Creating Use of Different Representations as an Effective Means to Promote Cognitive Interest, Flexibility, Creative Thinking, and deeper understanding in the Teaching of Calculus

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Abstract: *In this paper, we discuss some ways of developing intrinsic interest in mathematics in the course of teaching of calculus for engineering students. Our experience shows that the use of different representations for solving various problems promotes cognitive interest, creative thinking and deeper understanding of mathematics and assists essentially in many fields of knowledge. The examples in the paper are chosen from non-routine problems associated with understanding of two-way correspondence between formulas and graphs in rectangular and polar coordinate systems.*

Keywords: Calculus teaching, different representations, creative thinking, exercise- based learning, problems-based learning, equations and their graphs, rectangular and polar coordinate systems, cognitive motivation, challenge problems, surprising answers.

INTRODUCTION

Our long-term experience in teaching of basic mathematical courses for engineering students indicates that many of them lack cognitive motivation and have low interest in mathematical learning [1]. The students often see this discipline as a boring one, not too important for their future work and think that they are required to learn it only because it is a part of the curriculum. A similar observation was noted by many educators from different countries [12], [17], [21], [22]. This situation in many cases follows from previous learning in the secondary and high school, where the basic requirement was to solve a large amount of dull technical exercises [9], while solution of challenging problems connected with interesting real-life applications was not emphasized enough. Many educational researches deal with this problem, [18], [25].

We are trying to change this attitude of engineering students to mathematics taking into account the well-known phrases: "*Education is the kindling of a flame, not the filling of a vessel*" (Socrates) and "*Study without desire, spoils the memory and nothing is retained of what was taken in*" (Leonardo da Vinci). Therefore, developing intrinsic interest in mathematical approaches to various problems and promoting the feeling of beauty of unexpected short, logical solutions of complicated tasks, should be an important goal of the lecturers [1], [2], [19]. We try to achieve this in a number of ways. In this article we report about our experience in increasing of interest and cognitive motivation of novice engineering students within the framework of teaching the theme "equations and graphs" in the first year Calculus course and simultaneously enhancing the students comprehension of main mathematical ideas [4], [5], [20] .

The question is how to create intrinsic cognitive interest in mathematics learning among first year engineering students [16], [17]. Most of them have previous experience of "exercise-based learning" only - from examples demonstrated by the teacher, or solving a long series of exercises of the same type. To a certain extent, it was reasonable in the "pre-computer" age, when for many professions it was enough to acquire only routine skills for solving standard problems. In the digital age, the motivation to learn mathematics has decreased because now computer programs allow implementation of almost all mathematical techniques, including very complicated symbolic calculations. On the other hand, we see great opportunities in using modern technology to stimulate the student's interest and reveal the beauty of mathematics as a universal tool for solving different problems in many fields of knowledge [17], [19]. In our teaching, we always emphasize Galileo's statement: "*The Book of Nature is written in the Language of Mathematics*." We think that our mission is to bring to the understanding of every student learning STEM, the deep meaning of these words. To our regret, most novice students do not understand this statement and are not even familiar with it. In our opinion, if the students will really understand the vast possibilities that the language of mathematics can offer, it will contribute a lot to their motivation for studying mathematics.

PROBLEMS IN TEACHING GRAPHS OF EQUATIONS IN RECTANGULAR COORDINATE SYSTEM

In this study, we examined the effect of teaching equations and their graphs in the rectangular and polar coordinate systems in a way that promote cognitive activities of students, as well as enhances their interest in the subject using modern technology [19] to see the strength and beauty of simple mathematical formulas that describe complicated objects. This theme, whose understanding is very essential for mathematics learning and teaching, has been investigated by various authors ([26], [20], [6]-[8].

Why did we choose this subject? It is not familiar to both "strong and weak" novice students, does not require a great deal of "unknown knowledge", and so is suitable for heterogeneous groups. In addition, we chose this topic because it is essential for engineering students and allows the demonstration of the possibilities of modern computer applications, as well as the creative thinking for effective hand drawing, based on the theoretical understanding of the topic. The important additional goal is to teach students how to use their previous knowledge and cognitive tools that they already possess, as well as technological tools (calculator, computer and applications) to deal with previously unknown problems. We try to use different representations (verbal, numerical, symbolic, visual, and real and computer model presentations) in order to create the deep and stable understanding of the issue under study [3], [10], [11], [13]-[15], [24].

In the first lectures in the Calculus course, we mention the basic mathematical means for locating points and defining curves and figures in a plane, the rectangular (Cartesian) coordinate system, and a polar coordinate system, and give examples for understanding the relationships between formulas and graphs.

