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Editorial

Colleagues, we have two more Aha!Moments submitted to our collection, one from Korea in the geometrical context, and another from Poland during the process of understanding the concept of unknown while learning linear equations. We present them bare, without yet any attempt at interpretation. Soon we will interpret the whole collection to see how Piagetian theories and Koestler theory understand them.

Next we have an interesting article from Lehigh University in Bethlehem, PA which investigates the CCMS professional development for pre-service teachers' impact upon teachers beliefs. Unfortunately, Common Core curriculum has been a bit compromised due to the overemphasis on testing and the inability of testing industry to guarantee technological support for that testing. Lynn Columba and Megan Stotz, in their excellent and optimistic presentation, show that such an impact indeed exists. Our own belief is that unless such a PD is closely connected to practice, even for pre-service teachers, it will not leave lasting impression. It is an appropriate moment then to introduce our "refurbished" Teaching-Research/NYCity methodology (TR/NYCity Model), which with the incorporation of Koestler's bisociativity theory grew in the Chapter 1 of the Creative Enterprise of Mathematics Teaching Research book published recently by Sense Publishers in Netherland announced on the MTRJ website.

As immediate examples of the Teaching-Research/NYCity model we present two papers coming from technical fields at Hostos CC, Mathematics Department and Radiology Unit of the Urban Health Department. Both of them originated through the reflection on teaching mathematics at Hostos CC and propose new approaches based on that reflection, first proposes a method of integrating trigonometric integrals without the use of sec, cosec, and cotan, but solely using sin and cos functions. As the authors, Terry Brenner and Juan Lacay say *By concentrating on cosine, sine and tangent rather than all six trigonometric functions, you will attain a better understanding with less clutter in your mind*. The second paper, by Jarek Stelmark addresses difficulties in understanding inverse square law by students radiology. He supported the concept of the Inverse Square Law by 3 labs exercise for student showing a very direct connection between the law, the time of exposure to the radiation and involved mathematics. He noted increase of understanding by a pre-test/post-test method.

List of content

Aha!Moment from Korea.
Personal communication
Bronislaw Czarnocha

I met the team from Korea: ***Yoon, Sangjoon*;Oh, Kookhwan; Oh, Yaerin; Bae, Mi Seon; Kim, Doyen; Kwon, Oh Nam*** in Szeged, Hungary' PME 40 during their presentation of the paper that caught my attention: **ANALYSIS ON THE MENTAL STRUCTURE OF STUDENTS LEARNING GEOMETRY: Based on APOS Theory.** I know a bit about APOS theory; so I listened and, since from their talk it was clear they were describing the formation of the new schema, I asked whether they did in fact noticed an Aha!Moment in the process
Below is the answer to my question

We entirely concur with your opinion about beginning of the Schema construction.

We could convince that the student did further operation on object.

And Yes! we observed Aha! moments by students.

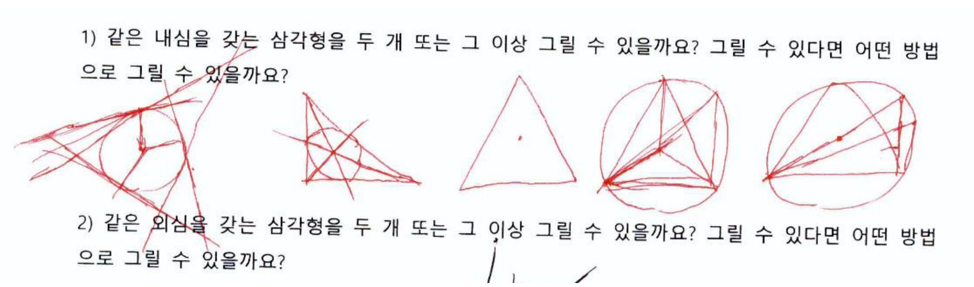
Especially, at the begining, student-1 didn't know the solution of interview question : finding two more triangles with the same incenter or circumcenter.(ppt # 9, solution with 'red')

Finding solution processes of student-1 are as follows

1) He had no ideas about the question.

2) He recalled that all the tringles in a certain circle, of which one side be a diameter of a certain circle, are right triangles.

3) He associated these with "a" circumscribed circle of right triangles.



4) He found that it is

possible to draw triangles in a circle with not only right angle but also acute angle and obtuse angle. (Aha!)

5) He extended this ideal!!! But he could answer the question about 'only' circumcenter. (So the triangles with same circumcener, drawn by student-1 in ppt #9, have always a common side.)

6) And then, he applied the idea to the question about inscribed circle.(Aha!!) (As you see in my attached file of 'answering of student-1', the triangles with same incener have also a common side)

7) Then, "I didn't know the solution before, but I JUST find it! Wow, (it is) really wonderful (for me to solve it like this way!) ", he said.

Aha!Moment from Poland
An elephant – or what use can be made of metonymy?
Celina Kadej, Matematyka #2, 1999

Linear equations with one unknown can be solved already by students in the elementary school. Those are simple equations and students often formulate them by themselves while solving word problems. Sometimes the problems lead to equations a bit more complex than the elementary additive equations of the type x + a = b.

I have had an opportunity to listen to the discussion of two enthusiastic students solving a standard word problem: *The sum of two numbers is 76. One of the numbers is 12 more than the other. Find both numbers.* It was a problem from Semadeni's set of problems for the 3rd grade and one had to solve it using equations and that's where the difficulty appeared:

Przemek (read Pshemik) wrote the equation: x + (x+12) = 76. To solve it was a bit of a problem for him, but still he dealt with it. He drew an interval and then a following dialog had taken place [between him and his friend Bart]:
P: *That is that number*: he extended this interval by almost the same length, and the another one like that.
And this is that number plus 12
B*: and this all together is equal to 76…*
P*: No, this is an equation, d'you understand…*
B could not accept it…
B: *Why did you draw this interval? You don't know yet what it's supposed to be?*
P: *That's not important.*
B: *Why 76?*

P: '*cause that's what is in the problem*
B*: that iks, that iks add 12 and that's supposed to be 76..?*
P: *Look instead of iks there is a little square in the book* – P showed the little square in the book.
B: *Aha, but here, here is written something else*
P: *But it could be as here. And now I am inputting a number into this square.*
B: *A number?! Why into the square?*
P: *No, it's into the window. Into this window I input the number which comes out here.*
B: *But here is a square* – B insisted.
P: *It's not a square but a window, and one inputs the numbers into that window.*
B: *How so*?:
P: *Two windows are equal 64, one window is equal 32. Well, now, you subtract 12 from both sides,* and *you see that the two windows are equal to 64.*
B: *But are there numbers in the windows*?
P: *Two windows are 64, so one window is 32*
B: *Window!?*
P: *That's right, a window. Look here:* ***an elephant*** *and* ***an elephant*** *is equal* **64***.* *Therefore what is* ***one elephant*** *equal to?* ***Two elephants*** *are equal* **64***. So,* ***one elephant*** *is* ***equal*** *to what*?
B: ***An elephant****? Hmm, I see.* **One elephant equals 32***. I understand now… so now the equation…*
P: *If* ***two elephants*** *are equal* ***60****, then* ***one elephant*** *is* ***equal*** *what?*
B: An elephant?, ok, one elephant equals 30. I see it now…..Now equation…………..**aaaaaaa**

Shifting Preservice Teacher Beliefs: The Power of the Common Core State Standards in Mathematics

Lynn Columba, Associate Professor and **Megan Stotz,** Doctoral Candidate
Lehigh University, Bethlehem, PA, USA

Abstract

This study explored preservice teachers' beliefs about mathematics teaching and if the CCSSM has the power to shift these beliefs. Teacher beliefs are key determinants of instructional practices and classroom environments (Thompson, 1992). The professional development model designed by Loucks-Horsley, Hewson, Love, and Stiles (1998) was implemented using the: Set Goals, Plan, Do and Reflect which aligns with the National Research Council's (1999) report on the science of learning describing several important themes on how teachers learn and change. Based on survey analysis and personal essays about their belief on teaching mathematics the largest growth was found in the teachers acquiring an understanding of how and why the CCSSM are important in teaching.

Keywords: Beliefs, elementary preservice teachers, Common Core State Standards in Mathematics

Teacher beliefs are key determinants of instructional practices and classroom environments (Thompson, 1992). These beliefs are often called a teaching philosophy or core values. As teachers clarify and articulate their beliefs, they become the guiding principles upon which their planning and decision making is based. As a leader in mathematics education, understanding the role beliefs play in the work of teachers is crucial to providing targeted support and direction for teachers of mathematics. Implementation of the Common Core State Standards for Mathematics (CCSSM) (2010), in particular the Standards for Mathematical Practice, requires that teachers have both the will and capacity to facilitate instruction that enables students to reason critically and make sense of the mathematics. The ability of a leader to build teacher capacity of pedagogical practices that support the CCSSM is dependent on a teacher's beliefs of teaching, learning, and mathematics (NCSM, 2013).

Many teacher education programs are oriented around the concept of "reflective teaching" in order to prepare teachers to become reflective decision makers. The roots of reflective thinking go back to John Dewey (1916). In analyzing Dewey's definition, the first step is the "meaning making process" and in our context that would entail making sense of the CCSSM. Dewey's definition of reflection as a rigorous way of thinking is complex; he uses 30 unique, specialized terms. This can be simplified to describe the learner's movement from a state of disequilibrium to a state of equilibrium. Dewey believed that reflection took place in a community where one had to express themselves to others. Also, he believed in the "affective dimension," or the attitudes that a teacher brings to bear on the art of reflection. The phrase "experience plus reflection equals growth" (1916) is attributed to John Dewey. A common practice throughout most teaching careers is to write a "philosophy of teaching." Dewey defined philosophy as the *general theory of education* (Dewey, 1916, p. 383) or why do I teach the way I do?

Leaders in mathematics education classrooms play vital roles with respect to encouraging implementation of the CCSSM and the standards for mathematical practice. In the mathematics methods classrooms, instructors can highlight the power of teaching for deep mathematical ideas and that these methods may look different from those that students learned when they were in school. Often this conceptual approach includes visual models and multiple representations. Leaders must create the experiences and opportunities for reflection that allow teachers to examine their beliefs and how these beliefs align with the expectations of the CCSSM. This role of the mathematics education leader begins in the preparation of preservice teachers.

Review of the Literature

Just as students do not enter the classroom with a *tabula rasa*, mathematics teachers similarly enter the profession with their own knowledge, attitudes, experiences, and beliefs about teaching, learning, and mathematics. Each of these components contributes to the manner in which a teacher approaches instruction and the type of learning environment created. Teacher beliefs and the learning environment merge in the mathematics classroom (Thompson, 1992; Hoyles, 1992; Skott, 2001; Guskey, 1986). Thus, a teacher's belief can influence his or her approach to teaching mathematics (NCSM, 2013). Beliefs teachers' have about students can result in certain populations having limited access to the high level of rigor of mathematics content. "Based on their concept of *students' needs,* teachers select which parts of the reform documents are appropriate for their students" (Sztajin, 2003, p. 53). Undoubtedly, this privileged position of a certain type of mathematical knowledge in society affects the teaching and learning of the subject. The overall level of mathematical content must be raised, and the difference in societal groups must be eliminated. One of the first steps towards change is the belief of teachers.

To help solve this problem Battista (1994) recommends that teacher education institutions need to offer numerous mathematics courses for teachers that treat mathematics as sense-making, not rule following. Teachers should "learn mathematics in a manner that encourages active engagement with mathematical ideas" (Battista, 1994, p. 470). Learning to teach to new standards is not easy and requires time. According to Darling-Hammond and Ball (1998), many teachers must face their deeply held beliefs about learning and knowledge and must reconsider their assumptions about students when instructing with new standards. Even if teachers' beliefs align with new reforms, such as CCSSM, they often must develop new ways of teaching and assessing their work. Fortunately, when teachers' beliefs change, the result is new ways of viewing their instructional practice (Lambert, 2002).

"Thoughtful analyses of the nature of the relationship between beliefs and practice suggest that belief systems are dynamic, permeable mental structures, susceptible to change in light of experience" (Thompson, 1992, p. 149). The National Research Council's [NRC] (1999) report on the science of learning describes several important themes on how teachers learn and change. First, teachers need a strong foundation in the core content. Having a deep understanding of the core content allows teachers to then be skilled in how to make decisions about what students understand, need to understand, and how they can impart that knowledge. "Teachers who know a lot about teaching and learning, and who work in environments that allow them to know students well, are the critical elements of successful learning" (Darling-Hammond, 1998, p. 6). Learners are aided by self-monitoring and analysis of what and how they are learning. Thus teacher learning is enhanced by interactions that encourage them to articulate their views, challenge those of others, and begin to understand professional learning communities.

Similarly, the professional development design by Loucks-Horsley, Hewson, Love, and Stiles (1998) in Figure 1, suggests a framework for use as a guide to the process of designing and providing quality professional development. The original design emerged from collaborative reflection with outstanding professional developers about their programs for both mathematics and science teachers. However, as professional development "designers," they felt strongly that professional development was not about importing models but about a process of thoughtful, conscious decision-making. Figure 2, is a modified framework (Loucks,-Horsley, Love, Stiles, Mundry, & Hewson, 2003) with a major difference being a tighter link among standards and a vision for student learning and analysis of student learning. Also, a change from the word *reflect* to the word *evaluate* to signal the importance of rigorous evaluation of professional development and reflection is still a vital part of professional development design. The framework in Figure 2 is also a generic planning sequence, incorporating the following actions: committing to a vision and a set of standards, analyzing student learning data, goal setting, planning, doing, and evaluating. This framework describes professional design at its best. It is not a sequence of steps or a recipe to be followed, but rather a tool to alert planners to important bases to cover and to stimulate reflection and refinement. Both frameworks guided the preparation of the methods course.

Figure 1. Original Professional Development Design Framework (Loucks-Horsley, Hewson, Love, & Stiles, 1998).

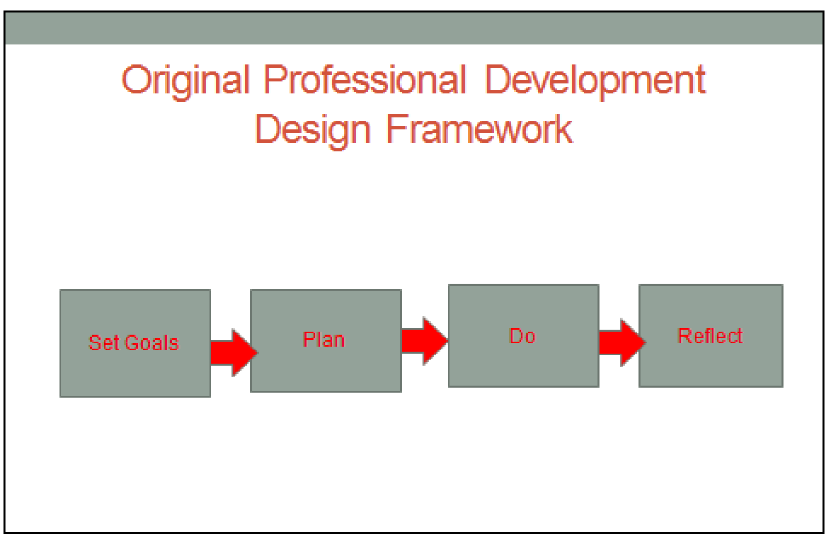
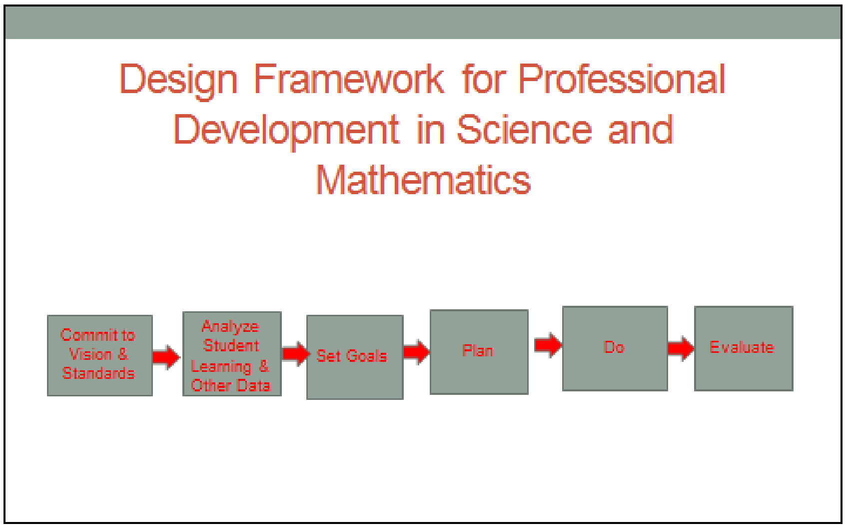


Figure 2. Design Framework for Professional Development in Science and Mathematics (Loucks-Horsley, Love, Stiles, Mundry, & Hewson, 2003).



