



Editorial from Malgorzata Marciniak

The current volume, VOL 10 no 3-4, of the Mathematics Teaching Research Journal contains a variety of topics that can serve as an inspiration for possible directions of own research in the classroom. All themes are well known and well established in a broad literature but bringing them to a particular locality of own college and own classroom, always gives them a new flavor. The reader will find an Inquiry Based Learning (IBL) technique, a report from using RStudio in a statistics class, an analysis of procedural understanding, a book review, and descriptions of techniques used in calculus classes. The first one is related to the creative thinking and cognitive interest, and the second describes grading individual students in a team.

The volume begins with an article by Jae Ki Lee from BMCC about using Inquiry Based Learning while teaching complex numbers. The author describes his class experiment where students find ways of computing the powers of the imaginary number i and comparing answers with each other to assess their results. In the conclusion it is stated that IBL encouraged students' curiosity and enhanced their learning experience. As the author pointed out students were eager to share their findings with each other and reacted with enthusiasm to the assignment. This attitude is rarely observed during traditional lecture-style teaching.

The second article, by Nkechi Agwu and Piotr Bialas from BMCC, calls for using a free software R or RStudio as a calculator in an Introductory Statistics class. Learning objectives contain producing and interpreting summary statistics, graphing and analyzing a scatterplot including the regression line, creating various graphs such as histograms and bar plots, and creating multiple graphs in multiple frames for comparison. The authors performed their work in five sessions assessing creativity and learning outcomes. Each session is presented in the article and contains all details to adopt it in another classroom. The authors indicated in the introduction that they learned RStudio during a two half-day session at the International Conference on Teaching Statistics and were convinced about usefulness of the software.

Jair Aguilar and James A. Telese from the University of Texas, Rio Grande Valley present their work related to pre-service teachers who go through assignments and are graded based on three types of competencies: procedural fluency, conceptual understanding, and problem solving. The rubric is provided to describe five levels of each competency. The authors analyze statistical data obtained during the experiment and draw conclusions about the self-faith gained by students who participated in the experiment. Problems solved by their students sound comical, for example: "The people of Freckleham are interesting creatures. Every Frecklehammer is different from the

other and has at least one freckle and one hair but no more than three freckles and three hairs. Make a list of all of the different Frecklehammers ...,” but they are basic combinatorial problems.

“The Outer Limits of Reason” by Noson Yanofsky is reviewed by Malgorzata Marciniak as an instance of a textbook written by CUNY faculty for use in the classroom. The book offers engaging and stimulating discussions suitable for mathematicians, computer scientists, philosophers, and anybody interested in philosophy of science. The discussion contains numerous classical topics such as mind games, paradoxes or computing complexities, but takes them into another level by offering a deep and insightful analysis of what is possible and what is impossible within them. The book is highly recommended for readers who enjoy engaging their thoughts in philosophical considerations about science or mathematics.

Creative thinking is a theme of the article by Miriam Dagan, Pavel Satianov, and Mina Teicher from Israel. The authors present their teaching techniques applied in calculus classes taught for engineers. As a motivation for their work they provide an argument that in the age of technology computational math skills can be performed by machines which will never be programmed to solve new problems. That is why students need to be trained to grow cognitive skills to perform this task. One of the most fascinating aspects of this article presents, with images, how to use an electric tape (or any other tape) to explain the idea of measuring angles in radians.

The article by Jeff Ford about team-based learning presents a solution for a dilemma we often face while grading team-based work. Here the author applies an iterative process of adjustments to a method found in the literature. The most intriguing part of this article is about introducing peer review reports among students and including the results as a factor in a course grade. Since students work with each other on daily basis, they keep record of attendance for each other and accept or reject each other’s excuses for absences. An ample set of sample documents is contained in the appendices to illustrate the work beyond the idea.

Malgorzata Marciniak

Managing Editor of MTRJ



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Exploring students' discoveries based on Inquiry-Based Learning Strategy

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INTRODUCTION

Inquiry-Based Learning (IBL) strategy is a student-centered learning method (Lee 2011). Recent studies show that student-centered learning methods are effective at motivating students and enhancing their learning experience (Boaler 2016; Kim & Lee, 2011). The IBL strategy increases students' curiosity, and it helps students explore problem-solving strategies (Von Reness & Ecke, 2017). Thus, researchers acknowledge that the IBL strategy leads to many positive student outcomes in undergraduate mathematics (Kuster, G., Johnson, E., Keene, K., & Andrews-Larson, C 2018). In addition, students' self-led exploration allows them to discover various approaches to solving problems. However, most of the IBL research was conducted at four-year institutions with upper-level mathematics courses (Von Reness & Ecke 2017). This paper applies the same conceptual approach to learning in a community college setting, an experiment which has not done before. This experiment demonstrates how the IBL setting helps community college students to develop and enhance their learning. Also, it demonstrates how community college students get engaged and discover their own problem-solving strategies in IBL setting class. This study was conducted in a college credit bearing class precalculus, focusing on the topic of developing approaches to simplify powers of i .

THE PURPOSE OF THE STUDY

The main purpose of the study is to explore if the IBL strategy encourages students' learning in a community college. Although various researchers demonstrated that student learning significantly improved in IBL setting, it is not a well-known or widely known instructional strategy at community colleges. Therefore, this study seeks to apply the IBL setting in a community college to assess whether or not a similar result can be expected compared to other existing studies. The following research questions support the aim of the study:

1. Can it be possible to have similar expectations, such of that developing curiosity, and enhancing student learning both at four-year institutions as well as at community colleges?
2. Can the IBL setting enhance students' appetite for self-led learning, and incentivize them to find varied approaches to finding solutions?

3. Can the IBL method increase student engagement with other students and increase collaboration?

Simplifying Powers of i

The imaginary number is defined to be of the form $a+bi$, where $i = \sqrt{-1}$, and $a, b \in \mathbb{R}$. Thus, any non-real values, such as $\sqrt{-3}$, $\sqrt{-5}$, and $\sqrt{-9}$ are indicated as $i\sqrt{3}$, $i\sqrt{5}$, and $3i$. It is because all those values can be expressed as $\sqrt{3} \cdot \sqrt{-1}$, $\sqrt{5} \cdot \sqrt{-1}$ and $\sqrt{9} \cdot \sqrt{-1}$. (Miller and Gerken2015; Larson, 2017). One of the topics in the college algebra or precalculus is simplifying the powers of i . The concept is basically defined based on $i^2 = -1$, and developed others:

$$\begin{aligned}
 i &= i \\
 i^2 &= -1 \\
 i^3 &= -i \\
 i^4 &= 1 \\
 i^5 &= i \\
 i^6 &= -1 \\
 i^7 &= -i \\
 i^8 &= 1 \\
 &\vdots
 \end{aligned}$$

Figure 1: Patterns of simplifying power of i .

Based on these patterns, students are asked to simplify higher power of i and define equivalent value. For example, $i^{11} = -i$, and $i^{10} = -1$. The main aim of this experiment is to explore how many different approaches to simplifying the power of i community college students can find. Does IBL setting increase students learning appetite, and assist them at finding multiple approaches for problem-solving?

TYPICAL APPROACHES TO SIMPLIFYING POWERS OF i

Based on the pattern of the power of i , the most well-known approach is dividing the exponent by four. It is because the pattern is rotated with four different values; i , -1 , $-i$, and 1 . Hence, if the

remainder is 0, then the answer is 1; if the remainder is 1, then the answer is i ; if the remainder is 2, then the answer is -1 ; and, finally, if the remainder is 3, then the answer is $-i$.

Another well-known approach is dividing the exponent by two. In this case, the remainder is always either 0 or -1 . Thus, we need to look at the dividend. If the dividend is odd, and the remainder is zero, then the answer is -1 , but if the remainder is 1, then the answer is $-i$. If the dividend is even, and no remainder, then the answer is 1, and if the remainder is 1, then the answer is i .

Another less frequently taught approach is to use the multiple of 100. Whenever the exponent is more than a three-digit number, the last two digits need to be analyzed. For example, i^{23411} , the exponent is more than a three-digit number. The exponent 23411 can be re-written as $23400+11$. Now, the expression becomes $i^{23400} \cdot i^{11}$. Since 23400 is a multiple of 4, i^{23400} is equal to 1. It implies that $i^{23400} \cdot i^{11} = (i^{100})^{234} \cdot i^{11} = ((i^4)^{25})^{234} \cdot i^{11} = 1 \cdot i^{11} = i^8 \cdot i^3 = -1$.

METHODOLOGY

The study demonstrates how the instructor taught the topic “Complex Number” based on IBL strategy. The researcher takes the role of the instructor and teaches the topic. There are twenty-five students participating in the study: thirteen males and twelve females. It is the first time IBL research is being applied in a mathematics class at a community college, so it doesn’t compare students learning between the IBL group and an ordinary null group.

A new IBL activity needs to be created because the existing IBL activities are based on upper-level college mathematics classes. The researcher created this IBL experiment to suit a community college-level mathematics course. Among the complex topics, this article focuses on the topic “Simplifying Powers of i .”

Teaching Simplifying Powers of i based on IBL Strategy

The instructor prepares two different IBL-based experiments, one; individual and the other; group activities.

Before the beginning of the first activity, the class discussed the pattern of the power of i . As shown in Figure 2, the instructor provided various flash cards, so that each student received a unique problem to simplifying the power of i .

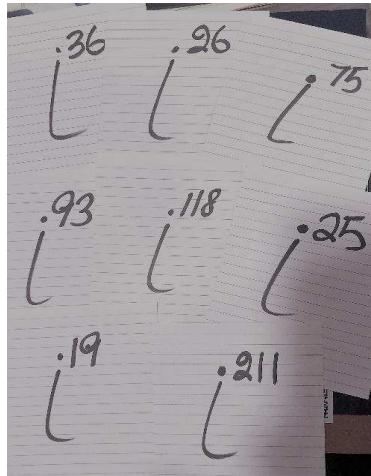


Figure 2: Activity cards for simplifying power of i

The instructor provided four quadrant clues i , -1 , $-i$, and 1 . Each student evaluated his/her assigned problem and found the right quadrant and answer. All students were asked to discuss with their quadrant peers who picked the same answers and verified whether each of them found the correct answer or not. If not, they should re-evaluate their questions, and move to another place where their modified answer is located. As soon as all the students chose the correct position, the instructor asked students to discuss their approach to problem-solving.

Quadrant 1: i	Quadrant 2: -1
Quadrant 3: $-i$	Quadrant 4: 1

Figure 3: Four Quadrant Slues i , -1 , $-i$, and 1

The second activity was to evaluate the multi-digits power of i . At this time, students were grouped and discussed the problem with their group members. The instructor set up a speaker for each group, and the speaker is the only person to share their group answer. For more active group collaboration, the instructor clarified that the speaker of each group present and discuss his/her group discussion with other groups.

STUDENT DISCOVERIES

Activity 1

All twenty-five students received different cards. Cards were randomly distributed, and students were asked to evaluate the higher power of i . Based on this activity, students independently found two different ways to solve the problem. The first approach was based on dividing the exponent by four and analyzed the remainder. Hence, the given power of $i: i^n$, students divide the exponent n by four, and if there is no remainder, then the answer is 1, the remainder is 1, then i , the remainder is 2, then -1 , and the remainder is 3 then $-i$. Another approach divided the power n by two. In this case, if the quotient is odd, and no remainder, then the answer is -1 ; if the remainder is 1, then the answer is $-i$. If the quotient is even, and no remainder, then the answer is 1 and if the remainder is 1, then answer is i . Those are typical approaches that are introduced in various textbooks. However, no other approaches were found based on this activity.

Activity 2

In the second activity, students were grouped by five or six people and discussed the problem. The instructor randomly selected the speaker from each group, and he strictly communicated with those speakers. Each group must share and verify their answer with the speaker. Each speaker presents his/her group answer. Students in other groups can ask questions to other groups, but only the speaker from each group can answer a given question.

The instructor provided a multi-digit power of i for the second activity. For example: $i^{20123420}$, $i^{1985401}$. Most of the students used either dividing the exponent by four or by two. Among those approaches, the students that used the divisor four struggled to figure out the answer. But, the students' groups, who divided n by the divisor two, got the answer more promptly. The following dialogue demonstrates how they showed the work.

Group A: If the power is odd, then the exponent is needed to subtract 1 from the total power and rewrite the expression as $i \cdot i^n$.

For example, $i^{1985401} = i \cdot i^{1985400}$. After that, just divide the exponent value 1985400 by 2, then it is 992700 which is even, so the answer is i . Another example, $i^{20123420}$, This problem is already even power, so they just divided n by 2. $20123420/2=10061710$. Since the division is even, the answer is 1.

However, the groups that divided the power by four couldn't get the answer promptly. So, comparing to the algorithm between dividing the power by two and four, dividing by two seemed easier to solve.

In a moment later, one student suggested a simpler property based on dividing the exponent n by four. Here is how he describe to simplifying the power of i .

Student B: If the power n is more than a three-digit number, we only care about the last two digits. For example, $i^{1985401}$. The exponent is 1985401. Although it is a big number, we only need to analyze the last two digits which is 01. Since 01 is not divisible by 4, and just remainder is 1, so the answer is i .

However, other students were confused and asked him again:

Other students: How come? Why?

Students B: Because the exponent 1985401, we can rewrite is as $1985400+01$, and 1985400 is a multiple of 100, and 100 is a multiple of 4, so it must be divided by 4 anyway. So, we only worry about two digits: 01.

The instructor: What is the reason that you wrote 1985401 as $1985400+01$? Why not $1985400+1$?

Students B: Because I want to leave out two digits.

Other students were still confused and started debating this question with him.

Other students: What do you mean 01 divides by 4? Isn't it 1 divide by 4?

Students B: Okay. Let me show a different example: $i^{10845631}$. As you see the exponent is 10845631. This exponent can be expressed as $10845600+31$. Since 10845600 is a multiple of 100, and 100 is a multiple of 4, it can be written as $1 \cdot i^{31}$.

The instructor: You need to clarify what the reason is that you can re-write $i^{10845631}$ as $1 \cdot i^{31}$.

