



Editorial from Małgorzata Marciniak

Addressing multifold issues of modern college education, we composed the current volume, Vol 11 no 1-2, so that it contains articles of various goals and themes. The readers can find here articles about quantitative reasoning, history of mathematics, book review of a biography of Emmy Noether, and a discussion essay with reflections about creativity being taken out from curriculum due to challenging assessment. Bonus contains another book review with a detailed report on the life of August Ferdinand Möbius.

The volume opens with an excellent exposition by **Paula Stickles** from Millikin University, a small liberal arts college located in Midwest. The article “Using Projects in a Quantitative Reasoning Course” describes project-oriented approach in a course directed to a general population of students. Her approach is motivated by the fact that students need to work on topics relevant to their majors, which should be reflected in class assignments and assessments. Individualized topics relevant to students’ major seem to make a perfect fit for those needs. The appendices contain sample projects. What is worth attention are the ways the projects are made relevant to students. To give an example, a project in financial mathematics that evaluates the down payments of a house contains an actual search through listings so students can find a house of their choice, its price, mortgage rates and then perform calculations for the data they found. Other projects follow a similar pattern and it is quite delightful to see so many creative ideas implemented in class work. The article contains sample projects in the attachments.

The article by **David M. Nabirahni, Brian R. Evans, and Ashley Persaud** with the title “Al-Khwarizmi (Algorithm) and the Development of Algebra” addresses classroom needs for a background material in the history of mathematics. The authors present the person of Al-Khwarizmi who gave a name to the modern word “algorithm” and is considered to be the author of the first known algebra book titled “*The Compendious Book on Calculation by Completion and Balancing*”. During his times Al-Khwarizmi was known as well for his works in geography, astronomy, and astrology, however not all of his books survived till our times. Aspects of this article can be used in high school and college classes to encourage students needs for additional context. Bringing biographies, historical and epistemological remarks to the classroom enriches the lectures by providing human aspect of the mathematical content.

Continuing the idea of bringing history of mathematics and biographies to readers attention, Roy Berglund prepared a book review of a biographical volume “**Emmy Noether**” authored by M. B. W. Tent. Even if Emmy was a daughter of a mathematics professor, she struggled during her life with her interest in an academic area which was quite unusual for a female during her times. We

learn from the book that WW II did not take much toll since Emmy was offered a position in the US; however, her health problems significantly shortened her life.

Bronislaw Czarnocha and William Baker in their “Notes From the Field: creativity kidnapped” discuss insufficient emphasis on creativity in modern education as a national, and possibly international issue. Assessment based on standardized tests scores does not favor creative approaches and simply ignores such unmeasurable skill as creativity. In the second part of the article the authors make a suggestion that they have an idea for a valid assessment of creativity, and they apply it on daily basis in their classes of remedial mathematics.

The second book review in the current issue was written in the honor of **August Ferdinand Möbius** and his influential work in astronomy and mathematics. Living on at the end of eighteenth and beginning of nineteenth century, Möbius experienced changes in Germany motivated by the *Aufklärung*, which had a great influence on his employment and profession.

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Using Projects in a Quantitative Reasoning Course

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Abstract: Many students have trouble communicating about mathematics and connecting mathematics to the outside world. Using projects is a way to encourage them to make the connections. We describe the integration of projects with real-world connections into a quantitative reasoning course.

At our small, private liberal arts institution in the Midwest the faculty have placed a value on quantitative reasoning (QR) for all students. As a result, all students are required to complete a QR course as part of their graduation requirements. There are a variety of courses on campus that meet the requirement, and many students meet the requirement simply by completing requirements for their major. However, there is a population of students who have no such requirement for their major, so they find themselves in the default QR course, Finite Mathematics.

The Finite Mathematics course typically covers four to five topics as determined by the instructor, and all instructors cover financial mathematics. Since students come from areas across campus the students in the course have majors ranging from theater to political science to sports management. With the wide of range of students, it is challenging to meet all of their interests. As a result, the authors decided to take the approach that what is done in class must be related back to the students' lives in some way. What better way to do that than have the students experience the mathematics? Thus, the idea of implementing projects as an assessment of the material was born.

BACKGROUND

One of students' greatest complaints about general education courses is they do not see the relevance of the information to their lives. So, the integration of projects into Finite Mathematics seeks to demonstrate the relevance of the mathematical topics to them. With the creation of each project, great thought has occurred in the selection of the scenario or context for each of the mathematical topics. It is important that not only does the context need to be real and relevant, it is also important that the context is of interest to the students (Middleton & Jansen, 2011; Sanchal & Sharma, 2017). When the context of problems is culturally relevant and relate to personal experiences, students are more engaged (Campbell, Adams, & Davis, 2007; Ukpokodu, 2011; National Council of Teachers of Mathematics, 2014).

For students who have not previously had to do more than regurgitate mathematical procedures, completing a mathematical project can be challenging. Teachers who implement problem-based tasks often encounter initial frustration and difficulties in their classrooms, but they find that the student learning and change in students' mathematical disposition is worth the effort (Marcus & Fey, 2003). In particular, when teachers can connect the mathematics to real-life problems, this helps in the development of student interest (Arthur, Owusu, Asiedu-Addo, & Arhin, 2018; Karakoç & Alacaci, 2015).

THE COURSE AND AN IDEA

In many traditional mathematics courses, a topic is presented and then students are given an examination to determine if the material was learned. While there are written assessments of varying length given throughout the semester in Finite Mathematics, after major assessments (exams) on the topics, a project is given as an alternative assessment form based on a portion of the most recent topic. Each project was assigned in the class meeting following the examination over the topic. Prior to distributing a project, the exams were returned so that students were able to identify any areas of weakness.

The purpose and requirements of each project was explained as well as the way in which the assignment would be assessed. Students were given the opportunity to ask questions to clarify the directions and requirements of the project. Students were then allowed to begin work on the project. There were multiple opportunities to meet with the instructor at designated times for further assistance or clarification of the project. The length and depth of the project dictated the amount of time students were allotted to complete the project. Times ranged from four days to a week.

In any given semester, the authors cover four to five subjects that always include financial mathematics. Additional topics may include statistics, geometry, sports statistics (introduction to data analytics), geometry, voting, probability, and sets. Although the students do not know what the project is until it is assigned, the instructor uses the project as a driving force in preparing materials for class and for motivation for the class to learn the information.

PROJECT EXAMPLES

As the mathematics topics vary between semesters, so do the projects. Also, the details are varied from one semester to another in order to keep them fresh and to avoid the re-use of a previous project. Following are representative examples of some of the projects implemented in Finite Mathematics.

Financial Mathematics

After the financial mathematics unit, the students are assigned a project that focuses on home mortgages. The authors vary the details of the financial mathematics project each semester, but the goal is to have students bring the idea of purchasing a home to life. All students can decide where they would like to live and then select their dream home. Students peruse real estate websites to find a house that is suitable to their liking. Once students decide on a house, they then need to find mortgage rates for a bank around the location of the house. The project includes finding interest rates for a traditional 30-year fixed rate loan as well as a shorter-term loan (10, 15, or 20 years). Once students have gathered the information from the bank(s), they use it to determine the monthly payment, total cost of the house over the life of the loan (excluding inspection costs, closing fees, etc.), and the interest paid for each interest rate at each bank. They then repeat the process of finding the amounts based on a fixed-percentage down payment given by their instructors. In addition, the students determine the down payment amount, the amount to be financed, and the total cost of the house (including the down payment and interest). (See Appendix A.)