Purpose of the Study and Research Questions

By linking the essential elements of teacher beliefs with learning, standards reform, and professional development that was garnered from the literature, the purpose of this study was to determine preservice teachers' beliefs about mathematics teaching and if the CCSSM has the power to shift these beliefs. Our investigation sought to answer the questions, does instructing students about the CCSSM's suggestions for improving mathematics instruction alter their beliefs about math instruction? If so, in what ways?

Methodology

Subjects

Data were obtained from eleven undergraduate and graduate students in a fifth year teacher certification program who were enrolled in a three-hour graduate credit elementary mathematics methods course for fourteen weeks. All of the students were seeking dual certification in PreK-4 general education and K-6 special education. Participants were asked on the first course meeting to complete the *Objectives of a Mathematics Methods Course* survey (adapted from Brahier, 2011) (See Appendix A. Objectives of a Mathematics Methods Course), received instruction for the next several weeks that included strategies to implement the CCSSM, and were then asked to retake the *Objectives* survey during the fourteenth meeting and complete a beliefs essay by the eighth meeting. All subjects participated in both pre- and post-implementations of the survey, as well as the essay. In order to protect the participants' real name the eleven undergraduate and graduate students have been assigned a letter name, such as Participant A.

Instruments

The *Objectives* survey (Brahier, 2011) included sixteen Likert scale questions designed to rate the importance of essential components in the mathematics methods course (See Appendix A. Objectives of a Mathematics Methods Course). The data was categorized as "Pairs" when analyzing the sixteen pre- and post-questions. For example, pre-post survey for question one is titled Pair One, question two is titled Pair Two, and so on. The survey's rating system ranged from 5-extremely important, 4-very important, 3-important, 2-somewhat important, 1-not very important, and 0-unnecessary. The *Objectives* survey was chosen because it clearly defined essential components found in the CCSSM standards documents. Also, *Objectives* defined in the instrument were categorized by the essential components that aligned with the course:

- The ability to describe the significance, general content, application in lesson planning of the standards documents of the National Council of Teachers of Mathematics [NCTM], which include *Principles and Standards for School Mathematics* (2000), State Standards, and *The Core Curriculum State Standards in Mathematics* (2010);

- The role that effective lesson planning has on classroom environment/management;
- The use of effective, researched-based strategies, such as cooperative learning strategies, higher level questioning in mathematics instruction, and literacy integration;
- The ability to illustrate how, when, and why to use a variety of assessment strategies to collect data regarding student academic progress and the development of dispositions toward mathematics;
- The ability to demonstrate how, when, and why to use technology to identify and maximize student learning; and,
- The ability to continue to develop a positive disposition toward the field of mathematics by becoming familiar with, and participate in, programs provided for continued professional growth in the field of mathematics education.

In addition to *Objectives* the participants wrote a beliefs essay during the seventh and eighth week of instruction. A beliefs essay is a subset of the students' philosophy of teaching, but this assignment was focused specifically on their beliefs about teaching mathematics and how the CCSSM will or will not affect their teaching of mathematics. All of the preservice teachers have a philosophy of education in their e-portfolio, which is one of the first assignments in the teacher certification program. The information in the beliefs essays were first categorized by the six essential components that paralleled with the *Objectives* survey mentioned above. Then, the essays were analyzed to determine if and how the CCSSM shifted their beliefs.

Procedure

After students completed the survey during the first course meeting, they were instructed following the NRC's (1999) learning principles regarding foundations in core content and Loucks-Horsley et al.'s (1998) professional development design framework (see Figure 1). The elementary mathematics methods course was developed with evidence-based, research-affirmed practices that included hands-on opportunities with technology and manipulative materials implemented in every class. The content and activities in the first six class sessions were intentionally designed to build a strong foundation in the core content and big ideas of mathematics education and an understanding of the CCSSM. *Elementary and Middle School Mathematics: Teaching Developmentally* (Van de Walle, Karp, & Bay-Williams, 2013) and *The Power of Picture Books in Teaching Math, Science, and Social Studies: PreK-8* (Author, 2009) were used in conjunction with the activities to support teachers when asking the essential question, "Why do I teach the way I do?"

Participants worked in collaborative groups with active learning strategies and were provided opportunities to make connections to CCSSM as part of each of the fourteen course sessions. The purpose for this learning environment was meant to offer the participants authentic opportunities on how to make decisions about what PreK-4 students know and will need to know mathematically, as well as how they can teach that information in a diverse mathematics classroom. After the introduction of the course, using the Loucks-Horsley et al. framework for Professional Development in Science and Mathematics (2003) (Figure 2), the instructors emphasized commitment to a vision and Standards, analyzing student learning, and setting goals as foundational portion of the course. To enhance their dialogue and strengthen connections between their beliefs in mathematics teaching and the CCSSM, during week seven and eight the participants viewed a webinar called New Resources for Illustrating the Mathematical Practices (Carnegie Learning, 2012). The webinar explains the CCSSM Standards and provides examples of real life classrooms modeling the connection between their personal beliefs and the standards reform. To understand the extent of any shifts that may have occurred, a brief essay about participants' beliefs about the CCSSM and its potential influence on their future classroom teaching was completed on the eighth week and analyzed qualitatively to determine if and how the CCSSM shifted their beliefs.

Furthermore, knowing that teacher learning is enhanced by interactions that encourage them to articulate and defend their beliefs, the remaining six course sessions were designed to target specific mathematical topics such as early numeracy, data and measurement, geometry, and algebraic thinking. Participants were encouraged to download and use *The Common Core* (Mastery Connect, 2011) application for their electronic tablet during collaborative group discussions. At this stage of development of the students, Figure 2 (Loucks-Horsley et al., 2003) continued to be the guiding model with an emphasis on plan, do and evaluate.

Students were then reissued the *Objectives* survey to determine whether or not their beliefs about the CCSSM's suggestions from improving mathematics instruction shifted following fourteen weeks of coursework.

Results

When identifying whether the CCSSM have the power to shift beliefs for improving mathematics instruction and in what ways, the descriptive mean change from the pre- and post- *Objectives of a Math Methods Course* survey were analyzed using paired sample *t*-tests. Only SPSS descriptive mean statistics were run to identify an increase or decrease in survey scores due to the small sample size. Furthermore, if the CCSSM have the power to shift beliefs for improving mathematics instruction, the qualitative data acquired from the participant's beliefs essay are used to explain how teachers' beliefs align with the CCSSM.

Based on the survey results from the paired sample *t*-test statistics (Table 1), there were significant positive differences in the participants' beliefs of the importance of mathematical objectives for the methods course from the first to the fourteenth course session. In particular, there was a mean increase in thirteen of the sixteen questions. Pair 1 and Pair 2, which identified the importance of learning how to describe the significance, general content, and application in lesson planning of the Standards documents of NCTM, State Standards, and CCSSM had a positive mean increase of 0.91

points (Pair 1) and 0.73 points (Pair 2). More importantly, as shown in Table 2, Pair 1 which directly targeted CCSSM, shows a positive statistical significance (p<.001).

Albeit, qualitatively ten out of the eleven participants beliefs about teaching mathematics were shifted due to the power of the CCSSM, Participant J was more reluctant to change. He believed, “Even if the Common Core Standards help to reinforce my philosophy, [he] thinks there will still be struggles with integrating them into education. The standards for each state have taken a long time to become an integrated part of education, and now that they are going to add a new set of standards, it is going to take more time to adjust to these standards. Even though the Common Core Standards have fewer standards, and are supposed to be clearer than what is currently published, it will still take time to integrate them.” Participant O’s thoughts on the other hand mirror those of the remaining ten preservice teachers, “Now that I have taken the time to really go through the goals of each CCSSM and explore the NCSM website, I am more aware of the various angles (no pun intended!) from which you can teach mathematics in the classroom. The best practices include those that teach students skills and strategies that forgo classroom instruction, and extend into the use of basic processing skills in society. These standards are geared toward producing effective citizens of the world. This starts by instructing students how to approach mathematical concepts in order to benefit their daily lives.”

While Pair 1 was the only item that was statistically significant, the questions that identified illustrating how, when, and why to use a variety of assessment strategies to collect data regarding student academic progress all had positive mean increases (Pair 10, Pair 12, and Pair 13 in Table 1) Considering how, when, and why the CCSSM influences teaching and learning via assessment Participant K thought, “In order for math to stay relevant to students, their prior knowledge, culture, language and lifestyle must be integrated into their instruction as well as their assessment tasks. The CCSSM helps to clearly define which skills and concepts should be focused on. The CCSSM believes that deep learning of concepts should be emphasized. This can be done by: encouraging students to use manipulatives, diagrams, technology and more to enhance their learning experience. Students should be encouraged to communicate their mathematical thoughts through writing, gestures, concrete objects such as drawings, etc." Additionally, Participant C demonstrated change in her belief system when she asserted, "I feel that overall my beliefs about teaching begin with the students, but I have began thinking more passionately how specific practices from the CCSSM and assessment can enhance my beliefs on teaching math."

The use of effective researched based strategies such as higher level questioning in mathematics instruction and literacy integration also showed positive mean increases (Pair 4, Pair 8, and Pair 14 in Table 1). As Participant G explains, "The *CCSSM* aim to facilitate confidence and growth of mathematics in the student population. Their practices suggest that teachers should educate students on how to work on and persevere to solve mathematics problems while focusing on precision. They continue to suggest that teachers do so by teaching reasoning and explaining skills, modeling and tool usage, and pattern/structure locating and generalization abilities. Each of these elements of the *CCSSM* stand to highlight what students already know, build on that understanding and use the information to grow as mathematicians."

Even though qualitatively the participants did not mention the importance of the development and continuation of positive dispositions toward the field of mathematics, there were positive mean increases according to the pre- and post-surveys (Pair 15 and Pair 16 in Table 1).

Table 1. Paired Samples Statistics.

Paired Samples Statistics

		Mean	N	Std. Deviation	Std. Error Mean
Pair 1	Pretest1	3.45	11	.522	.157
	Posttest1	4.36	11	.505	.152
Pair 2	Pretest2	3.27	11	.905	.273
	Posttest2	4.00	11	.775	.234
Pair 3	Pretest3	4.27	11	.905	.273
	Posttest3	4.73	11	.467	.141
Pair 4	Pretest4	3.64	11	1.027	.310
	Posttest4	3.91	11	1.044	.315
Pair 5	Pretest5	3.73	11	1.009	.304
	Posttest5	3.82	11	.874	.263
Pair 6	Pretest6	4.27	11	1.009	.304
	Posttest6	4.55	11	.522	.157
Pair 7	Pretest7	4.36	11	1.027	.310
	Posttest7	4.36	11	.924	.279
Pair 8	Pretest8	4.36	11	1.027	.310
	Posttest8	4.73	11	.467	.141
Pair 9	Pretest9	4.27	11	1.272	.384
	Posttest9	4.27	11	.647	.195
Pair 10	Pretest10	4.45	11	.934	.282
	Posttest10	4.73	11	.467	.141
Pair 11	Pretest11	4.18	11	1.250	.377
	Posttest11	4.18	11	.751	.226
Pair 12	Pretest12	4.18	11	1.079	.325
	Posttest12	4.55	11	.688	.207
Pair 13	Pretest13	3.82	11	1.250	.377
	Posttest13	4.18	11	.751	.226
Pair 14	Pretest14	4.18	11	.982	.296
	Posttest14	4.55	11	.688	.207
Pair 15	Pretest15	4.27	11	1.489	.449
	Posttest15	4.45	11	.522	.157
Pair 16	Pretest16	3.73	11	1.009	.304
	Posttest16	4.09	11	.701	.211

Table 2. Paired Samples Test.

Paired Samples Test

		Paired Differences					t	df	Sig. (2-tailed)
		Mean	Std. Deviation	Std. Error Mean	95% Confidence Interval of the Difference Lower	Upper			
Pair 1	Pretest1 - Posttest1	-.909	.539	.163	-1.271	-.547	-5.590	10	.000
Pair 2	Pretest2 - Posttest2	-.727	1.104	.333	-1.469	.014	-2.185	10	.054
Pair 3	Pretest3 - Posttest3	-.455	1.128	.340	-1.212	.303	-1.336	10	.211
Pair 4	Pretest4 - Posttest4	-.273	1.555	.469	-1.317	.772	-.582	10	.574
Pair 5	Pretest5 - Posttest5	-.091	1.300	.392	-.964	.783	-.232	10	.821
Pair 6	Pretest6 - Posttest6	-.273	1.191	.359	-1.073	.527	-.760	10	.465
Pair 7	Pretest7 - Posttest7	.000	1.095	.330	-.736	.736	.000	10	1.000
Pair 8	Pretest8 - Posttest8	-.364	1.027	.310	-1.054	.326	-1.174	10	.267
Pair 9	Pretest9 - Posttest9	.000	1.095	.330	-.736	.736	.000	10	1.000
Pair 10	Pretest10 - Posttest10	-.273	.905	.273	-.880	.335	-1.000	10	.341
Pair 11	Pretest11 - Posttest11	.000	1.183	.357	-.795	.795	.000	10	1.000
Pair 12	Pretest12 - Posttest12	-.364	.924	.279	-.985	.257	-1.305	10	.221
Pair 13	Pretest13 - Posttest13	-.364	1.362	.411	-1.279	.551	-.886	10	.397
Pair 14	Pretest14 - Posttest14	-.364	1.206	.364	-1.174	.447	-1.000	10	.341
Pair 15	Pretest15 - Posttest15	-.182	1.401	.423	-1.123	.760	-.430	10	.676
Pair 16	Pretest16 - Posttest16	-.364	1.206	.364	-1.174	.447	-1.000	10	.341

Discussion

The data validates the idea that the CCSSM does have the power to shift beliefs for improving mathematics instruction. Predominantly the shift in beliefs is found in the use of effective researched based strategies such as higher level questioning in mathematics instruction, literacy integration, illustrating how, when, and why to use a variety of assessment strategies regarding student academic progress, and the development of positive dispositions toward mathematics. By continuing to develop a positive disposition, or "beliefs," toward the field of mathematics these preservice teachers are more likely to present students with appropriately challenging mathematics while providing students with necessary supports to ensure learning of the content. Participant G said it best when affirming how her beliefs align with the CCSSM, "Teaching the set of skills required for math comprehension allows children to discover all that is math and build on that knowledge *ad infinitum*. When teaching reading or writing, students are educated on how to see patterns and decode or how to produce words and follow a basic process for recording and sharing ideas. Why would the math teaching process differ than that of any other basic life skill? Math is simply a mode of expression that describes the world in a way that all can understand despite culture or language. We must embrace the *CCSSM* and give today's children the toolkit they need to explain and understand the world around them. Those children are our future, and math needs to be a language with which they will succeed."

What became of particular interest when analyzing the data was identification of beliefs the preservice teachers already held towards mathematics and instruction, regardless if the CCSSM influenced them. Based on the pre- and post-survey analysis, students already held a belief in the importance of effective lesson planning on classroom environment-management, cooperative learning strategies, and technology in the classroom. Yet, because their beliefs in mathematics instruction shifted in other areas connected to these three topics, new beliefs emerged.

The questions identifying the importance of effective lesson planning on classroom environment/management (Pair 11 in Table 1) had a zero mean increase. This illustrates the participants already value the importance of successful design for learning. Good lesson planning is essential to the process of teaching and learning. A teacher who is prepared is well on their way to a successful instructional experience. However, even though they hold this foundational belief, the importance of creating lesson plans with the eight mathematics practices that focus of PreK-4 mathematics program became more important (Pair 3 in Table 1). This emerging knowledge of teaching with content in mind, rather than behavior in mind, indicates growth in the importance of teaching mathematics. Participant A supported this finding when he wrote, "While I believe CCSS are a great place for me as an educator to begin developing and planning, the traits and characteristics of my classroom and its students will be the driving force behind my implementation of particular practices. The benefit of using the CCSS as my guide, is knowing which direction to point my educational compass and steer my students towards." Additionally, Participant L began to realize how she can implement this objective in her classroom, "Overall, teachers need to teach with their students in mind so when formatting my lessons and units, I need to determine what practices and assessment should be focused on for my particular group of students. Students' success in mathematics is my ultimate goal."