Student B: $i^{10845631} = (i^{100})^{108456} \cdot i^{31} = ((i^4)^{25})^{108456} \cdot i^{31}$. Since $i^4 = 1$, it can be expressed as $1 \cdot i^{31}$. When you divide 31 by 4, the remainder is 3. So the answer is $-i$.

It is apparent that the student understands the exponent property: $a^m \cdot a^n = a^{m+n}$ clearly. Soon after other students tried the other question: $i^{20123420}$

Student C: So, you meant that we only divide 20 by 4 because other digits 20123400 is already a multiple of 4. Since the number 20 is a multiple of 4, so the answer is 1.

Student B: That's right.

Other students: That's awesome!!

Based on this activity, students were dedicated to working on simplifying the power of i and developed various ways to solve the problems.

OBSERVATIONS FROM THE EXPERIMENT

Based on the IBL activities of simplifying power of i , the experiment provides the following observations.

First of all, the IBL instructional strategy motivated students and increased their curiosity. Their curiosity encouraged them to collaborate with each other voluntarily, and all the students discussed the problems. During the discussion, students understood different methods for evaluating the power of i . and found dividing the power n by two is simpler than divide by four. Secondly, the study also found that the IBL strategy develops a deeper understanding of the topic. If the instructor provided a lecture on simplifying the power of i , it would be possible to instruct one method, i.e., either dividing the power n by two or four. It is because instructing various methodology may cause confusion for the particular student group; it becomes a reason that the class progress slows down in the lecture-based classroom environment. Therefore, instructors usually choose one method to teach the students. However, in the IBL class, students have developed up to three different approaches based on their own collaboration. One of the students, who found a new approach witnessed that he found the method while he was discussing with his groupmates. Overall, all students shared various approaches that they are developed together.

Finally, the IBL strategy allowed the instructor to cover more topics. Besides covering the topic of simplifying the power of i , the class covered combining, multiplying, and dividing complex numbers. Although those topics were not intended to be explored in this IBL experiment, students were already exploring topics outside of the scope of this experiment students were eager to volunteer to share their work with peers, and students were collaborating with each other to figure out the problems.

FURTHER STUDY

This IBL experiment demonstrated that IBL strategy develops students' curiosity and enhances their learning experience. So, it would be worthwhile for instructors to consider create lessons

based on IBL strategies. In addition, it would be valuable to compare IBL setting and ordinary lecture-style classes to document students learning outcomes and further study in this area.

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Using R/R Studio In An Introductory Statistics Course

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***Abstract:** In this paper we will share our experiences and approaches, based on **GAISE**¹ and **Project Mosaic**² to teaching Introductory Statistics class to students with no programming background. We think that R, open-source, and free programming language can be used as a “calculator” in teaching/discovering various statistical concepts in the realm of descriptive statistics and simple linear regression. Several examples of simple homework assignments will be presented; in addition, exemplary student report will be displayed.*

INTRODUCTION

In July 2014 one of the authors attended International Conference on Teaching Statistics, (ICOTS 9), held in Flagstaff, Arizona and enrolled in workshop *Statistics Using R and RStudio*. The workshop was created and conducted by Randall j. Purim, Nicholas J. Horton, and Daniel T. Kaplan and included introduction to R Studio, Mosaic, and R Markdown.

Two half-day sessions convinced attendant about necessary changes in ways of teaching an Introductory Statistics course. Main change involved using different kind of technology, different from Texas Instrument graphing calculators, Excel, SPSS, and even SAS packages.

R and R Studio computer application programs were on desk to be tried.

WHY R AND RSTUDIO ...?

Both computer application programs are **free** and open source entities; so, no more complaints from students about additional monetary expenditure for technology which in some cases was thought to be useless after completing the course.

¹ College Report 2016, (GAISE), Guidelines for Assessment and Instruction in Statistics Education

² Randall J. Pruim; Nicholas J. Horton; Daniel T. Kaplan; Project MOSAIC, *Start Teaching with R; Preliminary Edition*

R and R Studio were considered suitable for use because R seems to be intuitive and efficient tool in conjunction with R Studio when used in teaching the Introductory Statistics Course.

Moreover, R as an open source program can be used along with various packages to manipulate data and construct graphical displays that were not easily or ever obtainable using other technologies.

R/RStudio IN *Calculator* MODE

After numerous meetings and discussions authors agreed on trying R and RStudio in teaching an introductory course using mentioned above in a “calculator” mode.

Students were to use handout with assignment and R/R Studio package for calculations, then they were instructed to copy obtained results to their handout, notebook, homework assignment or special project, (typically in MS. Word format). Constructed graphical displays were to be saved in appropriate format to a desktop in order to be copied later. No working files in R/R Studio format such as *filename.r*, *syntax1.txt*, or work directory were to be saved. Students who intended to replicate procedures executed by R were suggested to **redo** original handout procedures for a particular assignment. The list of commands was frequently expanded from the original one submitted to students by the authors on day one.

Authors emphasized importance of working in groups of two, (voluntary grouping), use of help resource internal to R, and Internet resources such as YouTube, PDF's, .html files, and many others. The authors agreed on the following purposes.

LEARNING OBJECTIVES

The authors wanted to introduce R and RStudio software packages to Mat 150, Introductory Statistics students that would function primarily as a “calculator”, as well as, introduce the concept of density, and inform students how to read R output.

They intended to have students use these packages to produce and to interpret summary statistics for a single-variable or/and multiple-variable data set, as well as, to use these packages when working on simple regression topics to produce and analyze summary statistics related to a particular linear regression model and to graph/analyze a scatterplot including regression line.

In addition, they wanted students to produce simple graphical displays of one-variable, two-variable data set (e.g. bar plots, various histograms, different frequency polygons, boxplots, and multi-graphs presented in a single frame or multiple graphs in multiple frames presented at once for comparison purposes

Objectives that are NOT intended while teaching statistics with R and RStudio:

The authors did not intend to teach programming/programming techniques for programming purposes or for graphics only.

Neither they intend to teach proper saving of files including workspaces, complex coding, and complex graphics.

Expected benefits while teaching statistics with R or/and RStudio

Authors expected that students using R/RStudio as

a “calculator” in the introductory course would benefit as follows.

Students will be able to work/analyze a multiple-variable data set next to a single-variable/two-variable data set.

They would be able to expand their set of skills that are needed when comparing various data sets/variables’ distribution within data frame.

In addition, students will save time from number-crunching activities, and using various tables, (e.g. z-tables, t-tables).

Lastly, students will be able to work simultaneously with two programs: R/R Studio and MS. Word when preparing assignments or/and term papers and exams.

Both authors were present during instruction related to introduction to R and RStudio, facilitated group work and independent work of students. Discussions and *live* R computer outputs generated from instructor station presented by different students followed.

SETTINGS FOR MAT150 – 171 W CLASS

There were 22 mostly non-traditional students with gender-split almost 1:1.

The instruction took place in a computer laboratory, N553 where

each station included machine with R and RStudio installed and active Internet connection.

Instructor’s laptop was used by student-volunteer for demonstration of various activities.

Modality of Instruction in Mat 150-17W Class

The Mat 150-171W class met once a week every Sunday between the hours of 5:00 p.m.

and 8:30 p.m. in computer lab for five sessions over duration of the fall 2017 semester.

The class Mat150-171W is described as writing intensive section that meets the CUNY writing intensive graduation requirement.

Professor Nkechi Agwu was principal instructor of Introductory Statistics course. Topics included in R/R Studio students' activities were already taught to students and needed no reteaching by Professor Agwu or Professor Bialas.

Professor Piotr Bialas was visiting the course five times each for about 90 minutes in order to instruct students to use R/R Studio in their class.

Session 1 of Introduction to R/RStudio

During session 1 student's tasks consumed approximately 60 to 75 minutes of instruction.

Activity 1 lasted for 15 minutes, its intended goal was to reinforce students' knowledge about R/RStudio and possibility of uploading R/RStudio to personal computer's desktop or laptop.

Students used Internet in order to provide answers two questions: What is R/RStudio? And, how to download R and RStudio to a laptop/desktop computer? Finally, they shared they answers.

Activity 2 also lasted 15 minutes, when students had to determine how to get resources related to downloading R and R Studio. YouTube resources prevailed, and authors recommended Tutorial 1.1, Tutorial 1.2, Tutorial 1.3 from Series1 of *MarinStatsLectures about R*, (https://www.youtube.com/watch?v=cX532N_XLIs).

Activity 3 was allocated 30 minutes and intended to inform students about simple syntax needed to perform calculations selected by the authors.

Students obtained handout related to use of R/RStudio to complete series of calculations like that on (<http://www.pbialas.com/chapter-1.html>) They were to work in groups of two. Finally, self-selected students would provide answers using instructor laptop to project the syntax and answers on white board. Analyses of errors would follow.

Authors observed numerous instances of *aha*-moments students expressed during this activity with respect to application of parentheses, order of operation, and justifications for their use.

In addition, demonstration of calculations of an individual standardized *z-score* for a value in the data set, lead students to understanding that calculations of *z-scores* can be made with respect to all data values at once, students found those useful in detection of *unusual* data set values and also helpful in explanation of calculations related to the *Pearson's Linear Correlation Coefficient*.

Homework assignment for Session 1 involved uploading both packages, (R and R Studio), exploration of www.pbialas.com site, and practice of calculations like in CHAPTER 1 of the website.

Session 2 of Introduction to R/RStudio

Session 2 began with homework review when the authors observed, that large majority of students completed homework Assignment 1, and just very few were unable to download R/RStudio. Those who did not, were able to download the programs in class.

During Session 2 students' tasks consumed 45 to 75 minutes of instructional time. Students were instructed to visit www.pbialas.com website and review Assignment 0 as well as Model Solution to Assignment 0.

Assignment 0 was related to Simple Linear Regression topic involving bivariate data where students had to enter data by hand into R/R Studio file, follow instructions related to calculations, graphing, and write short interpretations of obtained results. The authors assisted students during Session 2 and noticed that some students were having *technical* difficulties related to copy-and-paste of the results obtained from R to MS. Word file.

Homework assignment for Session 2 was placed on www.pbialas.com and was similar to activities students worked on in class time. This website was developed for students' homework assignments and model problems. Students were expected to create a report in MS Word format by following detailed instructions provided online by the authors, (Assignment 1 Report).

Session 3 of Introduction to R/RStudio

Session 3 began with collection of Assignment 1 and homework review. Homework review included the list of issues students encountered while working on Assignment 1. Two students did not submit report on time and were given extension for completion of their homework.

During Session 3 students' assignment consumed between 45 and 75 minutes of instructional time. Students' activities were related to uploading a .csv file into R Studio. The authors provided short demonstration of the procedure along with explanation of each step. After the authors' presentation, students watched the YouTube [video about importing data from MS. Excel to R Studio](#). Finally, they worked in self-selected groups or individually on a *small* data set in MS. Excel format in order to import it into R Studio.

The authors observed that those students who completed work early offered their assistance to other students. Finally, self-selected students would provide answers using the instructor laptop to project the syntax and answers on white board. Analyses of errors followed.

Homework assignment for Session 3, Assignment 2 was placed on www.pbialas.com website. It involved uploading a .csv data set file, creation of a report in (MS. Word format) related to histograms and boxplots comparison topics using R Studio as "calculator" for calculations and graph(s). Students were directed to <http://www.pbialas.com/sunday-m1502.html> for instructions related to Assignment 2 and were given extended time to complete this assignment.

Session 4 of Introduction to R/RStudio

Session 4 begun with collection of Assignment 2 and homework review. Homework review included the list of issues students encountered while working on Assignment 2. Some students did not submit report on time.

During Session 4 Students continued to work on Assignment 2 Report in class in self-selected groups of two or individually for 45 to 75 minutes. Some students who completed their tasks offered again their assistance to other students. Students' completed assignment was to be re-submitted to the authors upon arrival to the next class session. (place the discussed issues)

Session 5

Session 5 begun with collection of Assignment 2 and homework review. Homework review included the list of issues students encountered while working on Assignment 2.

During Session 5 the authors reviewed with students four previous sessions related to R and R Studio packages and discussed issues students had while working on their respective assignments. One of the issues involved *to save or not to save* particular workspace while learning simple R syntax. In addition, one of the authors provided demonstrations of saving files in R and R Studio. Students agreed that at their stage of learning R/R Studio simplicity would prevail Further the authors discussed students' creativity with respect to their completed assignments.

ASSESSMENT OF STUDENTS CREATIVITY BY SESSION

The authors believe that in Introduction to R/R Studio in Introductory Statistics Course process of assessment of students' creativity is related to the following: originality of topics selected for the process, method of delivery (e.g. lecture, group-work, technology uses in the process-YouTube and more), originality of student's activities and homework assignments, and finally to use of Depth of Knowledge, (DOK) questioning levels.

In Session 1 of students' activities the authors observed various reactions of students to error messages produced by R/R Studio. Explanation of error messages by students and authors created numerous *I got it ...* moments to correctly executed commands, (e.g. need of parenthesis to preserve Order of Operation Rule and *what-if- the need* is violated ...). Students were expected to create their own problems; many did and shared their work using instructor's laptop via LCD projector.

The authors observed that *copy-and-paste and edit* idea when writing a syntax was immediately accepted and widely used by students through the reminder of all activities, (e.g. student's comment: *What a useful way to correct errors!*).

In Session2 students' activities the authors offered students *Assignment 0* and presented them with *Model Solution to Assignment 0* link. Students were expected to work with two programs at once;

perform computations using R Studio, then copy those into MS. Word file and provide narrative/interpretation of the computations in MS. Word file for each task in *Assignment 0*.

Homework for Session 2 involved creation of report, Assignment 1 Report. The authors observed that some students questioned the use of MS. Word format requested by the authors and were able to modify it successfully to their own format.

In addition, those students who wanted to replicate their calculations used in Assignment 1 Report found questionable the authors' suggestion: *do-not-save-file in-RStudio rule*. In class discussions and presentations convinced some students about usefulness of the *rule*, (*do-not-save rule*). This rule allowed students to avoid errors related to variable names, incorrect workspace location, and other aspects such as saving of graphical displays.

There were several students who decided to save their calculations to RStudio file and who attempted to use saved file to subsequent calculations. Later, they reported multiple error messages regarding syntax and obtained output including incorrect calculation(s). Among those students, one decided to save her calculations to *filename.r* format, she was able to access it, and use it multiple times for subsequent calculations correctly after consultations with one of the authors and her independent research online.