The students submit all their calculations as well as a written response. The write up is to include a short description of the house including the location and why the student selected it. Students are to be specific as to why they selected the location, layout/size of the house, etc., and discuss the features that caught their eye. Then, they identify what bank(s) they used for their loans and why they selected it (them). Their written response is to include the monthly payment amount found for the various interest rates and loan terms.

Approximately half the points for the project are based on correct calculations, 20% on a complete write up, 10% on correct and appropriate spelling/grammar in the write up, and the remaining approximately 20% is submitting the various required elements such as a printout of the house listing, bank interest rates, etc.

Statistics

After the completion of the statistics unit, the students are assigned a project where they collect data, organize it, and draw a conclusion. Students are tasked with the idea of being a business owner and opening a local eatery. (This class has a high population of business majors, so this is a popular idea.) Students create survey questions to sample the local population regarding their food preferences. They determine what sampling method to use as well as how they are going to present the survey (hard copy, digital, etc.). Once the students have collected their data, they must organize it to make sense of it to draw an informed conclusion. (See Appendix B.)

The students submit a write up in a proposal format to the local city council in an effort to obtain the required permits for opening the establishment. Within the write up they are to discuss their survey questions, how individuals were surveyed, question results, and a conclusion. They are to

include a discussion of the sampling method and a justification for the conclusion as well as at least two (total) graphs or tables displaying the statistical data.

Approximately 47% of the project is based on a complete write up with the required information and correct grammar/spelling, 20% on complete and correct graphs/tables, and the remaining is completing the required elements such as including at least three survey questions, surveying at least 30 individuals, etc.

Geometry

The geometric ideas covered in the course are not anything beyond what the students have seen before. Basic area, volume, and surface areas are reviewed with standard and nonstandard shapes. After the completion of the short unit, students revisit the idea of owning a home that might not be exactly what they want. As a result, the necessity of redoing the floors in the home arises. To reduce faculty workload and since not all home real estate listings include floor plans (or some rooms do not have dimensions), the students are given several different floor plans to choose from. (See Appendix C.)

Once students have selected their desired floor plan, they must determine how much flooring is necessary for each room. They then must decide what type of flooring to place in each room (carpet, tile, vinyl, hardwood, etc.). Once they have made these decisions, they must seek out cost information for the various types of flooring. They can do this from a variety of sources whether it be a big box do-it-yourself home store or a direct sell flooring company. Once they have gathered the information they use it to determine the cost of re-flooring each room as well as the total cost of re-flooring the house.

The students submit all their calculations as well as an itemized cost per room. The write up students submit must include a short description of which floor plan they chose and why they selected it. They must also identify where they got the pricing information for the flooring and include the total for redoing all the floors in the house. In the write up, they are also required to explain why they selected the type of flooring for each room (carpet versus hardwood, etc).

Approximately half the points for the project are based on correct calculations, 25% on a complete write up with correct spelling and grammar, and the remaining approximately 25% is submitting the various required elements such as printouts of the various flooring types.

STUDENT REACTION TO ASSESSMENT

At the end of a recent semester, the authors administered a survey asking students for their feedback on the projects. Despite a small sample ($n = 23$), the students' reactions were positive. Students offered comments such as follows:

- *It's a fun way to end the lesson.*

- *It gives students more practice and real world examples.*
- *I felt like it was a nice, relaxing activity after the test.*
- *The help of doing projects can knock out at least one bad exam grade.*
- *I strongly believe these help me learn more than studying for a test.*

In addition, students were asked about the timing of the assignment of the projects as well as the project weight in their overall grade. (The projects collectively were equivalent to one exam grade.) Over half the class indicated they preferred having the projects after an exam as opposed to before the exam. Over 25% of the students indicated it made no difference. Approximately 44% of the class felt the projects were weighted appropriately in their final grade whereas about 35% felt the projects should be equivalent to two exams.

Collectively, the students performed better on the projects than on the exams. This is not surprising as the projects were more focused on a smaller portion of a topic, they had more time to work through the project, and students could get assistance while working on the project. There were three students out of the 26 students in the class who did not complete all the projects. They were removed from the data set for exam and project grade comparisons for an n of 23. The overall exam average was approximately 84.1% while the overall project average was 89.7%. This was a statistically significant difference ($z = 2.127, p < 0.05$).

CONSLUSION

Despite some initial objections to having to write in a mathematics course, the projects have been well received in the Finite Mathematics course which is in line with Marcus and Fey's (2003) work. The projects continue to evolve from semester to semester as loopholes are identified in instructions or more emphasis or clarification is given to certain criteria. As students offer feedback as to what they find interesting or frustrating about the projects this information is reflected on and taken into consideration when creating a new version of the project. Additionally, the authors found that some students who struggled to assimilate large portions of information in preparation for exams were able to succeed with projects because of either a smaller amount of material or due to the real-life nature of the project.

It is hoped that other instructors of quantitative reasoning courses (and other courses) will incorporate projects in some form into their classes. Specifically, in a course that is considered a general education requirement, the real-life project give students the opportunity to view mathematics in context, and hopefully, come to appreciate the everyday use and need for mathematics.

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Appendix A, Financial Project

What is Your Dream?

It is time to find your dream home! You have no limit on your budget or location. So, think of what you want and go for it. You will determine your monthly payment, total cost of the house over the life of the loan (excluding inspection costs, closing fees, etc.), and the interest paid.

Nuts and Bolts

- Find your dream house and print out one page that contains the listing (with the price).

Highlight the price.

- Locate a bank (located in the same area as the house) and determine the interest rate for a fixed rate loan for 30 years. Print out this page, and highlight the bank name and interest rate.
- Locate a bank (located in the same area as the house) and determine the interest rate for a fixed rate loan for 10, 15, or 20 years. Print out this page, and highlight the bank name and interest

rate. *You may use the same bank for both interest rates.*

- * Determine the monthly payment, total cost of the house over the life of the loan (excluding inspection costs, closing fees, etc.), and the interest paid for *each interest rate at each bank*.

- * For each interest rate at each bank, assume you have the money to make an 18% down payment. Determine the down payment amount, the amount to be financed, the monthly payment, cost of the house over the life of the loan (excluding inspection costs, closing fees,

etc., AND down payment), the interest paid, and the total cost of the house (excluding inspection costs, closing fees, etc., but including down payment).

Your write up will include a short description of your house including the location and why you selected it. Be specific as to why you selected the location, layout/size of the house, etc. Discuss the features that caught your eye. Then, identify what bank(s) you used for your loans and why you selected it (them). Include the amounts you found for the two * items above. The paper should be typed and double-spaced using a 12-point font and 1-inch margins although the mathematics may be handwritten and attached. Grammar and spelling count.

Appendix B, Statistics Project

Sampling Satisfaction

You want to open a local eatery but there has been an influx of food businesses of late so you as a business owner must justify to the local city council why you should be given the required permits to open your eating establishment.

You are not sure what kind of food eatery to open. To be an effective owner you need to find out the public's opinion. You decide to survey the local public to see what their favorite food is and how often they go out to eat. You elect to create a survey to administer to a sample of the local population to find out their food preferences. The questions are up to you.

Your write up should be in a proposal format to the local city council. Within the write up you should discuss your survey questions, how individuals were surveyed, question results, and a conclusion that includes a discussion of the sampling method and a justification for the conclusion, as well as *at least two* (total) graphs or tables displaying the statistical data. The write up should be typed and double-spaced using a 12-point font and 1-inch margins. (The write up should be 1 – 2 pages.)

