Editorial from Bronisław Czarnocha, the Chief Editor.

The Spring Issue of MTRJ brings an interesting collection of papers, which includes new types of submissions:

1. We open the Book Reviews section calling for reports on any interesting new books in math and math ed, both in mathematics, mathematics texts and textbooks. We start with an interesting book on trigonometry by Brenner and Maysonet reviewed by the Managing Editor, Malgorzata Marciniak.
2. We open the section on reports from Math Musea, exhibitions and local conferences, starting with the Museum of Children Creativity in San Francisco.

The Spring'18 issue brings in two important papers about the old topic of procedural/conceptual divide, an intriguing paper on the assessment of peer mentorship (high school/middle school) within the program Boy With a Ball and two papers dealing with advanced mathematics teaching.

Procedural/conceptual divide is a very important old and unresolved theme in math ed. TR Team of the Bronx has dealt with the topic in the context of remedial algebra and use of language in mathematics classrooms (Baker and Czarnocha, 2016; Czarnocha 2016) showing that well positioned use of language in mathematics classrooms increases the conceptual understanding and, what's unusual, it impacts positively learning of ESL.

Procedural/conceptual divide is closely related to process/object duality of algebra as well to the qualitative/quantitative debate.

Alioune Khoule paper show us in a very precisely executed teaching experiment "Practice Does Not Always Make You Perfect" that a systematic and persistent emphasis on conceptual understanding of algebra brings in strong positive learning effects by increasing both conceptual and procedural understanding of the subject. Note a nicely designed assessment of understanding test showing, in contradiction to many standard arguments, that indeed one can measure understanding in mathematics quite precisely.

On the other hand, Kim Mahowsky's presentation A Procedural Mindset in a Conceptual World of

Mathematics looks upon the same divide from the point of view of the 3rd grade teacher trying to teach mathematics with the help of the CCSM scripted lesson guide. Mahovsky observes and comments upon teacher's pedagogical decisions as teacher is teaching following the script. She focuses on teacher's decisions in following or omitting the suggestions of the conceptually designed script. She finds out the procedurally oriented bias in the sequence of teacher's decisions.

Both papers address the same issue: the resistance to conceptual understanding both by elementary teachers and faculty professors to whom Khoule's paper is directed, among others. Both of the papers suggest several questions:

1. Why, despite the research evidence and the design of Common Course curriculum there is such a resistance to conceptual understanding of mathematics?

2. Is the procedural/conceptual framework correct framework for understanding the issues of the procedural/conceptual divide?
3. Is scripting lesson guide the best method to involve teachers in conceptual thinking about mathematics they teach?

Ping Ye, Whitney Richardson, Derek Allen, Aneta Galazka, William Fayson and Huijun Yi

in their presentation *Analyzing Student Data As A Measurement of Success for Boy With A Ball*

perform the assessment of the extremely successful in turning youth in high-risk back to school program Boy With the Ball. The program is based on peer mentoring (high school→middle school) and the paper shows us the method of quite precise assessment of the successes of the program. One of its central results turned out to be that a sense of connectedness with community-family plays an important role in the BWAB program.

Peer mentoring finds strong support in MTRJ. It shows success in US among the disadvantaged student population such as Hostos CC. TR Team of the Bronx conducted recently a teaching experiment on facilitation of the creativity of Aha!Moment and found out that peer mentors were the most successful in producing such moments.

The next two papers take the advanced mathematical thinking under their scrutiny.

Abram, Dagan, Satianov and Yoshpe point to the creativity inherent in introducing non-standard approaches to classical subjects such as extrema. Usually solved with the help of calculus, the authors introduce and *Use Algebraic Inequalities to Solve Extremum Problem*. There is a long standing tradition of algebraic approaches to classical calculus problems and authors' paper is an important extension of that tradition. The extension which is very rich in carefully constructed examples and applications.

On the other hand, Dimitric's paper *Feedback from Student Errors as a Tool in Teaching* connects the assessment emphasized by Peng et al above with the improvement of pedagogy, very much in the style of Teaching-Research, in the context of Calculus 3, Linear Algebra and Analytical Geometry. It's the presentation of a large scope grounded in critically assessed educational research literature and theories of learning. It ultimately focuses on the concept of the function and how to support it the best in advanced mathematics classrooms. It uses interesting metaphors such as "bread sticks" or "grinding machine" to make the concept of function closer to every day's experience of students.

Baker, Czarnocha (2016) From Arithmetic to Algebra: A Sequence of Theory-Based Tasks in B.Czarnocha, W. Baker, O. Dias, V. Prabhu (Eds) *Creative Enterprise of Mathematics Teaching-Research*. Sense Publishers, Netherlands

Czarnocha (2016) Algebra/English as a Second Language Teaching Experiment. in B.Czarnocha, W. Baker, O. Dias, V. Prabhu (Eds*) Creative Enterprise of Mathematics Teaching-Research. Sense Publishers*, Netherlands

Contents

Practice Does Not Always Make You Perfect 4

Alioune Khoule

A Procedural Mindset in a Conceptual World of Mathematics 16

Kimberly A. Mahovsky

Analyzing Student Data as a Measurement of Success for *Boy With A Ball* 35

Ping Ye, Whitney Richardson, Derek Allen, Aneta Galazka, William Fayson

Huijun Yi

Textbook reviews 46

Małgorzata Marciniak

A brief report from a visit at the Children's Creativity Museum in San Francisco on January 27, 2018 . 48

Małgorzata Marciniak

The Use of Algebraic Inequalities for Solving Differential Problems 54

Meirav Amram, Miriam Dagan, Michael Yoshpe, and Pavel Satianov

Feedback from students' errors as a tool in teaching 64

Radoslav Dimitrić

Practice Does Not Always Make You Perfect

Alioune Khoule

LaGuardia Community College, City University of New York

Abstract: *The study investigated the impact that conceptual understanding has on mathematics achievement of remedial mathematics students in a community college located in New York City. The study sample consisted of 105 remedial mathematics students from four elementary algebra sections. Two of these four sections were under Conceptual treatment. The other two sections were under procedural treatment and served as the control group. To measure subjects' conceptual and procedural knowledge, the participants completed two quizzes (a conceptual quiz and a procedural quiz) a week before the final exam. Students' mathematics achievements were measured using the conceptual quiz score, the procedural quiz score and the final examination score. The study found that the conceptual treatment group performed better on both quizzes, despite the fact that the procedural group practiced more procedural problems. The final examination score mean for the conceptual group was higher than the one for the procedural group.*

Keywords: Mathematics anxiety, Procedural knowledge, Conceptual knowledge

INTRODUCTION

Remedial mathematics has become an academic and career obstacle for many students, particularly community college students. In fact, it has become the largest single barrier to student advancement. Approximately 24 % of students who entered community college in the academic year of 2007-2008, the less likely that student is to ever complete college English or Math (Bailey, 2009). While the courses often do not qualify for college credit, students must nonetheless pay tuition for them.

A central goal for mathematics educators is to help students nurture their mathematics understanding. However, community colleges' teaching of Algebra is mostly procedural. In mathematics, it is important to know both the basic concepts and the correct procedures for problem solving. To overcome "negative attitudes" toward mathematics, educators should use "concrete manipulative materials" to form the connection between concrete learning and abstract thought (Taylor & Brooks, 1986, p. 10). Although The National Council of Teachers of Mathematics (1989) recommended that teachers emphasize the use of conceptual problems to help students understand mathematics subjects, to date, little research has been conducted regarding the impact that conceptual understanding in mathematics has on students' achievement at the community college level (NCTM, 1989). Rittle-Johnson & Alibali (1999) investigated the relationship between conceptual and procedural knowledge in children's learning. Their findings suggest that conceptual knowledge influences students' procedural knowledge and vice versa. However, the impact that conceptual knowledge has on procedural knowledge is greater than the reverse.

Research Question**:** Are there significant differences in mathematics achievements between students who have been exposed to a conceptual treatment and those taught in ways that emphasize procedures?

METHODOLOGY

Setting and Participants: In this study, the population consists of remedial mathematics students at LaGuardia Community College (LaGcc) located in New York City. In fall 2015, 10% of the students at LaGcc were white non-Hispanic, 37% were Hispanic, and 15% were Black and 38% other races. The Fall 2015 enrollment was about 18,623, of which 58% was female and 42% male. Among the 18,623 students in academic programs, 50% of them were non-native born students. LaGcc was selected because of my familiarity with the environment. However, the author did not teach any elementary section in which the data was gathered. The study was done throughout the second session of Fall 2016 that started from January 4 to February 16. This was a 6 weeks session including a final week. The elementary algebra courses including the ones that participated in this study met Monday to Thursday for 2 hours per day lecture, 2 hours computer lab and 2 hours tutoring lab per week. The instructors led the lecture and the computer lab sessions. Students used a mathematics platform, "educosoft", to complete their homeworks in the computer labs. These homeworks were mainly procedural. College assistants –students who were in their final college year- led the tutoring labs using worksheets that were prepared by the course coordinator.

The study sample consisted of 105 remedial mathematics students from four elementary algebra sections. Thus, the sampling frame met the following criteria: (a) potential subjects were elementary algebra (MAT 096) students in LaGcc (b) they were students enrolled in the four sections selected to participate in this study. The mean enrollment for each elementary algebra section was 30.

Research Design: For this study, we randomly selected two elementary algebra sections to a conceptual treatment and the other two sections were under procedural treatment. Participants' assignment to groups was not randomized. This means that a quasi-experimental design was used in this study. Because of lack of randomized design, we tried to select groups that were as similar as possible, so we could fairly compare them. For instance, participants in all groups were not repeating elementary algebra course. One procedural treatment section and one conceptual section were scheduled in the morning between 8am to 12pm. The other two sections were given in the afternoon between 12pm and 4 pm. Each of these sections met 2 hours a day from Monday to Thursday. The computer labs homeworks and tutoring lab worksheets were the same for all four sections. The instructors in all four sections were adjuncts from the Mathematics department and each has less than 2 years of teaching experiences. Two of the four sections were under conceptual treatment: instructors in these sections followed lesson plans that focused on concepts rather than procedures (figure 1). The other two sections were under procedural treatment. Instructors in the procedural treatment courses followed lesson plans that were focused on procedures rather than concepts (figure 2). The researcher prepared 7 conceptual lesson plans for the conceptual groups and 7 procedural lesson plans for the procedural groups. These lessons plans were split into 9 sessions throughout the semester. The researcher fully observed all four groups' sections – procedural groups sections and conceptual group sections - when instructors taught the lesson plans that we prepared. Before each class meeting, I met with the instructor for about 15 minutes to go over the lesson plan. This was done with all groups.

Figure 1: Conceptual Approach to Teaching Slope

Objective: The objective of this lesson plan is to help students gain a deeper understanding of slope and to be able to quantify it from a conceptual approach. This objective can be achieved through real life settings or experiences. By gaining a clear understanding of slopes, students will be able to appreciate how a concept such as slope is useful in understanding the world around us.

Methodology: A real life setting of skiing resorts is used to illustrate the concept. For the computational part, a triangle made of blocks is used. This lesson plan includes 6 stages.

Stage 1: Students are introduced to the rating level of ski runs. The ratings are: easy, moderate, steep, and vertical; a horizontal aspect is added here for completeness. The class is put into groups, and each group is asked to place the 6 different hills into the categories listed above by the ratings.
Stage 2: The class comes together and discusses why students placed the hills in each category. The idea of measuring steepness is introduced, and the teacher asks the groups to develop a way to measure steepness.
Stage 3: The groups are given a worksheet (See worksheet below) ….and …. to help them develop a formula to measure the steepness of given lines (using rise over run). They are asked to consider what lines have in common, what their differences are, and how a formula for steepness might be developed. Groups develop their own formula and then share them with the class.
Stage 4: The groups share their formulas with the class. The formulas are then tested on lines that go in different directions from which the class has been using.