In Session 3 and Session 4 students' activities the authors noticed that students in general understood the need of uploading a *.csv-type* of file into R/RStudio and some students were successful in this activity at their first attempt, (*aha moment*: no more typing multiple-variable data set(s) with *large* number of cases!). The authors observed that some students had to revisit topics involved in this activity related to histogram(s) and boxplot(s). They did it on their own, (extension was granted). It was their first time when they were exposed to analyses of a multi-variable data set including comparisons of distributions for selected variables.

Assignment 2 varied in quality and format. Some students followed directions with respect to format and presented their work early, no resubmission was necessary for those students. There were students who created their own format, (*viz.* copied console outputs and graphs into MS. Word formatted file, then answered questions). Some students requested more time for completion of Assignment 2 since quality of submitted homework was in question.

In Session 5, the authors summarized their experiences and pointed to numerous instances of students' creativity related to completion of assignments. For example, some students who did not want to replicate an assignment or part of it just printed R console content for themselves for future reference. Another student was able to learn on her own saving workspace procedure and demonstrated her ability to access saved file again. In addition, students were able to save graphical displays created in R/R Studio in various formats outside of R/ R Studio using copy-and-paste procedure.

WHAT ARE POSSIBLE IMPROVEMENTS TO USING R/RStudio IN AN INTRODUCTORY STATISTICS COURSE AS A “*Calculator*” APPROACH

The authors believe that introduction to R/RStudio in an introductory statistics course can take place in a computer laboratory classroom where all machines have *R* and *RStudio* programs uploaded or/and regular classroom equipped with access to desktop/laptop connection to LCD projector. The use of R/RStudio does not have to consume excessive amount of lecture time and can be left to discretion of instructor. Although, instruction related to R/RStudio should start early in the course and last through duration of the semester.

The authors find that students should have these programs installed on campus computer labs available for all after instruction time and access to trained in basic R/RStudio tutors in these settings. Newly minted tutors may be made of students who successfully completed the course, are interested in learning more about R and willing perform tutor function.

The authors believe that “*Less Volume, More Creativity*”³ approach is a key to successful introduction of R/RStudio to students. According to Randal Randal J. Pruim et.al ideas initial set of commands should be relatively small, coherent and powerful. In addition, students should be able to use R/RStudio to execute numerical summaries, graphical summaries, and create linear models.

The authors found that students liked to use Internet resources such as YouTube videos for instruction in R/RStudio as well as instructional tools in learning about selected statistical topics. Some students claimed that abundance of Internet resources may be a reason for not purchasing expensive textbook(s).

WHAT DOES THE FUTURE HOLD ...?

The authors believe that recent technological developments in computer science and data science cause already changes in ways introductory statistics courses are being taught. Such changes may include ability to work with *medium-size* and *large-size* data sets and ability to manipulate these sets in order to create graphical multivariable summaries, summaries that were not available in recent past few years.

The authors think that future challenges in teaching various statistics courses could be met with using appropriate teaching tools, one of them may be R with abundant number of packages and R Studio being used as computing device in a *calculator mode*.

The authors believe, that sooner or later some instructors may be tempted to utilize In-Cloud computing using R/R Studio in order to be independent of installation of R and RStudio packages.

³ Randall J. Pruim; Nicholas J. Horton; Daniel T. Kaplan; Project MOSAIC, *Start Teaching with R; Preliminary Edition*



Example of Student's Work-Assignment 1 Report

MAT 150-171W

Assignment 1 Student Worksheet

Open R Studio, write a command in console pane, execute it by pressing *ENTER* and then, copy and paste obtained answer to a box below.

REMEMBER to copy and paste command from console to this worksheet; see your model solution to Assignment 0 worksheet!!!

Simple Linear Regression Assignment 1

The data set below contains two variables: Study Time in minutes students spend on preparation for Quiz 2 and Grade in percent obtained on Quiz 2 (Assume that requirements for using linear regression model are satisfied.) Follow directions for the following tasks: #1-#17.

1. StudyT

```
studyT<-c(45,78,80,83,55,45,30,25,90,25,30,15,40,25,30)
```

2. Enter your data into R Studio console quiz 2

```
quiz2<-c(55,70,85,78,65,70,75,85,95,46,48,75,70,80,85)
```

3. Determine Summary Statistics for StudyT variable by typing into console

```
summary(studyT)
```

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
15.0	27.5	40.0	46.4	66.5	90.0

4. Determine Summary Statistics for SQuiz2 variable. Summary

```
summary(quiz2)
```

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
46.00	67.50	75.00	72.13	82.50	95.00

5. Determine standard deviation of study time variable

```
sd(studyT)
```

```
[1] 24.8619
```

6. Determine standard deviation of Quiz 2 variable

```
sd(quiz2)
```

```
[1] 14.0299
```

7. Determine correlation coefficient of Quiz 2 and study time variable

```
cor(studyT,quiz2)
```

```
[1] 0.3520541
```

8. Determine r-squared value

```
cor(studyT,quiz2)*cor(studyT,quiz2)
```

```
[1] 0.1239421
```

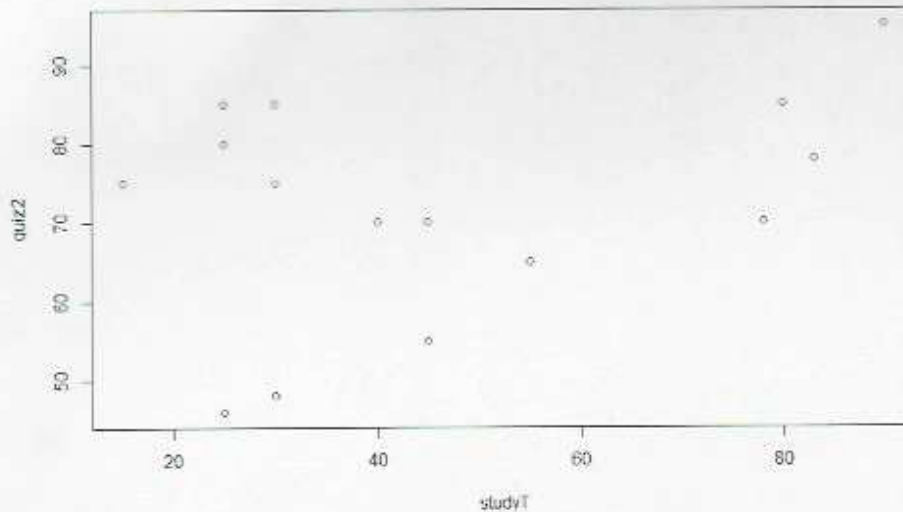
9. Explain what it (r-squared) means in context

```
[1] "just about 12.4% of variation in grade can be explained by variation in study time"
```

10. Graph scatterplot of Grade 2 vs. StudyT plot

Source: homework assignment from <http://www.pbialas.com/sunday-m150.html>

MAT 150-171W



11. Determine linear regression model equation lm1

```
lm1<-lm(quiz2~studyT);lm1
Call:
lm(formula = quiz2 ~ studyT)

Coefficients:
(Intercept)      studyT
  62.9151         0.1987
```

12. Write the equation of the linear model; $\hat{y} = b_0 + b_1x$

```
[1] "y-hat=62.9151+0.1987*x"
```

13. Write the equation of the linear model in context

```
[1] "averagequiz2grade=62.9151+0.1987*studytime"
```

14. Explain what the slope means in this context

```
[1] "for every minute change in study time average grade on quiz2 will change by 0.1987
%(just about 0.2%). OR for every 10 minutes increase in study time it is expected that
the grade will increase by 1.987%"
```

15. Explain what the y-intercept means in this context

```
[1] "average quiz2 grade for a student whose study time is ZERO minute, (e.g.Average quiz
22 grade=62.9151+0.1987*0, average quiz2 grade= 62.9151+0, average quiz2 grade=62.9151"
```

16. Estimate Quiz2 grade for a student who studied 30 minutes

```
"average quiz2 grade=62.9151+0.1987*study time, average quiz2 grade=62.9151+0.1987*30,
average quiz2 grade=68.8761"
```

17. Estimate Quiz2 grade for a student who studied 330 minutes and comment on it.

```
[1] "average quiz2 grade=62.9151+0.1987*study time, average quiz2 grade=62.9151+0.1987*
330, average quiz2 grade=128.4861" [1] "comment: 330 minutes is far from range of value
s for studyT variable, therefor the grade of 128.4861% is not realistic by common sense
measure"
```

Source: homework assignment from <http://www.pbias.com/sunday-m150.html>

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Mathematics Understanding of Elementary Pre-Service Teachers: The Analysis of their Procedural-Fluency, Conceptual-Understanding, and Problem-Solving Strategies

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Abstract: *Students in an elementary teacher preparation program at a Hispanic Serving Institution in deep South Texas were asked to solve non-routine, problem-solving activities. They were administered five tasks during one semester, as part of a mathematics methods course. Two experienced raters assessed the student's solutions to the non-routine problem-solving mathematical task using a mathematics understanding rubric that scores the Procedural Fluency (PF), Conceptual Understanding (CU), and Problem Solving/Strategic Competency (PS/SC). The research question was: What are the changes in procedural fluency, conceptual understanding, and problem solving-Strategic Competency in elementary preservice teachers after engaging in a series of non-routine problem-solving tasks? This is an ongoing research project, and preliminary results indicated that the teacher's candidates made improvements in each of the three measurements, demonstrating that they are able to successfully use procedures, and have adequate conceptual knowledge for problem solving.*

Keywords: Problem-Solving, Elementary, Preservice Teachers

Pre-service mathematics teachers' (PSMT) education programs are required to prepare candidate in both content and pedagogy (National Council of Teachers of Mathematics [NCTM], 2017). Mathematics education researchers have contended that it is very important for pre-service elementary teachers develop deep and connected understandings of mathematical ideas (Schoenfeld, 2007). One critical area is problem solving. Mathematics teachers are required to promote reasoning and problem solving with understanding among their students, while engaging them in productive discussions that elicit and enhance their learning acquisition (NCTM, 2014). Lam et al., (2013) suggested that problem solving should be infused in content courses. Thus, implying that problem solving is a critical ability to hone in mathematics. It is relevant that pre-service elementary teachers get involved in activities that foster their ability to engage in and teach problem solving (Olanoff, Kimani, & Masingila, 2009, p. 1299).

The purpose of this study was to examine pre-service elementary teachers' mathematical

understandings when regularly provided opportunities to solve non-routine mathematics problems. To this end, we present preliminary results of an ongoing research project that looks to answer the following question: What are the changes in procedural fluency, conceptual understanding, and problem solving-Strategic Competency in elementary preservice teachers after engaging in a series of non-routine problem-solving tasks?

PERSPECTIVES

Typically, teachers have their students solve problems after introducing concepts and procedures that follow examples and prescribed algorithms that require memorization, rather than creativity and strategic competence to solve non-routine problems (National Research Council [NRC], 2000; 2001). The NCTM (2017) considers important that teacher preparation programs should provide opportunities to challenge their mathematical knowledge and ability through the use of high cognitive demand mathematical tasks, involving problem solving and reasoning, and where they are challenged to explore different strategies and solutions paths. It is necessary to engage pre-service elementary teachers in non-routine problem-solving mathematical activities, in which they have the opportunity to “understand [and reason] about problem solving processes” (Koray et al., 2008, p. 1). Hence, elementary teacher education programs should include opportunities to develop conceptual understanding through the use of non-routine problem-solving tasks, which would “significantly influence how and what [they] teach, and how and what their students learn” (Olanoff et al., 2009, p. 1299).

Many teachers, and in particular elementary teachers, have expressed their discomfort when it is necessary to implement problem solving activities with their students (Wilburne, 2006). However, to reduce the anxiety that it may produce, it is necessary that teachers have experiences with solving non-routine problems that help them build their confidence and ability. Mastering the art of problem solving requires extra time, which often is considered a barrier to its implementation, and a disposition to understand the potential of teaching mathematics through problem solving. Similarly, elementary teachers have shown to be uncomfortable teaching mathematics (Wilburne, 2006). This can be attributed to teachers’ poor self-efficacy for teaching mathematics anxiety due to lack of knowledge or simply because of their negative attitudes toward mathematics (Bursal & Paznokas 2006).

Teacher-candidates need to experience and face the struggle of solving different types of problems, which develop, not only their mathematical concepts, but also their ability to address student solutions from different perspectives. Problem solving and reasoning were viewed as critical elements of mathematics teaching to the extent that Koellner, Jacobs & Borko, (2007) have incorporated a problem-solving approach into their design of a mathematics teacher professional development program, called the *Problem-Solving Cycle*. This involves teachers engaged in problem solving, video analysis of the implementation, and analyzing student work samples.

Historically, problem solving has been a part of the mathematical curriculum (Schoenfeld, 2011), and it becomes necessary to assess mathematics proficiency (Schoenfeld, 2007). Further, according to the NCTM (2012) problem solving skills are the main expectation of mathematics. Yet, teachers have difficulty implementing non-routine activities that are open-ended and require reasoning and problem-solving strategies. Phonapichat, Wongwanich, & Sujiva, (2014) argued that these may be due to the fact that teachers fail to connect real-life situations with the mathematical content, ask students to memorize algorithms and “keywords” to solve problems, do not deeply explain concepts behind textbooks problems, or they simple do not teach with understanding (p. 3171). All these affect students’ knowledge acquisition and comprehension, which is later reflected in poor achievement in mathematics. Therefore, preparing teachers candidates in the mathematical content –in particular in non-routine problems– and pedagogy needed are essential to have a “positive effect on [their] students’ learning” (Brabeck, et al, 2014, p. 5), as well as to increase their confident and self-efficacy.

METHODOLOGY

Setting

The study took place during the Spring semester of 2017, and an extension of the data collection is currently ongoing too. The participants were PSMT enrolled in a Hispanic Serving Institution in deep South Texas. The course in which the study took place had an enrollment of 28 PSMT, 100 percent were female. However, not all of them completed each of the tasks (See Table 3). This was due to students being absent when the task was administered. The tasks were administered at the start of class. They were allowed 15 minutes to individually solve the task. This was followed by sharing strategies and solutions with a partner. The instructor monitored the students as they were solving and discussing the tasks. The intent was to identify different solution strategies. Selected students were then asked to present their strategy to the whole group. During this time, connections were made by the instructor to the similarities and differences solution strategies. This allowed for all students to see different approaches that were mathematically efficient to less efficient solution methods.