Furthermore, the participants entered the learning experience with a firm belief of research-based strategies on cooperative learning, which resulted in a zero mean increase for Pair 9 (Table 1). Cooperative learning strategies such as think-pair-share, jigsaw, and group investigations are beneficial in the mathematics classroom because students develop a sense of community and commitment, which supports positive peer teaching. Another beneficial aspect of cooperative learning in the classroom is the increased amount of resources gained from a group. This will allow students to make better decisions and justify answers as they work through problems with more understanding and background information. "The *Common Core Standards* emphasis on learning and doing mathematics impacts positively on my beliefs about how to teach mathematics in the classroom" said Participant K. "For example, having a student show and justify his or her understanding of a problem proves comprehension much more than completion of a worksheet showing no depth of knowledge. The need for student engagement, discussion time, and extension opportunities is provided through the design of the activity's question and its reliance on conceptual understanding of key ideas.

Yet, the remaining learning strategies that discussed higher level questioning in mathematics instruction, and literacy integration both had positive mean increases (Pair 4, Pair 8, and Pair 14 in Table 1). This finding appears to indicate that preservice teachers understand how questioning and literature in the content area can be as useful a strategy as cooperative learning. In particular the use of questioning in cooperative groups was a prominent theme in the qualitative findings. Participant L realized, "In order to fully grasp math concepts, it is important that students will be able to problem solve and then explain their processes so they will be able to apply it to a different problem in the future. Therefore, unlike the speaker who highlighted problem solving and attending to precision, I will highlight problem solving and reasoning and explaining in my math units."

The final preexisting belief was about demonstrating when and why to use technology, which had a zero mean increase (Pair 7). However, Pair 5 and Pair 6, which both discuss how to use technology, had a positive mean increase (Table 1). This observation is rather eye opening in regards to supporting 21st century learners in the mathematics classroom. There is a vast difference between understanding when and why to use technology versus how to use technology in classroom instruction. Similar to the standards found in the CCSSM, knowing how to use technology means creating lessons that have active engagement, participation in groups, frequent interaction and feedback, and connections to the real world. Additionally, knowing how to use technology in the mathematics classroom offers educators effective ways to reach different types of learners and assess student understanding through multiple means, which supports teachers "growth mindset."

The data also supports previous work about professional development design. By constructing the fourteen week course sessions around the NRC's (1999) learning principles about foundations in core content and the Loucks-Horsley et al. framework (1998) Set Goals, Plan, Do and Evaluate design, it was observed that participants naturally began collaborating to explain or defend the rationale behind mathematical topics, learning strategies, how the standards influence lesson planning and assessment for each grade level, and why manipulatives, either concrete or virtual, were explicitly chosen. Essentially, the preservice teachers were learning and thinking in ways they believe their students should learn.

Changing Teacher Beliefs for CCSSM Implementation

Useful to mathematics leaders would be to determine which of the other pivotal documents in mathematics education field have on impact on beginning teachers' beliefs on implementing the CCSSM, such as the National Association of Education of Young Children's (NAEYC) statement, *CCSSM: Caution and Opportunity for Early Childhood*

Education and NCTM's statement on early childhood education. One of the next steps would be to provide several other classrooms of preservice teachers with the same instructional experiences to determine if the results can be duplicated or improved in those classrooms. Another useful task would be to provide the same quality experiences but with different pivotal documents, and compare the results in the impact on teachers' beliefs toward implementing the CCSSM.

As mathematics leaders it is essential to create learning environments that ensure teachers understand and have a strong foundation in the core content or big ideas in mathematics. Having a deep understanding of the CCSSM and eight principles in mathematics allows teachers to be proficient at decision making about what students know, need to know, and how they can impart that knowledge. This strong foundation supports the use of assessment, learning strategies, and integration of technology.

Most importantly, beliefs and practices must be considered holistically in understanding teaching and learning and in considering the professional learning opportunities for teachers (Bay-Williams & Karp, 2010). For meaningful and lasting change to occur, teachers need to engage in practical inquiry (Franke, Fennema, Carpenter, Ansell, & Behrend, 1998) and to move back and forth among a variety of settings to learn about new instructional strategies, to try them out in classrooms, and to reflect on what they observed in a collaborative setting (Borko, Mayfield, Marion, Flexer, & Cumbo, 1997). Whatever approach is used, it is clear beliefs and practices are linked, and emphasis on both is vital for professional development and a shift in teachers' beliefs.

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Appendix A

Objectives of a Mathematics Methods Course

The following are objectives for an elementary mathematic education course. For each of the following objectives, rate it on a scale of 0-5 (circle the number), using the following scale in terms of "what a student should learn in a mathematics education program to become a successful teacher":

5—Extremely Important, 4—Very Important, 3—Important, 2—Somewhat Important, 1—Not Very Important, 0—Unnecessary

1. Describe the significance and general content of the Standards documents of the National Council of Teachers of Mathematics—*Principles and Standards for School Mathematics* and *The Core Curriculum State Standards in Mathematics.*

0 1 2 3 4 5

2. Demonstrate an understanding of the philosophy and content of the Pennsylvania of Academic Standards for Mathematics, Pre-K-3 and Elementary.

0 1 2 3 4 5

3. Describe (and illustrate in lesson planning) how to make the eight mathematics practices the focus of Pre-K-4 mathematics program.

0 1 2 3 4 5

4. Describe popular learning theories that attempt to explain how students learn mathematics, such as the theory of Piaget (the constructivist viewpoint), direct instruction and inquiry-based learning.

0 1 2 3 4 5

5. Explain how research in mathematics and technology education is conduct, reported, and applied to reform in teaching and learning practices, with an emphasis on differentiating between appropriate and inappropriate us of technology.

0 1 2 3 4 5

6. Illustrate how to use technology (e.g., calculators, computer software, applets, interactive white board, and internet) and identify the benefits of technology to maximize student learning.

0 1 2 3 4 5

7. Identify, select and use appropriate technology resources to meet specific teaching and learning objectives.

0 1 2 3 4 5

8. Give examples of questioning strategies for the classroom that promote mathematical thinking and dialogue (discourse).

0 1 2 3 4 5

9. Use cooperative learning strategies in mathematics instruction.

0 1 2 3 4 5

10. Recognize the essential components of a lesson plan and prepare a mathematics lesson plan which includes outcomes, materials, a motivating activity, a structured sequence of experiences for the students, differentiated instruction, a logical closure, a planned extension, and a plan for assessment.

0 1 2 3 4 5

11. Describe a variety of strategies that teachers can use to promote positive classroom management and the role that effective lesson planning has on classroom environment.

0 1 2 3 4 5

12. Illustrate the ability to use a variety of assessment strategies to collect data, including electronic means, regarding student academic progress and the development of dispositions toward mathematics.

0 1 2 3 4 5

13. Explore a variety of ways in which teachers can gather field tested ideas for use in one's own classroom, including electronic sources.

0 1 2 3 4 5

14. Use literacy strategies and children's literature in mathematics instruction.

0 1 2 3 4 5

15. Continue to develop a positive disposition toward the field of mathematics.

0 1 2 3 4 5

16. Become familiar with and participate in programs provided for continued professional growth in the field of mathematics education, including the NCTM, PCTM, EPCTM, etc., including by means of the Internet and other electronic sources.

0 1 2 3 4 5

(Adapted from Daniel Brahier, 2011)

BRONISLAW CZARNOCHA

CHAPTER 1.1 TEACHING-RESEARCH NEW YORK CITY MODEL (TR/NY CITY)

TR/NYCity Model is the methodology for classroom investigations of learning, which synthetizes educational research with teaching practice. It is conducted simultaneously with teaching and the aim of improvement the learning by the teacher of the class in the same classroom, and beyond.

INTRODUCTION

TR/NYCity Model is based on the careful composition of ideas centred around Action Research (Lewin, 1946) with the ideas centred around the concept of the Teaching Experiment of the Vygotskian school in Russia, where it “grew out of the need to study changes occurring in mental structures under the influence of instruction” (Hunting, 1983). From Action Research we take its focus on the improvement of classroom practice by the classroom teacher and its cyclical instruction/analysis methodology, and from Vygotsky’s teaching experiment we take the idea of the large-scale experimental design based on a theory of learning and involving many sites – different classrooms (B. Czarnocha, 1999, Czarnocha and Prabhu, 2006). Vygotsky teaching experiment methodology introduced the possibility of viewing the classroom teacher as a member of a collaborative research team investigating the usefulness of research based classroom integration. The integration of these two distinct frameworks re-defines the profile of a teacher-researcher:

1. as an education professional whose classrooms are scientific laboratories, the overriding priority of which is to understand students’ mathematical development in order to utilize it for the betterment of the particular teaching and learning process;

2. who as a teacher can have the full intellectual access to the newest theoretical and practical advances in the educational field, knows how to apply, utilize and assess them in the classroom with the purpose of improving the level of students' understanding and mastery of the subject;
3. who as a researcher has a direct view of, and the contact with the raw material of the process of learning and development in the classroom, acts as a researcher in the context of the daily work and uses that process to design classroom improvement and derive new hypotheses and general theories on that basis.

The implicit vision underlying the profile is the conceptual and practical balance between researches and teaching, where both components of the educational profession are given equal value and significance; both the research knowledge of the researcher and the craft knowledge of the teacher are resources for the teacher-researcher.

Admittedly, the proposed profile is ambitious, yet it's doable, especially in the context of community colleges whose full time mathematics faculty have PhD level experience in mathematics, physics or engineering research and can relatively easily transfer those skills into classroom-based investigations of learning. On the other hand, given the progressing collapse of public education in US, the majority (80%) of freshman students who enter every semester into our colleges require remediation to be able to get to college level courses. The remediation starts on the level of arithmetic through algebra it constitutes 80% of our "bread and butter" courses. The placement into, and exit from remediation is decided by the university wide – standard exam. Consequently, the mathematics faculty of community colleges are intimately familiar with the issues of school mathematics. The composition of research skills with the craft knowledge of teaching elementary mathematics is at the basis of the formulation of TR/NYCity Model.

HISTORICAL BACKGROUND AND DEVELOPMENT OF TR/NYCITY MODEL.

Stenhouse TR Acts

TR/NYCity owns its formal origins to Action Research of Kurt Lewin (1946) and Teaching Experiment methodology of Vygotsky. TR/NYCity model finds its completion in the bisociation of Koestler (1964) leading to the Stenhouse TR acts (Rudduck and Hopkins, 1985).

Lewin proposed the Action Research methodology in the context of the quest for improvement of "group relations", a euphemism for interracial relations in US of 30ties and 40ties. He saw it as "…a comparative research on the conditions and effects of various forms of social action, and research leading to social action." His Action Research cycle consisted of the stages (or steps) of diagnosis with plan for action, implementation of action, its assessment providing at the same time the basis for "modifying overall plan" and leading to the next cycle. It was however Stenhouse who introduced Action Research methodology into education profession as teaching-research in the inaugural lecture at the University of East Anglia in 1979 presentation "Research as basis for teaching" – a theme whose importance has steadily grown till contemporary times. Already in early seventies of the last century he recognized that one of the possible explanations for the failure of research

> "…to contribute effectively to the growth of professional understanding and to the improvement of professional practice… was the reluctance of educational researchers to engage teachers as partners in, and critics of, the research results." (Rudduck and Hopkins, 1985).

The extracts from the transcripts of seminars with the part-time MA students reveal his understanding of Action Research in terms closely related to TR/NYCity model arrived at spontaneously through our work. He understood Action Research primarily as "the type of research in which the research act is necessarily a substantive act; that is an act of finding out has to be undertaken with an obligation to benefit others than research community" (p.57), in our case, students in ours, and other classrooms. However, it's the concept of "*an act* [which is] *at once an educational act and a research act*" (p.57), that completes a stage in our development of thinking technology, that is the process of integration of research and learning theories with the craft knowledge of the profession anchored in practice. The bisociative framework (see below) of TR acts produces new mental conceptions, the product of thinking technology. These conceptions (e.g. schema, ZPD, hidden analogy, bisociation) become part of the discourse within the community of teacher-researchers, tools to design methodology for improvement of classroom craft and for deepening one's research interest.

It is surprising Stenhouse did not utilize Action Research cycles. It could be because the curriculum research he envisioned as conducted by teachers, apart from case studies, was to test hypotheses arrived at by curriculum research outside of the teacher's classrooms (p.50).

The second root of our methodology is anchored in the methodology of the Teaching Experiment of Vygotsky, which had a professional research team together with teachers investigate the classroom and was conducted "…*to study changes occurring in mental structures under the influence of instruction*" (Hunting, 1983). Interestingly, introduction of professional research into classroom by Vygotsky and his co-workers in the thirties was the fulfilment of the first part of the Stenhouse's vision of the seventies who demanded "*In short, (1) real classrooms have to be our laboratories, and (2) they are in command of teachers, not researchers*" (p.127). For the second part of Stenhouse vision we propose classrooms, which are in the command of teacher-researchers as the synthesis of both methodological efforts.

The Teaching Experiment methodology reappeared in the work of Steffe and Cobb (1983) as a constructivist teaching experiment, which was appropriated by Czarnocha (1999) for teaching purposes in high school class of mathematics, already as a tool of a teacher. Czarnocha (1999) realized that the constructive teaching experiment can easily become a teacher's powerful didactic instrument when transformed into guided discovery method of teaching.

Design Science

The interest in the work of the professional practitioner of whom teacher is but one particular example has been steadily increasing in the second half of the previous century since the work of Herb Simon (1970), the Design of the Artificial. His work proposes the design as the "*principal mark that distinguishes the professions from sciences"* (p.55-58). Kemmis and McTaggart (2000) developed the principles of Action Research, while Schon (1983) investigated the concept of a Reflective Practitioner through the process of reflection-in-action. Both frameworks had found applications in the work of teachers and researchers through joint collaborations, however the research/practice gap hasn't been yet bridged.

The terms Design Experiment, Design Research or the Science of Design are often interchangeable and they refer to the professional design in different domains of human activities. It was introduced into research in Math Education by Ann Brown (1992), Collins (1992), and Whittmann (1995). Anne Brown had realized during her exceptional career that psychological laboratory can't provide the conditions of learning present in the complex environment of a classroom and transformed her activity as a researcher directly into that very classroom as the leading co-designer and investigator of the design in the complex classroom setting. In her own words: "As a design scientist in my field, I attempt to engineer innovative classroom environments and simultaneously conduct empirical studies of these innovations" (A. Brown, 1992). She provided this way one of the first prototypes of design experiments which, theoretically generalized by Cobb et al. (2003), "entail both "engineering" particular forms of learning and systematically studying those forms of learning within the context defined by means of supporting them…". The profession has followed her lead seeing the classroom design experiments as theory based and theory producing. Paul Cobb et al. (2003) assert that Design Experiments are conducted to develop theories, not merely to empirically tune what works. Design research paradigm treats design as a strategy for developing and refining theories (Edelson, 2002). Even Gravemeyer (2009) who defines "the general goal of Design Research to investigate the possibilities for educational improvement by bringing about and studying new forms of learning" hence stating it closer to substantive quality formulated by Stenhouse, yet he warns us that "great care has to be taken to ensure that the design experiment is based on prior research…" eliminating this way the designs anchored in prior practice.

Unfortunately, the educational research profession cuts itself off by these restrictions from the source of profound knowledge contained in the tacit and intuitive craft knowledge of the teachers. Clearly, if the goal is improvement of learning, a more general framework is needed which recognizes both education research and teaching practice as two approaches of comparable significance, value and status.

Frameworks of Inquiry and the Unity of Educational and Research Acts

We find such a framework within the three frameworks of inquiry identified by Margaret Eisenhart (1991): theoretical, practical, and conceptual (Lester, 2010). Following Eisenhart, Lester (2010) posits three types of frameworks used in Math Education, first, the theoretical framework based upon theory i.e. the constructivist, radical constructivist and social constructivist theories discussed second, a practical framework, "… which guides research by using 'what works' … this kind of research is not guided by formal theory but by the accumulated practice knowledge of practitioners and administrators, the findings of previous research, and often the viewpoints offered by public opinion" (p. 72). The third is a conceptual framework that can pull from various theories as well as educational practice.

The theoretical framework guides research activities by its reliance on a formal theory; that is, a theory that has been developed "on the theoretical, conceptual, and philosophical foundations" (Lester, 2010) by using an established, coherent explanation of certain sorts of phenomena and relationships—Piaget's theory of intellectual development and Vygotsky's theory. However, as soon as such a theory- based design undergoes a TR cycle, the initial determinative role of theory changes into the JiTR-approach (Just-in Time-Research; see below), which allows for the participation of craft knowledge based on the teaching experience in equally significant manner.