Nuts and Bolts

- Determine your sampling method and survey at least 30 individuals. Attach the survey results from all individuals. (You are familiar with five sampling methods. The choice is yours.)
- Attach any questionnaires that you may have used during the survey. (For example, if you gave any form to participants or showed them anything.)
- The survey must contain *at least* three questions.
- The graphs or tables may be generated from a website or Excel.
- Make a decision as to the type of eatery to open based on the sample data. Be sure to justify this decision in your proposal based on the survey results.

Appendix C, Geometry Project

Designing Your House

You've found your home, but before you move in you want to do some work. In order to ensure the final product is what you want, you will handpick all the materials.

Nuts and Bolts

- Determine which of the floor plans you will use for “your house”. Be sure to state this.
- For each room, determine how much flooring is needed for that room.
- For each room, determine whether you want hardwood flooring, tile, vinyl, or carpet.
- For each room, determine *specifically* what flooring you will use and identify the cost of the flooring. Compute the cost of flooring for the room.

****You must use at least three different (priced) floorings.**

*** Include the computation of the flooring needed for each room.**

*** Include an itemized list (typed) of the rooms with the cost for each room.**

*** Include a printout of *each* flooring used. Highlight the price and the name of the business.**

On the printout, identify what room the flooring is being used in.

Your write up should include a short description of which floor plan you chose and why you selected it. Identify where you got your pricing information for the flooring, and include the total for redoing all the floors in the house. In your write up, explain why you selected the type of flooring for each room. Do not forget the itemized cost list of the rooms. The paper should be typed and double-spaced using a 12-point font and 1-inch margins although the mathematics may be handwritten and attached. Grammar and spelling count.

Al-Khwarizmi (Algorithm) and the Development of Algebra

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Abstract: *The purpose of this article is to provide some background on the life and contributions of Muhammad ibn Musa al-Khwarizmi to the development of algebra, in particular. The authors would like to share this information with algebra teachers at the high school and college level because mathematics history has the potential to engage students and provide a “human face” to the subject. Consequently, this may lead to higher achievement for students. This article contributes to mathematics history and algebra.*

INTRODUCTION

Muhammad ibn Musa al-Khwarizmi *aka* Algorithm was an intellectual from the 8th through the 9th centuries who contributed significantly to the development of algebra (Evans, 2014). Al-Khwarizmi was born in 783 CE in Khwarizm in then Persia of Southwest Asia, which is presently the city of Khiva in Uzbekistan (Aksoy, 2016; Stewart, 2017). Al-Khwarizimi was highly influential in the development of algebra throughout Southwest and Central Asia, North Africa, and Europe, which thus subsequently influenced the development of algebra and medieval and modern mathematics throughout the world. It would serve any high school or college algebra instructor very well to be able to impart some of this historical knowledge to students in the classroom. The authors are interested in sharing background on al-Khwarizmi with other mathematics teachers since it is believed this will make mathematics more interesting and engaging for students, which may lead to higher level of achievement for students along with an increased appreciation for the contributions of Persian mathematics.

HISTORICAL BACKGROUND

As the Islamic Caliphates of Baghdad had only invaded the Khwarizm for mass conversion to Islam just a few years before his birth, and as in part evidenced by al-Khwarizmi's father's name Musa in then a predominantly Zoroastrian Persian world, some speculated him to be of Mizrahi Persian Jewish heritage. Moreover, although the newly established Islamic Omayyad followed by Abbasid caliphate dynasties had overshadowed the Persian and Egyptian Empires, and the Iberian peninsulas in Europe, and thus as a result Baghdad had become the center, and the Arabic language had become the *lingua franca* as English is today; nonetheless, the use of the Islamic world is a misnomer. It is true the Persians had by and large become Muslim but had and still continue to retain their historical multiethnic Persian/Iranian identity. Al-Khwarizmi's early studies took place in various prominent madrasas called Khiva, Kiat, and Gurgench (Aksoy, 2016). Al-Khwarizmi was able to study texts in multiple languages including Persian, Arabic, Syrian, and Sanskrit (Aksoy, 2016). The topics al-Khwarizmi studied in depth and later contributed toward their advancement were geometry, astronomy, and geography among others (Aksoy, 2016). Eventually, he became widely known as a top scholar of the region of Khorazm (Aksoy, 2016).

Al-Khwarizmi lived during an era called the Golden Age of Islam, between the 8th and 13th centuries (Evans, 2014; Faruqi, 2015). Contributing to the Persian Golden Age, al-Khwarizmi led a very pious and intellectual religious life (Faruqi, 2015). By 830 CE, he began studying under the reign of Caliph al-Ma'mun, who ruled from 813 to 833 CE (Ramsden, 2017). Al-Khwarizmi's studies took place in the *Bayt al-Hikma*, or the House of Wisdom, in Baghdad (Stewart, 2017). The House of Wisdom, founded by al-Ma'mun, contained a research library and observatory where many scholars studied (Aksoy, 2016). Al-Ma'mun envisioned the House of Wisdom as a place where all Greek texts could be translated into Arabic and or Persian (Aksoy, 2016). Baghdad became a central location for science and trade, which attracted many scholars mostly from Persia and as far as China and India (Stewart, 2017).

Al-Khwarizmi worked with many prolific mathematicians and scientists in the region, such as the Banu Musa brothers, also known as Sons of Moses, and Al-Farghani (Aksoy, 2016; Ramsden, 2017). He worked with others to calculate the circumference of the Earth and determined that the Earth was spherical. This conclusion was soon accepted by the scientists of this time (Aksoy, 2016).

Around 825 CE, al-Khwarizmi wrote the manuscript, *al-Jam' wa al-Tafrīq bi Hisāb al-Hind*, translated as *The Book of Addition and Subtraction According to the Hindu Calculation* (Evans, 2014; Gillispie, Holmes, Koertge, & Gale, 2008; Stewart, 2017). The text was al-Khwarizmi's most influential work and remained so for the several hundreds of years that followed (Stewart, 2017). During the medieval European era, the text was translated into Latin and shared widely because it showed Europeans a novel method to carry arithmetic operations. The original version of this text has been lost, but the Latin version, *Algorithmi de Numero Indorum*, still exists (Gutstein & Peterson, 2013). In this work, al-Khwarizmi introduced the decimal system that was created by Hindu mathematicians in the 6th century and added zero to the system to complete it (Aksoy, 2016; Baharuddin & Wan Abdullah, 2014). The concept of zero as both a number and placeholder should not be taken lightly. The concept of zero may be one of the most important ideas in mathematics as related to numeral systems. Evidence can be found for the development of zero both in Indian and Mayan mathematics with the Indian concept influencing the work of al-Khwarizmi and other scholars of the time.

Al-Khwarizmi explained how to add, subtract, multiply, and divide using this numeral system (Gillispie et al., 2008). Al-Khwarizmi provided solutions as sequenced steps, thereby introducing the concept of the algorithm and thus leading to the creation of the word algorithm the precursor for today's computing (Aksoy, 2016). The word "Algorithmi" in the Latin title of the work became known as "Algorismi" and the mathematical methods using the numeral system as described in the work became known as "algorithms" (Stewart, 2017). Among the Europeans, the phrase "dixit Algorismi," or "thus spoke al-Khwarizmi" became an arguing point in mathematical disagreements to send the message that any words written by al-Khwarizmi are final, true, and must be followed, not argued against (Stewart, 2017).