Data Collection and Analysis

PSMT’s solutions to the problem-solving tasks were analyzed during a fourteen-week spring semester of an elementary mathematics methods course. They were asked to individually solve each problem-solving task and then shared their solutions with others in small groups following this student work samples were selected for presentation to the class. The unit of analysis was the individual PSMT’s responses to the mathematical tasks. Table 1 presents the name of each task, mathematical content addressed, and the date of implementation.

The solutions of the PSMT varied depending on the type of problem. They used different solution pathways in response to these non-routine problem-solving activities.

The Freckleham People Problem

The people of Freckleham are interesting creatures. Every Frecklehamer is different from the other and has at least one freckle and one hair, but no more than three freckles and three hairs. Make a list of all the different Frecklehammers.

Make a list of all the different Frecklehammers.

The mayor of Freckleham decided to improve the manners of his townsfolk. He issued an order: When two Frecklehammers meet, the one with the most hairs or freckles will greet the other and say, "I have more _____ than you have. A Frecklehammer might say, "I have more freckles than you have," or a Frecklehammer might say, "I have more hairs than you have." Or a Frecklehammer might not be able to say anything at all.

At a town meeting of all the Frecklehammers, the greeting, "I have more _____ than you have" was heard many times. How many times?

Figure 1 shows three different sample solutions for the FreckleHammer (Treffers & Vonk, 1987) task, in which they were asked to find the number of times a FreckleHammer said "I have more freckles or hair than you have." PSMT one (Student 1) used a trial-and-error strategy, which resulted in a less clear and less efficient solution. PSMT two (Student 2) represented the data in a table, and later used an ordered pair to obtain a final solution. The PSMT three (Student 3) used a table and a tree diagram to represent the data.

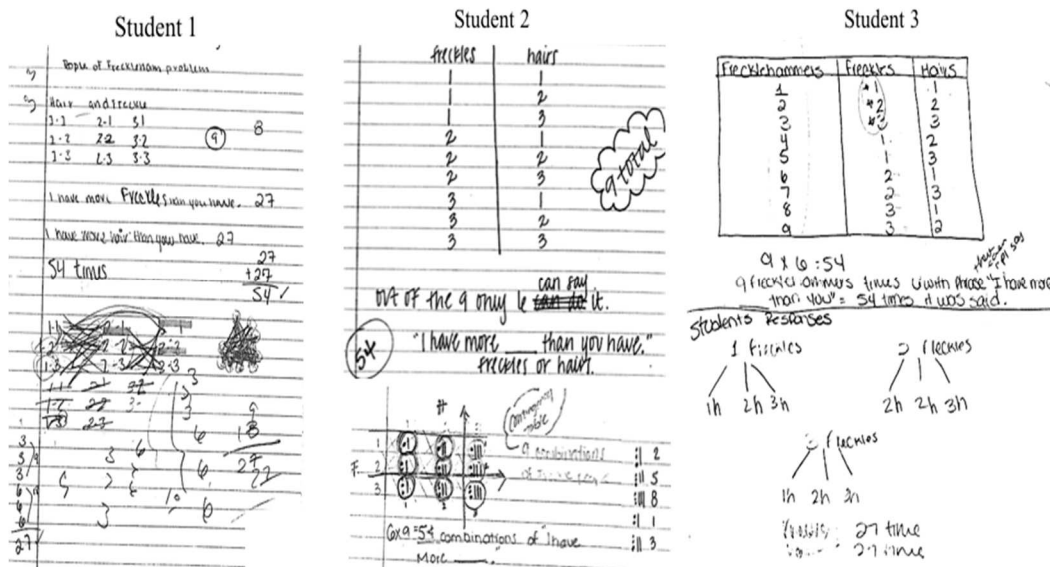


Figure 1. Example of student's solution for the FreckleHammer activity

Table 1

Task Names, Mathematical Content and Date of Administration

Task	Mathematical Content	Date Implemented
Freckle Hammer (Treffers & Vonk, 1987)	Logic	1/25/17
Rectangle Area*	Logic, area, perimeter	2/13/17
Vegetable Garden Fractions (Pelikan, DeJarnette, & Phelps, 2016, p. 332)	Fractions	2/20/17
Picking Pumpkins (www.mathwire.com)	Finding Patterns	3/22/17
Growing Caterpillar (Blanton, 2008)	Algebraic generalizations	4/12/17

* Retrieved from <https://sites.google.com/a/arlington.k12.ma.us/ms-tomilson-750-math/>

A rubric was designed that considers three mathematics proficiencies identified by The National Research Council [NRC] (2001): (a) Procedural Fluency (PF), (b) Conceptual Understanding (CU), and (c) Problem Solving-Strategic Competency (PS-SC). The reliability of the rubric was tested using Generalizability Theory (see Table 2). The G-coefficient for Conceptual Understanding was 0.86, Problem Solving was 0.88 and Procedural Understanding/Fluency was 0.92 (Telese, 1994). These coefficients indicated high reliability when rating each proficiency. The Mathematics Understandings Rubric (see table 3) was used to rate the solutions (Telese, 1994).

Table 2

Rubric's Generalizability Coefficients

Mathematical Proficiency	Generalize Coefficient
Conceptual Understanding	0.86

Procedural Fluency	0.92
Problem-solving/Strategy Competency	0.88

Table 3
Mathematics Understandings Rubric

Performance Level	Procedural Fluency	Conceptual Understanding	Problem Solving/Strategic Competency
0	No response	Lack of evidence to determine knowledge, or no attempt made	No response
1	Incorrect or very limited use of operations, more than one major error or omissions	Wide gaps in concept understanding, major errors made based on lack of conceptual knowledge	Unworkable approach, incorrect or no use of mathematical representations, poor use of estimation, evidence for lack of understanding
2	Some correct use of number operations but a major error or with several minor errors	Some evidence of conceptual understanding, but difficulty in using models, diagrams, and symbols for representing concepts or translating from one mode to another mode. Some evidence of the concept's properties	Appropriate approach, estimation used, implemented a strategy, possibly reasoned decision making, solution with observations
3	Appropriate use of number operations with possible slips or omissions, but without significant errors	Good evidence of conceptual knowledge. No major misconceptions; responses contain accurate use of models, diagrams, and symbols with evidence of translation from one mode to the other. Recognition of the meaning and interpretation of concepts. Some evidence of using concepts to verify or explain procedures	Workable approach, used estimation effectively, mathematical representation used appropriately, reasoned decision-making inferred, judge reasonableness of solution
4	Extended use of number operations	Clear understanding of concepts and associated procedures.	Efficient/sophisticated approach, estimation

without errors in calculations; appropriate use of models or representations

Effective use of diagrams, and symbols with broad translation from one mode or to another. Recognition of the meaning and interpretation of concepts to explain or verify procedures or conclusions

used effectively, extensive use of mathematical representations, explicit reasoned decision-making Solutions with connections, synthesis or abstraction

RESULTS

The students became more confident in themselves than at the beginning of the semester. They hesitant at the beginning to engage with the task and often asked, “Where do I start?” The sharing of solutions assisted those who were less able to develop a strategy. The less creative students became more creative in their problem-solving approaches. Over the course of the semester, students engaged in problem solving; as a result, their confidence for providing similar opportunities to their future students improved, for example one student noted, “I learned about how a simple math problem can be solved in many different ways, and how we can help our students in the classroom when that happens.” The student clearly indicates self-confidence in that when her future students are problem solving, she will know how to differentiate and compare strategies.

Inter-rater reliability was performed using percent agreement, where a difference of one was considered agreement. The raters conducted a calibration session prior to scoring. The percent agreement for each mathematics proficiency ratings had a low of 88 percent on the Vegetable Garden task’s conceptual understanding ratings to 100 percent on other tasks and proficiencies. Mean ratings were calculated for each of the three Mathematics Proficiencies from two raters. Table 4 presents the overall means and standard deviations for each task’s ratings for mathematics understandings.

Table 4

Tasks’ Means and Standard Deviations for Mathematics Understandings

Task Administered	Procedural Fluency	Conceptual Understanding	Problem Solving Strategic Comp.
Freckle Hammer n = 26	3.23 (0.59)	3.37 (0.37)	3.25 (0.67)
Rectangle Area n = 23	3.37 (0.53)	2.94 (0.73)	2.94 (0.79)
Vegetable Garden n =26	3.5 (0.50)	3.72 (0.40)	3.52 (0.59)

Pumpkin Patch n = 26	3.21 (0.45)	3.02 (0.48)	2.78 (0.51)
Growing Caterpillar n = 15	3.80 (0.32)	3.83 (0.52)	3.57 (0.70)
Total n = 116	3.40 (0.53)	3.34 (0.67)	3.34 (0.71)

Note: The number in parenthesis is the standard deviation.

When a MANOVA was conducted with the three dependent variables (PF, CU, and PS-CS) and the five tasks as the independent variables, the dependent variables were found to be highly correlated. A decision was made to conduct separate ANOVAs for each of the dependent variables because of the high correlation. The Levene's Test of Equality of Error Variances was found not to be statistically significant for Procedural Fluency and Problem Solving/Strategic Competency; hence, the Tukey Post Hoc test was conducted on the tasks. The Tamhane's Post Hoc test was used for Conceptual Understanding because equal error variance was not found. Statistically significant result was found for Procedural Fluency with $F(4, 111) = 4.80, p < 0.001, \eta^2_{partial} = 0.147$, for Conceptual Understanding with $F(4, 111) = 3.43, p < 0.001, \eta^2_{partial} = 0.262$, and for Problem Solving/Strategic Competency with $F(4, 111) = 6.56, p < 0.001, \eta^2_{partial} = 0.191$. The students' Procedural Fluency performance varied by task where they performed best on the 'Growing Caterpillar' task when compared to the 'Freckle Hammer' and 'Picking Pumpkins' tasks. Similar result were found for both Conceptual Understanding and Problem Solving/Strategic Competency ratings on the tasks. They performed best on the 'Growing Caterpillar' task when compared to the 'Rectangle' and 'Picking Pumpkins' tasks.

CONCLUSION

The PSMT displayed a variety of performance levels in relation to the task. The non-routine tasks were administered to provide practice at solving problems and to encourage modeling in order to demonstrate conceptual understanding. Also, meaningful mathematics discussions occurred to foster mathematics understanding. A limitation of the study was the small number of solutions to the 'Growing Caterpillar' task. The high performance may have been due to the smaller sample. Generally, the PSMTs scored higher on the 'Growing Caterpillar' where they had to express a pattern in algebraic terms, and the 'Vegetable Garden' task where they had to use knowledge of fractions. The PSMTs demonstrated a moderate level of Procedural Fluency. They could use appropriate number operations without significant errors, trending toward an extended use of number operations using appropriate models.

Regarding Conceptual Understanding, the 'Rectangle Area' task proved more challenging perhaps due the nature of the task being related to fractions, which had very little guidance embedded in the task to hint at a solution. They had to arrange areas and perimeters of rectangles

when given particular conditions and may have held weak content knowledge associated with area and perimeter. Generally, PSMT' rating for Conceptual Understanding revealed that they held no major misconceptions, used accurate models, diagrams, and symbols while being able to move from one representation to another and tended to use concepts to explain or verify procedures. Taking all the tasks together, their Problem Solving/Strategic Competency indicated that they used workable approaches, used estimation effectively, mathematics representations were used appropriately and demonstrated reasoned decision making while judging the reasonableness of the solution. However, the overall mean for Problem Solving/Strategic Competency indicates that problem solving ability could be improved to the extent that they should be able to use more efficient or sophisticated approaches to solving problems. Although Conceptual Understanding was rated consistently, the error variance indicated that the preservice teachers had a wide range of conceptual understanding related to the tasks.

Consequently, the preservice elementary mathematics teachers demonstrated that they are able to successfully use procedures, and they have adequate conceptual knowledge for problem solving. Their problem-solving's capabilities need sharpening to reach the efficient and sophisticated approach to problem solving. This would come about through enhanced content knowledge. As Olanoff et al., (2009) noted, teaching mathematics through problem solving requires the teacher to have deep understanding of the mathematics and need to anticipate different approaches. Improvements in the way the problems are chosen and used are necessary too. The non-routine tasks should be selected with a common thread, whether content, or for encouraging the development of generalizations. Similar to the results of Olanoff et al's study, it appeared that the process supported the teacher candidates' problem-solving ability.

Finally, to specifically answer the research question, it has been shown that overall the PSMTs ability in all three dimensions improved throughout course of the semester. However, there was inconsistent performance on the tasks from the first implementation to the last task in all three dimensions. This means that on some tasks the PSMTs performed lower than the task previously administrated. The result could be due to the nature of the tasks; some tasks are less challenging than others; thus affecting performance. This study will continue to examine PSMTs performance in all the dimensions of the rubric, and their beliefs and perceptions toward problem solving.

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APPENDIX A: Problem Solving Task

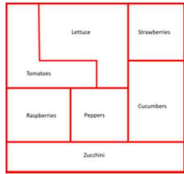
1. Freckle Hammer (Treffers & Vonk, 1987).

The people of Freckleham are interesting creatures. Every Frecklehammer is different from the other and has at least one freckle and one hair but no more than three freckles and three hairs. Make a list of all of the different Frecklehammers. The mayor of Freckleham decided to improve the manners of his townsfolk. He issued an order: When two Frecklehammers meet, the one with the most hairs or freckles will greet the other and say, "I have more _____ than you have." A Frecklehammer might say, "I have more freckles than you have," or a Frecklehammer might say, "I have more hairs than you have." Or a Frecklehammer might not be able to say anything at all. At a town meeting of all of the Frecklehammers, the greeting "I have more _____ than you have" was heard many times. How many times?