The Practical Framework is employed in what we refer to as 'action research' and as discussed, it has some common components with teaching-research.

> "For Scriven, [quoted in Lester (2010)] a practical framework guides research by using "what works" in the experience of doing something by those directly involved in it. This kind of framework is not informed by formal theory but by the accumulated practice knowledge of practitioners and administrators, the findings of previous research, and often the viewpoints offered by public opinion. Research questions are derived from this knowledge base and research results are used to support, extend, or revise the practice." (Lester 2010)

However, the distinction that we make with Lester's description of a practical framework and a framework for teaching research is that we, as researchers, view the goal of teaching-research to inquire into how theory and models of learning reflect upon what the teacher and student experience in the classroom. The question for the teacher researcher and supportive TR community is what needs to be transformed or changed in the existing theories or models in order to improve the fit between these frameworks and classroom practice?

The third and final framework considered by Lester is that of

> "a conceptual framework [that] is an argument that the concepts chosen for investigation, and any anticipated relationships among them, will be appropriate and useful given the research problem under investigation. Like theoretical frameworks, conceptual frameworks are based on previous research, but conceptual frameworks are built from an array of current and possibly far-ranging sources. The framework used may be based on different theories and various aspects of practitioner knowledge" (Lester, 2010).

We argue that amongst the three frameworks for research present in philosophy of education research only the conceptual framework allows for the possibility of bisociative synthesis between teaching and research through Stenhouse TR acts.

Of special importance in working with conceptual frameworks is the notion of *justification*. A conceptual framework is an argument including different points of view and culminating in a series of reasons for adopting some points and not others. The adopted ideas or concepts then serve as guides: to collecting data, and/ or to ways in which the data from a particular study will be analysed and explained (Eisenhart, 1991). According to Lester (2010) "…too often educational researchers are concerned with coming up with "good explanations" but are not concerned enough with justifying why are they doing what they are doing…" (p.73).

Our insistence on the balance between research and teaching practice, the basis for the unified Stenhouse TR acts, finds its justification and fulfilment in the bisociation of Koestler (1964) that is in "*a spontaneous leap of insight which connects previously unconnected matrices of experience*" (p. 45). A bisociative framework is the framework composed of "two unconnected matrices of experience" where one may find a "hidden analogy" – the content of insight (Chapter 1.2). Given the persistent divide and absence of deep connections between research and teaching practice, TR/NYCity constitutes a bisociative framework composed of "unconnected [in general] matrices of experience" of teaching and research, within which one can expect high degree of creativity on the part of the teacher-researcher through leaps of insight leading to the unified Stenhouse acts defined above. The process of coordination of TR/NYCity with Koestler bisociation theory is the guiding theme of Unit 2: Creative Learning Environment. Unit 2 presents the search for classroom creativity by Vrunda Prabhu during which this coordination has taken place revealing "hidden analogy" between Koestler theory and Prabhu's teaching practice.

We can state now a new definition of TR/NYCity methodology:

TR/NYCity Model is the conceptual bisociative framework of Design Research conducted by the classroom teacher, whose aim is to improve the process of learning in the classroom, and beyond – the characteristic of its "substantive nature".

TR bisociative framework facilitates integration or, still better, synthesis of practice and research through instances or sequences of instances of Stenhouse acts which are "at once an educational act and a research acts" (Rudduck and Hopkins, p.57). In what follows we will call them Stenhouse TR acts. The Stenhouse TR acts are the foundation stones of "thinking technology" discussed below within which their unity is naturally positioned. The facilitation of longer or shorter instances of Stenhouse TR acts can be reached from either teaching practice or from application of research to practice, as well as from both simultaneously. The "skeletal structure" (Eisenhart, 1991) of the TR/NYCity conceptual framework can be obtained as requirements and conclusions from the definition.

We discuss different designs of teaching experiments and TR investigations in Unit 4, The Teacher as a Designer of Instruction: TR Design, while in Chapter 3.2 we discuss "nuts and bolts" of classroom teaching experiment. The Introduction to Unit 4 develops the "skeletal structure" of TR/NYCity as the consequence of the definition.

TEACHING-RESEARCH CYCLE (TR CYCLE)

Just-in-Time Teaching (JiTT) and Just-in-Time Research (JiTR)

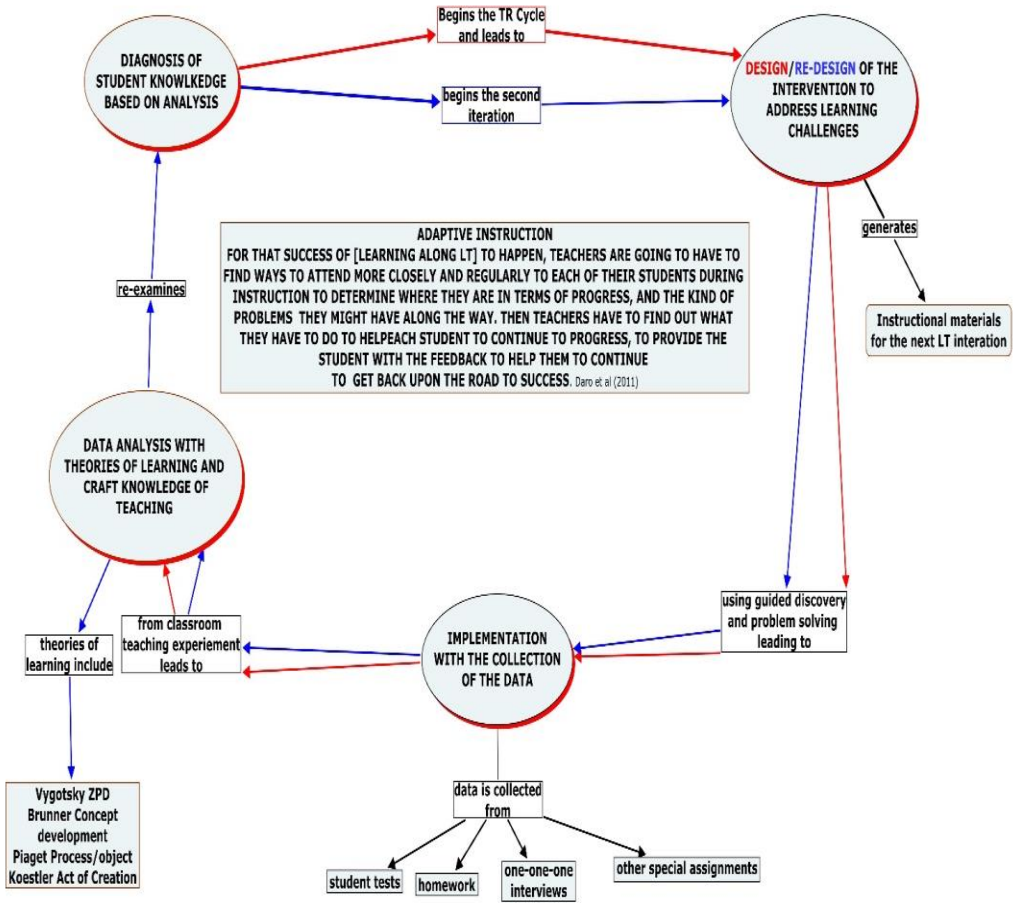
Teaching-Research cycle is the fundamental instrument in our work, which allows for the smooth integration of research and teaching practice within our conceptual framework. The difference from other similar cycles of Action Research or of the Design Experiment (Cobb et al., 2003) is simple: it allows the teacher-researcher to enter the classroom investigation from either of both directions, from research and from teaching. There is however, an important methodological trade off: whereas a Design Experiment researcher prepares the design of classroom intervention on the basis of prior research, the teacher-research uses Just-in-Time approach, that is research literature consultation takes place during the TR cycle, generally at the Analysis and Refinement nodes, when we either compare the results to assumed theory of learning, or when we search for adequate theoretical framework to understand the learning situation, or in any other unclear classroom situation.

Just-in-Time Teaching (JiTT) as expressed by Novak et al. (1999) is a teaching and learning strategy based on the interaction between web-based study assignments and an active learner classroom. Students respond electronically to carefully constructed web-based assignments which are due shortly before class, and the instructor reads the student submissions "just-in-time" to adjust the classroom lesson to suit the students' needs. Thus, the heart of JiTT is the "feedback loop" formed by the students' outside-of-class preparation that fundamentally affects what happens during the subsequent in-class time together. JiTT has been used well together with Peer Leader methodology (Mazur and Watkins, 2009).

Analogically, Just-in-Time Research (JiTR) is research and teaching strategy based on the "feedback loop" formed between the didactic difficulties in the classroom encountered by a teacher who turns to educational research results that may throw light into the nature of these difficulties. At this moment, the classroom teacher makes contact with the bisociative framework of TR/NYCity model.

Anchoring TR in TR cycle.

Fig. 1. The TR Cycle



It is in the introduction of educational research into the classroom that we differ from Action Research. The JiTR approach differs from standard educational research in that theory is repositioned from being a required foundation to the Just –in-Time solution for didactic difficulties in the mathematics classroom.

William J. Harrington, describing his work of a teacher-as-researcher in Laura R. Van Zoest (2006) states that, "Teachers do informal research in their classrooms all the time. We try a new lesson activity, form of evaluation, seating arrangement,

grouping of students, or style of teaching. We assess, reflect, modify, and try again, as we consider the perceived consequences of changes we made." Hence, there is a natural pathway that extends these informal activities into systematic research, offered by the TR/NYCity model that successively progresses along *Teaching-Research* (TR) cycles of diagnosis, design of instruction in response to diagnosis, collection of relevant data and its analysis, and, ultimately, with the help of relevant external research results through JiTR approach, the redesign of interventions. The TR cycle, the explicit generalization of Action Research principles in the classroom, is particularly well fit into our work because of our work's naturally cyclic structure via semesters or academic years. Since every teacher has an option of repeating to teach the same course to a new cohort of students, the TR cycle allows for the continuous process of classroom investigations of the same research question during consecutive semesters. The sequential iteration of TR cycles is one of the main methodological research tools of the TR/NYCity Model facilitating the process of integration of teaching and research into a new unit of professional classroom activity, teaching-research.

TR/NYCity requires a minimum of two full TR cycles within a context of a single teaching experiment to fulfil the requirement of improvement of instruction. In its insistence on the improvement of learning through cycle iteration, TR/NYCity incorporates and generalizes the principles of Japanese and Chinese Lesson studies (Huang and Bao, 2006).

Consequently, every teaching experiment of the TR/NYCity Model has a main teaching-research question, composed of two sub-questions:

– What is the state of the students' knowledge under the impact of the new intervention?
– How to improve that state of knowledge?

The duration of the TR cycle can vary depending on intervention. In can last a year, a semester, and a couple of days or even one class. In its rudimentary form we can find it even in teacher-student inquiry dialogs (see example in Chapter 4.1).

The bisociative creativity of the teacher reaches its fulfilment during this period of reflection and redesign spurred by the simultaneous consideration of data analysis results, relevant teaching experience, relevant JiTR results from professional literature and appropriate theories of learning or conceptual development. It is precisely at this moment when the new teaching-research hypotheses are formed, leading to new theories and investigations. The focus of this teaching-research activity is the investigation of student learning followed by the design of teaching, whose effectiveness is often investigated in the subsequent TR cycle.

Instructional Adaptability of the TR/NYCity Model via TR Cycle

The increased degree of flexibility created by this integration of teaching and research within a single "tool box" helps teachers reach new levels of instructional adaptability to student learning needs. In fact, the comparison of the adaptive instruction described by (Daro et al., 2011) with the TR cycle reveals a very high degree of correspondence:

> For that [success of LT framework] to happen, teachers are going to have to find ways to attend more closely and regularly to each of their students during instruction to determine where they are in their progress toward meeting the standards, and the kinds of problems they might be having along the way. Then teachers must use that information to decide what to do to help each student continue to progress, to provide students with feedback, and help them overcome their particular problems to get back on a path toward success. This is what is known as adaptive instruction and it is what practice must look like in a standards-based system.

Every TR cycle consists of the following components:

– (1) The design of the instruction/intervention, in response to the diagnosis of student knowledge,
– (2) Classroom implementation during an adequate instructional period and collection of data; this incorporates problem-solving, guided discovery classroom discourse and design of interventions for diagnosed difficulties,
– (3) Analysis of the data, in reference to existing experimental classroom data, appealing to the general theory of learning through J-i-T approach and the teacher-researcher's professional craft knowledge,
– (4→1) Design of the refined instruction based on the analysis of the data obtained in steps 1 through 3, leading to the hypothesized improvement of learning. The symbol "4→1" is intended to convey that the 4th step in the cycle is equivalent to going back to the 1st step in the cycle.

As a result, every such 1→2→3→4→1 is an instance of adaptive instruction– finding the level of students' understanding through tests, homework assignments and one-on-one interviews, responding to the difficulties by the re-design of the intervention, implementation and assessment. Consequently, the TR cycle is called for, as the theoretical framework of the teacher's work in a mathematics classroom driven by the Common Core Standards. Transformations of the teacher's pedagogy and improvements, based on research and evidence, have to take place exactly within such a framework. Chapters 4.2, 4.4 and 4.5 provide detailed examples of two (or more) full cycles of such an approach.

Generalization in TR/NYCity Model .

One of the central questions asked of frameworks related to action research is the question about the generality of our assertions. How general is TR/NYCity? Why and how that what we understand in the Bronx, has any bearing anywhere else? In terms of the original definition at the beginning of the chapter, what is the nature of the word "beyond" in that definition? TR/NYCity has three ways to generalize its findings: By coordination with a theory whose correctness has been asserted in the profession. If we coordinate our findings with a theory, then they acquire degree of generality afforded to the theory, that is one can draw conclusions from the findings in terms of the coordinated theory of learning. These conclusions might be relevant, with proper modifications to any classroom situation to which that theory applies. By running an artefact used in a TR investigation through many iterations with different cohorts of students. As a result, the artefact acquires large degree of generality, which provides the basis for its application to different new situations (Chapter 2.2).A special window of generalizations opens up when we consider student populations with similar socio-economic status to the one in the Bronx. The similarity of the socio-economic status results in similar cognitive/affective challenges experienced by students to which similar adaptive interventions are needed (Kitchen et al.) The successful generalization of TR/NYCity artefacts has been reached amongst Indian Dalits (downtrodden) of Tamil Nadu (Chapters 2.2 and 5.3.1) and in Poland amongst rural students of Southern Poland (Czarnocha, 2008). The discussion of artefacts in the context of Design Research (Unit 4) brings forth an important clarification that its generalization can be obtained by expanding its application to similar student populations.

Thinking Technology

The dictionary definition of technology is "the application of scientific knowledge for practical purposes, especially in industry." Thinking technology in TR/NYCity model is the process of integration of research results and framework with craft knowledge of the teacher. This spontaneous process inherent for TR/NYCity model finds its elegant expression in Koestler bisociation theory and Stenhouse TR acts.

It is a very subtle process, in which scientific concepts such as "hidden analogy" of Koestler become the critical tools, metaphors with the help of which we start to identify classroom situations, the term becomes a phrase with the help of which we, members of the TR team start communicate with each other in our own new language. In fact, by making the connection between scientific meaning and classroom situation we create the analogy between two generally separate matrices of thinking – hence the connection itself is a new bisociation, a possibility of new meaning.

One could conjecture that any process of coordination (as distinct from application) of a theory of learning with elements of teaching practice is the bisociative creative process during which new connections and therefore new meanings are made.

The process of coordinating research and teaching practice is facilitated by the duality inherent in the teacher-researcher work (Malara and Zan, 2002). The practice of teaching-research duality creates a new mental attitude promoting a novel design of instructional methodologies while, at the same time, requiring an investigative probe into student thinking, on the basis of which consequential teaching and research decisions are made. This duality is explored deeper in Units 2 and 4. The exploration together with utilization of the duality is conducted by the classroom teacher-researcher. In this process, teachers are not solely engaged in research on learning, they are also engaged in the transformation of teaching on the basis of, and through that research. This means that they do not simply incorporate the results of research into their teaching practice but rather allow methods of research to become the methods of teaching leading to Stenhouse TR acts. Thus the route towards Stenhause TR acts is through the process of integrating research knowledge and craft knowledge in practice of teaching. In this process, teachers do not switch into a role of researcher, instead, they oscillate between the role of a teacher and the role of a researcher and fuse their efforts toward a new unit of professional activity – bisociative teaching-research with its Stenhouse TR acts.