Around 830 CE, al-Khwarizmi wrote another significant mathematical work in an effort to give the Persian hemisphere and Muslims mathematical aids to solve issues of inheritance, partition, lawsuits, legacies, and trade (Stewart, 2017). His work, *al-Kitāb al-mukhtasar fī ḥasab al-jabr wa-l-muqābala*, translated as *The Compendious Book on Calculation by Completion and Balancing*, is considered to be the first book on algebra (Rashed, 2015; Stewart, 2017). At this time, algebra was solely the operation of restoring an amount that was subtracted when solving for an unknown (Hannah, 2015). Al-Khwarizmi began the book by writing about six algebraic equations of the first and second degrees in which linear and quadratic equations can be reduced in order to find a solution (Gillispie et al., 2008). The solutions for

reducing any problem to these equations involve the operations of balancing and completion (Gillispie et al., 2008). A geometric model for a quadratic case is that x^2 can be thought of as a square with side of x units. More complex quadratic equations can also be thought about from a geometric perspective by added additional rectangles such as $2x$ with length of 2 units and width of x units to model $x^2 + 2x$.

The word “al-muqabala” in the title of the book referred to the operation of balancing. The word algebra comes from “al-Jabr” (literally means to *enforce*) in the title of the book, which al-Khwarizmi used to describe the operation of completion (Aksoy, 2016; Faruqi, 2015). In the second part of the book, al-Khwarizmi writes about mensuration. He provides steps for solving the area of plane figures such as the circle and solving the volume of solids such as the truncated pyramid (Gillispie et al., 2008). The third part of the book is the longest and consists of solved problems regarding legacies. The solutions involve arithmetic and simple linear equations. However, knowledge on Islamic juris prudence on inheritance laws is needed in order to understand the problems (Gillispie et al., 2008).

In conjunction with the algebraic solutions of the equations in al-Khwarizmi’s book, there are geometric proofs (Aksoy, 2016). Al-Khwarizmi was the first scholar to provide geometric proofs to quadratic equations (Aksoy, 2016). Al-Khwarizmi’s original ideas in this text, such as those pertaining to geometry, are believed to be inspired by Euclid’s *Elements* (Ramsden, 2017). In all of his mathematical texts, al-Khwarizmi did not use the standard mathematical symbols we use today (Stewart, 2017). Expressions and solutions were described verbally in sentences without symbols. Al-Khwarizmi used the word “unit” to describe a number, “ x ” or to describe a root, and the word square “Morabba” to describe x^2 (Aksoy, 2016). For example, the equation $x^2 + x = 12$ would be expressed as “square plus root equals twelve units” (Stewart, 2017, p. 31). In his equations, al-Khwarizmi only used whole numbers, and the solutions only included positive numbers (Aksoy, 2016).

Not only is al-Khwarizmi recognized for his seminal contribution to mathematics, but that he is also regarded for his contribution to the field of astronomy. Al-Khwarizmi was one of the first scholars to draw the world map and create an astronomical table (Baharuddin & Wan Abdullah, 2014; Faruqi, 2015). His schematic tabulation was used to find the positions of stars and planets using calculations (Faruqi, 2015). Al-Khwarizmi’s most famous astronomical work is *Zij al-Sindhind*, translated as *Astronomical Table of the Sindhind* (Gillispie et al., 2008; Stewart, 2017). The table was written around 820 CE and was in part based on the Indian methods of studying astronomy (Stewart, 2017; Yazdi, 2011). The text includes astronomical tables for determining the magnitudes of solar eclipses and finding solar and lunar velocities and apparent diameters (Yazdi, 2011).

Since al-Khwarizmi was interested in astronomy, he wrote a treatise on the Jewish calendar titled *Risala fi istikhraj ta’rikh al-yahud*, translated as *Extraction of the Jewish Era* (Gillispie et al., 2008; Stewart, 2017). In this text, al-Khwarizmi explains features of the Jewish calendar such as the 19-year Metonic cycle, the process of determining which day Tishri should fall on, and the steps for using the calendar to figure out the mean longitude of the sun (Gillispie et al., 2008). The text also includes a calculation of the time period between the Jewish era and the Seleucid era (Gillispie et al., 2008). It took three more centuries when another Persian mathematician-philosopher-poet Omar Khayyam corrected the solar calendar called *Jalali* down to 365 days and six hours (Akrami, 2017). Al-Khwarizmi also wrote two texts on the astrolabe, *Kitab ‘amal al-asturlab*, translated as *Book on the Construction of the Astrolabe*, and *Kitab ‘amal bi’l-asturlab*, translated as *Book on the Operation of the Astrolabe* (Gillispie et al., 2008).

Al-Khwarizmi's geographical text, *Kitab surat al-ardz*, translated as *Book of the Velocity of the Earth*, was written in 833 CE (Stewart, 2017). In this text, al-Khwarizmi provided latitudes and longitudes for 2,402 places and divided the places into six sections which are seas, mountains, islands, rivers, cities, and regions. In each section, all of the places are arranged within certain climata (Gillispie et al., 2008; Ramsden, 2017). The climata are seven sections of the world divided by longitude as seen in ancient Greek works (Gillispie et al., 2008). Al-Khwarizmi discerned information from Ptolemy's geographical work, but al-Khwarizmi's geography on Southwest Asia was more detailed and more accurate than Ptolemy's work (Gillispie et al., 2008; Ramsden, 2017). Ptolemy's length of the Mediterranean Sea was too long, and al-Khwarizmi corrected it (Stewart, 2017). Ptolemy surrounded the Atlantic and Indian oceans by land, portraying them as seas. Al-Khwarizmi in contrast, did not border the oceans by land (Stewart, 2017).

Al-Khwarizmi wrote another compendium that did not survive called *Kitab al-ta'rikh*, translated as the *Chronicle*; it was a record of events that took place in Southwest Asia in accordance to astrology (Gillispie et al., 2008). It has been noted that al-Khwarizmi used the astrological methods of this text to figure out the hour in which the prophet Mohammed was born according to the astrological events of his life (Gillispie et al., 2008). Another text that did not survive is *Kitab al-rukhnama*, translated as *On the Sundial* (Gillispie et al., 2008). Only the title of this text is known, but the subject of the title appears to match al-Khwarizmi's interests (Gillispie et al., 2008).

CONCLUSION

Al-Khwarizmi lived until 850 CE, but his work lived on much longer (Evans, 2014; Ramsden, 2017). Other Persian philosophers who followed al-Khwarizmi and his work were: Farabi Omar Khayyam, Nasir-E-Din Tusi, Avicenna, Razes, Averroes, Biruni, and al-Kindi, to name a few. There are few historical figures who carry as much influence on the development of mathematics, and nearly none on the development of algebra, as al-Khwarizmi had achieved.

The first two authors are faculty in the chemistry and mathematics departments at the college level, respectively, and both integrate the life of al-Khwarizmi into their own instruction. Additionally, the second author teaches the history of mathematics and spends time connecting the development of algebra to al-Khwarizmi. It is recommended that high school and college instructors of algebra incorporate some of the historical knowledge within instruction in order to motivate students, to inspire them to feel its relevance, and to place a human face on the subject; otherwise, many young students do not directly correlate the relevance of mathematics to their own lives.

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Book Review for *EMMY NOETHER – the mother of modern algebra* by M.
B. W. Tent

Roy Berglund

City Tech, Borough of Manhattan Community College of CUNY

Title: Emmy Noether

Author: M. B. W. Tent

Publisher: A. K. Peters, Ltd.

Publication Date: 2008

Number of Pages: 177

Format: Hardcover

Special Features: Glossary

Price: \$29.00

ISBN: 978-1-56881-430-8

Reviewed on April 26, 2019

The story of Emmy Noether is one of “little unknown girl makes good and becomes famous”. Not a lot is known about Noether’s early life growing up, but it is abundantly clear that she was blessed with strong family ties in secure circumstances. Such an environment tends to foster success for the individual, whenever one or both parents are supportive and expose the child to positive cultural and educational influences. The author chose to describe Noether’s biographical life rather than her mathematical accomplishments. In consequence Noether’s achievements are lightly mentioned, but they are presented without any mathematical details. It would be illuminating to follow Noether’s thinking through her discoveries and to track her development to mathematical maturity.

