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Using Algebraic Inequalities to Solve Extremum Problems

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Abstract

The use of nonroutine tools in mathematical problem-solving is very beneficial: it encourages creativity, enables approaches that are easier and faster, and allows simplification instead of a tedious work. In this paper we present one such situation: we discuss the use of algebraic tools in the solution of extremum problems. These questions, which at first seem to have one classical solution, in fact can be solved using the approach of algebraic tools, such as Mean Inequalities. We lay out in detail the benefits of nonroutine solutions to problems, including some positive results according to the practical experience of the authors. Then we give several examples in which the usefulness of this doctrine is demonstrated, including some graphical illustrations. These materials are aimed at teachers, lecturers, and students, and are designed for them to find out about nonstandard teaching and problem-solving methods.

Keywords: algebraic inequalities; calculus learning; challenging problems; visual representation

1 Introduction

The use of nonstandard tools in problem-solving is essential in the teaching of mathematics – both for creativity and inspiration, as well as for dealing with hard problems with accessible and non-cumbersome solutions. In the following material, an explanation is presented for unique and nonroutine solutions to problems arising from analysis, with the added use of algebraic tools. We examine why the use of these techniques is necessary in high school and undergraduate studies. Then, explicit examples of problems where an algebraic solution is of use are given, to demonstrate this argument.

In most analysis courses in undergraduate studies, the approach for solving extremum problems, such as classical min-max problems, is straightforward and is accomplished by calculating the derivative, then equating it to zero and checking for appropriate solutions. This leads to a robotic way of thinking and discourages creativity. To this end, we present a different and refreshing approach for solving min-max problems, with the use of algebraic inequalities. The nonstandard methodology intrigues the student, and allows him to learn new techniques and different approaches.

The use of algebraic tools in the research of analysis is not new. Some previous works have focused upon this subject, especially on the development of general mean-inequalities and their use in order to solve problems from a variety of topics. In a book by [3], the author deals in detail with mean-inequalities and holds various useful propositions and examples. A generalization of mean-inequalities was conducted by [1]. On different approaches to algebraic inequalities, see the work of [4].

This paper is one in a series of papers that try to present unique topics that are unpopular among lecturers and teachers, and are unknown to students. These include the authors' previous work on the staircase functions ([2]), as well as the paper on the min-max functions ([5]). These functions are complicated and sometimes create teacher confusion when trying to determine how they should be taught. In these papers, as well as in the following, we offer some solutions and new insight into such problems and subjects.

One of the special and distinctive methods for emphasizing our point is the use of graphical tools. The two papers by [6, 7] describe the cognitive development in mathematics and its relations with visual imaginary; this development enables researchers and students to use graph programs for the functions. In their paper, [8] justify using such a professional program. In order to automatically present the accurate drawing of the resulted function and to visualize our results, we use several `Python`™mathematical and graphical libraries, such as `Numpy` and `Matplotlib`.

The paper is arranged as follows. In the next section, we give a general preview, in which we explain why the use of distinctive tools and examples, which hold different approaches and use creative thinking, can lead to a better teaching process. We end that section with some concrete detail about the success of the use of algebraic inequalities in calculus examinations. Then, in the following section, we give several sets of unique examples from different subjects, in order to demonstrate our principal idea.

2 Mathematics teaching with the use of special tools and special examples

In this section we provide a justification for the use of nonroutine tools and extreme examples while teaching fundamental subjects in high school and undergraduate mathematics. This methodology is relevant not only in the mathematical environment – it applies as well to other mathematics-related disciplines such as physics and engineering. After outlining the methodology, we concentrate on our specific subject, *the solution of extremum problems through the use of algebraic inequalities*. We show that a creative and refreshing approach to the solution of problems from the standpoint of analysis, with known straightforward solutions, can be beneficial to both student and teacher.

In general, the teaching of mathematical subjects is a challenging task – one must choose carefully which subjects to include, with respect to the audience and to the level of the students. Then, the teacher must pass the material in the correct order with the right amount of explanations, justifications, and examples. Finally, at the end of the teaching process, comes the examination of the knowledge acquired in the classroom: the teacher must create a challenging but fair test, to summarize the teaching procedure for the student.

According to our experience, some teachers and lecturers choose an easy solution – they take a known set of classical and banal propositions and examples, and pass them on to the students, over and over again throughout the years. The advantages of this methodology are obvious – it is easy to teach, easy to examine, and the needed material is passed on. The biggest disadvantage of this approach is that it is based on the lack of creativity and diversification: the students get a very static image of the subject, and cannot develop the innovative thinking needed for solving problems.