TR/NYCITY AND THE DISCOVERY METHOD OF TEACHING.

The discovery method of teaching has been the preferred instructional method by the teacher-research team working with and developing TR/NYCity methodology since its inception. The Discovery method of teaching has a fundamental role in the TR/NYCity model. This method was introduced into TR/NYCity via the Texan Discovery method created and formulated by R. L. Moore, a topologist brought up by the Chicago school of mathematical thought of the thirties. B. Czarnocha and V. Prabhu adopted this method during their NSF grant in calculus 2002-2006. However, our understanding of its role in TR classrooms came with time through many TR investigations and teaching experiments. Using different approaches such a "guided discovery method", "inquiry method" or "inquiry leading to discovery", it has appealed to our imagination and practice as teacher-researchers because with its help we could lay bare student authentic thinking for our investigations.

On the one hand, from the educational aspect Discovery method provides learning environment best suitable for facilitation of bisociation. According to Koestler (1964) subjective, individual bisociation are more often encountered in the condition of "untutored learning". The Discovery method is one of the closest classroom approximations of this condition. This approach to teaching relies on designing situations and using techniques, which allow the student to participate in the discovery of mathematical knowledge. These are authentic moments of discovery with respect to student's own knowledge, which in the further development of methodology are related to subjective Aha! Moments of Arthur Koestler (Chapter 1.2).

On the other hand, from the research point of view, it is the best instrument, which opens the nature of student thinking to us, teacher-researchers for investigation through careful interaction. It allows us to investigate and to extend the scope of students' ZPD, to help in eliminating misconception as well as in facilitating bisociations. Thus the process of TR together with Discovery method of teaching constitutes an extended in time Stenhouse TR act.

Creativity: From Bathos to Pathos – From Habit to Originality

The institution of creativity as the structural component generated within the learning environment provided by teaching-research has significant consequences beyond its cognitive importance.

Vrunda Prabhu has found out (Chapter 2.4) that student success in her classroom depended on three closely connected components of (i) cognition, (ii) motivation and (iii) self-regulated student learning (Prabhu, 2006). More specifically, when creativity is explicitly nurtured and facilitated in a mathematics classroom in the context of such an integrated learning environment, it can transform the habit of distaste toward mathematics into mathematical originality supporting Koestler's assertion that "creativity means breaking up habits and joining the fragments into new synthesis" (p. 619). Moreover, according to Koestler:

> The creative act, by connecting previously unrelated dimensions of experience, enables him [the inquirer] to attain a higher level of mental evolution. It is an act of liberation – the defeat of habit by originality.

Habitual dislike of mathematics is, at present, one of the main student obstacles for success in mathematics learning that could be eliminated with the help of that "act of liberation" providing a pathway from Bathos to Pathos, using Koestler metaphor (p. 96).

Summary of the argument

To summarize the argument, TR/NYCity is the generalization of Action research and of the Design experiment methodology (Design experiment methodology is seen here as the further development of the Teaching Experiment of Vygotsky school in Russia). In its original vision it was seen as the bridge between the two methodologies, which eliminates the limitations of both – a new integrative conceptual framework. By the same token, TR/NYCity is designed specifically to bridge the gap between research and teaching practice – one of the fundamental obstacles in the effective transformation of mathematics education. The need for such a bridge was indicated by the report of US National Research Council, How People Learn-Bridging Research and Practice (Donovan et al., 1999). We review below essential components of the research/teaching practice gap in our profession as seen by contemporary reports.

GAP BETWEEN RESEARCH AND PRACTICE

English (2010a) notes that the complexity of educational theory and philosophy, has lead to a gap between educators and researcher based upon concerns about the relevancy of such philosophies to educational practice,

> “The elevation of theory and philosophy in mathematics education scholarship could be considered somewhat contradictory to the growing concerns for enhancing the relevance and usefulness of research in mathematics education. These concerns reflect an apparent scepticism that theory-driven research can be relevant to and improve the teaching and learning of mathematics in the classroom. Such scepticism is not surprising...claims that theoretical considerations have limited application in the reality of the classroom or other learning contexts have been numerous...it remains one of our many challenges to demonstrate how theoretical and philosophical considerations can enhance the teaching and learning of mathematics in the classroom...” (p.66).

Harel (2010) and Lester (2010) both note that government funding agencies and panels created to direct government research efforts are increasing restricting their attention to quasi experimental-control group efforts with a goal of what works i.e. action research. They advance the hypothesis that more attention to research frameworks would perhaps counter the ideology that all research should be practical-statistical i.e. scientific based methodology based upon a p value indicating success or failure i.e. ‘what works.’ Harel’s

(2010) claim that attention to frameworks is lacking in educational research is due in part to his belief that there exists "...a feeling on the part of many researchers that they are not qualified to engage in work involving theoretical and philosophical considerations."(p.88-89) The issue that arises for those of us advocating for a more active role of teachers in integrating educational research and craft is that, if researchers feel they are not qualified then how much more likely those teachers feel unqualified. That is, how can practical research methodology such as that used in action research be expected to integrate theory and practice in a meaningful way when its practitioners may feel unqualified to engage in theoretical considerations? This question is particularly relevant to us because we strongly believe in order for reform efforts, indeed, any research based pedagogy to actually improve education there must be a sustained effort in the school and that any such effort must involve the teacher and the researcher working together or a teacher-researcher to determine what works as well as to reflect upon why it does or does not work from both a practical craft level as well as through the lens of theoretical framework.

Another reason reform effort to improve mathematics education through theoretical considerations has floundered is that mathematical education theories are often appear impractical to the craft practitioner to implement i.e. theories that provide little guidance for instructional design but within the research community there is often contradictory positions about such efforts. The result is that reform efforts and counter reactionary movements tend to arise and disappear like last year's fashion statements. Sriramen and English (2010) comment on an early attempt by mathematicians to change traditional mathematics called New Math which in the 50's and 60's tried to change the rigidity of traditional mathematic through a top down approach to pedagogical change. "One must understand that the intentions of mathematicians such as Max Beberman and Edward Begle was to change the mindless rigidity of traditional mathematics. They did so by emphasizing the whys and the deeper structures of mathematics rather than the how's but in hindsight…it seems futile to impose a top-down approach to the implementation of the New Math approach…" (p.21). Goldin (2003) notes how behaviourism led to a back to basics counter movement within mathematics education: "behaviourism was fuelling the 'back to basics' counterrevolution to the 'new mathematics', which had been largely a mathematician-led movement. School curricular objectives were being rewritten across the USA to decompose them into discrete, testable behaviours" (p.192). Goldin (2003) also notes that constructivism has more recently displaced this back to basic reactionary movement. "Radical constructivism helped overthrow dismissive behaviourism, rendering not only legitimate but highly desirable the qualitative study of students' individual reasoning processes and discussions of their internal cognitions" (p.196). Yet he warns that the excessive of radical constructivism will render it impractical and unsuitable "Constructivists excluded the very possibility of 'objective' knowledge about the real world, focusing solely on individuals' 'experiential world'" (p.193).

The point being that a top-down approach to educational reform by research experts has not succeeded and we venture will never succeed without first teacher buy in, but this is not near enough, in order for the craft practitioner to continue to implement reform methodology and to design instruction based upon theory, when the researcher goes back to academia the teacher must internalize the theory and even more how such theory relates to design of instruction. Yet we consider that even this is not enough to sustain reform efforts especially with underserved populations that demonstrate serious negative affect with mathematics. The approach to educational research in which experiments have a beginning and an end is founded upon an underlying assumption that some truth can be found that will dramatically change educational practice. This assumption needs to be re-evaluated if educational craft practice is to actualize the benefits of research. We consider that a constant collaboration between educational researchers and teachers is needed and provides the best hope of actualizing change in educational practice to close widening gap between research and theory and the scepticism it has caused. Boote (2010) comments on the need for continual teacher development based upon design research in improving educational practice: “Indeed, the professional development of all participants may be more important and sustaining than the educational practices developed or the artefacts and knowledge gained” (p.164). Examples of such an international professional development of teacher-researchers based on TR/NYCity methodology are discussed in the Unit 5.

THE COMPARISON BETWEEN TEACHING-RESEARCH AND DESIGN-BASED RESEARCH

The discussion in this section is the continuation of the theme found in the section *Frameworks of Inquiry and the Unity of Educational and Research Acts*, which gets further clarification in the Introduction to Unit 4. Our aim here is to provide a detailed comparison between theoretical and practical frameworks as seen from the point of view of TR/NYCity, which we see as the conceptual framework creating the bridge between the two via TR cycle.

Research, in particular, design-based research	**Teaching-Research, in particular TR/NYCity Model**
Theory driven: (EDUCATIONAL PSYCHOLOGIST, 39(4), 199–201 Copyright © 2004, Lawrence Erlbaum Associates, Inc. William A. Sandoval, Philip Bell Design-Based Research Methods for Studying Learning in Context: Introduction.) Design-based research can contribute to theoretical understanding of learning in complex settings. Each of the articles by Sandoval, Tabak, and Joseph reveal how the design of complex interventions is an explicitly theory-driven activity.	**Practice driven:** (*Professional Development of Teacher-Researchers*, Rzeszow University, Poland, 2008) (*Teaching Experiment NYCity Method*. 2004) Teaching-research is grounded in the craft knowledge of teachers that provides the initial source and motivation for classroom research; it then leads to the practice-based design. Its aim is the improvement of learning in the classroom as well as beyond.
Use of Theories of Learning in Design-Based Research: (Educational Researcher, Vol. 32, No. 1,	**Use of Theories of Learning in Teaching-Research:** (Dydaktyka Matematyki, 2006, v.29,

pp. 5–8), (Design-Based Research: An Emerging Paradigm for Educational Inquiry by The Design-Based Research Collective, 2003) In addition, the design of innovations enables us to create learning conditions that learning theory suggests are productive, but that are not commonly practiced or are not well understood.	Poland, Teaching-Research NYCity Model. B. Czarnocha, V. Prabhu) The design of innovation enables the teacher-researcher to create the Creative Learning Environment based on teacher's craft knowledge, which improves learning in the classroom and transforms habits such as misconceptions, into student originality (Koestler, 1964). Learning theories are used as needed to support teachers' craft knowledge.
Focus of the Teaching Experiment in Design-Based Research: (Journal for Research in Mathematics Education. 14(2) pp.83-94, 1983, Cobb, P. and Steffe, L. P., The Constructivist Researcher as Teacher and Mod el Builder) Cobb and Steffe assert that the interest of a researcher during the teaching experiment in the classroom is "in hypothesizing what the child <u>might</u> learn and finding [as a teacher] ways and means of fostering that learning".	**Focus of the Teaching Experiment in Teaching-Research:** Proceedings of the epiSTEME Conference, Bombay, Homi Bhabha Institute, 2007, B. Czarnocha, V. Prabhu Teaching-Research and Design Experiment – Two Methodologies of Integrating Research and Classroom Practice) …The interest of a teacher-researcher is to formulate ways and means to foster what a student <u>needs</u> to learn in order to reach a particular moment of discovery or to master a particular concept of the curriculum (Czarnocha, 1999). Since, however, "such moments occur only within students' autonomous cognitive structures, the [constructivist] teacher has to investigate these structures during a particular instructional sequence [in order to be of help to the students]. In this capacity, he or she acts as a researcher".
Use of Iteration in design-based research: (ICLS, 1, pp.968-975, 2010, Confrey, J., Maloney, A., The	Use of Iteration in TR/NYCity model: Step 1: Process of iteration, starting with the first iteration designed on the basis of teaching practice.

construction, refinement and early validation of the equi-partitioning Learning Trajectory) …articulating, refining and validating is an "iterative process of research synthesis and empirical investigations involving" many types of evidence. Step 1: Meta-research of the concept to create the prototype. Step 2: Iterative refinement of the prototype	Step 2: Incorporation of research results as needed in between consecutive iterations. It is the concept of iteration of the design from semester to semester together with the related refinement that can bring in now relevant research results illuminating the classroom situation or providing help in the design of appropriate set of assignments.

The TR cycle through its natural iteration of teacher's activity from semester to semester provides the opportunity to move beyond the narrow " chicken or the egg" question of "What is the primary, or the more important realm, — research or practice?" and to creatively integrate design-based practice and design based research (see Unit 4).

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AFFILIATIONS

Bronislaw Czarnocha, Ph.D.

Mathematics Department

Eugenio Maria de Hostos Community College

City University of New York

Getting through Calculus without using the Trigonometric functions Cosecant, Secant, and Cotangent

Terrence Brenner, Juan Lacay,
Hostos Community College

Abstract: We demonstrate how a student could approach problems containing Cosecant, Secant, and Cotangent using only Sine, Cosine and Arctangent.

Introduction

Trigonometric functions are widely used in most branches of mathematics as well as in solving real-world problems. When using trigonometric functions, it is often of value to change a trigonometric expression from one form to an equivalent form by the use of identities. Mathopenref states "Of the six possible trigonometric functions, secant, cotangent, and cosecant, are rarely used. In fact, most calculators have no button for them, and software function libraries do not include them. They can be easily replaced with derivations of the more common three: sin, cos and tan (2009) ". Axler (2013) states "Many books place too much emphasis on secant, cosecant and cotangent. You will rarely need to know anything about these functions beyond their definitions. Whenever you encounter one of the functions, simply replace it by its definition in terms of cosine, sine and tangent and use your knowledge of those familiar functions. By concentrating on cosine, sine and tangent rather than all six trigonometric functions, you will attain a better understanding with less clutter in your mind". We demonstrate how a student could approach problems containing sec x , csc x , cot x, $\sec^{-1}$ x, $\csc^{-1}$ x and $\cot^{-1}$ x using sin x, cos x and $\tan^{-1}$x . All the students would need to know are the trigonometric identities and how to use them. For example, if they have csc x they would change it to $\frac{1}{\sin(x)}$. This is how a group of our best students in our calculus I and II classes (engineering students) actually did the problems on their tests. These students changed sec x , csc x and cot x into expressions using sin x and cos x then proceeded to do the problems. We start this paper by discussing students" actual answers to our test questions. We next discuss the usual way of evaluating the integral of sec x and show the answer in terms of sin x and cos x. We include a method of partial fractions that also uses only sin x and cos x After this we show how to find the derivatives of $\sec^{-1}$ x , $\csc^{-1}$ x and $\cot^{-1}$ x using only sine and cosine. It seems the real purpose of $\sec^{-1}$ x is only for the integral $\int \frac{dx}{x\sqrt{x^2-1}}$. We will show four more answers to this integral, two using arctan ($\tan^{-1}\left(\sqrt{x^2-1}\right)+c$, $2\tan^{-1}(x+\sqrt{x^2-1})$ +c) , one using arcsin ($-\sin^{-1}\left(\frac{1}{x}\right)$ +c) and the other arccos ($cos^{-1}\left(\frac{1}{x}\right)+c$) . We then reference some of the previous literature on the different techniques of integration that use sine and cosine only. Finally we discuss the only two places where students still use cosecant in the real world, they are offset bends that electricians use and radar.

We taught our calculus I and II classes the traditional way using sec(x), csc(x), and cot(x). If a student did not have a TI-89 or TI-Nspire CAS graphing calculator, we loaned the students the TI-89 for the semester. Students were allowed to use the calculator on all the exams. We had a group of our best students in calculus I and II (all engineering students) who changed every problem that had sec x, csc x, cot x into a problem in terms of sin x and cos x, and they were able to do calculus I and II without sec x, csc x and cot x. We show how some of our students used sec x, csc x, cot x to do the problems and then show how our students who did not using sec x, csc x, cot x did the problems.

We gave the following problem on a test: find $\frac{dy}{dx}$ for y=$\frac{\sec(x)}{\cot(6x)}$. We were expecting the students to get $\frac{sec(x)\tan(x)\cot(6x)+6csc^2(6x)\sec(x)}{cot^2(6x)}$.