Emmy’s father, Max, was a mathematics professor in the college at Erlangen. Both parents tended to channel their daughter into preparation for domestic life as a wife and mother. However, Emmy was bright child, and solved puzzles and riddles of all kinds from an early age. She continually questioned her designated role as a girl, and queried her father why she, too, should not be a mathematician, ever resisting the rote of learning French, or of a musical instrument. One day she asked her father “what is algebra?”, and he provided her with some elementary examples, which stimulated her interest. They agreed that she could learn algebra. Emmy attended a girl’s school

(Tochterschule), but she wanted to learn what her brothers were learning. Her father's colleague at his college, Professor Paul Gordon, encouraged her study of algebra, and suggested that she could audit courses at the college. Generally, women were not permitted to attend university in nineteenth century Germany.

Noether earned a teacher's exam to teach English and French, but she wanted to teach mathematics and go on to receive an advanced degree from Goettingen. The success of mathematicians Kovalevsky in Poland and Young in England was an inspiration to her. Emmy was especially eager to follow Klein's work in formalizing algebra into an axiomatic subject. For the present she had to obtain her Ph.D. at Erlangen under her father and Paul Gordon and was the first young woman to do so.

Her work was entirely conceptual: commutative rings, set theory, and symmetry dealing with algebraic objects rather than specific equations, or numerical examples. Noether also worked with Ernst Fischer at Erlangen. Her parents proposed that she and Fischer become personally interested in one another, but Emmy declined. She apparently accepted that she was not gifted with good looks or many social graces, preferring to reserve all her energy and effort for mathematics. Her father intervened and brought her to Goettingen to visit with Hilbert and Klein. Hilbert obtained for Emmy an unpaid teaching position at Goettingen under his name. By 1915 Emmy became known as a serious mathematician in her own right, after her work in commutative algebra. During and after World War I, German scholars became increasingly isolated from academia in other countries.

Noether's work on Einstein's general theory of relativity helped formulate the mathematical basis for the theory. Also, she edited the mathematics journal "Mathematische Annalen". During this period Emil Artin came to Goettingen and studied with Emmy. She then obtained a position of Extraordinarius at Goettingen without salary but was able to mentor doctoral students. One of her post-doctoral students was Van der Waerden. P. S. Alexandroff came to Goettingen from Russia in 1923 to lecture, and to work with Noether. The period of the late 1920s and early 1930s was a wonderful one for the community of professors and students. At this time Emmy was enthused by Soviet Socialism in principle.

During the 1930s German politics changed when the Nazis took over the country, and a number of professors were expelled because of their ancestry. Fortunately for Noether, she was offered a position at Bryn Mawr College in Pennsylvania, when she realized she could no longer remain in Germany. Success was her companion for her remaining years at Bryn Mawr until April 1935 when she started to feel ill and was operated on for an ovarian cyst, dying from the complications of surgery at age 53. The world was deprived of great woman mathematician.

Emmy Noether-the Mother of Modern Algebra is an enjoyable book and quite readable by non-mathematicians. Younger school-age readers might be stimulated to learn more about the subject.

Notes from the Field: creativity kidnapped. Discussion essay

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Abstract: *The essay discusses the process through which the subject of creativity has been limited in scope by the imposition of external to creativity professional standards. The primary focus of creativity research in mathematics is on the relationship of creativity with giftedness, which habitually is determined by SAT and other scholastic scores. This focus eliminates attention to the creativity of the ‘rank and file’ learners to the degree that almost nothing is known about creativity of ‘normal’ students, and in particular, about the creativity of underrepresented. That’s the content of Creativity Kidnapped. The second part of the essay offers the avenue for Liberation of Creativity, which is anchored in the concept of bisociation by Koestler, (1964), that is in the creativity of the Aha!Moment. Aha!Moment is widely known, common experience of creative insight.*

SECTION 1. INTRODUCTION

The presented essay is the outgrowth of the recent work of the Teaching-Research Team (TRT) of the Bronx focused on preparing the conference grant Creativity in STEM for the NSF INCLUDES program (Dear Colleague Letter 17-111). The NSF INCLUDES program is a recent progressive initiative of NSF whose aim is to increase participation of women, minorities and working class students in STEM industry. After two yearly solicitations, the program’s portfolio has around hundred successful awards for the pilot projects from around the nation. Each pilot project explores a new avenue of teaching and/or organization which hopefully will lead to the increase of participation of ‘underrepresented and underserved’ in STEM careers.

The TR Team of the Bronx has been exploring the creativity of Aha!Moment in mathematics classrooms for the last 8 years. Our attention was attracted to the NSF INCLUDES portfolio of awards because none offered mathematical, scientific or engineering creativity as a fundamental motivational and cognitive component for the dreamt about increase of participation. 8% of the projects listed creativity in their abstracts as an interesting by-product of the project. The dissonance between the absence of attention to STEM creativity by educators and the explicit need has been expressed more often by the industry as:

- ARMONK, NY - 18 May 2010: According to a major new IBM (NYSE: IBM) survey of more than 1,500 Chief Executive Officers from 60 countries and 33 industries worldwide,

chief executives believe that – **more than rigor, management discipline, integrity or even vision** –successfully navigating an increasing complex world **will require creativity**.

- Jul 31, 2015 *Creativity in Engineering*, SpringerLink noted that Creativity is concerned with the generation of effective and novel solutions to problems. Engineering is concerned more specifically with generating technological solutions to problems. ... **Engineering, in short, is fundamentally a process of creative problem solving**. Cropley (2015) asserts that, “*Because creativity is concerned with the generation of effective, novel solutions, creativity and engineering are, in essence, two sides of the same coin*”. (p.2).
- Cooper and Hevearlo (2013) inform that according to the National Academy of Engineering, students need to begin associating the possibilities in STEM fields with the need for creativity and real-world problem-solving skills.

offered us rich food for critical thought which produced remarks below.

The approach we take here is aptly characterized by mathematics researchers as critical mathematics pedagogy, which is an approach to mathematics education that includes a practical and philosophical commitment to liberation. (Tutak et al, 2011) Approaches that involve critical mathematics pedagogy give special attention to the social, political, cultural and economic contexts of oppression, as they can be understood through mathematics (Frankenstein, 1983). They also analyse the role that mathematics plays in producing and maintaining potentially oppressive social, political, cultural or economic structures (Skovmose, 1994). Critical mathematics pedagogy demands that critique is connected to action promoting more just and equitable social, political or economic reform.

Thus, in Section 2 we describe the process by which educational profession had ‘kidnapped’ the knowledge about creativity for the benefit of giftedness, which is traditionally defined by the high scores in high stake tests such as SAT scores. The kidnapping of creativity left the researchers and practitioners without an approach which can address the creativity of “underrepresented and underserved”. The Section 3 focuses on the process of liberating creativity with the help of bisociation theory of Arthur Koestler, as the theory of “creativity of and for all”.

SECTION 2. CREATIVITY KIDNAPPED

The assault on creativity started in 2001 when Anderson, L & Krathwohl, D. with their team published the revised Bloom taxonomy based on the new research conducted between 1995-2000. The process resulted in significant changes in the original Bloom (1956) taxonomy. The comparison of the two is shown in Fig. 1 below. There are three essential changes in the revised taxonomy: (1) use of verbs instead of nouns, (2) changing Synthesis to Creativity and (3) the change of order between from Synthesis → Evaluation in the original Bloom’s taxonomy to the Evaluation → Creativity in the revised one.