Even though the students' awareness of the rectangular coordinate system from their previous studies, we indicate that most of them have difficulties in understanding and sketching graphs of simple equations, not exposed in previous studies [20]. An example is the question of what the graphs of such equations are:

1) $\sqrt{(x-2)}+\sqrt{(y+1)}=0;$ 2) $(x-2)(y+1)=0;$ 3) $x^2 - y^2 = 0;$

4) $\sqrt{x}\sqrt{-y}=0;$ 5) $\sqrt{1 - x^2}\sqrt{1-y^2} = 0;$ 6) $\max(|x|,|y|)=1.$

Only a small percent of the students gives correct answers without the teacher's prompt (even to the questions 1-3). This indicates a lack of understanding of the relationship between formulas and graphs and an incompleteness of their thinking. They try to use their “memory” but are unsuccessful.

Here are some teachers' questions to guide the students' thinking:

1. What is a graph (locus) of equations?
2. How can you check that a certain point (a, b) belongs to the graph of an equation?
3. Can you specify several points belonging to the graph of a given equation?
4. Can you give more points of the graph?
5. What is now your suggestion about the graph of given equations?
6. What is the reasoning of your suggestion about the graph?

By such questions, we try to activate a student's activity and to form his/her independent thinking in solving new problems.

We also asked the reverse type of questions. For example:

"Give an equation whose graph consists of only two points (1,1), (-1,1) ".

One of the possible answers is: $(x^2-1)^2+(y-1)^2=0.$

We also try to stimulate creative thinking of our students by questions such as:

<u>Problem 1.</u> Construct a formula, whose graph is similar to figure 1.

Hint 1. Can you find formulas for functions with graphs presented in figure 2?

Hint 2. How can we use the formulas whose graphs were found in figure 2 to construct the formula of the graph in figure 1?

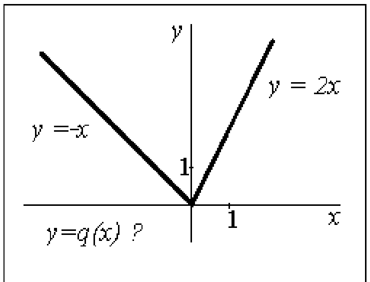

Figure 1

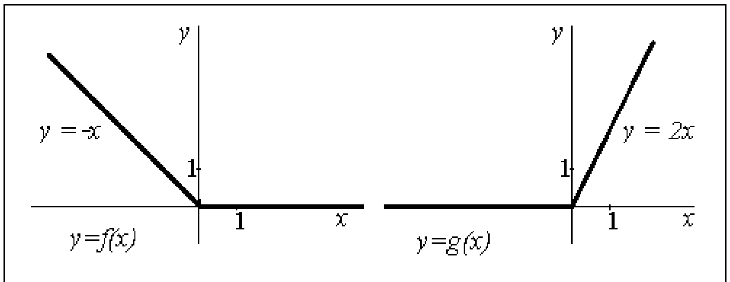

Figure 2

The possible solution using ‘min’ and ‘max’ functions [23]:

$$f(x)=\max(-x,0)=\frac{-x+|x|}{2};\qquad g(x)=\max(2x,0)=\frac{2x+|2x|}{2}=x+|x|;$$

$$\left[\textit{here we used the well-known formula}: \max(a,b)=\frac{a+b+|a-b|}{2}\right]$$

$$q(x)=f(x)+g(x)=\frac{x+3|x|}{2}.$$

Our intention is to change the routine schema to which students gets accustomed: “The sample example, then the teacher’s solution, then using the example of samples, and exercises repeated many times”. We prefer teaching focusing in another direction: “A new question, then an unaided thinking problem with hints of the teacher (if needed), then answers, afterwards discussion and generalization (if possible)”.

We often try to ask new questions or demonstrate examples with “surprising” answers:

<u>Problem 2.</u> Find the point $P(a,a^2)$ of the graph $y=x^2$ without any measuring or calculations (See solution in figure 3).

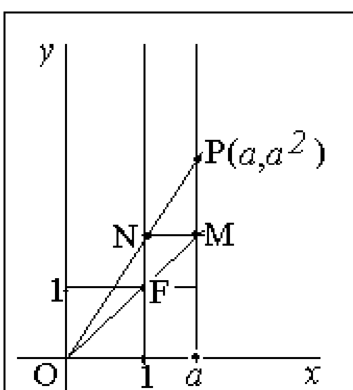

Figure 3

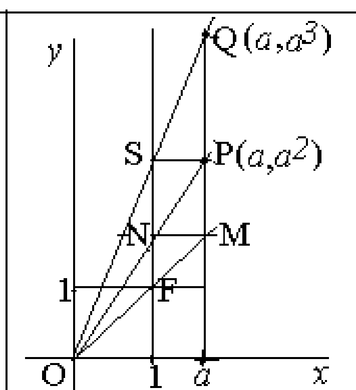

Figure 4

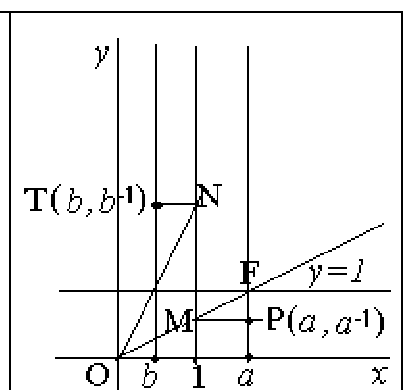

Figure 5

Try to justify this solution.