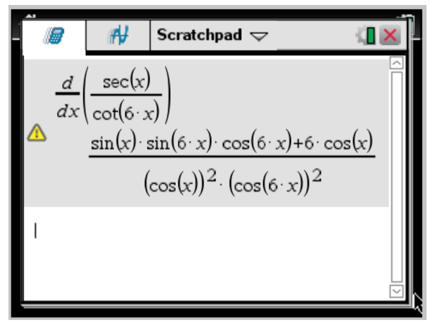
The majority of the students did get this result on the test. There was a different group of students (all were engineering students) who changed the problem to y= $\frac{\sin(6x)}{\cos(x)\cos(6x)}$

These students got the answer

$$\frac{6\cos(6x)\cos(x)\cos(6x) - (-(\sin(x)\cos(6x) - 6\sin(6x)\cos(x))\sin(6x)}{cos^2(x)cos^2(6x)}.$$

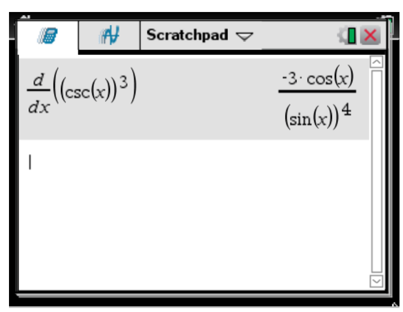
Some of these students simplified their answer to: $\frac{\sin(x)\sin(6x)\cos(6x)+6\cos(x)}{cos^2(x)cos^2(6x)}$.

Figure 1 is a screenshot from the TI-Nspire CAS graphing calculator.



Another example we had on a test was as follows: find $\frac{dy}{dx}$ for y= $\csc^3(x)$. We were expecting the answer $-3\csc^3(x)\cot(x)$. These same students who changed the previous problem to use only sine and cosine changed the problem to y= $\frac{1}{sin^3(x)}$ = $\sin^{-3}(x)$. They then had $\frac{dy}{dx}$ = $-3\sin^{-4}(x)\cos(x)$= - $\frac{3\cos(x)}{sin^4(x)}$.

Figure 2 is a screenshot from the TI-Nspire CAS graphing calculator.



On yet another test, we gave the problem: $\lim_{x\to0}\frac{\csc(x)}{1+\cot(x)}$ We were expecting students would do the problem in the following way: $\lim_{x\to0}\frac{\csc(x)}{1+\cot(x)}$ = $\lim_{x\to0}\frac{-\cot(x)s\csc(x)}{-csc^2(x)}$ =$\lim_{x\to0}\frac{\cot(x)}{\csc(x)}$. Most of the students would continue using L'Hospital's rule until they got lost if they did not realize that $\frac{\cot(x)}{\csc(x)}$ =cos(x).

These same students who changed the previous problem to use only sine and cosine did the following: $\lim_{x\to 0}\frac{\csc(x)}{1+\cot(x)} = \lim_{x\to 0}\frac{-\frac{\cos(x)}{\sin^2(x)}}{\frac{-1}{\sin^2(x)}} = \lim_{x\to 0}\cos(x) = 1$.

One problem we discussed in our classes was $\int \sec(x)dx$. The traditional answer is: ln[sec(x)+tan(x)] . We then asked our students to put $\int \sec(x)dx$ directly into the calculator. The result they got was $\ln\left(\frac{-|\cos(x)|}{\sin(x)-1}\right)$.

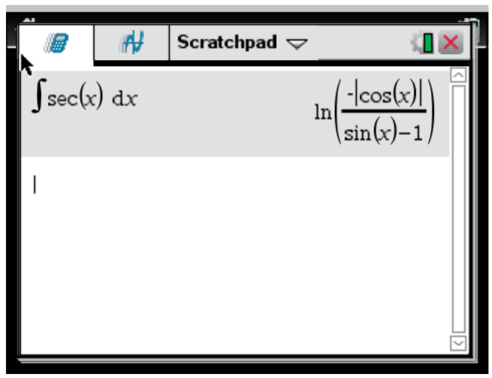


Figure 3 is a screenshot from the TI-Nspire CAS graphing calculator

They asked if this was correct and we told them to prove that | sec(x)+tan(x) | = $\left|\left(\frac{\cos(x)}{\sin(x)-1}\right)\right|$ using only the basic identities. We now explain how to get the answer $\ln\left(\frac{-|\cos(x)|}{\sin(x)-1}\right)$ using just sin(x) and cos (x).

$$\int \sec(x)\,dx = \int \frac{1}{\cos(x)}dx$$

$$= \int \frac{\frac{-1}{\sin(x)-1}}{\frac{-\cos(x)}{\sin(x)-1}}dx$$

$$= \ln\left(\frac{-\cos(x)}{\sin(x)-1}\right) + c$$ (let u = $\frac{-\cos(x)}{\sin(x)-1}$ and use $\int\frac{du}{u}$ =ln(u)) .

Chen and Fulford (2004) solve the integral of $\int \sec\theta d\theta$, by first replacing $sec\theta \ with \frac{1}{cos\theta}$ and then use partial fraction as follows:

$$\int \sec\theta d\theta = \int \frac{1}{cos\theta}d\theta.$$

$$= \int \frac{cos\theta}{cos^2\theta}d\theta.$$

$$= \int \frac{cos\theta}{1-sin^2\theta}d\theta$$. Using the substitution $y = sin\theta,\ dy = cos\theta d\theta$ yields

$$= \int \frac{1}{1-y^2}dy = \frac{1}{2}\int\left[\frac{1}{y+1} - \frac{1}{y-1}\right]dy$$

$$= \frac{1}{2}[ln|y+1|] - ln|y-1| + C$$, then, we have

$$= \frac{1}{2}ln\left|\frac{y+1}{y-1}\right| + C$$

$$= \frac{1}{2}ln\left|\frac{sin\theta+1}{sin\theta-1}\right| + C.$$

We leave it to the reader to verify that $\left(\frac{-|\cos(x)|}{\sin(x)-1}\right) = \sqrt{\frac{sin\theta+1}{sin\theta-1}}$. This would be an interesting problem to put on an exam, probably as extra credit.

Similarly, we can integrate $\int \csc\theta d\theta$ using a similar technique of partial fractions as the integration of $\int \sec\theta d\theta$ above. However, Weierstrass' half-angle substitution is a useful technique to integrate $\int \csc\theta d\theta$. Steward (1995) states, "Karl Weierstrass (1815-1897) noticed that the substitution $t = tan\frac{x}{2}$ will convert any rational function of $sinx$ and $\cos x$ into an ordinary rational function (p. 465)."

$\int csc\theta d\theta = \int \frac{1}{sin\theta} d\theta$. Let $tan\frac{\theta}{2} = t, \theta = 2tan^{-1}t\,, and\ d\theta = \frac{2}{1+t^2}dt$. By substitution,

$$= \int \frac{1}{\frac{2t}{1+t^2}} * \frac{2}{1+t^2} dt$$

$$= \int \frac{dt}{t} = ln|t| + C$$

$$= ln\left|tan\frac{\theta}{2}\right| + C.$$

(The authors leave it to the reader to prove the identity $\frac{2tan\frac{\theta}{2}}{1+\left(tan\frac{\theta}{2}\right)^2}$ $=\sin\theta$, hint use $\sin\theta = \sin\left(\frac{\theta}{2}+\frac{\theta}{2}\right)$ or use $\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)}$)

For integrals of $\int sec^m\theta d\theta$ with higher powers of m, we can apply the same technique of partial fractions. However, the algebra involved can be lengthy at best.

For the case of m=3, we can first rewrite $sec^3\theta = sec\theta sec^2\theta$ and perform integration by parts. However, by using the reciprocal function we can achieve the same answer and thus avoid the use of the secant as follows:

$$\int sec^3\theta d\theta = \int \frac{1}{cos^3\theta} d\theta$$

$$= \int \frac{cos\theta}{cos^4\theta} d\theta$$

$$= \int \frac{cos\theta}{[1-sin^2\theta]^2} d\theta \text{ Let } y = sin\theta, dy = cos\theta d\theta\text{, thus}$$

$$= \int \frac{1}{[1-y^2]^2} dy$$

$$= \int \left[\frac{1}{2}\left(\frac{1}{y-1} - \frac{1}{y+1}\right)\right]^2 dy$$

$=\frac{1}{4}\int\left[\frac{1}{(y-1)^2}-\frac{1}{y-1}+\frac{1}{y+1}+\frac{1}{(y+1)^2}\right]$ After integrating the integral above and making the appropriate substitutions, we have:

$$=\frac{1}{4}\left[-\frac{1}{y-1}-\frac{1}{y+1}+ln|y+1|-ln|y-1|\right]+C$$
$$=\frac{1}{4}\left[-\frac{2}{y^2-1}+ln\left|\frac{y+1}{y-1}\right|\right]+C$$
$$=\frac{1}{2}\frac{y}{1-y^2}+\frac{1}{4}ln\left|\frac{y+1}{y-1}\right|+C$$
$$=\frac{sin\theta}{2cos^2\theta}+\frac{1}{4}ln\left|\frac{sin\theta+1}{sin\theta-1}\right|+C$$

For a more lengthy coverage of the integrals of the type $\int csc^m\theta\, sec^n\theta d\theta$, Chen and. Fulford use $\int\frac{1}{sin^m\,\theta cos^n\theta}d\theta$. (c.f. Chen & Fulford ,2004)

The authors will now show how to find the derivative of $\sec^{-1}(x)$, $\csc^{-1}(x)$ and $\cot^{-1}(x)$, using only sine and cosine. We proceed in the following way:

$y=\sec^{-1}(x)=cos^{-1}\left(\frac{1}{x}\right)$

$\frac{1}{x}=\cos(y)$ then

$x=\frac{1}{\cos(y)}$. We next differentiate to get

$$1=\frac{\sin(y)}{cos^2(y)}\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{cos^2(y)}{\sin(y)}$$
$$=\frac{cos^2(y)}{\sqrt{sin^2(y)}}$$
$$=\frac{cos^2(y)}{\sqrt{1-cos^2(y)}}$$
$$=\frac{\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}}$$
$$=\frac{1}{x\sqrt{x^2-1}}.$$

To find the derivative of $\csc^{-1}(x)$ using only sine and cosine we proceed in the following way:

$y=\csc^{-1}(x)=sin^{-1}\left(\frac{1}{x}\right)$

$\frac{1}{x}=\sin(y)$ then

$x=\frac{1}{\sin(y)}$. We next differentiate to get

$$1=\frac{-\cos(y)}{sin^2(y)}\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{-sin^2(y)}{\cos(y)}$$

$=\frac{-sin^2(y)}{\sqrt{cos^2(y)}}$

$=\frac{-sin^2(y)}{\sqrt{1-sin^2(y)}}$

$=\frac{-\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}}$

$=-\frac{1}{x\sqrt{x^2-1}}.$

To find the derivative of $\cot^{-1}(x)$ using only sine and cosine we proceed in the following way:

$y=\cot^{-1}(x)=tan^{-1}\left(\frac{1}{x}\right)$

$\frac{1}{x}=\tan(y)$ then

$x=\frac{1}{\tan(y)}$.

$x=\frac{\cos(y)}{\sin(y)}$. We next differentiate to get

$1=\frac{-\sin(y)\sin(y)-\cos(y)\cos(y)}{sin^2(y)}\frac{dy}{dx}$

$1=\left(-1-\frac{cos^2(y)}{sin^2(y)}\right)\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{-1}{1+\frac{cos^2(y)}{sin^2(y)}}$

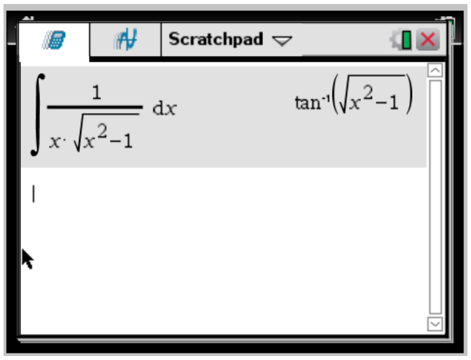
$=\frac{-1}{x^2+1}.$

We now present an integral with five answers. The integral is: $\int\frac{dx}{x\sqrt{x^2-1}}$.

Here are the different possible answers

- $\sec^{-1}(x)+c$ (1)
- $cos^{-1}\left(\frac{1}{x}\right)+c$ (2)
- $\tan^{-1}(\sqrt{x^2-1})+c$ (3)
- $-\sin^{-1}\left(\frac{1}{x}\right)+c$ (4)
- $2\tan^{-1}(x+\sqrt{x^2-1})+c$ (5)

Figure 4 is a screenshot from the TI-Nspire CAS graphing calculator.



Formula (1) is in any standard calculus book, formula (2) the authors derived earlier in this paper (when we showed how to find the derivative of $cos^{-1}\left(\frac{1}{x}\right)$), formula (3) is from the calculator, formula (4) we discovered when trying to derive (5), (5) is from Fulling (2005).

For formula (3) we use $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\frac{x}{\sqrt{x^2-1}}}{1+x^2-1}dx$

$$= \int \frac{\frac{x}{\sqrt{x^2-1}}}{1+\left(\sqrt{x^2-1}\right)^2}dx$$

$=\tan^{-1}\left(\sqrt{x^2-1}\right)$ +c (let u $=\sqrt{x^2-1}$ and use $\int \frac{du}{1+u^2} =$ $\tan^{-1}(u)$).

For formula (4) we use $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{dx}{x\sqrt{x^2(1-\frac{1}{x^2})}}$

$$= \int \frac{\frac{1}{x^2}}{\sqrt{1-\frac{1}{x^2}}}dx$$

$$= \int \frac{-\left(-\frac{1}{x^2}\right)}{\sqrt{1-\frac{1}{x^2}}}dx$$

$= -\sin^{-1}\left(\frac{1}{x}\right) + c$ (let u $=\frac{1}{x}$, then du= $-\frac{1}{x^2}$ dx and use $\int \frac{-du}{\sqrt{1-u^2}}$).

Fulling (2005) uses hyperbolic trigonometric functions to solve the $\int \frac{dx}{x\sqrt{x^2-1}}$, but we get this formula without the use of hyperbolic trigonometric functions. We do it in the following way

For formula (5) we use $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{1}{\sqrt{x^2-1}}\frac{2}{2x}dx$

$$= \int \frac{2(x+\sqrt{x^2-1})}{\sqrt{x^2-1}(2x)(x+\sqrt{x^2-1})}dx$$

$$= \int \frac{2(x+\sqrt{x^2-1})}{\sqrt{x^2-1}(2x^2+2x\sqrt{x^2-1})}dx$$

$$= \int \frac{2(x+\sqrt{x^2-1})}{\sqrt{x^2-1}(x^2+1+2x\sqrt{x^2-1}+x^2-1)}dx$$

$$= \int \frac{2(x+\sqrt{x^2-1})}{\sqrt{x^2-1}\left(1+\left(x+\sqrt{x^2-1}\right)^2\right)}dx$$

$$= \int \frac{\frac{2(x+\sqrt{x^2-1})}{\sqrt{x^2-1}}}{1+\left(x+\sqrt{x^2-1}\right)^2}dx$$

$$= 2\tan^{-1}\left(x+\sqrt{x^2-1}\right)+c$$

(Let $u = x+\sqrt{x^2-1}$, then $du = \frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}dx$ and use $\int \frac{2du}{1+u^2}$).

We now show that all five answers differ by a constant, and in fact three are equal.

For (2) we recall that $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$.

For (3) we see that $\sec^{-1}(x) - \tan^{-1}\left(\sqrt{x^2-1}\right) = 0$.

For (4) $\sec^{-1}(x) - \left(-\sin^{-1}\left(\frac{1}{x}\right)\right) = \sec^{-1}(x) + \csc^{-1}(x) = \frac{\pi}{2}$.

(5) Involves more work. We use $\tan\left(\frac{a}{2}\right) = \frac{\sin(a)}{1+\cos(a)}$, then $\cot\left(\frac{a}{2}\right) = \frac{1+\cos(a)}{\sin(a)}$, and let $\theta = \sec^{-1}(x)$.

$\cot\left(\frac{\frac{\pi}{2}-\theta}{2}\right) = \frac{1+\cos(\frac{\pi}{2}-\theta)}{\sin(\frac{\pi}{2}-\theta)} = \frac{1+\frac{\sqrt{x^2-1}}{x}}{\frac{1}{x}} = x+\sqrt{x^2-1}$, then

$\tan\left(\frac{\pi}{2}-\left(\frac{\frac{\pi}{2}-\theta}{2}\right)\right) = \cot\left(\frac{\frac{\pi}{2}-\theta}{2}\right) = x+\sqrt{x^2-1}$. This leads to

$\frac{\pi}{4}+\frac{\theta}{2} = \tan^{-1}\left(x+\sqrt{x^2-1}\right)$. We conclude that

$\sec^{-1}(x) - 2\tan^{-1}\left(x+\sqrt{x^2-1}\right) = -\frac{\pi}{2}$.