One could argue that the priority of verb (or process) over the noun (the product) is equally biased as the priority of noun over the verb we see in the original taxonomy; both are necessary for the

full description of the cognitive content. We know, following Sfard (1992) that the object/process duality is one of central concepts in learning algebra.

However, the main issue as it relates to kidnapping creativity is not in the point (1) nor (2) - creativity often involves, of course, synthesis. The kidnapping process starts in the point (3) – the change of order between Evaluation and Synthesis/Creativity in the revised taxonomy. Note that Evaluation in the original taxonomy, which comes after Synthesis is characterized by the statement “*Judge value of [obtained] material for a given purpose*” while the evaluation in the revised one introduces a different element, significantly, before reaching creativity of the pyramid: “*make judgements based on criteria and standards*”. In other words, whereas Synthesis/creativity was the matter solely between the creator and the purpose of the creation in Bloom’s taxonomy, the revised taxonomy imposed the limitation upon creativity by requiring it to be in agreement with professional standards and criteria outside of the creative process. The adherence to the standards and rules was placed in advance of promoting learner’s thinking to the creative top of the revised taxonomy. These extraneous standards or criteria often serve as a filter.

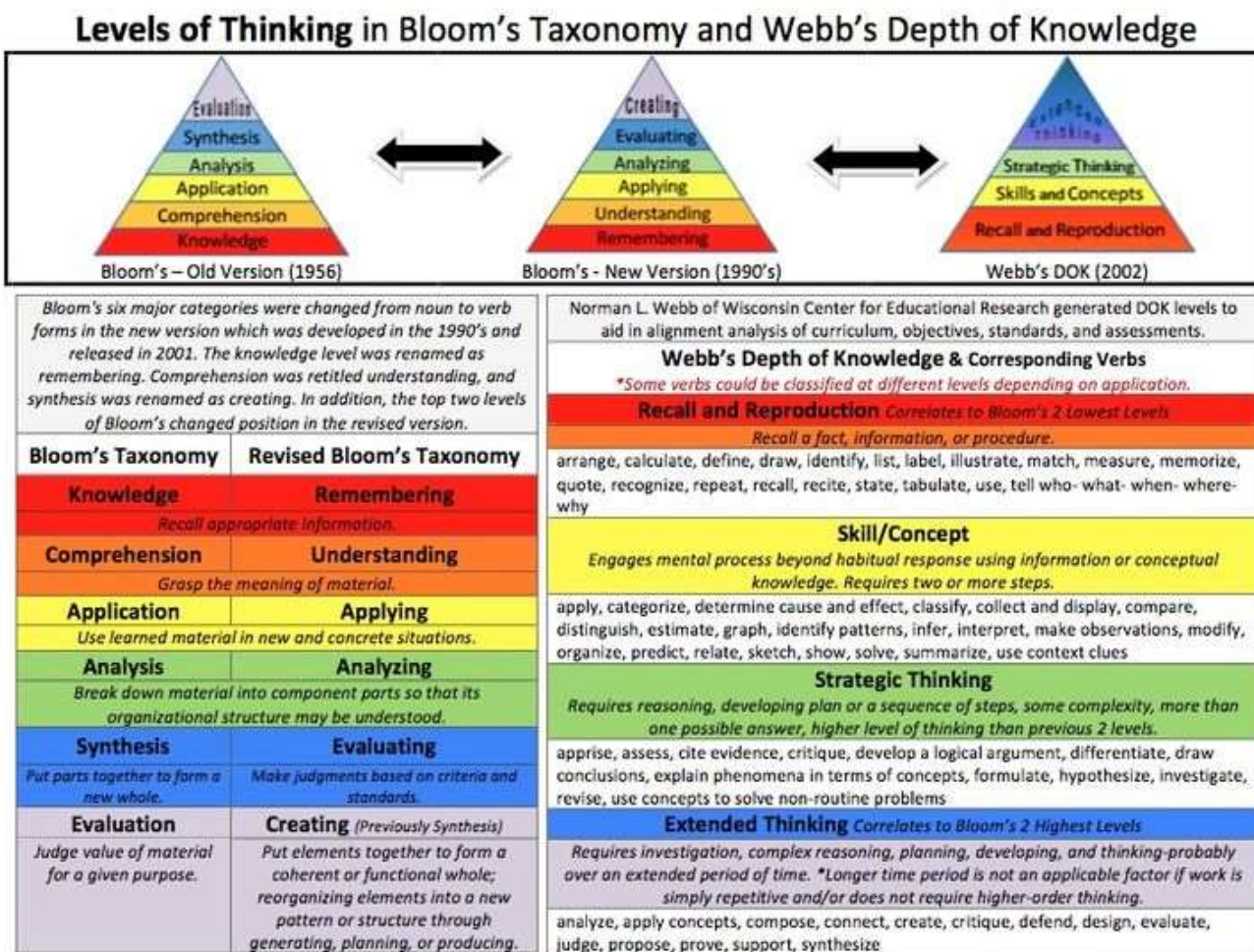


Fig. 1

How does this change impact our own profession of mathematics creativity research?

The creativity research in mathematics education has been focused on the relationship between creativity and giftedness. The majority of recent publications on creativity (Leikin and Koichu, 2009; Sriraman and Lee, 2011; Leikin and Sriraman, 2016) emphasize the creativity of the gifted. According to Wagner and Zimmerman (1986) giftedness is identified by high achievement in two tests: The Scholastic Aptitude Test (SAT) and the HTMB, a set of seven problems designed especially for talent search.

We see here clearly how the role of standards external to creativity is impacting research on creativity in mathematics education.

The work of Sriraman (2004) confirms the SAT paradigm in his choice of four gifted students for research on the creativity of their giftedness by pointing out that all of them scored in the first one percentile of the Stanford Achievement Test. This means that giftedness is traditionally found among students who are very good in school measures of achievement in mathematics. This approach filters out those students who might be creatively gifted, but who are not fluent in mathematics language and procedures. What does it mean to be mathematically creative, but not fluent in mathematical language? It means finding oneself in a remedial classroom filled with many other talented students who either don't know their own creative gift or are destroying it under emotional or material poverty stress. And that's why Prabhu (2016) notes on the basis of her teaching-research experience that *"the creativity in teaching remedial mathematics is teaching gifted students how to access their own creativity."*

How is creativity being blocked here in relationship to classroom instruction? There are at least two blocking spots. The first is within the realm of critical thinking. A lot of what we do in the classroom is supporting the interaction of ideas between students. In such a situation the importance of peer interaction or teacher student interaction may outweigh the need to verify if the ideas considered are up to professional standards. The second place I can see this interference is when you are trying to get students to involve their spontaneous intuitive knowledge and bring it to bear on a problem this requires conscious thought on semi-conscious actions. This process is helped along by critical analysis especially as the situation presented moves out of their intuitive comfort zone.

Keyung Hee Kim (2012; 2008) has made recently a significant and relevant observation of the decrease in Creative Thinking Scores on the Torrance Test of Creative Thinking. The significant decrease of Fluency and Originality scores was observed between 1990 and 2008. The largest decrease was for the kindergarteners through third graders; the second largest decrease was for four through six graders (p.292). These very students have now become the majority of the population entering community colleges across the nation. Kim (2008) reviews the studies and theories that have shown that once underachievers are placed in an environment that fosters their creative needs with motivation, mentors, understanding, freedom, and responsibility, they can become highly productive. She notes that *"many gifted students are underachievers and up to 30%*

of high school dropouts may be highly gifted.” Naturally that percent is higher in the community colleges where former high school dropout students enter through GED exams. Thus, similarly, to Prabhu (2016), Kim suggests that once “underrepresented” are surrounded by the creative learning environment, their motivation for learning and understanding of STEM disciplines increases.