To this end, the use of nonstandard methods and examples is crucial for succeeding in the teaching process. It is the teacher's job to bequeath his knowledge to his students, but also to allow them to strive for different and unique approaches. The capability of looking at problems from a different point of view can be beneficial by several means: first, one can find an easier and more elegant way for solving known problems; second, when dealing with difficult problems for which there does not seem to exist a straightforward solution, creative thinking can lead to success. Further reading upon this approach can be found in [9].

Of course we do not suggest that the teaching of classical and basic subjects is unnecessary; the fundamental and core themes and tools in the various areas of mathematics must be passed on to the students. Rather, we explain why showing **other** approaches should be required of teachers and lecturers, for a variety of reasons: for a better understanding of the materials; for providing the tools that are needed in the real world of problem-solving; and for making the learning process more interesting and beneficial.

The above assertion is also true in the case of solutions for extremum problems. There is indeed the square method for solving min-max problems, in one variable as well as in several variables, and this suffices to give the students enough information to solve such problems. However, in some cases, the use of known algebraic inequalities can be helpful; the solution gets much easier, and a new creative approach is discovered.

One of the interesting algebraic tools, which gives birth to some useful algebraic inequalities, is focused on the definitions of different *means*. By a broader definition, a *mean* is a number $M(x_1, \ldots, x_n)$ corresponding to a set of numbers $x_1, \ldots, x_n$, that satisfies

$$\min(x_1, \ldots, x_n) \leq M(x_1, \ldots, x_n) \leq \max(x_1, \ldots, x_n).$$

Unfortunately, aside from the arithmetic mean, which is very popular, students do not encounter other means. Some of the lesser known of these are the *harmonic mean*:

$$\frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}, \tag{1}$$

and the *quadratic mean*:

$$\sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}. \tag{2}$$

The uses of these means are more heavily emphasized in different parts in mathematics and its applications (such as calculus, geometry, physics, finances), and their recognition is essential for the young student.

After defining the basic means, we can present one of the classical algebraic inequalities; the inequality of quadratic, arithmetic, geometric, and harmonic means (QM-AM-GM-HM inequality):

$$\sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} \geq \frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}. \tag{3}$$

This inequality, in which equality holds if and only if all terms are equal ($x_1 = \cdots = x_n$), holds a vast power in the solution of various problems. For the proof of these statements, as well as further research of the properties of means, see [3]. Some teachers and lecturers do not emphasize these methods, and thus their students lack training in the use of such useful tools. Moreover, techniques such as these can be used to simplify some min-max problems in several variables, to the level that even a high school student can solve; this will be shown below.

In the experience of the authors, the use of unique and refreshing algebraic tools in the classroom is beneficial. The student is presented with a different approach, and is somewhat intrigued by its creativity; this increases the student's will to participate and contribute to the classroom, and the student finds a beautiful method. These tools will also help him later on: in the short term, for solving similar problems, and in the long term, for use in solving general problems from analysis, in which standard solutions are too cumbersome, or even out of reach. Indeed, one of the authors has used such methods repeatedly in the teaching of the first and second calculus courses for engineers. In one such experiment, two groups of 60 students were given a first analysis course. The first one was taught the technique of solving extremum problems using algebraic inequalities (such as the AM-GM inequality), as well as other standard techniques (involving derivation). The second group was not taught the techniques involving algebraic inequalities, but only the standard methods. Then, given an exam that contained an extremum problem that can be solved using algebraic techniques, the results were as follows. From the first group, 75% of the students solved the problem correctly, out of which 85% used algebraic inequalities in their solution. In the second group, only 30% of the students solved the problem correctly.

3 Examples

After seeing the importance of the presented subject, we give several particular examples for an unusual solution of problems in analysis, with applications to other fields as well.

3.1 min-max problems for functions of one variable

In this section we examine several examples of nonstandard solution methods for min-max problems of one variable functions.

First, let $f_1(x) = x + \frac{4}{x}$, which is an odd function; we want to find the minimal value of f_1 in the domain $x > 0$. One could, using the classical way, find the derivation of f_1 and equate it to zero. Instead, using the AM-GM inequality, as in 3, we see that

$$x + \frac{4}{x} \geq 2 \cdot \sqrt{x \cdot \frac{4}{x}} = 4.$$

Equality in the above occurs when $x = \frac{4}{x}$, meaning that $x = 2$, so the minimal point of f_1 in the domain $x > 0$ is $(2, 4)$.