Math Topics

Geometric context: Slope analysis and interpretation, calculation of slopes, ordered pairs, graphing lines.

Tools

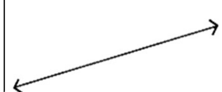
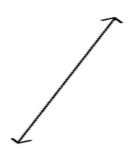
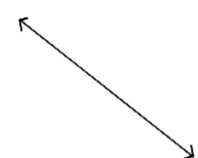
Multimedia projector, Computer, educational blocks, Picture of Ski resorts

Time 60 minutes

When to Introduce

This approach should be introduced to students at their first exposure to equation of a line.

Teacher's response: Teacher brings out a triangle made of blocks to illustrate how slopes differ in size, using the pictures, the lines and the number line to show how the sign of the slope could be generated. At this point, the notion of first quadrant and second quadrant of the Cartesian plane may be used. Teacher will then generalize students answers to come up with the formula: Slope = $(y_2-y_1) / (x_2-x_1)$ given two points (x_1, y_1) and (x_2,y_2).
Mini-Experiment 1: Now that you've developed a formula for finding the slope of the line, you are to use your transparent graph chart to find the slope of the lines below.

 Slope is________	Slope is ________ 	Slope is _________ 	 Slope is _________

Mini-Experiment 2: Now that you've learned how to find the slope of a line, we're going to switch things up. You are going to be given a slope, and you need to graph a line with that given slope. There are infinite number of lines that all have the same slope, so there's no one-way to do this. Instructors will ask some students to share their answers with the class.
Stage 5 (practice): Students get to practice the following problem using the slope formula.
Problem: Find the slope of the line passing through:
a. (1, 2) and (3, 3) b. (3, 4) and (-3,-6) c. (-4, -2) and (-1,2) d. (3,2) and (3,7)
Have volunteer students share their answers with the class
Stage 6: Students will be asked to find two points from the given equations to find the slope a. $y = 2x+1$ b. $y = 2$ c. $x=3$
Students share their findings and discuss about easier ways to find the slope (given an equation) without using points. The goal of this assignment is to come up with the slope intercept form $y = mx + b$ (m is the slope and b is the y-intercept).

Figure 2: Procedural Approach to Teaching Slope

Objective: Students will be able to develop an accurate formula for finding the slope of line.

Methodology: The given formula of slope is used to solve problem. This lesson plan includes 6 stages.

Stage 1 (10 minutes): The students are introduced to definition of slope: In coordinate plane, the slope of a straight line is defined by the change in y divided by the change in x.
Slope = Change in y / Change in X = $(y_2 - y_1)/(x_2 - x_1)$
Give an example applying the formula above
Stage 2 (15minutes): Students are put into groups to practice using the slope formula.
Practice problem: Graph each pair of given points below then find their slopes. (use separate graphs) a. (1,2) and (3,3) b. (3,4) and (-3,-6) c. (-1,-2) and (-1,2) d. (-3,2) and (0,7) e. (1,2) and (0,2) f. (1,2) and (1,7)
Stage 3: (10 minutes) Students are asked to go the board to share their results with the rest of the class.
Stage 4: (10 minutes) Now that we have a slope and a line from each given pair, let's have a discussion about when the slope is positive, negative , zero or undefined.

Stage 5 (7 minutes) Instructor introduces the slope intercepts formula: y = mx + b and how to find the slope using the slope-intercept form.
Example: Instructor solves the followings: a. y= 2x+1 b. y=2 c. x =3
Stage 6 (8 minutes)
Students solve the following problem and share their results with the class.
Practice Problem: Find the slope of the following equations: a. y=3x+1 b. 2y=x-1 c. 3x+2y =2 d. x+2=2 e. 2y+1=0

Math Topics

Geometric context: Slope formula and interpretation, calculation of slopes, ordered pairs, graphing lines.

Tools

Multimedia projector, Computer.

Time 60 minutes

When to Introduce

This approach should be introduced to students at their first exposure to equation of a line.

Figure 3: Conceptual and Procedural Quiz

Conceptual Quiz (25 min)	Procedural Quiz (25 min)
1. Explain why the equation x+ 1 = x+3 has no solution. 2. Explain the difference (in terms of the solutions) between x+1 =3 and x+1>3 3. Do you agree or disagree with the following statement? $x^2 = -1$ has no solution. Explain your answer 4. x (x-1) = 1 implies x=1 and x-1=1. Do you agree or disagree ? Explain your answer. 5. Explain the following statement: The graph of a function can have infinite x intercepts and at most one y intercept. 6. Explain the following: $\lvert x \rvert < -2$ has no solution. 7. The system of equations 2x+5y = 6 and 2x+5y = 5 has no solution. 8. The equation y = 2 has a slope of 0 (m = 0) and x = 2 has an undefined slope. Explain both cases. 9. You want to rent a car for your coming vacation. One rental agency charges a flat fee of $55 per day, while another charge $10 per day plus 20 cents for each mile driven. You expect to drive an average of 150 miles a day during your vacation. How much more money will you spend per day if you use the first rental agency? 10. $-x < 3$ implies that $x > -3$. Explain the change of the symbol < to >.	1. Solve the equation x+ 1 = x+3 2. Solve the inequality 2x + 1 < 3x-1 3. Solve $2x^2 = 18$ 4. Solve $x^2-2x-1=0$ 5. Find the intercepts of the equation: 3x+5y = 15 6. Solve and graph the solution: $\lvert x+1 \rvert < 2$ 7. Solve the system of equations: x+5y = 6 and 2x+5y = 5 8. Find the slope of the equation 3x+5y = 15 9. Find the equation of the line containing the points (2, 3) and (4, -1). 10. Evaluate f (–1) for the function $f(x) = x^2-2x+1$

Note: Each quiz is worth 10 pts and 25minutes were allowed to complete each one of them

Instruments: The participants in both groups completed a conceptual quiz and a procedural quiz (Table 3) a week before the final Exam. One is approached conceptually and the other was focused on procedural skills. The purpose of the conceptual quiz was to explore how well students understand the subject matter. On the other hand, the procedural quiz was to determine how well students know the rules or procedures for solving mathematic problems. Participants were given 50 minutes to complete both the procedural and conceptual quiz. Each quiz was worth 10 points. To measure achievement, participants' final exam scores were also collected from course instructors. The final exam is a departmental exam that has 25 questions which are more procedural questions. Students are given 2 hours to complete the final exam. The researcher graded all quizzes and the final exams were grade by the instructors.

A T-test was used to explore whether the final exam score mean differed significantly between the conceptual and procedural group. The final exam score was set to be the response variable for the analysis of the variance (ANOVA).

RESULTS AND DATA ANALYSIS

First, summary data (Table 1) on quiz scores (conceptual and procedural quizzes), and final exam scores for all respondents are described. The factor analyses of these variables were reported in order to compare the factor structure of these tests for both the conceptual and procedural group.

Table 1: Descriptive Statistics for all students' final examination score and quizzes (N=105)

	Minimum	Maximum	Mean	Std. Deviation
Conceptual-quiz	2	10	5.69	1.867
Procedural-quiz	3	10	6.47	2.024
Final-Exam-Score	6	96	65.00	13.516

**Quizzes are out of 10 and the final exam score is out of 100.*

Table 2: Descriptive Statistics for Conceptual treatment group (N=53)

	Minimum	Maximum	Mean	Std. Deviation
Conceptual-quiz	4	10	6.81	1.532
Procedural-quiz	3	10	7.08	2.111
Final-Exam-Score	35	96	69.09	12.023

**Quizzes are out of 10 and the final exam score is out of 100.*

The results of Table 3 below show that subjects in the procedural treatment had a final exam score mean of 60.83 which was lower than the final exam score average of all respondents (65) shown in Table 1. For this group, the conceptual quiz average (4.54) and procedural quiz average (5.85) were also lower.

Table 2 shows that subjects in the conceptual treatment averages on conceptual quiz (6.81), procedural quiz (7.08) and final examination (69.09) are higher than the averages of all respondents shown on table 1.

Table 2 and Table 3 show that students under conceptual treatment performed better on both quizzes and the final examination.

A set of tests of normality (Normal QQ plot, Kolmogorov-Smirnov test for normality) was performed on each group and all variables appear to be normally distributed (figure 4 and figure 5).

Figure 4 Normal Q-Q Plot for conceptual group using final exam score

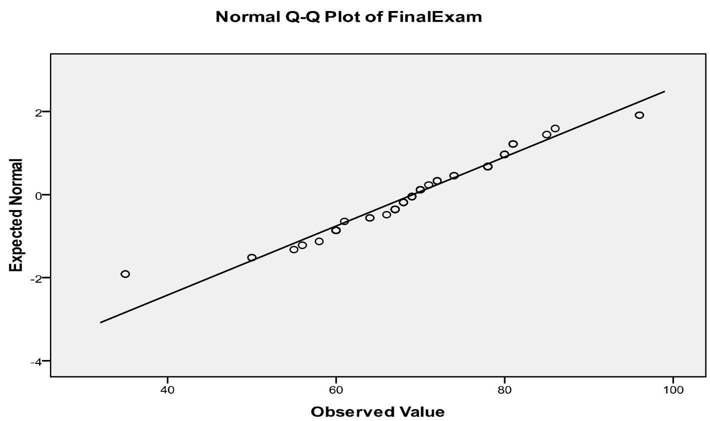


Figure 5 Normal Q-Q Plot for procedural group using final exam score

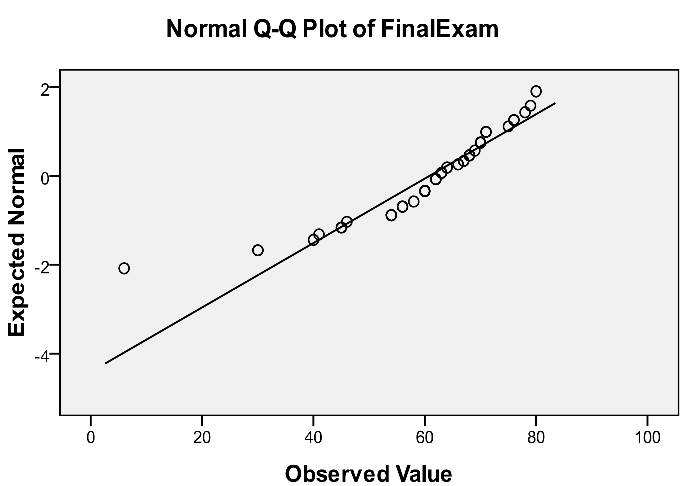


Table 3: Descriptive Statistics for Procedural treatment group (N=53)

	Minimum	Maximum	Mean	Std. Deviation
Conceptual-quiz	2	8	4.54	1.434
Procedural-quiz	3	10	5.85	1.742
Final-Exam-Score	6	80	60.83	13.784

**Quizzes are out of 10 and the final exam score is out of 100.*

First, The Levene's variance test shown in Table 4 for equality of variance displays a significance level of 0.666, which is greater than 0.05. Here, the Null hypothesis of equal variance is not rejected. There is enough evidence to say that the variance on the final exam score achievement for the two groups is statistically different. This is rather expected because scores on the final examination are dependent on how students have been taught throughout the semester and are relatively predictable.

Table 4: Independent sample test for Final Exam Score

	Levene's Test for Equality of Variance		T.test for Equality of Means						
								95% Confidence Interval of the Difference	
Final-Exam Score	F	Sig.	t	df	Sig.	Mean Diff	Std.Err Diff	Lower	Upper
Equal Variance	.188	.666	3.277	103	.001	8.267	2.523	3.264	13.271
Unequal Variance			3.273	100.58	.001	8.267	2.526	3.256	13.279

Second, the significance level of the test for equality of means (Table 4) is 0.001, which is less than 0.05. Here, the Null hypothesis will be rejected. There is sufficient evidence to say that there is a difference between the mean final examinations of the two groups. In other words, the teaching methods to which students are exposed, whether they emphasize conceptual understanding or use a procedural approach, affected the outcome of students' final exam scores.