2. Rectangle Area.

Five rectangles are arranged from the least to the greatest area and named A, B, C, D, and E in order of increasing area. All dimensions are whole numbers, and no two rectangles have the same area. Determine the dimensions of all five rectangles using the following clues: The median area is 15 units². Rectangles B and D are squares. Rectangles C and D have the same perimeter. Rectangles A, B, and C have the same length. Rectangles D and E have the same length. Rectangles C and E have the same width.

3. Vegetable Garden Fractions (Pelikan, DeJarnette, & Phelps, 2016, p. 332).



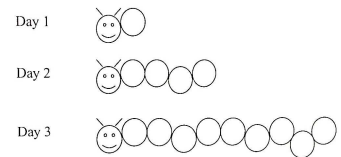
In the plot, what fraction of the garden is composed of Lettuce? What fraction of the garden do zucchini and cucumbers together use?

4. Picking Pumpkins (www.mathwire.com).

Allie was picking pumpkins for her school. She picked one pumpkin on the first day. She picked two pumpkins on each of the next two days. Allie picked three pumpkins for three days. Next she picked four pumpkins a day for four days and so on. If Allie continues this pumpkin picking schedule, on what day will she first pick 6 pumpkins? How many pumpkins will she have picked altogether for her school when she completes that first 6th day?

5. The Growing Caterpillar (Blanton, 2008).

A caterpillar grows according to the chart. If this continues, how long will the caterpillar be on Day 4? Day 5? Day 100? Day x? (Measure length by the number of circle body parts.)



APPENDIX B: Student work sample

1. Freckle Hammer.

Fleekchomms	Fleekrs	Hows	P. hic 3/3/2
1	1	1	
2	2	2	
3	3	3	
4	4	2	
5	1	3	
6	2	1	
7	2	3	
8	3	1	
9	3	2	

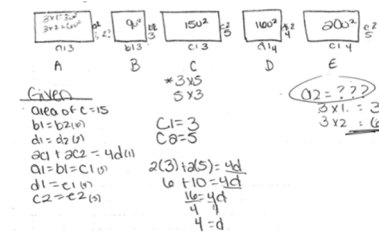
$9 \times 6 = 54$
 9 fleekchomms times 6 with phrase "I have more than you" = 54 times it was said.

Students Responses:

- 1 fleekies
 - 1h 2h 3h
- 2 fleekies
 - 2h 2h 3h
- 3 fleekies
 - 1h 2h 3h

4 fleekies: 27 time
 5 fleekies: 27 time

2. Rectangle Area.



A: $a_1 \times b_1 = a_2 \times b_2$
 $a_1 = 3, b_1 = 5, a_2 = 4, b_2 = 3.75$

B: $a_1 \times b_1 = a_2 \times b_2$
 $a_1 = 4, b_1 = 3, a_2 = 3, b_2 = 5.33$

C: $a_1 \times b_1 = a_2 \times b_2$
 $a_1 = 3, b_1 = 5, a_2 = 4, b_2 = 3.75$

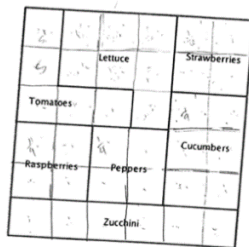
D: $a_1 \times b_1 = a_2 \times b_2$
 $a_1 = 4, b_1 = 3, a_2 = 3, b_2 = 5.33$

E: $a_1 \times b_1 = a_2 \times b_2$
 $a_1 = 3, b_1 = 5, a_2 = 4, b_2 = 3.75$

General: Area of C = 15
 $b_1 = b_2 = 5$
 $d_1 = d_2 = 3$
 $a_1 \times a_2 = 4d_1$
 $a_1 = b_1 = c_1 = 3$
 $d_1 = c_1 = 3$
 $c_2 = c_2 = 3$

$2(3) \times 5 = 4d$
 $6 \times 5 = 4d$
 $30 = 4d$
 $4 = d$

3. Vegetable Garden Fractions.



Lettuce: $\frac{4}{36}$
 Zucchini: $\frac{4}{36}$

4. Picking Pumpkins

the sake of the united experience of the composite mind. However, when our mind constructions are used for the purpose of highly formalized practices, such as science or mathematics, we are forced to face these paradoxes to understand the limitations of our work.

Noson S. Yanofsky in his book “The Outer Limits of Reason” makes an attempt to define the limitations of human and computer languages, calculations, and procedures. He is successful in creating a text that can be read many times to offer a manifold of topics and their interconnections.

I found the book incredibly inspiring, intriguing and stimulating, and it seems that philosophers and thinkers of all kinds would benefit from reading it. The book contains carefully crafted discussions of which tasks can and which cannot be performed within the limitations imposed on the performance by the performers. In mathematics these limitations have been changing throughout the centuries, beginning with the ruler and compass for the ancient Greeks and ending with programming for modern scientists. Regardless of the times and the set ups, the limits of reason are always present and affect the way we think, live and work.

The book has a structure of an academic publication and is divided into ten chapters with each chapter being divided into several sections and subsections. Every chapter begins with generous mottos that reflect on the concepts to be discussed in the following text. Every chapter closes with a section that provides a follow up of the themes touched upon in a form of a further reading. It is not just a list of volumes related to the topic, but each paragraph gives a connection and detailed description of exactly which themes are expanded. Since the book is used as a textbook, this additional section encourages students to read beyond the volume, but the way this encouragement is performed, appears to be highly effective.

“The Outer Limits of Reason” begins with analysis of a problem of covering the chess board with domino tiles, which is a classic in the world of puzzles and games. Chapter 2 is a collection of various well known or less known paradoxes including Epimenides’ paradox, Quine’s sentence, barber paradox, heterological paradox, Russell’s paradox, Yablo’s paradox, “interesting number” paradox, Berry phrase, etc. Chapter 3 discusses dilemmas of identity for objects and humans with an example of the ship of Theseus, asking whether a ship of Theseus can remain under its name after replacing its planks? Similarly, do humans retain their identity after seven years even if all cells of the body are replaced? This is quite engaging topic studied intensely by western and eastern philosophers. For example, the dilemma of existence was intensely studied by Buddhist masters who introduced the Middle Way saying that the composite things (as we are, and the ship of Theseus is) neither exist nor non-exist and making a distinction between the “inherent existence” and “ultimate existence.”

What is particular about the book is that the author does not simply present and discuss these paradoxes but makes them into a starting point of a discussion about a nature of things. For example, the Zeno’s paradox about taking a step that is equal to a half of a given distance is a starting point of a discussion about ability to measure distances of the lengths not smaller than the

Planck's length. Examples of modern science and popular culture are discussed throughout the book. A game of the Monty Hall Show gives rise to a dispute about the strategy of knowing others' strategy and using it for creating our own strategy (The situation resembles the "tit for tat" strategy or the prisoners dilemma). Fuzzy logic is motivated by examples of vague definitions of being bald or having uncertainty of how many grains can be taken away from a heap of grains, so it remains a heap? Chapter 4 contains proofs that the set of rational numbers is equinumerous to the set of natural numbers. At the same time, it discusses the original proof of George Cantor that the set of rational numbers is not equinumerous with these sets. The author brings to this chapter the axioms of the Zermelo-Frenkel set theory and the axiom of choice, and poses the questions whether mathematical theorems are discovered or invented?

Classical problems from computer science are presented in chapter 5: computing complexities, finding paths on Konigsberg bridges, traveling salesman problem, Hamiltonian cycle problem, set partition, etc. Here the author goes one step beyond the standard disputes and presents superexponential problem and PSPACE problems that are becoming even more relevant during the times of the Big Data. The importance of the dilemma of the complexity of a problem "P vs. NP" was recognized by mathematicians and listed as one of the seven Millennium Problems (<http://www.claymath.org/millennium-problems>). Computing impossibilities such as Halting Problem or Zero Program Problem and their interconnections are analyzed in chapter 6. Here the author discusses touchy topic whether machines will ever be able to mimic the actions of the human mind, pointing out that humans have self-reference and consciousness, which are currently not programmable, but the future may bring surprises in that matter.

Chapter 7 carries elements of quantum mechanics and relativity theory and discusses whether the two theories can be united into one, but the main theme of this chapter is predictability of a system. With an example of the three-body problem the book unfolds the problem of chaotic systems which simply do not follow the same arrangements as the tame systems such as the two-body problem.

A clear distinction between science and non-science is discussed in Chapter 8 together with connections among the mind, science, mathematics, and the universe. Here three fundamental questions are brought to the surface and analyzed carefully:

1. Why is there any structure in the universe?
2. Why is the structure that exists capable of sustaining life?
3. Why did this life sustaining structure generate a creature with enough intelligence to understand the structure?

In the light of the issues with the observer touched upon in chapter 7 these questions pose quite a mind dilemma whether humans as evolving creatures are on the path of reaching the enlightening purpose of observing the universe. But it seems that we are still far from understanding it sufficiently. Especially because in the presence of an intelligent observer who can observe the

universe, who would observe the observer? The chapter concludes with a discussion of metascience and its conservation laws: momentum, angular momentum, energy that correspond to symmetries of the system: place, orientation, and time. Limitations of mathematics are presented in Chapter 9. Often self-imposed such as already mentioned Greek School of Mathematics' constructions with a ruler and a compass but sometimes generated by the limitations of its structure as in case of arithmetic. Thus, Gödel incompleteness theorem is carefully presented with its implication of proving consistency of one system in a another, stronger system.

With a large quantity of topics touched upon in the book the author makes in the last chapter a brilliant decision to provide a summary of previous divagations. Here all limitations are classified in terms of the kind of limitations they admit. Paradoxes are placed in a table and characterized in terms of objects of self-reference and the type of consequences. At the end of the book the definition of reason is provided and discussed in detail.

Everybody who is interested in philosophy of science will benefit from reading this book, which is packed with exciting topics. Choosing just a few to reflect on the entire content is really hard.

To close, I would suggest that the author includes exercises and open problems related to the topic in the future editions of the book. The readers will highly benefit from having their own walk of mind parallel to the dispute presented in the book.

Creating Use of Different Representations as an Effective Means to Promote Cognitive Interest, Flexibility, Creative Thinking, and deeper understanding in the Teaching of Calculus

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Abstract: *In this paper, we discuss some ways of developing intrinsic interest in mathematics in the course of teaching of calculus for engineering students. Our experience shows that the use of different representations for solving various problems promotes cognitive interest, creative thinking and deeper understanding of mathematics and assists essentially in many fields of knowledge. The examples in the paper are chosen from non-routine problems associated with understanding of two-way correspondence between formulas and graphs in rectangular and polar coordinate systems.*

Keywords: Calculus teaching, different representations, creative thinking, exercise- based learning, problems-based learning, equations and their graphs, rectangular and polar coordinate systems, cognitive motivation, challenge problems, surprising answers.

INTRODUCTION

Our long-term experience in teaching of basic mathematical courses for engineering students indicates that many of them lack cognitive motivation and have low interest in mathematical learning [1]. The students often see this discipline as a boring one, not too important for their future work and think that they are required to learn it only because it is a part of the curriculum. A similar observation was noted by many educators from different countries [12], [17], [21], [22]. This situation in many cases follows from previous learning in the secondary and high school, where the basic requirement was to solve a large amount of dull technical exercises [9], while solution of challenging problems connected with interesting real-life applications was not emphasized enough. Many educational researches deal with this problem, [18], [25].

We are trying to change this attitude of engineering students to mathematics taking into account the well-known phrases: "*Education is the kindling of a flame, not the filling of a vessel*" (Socrates) and "*Study without desire, spoils the memory and nothing is retained of what was taken in*" (Leonardo da Vinci). Therefore, developing intrinsic interest in mathematical approaches to various problems and promoting the feeling of beauty of unexpected short, logical solutions of complicated tasks, should be an important goal of the lecturers [1], [2], [19]. We try to achieve this in a number of ways. In this article we report about our experience in increasing of interest and cognitive motivation of novice engineering students within the framework of teaching the theme "equations and graphs" in the first year Calculus course and simultaneously enhancing the students comprehension of main mathematical ideas [4], [5], [20].

The question is how to create intrinsic cognitive interest in mathematics learning among first year engineering students [16], [17]. Most of them have previous experience of "exercise-based learning" only - from examples demonstrated by the teacher, or solving a long series of exercises of the same type. To a certain extent, it was reasonable in the "pre-computer" age, when for many professions it was enough to acquire only routine skills for solving standard problems. In the digital age, the motivation to learn mathematics has decreased because now computer programs allow implementation of almost all mathematical techniques, including very complicated symbolic calculations. On the other hand, we see great opportunities in using modern technology to stimulate the student's interest and reveal the beauty of mathematics as a universal tool for solving different problems in many fields of knowledge [17], [19]. In our teaching, we always emphasize Galileo's statement: "*The Book of Nature is written in the Language of Mathematics.*" We think that our mission is to bring to the understanding of every student learning STEM, the deep meaning of these words. To our regret, most novice students do not understand this statement and are not even familiar with it. In our opinion, if the students will really understand the vast possibilities that the language of mathematics can offer, it will contribute a lot to their motivation for studying mathematics.

PROBLEMS IN TEACHING GRAPHS OF EQUATIONS IN RECTANGULAR COORDINATE SYSTEM

In this study, we examined the effect of teaching equations and their graphs in the rectangular and polar coordinate systems in a way that promote cognitive activities of students, as well as enhances their interest in the subject using modern technology [19] to see the strength and beauty of simple mathematical formulas that describe complicated objects. This theme, whose understanding is very essential for mathematics learning and teaching, has been investigated by various authors ([26], [20], [6]-[8]).

Why did we choose this subject? It is not familiar to both "strong and weak" novice students, does not require a great deal of "unknown knowledge", and so is suitable for heterogeneous groups. In addition, we chose this topic because it is essential for engineering students and allows the demonstration of the possibilities of modern computer applications, as well as the creative thinking for effective hand drawing, based on the theoretical understanding of the topic. The important additional goal is to teach students how to use their previous knowledge and cognitive tools that they already possess, as well as technological tools (calculator, computer and applications) to deal with previously unknown problems. We try to use different representations (verbal, numerical, symbolic, visual, and real and computer model presentations) in order to create the deep and stable understanding of the issue under study [3], [10], [11], [13]-[15], [24].

In the first lectures in the Calculus course, we mention the basic mathematical means for locating points and defining curves and figures in a plane, the rectangular (Cartesian) coordinate system, and a polar coordinate system, and give examples for understanding the relationships between formulas and graphs.