For an example that is a bit more complicated, consider $f_2(x) = x^2 + \frac{8}{x^3}$, in the domain $x > 0$. We want to find the minimal value of f_2 now; the previous use of the AM-GM inequality will not work, because the multiplication of the summands is not independent of x; however, writing

$$f_2(x) = 5 \cdot \frac{\frac{x^2}{3} + \frac{x^2}{3} + \frac{x^2}{3} + \frac{4}{x^3} + \frac{4}{x^3}}{5},$$

and using the AM-GM inequality with the summands as above, we get that $f_2(x) \geq 5 \cdot \sqrt[5]{\frac{16}{27}}$. Again, equality holds when all the summands are equal, that is $\frac{x^2}{3} = \frac{4}{x^3}$, or $x = \sqrt[5]{12}$, so the minimal point is $(\sqrt[5]{12}, 5 \cdot \sqrt[5]{\frac{16}{27}})$.

To summarize this set of examples, we give a generalization of the above, by letting $f_3(x) = ax^n + \frac{b}{x^m}$, with $m, n \in \mathbb{N}$, and $a, b \in \mathbb{R}$ positive. To use the AM-GM inequality, we split the two summands of f_3 info the following sums:

$$f_3(x) = \sum_1^m am^{-1}x^n + \sum_1^n bn^{-1}\frac{1}{x^m}.$$

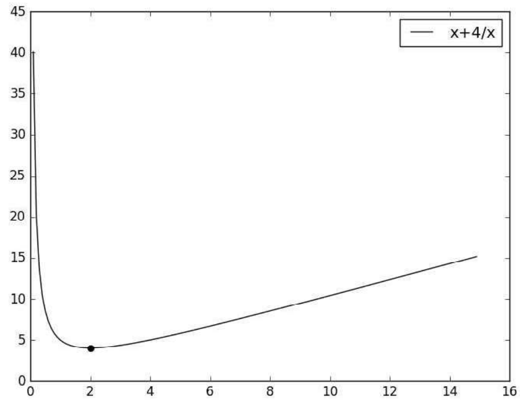
Then, by the AM-GM inequality:

$$f_3(x) \geq (n+m) \sqrt[n+m]{(am^{-1}x^n)^m \cdot (bn^{-1}x^{-m})^n} = (n+m) \sqrt[n+m]{\frac{a^m b^n}{m^m n^n}};$$

The minimal value is achieved when all the summands are equal, i.e., when $am^{-1}x^n = bn^{-1}x^{-m}$, which is at the point

$$x = \sqrt[n+m]{\frac{bm}{an}}.$$

The reader can insert several different values m, n in the function f_3, and see that the result is as presented in the point above. In 1, we give the graphs of $f_2(x)$ and $f_4(x) = 2x^2 + \frac{1}{x^6}$, and show that the minimum is indeed achieved in the point, as above.



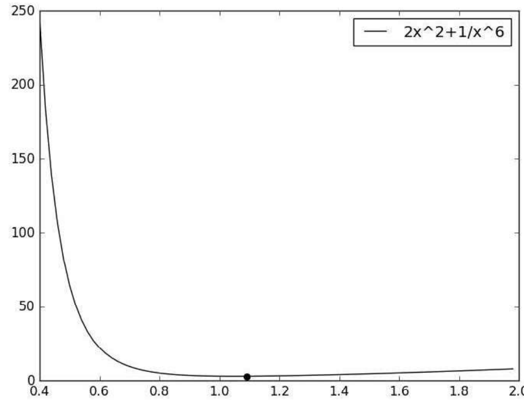


Figure 1: Graphs of the functions $f_2(x)$ and $f_4(x)$, and their minimal values

3.2 Geometric min-max problems

In this subsection we present some geometric min-max problems and their solution through the use of algebraic inequalities.

For the first example, we will find the triangle with the least possible area, which is bounded by the first quadrant and a descending line that goes through a given point $(a, b) \neq (0, 0)$.

Let m be the slope of the line ($m < 0$), which we regard as a variable. Thus, if the line is denoted by $y = mx + n$, we have that $n = b - ma$, and if $mx + n = 0$, we have $x = \frac{-n}{m} = \frac{ma-b}{m}$. Therefore, the area of the desired triangle is $S(m) = \frac{1}{2}(b - ma)\frac{ma-b}{m}$, or after a simplification:

$$S(m) = \frac{1}{2}\left(2ab - a^2m - \frac{b^2}{m}\right).$$

For $S(m)$ to be maximal, we must have that $a^2(-m) + \frac{b^2}{-m}$ is maximal. Using the AM-GM inequality:

$$a^2(-m) + \frac{b^2}{-m} \geq 2\sqrt{b^2a^2} = 2ab.$$

For an equality, we must have that the summands are equal, that is $\frac{b^2}{m} = a^2m$, so that $m = -\frac{b}{a}$ (recall that m is negative). In this situation, we get that the line is of the form $y = -\frac{b}{a}x + 2b$, and the minimal area is $S_{\min} = 2ab$. An example, with the values $a = 2, b = 4$, can be seen in 2.