Table 5: Group Statistics using Final exam score

	Group	N	Mean	Std. Deviation	Std. Error Mean
Final-Exam Score	Conc	53	69.09	12.023	1.651
	Proc	52	60.83	13.784	1.911

As it can be seen in the group's statistics tables (Table 5), students exposed to teaching methods that emphasized conceptual understanding performed better than students who were taught in ways that emphasized procedures.

The specific group is retained as expected in the model with a p-value of less than 0.05. The relevance of the group has been established since the beginning, and with the coefficients, to estimate a final exam score, the formula* stated below Table 6 was used.

Table 6. Coefficients of the final-exam-score model

Model		Unstandardized Coef		Standardized Coef	T	Sig.
		B	Std. Error	Beta		
1	(Constant)	27.296	4.219		6.470	.000
	conceptual-quiz	3.602	.886	.498	4.066	.000
	Procedual-quiz	2.923	.517	.438	5.657	.000
	Groupdummy	-2.880	6.565	-.107	-.439	.662
	Conceptconc	-.362	1.105	-.096	-.328	.744
	Conceptconc	-.362	1.105	-.096	-.328	.744

*Estimated-FinalExam = 28.157+2.918*proceduralquiz + 3.419* conceptualquiz - 4.888*groupdummy

CONCLUSION

The performance of community college students in the subject of mathematics has raised concerns for decades. Several measures have been taken to address this problem. Mathematics departments and remedial course instructors are constantly working to teach these courses in ways that will better help students understand the subject matter. At LaGuardia Community College, remedial non-credit courses constitute nearly 65% of all courses offered by the Mathematics department. Placement in a specific level in these courses is based on a university placement test. Unfortunately, it is not uncommon for students to repeatedly fail these remedial courses, sometimes up to three times. Unfortunately, these courses are mandatory and can prevent students from obtaining a college degree.

The study explored the relationship between mathematics achievement and conceptual understanding. It examined the mathematics achievement difference between conceptual method and procedural method of teaching. . Four variables were used to answer this question: the final examination score, the procedural quiz, the conceptual quiz, and the conceptual groups (groupdummy). The final examination score was the response variable.

The ANOVA test revealed that the relationship between conceptual quiz, procedural quiz, group (groupdummy) and final examination (response variable) was statistically significant. The results revealed that the conceptual groups outperformed the procedural groups on the conceptual quiz. The conceptual quiz average score for the conceptual groups (6.81 out of 10) was higher than the one for the procedural (4.54 out of 10). The conceptual quiz questions were not based on problem solving. These questions were designed to test students' knowledge of the subject matter. As the results illustrated, the conceptual groups had a better understanding of the subject matter. The conceptual groups also performed better on the procedural quiz, despite the fact that the procedural groups practiced more procedural problems than the conceptual groups and was exposed to a procedural treatment. The procedural quiz average score for the conceptual group was 7.08/10 and the procedural group's average was 5.85/10. The procedural groups' lower average score on the procedural quiz indicates that the conceptual treatment provided a more flexible understanding of mathematics, which allowed them to utilize their knowledge as a tool to solve problems. In other words, conceptual groups were more able to reason logically, formulate, represent, and solve mathematical problems. This finding supports Brownell's (1973) idea: "the greater the degree of understanding, the less the amount of practice necessary to promote and to fix learning" (p.188). These findings also support the NCTM's reforms (1989) in mathematics education, which argued that teachers should inculcate conceptual understanding before approaching procedural knowledge.

For the same reasons, the final examination score mean for the conceptual groups was higher than the one for the procedural groups by a margin of approximately 10%. This should not be a surprise, since the final examination format was similar to the procedural quiz. These results demonstrate that conceptual understanding is an important component of proficiency, along with factual knowledge and procedural facility.

Overall, this study statistically demonstrates that the use of conceptual method of teaching plays greater role on improving students' mathematics achievement and understanding of the subject matter. The rote memorization common in the traditional method of teaching mathematics focuses mainly on mastering rules. The procedural method, while sacrificing attention to concepts and when applied alone, is easy to forget or hard to remember; therefore, it is often associated with pain and frustration for students. While taking the examination, students must recall their lessons and the material they studied, a technique that generates rote learning, disabling students from performing well.

This study reveals that students achieve higher scores in mathematics when they are engaged in exploring and thinking rather than engaging only in rote learning of rules and procedures. In fact, these conceptual and active methods help students build the necessary confidence to learn new mathematical concepts. The question then still remains: In what way can these initiatives and instructional strategies be implemented in remedial mathematics classes in order to improve students' mathematics achievement.

REFERENCES

Brownell, W. A. (1973). *Meaning and skill—Maintaining the balance*. In F. J. Crosswhite, J. L.

Bailey, T. (2009). Challenge and opportunity: Rethinking the role and function of developmental education in community college.New Directions for Community Colleges,145, 1130.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Rittle-Johnson, B., & Alibali, M.W. (1999). *Conceptual and Procedural Knowledge of Mathematics:Does One Lead to the Other?* Journal of Educational Psychology, Vol.91, 175-189

Taylor, L., & Brooks, K. (1986). *Building Math Confidence by Overcoming Math anxiety*. Adult Literacy and basic Education, v10.

A Procedural Mindset in a Conceptual World of Mathematics

Kimberly A. Mahovsky

University of Northern Colorado, Greeley, Colorado, United States

Abstract: A procedural mindset is one of the most significant obstacles mathematics teachers confront when teaching for conceptual understanding. This educational criticism and connoisseurship responds to one first-year teacher's experience shifting to a more conceptual mindset to mirror the Common Core State Standards and examines three elementary teachers observed over an eight-week time frame before and after the administration of high stakes testing. Based on the analysis of 168 activities that were included in the lessons but were **not** part of the intended lesson, 68 percent of the activities taught were procedural, and 31 percent were conceptual. This article discusses the implications for students by these pedagogical choices made by the teachers.

Keywords: procedural, conceptual, common core standards, pedagogical choices

A Procedural Mindset in a Conceptual World of Mathematics

Mathematics teachers today face many obstacles when teaching mathematics conceptually, but the most substantial obstacle is mostly around the planned, and unplanned pedagogical decisions teachers make in every mathematics lesson. This article responds to one first-year elementary teacher's experience shifting to a more conceptual mindset to mirror the Common Core State Standards in an article titled Letting Go by Kevin Junod (2017). Additionally, the qualitative research presented elaborates on Junod's (2017) perspectives using educational criticism and connoisseurship. As this teacher admitted,

> During my first year of teaching, my teammates and I taught our students computations. We thought having our students simply solve a calculation correctly was true understanding of mathematics. Teaching just computations to rote memory was easy to teach and felt rewarding (Junod, 2017).

To better understand a teacher's mathematical practices, it is essential to understand a teacher's knowledge base and where a teacher's pedagogical decisions originate. This educational criticism addresses the following research questions:

Research Question 1: What pedagogical choices do third-grade mathematics teachers make to prepare their students for high stakes testing under the Common Core State Standards?
Research Question 2: What implications are there for students by these pedagogical choices made by the teachers?
Research Question 3: What are the challenges and opportunities of a teacher's pedagogical choices?

Grounded in Literature

Teachers are now being asked to instruct students in ways that were neither taught during preservice training nor learned as a student themselves in early grades (Heitin, 2014). Therefore, Beckmann (2010) contends that this lack of scholarly ability sometimes makes mathematics teachers uninterested in considering new ideas or instructional strategies. School districts/counties are now rewriting their curriculums or adopting curriculums like *Bridges* to help teachers mirror the expectations of the Common Core State Standards (CCSS) which has been adopted verbatim, partially, or with modifications by forty-one states, the District of Columbia, four territories and the Department of Defense Education Activity (Do DEA). However, the states Indiana, Oklahoma, and South Carolina withdrew from adopting the CCSS in 2014 alone. Osborne (2015) asserts that what was once merely a research-based explanation for the present transformation of the public school educational system has progressed into an outright demand for a shift to the Common Core State Standards. Teachers are the key to enacting the standards as they are designed and, for this reason, teachers need to know the mathematics well and how to teach the standards in captivating and compelling ways, but they are not given the support or the direction to be successful in this endeavor (Beckmann, 2010). Therefore, the CCSS for Mathematics are arduous and put high demands on the teachers due to their conceptual nature; as a result, teaching becomes more routine than authentic where students are learning through exercise-oriented lessons versus inquiry-based lessons due to high demands related to test scores and the time-intensive nature of teaching conceptually (Beckmann, 2010). As Junod (2017) admits, "I failed my students when they solved story problems or any problem that did not have an obvious equation to solve. My students were unsuccessful at these problems because the problems were too complex for them to identify the process to the solution". Teaching mathematics conceptually requires attending to student's questions, anticipating obstacles, capitalizing on opportunities, making connections, and providing enrichment beyond the immediate tasks (Rowland & Zazkis, 2013). As such, teachers do not feel they have the time to allow their students to persevere in problems solving, so the alternative is to teach the mathematics procedurally. In essence, scripts, tests, and textbooks cannot continue to take a teacher's place in the classroom instruction, but a teacher must apply freedom and inventiveness that the teaching profession demands (Vecellio, 2013). Dewey (1938) contends that teachers must have a sympathetic understanding of students as individuals giving the teacher an idea of what is actually going on in the minds of those learning, Therefore, a teacher's interest shifts towards a more profound knowledge of a student's mathematical understanding and also more towards concrete mathematics lessons (Ticha & Hospesova, 2006).

In two publications by the National Council of Teachers of Mathematics (NCTM): Improving practice, Improving Student Learning (NCTM, 2007) and the Principles and Standards of School Mathematics (NCTM, 2000), the council provides a vision for mathematics pedagogy, however, there is a disparity between NCTM's vision of mathematics pedagogy and what is actually occurring during mathematics instruction across the United States (Walkowiak et al., 2014). According to Osborne (2015), the new standards, both Common Core and NCTM, go deeper requiring a more refined skill set for teaching based on relevant pedagogical content knowledge. A teacher's motivation to refine their teaching methods can be fueled by the support they receive to create sociomathematical norms or mathematical discussions that are specific to students' mathematical activity (Kazemi & Stipek, 2001). These norms are what regulate mathematical argumentation and influence the potential learning opportunities for both the students and the teacher (Yackel & Cobb, 1996). In the study conducted by Wilson and Downs (2014), many of the teachers pushed for the use of gradual release and modeling to support students in learning mathematical concepts and grappled with slowing releasing students to work on their own and modeling to support their students with learning mathematical concepts to differentiate their instruction. However, these strategies will most likely be ineffective at advancing attainment of mathematical knowledge and claim that these approaches provide only narrow opportunities for students to gain competence in math standards (Wilson & Downs, 2014). As Junod (2017) eventually realized is that he needed to let go of the ideology that students needed to learn procedurally through repeated practice of problems and that he needed to focus on problem-solving skills to succeed on high-stakes testing. He had to learn that it was okay to let his students fail on problems repeatedly to the point of frustration allowing them an opportunity to improve on their problem-solving skills. Junod (2017) concluded that allowing his students to struggle was not enough, but he had to anticipate their struggles and create guiding questions that would support them in solving the problems presented to be successful in solving unfamiliar problems presented on the high-stakes test. Vogler and Burton (2010) question whether or not teachers are genuinely providing enough standard-based instruction practices to allow students to make sense of mathematics in meaningful ways and the ability to apply higher-order thinking strategies to solve mathematical problems. Vecellio (2013) asserts that educators need to prepare students for this world which requires teachers to take on interdisciplinary issues, projects and problem situations that present themselves during mathematics instruction. For this reason, it is critical to understand the dynamics of pedagogical practices in mathematics and how to find a balance between conceptual and procedural mathematics instruction with effective questioning.