Even though the students' awareness of the rectangular coordinate system from their previous studies, we indicate that most of them have difficulties in understanding and sketching graphs of simple equations, not exposed in previous studies [20]. An example is the question of what the graphs of such equations are:

$$1) \sqrt{(x-2)} + \sqrt{(y+1)} = 0; \quad 2) (x-2)(y+1) = 0; \quad 3) x^2 - y^2 = 0;$$

$$4) \sqrt{x}\sqrt{-y} = 0; \quad 5) \sqrt{1-x^2}\sqrt{1-y^2} = 0; \quad 6) \max(|x|, |y|) = 1.$$

Only a small percent of the students gives correct answers without the teacher's prompt (even to the questions 1-3). This indicates a lack of understanding of the relationship between formulas and graphs and an incompleteness of their thinking. They try to use their "memory" but are unsuccessful.

Here are some teachers' questions to guide the students' thinking:

1. What is a graph (locus) of equations?
2. How can you check that a certain point (a, b) belongs to the graph of an equation?
3. Can you specify several points belonging to the graph of a given equation?
4. Can you give more points of the graph?
5. What is now your suggestion about the graph of given equations?
6. What is the reasoning of your suggestion about the graph?

By such questions, we try to activate a student's activity and to form his/her independent thinking in solving new problems.

We also asked the reverse type of questions. For example:

"Give an equation whose graph consists of only two points $(1,1)$, $(-1,1)$ ".

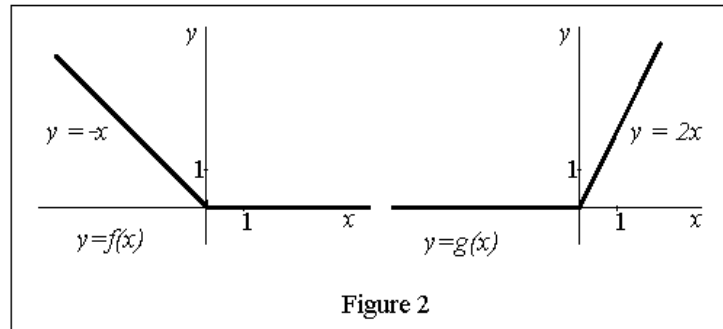
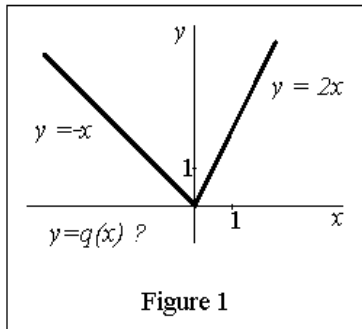
One of the possible answers is: $(x^2 - 1)^2 + (y - 1)^2 = 0$.

We also try to stimulate creative thinking of our students by questions such as:

Problem 1. Construct a formula, whose graph is similar to figure 1.

Hint 1. Can you find formulas for functions with graphs presented in figure 2?

Hint 2. How can we use the formulas whose graphs were found in figure 2 to construct the formula of the graph in figure 1?



The possible solution using ‘min’ and ‘max’ functions [23]:

$$f(x) = \max(-x, 0) = \frac{-x + |x|}{2}; \quad g(x) = \max(2x, 0) = \frac{2x + |2x|}{2} = x + |x|;$$

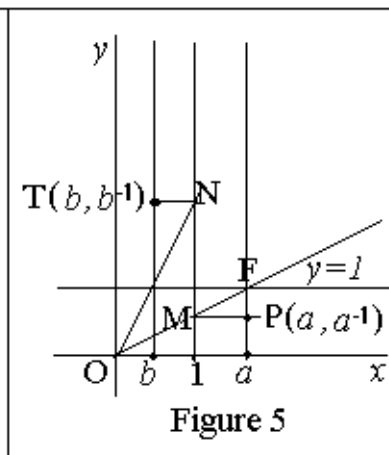
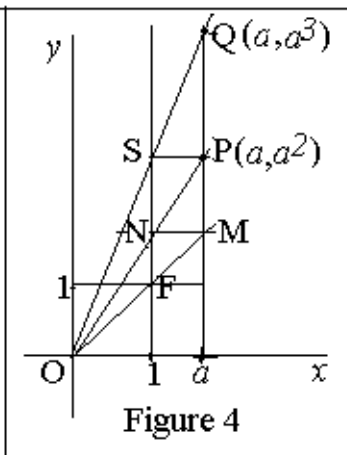
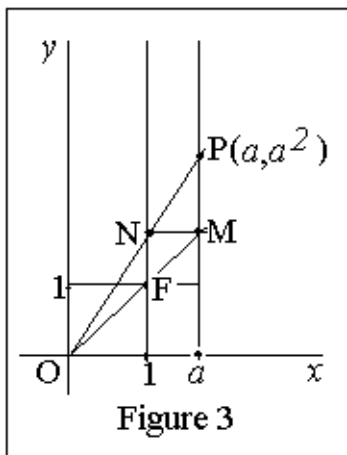
$$\left[\text{here we used the well-known formula : } \max(a, b) = \frac{a + b + |a - b|}{2} \right]$$

$$q(x) = f(x) + g(x) = \frac{x + 3|x|}{2}.$$

Our intention is to change the routine schema to which students gets accustomed: “The sample example, then the teacher’s solution, then using the example of samples, and exercises repeated many times”. We prefer teaching focusing in another direction: “A new question, then an unaided thinking problem with hints of the teacher (if needed), then answers, afterwards discussion and generalization (if possible)”.

We often try to ask new questions or demonstrate examples with “surprising” answers:

Problem 2. Find the point $P(a, a^2)$ of the graph $y = x^2$ without any measuring or calculations (See solution in figure 3).

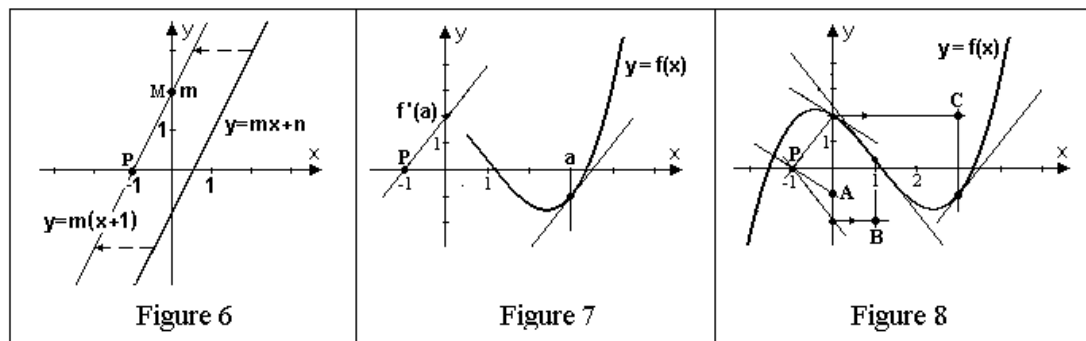


Try to justify this solution.

Problem 3. Generalization of problem 2: How can we construct the point $Q(a, a^n)$ ($n \in \mathbb{N}$)? (See solution for $n = 3$ in figure 4)

Problem 4. Think about the construction of the points of the graph $y = x^{-1}$

An important concept in the Calculus course is the slope of a straight line. This concept should be familiar to students from High School. Nevertheless, for the question “How can we find the slope of the straight line drawn on the rectangular coordinate system without measuring and calculations?” This question asked many times of many students, but none of them gave the proper answer. The graphical method of finding a slope shown in figure 6. Every time this simple construction was seen surprise and sometimes even with delight. (Note that this method is convenient for plotting a graph of a derivative of the function given by its graph (Figures 7, 8)).



Note that the widespread opinion among students and some teachers too is that the slope m of the straight line is equal $tg(\theta)$, where θ is the angle of inclination of this line to the x -axis. This statement is true only in the case of equal scales on the x and y -axis (e.g. the length of the segment $[0,1]$ on x -axis is the same as on the y -axis). This makes no sense in the case of graphs of physical dependencies (for example $s=s(t)$ where the time axis, marks seconds, and the distance axis, marks meters. This formula, $m = tg(\theta)$ gives an incorrect result for most graphs of “abstract functions” obtained by computer applications and in the case of hand-drawn graphics as well. In this occasion, we want to give an example of a question that one of the students asked the teacher about the task of sketching the graph of the function $y = x^2 + 50$: “How can I do this because the graph does not fit on my drawing?” The reason was that the student marks 1 on the x and y -axis at the same distance from the origin point of the coordinate system. In this connection, we always say to students “Fit the scales to the given graphic task”. We also want to add that the graphic method for finding a slope of a straight line, described in figure 6, fits for all the above cases.

PROBLEMS IN TEACHING GRAPHS OF EQUATIONS IN THE POLAR COORDINATE SYSTEM

During the first time mentioning the polar system in Calculus, we usually make sure that this topic is completely unfamiliar to most of the novice students. We introduce this system in connection with trigonometric functions and with the need for radian measure of angles. To clarify the state of the initial knowledge in this subject, we often write on the class-board the question: " $\sin(90)=?$ " and ask students what the answer is. Each time we get the same answer: $\sin(90)=1$. Afterwards, we ask the following question: "Check the answer on your calculator. Is it the same for all students in the class?" The next question is: "How many different answers can a calculator give for $\sin(90)$ computation?" After that, we discuss the three options for measuring the angles programmed in the calculator "degree, radian, grad" and how they are defined. This theme is new and is of interest to students. Since the radian measurement of angles is prominent, we deal with it in detail.

The radian measurement of angles little known to students and requires special attention in the formation of this concept. We use geometrical and real three- dimensional circle representations of this method of measuring angles (figure 11).

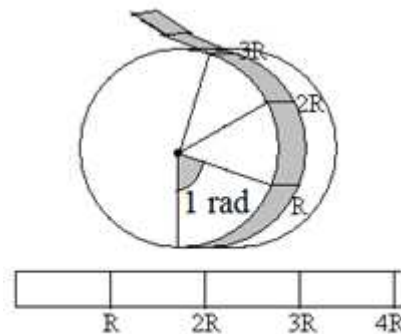


Figure 11

We explain that in the study of trigonometric functions, the concept of rotation of a point along the circle is important, and the value of rotation measured in radians.

We ask the students a few questions to make sure they understand this concept. For example:

Problem 5. The car drives along a ring road with a diameter of 10 km and performs a turn of 8 radians. *a)* What distance along the road did the car cover? *b)* What is the length of an arc of a θ radian in a circle with radius R ?

Each time we as lecturers are convinced that a formal concept definition is not quite enough to form a correct concept image for the students [27].

After a minimum preparation and definition of Polar coordinates on the plane, we turn to the graphs of equations in this system and it is a new and intriguing theme for all of the students. We begin with the simplest formula $r = \theta$ and ask students to answer the following question:

Problem 6. What does the equation's $r = \theta$ graph look like?

The first answers are usually: “a straight line”, “a circle”, or “we have not solved such problems”.

Then the lecturer suggests that the students draw the graph by themselves based on all their own knowledge. However, in almost all cases, students do not know how to start. We are giving a few hints to direct the thinking of students. Following are possible hints:

1. What is a graph of an equation? (The graph of an equation in two variables is the set of all points and only these points, whose coordinates satisfy the equation);
2. In the equation $r = \theta$ what is “ r ” and what is “ θ ”?
3. Can you plot a point with coordinates $r = 1, \theta = 2$? Does this point belong to the graph of the equation $r = \theta$?
4. Can you specify a few points belonging to the graph of the equation $r = \theta$?
5. Can you plot the graph using the following table of values?

θ :	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$3\pi/4$	π	$3\pi/2$	2π
r :	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$3\pi/4$	π	$3\pi/2$	2π

Plotting of these points and finding that the curve is a spiral is always of great interest to students.

Note that it is important to make the students feel that they can to initiate and develop independent thinking by themselves.

To reinforce the understanding of the radian measure of angles and as an example of creative engineering thinking, we demonstrate to students the next simple construction for drawing the curve $r = \theta$ (Archimedes’ spiral) that does not require any measuring or calculations.

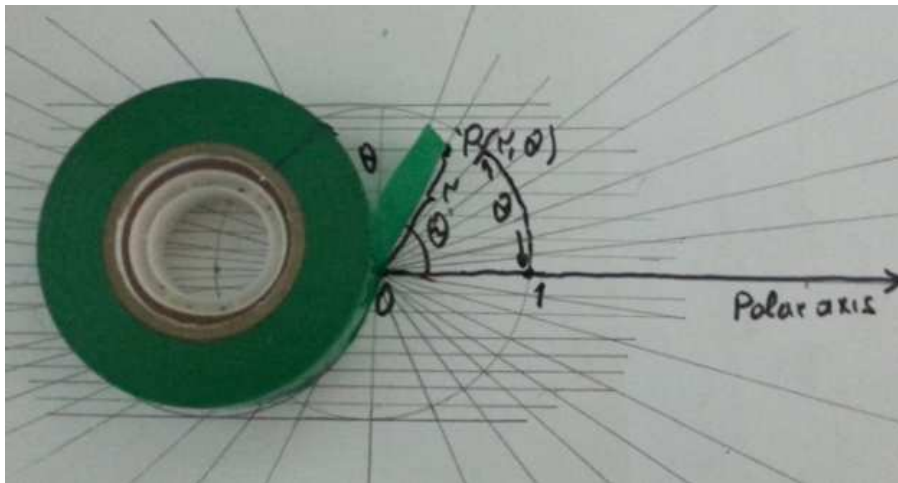


Figure 12. The electric tape model for drawing the curve $r = \theta$

Problem 7. What does the graph of the equation $r = \sin(3\theta)$ look like?

We give time to students to reach their own conclusions. Then we give the appropriate hints if necessary.

We strongly encourage students to use computer graphical applications to verify the responses received, encourage them to use technology and enhancing digital literacy for improving their understanding [17], [19].

In the second mid- test of the semester, we asked the following question: draw the graph of the equation $r = \frac{1}{\theta}$: 85% of examinees give correct answers.