The imposition of professional standards upon the definition of creativity have eliminated a sizable sample of creative ‘rank and file’ students from attention of our professional research. In fact, Sriraman et al (2011) point out understanding “*The role of creativity within mathematics education with students who do not consider themselves gifted is essentially non-existent* (p.120).” Chamberlin (2013) asserts that “*Missing is information on what initiatives are in place to develop and facilitate mathematical creativity in underserved and under-identified populations. This type of discussion would be informative to the field of gifted education and counter the criticism that field is not inclusive.* P.856)”. We not only don’t know the nature of creativity of ‘rank and file’ students, but we also don’t have tools to investigate it. The tools developed in the context of mathematical giftedness cannot fit the tools needed here (Czarnocha et al, 2016).

SECTION 3. CREATIVITY LIBERATED

The act of creation is

The act of liberation:

The defeat of habit by originality.

Arthur Koestler, *The Act of Creation*

The search for the appropriate approach to creativity which can take the creativity of the “underrepresented” into account, or the search for “creativity of and for all” is on.

Our own team, the Teaching-Research Team of the Bronx has focused attention on the commonly known Aha!Moment called also Eureka experience as the manifestation of creativity in our classes and in their communities. Vrunda Prabhu (2016) coordinated the bisociation theory of Aha!Moment Koestler (1964) with classroom events in remedial mathematics, showing, together with Liljedahl (2013), its high positive motivational and cognitive value. The definition of bisociation as *the spontaneous leap of insight which connects two or more unconnected frame of reference*, (Koestler, p.45) and makes us experience reality along two planes at once, offers hints how to facilitate Aha!Moment (Czarnocha and Baker, 2016) and how to measure the depth of knowledge (DoK) reached during the moment of insight (Czarnocha, 2018). The DoK assessment is based upon the coordination of bisociativity with theories of learning by Piaget, in particular, PG Triad of conceptual development (Piaget and Garcia, 1987). Baker (2016) had demonstrated that two central components of reflective abstraction, interiorization and transcendence are reachable, in the context of problem solving, through the insight of Aha!Moment.

Interiorization defines the distinction between students who appear to understand during the social classroom learning experience and those who do not internalize and hence suffer from the so called ‘next day effect.’

The question here is how to free creativity from extraneous standards and criteria. Focus on the nature of the creative insight suggests itself as helpful in that very respect. Since the intrinsic quality of such an insight is connecting unconnected frames of reference or matrices of thought, that is building a schema of thinking, the natural approach is to look upon it from the viewpoint of the development of the schema. That’s why PG Triad as the theory of schema development is so useful as a DoK assessment tool. It measures content of insight – the change in understanding during the ‘leap’ of insight.

The analysis of Aha!Moments reached by students in remedial mathematics classrooms (Czarnocha, 2018) shows the inadequacy of standard taxonomies of Bloom, Anderson and Krathwohl revised taxonomy of Bloom as well as following them DoK instrument by Webb (2002) due to their static character. Absence of the dynamical instrument tracing the development of conceptual understanding from one level to another makes it difficult to analyse dynamics of insight. The process of understanding present during an Aha!Moment may involve several dynamic cycles of movement from elementary understanding of mathematical concepts through their analysis to the creative synthesis, all in an instant. Coordination of bisociativity with reflective abstraction offers exactly the instrument of analysis responding to the dynamics inherent in the insight. Reflective abstraction is used in educational research to assess and analyse student learning. While most authors use of this concept reflect Piaget definition, they tend to vary in application and interpretation. Koestler work on creativity, in particular on bisociation provides a lens through which to interpret the work of Piaget on schema development though the creation of meaning i.e., schema.

One of the central observation which allows and promotes the investigations of creativity of every student has been brought to light by Shriki (2010) who indicates two of the main reasons for the absence of research into creativity of the rank and file: “(1) *The significance of creativity in school mathematics is often minimized because it is not formally assessed on standardized tests, which are designed to measure mathematical learning.* (2) *The problem with relating to students’ work as ‘creative’ is rooted in the definition of creativity as a useful, novel, or unique product...Although according to the traditional view of creativity; students’ work would not be considered as creative, the researchers agree that students’ discovery may still be considered creative if we examine the issue of creativity from a personal point of view, namely, whether the students’ discoveries were new for them.*”

This important broadening of the domain of creativity to include its subjective component is supported by several statements by mathematicians and scientists based on experience:

- (1) According to French mathematician Hadamard: “*Between the work of the student who tries to solve a problem in geometry or algebra and a **work of invention**, one can say that there*

is only the difference of degree, the difference of a level, both works being of similar nature” Hadamard, (1945, p.104). Similar assertion is known of Polya (1981).

- (2) Applebaum and Saul, (2009) observe *“a remedial algebra student can exhibit creativity as often and as clearly as an advanced calculus student”* after which they assert (Assertion III) *“Creativity in mathematics can be found at any level of the subject matter, and at any level of mastery.”*
- (3) Koestler asserts *“minor subjective bisociative processes...are the vehicles of untutored learning (p.658).”* The contemporary meaning of *“untutored learning”* is given in Larkin (2005), who realizes that *“the active process of creativity in learning is not something we do to the student. Rather, learning through the creative experience is something students can and must experience for themselves”*. The closest classroom approximation to such conditions of untutored learning, allowing students to ‘experience creativity for themselves’ is the Discovery method of teaching in many of its variations, i.e., inquiry and/or guided discovery, problem solving, project-based learning, or faculty/student research.

The TR Team of the Bronx has studied bisociative creativity within the framework of problem solving and inquiry-based learning focusing on the synthesis of planes of reference. For example, through the coordination of different actions, the reversal or a process of the application of an action to more structural or abstract object outside the students’ comfort zone, and the comparison of different strategies-actions to solve a problem. Questions that arise are (1) where does creativity come into play within critical thinking? and (2) when does bisociative experiences within a social situation lead to internalization of knowledge i.e., the next day effect.

The point of view of Shriki (2010) is closely connected with the terms “c-creativity” and “C-creativity” referring to the creative novelty being new to the individual and the novelty being new to the larger human community. That distinction allows to free creativity from the grasp of giftedness as defined by SAT scores and those of other similar instruments, and redirect research and practice attention to creativity of every individual.

SECTION 4. CONCLUSIONS

We have sketched here an argument, which strings together seemingly unconnected decisions concerning the subject of creativity:

- changes in Bloom’s taxonomy,
- the choice of criteria for giftedness within academia together with the focus of professional research attention on creativity of giftedness so defined;
- absence of research knowledge of the creativity of ‘rank and file’;

- absence of emphasis on STEM creativity in the NSF INCLUDES pilot projects, whose aim is to increase STEM participation of the “underrepresented”.

The common theme for these decisions is participation in the systematic and systemic process of eliminating access to creativity from the ‘rank and file’ learners by imposing, consciously or unconsciously, criteria extraneous to creativity, which limit creativity research and teaching practice. In response, the TR Team of the Bronx (Prabhu and Czarnocha, 2014) proposed the bisociation theory of Aha!Moment of Koestler (1964) as the tool for the democratization of creativity.

As a final comment we would like to offer the following observation. Both decrease in creativity observed by Kim (2012) and work leading to the revised Bloom taxonomy started in the first half of the nineties decade of the last century. That is also the decade of the final world-wide victory of uniform globalization, which eliminated previous highly disconnected bipolar world. Since the conditions for the creativity of Aha!Moment is the gap between two or more separate frames of reference one can conjecture that the process of kidnapping creativity was initiated exactly with the global uniformity of globalization.