Pedagogical Practices in Mathematics

Shulman (1984) conceptualizes pedagogical content knowledge and defines it as a "blending of content and pedagogy into an understanding of how particular topics, problems, issues are organized, represented, and adapted to the diverse interests and abilities of learners and presented for instruction" (p. 8). Vecellio (2013) suggests that this is part of what means to become a pedagogue. The pedagogue examines all the standards to see how they figure into problems and processes of the world, and then teach those standards in rich, powerful, and authentic ways which include using a variety of texts and other media during instruction. (Vecellio, 2013). Mason and Davis (2013) attest that a teacher who is mindful not only of the subject matter but the pedagogical facets of mathematics is in a better position to direct a student's attention to what is essential, to choices available, and to criteria that might be applied when working with students on exercises and worked examples.

A teacher may have their personal theories of learning, interpretation of their own brand of mathematics, teaching styles, and so forth, but the major factors in a teachers' capacities to engage flexibly and productively with their students is their capacity for in-the-moment pedagogy (Mason & Davis, 2013). In-the-moment pedagogy is described by Mason and Davis (2013) as the scope and range of a teacher's mathematical thinking, the pedagogical strategies associated with this thinking, and the academic procedures that come to mind in the moment. Similarly, Dewey (1938) describes his principle of continuity of experience which rests upon the fact of habit. The basic characteristic of habit is that every experience enacted and undergone modifies the one who acts and undergoes, whether we wish it or not, the quality of subsequent experiences (Dewey, 1938). To put it another way, the principle of continuity of experience means that every experience in the classroom both takes up something from prior activities or discussions and modifies the quality of those activities or discussions that follow it (Dewey, 1938). This in-the-moment pedagogy often causes confusion and tension for teachers as they try to navigate the selection of instructional approaches to address particular mathematical standards (Wilson & Downs, 2014). Persevering in problem-solving and critiquing others' mathematical arguments are some behaviors and skills that would be utilized in these classroom practices to grow pedagogical competence (Wilson & Downs, 2014). In either case, the Department of Education (2008) concludes that the sweeping recommendation that instruction be entirely student-centered or teacher directed is not supported by research and should be rescinded if such research does exist. "High-quality research does not support the exclusive use of either approach" (Department of Education, 2008, p. xxii). Therefore, there would be a pedagogical struggle among teachers to find balance among procedural and conceptual learning in mathematics.

Procedural and Conceptual Understanding in Mathematics

Procedural knowledge is made up of two specific parts: One part is composed of the formal language of mathematics which includes an awareness of symbols used to represent mathematical ideas and the syntactic rules for writing symbols in an appropriate form; the second part consists of rules, algorithms, and procedures used to solve mathematical problems (Hiebert & Lefevre, 1986). According to Hiebert and Lefevre (1986), the key feature to procedural knowledge is that the procedures are executed in a predetermined linear sequence. When teaching for procedural knowledge, the teacher ensures that students get to automaticity (Willingham, Winter 2009-2010). The teacher explains to students that memorizing procedures and facts is necessary because it frees the mind to think about concepts (Willingham, Winter 2009-2010).

Conceptual understanding, on the other hand, is achieved by the building of relationships between pieces of information (Hiebert & Lefevre, 1986). The joining process emerges between two pieces of information that are already stored in a memory or between an actual piece of knowledge and one that is just learned (Hiebert & Lefevre, 1986). These relationships encumber the individual facts and schemes so that all the pieces of information are now linked to some network (Hiebert & Lefevre, 1986). Sociomathematical norms that characterize or promote reasoning based upon conceptual knowledge include: (a) the explanation that a student gives for a problem consists of a mathematical argument and not just a procedural description; (b) the mathematical thinking demonstrates understanding relationships among multiple strategies; (c) errors made give students opportunities to reconceptualize a problem, explore disparities in solutions, and attempt different strategies, and (d) working collaboratively employs individual accountability and reaching agreement through mathematical discussions. Between conceptual knowledge and procedural knowledge, conceptual knowledge is the most difficult to obtain because this knowledge is not easily acquired since a teacher cannot pour concepts directly into a student's head, but instead, new concepts must build upon knowledge that already exists within a student (Willingham, Winter 2009-2010). Drawing connections among mathematical topics, as Hiebert & Lefevere (1986) describes, deepens conceptual knowledge, but, unfortunately, it is one of the desired outcomes that is rarely met in the mathematics classrooms across the United States (Willingham, Winter 2009-2010). This outcome is primarily due to the conflict between the Common Core and NCTM standards requiring students to persevere in problem solving and the procedural mindset of teachers who feel that in order for their students to succeed on the high-stakes testing their only choice is to push for procedural versus conceptual understanding.

Finding Balance: Experience that Leads to Growth

According to the Department of Education (2008), if mathematical ideas are taught using real-world situations, then students will improve their performance on assessments that involve similar real-world problems. Vecellio (2013) asserts that educators need to prepare students for this world which requires teachers to take on interdisciplinary issues, projects and problem situations that present themselves during mathematics instruction daily. Therefore, it is the responsibility of the educator to see the direction an experience is heading and follow it (Dewey, 1938). Being aware that students often become more engaged with the real-world aspects of math problems rather than the mathematical concept intended helps teachers anticipate student responses and prepares a mechanism to embody the line of thinking into the math concept being considered (Inoue & Buczynski, 2010). Therefore, students are encouraged to learn, discover, understand or solve problems on their own by experimenting and evaluating possible answers by trial and error, so they have a higher problem-solving performance than those students taught only one way to solve a problem (Bruun, 2013). Rowland and Zazkis (2013) believe that this extended exposure to mathematics serves as a support structure for a teacher's readiness to hypothesize, experiment, to take risks, and, most importantly, take advantage of unforeseen opportunities that arise during instruction. One of the primary responsibilities of an educator is that they are not only aware of the main essence of shaping the experience by the surrounding conditions, but that the teachers also recognize what surroundings are conducive to having experiences that lead to growth (Dewey, 1938). Dewey (1938) asserts that, above all, teachers should know how to use their surroundings, both physical and social, that exist during a lesson so that teachers can acquire all that students have to share in order to build up experiences that are worthwhile. To understand the lived experiences of students, teachers must explore more deeply into what "knowing-in-the-moment can be like in order to suggest how teachers can develop that knowing, without losing the essential complexity of the phenomenon" (Mason & Davis, 2013, p. 188). Mostly, this has to do with teachers being mathematical with and in front of their students (Mason & Davis, 2013). Therefore, the most crucial attitude a teacher can form with students is the desire to go on learning (Dewey, 1938). For this reason, it is essential for mathematics education shift from the basic transfer of information, instructions, and algorithms to grasping, acting, experiencing, and developing a desire for life-long learning (Ticha & Hospesova, 2006). This requires a change in the teacher's role in promoting new schemes and demanding more from students through differing strategies.

Summary

The Common Core State Standards go more in-depth and require a more refined skill set based on relevant pedagogical content knowledge (Osborne, 2015). It is optimal for a teacher to be cognizant of the mathematical subject matter as well as the pedagogical facets of mathematics to direct students to what is essential and the varying ways problems can be solved (Mason & Davis, 2013). Therefore, a teacher's motivation can be fueled by the support they receive to create sociomathematical norms in their classroom (Kazemi & Stipek, 2001). This is where conceptual versus procedural understanding of mathematics is pivotal. Procedural knowledge is when procedures are followed in a predetermined linear sequence ensuring automaticity. However, conceptual understanding is gained by the students building relationships between pieces of information acquired over time. Drawing connections among mathematical concepts are what deepens conceptual knowledge. The Common Core and NCTM standards require this connection, but teachers do not have the training or support to be successful even when given a scripted curriculum. It is essential to understand the specifics of a student learning procedurally and conceptually in a mathematics classroom. More importantly, there are several lessons that can be learned from conducting an educational criticism and connoisseurship to study the pedagogical decisions teachers make during mathematics instruction and how these decisions affect a student's ability to conceptualize math concepts.

Method

In the 1960s, Eisner (1991) developed a method of qualitative research that he named "educational criticism and connoisseurship". Using the arts as the basis for his thinking, Eisner came to discern connoisseurship as the art of appreciation and criticism as the art of disclosure (p xi). He formulated a manner in which one can disclose what one learned through his and her connoisseurship as description, interpretation, evaluation and thematics. Thus, an educational criticism and connoisseurship was used to examine mathematics instruction to develop an in-depth understanding of the pedagogical choices third-grade mathematics teachers make to prepare their students for high stakes testing in three different suburban school settings within a large district in one of the states of the Rocky Mountain West. The purpose was to provide a vivid picture of how teachers were challenged with making pedagogical decisions while teaching within a conceptual framework and how students' learning was impacted.

Four Dimensions of Educational Criticism and Connoisseurship

Educational Criticism has four dimensions through which the critic disseminates his/her observations: *description, interpretation, evaluation, and thematics* (Eisner, 1991). When *describing*, the critic uses narrative to portray what is essential from the profound qualities of the experience. The goal is to express what it would feel like to be in the environment in which the researcher is trying to portray, and in doing so helps the reader know the environment (Eisner, 1991). The critic cannot attend to everything in an educational setting, but instead provides those elements that help the reader participate vicariously in the experience to understand and critique the interpretations made by the critic based upon these observations. Through *interpretation*, the critic explores the meanings and consequences of the educational setting observed. According to Eisner (1991), the goal is to illuminate "the potential consequences of practices observed and reasons that account for what's been seen" (p. 95). As mentioned earlier, theory, experience, and various viewpoints influence how the critic interprets the educational events described; therefore, there is not a right or wrong interpretation. Instead, the interpretation provides a means of exchange between the critic and the reader to develop a concept of reality. The *evaluative* dimension examines the educational significance of the description and interpretation. The aim of education "is not merely to change students, but to enhance their lives" (Eisner, 1991). The evaluation piece helps to discern if the educational experience observed has met this objective. Since the researcher makes choices in what he/she pays attention to, the values that guide the observation also inform the evaluative dimension and imbue what is written. Lastly, *thematics* afford the reader with the larger lessons a criticism has to offer (Eisner, 2002). "That is, every particular is also a sample of a larger class. In this sense, what has been learned about a particular can have relevance for the class to which it belongs" (Eisner, 1991). The researcher's aim is to provide readers with novel theories or guides to help them understand and appraise the pedagogical choices third-grade teachers make to prepare their students for high stakes teasing under the Common Core Standards. By studying the practices of three teachers in this study, the researcher has developed a powerful understanding of the implications there are for students by these pedagogical choices. The researcher's goal is to provide a vivid description, interpretation, and analysis of these pedagogical decisions to guide stakeholders to understand these pedagogical decisions in hopes of improving mathematical practice.

Validity in Educational Criticism and Connoisseurship

According to Eisner (1991), validity in qualitative research and educational criticism and connoisseurship categorically relies on three different criteria: (a) structural corroboration; the "means through which multiple types of data are related to each other to support or contradict the interpretation and evaluation of a state of affairs" (p. 110); (b) consensual validation: "at base, agreement among competent others that the description, interpretation, evaluation, and thematics of an educational situation are right" (p. 112); and (c) referential adequacy: the "extent to which a reader is able to locate in its subject matter the qualities the critic addressed and the meanings he or she ascribes to them…when readers are able to see what they would have missed without the critic's observation" (p. 114). Therefore, it was essential to have observations of mathematics lessons, but also interviews before and after the observation window, as well as teacher reflections of the lessons observed each offering additional details and clarifications to the observations.

Research Setting

Observing three third-grade classroom teachers at three different elementary schools for a total of 30 hours allowed for an in-depth understanding of the decisions teachers made in preparing students for high stakes testing under the Common Core State Standards and the factors that led to these decisions. Criterion sampling was used so that three third-grade teachers in the Red River District (pseudonym) could be studied. The reasoning behind studying third-grade, in particular, was because the pressure is more significant in third-grade classrooms versus second-grade classrooms due to second-grade scores not being counted when determining a school's accountability rating (Plank & Condliffe, 2013).