In the next example, we will find the cuboid with maximal volume that can be bounded inside the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$. For that, let (x, y, z) be the vertex of

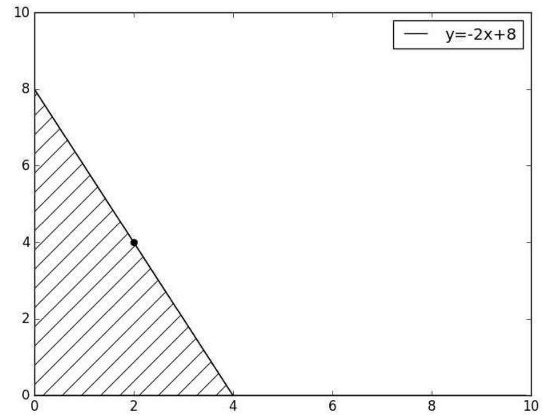


Figure 2: A triangle with minimal area, bounded by the axes and the line $y = -2x + 8$ in the first quadrate

the cuboid, with all positive parameters. Then, the length of the sides of the cuboid are $2x$, $2y$, $2z$, and thus the volume is $V(x, y, z) = 8xyz$. Now, using the AM-GM inequality on the ellipsoid formula, we get

$$1 = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} \geq 3\sqrt[3]{\frac{x^2}{4} \cdot \frac{y^2}{9} \cdot \frac{z^2}{25}},$$

and therefore

$$V(x, y, z) = xyz \leq \frac{10}{\sqrt{3}}.$$

Thus, by the maximality property of the AM-GM inequality, the maximal possible value for V is $\frac{10}{\sqrt{3}}$ which is achieved where $\frac{x^2}{4} = \frac{y^2}{9} = \frac{z^2}{25}$. Therefore, we get that $\frac{x^2}{4} = \frac{y^2}{9} = \frac{z^2}{25} = \frac{1}{3}$ so $(x, y, z) = \left(\frac{2}{\sqrt{3}}, \sqrt{3}, \frac{5}{\sqrt{3}}\right)$, so the vertices of the cuboid are $\left(\pm\frac{2}{\sqrt{3}}, \pm\sqrt{3}, \pm\frac{5}{\sqrt{3}}\right)$. The two shapes can be seen in 3.

For the last example in this subsection, we solve the following problem. From a given square paper with a side of length a, one can create a box without a cover, in the following manner: create 4 small squares of the same length x at the corners of the square, then fold them upwards into a box. For this box, we want to find the value x so that the volume is maximal. By the construction described above, the sides of this open box are $x, a - 2x, a - 2x$, so that the volume $V(x)$ is given by the formula

$$V(x) = x \cdot (a - 2x)^2.$$

Now, using the AM-GM inequality, we have that

$$\frac{2a}{3} = \frac{(a - 2x) + (a - 2x) + 4x}{3} \geq \sqrt[3]{(a - 2x)(a - 2x) \cdot 4x},$$

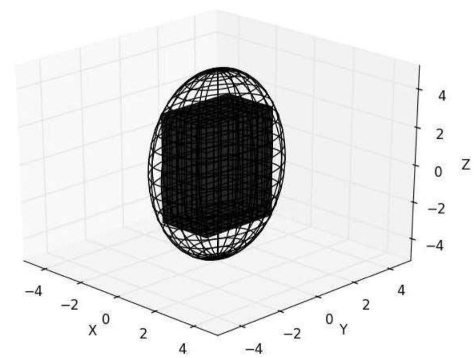


Figure 3: A cuboid bounded in a given ellipsoid, such that its volume is maximal

After a simplification we get:

$$V(x) = (a-2x)^2 \cdot x \leq \frac{2a^3}{27}.$$

Therefore, the maximal volume of the constructed box is $\frac{2a^3}{27}$, which (by the maximality of the AM-GM inequality) is achieved as the point where $a - 2x = 4x$, meaning $x = \frac{a}{6}$.