Several mathematical papers have been published to address the problem of solving trigonometric integrals by nontraditional methods, namely, using trigonometric substitution. In a paper recently published in *The College Mathematics Journal*, Fulling(2005) points out the need to bring to closure, in the traditional sense, the teaching of trigonometric integrals from trigonometric substitution to hyperbolic substitution. Fulling (2005) states “one might have expected that after a decade of calculus reform, the secant function and its inverse would have been de-emphasized to the point, along with its even less useful siblings, cosecant, cotangent, and their inverses(p. 381).” Furthermore, Fulling (2005) states that he hopes “to convince the reader that there is nothing that the secant and its inverse secant do in the traditional techniques of integration chapter that cannot be done better by the hyperbolic sine and cosine and their inverses. It is time for sec, csc, cot, $\sec^{-1}$, $\csc^{-1}$, $\cot^{-1}$ to be retired from our calculus syllabus(p. 382)". While Fulling (2005) uses hyperbolic substitution to solve trigonometric integrals, Velleman (2002) uses combinatorics identities involving binomial coefficients for integrals of the type $\int \sec^{2n+1} x\,dx$, while Wu (2008) uses integration by parts to obtain a recursive relation for the same integral form. In an extended form of the previous integral solution by Velleman (2002), Cheng and Fulford (2004) use parametric differentiation to obtain partial fraction decomposition for integrals of the type $\int \frac{dx}{\sin^m x \cos^n x}$.

While the primary objective of this paper is teaching calculus without using the trigonometric functions cosecant, secant, and cotangent**,** we note that in highly specialized areas students still need to be familiar with the terminology used. We now give two examples where cosecant is still used. One such specialized field in electromagnetics is Radar theory. Radar (antenna) is an acronym derived from the words radio, detection, and range. It refers to the method of using electromagnetic waves to detect the existence of objects at a distance. The energy emitted from an antenna forms a field having a particular radiation pattern. A radiation pattern is a way of mapping the radiated energy from an antenna. This energy is measured at different angles at a constant distance from the antenna. The characteristic of this pattern depends on the type of antenna used. Kai Chen (2004) states "The basic role of the radar antenna is to act as a transducer between the free space and the electromagnetic wave sources or receivers. During transmission, it is used to concentrate the radiated energy into a shaped beam or in a desired direction. During reception, the radar is used to collect the echo signal and deliver it to the receiver (p.676)." Wolff (2006) states "Antennae with cosecant squared pattern are special designed for air-surveillance radar sets. These permit an adapted distribution of the radiation in the beam and causing a more ideal space scanning. The cosecant squared pattern is a means of achieving a more uniform signal strength at the input of the receiver as a target moves with a constant height within the beam". Another highly specialized area is an electrician who needs to make offset bends. Porcupine press (1998-2012) states that "Offset bends are used to move a run of conduit from one plane to another. An offset is normally used to bend the conduit around an obstruction, or to relocate the conduit close to a structural member to make it easier to fasten the conduit. A trigonometric function, the Cosecant, is used to determine the distance between the centers of the two bends used to make the offset. " Google (2012) states that "the cosecant for any given angle of bend may be found by dividing the distance between bends by the depth of offset or saddle. It is basic trigonometry that multiplying the cosecant of a given angle by the length of the opposite side of a right triangle gives the length of the hypotenuse of that right triangle. Thus, for a given range of angles there is a corresponding range of cosecants for the given angles in degrees."

Conclusion

In this paper, we demonstrated how to integrate and differentiate a wide variety of trigonometric functions without using the traditional method of using the reciprocal trigonometric identities discussed in our paper. Our method strictly relies on the exclusive use of the sine, cosine, and arctangent, without using the trigonometric functions cosecant, secant, and cotangent. We do not claim that this method will always be more efficient than other methods used in traditional calculus courses. However, we showed how to use modern technology (graphing calculator) not only to verify our results, but also to find new insights to the existing methods, as illustrated by our pictorial results. We also showed that cosecant is used in very specialized areas and students not entering these fields do not need secant, cosecant or cotangent. The occurrence of the sine, cosine and tangent in formulas makes the use of secant, cosecant and cotangent obsolete as the use of modern technology (graphing calculator) clearly shows.

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The Analog Imaging Visual Cues to Enhance Understanding of Inverse and Direct Square Laws in Digital Imaging.

Jarek Stelmark, MBA
Hostos Community College
500 Grand Concourse
Bronx, NY 10451
Telephone: 718 518 4123
Email: jstelmark@hostos.cuny.edu

Abstract

The Analog Imaging Visual Cues to Enhance

Understanding of Inverse and Direct Square Laws in Digital Imaging.

In this paper the author's goal was to evaluate if the analog based imaging visual cues help radiologic technology students master the inverse and direct square law problem solving skills in digital computed radiography cassette and direct radiography cassetteless radiography. The radiographic conventional analog systems respond with visual hints to increase and decrease in radiographic technique factors unlike digital systems. As a result, the author disabled the brightness and contrast correcting algorithm in digital system allowing it to respond with varying degree of brightness and contrast to changes in distance and mAs. Students in digital imaging evaluate images based on their numeric exposure index and degree of quantum noise. Allowing students to see different amount of brightness and contrast during digital radiographic procedures induced better understanding of the inverse and direct square law concepts. Consequently, students improved their computational fluency when formulating radiographic techniques and gained self-confidence and deeper interest in the more challenging material.

The American Society of Radiologic Technologists' radiography core curriculum and the Joint Review Committee on Education in Radiologic Technology's Standards for an Accredited Educational Program in Radiologic Sciences enunciate that good problem solving and critical thinking skills are absolutely essential in the effective practice of radiologic technology (ASRT, 2011; JRCERT, 2010). In addition to possessing these higher skills, technologists must have a good computational fluency to expediently modify their radiographic technique in response to a changing clinical situation. Unfortunately, the vast majority of the first semester radiologic technology students occupy the lower end of the mathematical skills continuum and a small minority places at the top. This is consistent with the observation that students in the inner- urban classrooms span a broad range of mathematical skills and abilities (Oakley, 2003). The duty of the radiologic educators is to span these gaps in the math skills.

Radiographic Quality in Analog and Digital Imaging

A quality radiographic image accurately represents the anatomic area of interest, and information is well visualized for diagnosis (Fauber, 2012). Radiographic images can be acquired from two different types of image receptors: digital and analog film-screen. The process of creating the image by applying radiation is the same for digital and analog systems, however, processing, and display vary greatly.

The primary factor that affects the amount of brightness or density produced in an image is the amount or quantity of radiation reaching the image receptor. However, the quantity of radiation reaching the image receptor has less pronounced effect on the brightness of a digital image because of image correcting algorithm called autorescaling.

The quantity of radiation reaching an analog film-screen image receptor has a direct effect on the amount of density or darkness produced in a film image. (Bushong, 2016).

Figure 1 shows changes in density in analog system in response to different amount of radiation. Underexposed radiograph has low radiographic density and overexposed radiograph has excessive radiographic density.

Figure 1 Image Density in Analog Imaging

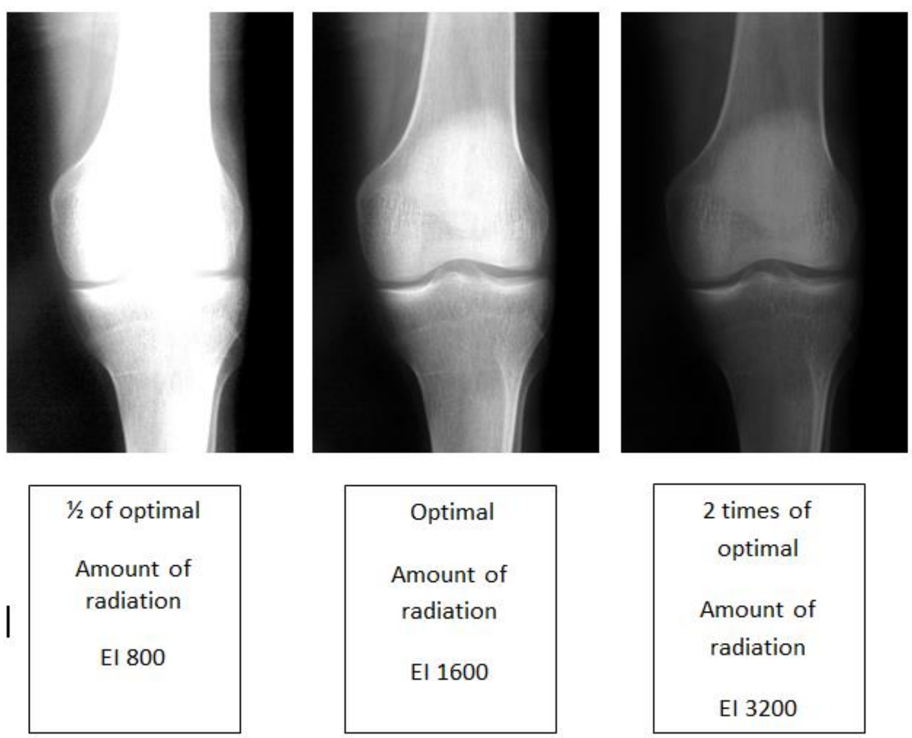
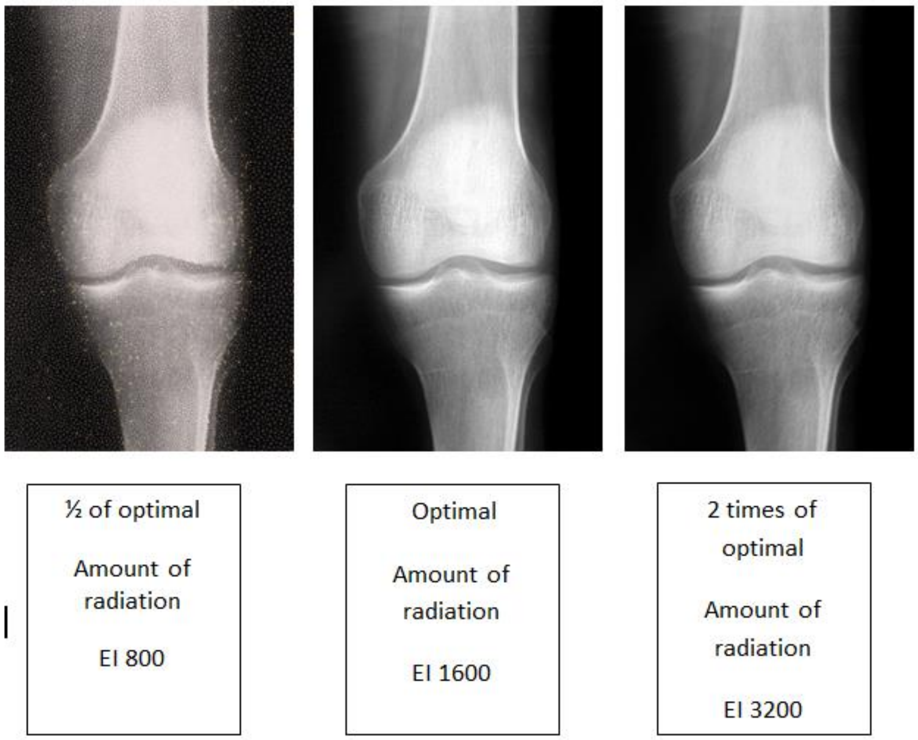


Figure 2 shows three radiographs exposed to different amount of radiation. All radiographs have very similar brightness and contrast because of the autocorrecting algorithm. However, the underexposed radiograph has more quantum noise which manifests itself as graininess. In addition, all images have different exposure indices (EI). Exposure index tells us the amount of radiation that was absorbed by the image receptor.

Figure 2 Image Brightness, Noise and Exposure Index in Digital Imaginning



In digital cassette systems, the exposure index value represents the amount of radiation to the imaging plate, and the values are vendor specific. Fuji and Konica use sensitivity (S) numbers, and the value is inversely related to the amount of radiation to the plate. A 200 S number is equal to 1 mR of exposure to the plate. If the S number increases from 200 to 400, this would indicate a decrease in exposure to the IR by half. Conversely, a decrease in the S number from 200 to 100 would indicate an increase in exposure to the IR by a factor of 2, or doubling of the exposure. Carestream (Kodak) uses exposure index (EI) numbers; the value is directly related to the exposure to the plate, and the changes are logarithmic expressions. For example, a change in EI from 2000 to 2300, a difference of 300, is equal to a factor of 2 and represents twice as much exposure to the plate. Agfa uses log median (lgM) numbers; the value is directly related to exposure to the plate, and changes are also logarithmic expressions. For example, a change in lgM from 2.5 to 2.8, a change of 0.3, is equal to a factor of 2 and represents twice as much exposure to the IR. Optimal ranges of the exposure indices values are vendor specific and vary among the types of procedures, such as abdomen and chest imaging versus extremity imaging (Fauber, et al.)

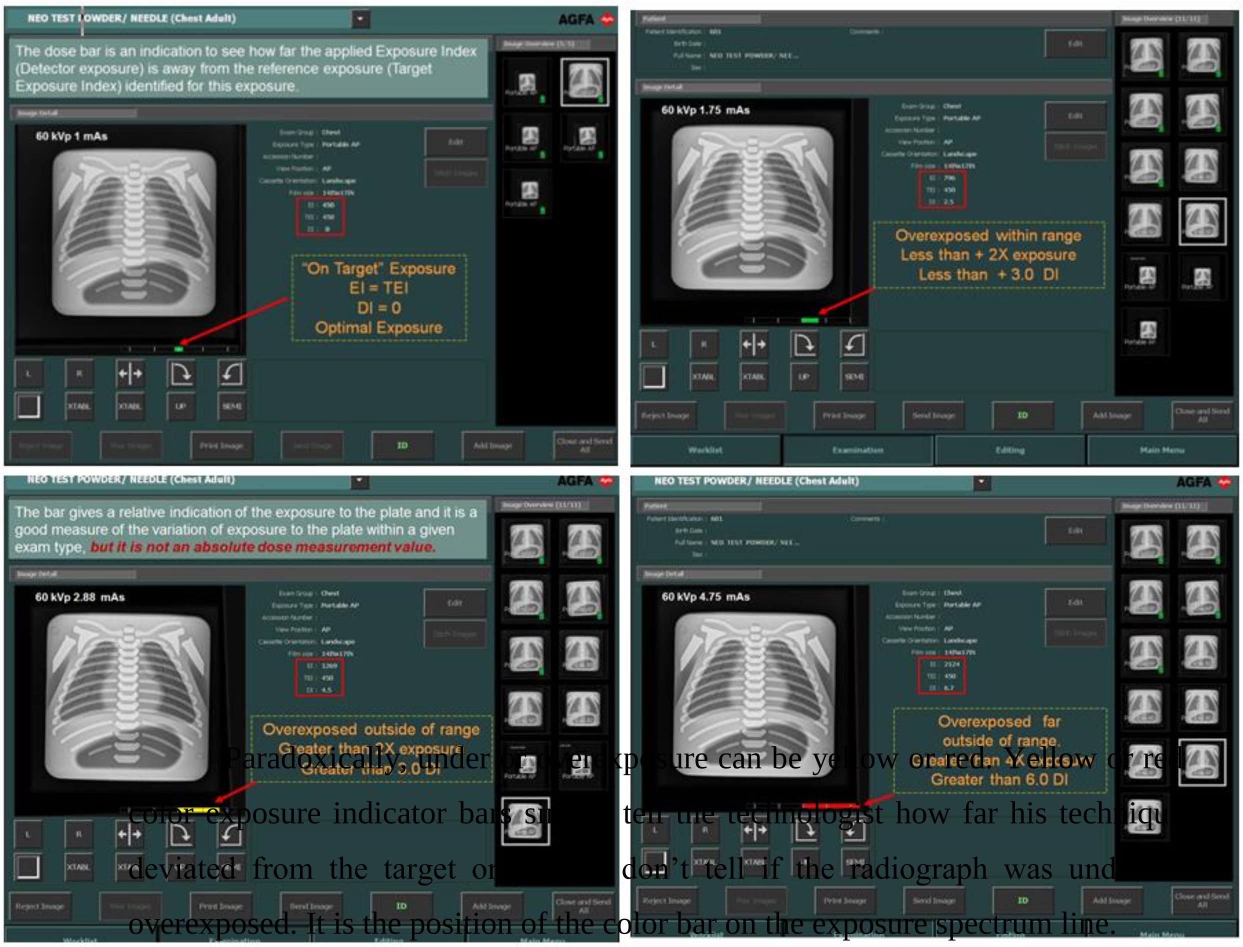
The direct radiography cassette-less systems also use exposure indices that are vendor specific. For example, Agfa use exposure index EI. Agfa EI exposure is different from Care-stream cassette exposure index system. The exposure to an image receptor consists of three clues: target exposure index (TEI), exposure index (EI), and deviation index (DI). Exposure index is linear in relation to detector doseAs exposure to the plate increases, the Exposure Index increases. Target Exposure Index is the reference exposure index for a particular exposure. It can be determined by statistical averaging (50 exposures) and preferred scenario can be pre-set (fixed) by the user. Deviation Index Expresses how far the exposure is away from a reference value and provides a relative indication for under/over exposure three deviation units equals 2x exposure or ½ exposure (+3 or -3) (Gibbs, 2012).

