals of the type $\int \sec^{2n+1} x\,dx$, while Wu (2008) uses integration by parts to obtain a recursive relation for the same integral form. In an extended form of the previous integral solution by Velleman (2002), Cheng and Fulford (2004) use parametric differentiation to obtain partial fraction decomposition for integrals of the type $\int \frac{dx}{\sin^m x \cos^n x}$.

While the primary objective of this paper is teaching calculus without using the trigonometric functions cosecant, secant, and cotangent**,** we note that in highly specialized areas students still need to be familiar with the terminology used. We now give two examples where cosecant is still used. One such specialized field in electromagnetics is Radar theory. Radar (antenna) is an acronym derived from the words radio, detection, and range. It refers to the method of using electromagnetic waves to detect the existence of objects at a distance. The energy emitted from an antenna forms a field having a particular radiation pattern. A radiation pattern is a way of mapping the radiated energy from an antenna. This energy is measured at different angles at a constant distance from the antenna. The characteristic of this pattern depends on the type of antenna used. Kai Chen (2004) states "The basic role of the radar antenna is to act as a transducer between the free space and the electromagnetic wave sources or receivers. During transmission, it is used to concentrate the radiated energy into a shaped beam or in a desired direction. During reception, the radar is used to collect the echo signal and deliver it to the receiver (p.676)." Wolff (2006) states "Antennae with cosecant squared pattern are special designed for air-surveillance radar sets. These permit an adapted distribution of the radiation in the beam and causing a more ideal space scanning. The cosecant squared pattern is a means of achieving a more uniform signal strength at the input of the receiver as a target moves with a constant height within the beam". Another highly specialized area is an electrician who needs to make offset bends. Porcupine press (1998-2012) states that "Offset bends are used to move a run of conduit from one plane to another. An offset is normally used to bend the conduit around an obstruction, or to relocate the conduit close to a structural member to make it easier to fasten the conduit. A trigonometric function, the Cosecant, is used to determine the distance between the centers of the two bends used to make the offset. " Google (2012) states that "the cosecant for any given angle of bend may be found by dividing the distance between bends by the depth of offset or saddle. It is basic trigonometry that multiplying the cosecant of a given angle by the length of the opposite side of a right triangle gives the length of the hypotenuse of that right triangle. Thus, for a given range of angles there is a corresponding range of cosecants for the given angles in degrees."

Conclusion

In this paper, we demonstrated how to integrate and differentiate a wide variety of trigonometric functions without using the traditional method of using the reciprocal trigonometric identities discussed in our paper. Our method strictly relies on the exclusive use of the sine, cosine, and arctangent, without using the trigonometric functions cosecant, secant, and cotangent. We do not claim that this method will always be more efficient than other methods used in traditional calculus courses. However, we showed how to use modern technology (graphing calculator) not only to verify our results, but also to find new insights to the existing methods, as illustrated by our pictorial results. We also showed that cosecant is used in very specialized areas and students not entering these fields do not need secant, cosecant or cotangent. The occurrence of the sine, cosine and tangent in formulas makes the use of secant, cosecant and cotangent obsolete as the use of modern technology (graphing calculator) clearly shows.

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