3.3 min-max problems for functions of two variables subject to a condition

This subsection is devoted to solving min-max problems for functions of two variables, by again using algebraic inequalities. As opposed to the problems presented in the previous subsection, in the two-variable case, the problems are more difficult – this is due to the possibility of limiting the solution to a given domain. In such a case, the classical method for solving these problems is the *method of Lagrange multipliers*, which is a tedious procedure that can sometimes be avoided; therefore, these problems can be presented to high school students, without the use of heavy subjects from analysis.

For the first example, let $f(x, y, z) = x^2y^3z$; find its maximal value, subject to the condition $x + y + z = P$ $(P \neq 0)$.

First, we present the classical solution, using the method of Lagrange multipliers: define

$$h(x, y, z) = x^2y^3z - \lambda(x + y + z - P);$$

we want to find the zero point of the equation $\nabla h = 0$, which gives, in addition to the original constraint, the three equations:

$$2xy^3z - \lambda = 0 \tag{4}$$

$$3x^2y^2z - \lambda = 0 \tag{5}$$

$$x^2y^3 - \lambda = 0 \tag{6}$$

There are 3 generic solutions for these equations:

1. $(0, y, P - y)$ for $y \in \mathbb{R}$, in which case $\lambda = 0$;
2. $(x, 0, P - x)$ for $x \in \mathbb{R}$, in which case $\lambda = 0$;
3. $\left(\frac{P}{3}, \frac{P}{2}, \frac{P}{6}\right)$, in which case $\lambda = \frac{P^5}{3^2 \cdot 2^3}$.

Recall that we want the maximal value of f; we see that putting the first two points gives $f = 0$, where the third one gives $f = \frac{P^6}{3^2 \cdot 2^3 \cdot 6} \geq 0$, and therefore this point is the unique maximal point, with maximal value $\frac{P^6}{3^2 \cdot 2^3 \cdot 6}$.

Now, we solve the problem using algebraic tools. For that, note that we can write $P = \frac{x}{2} + \frac{x}{2} + \frac{y}{3} + \frac{y}{3} + \frac{y}{3} + z$, and then using the AM-GM inequality we get that

$$P \geq 6 \cdot \sqrt[6]{\frac{x^2y^3z}{108}};$$

rephrasing the above, we get

$$f(x, y, z) = x^2y^3z \leq \frac{P^6}{2 \cdot 6^3}.$$

By the maximality property of the AM-GM inequality, f is maximal precisely where the summands in the above are equal. Therefore the maximal point is achieved where $\frac{x}{2} = \frac{y}{3} = z$, which means that $z = \frac{p}{6}$ and so the maximal point is $(\frac{P}{3}, \frac{P}{2}, \frac{P}{6})$, with the maximal value $\frac{P^6}{2 \cdot 6^3}$.

One notes the obvious difference between the two solutions presented above – the algebraic one is much simpler, and can be used without involving advanced tools from analysis.

Just as in the previous subsection, this example can be generalized to a more abstract scenario, for the function $x^my^nz^k$. We leave the details for the reader; the final maximal point is

$$\left(\frac{mP}{m+n+k}, \frac{nP}{m+n+k}, \frac{kP}{m+n+k}\right) = \frac{P}{m+n+k}(m, n, k),$$

with the maximal value

$$\frac{m^m n^n k^k}{(m+n+k)^{m+n+k}} P^{m+n+k}.$$

The last example we present is quite intriguing: to find the measurements for a cuboid without a top cover, with a given surface area, such that its volume is maximal.

In order to solve this problem, let a, b, h be the measurements; we want to find the maximal value for $V(a, b, h) = abh$, with the condition $ab + 2ah + 2bh = S$, where S is given. Using the AM-GM inequality, we get

$$\frac{ab + 2ah + 2bh}{3} \geq \sqrt[3]{ab \cdot 2ah \cdot 2bh}$$

so that

$$\frac{S^3}{108} \geq a^2b^2h^2 = V^2(a, b, h).$$

Now, V is maximal if and only if V^2 is maximal; for V^2 to be maximal, using the maximality property of the AM-GM inequality, we must have an equality of the summands, so that $ab = 2ah = 2bh$, or $a = b = 2h$, so the desired point is $(2h, 2h, h)$, where $12h^2 = S$; that is $\left(\sqrt{\frac{S}{3}}, \sqrt{\frac{S}{3}}, \sqrt{\frac{S}{12}}\right)$.

Note that if we want the cuboid to have a top cover, the condition becomes $2ab + 2ah + 2bh = S$, and then by the same method as above, $a = b = h$, so we get a cube.

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