In addition, Agfa introduced exposure color coding system to help technologist evaluate his technical factors. Green color indicates optical radiographic technique, yellow color means caution when patient is overexposed, however, when patient is underexposed it may require repeat. Red color indicates that patient is grossly under or overexposed and repeat radiograph is mandated.

Figure 3 Agfa Exposure Index table

Exposure Index	Deviation Index - DI	Correction Needed
Over Exposed	**4**	**Reduce mAs in half***
800	3	None - Caution
	2	None
	1	None
AIM 400	**0**	**None**
	-1	None
	-2	None
200	-3	Possible Repeat
Under Exposed	**-4**	**Double mAs***

Figure 4 Agfa Exposure Color Coding and Radiographs



Paradoxically, under or overexposure can be yellow or red. Yellow or red color exposure indicator bars simply tell the technologist how far his technique deviated from the target or [illegible] don't tell if the radiograph was und[illegible] overexposed. It is the position of the color bar on the exposure spectrum line.

As we can see, Technologists must traverse this labyrinth of different digital exposure factors to generate diagnostic quality images. It must be noted that even the same radiology department has radiographic units that utilize different exposure index system. Every clinical situation has a plethora of patient variables and different exposure indices only add to the complexity of the situation.

Background of the Study

The institution involved in this research project is a radiologic technology program at a two-year public, open admission institution that is part of a city wide system. The total college current student enrollment is 6000 students. Close to 90% of matriculating students enter needing at least one remedial course in reading, writing or mathematics. Consequently, difficulty in completing mathematics courses is a major contributing factor to low graduation rates. The radiologic program enrolls 50 students every fall for the past three years. Students entering the clinical phase of the program must have completed at least seven general education courses with the GPA of at least 3.0. Despite these stringent entrance requirements program has experienced a graduation rate of 54%, which is higher than the overall college graduation rate of 25%, but is lower than the other radiologic technology programs within the same system. Recent accrediting agency evaluation process revealed that passing rate benchmarks for the program were set too low. Deficiencies in math skills were identified as one of the major contributing factors to the low program student retention rate as well. The basic math skill tests administered by the author during the first radiologic Science 1 class during the fall semester of 2016 revealed that vast majority of students were entering the program with the very weak math skills. Consequently, one of the least understood concepts in the radiologic science was the inverse s and direct square laws. The knowledge of these concepts is critical in the radiographic technique formulation and the correct amount of radiation applied to the patient.

The program had had a long history teaching using conventional analog systems. However, recent publication by American Registry of Radiologic Technologist (ARRT, 2016) of the didactic requirements indicated that the knowledge of analog system concepts will no longer be tested on the Radiography Registry Examination. As a result, the faculty decided not to use darkroom and analog processor and not to teach analog imaging anymore.

The purpose of this pilot study was to learn if the students would improve from pre to posttest after conducting Direct Square and Inverse Square Law laboratory exercises. Concretely, the research questions were as follows:

1. Did the laboratory exercises without the correcting algorithm help the students better understand the concept of Inverse Square law?

2. Can the students compensate radiographic technique in response to changes in the distance from the source of radiation to an image receptor utilizing Direct Square law?

3. Can the students identify digital images with the highest amount of quantum noise?

4. Which group of students benefited the most from the laboratory exercises?

Methods

Experiment 1

Inverse Square Law

Students conducted experiment one "Direct Square Law" in the live radiographic lab. Experiment was started with the lab discussion in the form of the soft scaffolding session. The experiment consisted of three parts. Radiographic technique was provided to all students for all six exposures. In part one, instructor exposed ionization chamber at 18" distance and record the reading in the Dosimeter sheet. Students were asked to calculate the exposure intensity at 36" and 72". Furthermore, instructor exposed the dosimeter at those distances and the reading was compared to students' calculated values.

In part two, students exposed natural bone knee phantom at 18", 36", and 72" distance from the target of an x-ray tube to the digital cassette image receptor. However, these three images were processed without brightness and contrast correcting algorithm.

Image two was exposed with correct technical factor and possessed optimal brightness. Image one was too dark because shorter distance and image three was too light and possessed the most quantum noise. Students were able to observe that as the image receptor was moved farther away from the radiation source the image brightness increased.

In part three, the same three exposures were taken like in part two but images were processed using correcting algorithm. No changes in brightness or contrast were observed. However, image taken at 72" distance displayed significant amount of noise.

After the completion of the lab exercises students were instructed to discuss their impressions and findings in their respective lab groups. Finally, instructor engaged students in the soft scaffolding session during which students were asked to explain their findings, impressions, and conclusions.

Experiment 2
Direct Square Law

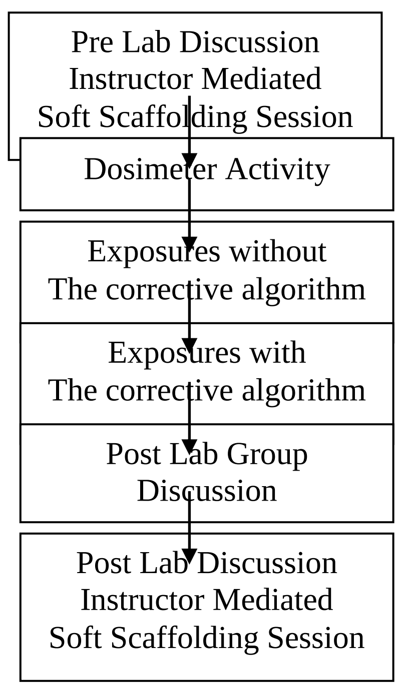
Students conducted experiment one "Direct Square Law" in the live radiographic lab. Experiment was started with the lab discussion in the form of the soft scaffolding session. The experiment consisted of three parts. Radiographic technique was not provided to students for any of the six exposures. In part one, instructor exposed ionization chamber at 18" distance and recorded the reading in the Dosimeter sheet. Students were asked to calculate new technique (mAs) in order to maintain exposure intensity at the image receptor at 36" and 72" distances. Furthermore, instructor exposed the dosimeter with compensated technique at those distances and the reading was compared to students; calculated values.

In part two, students exposed natural bone knee phantom at 18", 36", and 72" distance from the target of an x-ray tube to the digital cassette image receptor with compensated technique. However, these three images were processed without brightness and contrast correcting algorithm. All images possessed the same brightness when radiographic technique was compensated utilizing Direct Square Law. Students were able to observe that as the image receptor was moved farther away from the radiation source the image brightness was maintained when mAs was increased.

In part three, the same three exposures were taken like in part two but images were processed using correcting algorithm. No changes in brightness or contrast were observed. The same amount of quantum noise was observed because the radiographic technique was modified in response to a changing distance.

After the completion of the lab exercises students were instructed to discuss their impressions and findings in their respective lab groups. Finally, instructor engaged students in the soft scaffolding session during which students were asked to explain their findings, impressions, and conclusions.

Figure 5
Inverse Square and Direct Law Lab Exercise Learning Schema



Pre- and Post-project Test

The pretest and post test scores ranged from 0 to 100. The pretest was administered during the tenth Radiologic Science 1 class. The post test was administered after 4 weeks later after the completion of the Direct and Inverse Square law laboratory exercises. There were two versions of the tests, with similar items of equivalent difficulty. Versions were counterbalanced across the participants. Initial comparison indicated that the tests were similar in difficulty. The tests included problems that involved the inverse square law and direct square law problems taught in the class and reinforced in the laboratory exercises.

Participants

No comparison group was included in this study. The study involved the convenience sample of radiologic technology students enrolled in Radiologic Science 1 during the fall of 2016 who completed the pre and posttests, Frequency statistics for the respondents (N=48) indicated female population of 44% (n=21) and a male population of 56% (n=27) for the initial pretest. There was an attrition rate of 7 due to the withdrawal from the course. Therefore the withdrawn student's pre test scores were excluded from the evaluation. Final respondent population (N=41) was 49% (n=20) female and 51% (n=21) male.

Results and discussion

For all of the following analysis, students were divided into those who scored above or below 75% on the pre-test and pos-test. The reason to select 75% as the threshold was that American Registry of Radiologic Technologists uses 75% as the minimum passing score. Therefore, students who received above 75% in the pre-test were given group A for example. That group remained the same when examining the post test scores.

The study focused on the performance of the students on two pre and post test questions:

Question 1*: The following radiographic technique 85 kVp and 40 mAs produces an exposure of 200 mR at a source to image receptor distance (SID) of 100 cm. What would the exposure be at an SID of 100 cm if the same technical factors were utilized?*

This question introduced two extra technical factors that are irrelevant to the calculation of the intensity of radiation: kilovoltage peak (kVp) and mAs. These two factors acted as distractors. 40% of students who scored overall below 75% on the pretest answer this question correctly, 77% of students who scored overall above 75% on the pretest answered this question correctly. The improvement on the post test was noted with both groups. 66% of students who scored overall below 75% answered this question correctly and 91% of students who scored overall above 75% answered this question correctly.

Question 2*: If an instrument positioned 100 cm from a point source of radiation is moved 50 cm closer to the source, the radiation intensity will increase or decrease by what factor?*

67% percent of students who scored overall below 75% on the pretest answer this question correctly, 85% of students who scored overall above 75% on the pretest answered this question correctly. The improvement on the post test was noted with both groups. 87% of students who scored overall below 75% answered this question correctly and 94% of students who scored overall above 75% answered this question correctly.

Table 2

Percentage of correct answers on questions 1 and 2

		Question 1	*Question2*
Pre test	Under 75%	40% N=7	67% N= 11
	Over 75%	77% N= 20	85% N=22
Post test	Under 75%	66% N=9	87% N=14
	Over 75%	95% N=22	94% N=24

Figure 5 Percentage of correct answers on question 1

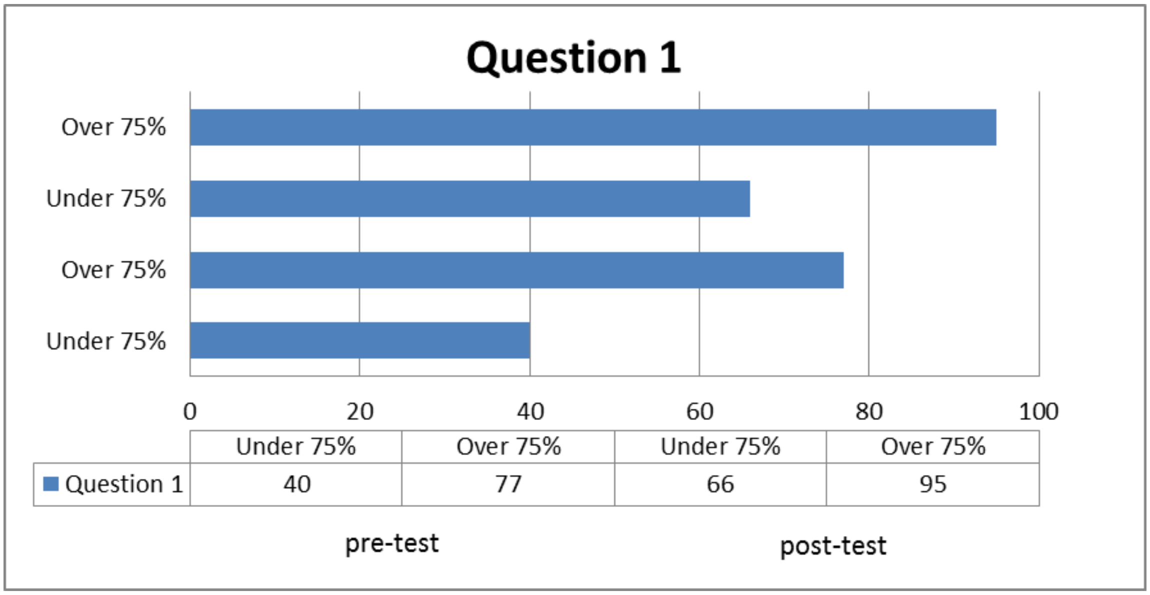


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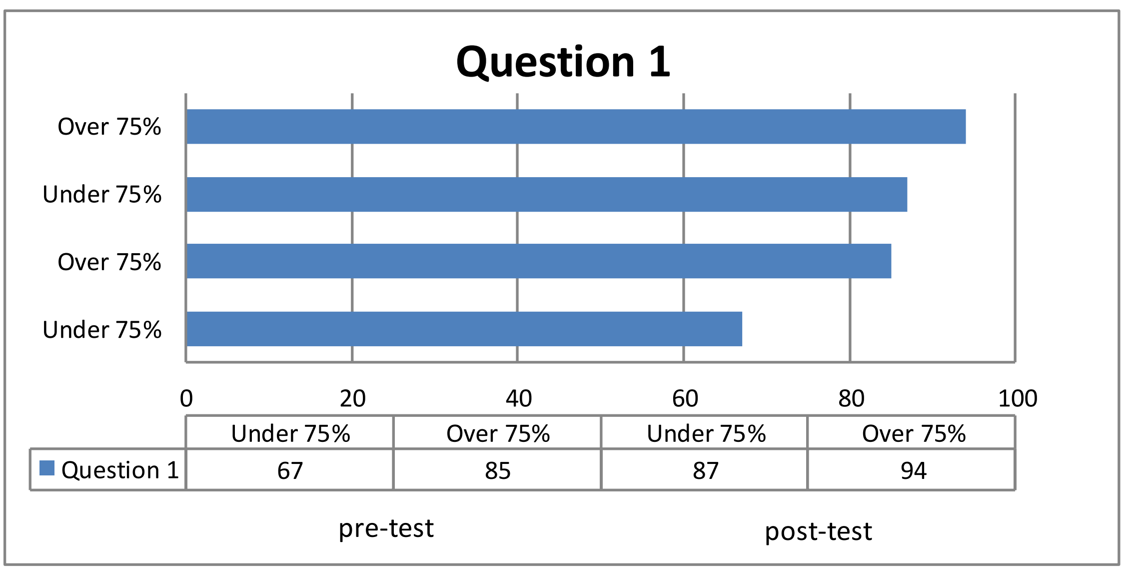
Figure 6 Percentage of correct answers on question 2



Discussion and Conclusion

All students enjoyed conducting the lab experiments. The experiments helped the most the low performing students. They needed that visual stimulation to reach the "eureka" moment. Quantum noise is a confusing concept in digital imaging and many students don't understand its manifestation on the digital radiographic image. Changing image brightness in response to changing distance enabled a lot of students to understand Inverse Square Law. On the other hand, when performing Direct Square law, students understood that technical factor compensation led to maintaining the same image brightness.

"Mathematics is a subject that allows for precise thinking, but when that precise thinking is combined with creativity, openness, visualization, and flexibility, the mathematics comes alive." (Boaler, 2016)

Researchers found that training students through visual representations improved students' math performance significantly, even on numerical math, and that the visual training helped students more than numerical training (Park & Branon, 2013)

"Based upon research outcomes, the effective use of visuals can decrease learning time, improve comprehension, enhance retrieval, and increase retention. In addition, the many testimonials I hear from my students and readers weigh heavily in my mind as support for the benefits of learning through visuals. I hear it often and still I can't hear it enough times . . . by retrieving a visual cue presented on the pages of a book or on the slides of a lecture presentation, a learner is able to accurately retrieve the content associated with the visual." **(**Kouyoumdjian, 2012)

Advantages of using visual cues in digital radiography:

1. Lab exercises provide multisensory stimulation: visual, auditory, tactile and kinesthetic, which help students with different learning styles to better understand difficult concepts.

2. Visual cues are transferred to long term memory where they are indelibly etched and can be later retrieved.

Disadvantages:

1. Requires software engineer to disable image correcting algorithm.
2. Students may erroneously expect that digital image brightness varies in response to radiation exposure.

Limitations

Since a convenience sample was used in this study, generalizations are limited outside of the target institution's radiologic technology program. A larger sample size with a control group could allow more general conclusions about the benefits of the visual cues in digital radiography.

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