A week later, a questionnaire was given to students about their feelings regarding the subject 90% answered that they understood the subject well and asked for new and intriguing questions. They also noted that the graphical applications helped them understand and test their answers and their self-confidence increased.

CONCLUSIONS

Our long-term experience shows that the use of such questions as mentioned earlier, their formulations and their solutions, when used in conjunction with transitions from one representations (verbal, symbol, graphical, numerical, computer, physical, real-life) to another, increases students' interest in mathematics learning and contributes to a deeper understanding the subject being studied.

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Blending Team-Based learning with Standards-Based Grading in a Calculus Classroom

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Abstract: *We present here a Teaching Practice paper explaining how to prepare a course structure that implements both Team-Based Learning and Standards-Based Grading. The main goals of this course design were to increase student engagement, decrease mathematical anxiety, and more adequately prepare students for future courses. Team-based Learning allowed us to remove the majority of lecture from the classroom, leaving time for exploratory activities. Standards-Based Grading allowed for reduced subjectivity in grading, while ensuring students were accountable for their own success and more well-prepared for future courses. These teaching methods were combined in a first-semester calculus course, for two sections of 30 students, at a small mid-western liberal arts college. The method is covered in detail, including samples of how the student's readiness and mastery were assessed. Students reported feeling more engaged in the classroom, more accountable for their own success, and more likely to prefer future courses where this method would be employed. Reports from faculty teaching courses in subsequent semesters indicated students who had been involved in these courses were better prepared than their counterparts.*

INTRODUCTION

As instructors, we constantly seek ways to improve our teaching. To that end, we present a blending of techniques that overcome a variety of issues faced by students. The problems with student engagement, and the need for students to take a pro-active role in the classroom have been explored in detail, for example Halmos, Moise, and Piranian (1975). We incorporate Team-Based Learning (TBL) and Standards-Based Grading (SBG) and our techniques to try and address some of these issues. The goals of this method are four-fold. The first is Readiness. We seek to ensure that students take responsibility for preparing to learn the material, and to familiarize themselves with the importance of reading in their discipline. The second is Inquiry and Struggle. Students need to be given a framework in which they can work on guided exercises, with the emphasis being on critical thinking and the application of effort, rather than specific algorithms. The third is Productive Failure. The readiness and inquiry must lead to a situation in which students can safely

fail, and where learning from that failure is built into the course. The fourth and final stage is Mastery, when a student is able to demonstrate full command of the course materials.

With any course redesign, we must start with the issues we need to correct. These are covered in the first section we discuss the literature relevant to our two methods, and the evidence of their efficacy. In our second section, we discuss in detail the implementation of the method, and provide examples of the materials. In the third section, we present the results of this implementation, and include student feedback on the method.

LITERATURE REVIEW

Team-Based Learning

The first issue we seek to address is student engagement. How can we ensure that students feel that their time in class is valuable? We seek to improve this with active-learning strategies that keep students engaged in the classroom. In practice, this means minimizing lectures, and asking students to work in team to solve complicated problems. The majority of our method here comes from the framework in Michaelsen, Bauman-Knight, and Fink (2004). Evidence suggests that traditional, teacher-focused methods, do not enable all students to engage with the material Hake (1998). We use TBL to make the change to a more student-focused method. The literature indicates that TBL can improve test scores, course grades, and student satisfaction Sisk (2011). We also know that TBL students tend to be more prepared for class Allen, Copeland, and Franks A.S (2013) and that students report learning more content Altintas, Altintas, and Caglar (2014).

We also seek to address student accountability. Students need to learn how to learn. They need to be prepared when they arrive in the classroom, and a mechanism needs to be in place to ensure that readiness is the student's responsibility. This contrasts with traditional lectures, where the students often feel it is the instructor's job to get them ready to learn. The TBL methods in Miahaelsen and Sweet (2008), Miahaelsen and Sweet (2011) ensure the students are accountable for their preparation.

Standards-Based Grading

Further accountability is passed to the students in how their grades are assigned. Students tend to see their grade as starting at an "A", and being lowered each time they make an error. By changing this so that the student sees grades as something they earn, rather than something the instructor disseminates, accountability increases.

Standards-Based Grading allows student to be graded based on demonstrated proficiency in meeting a clearly articulated set of learning objectives Reeves (2011). It is a formative assessment, where the final product of a student's participation is prioritized, rather than a summation of grades awarded at various stages in the course. This allows for students to fail in a way that is safe and

productive. The idea of productive failure Kapur, Dickson, and Yhing (2010) has been shown to improve student's ability to solve mathematical problems. In both well-posed and poorly-posed problems, students who have had experience with productive failure outperformed students in more traditional lecture courses.

Productive failure is also key in increasing accountability. It is the student's decision to make a failure productive. Structuring the course so that students can use an error as a learning opportunity, rather than something that merits punishment, can be encouraging for students. It requires accountability, as making the effort to learn from a failure can only be done by the student.

Finally, we seek to remedy issues with student's preparedness for subsequent courses. Through experience, we have learned that there is a great difference between a student who has earned 70% of the points available in a course, and a student who has mastered 70% of the content in a course. The issue is that both of those students would receive a "C" grade, and would be able to register for the subsequent course. Our argument is that only the second student is truly prepared for the next course in the sequence. It is our intention to design a multi-semester study, where students from SBG and traditional courses are followed through subsequent semesters and their retention is tested.

With these goals in mind, we redesigned a Calculus I classroom, using Team-Based Learning to make the course as active as possible, and Standards-Based Grading, to ensure accurate measurement of student mastery. As the goals of the course redesign were multi-faceted, we determined that a single approach would not achieve all of the desired results. For this reason, we blended the two approaches. The pedagogy is described in the next section.

METHODS

Team-Based Learning

Our implementation of Team-Based Learning (TBL) involves four major components, similar to Michaelsen et al. (2004). The main difference is that we do not use simultaneous reporting on our classroom, preferring to let teams work at their own pace. This is also a factor of having the students for only 50 minutes per class period, where regular breaks for simultaneous reporting were not feasible with the time constraint. The four components we take from Michaelsen et al. (2004) are:

1. Setting the Stage and Assigning Teams
2. Individual and Team Readiness Assessments

3. Challenging Daily Team Activities

4. Peer Evaluations

Setting the stage on the first day is crucial to success. Students are generally unfamiliar with TBL, and will complain when they realize lectures are not generally part of the course. It seems counter-intuitive that these complaints would exist, since students generally also complain about lectures. Explaining the course structure in detail, as well as the instructor's motivation for the alternative method, can be helpful in getting students invested in the method earlier.

The first component of a TBL classroom is the formation of teams. These should be groups of students that are assigned by the instructor, and which will last for the entire term. It is helpful at the beginning of the term to use the work "team" rather than "group". Students tend to have preconceived notions about what happens when groups are used in the classroom, and most of those notions are negative. It is common to find high-achieving students who are concerned that other team members will earn credit for work they did not complete. This method actively prevents that, so you can assure the students that this is not true.

The second component of the TBL classroom is student's responsibility for readiness. This is achieved through Readiness Assessments. Students are assigned preparation work to be completed before the class meets. This might include readings, videos, or inquiry-based questions.

Assessments are done twice. For the first attempt, the students are to complete the assessments on their own, hence the term Individual Readiness Assessment (iRAT). We have had success using a product called ZipGrade to score these assessments quickly. A sample is provided in the appendix. Once all students have completed the iRAT, the same assessment is taken in teams. This is the Team Readiness Assessment (tRAT). We have had success using the IF-AT scratch-off response cards for this, and an example is provided in the appendix.

The use of this method has been in a course with 4 50-minute periods per week. As such, an entire day is devoted to Readiness Assessments. The RATs are generally completed within 50 minutes. The instructor can spend the rest of the time delivering a short lecture on the components of the RAT where students had difficulty. This is where the use of an electronic grading system is helpful. By getting immediate feedback on the problems where some or most of the class made an error, the instructor can tailor the mini-lecture to the problem areas. Items where most or all of the students answered correctly were learned via the readings, and do not need to be addressed in the lecture.

Beginning on the day following the RATs, the student start working on Team Activities. These should be challenging questions, above and beyond the typical rote practice in a Calculus textbook. Sample daily activities are in the appendix.

There are two components to encourage student accountability in the classroom. The first is that the teams are responsible for assigning themselves credit for participation each day. Each team is given a sign in sheet, and the authority to sign in an absent teammate. If a student needs to miss class, they inform their teammates, and if the teammates feel the absence is warranted, they can sign their absent teammate in. The instructor only assigns points based on the self-reported participation.

The second component is a peer evaluation. This starts on the first day of class, when the teams are assigned. Each team is asked to create a team contract, where effective team member behaviors are laid out. These contracts are the basis for two peer evaluations done during the semester. The first is done at midterms. Each student is given an electronic survey, where they are allowed to assign 100 points per member of their team (less themselves) to show how much they feel each other member is contributing. For example, if the team has 5 members, each student is given 400 points. They could assign the values as 100, 100, 100, 100 or as 110, 90, 60, 140, or anywhere in between. Each score must be accompanied by a comment on how a team member makes a positive contribution, and a comment on how a team member could improve their contribution. These are anonymized and distributed to the students. All critical feedback must specifically address a team behavior that was agreed upon when the team contract was written.

The purpose of the mid-term evaluations is to give students a sense of how their contribution to the team is perceived. An evaluation is also required on the final day of the course. This does not include any comments, just a raw score. A student's raw score is multiplied, as a percent, by the total team points they have earned. For example, if the team had earned 100 points, but a student's average peer evaluation was only 70, that student would only earn 70 points in their final grade calculation. It is the intention that a poor mid-term evaluation will motivate those students to correct the offending behavior, and that their final evaluation score will improve. This has almost universally been the case in the courses we have taught.

Standards-Based Grading

The second component to this teaching method is Standards-Based Grading (SBG). Readiness and inquiry are addressed by TBL, and productive failure and mastery are addressed by SBG.

SBG begins with a clear set of learning objectives for the course, which are shared with the student. Methods for constructing these follow Guskey and Bailey (2001) and Marzano and Haystead (2008). The instructor must be transparent about what is to be expected, and what grade

will be earned for meeting those expectations. In our case, we had 38 learning objectives, which are listed in the Appendix.

We chose to have standards assessments every 2 weeks. On an assessment day, each student was provided a folder that listed all possible objectives, and contained problems to be completed, covering all objectives up to that point. Objectives were graded on a ✓ or × basis. A ✓ indicated that the work satisfied the desired learning objective. An × indicated that the work was not satisfactory, and indicated that the problem was to be attempted again. The student folder also contained a spreadsheet with the instructor's signature next to each satisfied objective. Once an objective was satisfied on two separate occasions, it was considered mastered. On each assessment, a student would only attempt problems they had not yet mastered.

In order to make the failure to meet an objective productive, students had the opportunity to revise and resubmit their work. This could be done two ways. The first is that each learning objective covered in the course appeared on each standards assessment. Therefore, even if a student missed an objective on an early assessment, that objective would remain on all subsequent assessments. The second was to use a Revision Request form. This is located in the Appendix. A Revision Request required that the student visit the instructor in office hours. Prior to the appointment, the student was to have revised the missed work, and written a statement explaining both their misunderstanding of the original problem, and what they had done to learn from the mistake. A student who had done this was given a new problem, to be done at the board in front of the instructor.

Successful completion of this board work resulted in the grade change from × to ✓.

Students were provided with a great deal of sample and practice problems to assist them in mastering the content, but none of this work was required to be submitted for a grade. The students were forced to be accountable for their own learning, and to make the decision to prepare for the assessments, and to learn from their failures.

RESULTS

Calculus Understanding

Over the course of the semester, student scored an average of 7.2 on the iRATs and 8.7 on the tRATs. The individual scores rose from an average of 6.3 on the first assessment to 8.6 on the final assessment. Team scores rose from 8.1 to 9.4 in the same time period. Students seemed to become more proficient at reading the course material, and preparing for the assessments. The average student mastered 79% of the course objectives fully.

Unfortunately, only full mastery was recorded, so data on partial mastery is not available. This oversight will be corrected in future data collection. 10% of the students did not pass the course. Among students who did pass the course, 84% of course objectives were mastered.

Students self-reported engagement

Overall, the method appeared successful to the instructor. Student engagement was high, attendance was nearly perfect, and student feedback at the end of the term was primarily positive. The instructor is currently using the method a second time, with revisions as suggested by the students from the first iteration. A selection of positive and constructive student comments are included in the appendix. Anecdotal evidence from the instructor of the Calculus II course during the following semester indicated that the students who had used the TBL/SBG approach were well prepared to succeed.

As a whole, the method improves student engagement, encourages active learning, and teaches a variety of soft skills outside of just the course content. It is the instructor's intention to continue using this method. We would like to gather data in the future on the performance of students in future courses. We would also like to gather data on student's mathematics anxiety and mathematical self-efficacy, to study how this is affected by our method.

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Appendix A

Example Syllabus

CALCULUS I

This syllabus is subject to revision. All changes will be posted in the learning management system, and will be communicated to students via email.

Course Description: Limits, continuity, the derivative and its applications, and the integral and its applications.

Learning Outcomes: At the completion of this course, students will be able to:

1. Understand the concept of a limit, and compute limits of functions.
 2. Understand the concept of a derivative, and be able to compute the derivatives of various functions.
 3. Apply limits and the derivative to applications in physics and engineering.
 4. Understand the concept of an integral, and be able to compute the integral of various functions.
- **Standards Assessments:** At the end of each two-week module, you will have a standards assessment. These are chances for you to demonstrate that you fully grasp a learning objective of the course. Standards will be graded as Correct (✓) or Incorrect (✗). There are 38 learning objectives in the course. All standards covered up to that point in the course will be available on each assessment. You may choose which problems you complete. Each

objective must be completed twice, on separate assessments, in order to count as mastered. Certain more complicated problems may count as covering multiple objectives.