Problem 3. Generalization of problem 2: How can we construct the point $Q(a,a^n)$ $(n \in N)$? (See solution for $n = 3$ in figure 4)

Problem 4. Think about the construction of the points of the graph $y = x^{-1}$

An important concept in the Calculus course is the slope of a straight line. This concept should be familiar to students from High School. Nevertheless, for the question "How can we find the slope of the straight line drawn on the rectangular coordinate system without measuring and calculations?" This question asked many times of many students, but none of them gave the proper answer. The graphical method of finding a slope shown in figure 6. Every time this simple construction was seen surprise and sometimes even with delight. (Note that this method is convenient for plotting a graph of a derivative of the function given by its graph (Figures 7, 8)).

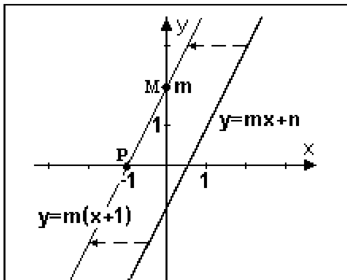

Figure 6

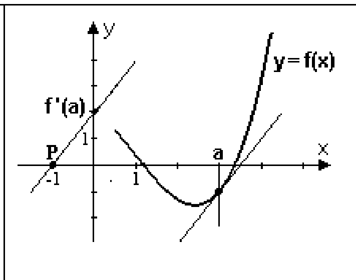

Figure 7

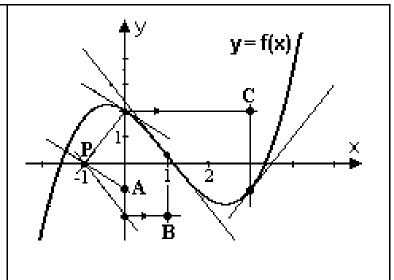

Figure 8

Note that the widespread opinion among students and some teachers too is that the slope m of the straight line is equal $tg(\theta)$, where θ is the angle of inclination of this line to the x-axis. This statement is true only in the case of equal scales on the x and y-axis (e.g. the length of the segment [0,1] on x-axis is the same as on the y-axis). This makes no sense in the case of graphs of physical dependencies (for example $s=s(t)$ where the time axis, marks seconds, and the distance axis, marks meters. This formula, $m = tg(\theta)$ gives an incorrect result for most graphs of "abstract functions" obtained by computer applications and in the case of hand-drawn graphics as well. In this occasion, we want to give an example of a question that one of the students asked the teacher about the task of sketching the graph of the function $y = x^2 + 50$: "How can I do this because the graph does not fit on my drawing?" The reason was that the student marks 1 on the x and y-axis at the same distance from the origin point of the coordinate system. In this connection, we always say to students "Fit the scales to the given graphic task". We also want to add that the graphic method for finding a slope of a straight line, described in figure 6, fits for all the above cases.

PROBLEMS IN TEACHING GRAPHS OF EQUATIONS IN THE POLAR CCOORDINATE SYSTEM

During the first time mentioning the polar system in Calculus, we usually make sure that this topic is completely unfamiliar to most of the novice students. We introduce this system in connection with trigonometric functions and with the need for radian measure of angles. To clarify the state of the initial knowledge in this subject, we often write on the class-board the question: "sin(90)=?" and ask students what the answer is. Each time we get the same answer: sin(90)=1. Afterwards, we ask the following question: "Check the answer on your calculator. Is it the same for all students in the class?" The next question is: "How many different answers can a calculator give for sin(90) computation?" After that, we discuss the three options for measuring the angles programmed in the calculator "degree, radian, grad" and how they are defined. This theme is new and is of interest to students. Since the radian measurement of angles is prominent, we deal with it in detail.

The radian measurement of angles little known to students and requires special attention in the formation of this concept. We use geometrical and real three- dimensional circle representations of this method of measuring angles (figure 11).

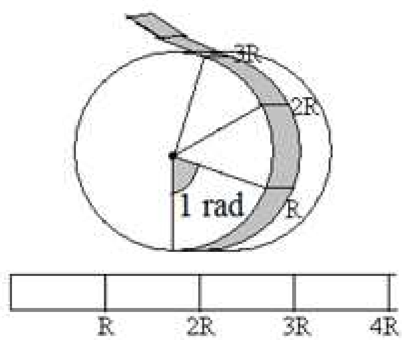

Figure 11

We explain that in the study of trigonometric functions, the concept of rotation of a point along the circle is important, and the value of rotation measured in radians.