Data Collection Methods

Within the three schools in which the third-grade teachers reside, the researcher conducted a pre and post interview with each teacher around the observation window and also reviewed lesson plans weekly. Most importantly, the researcher collected data related to the training and professional development provided to prepare the teachers for the teaching of the Common Core State Standards and, in particular, the new conceptual mathematics curriculum, Bridges in Mathematics, introduced this year. The observation window was six weeks prior to the administration of the high stakes assessment and two weeks following the completion of the high stakes assessment. Each teacher was observed weekly to record descriptive and reflective notes during the mathematics lesson. Seeing student reactions to lessons taught and how they interacted with other students and the teacher allowed the researcher to answer the research question around implications for the students by these decisions and how these pedagogical choices influenced the students understanding of mathematical concepts.

Data Analysis Methods

Once written and recorded observations were collected up to the point of the administration of high stakes testing and also collected a few weeks after the assessment, observation notes, lesson plans from the actual curriculum, documents used during each lesson and journal entries were put in chronological order and grouped by teacher. The key was to have a complete picture of the lesson observation from what was actually taught to what was the intended lesson from the curriculum. Once the observations, curriculum lesson plans, supplemental lesson documents, journal entries, lesson plans were put in chronological order, they were read four times using a different type of lens for each reading. During the first reading, the researcher focused on items/activities observed during the lesson that directly correlated to the intended lesson plan in the curriculum. The researcher then focused on items that were observed that were **not** part of the intended lesson plan in the curriculum. For the second reading, the researcher analyzed those items/activities that were **not** part of the intended lesson plan from the intended curriculum to distinguish if the item/activity was procedural or conceptual in nature. As a result, those items/activities that were procedural in nature that were not part of the intended curriculum and those items/activities that were conceptual in nature that were not part of the intended curriculum became apparent. This made it very easy for the researcher to see what pedagogical decisions the teacher was making outside of the intended curriculum and if they were procedural or conceptual in nature. Also, analyzing how much the teacher followed the intended curriculum or modified it to her own pedagogy was equally important. The researched determined how many items/activities were not part of the intended curriculum and determined the percentage of those activities that were procedural and conceptual in nature based on the teacher's pedagogical decisions. The third reading focused on if the pedagogical decisions made by the teacher throughout the entire lesson impacted the intended skills or concepts from the intended curriculum lesson in the teacher handbook. The researcher determined what pedagogical decisions made by the teacher during the observations impacted the intended skills/objectives by highlighting the skills/objectives that were covered by the observed lesson on the curriculum lesson plan and those skills/objectives **not** meet by the pedagogical decisions of the teacher during the observation. The fourth reading involved highlighting those activities/items that were actually left out of the intended curriculum lesson. Those items/activities that were left out of the intended curriculum that were procedural and conceptual in nature conceptual were noted. This gave the researcher a visible picture of those pedagogical decisions made by the teacher during the observations that were procedural or conceptual in nature. While reading the interviews, it was essential to disregard the actual question asked so the researcher could focus on what is actually said. Interviews were kept separate and used to analyze for teacher reflection to get an overarching picture of the teachers' feelings about high-stakes testing and the pressures of teaching a new curriculum this year. The journals filled out by the teachers for each lesson observed were read in conjunction with the lesson plans and observations.

Findings

After identifying 186 items/activities that were **not** part of the intended lesson plan provided in the teacher handbook, the researcher determined which of those items/activities were procedural or conceptual in nature. After analyzing each item/activity, 68 percent of these items/activities were deemed procedural in nature, and 31 percent of these items/activities were deemed conceptual in nature. Based on this vast percentage difference between those items/activities added by the teacher that was procedural versus conceptual illustrates a teacher's insecurities with teaching conceptually or even their preference for teaching procedurally. With the new curriculum mainly using strategies to build conceptual understanding, it was not surprising to see such a gap in percentages. If a teacher is not confident in her ability to teach a concept conceptually, she will resort to what she already knows. This idea is reinforced by Mason and Davis (2013) in that teachers seem to misunderstand the vital connection between mathematical awareness and in-the-moment pedagogy to be successful in teaching mathematics for conceptual understanding. The vignette below illustrates a lesson that advocates for a conceptual approach through a lesson to be followed in the teacher handbook, but she finds herself teaching the lesson strictly for procedural understanding focusing strictly on the algorithm for area and perimeter.

> The teacher stands by the pocket chart in 'number corner' as the students sit on the United States carpet with their eyes glued to the teacher as she reviews the terms area and perimeter and asks the students to tell her how they are alike and different. A large number of students raise their hands in hopes that the teacher calls on them. As the teacher calls on one student to give an attribute, the students anxiously wait for her to call on another student to add to his response. The teacher reads the term on the card and begins to chant, "Perimeter is—you add the sides up!" The students repeat

> the chant with her as she puts the card back in the pocket chart. She quickly pulls the next card out of the pocket chart and displays it to the students. Immediately hands bolt up to the ceiling as the teacher reads the term on the card, “Area.” The students immediately start performing motions with their hands with their palms up as if they are lifting a box and then immediately makes a cross with their forearms and then raises the imaginary box to the ceiling. The teacher then repeats the chant as the students follow, “Area is—base times height.” The students follow along as the teacher repeats the chant and makes the hand motions to match. She repeats this three more times in hopes that the students would repeat the chant in sync but is greatly disappointed. She then says, “Perimeter is—you add the sides up.” The students continue to mimic the teacher’s hand motions as they sing chorally the algorithms to find the area and perimeter of a geometric figure. The students are then asked to return to their desks as the teacher saunters over to the Smartboard data projector to project a square centimeter. The students slowly begin to settle in their desks as chairs begin clanking against desks as they get comfortable. The teacher writes on the paper projected on the Smartboard—Perimeter: 1 + 1 + 1 + 1 = 4 cm. She then proceeds by writing the algorithm for finding the area of the square centimeter—Area: 1 x 1= 1 cm^2. The students watch as the teacher displays her calculations on the Smartboard.

It was interesting to observe the teacher using the notation cm^2 versus writing square centimeters. The students do not learn about exponents in third-grade, but the teacher made a pedagogical choice to display this notation. What was even more interesting was that the students did not even question what it meant. This indicated that the students were used to being fed information instead of discovery.

> She continues by writing the word ‘dimensions:’ with 1 by 1 written underneath it. She places an arrow pointing to the 1 and writes width under it and then does the same for the other numeral 1 and writes height below the arrow. The students continue to watch the teacher display writing on the Smartboard in utter silence. You could almost hear the gears moving in their heads as they try to make sense of what the teacher is displaying on the Smartboard. No hands are raised as she proceeds to put up a strip that is 10 centimeters long. She asks the students, “How would we find the perimeter of this rectangle?” The students sit in silence as four hands raise timidly to be called upon. The teacher calls on one student to answer her question. The student answers, “You would add 10 + 1 and 10 + 1.” The teacher quickly writes this down to be displayed on the Smartboard and then moves on to ask the students how to find the area of the rectangle. A very excited boy leaps from his chair to raise his hand. The teacher patiently waits to call on him in hopes that other students would raise their hands. She calls on the student, and he responds, “1 x 10 = 10”. The teacher immediately asks, “10 what? We don’t want a naked number!” Then several students chuckle as the student answers, “square centimeters.” Most of the students look bewildered as the teacher asks, “Could we use this to measure something if we did not have a ruler?” Most of the class chorally answers, “Yes!” The teacher immediately asks, “Why?” She then answers her own question without calling on the students, “Because we know that each square is a centimeter.” The students are then instructed to open their student workbooks to page 206.

The teacher began the lesson conceptually as intended by the teacher handbook but quickly strayed teaching procedurally by focusing on the algorithm for area and perimeter. The teacher was to ask student pairs to compare and contrast the two terms by how they are alike and different. In introducing these terms, the teacher immediately felt the need to sing the chants associated with the algorithms. Next, the students were to share their ideas with the class than with their partner. From there, the students were to get out a ruler and a set of base ten pieces setting aside one small square unit. The teacher was to pose the following questions about the square unit and have students use the centimeter side of their ruler to help find the answers: What are its dimensions? What is its perimeter? What is its area? From there, everyone was to proceed to measure the base ten strip using his or her rulers to help find the dimensions, perimeter, and area of the strip in metric units. Per the observation, no manipulatives were passed out. The next activity in the handbook instructed the teacher to ask the students what they notice about measuring area as compared to measuring perimeter determining how they are similar and different. As indicated by the lesson plan in the teacher handbook, the students were to discover the difference between area and perimeter by measuring a square unit and base ten strips on their own. The teacher chose to use procedural activities to move the lesson along not allowing the students to conceptually understand area and perimeter through measurement. At the very beginning of the lesson, the teacher's intentions were going in the right direction, but feeding the students the information was how she chose to continue the lesson. This could have been a lesson within a series of lessons discovering area and perimeter. However, if the teacher introduced the algorithm during the very first lesson of the series, it makes sense as to why she did not expand the discussion in this particular lesson since it would be redundant. This is another downfall of a teacher not following a curriculum as instructed. If the lessons were written to build upon each other each day to build conceptual understanding, and the teacher makes pedagogical choices to change the lesson to be procedural in nature, then the lessons that follow would not serve their purpose. Figuratively speaking—she would be putting the cart before the horse. The following vignette illustrates how a teacher's intentions were sincere in fostering conceptual understanding of area and perimeter, but, in actuality, she was narrowing the scope of strategies to a few students' ideas.

> The teacher has already displayed an example of a table in the shape of a backwards L on the Smartboard (see figure 1) that has an area of thirty square centimeters. The problem at the top of the page reads—*One day when Emery was in town doing errands, the Goat Twins, Zachary and Whackery, decided to surprise him. They got some of Emery's small square tables out of the shed and arranged them in an unusual way. When Emery got home, he laughed at the twins and said, "OK, if the two of you are so smart, can you tell me the area of this big new table you've arranged?" Help the twins. How can they find the area of the table they made without having to count all the small squares?* The teacher allows one student to share a way to divide the table into sections to calculate its area. The teacher now asks the students to get out their journal and label the page 'Goat Twins Table'. The teacher asks, "Does someone have a different way?" The teacher calls on Taylor

and asks the students to write 'Taylor's Way' in their journal. The teacher is modeling this on the Smartboard. The student then proceeds to tell the class that he shaded this part (pointing to the 6x3 section of the L shape) and this square (pointing to the 3x4 section of the L shape) [see figure 2]. The students rush madly to replicate the teacher's notations from the Smartboard into their own journals. The teacher asks, "What is your equation?" The student replies reluctantly, "3x6=18, 4x3=12, so 18+12= 30 units ^{2}. The teacher is working ahead of Calvin as the students rush to write down Calvin's way. The teacher responds to Calvin, "Good, we got the same answer, University so that way works! Who wants to show another way? Who has another way?" Several hands raise in the air in hopes of seeing their name in lights up on the Smartboard. This time Jordan is called upon to answer. She jumps out of her seat rushing over to stand next to the teacher who is displaying the same L shaped table with no markings on the Smartboard (see figure 1). She excitedly tells the class, "I did the bottom 2x3, the middle 2x3, the top left 1x3, then the top middle 2x3 and then the far right 2x3 and then finally 1x3 for this last section." The teacher colors in each section identified by Julie in a different color separating each with a thick dark line (see figure 3). The teacher coordinates the color of each section with the multiplication sentence to match by the diagram. The teacher says, "There are many ways to split this table up. There may be a more efficient way of splitting it up instead of counting each one. So, what is the equation for Jordan's way?" She continues with Jordan's way by finding the product of each multiplication sentence. She asks Jordan, "What do we do next?" She responds, "Well, I added them up as I went." The teacher replied, "But you actually added them all up, right?" Jordan replies, "Yes!" Suddenly, the teacher announces, "Well, I am going to do it this way. 3+3=6 and 6x4=24 so 6+24=30 units^2. The teacher explored one more student's way to split up the table finding the area and then abruptly asks the students to complete page 212 in their student book.