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Book Review for *MÖBIUS AND HIS BAND* edited by J. Fauvel, R. Flood, and R. Wilson

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Although this informative book is a collection of essays by six authors, each with a different perspective of Möbius's contributions, there is no disruption in the historical theme, rather a consistency that is quite seamless; the chapters can be read in any order without loss of continuity. The final chapter contains research done by others in subsequent years, such as optimization in Morse Theory, dynamical systems, chaos, relativity and quantum mechanics, which were not yet discovered in the age of Möbius.

August Ferdinand Möbius was born on 17 November 1790 and died on 26 September 1868 in an area of Saxony between Leipzig and Jena, between Prussia to the North and the Hapsburg Empire to the South. Thus, Möbius considered himself a Saxon and not a German. At this time the development of mathematics to higher levels paralleled the rise of Germany from independent states, through invasions, wars and revolutions, to an empire under the military might of Prussia of Frederick the Great, while collaterally France was engaged in a revolution that utilized all of its resources, with little emphasis on the arts and mathematics.

This was an age of transition in arts, sciences and politics throughout most of Europe. Mozart (an Austrian), Goethe, Gauss and Beethoven were flourishing, the French revolution standardized weights and measures into the Metric system, as Napoleon Bonaparte came to power, and most of Europe was at war.

Not much is known about Möbius's early life and education. However, he entered the University of Leipzig at age 18, and first studied law but changed to mathematics, physics and astronomy (under Karl Mollweide). Möbius visited Göttingen to study astronomy under Gauss, who was the director of the observatory there. He later visited Halle, where he studied mathematics under Johann Pfaff, who had been the teacher of Gauss.

Möbius considered joining the Prussian army, because his life as a teacher met with little success. Students attended only if the lectures were free, and also because in his unsatisfying role as a mathematics teacher he worked only with low-level students on calculations. Because he was not Prussian, he bolstered his Saxon pride, avoided military service and returned to finish his habilitation thesis. In 1816 was appointed Professor of Astronomy at the University of Leipzig, where he remained for the rest of his life. In 1848 he was promoted to Director of the Observatory. Thus, Möbius spent most of his life as an astronomer, although he is remembered for his mathematics.

At the turn of the 19th century France promoted mathematics significantly, both for social advancement and as service to the state. Lagrange at the Ecole Polytechnique, along with Laplace, Monge, and Legendre were doing everything from teaching to research. Germany had nothing like this, but after Napoleon's defeat of Germany at the Battle of Jena in 1806, Germany developed a new national pride mainly through the University of Berlin, where the research-oriented professional philosophy emerged, especially for mathematics, and constituted a new unique approach to professional mathematics enduring into contemporary times that involved teaching, research and graduate seminars, thereby creating new knowledge as well as teaching it, and with research taking prominence. Prior to this time teaching and research were separate and distinct disciplines that did not interact. It is important to note the constant alternation in mathematics evolution of France, then Germany, as a result of war and politics.

In the 19th century Germany took over the mathematical leadership from France, the most important German mathematician being Gauss. The increased rigor in German schools, with teachers doing research as well as teaching, resulted in mathematics becoming a subject in its own right and as an independent discipline in the university. High schools, or *Gymnasien*, followed the same protocol: the professionalization of teachers, and the institutionalization of mathematics as an independent or autonomous discipline. Student seminars were held once a week, and provided students the opportunity to present a topic, to be criticized and guided by their professors. This introduced students to research publications and research techniques, resulting in increased emphasis on pure mathematics.

The *Schulprogramme* stimulated teachers in the lower schools to participate by publishing material on basic topics. A return to basic synthetic geometry (J. Steiner) began the resurgence of geometry versus arithmetization, with emphasis on descriptive and projective geometry. The priority in English and Scottish universities was to teach, not to conduct research, whereas in German universities, research by means of the Ph.D. programs was to give all candidates some research exposure. Hence, German replaced Latin as the academic language, due to the proliferation of German science periodicals beginning with the appearance of Crelle's *Journal for Pure and Applied Mathematics* in the 1820s, followed by other journals to make research results more accessible.

Halley's comet appeared once again in 1835 and spurred Möbius to produce two popular treatises which analyzed the comet's orbit and the wider laws of astronomy in order to perfect the knowledge of solar system dynamics, including a more accurate positioning based on Newton's laws of motion. Möbius's work *Mechanik des Himmels* (1843) presented basic information of celestial mechanics to the amateur. Heinrich Olbers of Bremen was the foremost comet expert and discovered several comets as an amateur astronomer. The observatory at Königsberg was to be headed by Friedrich Bessel, who was both a superlative theoretician and a meticulous observer, and these observatories were state established by the local regent, or king of Prussia. Möbius was appointed astronomer (observer), and later director in 1848 at Leipzig University.

Although Möbius was not an astronomer by vocation, his own mathematical ideas and researches were conducted in a context of intense astronomical activity, taking place in the discipline in which he held a professorial chair. Thus, Möbius had access to the latest developments in astronomy and was able to relate this to his teaching. This confluence of talented astronomers, scientists and instrument makers continued up to 1914 in which Germany's role was paramount.

In his day Möbius was most known for astronomy, but the work that established him among mathematicians was his work in geometry and mechanics. Geometry diminished in importance in the 18th century, eclipsed by algebra and calculus; Lagrange codified mechanics in terms of calculus. However, Monge, as head of the Ecole Polytechnique and a geometer, brought geometry back as the core of the curriculum. Jean Poncelet created projective geometry. Louis Poincaré gave a geometrical description of the way a body rotates. Möbius independently studied mechanics by means of geometry and algebra. German geometers Plücker and Hesse championed the geometric approach to mechanics.

Möbius is best known for his Möbius band or strip, conceived from his study and prize for the geometric theory of polyhedrons. His other discoveries include the Möbius function, inversion formula, transformation and the Möbius net.

In 1827 Möbius published a book on barycentric calculus, introducing barycentric coordinates, to describe lines and conics and their transformations. Möbius solved the center of gravity problems by recourse to law of the lever, which in three dimensions becomes the barycenter of the object.

In the 1830s Möbius studied statics, considering questions such as “what effect do certain forces have on the behavior of a rigid body?”, and then producing a book on the subject.

Möbius had influence and contributions to topology, although in his time, there was no such discipline, the Möbius band notwithstanding. In the 18th century Euler related the vertices, edges and faces of pyramids, prisms, crystals of various kinds to produce the formula $V-E+F=2$. However, the Swiss mathematician Simon Lhuillier showed that in some cases Euler’s formula is incorrect, since this depends on the number of ‘holes’ in a solid. Möbius described and explained the one-sided notion of his band as non-orientability. Another mathematician, Johann Listing, who studied and worked with Gauss, and from whom his ideas of topology derived, consequently published a book on the generalization of Euler’s theorem on polyhedrons in 1861, in which he described the Möbius band. Riemann’s study of many-valued functions, described geometrically as Riemann surfaces, led to Felix Klein’s concept of automorphic functions. Thus, two main threads of topology emerged: the analytic and the algebraic. The prominent ideas that interested Möbius were topology, symmetry, and celestial mechanics: topology by means of the Möbius strip; symmetry in crystallography; celestial mechanics as in planetary motion. Möbius’s contribution was doing mathematics effectively and concentrating on what’s important, which is the essence of his modern legacy.

This review has tried to demonstrate the interdependency of scientific research and the conditions of the society within which it occurs, since the scientist and his society do not operate in mutually exclusive realms. This book should be of particular interest to scientists and sociologists pursuing the history of science and technology.