We ask the students a few questions to make sure they understand this concept. For example:

<u>Problem 5</u>. The car drives along a ring road with a diameter of 10 km and performs a turn of 8 radians. *a*) What distance along the road did the car cover? *b*) What is the length of an arc of a θ radian in a circle with radius R?

Each time we as lecturers are convinced that a formal concept definition is not quite enough to form a correct concept image for the students [27].

After a minimum preparation and definition of Polar coordinates on the plane, we turn to the graphs of equations in this system and it is a new and intriguing theme for all of the students. We begin with the simplest formula $r=\theta$ and ask students to answer the following question:

Problem 6. What does the equation's $r=\theta$ graph look like?

The first answers are usually: “a straight line”, “a circle”, or “we have not solved such problems”.

Then the lecturer suggests that the students draw the graph by themselves based on all their own knowledge. However, in almost all cases, students do not know how to start. We are giving a few hints to direct the thinking of students. Following are possible hints:

1. What is a graph of an equation? (The graph of an equation in two variables is the set of all points and only these points, whose coordinates satisfy the equation);
2. In the equation $r=\theta$ what is “r” and what is “θ”?
3. Can you plot a point with coordinates $r=1, \theta=2$? Does this point belong to the graph of the equation $r=\theta$?
4. Can you specify a few points belonging to the graph of the equation $r=\theta$?
5. Can you plot the graph using the following table of values?

$$\begin{array}{llllllllll} \theta: & 0 & \pi/6 & \pi/4 & \pi/3 & \pi/2 & 3\pi/4 & \pi & 3\pi/2 & 2\pi \\ r: & 0 & \pi/6 & \pi/4 & \pi/3 & \pi/2 & 3\pi/4 & \pi & 3\pi/2 & 2\pi \end{array}$$

Plotting of these points and finding that the curve is a spiral is always of great interest to students.

Note that it is important to make the students feel that they can to initiate and develop independent thinking by themselves.

To reinforce the understanding of the radian measure of angles and as an example of creative engineering thinking, we demonstrate to students the next simple construction for drawing the curve $r=\theta$ (Archimedes’ spiral) that does not require any measuring or calculations.

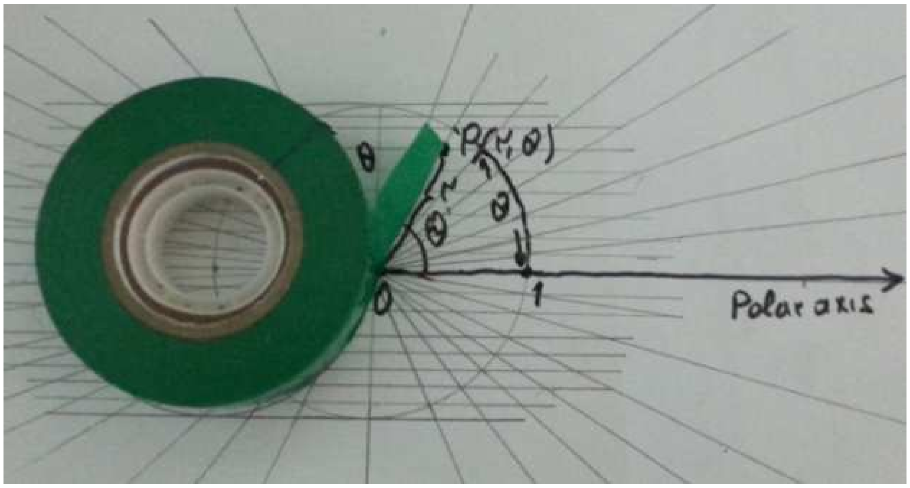

Figure 12. The electric tape model for drawing the curve $r = \theta$

Problem 7. What does the graph of the equation $r = \sin(3\theta)$ look like?

We give time to students to reach their own conclusions. Then we give the appropriate hints if necessary.

We strongly encourage students to use computer graphical applications to verify the responses received, encourage them to use technology and enhancing digital literacy for improving their understanding [17], [19].

In the second mid- test of the semester, we asked the following question: draw the graph of the equation $r = \frac{1}{\theta}$: 85% of examinees give correct answers.

A week later, a questionnaire was given to students about their feelings regarding the subject 90% answered that they understood the subject well and asked for new and intriguing questions. They also noted that the graphical applications helped them understand and test their answers and their self-confidence increased.

CONCLUSIONS

Our long-term experience shows that the use of such questions as mentioned earlier, their formulations and their solutions, when used in conjunction with transitions from one representations (verbal, symbol, graphical, numerical, computer, physical, real-life) to another, increases students' interest in mathematics learning and contributes to a deeper understanding the subject being studied.

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