- **Team-Based Learning:** This course will be using a team-based learning (TBL) approach. TBL encourages self-directed learning and will help teach you how to apply what you learn in a collaborative environment. TBL requires you to be prepared for and attend classes. Using the TBL method will allow us to avoid long lectures so that we can dig into the more complicated business of critical thinking about what we are learning. The course components will include the following:
- **Readiness Assessments (RA):** At the beginning of classes where there is an assigned reading due, we will have a closed-note Readiness Assessment (10 items). You will take this RA twice; you will complete it once as an individual (iRA), and then as a team (tRA). These quizzes will be scored immediately so you can appeal your scores on the RA if you believe the key is wrong or there is another correct answer listed. If you are absent, you will receive a zero for both the iRA and the tRA for that day; however, I will also drop your lowest iRA and tRA at the end of the year, so you can miss one without penalty. I do not allow makeups for these assessments.
- **Team Activities:** In most classes, we will have one or more application activities where you will need to work with your team to devise the best solution or approach to a selection problem. You will not need to work on these assignments outside of the classroom, although completing the required readings will be imperative for your success on these assignments. The grading for this will assess whether you have appropriately applied key concepts you've learned to the problem. You will sign in to a sheet with your team each day to indicate that you've completed the Activity. Each daily activity is worth 3 points.
- **Peer Evaluations:** Twice during the course, you will complete a peer review. These will be anonymous, but they will be shared with your teammates. The goal of this review is to give you an opportunity to provide constructive feedback to your team members. Your average rating on the second review will serve as a multiplier on your team performance score. If you do not complete the reviews, you receive a score of zero for your own review, which will lead to a failing grade in the course.

Grading

Desired Grade	Objectives Mastered	Desired Grade	Objectives Mastered
A	35	C	27

A-	34	C-	26
B+	33	D+	25
B	31	D	23
B-	30	D-	22
C+	29	F	≤ 22

There are a total of 196 points available for Readiness Assessments and Team Activities.

Grades are capped at the team score percentage (90% for an **A**, 80% for a **B**, etc.). A student whose team grade exceeds their standards grade, will have their grade raised by 1/3. For example, if a student mastered 33 objectives, but scored over 90% on the team grade, that student would receive an **A-**. On the other hand, if a student mastered 35 objectives, but only scored 70% on the team grade, that student would receive a **C**.

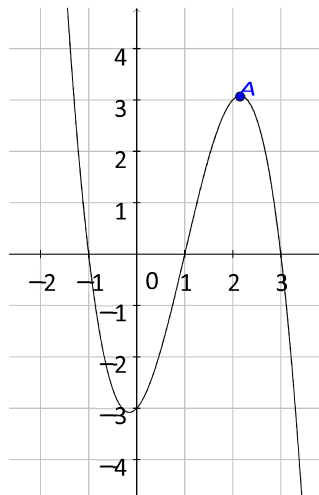
Appendix B Learning Objectives

1. Evaluate a limit using the limit laws.
2. Evaluate a limit at infinity.
3. Evaluate an infinite limit.
4. Use limits to determine the horizontal and vertical asymptotes of a function.
5. Show, via the definition, that a function is continuous at a point.
6. Apply the Intermediate Value Theorem to show a function has a zero on a given interval.
7. Calculate the derivative of a polynomial function directly from the definition.
8. Calculate the derivative of a polynomial function using the power rule.
9. Calculate the derivative of a trigonometric function.
10. Calculate the derivative of a function using the product rule.
11. Calculate the derivative of a function using the quotient rule.
12. Calculate the derivative of a function using the chain rule.
13. Calculate the derivative of a function using a combination of the power, product, quotient, and/or chain rule.
14. Find the equation of a tangent line to a curve at a given point.
15. Correctly find the derivative of an implicit function.
16. Correctly set up a problem involving at least two related rates.
17. Solve a problem involving at least two related rates.
18. Identify the intervals on which a function is increasing and/or decreasing.

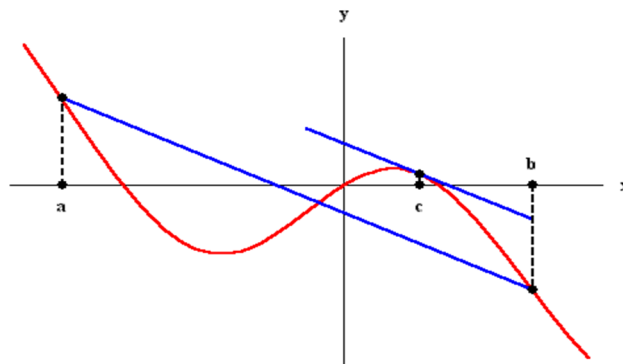
19. Identify the intervals on which a function is concave up and/or concave down.
20. Find all critical values of a function.
21. Use the 1st derivative test to classify extrema of a function.
22. Use the 2nd derivative test to classify extrema of a function.
23. Apply the Extreme Value Theorem to a problem.
24. Apply the Mean Value Theorem to a problem.
25. Correctly set up an optimization problem using the methods of Calculus.
26. Solve an optimization problem using the methods of Calculus.
27. Calculate an antiderivative of a polynomial function.
28. Calculate an antiderivative of a trigonometric function.
29. Use a finite summation to approximate the area under a curve.
30. Calculate a definite integral using the Riemann Sum.
31. Evaluate a definite integral using the Fundamental Theorem of Calculus.
32. Evaluate an indefinite integral.
33. Evaluate an indefinite integral using substitution.
34. Calculate the derivative of a logarithmic function.
35. Calculate the derivative of an exponential function.
36. Calculate a derivative using logarithmic differentiation.
37. Calculate the derivative of an inverse trigonometric function.
38. Evaluate a limit using L'Hospital's rule.

Appendix C Example Readiness Assessment

1. The point A on this graph would be a:



- (a) Local maximum
 - (b) Local minimum
 - (c) Absolute maximum
 - (d) Absolute minimum
2. At any point where a function is increasing, the derivative must be:
 - (a) Zero
 - (b) Non-existent
 - (c) Negative
 - (d) Positive
 3. The Extreme Value Theorem can be used to find out what information?
 - (a) The absolute max/min of a function, for all values.
 - (b) The derivative of a function
 - (c) Where a function is continuous.
 - (d) The absolute max/min of a function, on a closed interval.
 4. The Extreme Value Theorem has a familiar hypothesis about which kind of function it can apply to. What is that hypothesis?
 - (a) The function must be increasing.
 - (b) The function must be decreasing.
 - (c) The function must be continuous.
 - (d) The function must be differentiable.
 5. A critical number of a function f is a number c in the domain where which of these happens?
 - (a) The function is differentiable at c .
 - (b) $f'(c) = 0$.
 - (c) The function is not defined at c .
 - (d) $f'(c) = 0$, or it does not exist.
 6. The following image is often shown with the Mean Value Theorem. What can we assume about the derivative of the function f when $x = c$ and the slope of the line connecting the points where $x = a$ and $x = b$?



- (a) The derivative is zero when $x = c$.

- (b) All values of the derivative exist between a and b
 - (c) There is a local maximum when $x = c$
 - (d) The derivative at $x = c$ is the same as the slope of the line from $x = a$ to $x = b$.
7. What happens at a point of inflection?
- (a) The function has a local maximum.
 - (b) The function has a local minimum.
 - (c) The graph changes concavity.
 - (d) The graph changes from increasing to decreasing.
8. Which of these would be described as an optimization problem?
- (a) Find the relationship between the change in volume of a sphere and the change in its radius.
 - (b) Proving a real zero of an equation exists over a certain interval.
 - (c) A business owner wanting to maximize profits and minimize costs.
 - (d) Finding the slope of the tangent line to a curve at a specific point.
9. Optimization problems always require that we find which of the following?
- (a) Absolute maximum or minimum of a function.
 - (b) Local maximum or minimum of a function.
 - (c) Intervals where a function is increasing or decreasing.
 - (d) Intervals where a function is concave up or concave down.
10. At any point where a function is decreasing, the derivative must be:
- (a) Zero
 - (b) Non-existent
 - (c) Negative
 - (d) Positive




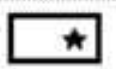












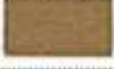






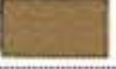




**Appendix D IF-AT and ZipGrade
Answer Sheets**

IMMEDIATE FEEDBACK ASSESSMENT TECHNIQUE (IF AT®)

Name _____ Test # _____

Subject _____ Total _____

SCRATCH OFF COVERING TO EXPOSE ANSWER

	A	B	C	D	Score
1.					_____
2.					_____
3.					_____
4.					_____
5.					_____
6.					_____
7.					_____

Name			
Date		Period	

	A	B	C	D	E		A	B	C	D	E
1	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>		11	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
2	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>		12	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
3	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>		13	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
4	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>		14	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
5	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>		15	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
6	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>		16	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
7	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>		17	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
8	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>		18	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
9	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>		19	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
10	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>		20	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>

Test Version: A B C D

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Appendix E

Sample Team Activities

Applying Differentiation Techniques

1. How do we find the equation of the tangent line to a curve at a point?
 - First, make sure the team recalls the point-slope formula for the equation of a line. We should be able to find the equation of a straight line using only the slope, and some arbitrary point on that line.
 - Given the curve $y = x^2$, what is the equation of the tangent line when $x = 3$?
 - Given the curve $f(x) = \sin(x) - \cos(x)$, find the equation of the tangent line at the point $(0, -1)$.
2. How do we linearly approximate functions with a derivative?
3. Suppose we want to find the approximate value of $\sqrt[3]{8.03}$
 - Can we find the exact value of $\sqrt[3]{8.03}$ without a calculator?
 - Find the equation of the tangent line to the curve $y = \sqrt[3]{x}$ when $x = 8$.
 - Find the value of the point on this tangent line when $x = 1.03$. Compare this to the result on your calculator for finding $\sqrt[3]{1.03}$.
4. How could we use this idea to find an approximate value for $\sqrt[3]{31.94}$?
5. Can we use the tangent line to help approximate a zero of a function?
 - Start with the equation $x^3 + 2x - 1 = 0$.
 - It's tough to find an exact solution to this. First, show that the function $f(x) = x^3 + 2x - 1$ has at least one real zero on the interval $[0, 1]$.
 - Find the equation of the tangent line to the function $f(x)$ when $x = 1$. We'll need some notation, so let $x_0 = 1$
 - Where does that tangent line cross the x -axis? Label that point x_1 .
 - Now find the equation of the tangent line to the curve when $x = x_1$. Where does that line cross the x -axis? Label that point x_2 .
 - If you were to keep doing this, make a conjecture about what would happen with the sequence of points x_0, x_1, x_2, \dots

Appendix F

Sample Standards Assessment

1. Possible Objectives: 1

Evaluate the limit, or determine that it does not exist.

$$\lim_{x \rightarrow 2} \frac{x^2 + 6x + 8}{x + 2}$$

2. Possible Objectives: 2

Evaluate the limit, or determine that it does not exist.

$$\lim_{x \rightarrow \infty} \frac{x^2 + 5x - 1}{5x^2 + 4}$$

3. Possible Objectives: 1, 3

Evaluate the limit, or determine that it does not exist.

$$\lim_{x \rightarrow 3} \frac{2x}{3 - x}$$

4. Possible Objectives: 1, 2, 3, 4

Use limits to find all vertical and horizontal asymptotes of this function.

$$f(x) = \frac{3x^2 - 4}{2x^2 - 2}$$

5. Possible Objectives: 1, 5

Use the definition to show that the given function is **not** continuous at the point $x = 1$.

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \neq 5 \\ 3 & \text{if } x = 5 \end{cases}$$

6. Possible Objectives: 6

Apply the Intermediate Value Theorem to show that the function has a root in the interval $[0, \pi]$. State any hypothesis you use.

$$f(x) = \cos(x)$$

7. Possible Objectives: 1, 7

Find the derivative of the following function, using the limit definition.

$$f(x) = 5 - 2x$$

8. Possible Objectives: 8

Find the derivative of

$$y = \frac{2x^2}{x^{1/2}}$$

without using the quotient rule.

9. Possible Objectives: 9 Given $y = \sec(\theta)$, find $\frac{dy}{d\theta}$

10. Possible Objectives: 8, 9, 10

Find the first derivative of

$$f(x) = \tan(x)(3x - x)$$

11. Possible Objectives: 8, 9, 11

Find the first derivative of

$$f(x) = \frac{\tan(x)}{3x-5}$$

12. Possible Objectives: 8, 9, 12

Find the first derivative of

$$f(x) = \tan(3x - 5)$$

13. Possible Objectives: 8, 9, 10, 11, 12, 13 Find the first derivative of

$$f(x) = \frac{\tan(3x^2)(2 - \sqrt{x})}{\cos(x + 5)}$$

14. Possible Objectives: 8, 9, 10, 11, 12, 13, 14

Find the equation of the tangent line to the curve $y = x^3 + \cos(x)$ at the point $x = 0$.

15. Possible Objectives: 8, 9, 10, 11, 12, 13, 15

Find $\frac{dy}{dx}$ for $xy = 2y^3$

16. Possible Objectives: 8, 9, 10, 11, 12, 13, 15, 16, 17

A 15 foot ladder is resting against the wall. The bottom is initially 10 feet away from the wall and is being pushed towards the wall at a rate of 1/4 ft/sec. How fast is the top of the ladder moving up the wall 12 seconds after we start pushing? (Note that this can count for objectives 16 and 17. Be very clear with your setup)

Appendix G

Selection of Student Feedback

- Was completely honest and consistent with the goal to understand how we know what we learned versus being able to get an answer. Understanding the method was more important than finding the most simplified answer. It left me understanding how and why we made the computations we did.
- The team based learning was much more effective for me than any other strategy used on me before.
- I would say this style of learning was not very rigorous in comparison to the other classes, but I feel like I learned the material I needed to. I am worried about making the shift next semester back to a normal learning style.
- I liked the core objectives and supplemental objectives sheet so we could clearly see what we have accomplished and what we need to work on.
- At the beginning of the course I was confused and didn't understand a lot of the stuff we were doing, but after going to office hours and asking for help, my grade improved.

- I can really see why the way my instructor teaches the class is helpful. I liked the groups and the backward class style.
- I like the group set up of things, but mini lectures with examples when introducing a new topic would be helpful
- I like the style of having second chances on standards tests. It allows me as a student to focus on learning versus stressing about a perfect score on a test.
- Very good course. Much better taking calculus this time with the team based approach. The class is set up very nicely.
- The team based learning was a great idea, I liked bouncing ideas off of one another and there was usually someone in the group that understood what was going on if nobody else didn't.