Figure 1

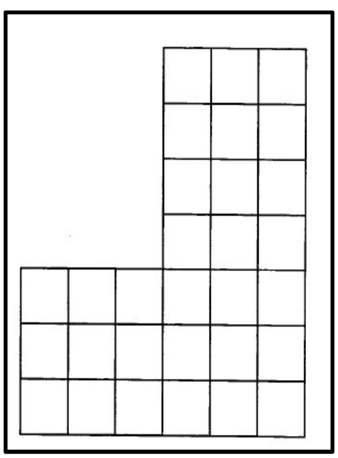


Figure 2

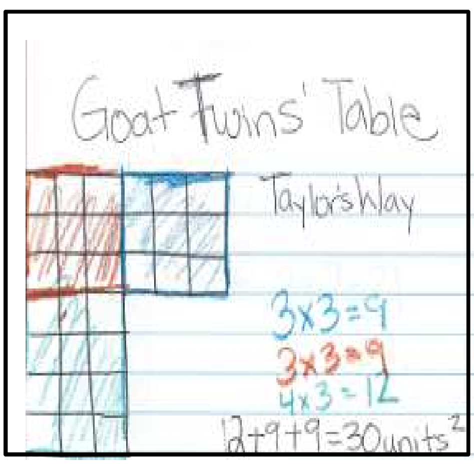
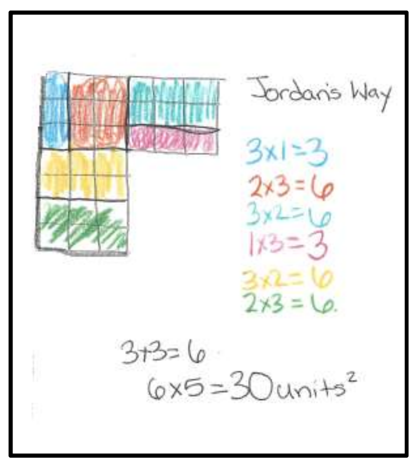


Figure 3



The teacher's intention was to explore different strategies for finding the area of an irregular shape. She shared with the students four different ways to split up the table to find its area given by four students. The crucial aspect missing out of this discussion was allowing students to disagree with the division of the shape and a discussion about a better alternative. Instead, the teacher made a pedagogical choice to keep the discussion between her and one other student at all times. Also, the teacher handbook instructed the teachers to give the students time to solve the problem on their own and share in pairs before sharing with the class. As each strategy was shared by a student, the students should have been encouraged to disagree and share a potentially more straightforward way. Instead, the teacher shared **her** way of solving the problem without any further discussion. Also, she did not give Jordan any validation of her strategy of adding as she went as if Jordan's way was discouraged. The teacher was so set on having an equation to solve, that she could not see that Jordan's way was just as efficient. In the teacher's reflection of this lesson collected the following week, she enjoyed watching the students figure out the area of the table as many were as possible, but she would have prepared the table array in advance figuring that her student could not draw an accurate array which caused disorganization and lack of flow. The teacher also felt that the majority of her students could come up with at least one way of finding the area, however, they struggled with the lack of organization. These reflections are collected and not discussed. This reflection highlights the teacher's inability to allow students to persevere in problem solving and how the accuracy and organization of her students' arrays overpowered the need for students to have a conceptual understanding of area and perimeter.

Discussion

Based on this analysis of 168 activities/items that were included in the lessons but were **not** part of the intended lesson provided in the teacher handbook, 68 percent of the items/activities taught to students during observations were procedural in nature and 31 percent of these items/activities were conceptual in nature. This leads to the question: What implications are there for students by these pedagogical choices made by the teachers? According to Junod (2017), the implications center around a teacher's ability to let go of their procedural mindset allowing students the time and space to persevere in problem solving without supplying a procedure to follow. When teachers make a pedagogical decision to merely push for solving a computation correctly to help students understand mathematics, they have taken away a student's ability to solve real-world problems out of the context of a number sentence. The result is that students find word problems to be too complex and are unable to identify the process to the correct solution on their own. They also struggle with the perseverance to problem solve. The reality is that many teachers find themselves switching to telling and explaining when things are not going as planned during a mathematics lesson (Mason & Davis, 2013). The teachers in this study found it difficult to let students think and problem solve on their own and were insistent on teaching the most efficient strategy to solve the problem.

According to the new standards, students should be taught to persevere in problem-solving enabling them to come up with varying solutions to problems. Without this struggle, students lose the opportunity to discuss their thinking and reason with other students when conceptual items/activities are not incorporated. Learning new mathematical concepts hinges on what a student already knows, so as students advance, new concepts will depend more on conceptual knowledge (Willingham, Winter 2009-2010). For instance, correct conceptual understanding of the equal sign in a number sentence depends on understanding algebraic equations, so if students fail to gain conceptual understanding, it becomes more difficult for students to catch up since new conceptual knowledge depends on the old. Thus, students become more likely to just memorize algorithms and apply them without understanding (Willingham, Winter 2009-2010). This was clearly evident in this study with the teachers' use of chants in order for students to memorize the algorithms when conceptual understanding was not gained first. Perhaps chants would be acceptable once algorithms are introduced at the very end of a series of lessons around a single mathematical concept, but not before. One explanation for this pedagogical decision might be that the teachers did not understand the importance of the item/activity presented in the teacher's handbook and, therefore, chose to adapt it or skip the activity altogether. However, according to the teacher reflections collected after each observation, the teachers did not mention why they skipped activities and reiterated that they thought their lessons were successful. Mathematics educators need to know whether or not an item/activity measures conceptual knowledge, but in order for the teacher to be successful, the teacher needs to know whether the students understand the definition of a concept or if they just memorized it (Star & Stylianides, 2013). Another possibility that would lead to this pedagogical mindset is that teaching procedurally has just become habit, especially for the more experienced teachers. The basic characteristic of habit is that every experience enacted and undergone modifies the one who acts and undergoes, whether we wish it or not, the quality of subsequent experiences. Therefore, every experience in the classroom both takes up something from prior activities or discussions and modifies the quality of those activities or discussions that follow it (Dewey, 1938). This in-the-moment pedagogy often causes confusion and tension for teachers as they try to navigate the selection of instructional approaches to address particular mathematical standards (Wilson & Downs, 2014). According to survey data collected by Vogler and Burton (2010), the teachers indicated they use a balance of standards-based and traditional instructional practices, but it was unclear the amount of time and focus used on both. However, these practices tend to lend themselves to either traditional or standards-based instruction which, in reality, lies within the intention of the teacher and cannot be measured by a survey (Vogler & Burton, 2010). This is the main reason the researcher chose observation of mathematics lessons being taught as being critical for this study. The researcher found discrepancies between what the teacher described in their reflections they were doing during instruction either conceptually or procedurally and what was reality. One explanation might be that these teachers were not allowed opportunities to observe mathematics' lessons taught conceptually for this curriculum during their professional development and what it looks like for the teacher and the students. According to the participants in the study, they only received a two-day training for this new curriculum which included only a review of the teacher's manuals and not videos of lessons being taught conceptually with students. With this new mathematics program having a conceptual framework, the teachers would need time in the classroom to conduct lessons ensuring a follow-up session with the trainers to address teachers' questions or concerns regarding the program. According to the teachers in the study, they did not receive the help or support necessary to implement this new time demanding technique of guide discovery or conceptual understanding. Additionally, if the teacher does not have an understanding of the way these ideas build across grades, it is possible for a teacher with good intentions to prematurely introduce an algorithm, thus, preventing the development of a deeper understanding of the concept (Wilson & Downs, 2014). Therefore, following a conceptual lesson intended in the teacher handbook is recommended until a teacher feels confident that it will not change future objectives or activities within the curriculum. The researcher cannot conclusively state that the suggestions above will rectify the lack of conceptual learning during mathematics lessons, but it would lead teachers to challenge a procedural mindset in order for students to find a balance when modeling strategies to students. The wait time necessary to allow students to persevere in their problem-solving skills is important to comprehend. As asserted by Rowland and Zazkis (2013), teaching mathematics conceptually requires attending to students' questions, anticipating obstacles, capitalizing on opportunities, making connections and providing enrichment beyond the immediate tasks. Ultimately, there is no place for a procedural mindset in a mathematics classroom when the new standards require a balance of both procedural fluency and conceptual understanding.

Contribution to the Field

This educational criticism and connoisseurship highlights how a teacher's procedural mindset hinders the ability to embrace conceptual learning of mathematics and, therefore, does not release a student to learn mathematics conceptually. This is compounded by the pressures of high-stakes testing. Teachers must resist the urge to teach mathematics "the way it has always been done." In order to reach all students in the classroom, the teacher must allow students to persevere in problem solving and explore strategies on their own. Only then will students be able to build skills necessary for success with unknown problems presented during high-stakes testing. This goes against everything experienced teachers have been taught and requires trust in the process of the struggle. This does not require a teacher to abandon their mathematics curriculum assigned by their district or school but modify group and independent practice with the use of real-world problems. For example, if a teacher is working in an urban school where a worksheet to be assigned involves the addition of three-digit numbers, several different word problems can be created on chart paper diversifying each problem to meet the needs of every student or group in the classroom. Groups would then solve the problems using appropriate manipulatives for the skill and would display their strategy on the chart paper. The teacher can monitor each group's progress using a rubric to score their problem solving, reasoning, communication, connections, and representations ensuring accountability. Rubrics can be customized to focus on certain skills and understandings. Group work allows students to work with others to persevere in problem solving and argue constructively their strategies they experiment with to solve the problems. The key is allowing students time to problem solve on their own versus telling them how to solve problems procedurally. Teaching students strategies would be appropriate if they cannot offer any strategies on their own. Gauging what students know through these types of activities before teaching a mathematics concept procedurally is optimal. In a whole group setting, you can introduce a word problem and work on it as a group having individual students share their strategies on the document camera using manipulatives, drawings or computations until all strategies are exhausted by the students. This allows the teacher to ask essential questions and encourage students to argue their answer without the correct answer being given away at this point. The Common Core Standards were developed to enhance a mathematics curriculum to instill a desire for students to problem solve thus increasing their success on unfamiliar problems given on high-stakes testing. Teachers must have faith in their students and allow them to persevere without being told the steps necessary to solve problems. It has become habit to "feed" students strategies and steps needed to solve problems. The greatest gift that can be given to students is the ability to think and problem solve. This is not only a 21st-century skill but what all future employers look for in their applicants. We are doing our students a disservice if we do not teach them to think. A conceptual curriculum can be given to teachers along with weeks of training, but if the procedural mindset remains, there will be no change in thinking. This study highlights that teaching procedurally, in most cases, is second nature and perseveres even after teachers are given a conceptual mathematics curriculum and training; it is almost a subconscious act. Recognizing that a procedural mindset exists is the first step towards embracing the conceptual teaching of mathematics.

References

Beckmann, S. (2010). From the common core to a community of all mathematics teachers. *Mathematics Educator, 20*(2), 3-9. Retrieved from http://du.summon.serialssolutions.com/2.0.0/link/0/

Bruun, F. (2013). Elementary teachers' perspectives of mathematics problem-solving strategies. *The Mathematics Educator, 23*(1), 45-59.

Department of Education, W., DC. (2008). *Foundations for success: The final report of the national mathematics advisory panel* Distributed by ERIC Clearinghouse. Retrieved from http://du.summon.serialssolutions.com/2.0.0/link/0/

Dewey, J. (1938). *Experience and education*. New York: The Macmillan Company.

Eisner, E. W. (2002). *The educational imagination: On the design and evaluation of school programs* (3rd ed.). Upper Saddle River, N.J.: Prentice Hall.

Eisner, E. W. (1991). *The enlightened eye: Qualitative inquiry and the enhancement of educational practice*. New York, N.Y: Toronto.

Heitin, L. (2014, 03/26). Elementary math teachers rethink pedagogy under the common core. Message posted to http://blogs.edweek.org/edweek/curriculum/2014/03/with_common_core_early_math_te.html?qs=elementary+math+teachers+rethink+pedagogy+under+the+common+core

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum.

Inoue, N., & Buczynski, S. (2010). You asked open-ended questions, now what? understanding the nature of stumbling blocks in teaching inquiry lessons. *Mathematics Educator, 20*(2), 10-23.

Junod, K. (2017). Letting Go. E-Journal of Colorado Mathematics Teacher, Spring 2017, 20.

Junod, K. (2017). Letting Go. E-Journal of Colorado Mathematics Teacher, Spring 2017, 20.

Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *The Elementary School Journal, 102*(1), 59-80. doi:10.1086/499693

Martin, T. S., & Speer, W. R. (2009). *Mathematics teaching today: Improving practice, improving student learning* National Council of Teachers of Mathematics, Inc. Retrieved from http://du.summon.serialssolutions.com/2.0.0/link/0/

Mason, J., & Davis, B. (2013). The importance of teachers' mathematical awareness for in-the-moment pedagogy. *Canadian Journal of Science, Mathematics & Technology Education, 13*(2), 182-197. doi:10.1080/14926156.2013.784830

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* National Council of Teachers of Mathematics.

Osborne, J. H. (2015). The charge is ours: Teaching elementary school mathematics through best practices and common core initiatives. *Delta Kappa Gamma Bulletin, 82*(1), 23-25.

Plank, S. B., & Condliffe, B. F. (2013). Pressures of the season: An examination of classroom quality and high-stakes accountability. *American Educational Research Journal, 50*(5), 1152-1182.

Rowland, T., & Zazkis, R. (2013). Contingency in the mathematics classroom: Opportunities taken and opportunities missed. *Canadian Journal of Science, Mathematics and Technology Education, 13*(2), 137-153.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1. Retrieved from http://du.summon.serialssolutions.com/2.0.0/link/0/

Star, J. R., & Stylianides, G. J. (2013). Procedural and conceptual knowledge: Exploring the gap between knowledge type and knowledge quality. *Canadian Journal of Science, Mathematics and Technology Education, 13*(2), 169.

Ticha, M., & Hospesova, A. (2006). Qualified pedagogical reflection as a way to improve mathematics education. *Journal of Mathematics Teacher Education, 9*(2), 129-156. doi:10.1007/s10857-006-6893-7

Vecellio, S. (2013). How shall I teach thee?: An uncommon pedagogy for the common core. *Schools: Studies in Education, 10*(2), 222-241.

Vogler, K. E., & Burton, M. (2010). Mathematics teachers' instructional practices in an era of high-stakes testing. *School Science and Mathematics, 110*(5), 247. Retrieved from http://du.summon.serialssolutions.com/2.0.0/link/0/

Walkowiak, T., Berry, R., Meyer, J., Rimm-Kaufman, S., & Ottmar, E. (2014). Introducing an observational measure of standards-based mathematics teaching practices: Evidence of validity and score reliability. *Educational Studies in Mathematics, 85*(1), 109-128. doi:10.1007/s10649-013-9499-x

Willingham, D. T. (Winter 2009-2010). Ask the cognitive scientist: Is it true that some people just can't do math? *American Educator,* 14-19.

Wilson, P., & Downs, H. (2014). Supporting mathematics teachers in the common core implementation. *AASA Journal of Scholarship and Practice, 11*(1), 38-47.

Yackel, E. & Cobb P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458-477). Retrieved from http://www.jstor.org/stable/749877

Analyzing Student Data as a Measurement of Success for *Boy With A Ball*

Ping Ye, Whitney Richardson, Derek Allen, Aneta Galazka, William Fayson

University of North Georgia, USA

Huijun Yi

Troy University, USA

Contact Email*: ping.ye@ung.edu

Abstract: *This paper analyzes survey responses and student outcomes from the Boy With A Ball (BWAB) program. Decision trees are applied to analyze the Hemingway survey and a set of selection criteria is constructed to identify a group of high-risk students who need help. Meanwhile, pivot tables and multiple linear regressions are utilized to evaluate the mentoring program. The goal is to measure how the program has been executed by the BWAB team and determine whether it will be of great value to the investors. Not only are student academic performances used for evaluating the BWAB mentoring program, but also internal student characteristics are considered. After analyzing student data from both the survey responses and the student academic records, it shows that a sense of connectedness with community-family plays an important role in the BWAB program.*

Keywords: Hemingway, attendance analysis, student characteristics**,** connectedness

1. Introduction

Boy With A Ball (BWAB) is a non-profit organization which works to make cities better places by reaching young people and equipping them to be leaders capable of transforming their communities. The BWAB program works in multiple locations, both across the nation and globally, with the goal of developing troubled youth and thus develop communities. Team members and volunteers choose a city's most hard to reach neighborhoods and come together in a number of outreach methods that build relationships with every resident in an economically disadvantaged community. Team members, volunteers, and community members then provide mentoring relationships to young people and their family members who are facing moments of crisis. As community members are identified as facing similar situations, supportive small groups are formed to meet in community homes. Often, these groups include mothers' group, children's group, tutoring center, job skills training group, ESL group and leaders' group. As these teams and volunteers pour out continual outreach that builds on each other, the community is powerfully developed. This paper works on student data for a cross-age mentoring program such that it pairs middle and high school students together with the main goal of preventing both from dropping out of school. One success story of this program includes a school labeled as a "drop out factory" in 2007. Three years later, the BWAB program intervened, targeting students with few or no extra-curricular activities or students considered at high-risk for dropping out of school. This intervention resulted in a 95 percent graduation rate, with most students continuing their education in college.

"The Hemingway" is a questionnaire given to students both as they enter and as they exit the BWAB program. This 57-question survey aims to show the overall level of connectedness of the student. These responses along with the participants' grades, number of absences, and number of discipline referrals are meant to improve with program participation. Improvement in these four areas leads to the overall development of both the participants and their communities.

With the support of PIC Math grant (Preparation for Industrial Careers in Mathematical Sciences), the research group helps BWAB to analyze the 2009-2014 St. Antonio data for the cross-age mentoring program. BWAB relocated their global headquarter from St. Antonio Texas to Atlanta Georgia in July 2013.

2. Experimental Section

The endeavor of the authors for this study was to learn how to design experiments, clean data and utilize R/R Studio/Excel in order to generate and interpret results. Data analyzed included the survey questions, the grades of the aforementioned four major areas of study, the number of absences from school, the frequency of discipline referrals, program attendance, and a range of other tracked data regarding the program's Participants. BWAB provided the authors this data via several Microsoft Excel spreadsheets. The authors also compiled the data into a single spreadsheet for easier reading and interpretation.

2.1 Hemingway Survey

The original "The Hemingway" survey has two versions: The short version contains 57 questions, while the long version has 31 extra questions besides those 57 ones. The authors focus on the 57 questions due to the limited sample size of 220 students. Here, the missing data is treated as survey response "Unclear". Students took the Hemingway survey before entering the BWAB mentoring program and after finishing the program.

2.2 Attendance Analysis

After preparing data from table "BWAB_Att" (Attendance table), simple linear regressions are applied. The goal is to check the effectiveness of the BWAB mentoring program for different student groups. The designated student groups are represented by the following:

AA - original middle school student mentees who joined during 2011 school year

BB - middle school student mentees who joined during 2012 school year

A - original high school student mentors who joined during 2011 school year

B – high school student mentors who joined during 2012 school year

2.3 Multiple Linear Regression

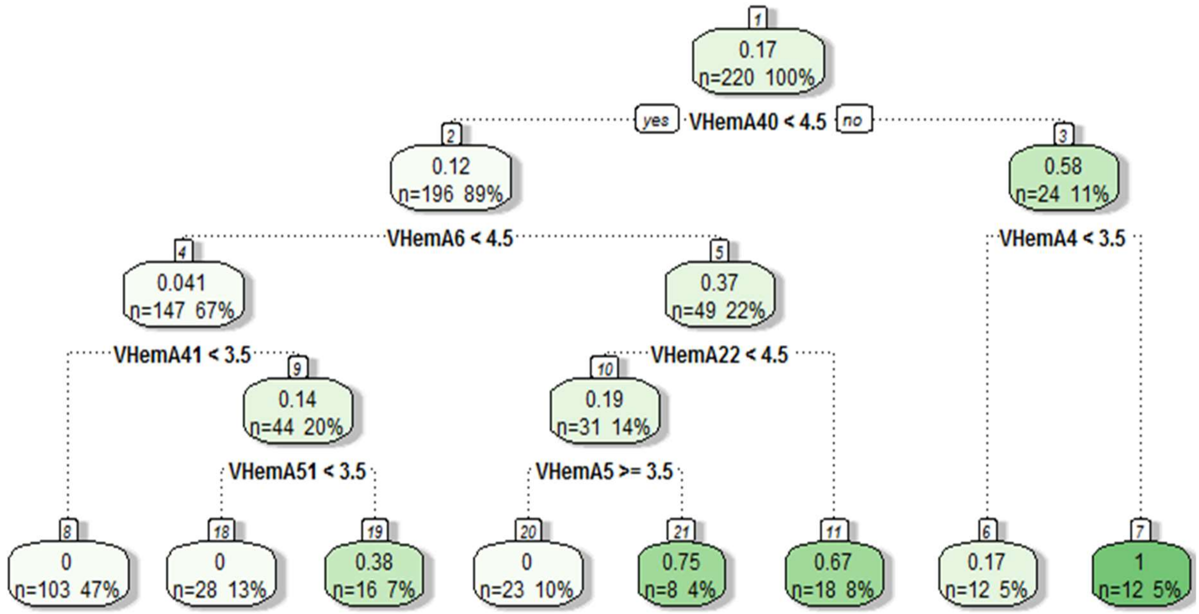
BWAB provided total 18 tables to the research team to investigate. The research team combined these tables into a master table and operated multiple linear regressions. The master table data frame includes the variables year, the grades in four disciplines (math, science, English and social science), attendance, absence and count of incidents. The goal is to identify the disciplines which affect incidents/absences the most.

3. Results and Discussion

3.1 Decision tree Results

3.1.1 Gini index decision tree for program status

The classification is binary: 0 is coded as taking the Hemingway survey before entering the BWAB program, while 1 is coded as taking the Hemingway survey after finishing the BWAB program. The Gini index tree is as follows:

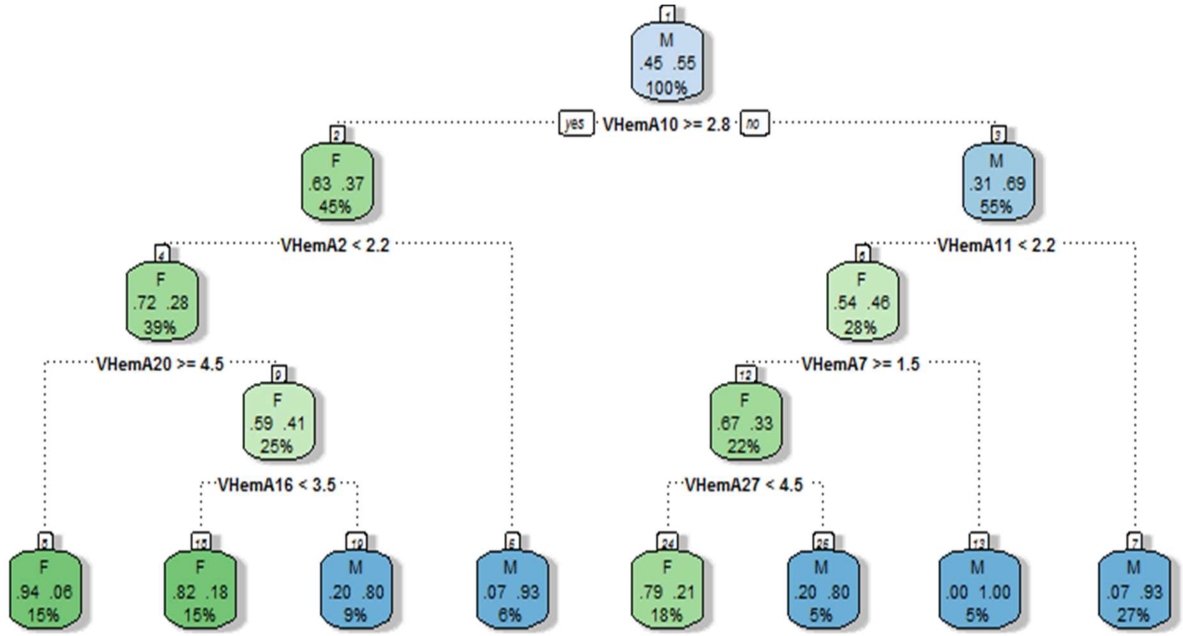


The variable selection from 57 survey questions shows the most important 7 survey questions as the following order:

1. Question # 40 “I often read when I have free time”;
2. Question #6 “I work hard at school”;
3. Question #4 “My family has fun together”;
4. Question #41 “I hang out a lot with kids in my neighborhood”;
5. Question #22 “Spending time with my friends is a big part of my life”;
6. Question # 51 “My neighborhood is boring”;
7. Question # 5 “I have a lot of fun with my brother(s) or sister(s)”.

The above result shows that after finishing the BWAB program, students are more likely to be connected with family members and friends so that they will feel more secured and being loved. The BWAB program improves internal student characteristics, such as knowledge, attitudes, beliefs, self-perception traits like self-esteem or self-efficacy. Since the most important 7 questions are shown on the above Gini index decision tree result, they provide an idea to design a short version survey to identify the high-risk student who needs help.

3.1.2 Gini index decision tree for genders
The classification is binary: male and female. The Gini index tree is as follows:



The variable selection from 57 survey questions shows the most important 7 survey questions for the differences between male and female students. The questions are in the following order:

1. Question # 10 "I enjoy spending time by myself reading";
2. Question #2 "Spending time with friends is not so important to me";
3. Question #11 "I spend a lot of time with kids around where I live";
4. Question #20 "I like to read";
5. Question #7 "My classmates often bother me";
6. Question # 16 "I enjoy being at school";
7. Question # 27 "I like working with my classmates".

In general, female students who took the survey are more likely to enjoy reading alone, they prefer to have less time at school; while male students are more likely to spend time with friends and are willing to spend more time both at school and after school with other kids.

3.2 Attendance Analysis

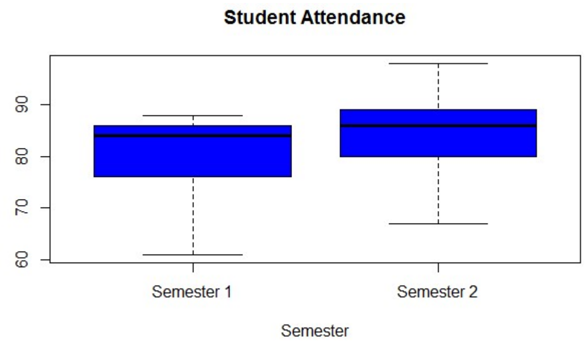
The research team tried to use simple linear regression to model the student attendance pattern.

Model:

$$y = \beta_1 x + \beta_0 \ ,$$

where y represents the counts of attendance, x refers to years in the program.

R-squared is found to be less than 0.01 after modelling the data, it indicates that simple linear regression model does not fit the data. Then the research team visualized the attendance data of 2009-2014 school years for both semesters by applying the following boxplot:



It shows that the attendance for semester2 is better than the attendance for semester1 in general during those school years. The research team decided to look into the details for attendance of different school years to figure out the impacts of the BWAB mentoring program.

3.2.1 Group AA

Row Labels	Average of PRES_SEM1	Average of PRES_SEM2	Increasing Rate
2009	78.00	83.23	6.7%
2010	74.85	79.28	5.9%
2011	78.00	79.69	5.9%
2012	67.60	70.84	4.8%
2013	62.28	73.17	17.5%
2014	66.98	66.38	-0.9%
Grand Total	**70.68**	**74.92**	

After using pivot table to compare the attendance results of two semesters for the group of middle school student mentees who started from 2011, it shows that this group has better attendance in general during semester2. During the school year 2014, the semester2 attendance slightly dropped due to the complexity that the BWAB headquarter was moved from St. Antonio to Atlanta. After attending the BWAB mentoring program, the attendance of middle school student mentees keeps increasing, especially during 2013 school year it increased dramatically due to new methods adding to the program. Therefore, the BWAB mentoring program has positive impacts on the middle school student mentees. It's evaluated by the increasing of attendance.

3.2.2 Group BB

Row Labels	Average of PRES_SEM1	Average of PRES_SEM2	Increasing Rate
2009	81.25	84.08	3.5%
2010	82.92	86.42	4.2%
2011	79.64	78.29	-1.7%
2012	69.87	78.47	12.3%
2013	53.05	60.37	13.8%
2014	46.47	46.26	-4.5%
Grand Total	**66.20**	**69.73**	

The pivot table results of both semesters for the group of middle school student mentees who joined during 2012 school year show that the BWAB mentoring program has positive impacts on the middle school student mentees since student attendance increased dramatically right after the joining year 2012. It's evaluated by the double digits increasing rates. It indicates that the mentoring program works extremely well to reduce the absences for this group.

3.2.3 Group A

Row Labels	Average of PRES_SEM1	Average of PRES_SEM2	Increasing Rate
2009	82.57	83.90	1.6%
2010	79.23	84.13	6.2%
2011	85.63	86.97	1.6%
2012	84.80	86.60	2.1%
2013	69.97	83.16	18.9%
2014	69.94	68.53	-2.0%
Grand Total	**77.66**	**81.20**	

After comparing the results for two semesters for the group of original high school student mentors started from 2011 school year, it shows that there are slight increases during 2011 and 2012 school years for the attendance. While there is a significant increase during the 2013 school year due to new methods added to the program. Therefore, the BWAB mentoring program has positive impacts on attendance of high school student mentors as well.

3.2.4 Group B

The sample size for the group of high school student mentors who joined during 2012 school year is 14, which is too small for modeling the attendance pattern. The attendance analysis for this student group is ignored.

3.3 Multiple Linear Regression Results

Model:

$$y = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_0 \qquad (*)$$

3.3.1 Count of discipline incidents versus four discipline grades

For the above model (*), y is the count of incidents, x_1 to x_4 represent math, science, English and social study grades, respectively.

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|) ***
(Intercept) 10.0075020  1.5576936   6.425 2.62e-09
M           -0.0122680  0.0259546  -0.473   0.6373
S           -0.0413438  0.0287558  -1.438   0.1530
E           -0.0002648  0.0279868  -0.009   0.9925
SS          -0.0582040  0.0254775  -2.285   0.0241 *

---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.752 on 123 degrees of freedom
  (53 observations deleted due to missingness)
Multiple R-squared:  0.2365,   Adjusted R-squared:  0.2117
F-statistic: 9.527 on 4 and 123 DF,  p-value: 9.625e-07
```

Here the p-value of the F-test shows that the model is statistical significant. Among four predicators the only p-value less than the significance level 0.05 is the one of the social study grades, which indicates that if the students in the BWAB program improve their social study grades, their count of discipline incidents may drop. Therefore, social study seems to have the largest impact on the count of discipline incidents. Hence BAWB may develop its mentoring program by focusing on improving students' social study grades in order to improve their behaviors at schools. The adjusted R-squared of this model is 0.2117 which indicates that it could be improved by other regression model. This will be part of the future work for the research team to investigate.

3.3.2 Absences versus four discipline grades

For the above model (*), where y is the number of absences, x_1 to x_4 represent math, science, English and social study grades, respectively.

```
Coefficients:
              Estimate Std. Error t value Pr(>|t|) ***
(Intercept) 46.581928   7.106649   6.555 1.38e-09
M           -0.184578   0.118412  -1.559   0.1216
S           -0.246966   0.112206  -2.201   0.0296 *
E            0.000493   0.127684   0.004   0.9969
SS          -0.048539   0.116236  -0.418   0.6770

---
Signif. codes:
0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 7.995 on 123 degrees of freedom
  (53 observations deleted due to missingness)
Multiple R-squared:  0.2298,   Adjusted R-squared:  0.2048
F-statistic: 9.176 on 4 and 123 DF,  p-value: 1.604e-06
```

Here the p-value of the F-test shows that the model is statistical significant. Among four predicators the only p-value less than the significance level 0.05 is the one of the science grades, which indicates that if the students in the BWAB program improve their science grades, their number of absences may drop. Therefore, science seems to have the largest positive impact on the class participation. As the social science, BWAB may also focus on improving students' science grades in order to improve their behaviors at schools. Here the adjusted R-squared is 0.2048 which indicates that this model could be improved by other regression model. This will be part of the future work for the research team to investigate.

4. Conclusions

By applying the decision tree, seven questions from the Hemingway survey could be used to redesign a short survey and to identify the students who need help. The results also show that the BWAB mentoring program connects students in the program to satisfy their safety and belonging needs. Attendance analysis results indicate the BWAB has positive impacts on the academic performance and behavior of the student participants. Multiple regression analysis reveals the relationship between the subjects of social science along with science and student behaviors at schools. Connectedness plays an important role once again.

Connectedness within the BWAB program allows students the potential to reach and satisfy a couple of levels through their program. Including safety and security, as well as love and belonging needs. Safety needs focus on the environment including, the home, school, and elsewhere. If a student is placed in a bad environment or an unsafe neighborhood, such conditions will hinder academic performance. Research suggests that the student will have trouble focusing on learning when he/she does not feel secure (Burleson and Thoron, 2014). In terms of love and belonging needs, these needs include friendships and relations. This can include a sense of belonging through clubs, volunteer groups, churches, or in this case, the BWAB program. It has been shown that students who feel connected are less likely to engage in risky behaviors, such as, alcohol or drug use as well as violence and gang involvement (McLeod, 2016).

BWAB is able to stay connected and fulfill this safety need for students within school by placing representatives from BWAB inside the schools that the students attend. These representatives act as mentors, who care for the students' best interests. For the love and belonging needs, BWAB allows these at-risk students a place where they can engage with other peers, mentors, and in various events throughout the community. This allows students the opportunity to establish relationships that can last after they have left the BWAB program. With the BWAB program being able to satisfy and guide students to meet these two needs, it gives the students a chance to reach a higher level on their own in terms of performance. These needs are referred to as the Esteem Needs.

Once students in the BWAB program feel as though they belong, they can begin to satisfy and develop positive feelings of self-worth and self-esteem. Esteem Needs involve one's confidence, self-worth, and sense of achievement (Perera, 2015). In terms of achievement, this includes school performance and the students, desire to see themselves achieve good grades. Self-esteem also goes hand in hand with motivation. If a student is not motivated, they are less likely to perform well in school or any activity for that matter. Given the record of the BWAB program, which includes increased academic performance and graduation rates for students who are part of the program, it appears as though the way in which they get these kids connected is working positively in satisfying Maslow's needs (Maslow, 1943) and overall improving the student participants' future potential.

Future Work

The student data analyzed included records from the program in one district in Texas and followed one set of students, who were in middle school and high school. It involved the first group of students participating in the program. Given that the program is expanding nationally and increasing student participation in already existing areas of operation, more data should be analyzed so that BWAB can compare consistency in the results. Meanwhile, the multiple regression shows higher social study and science grades correlate to better school behaviors. Other models using attendance/count of discipline incidents as dependent variable could be built to explore the connection between the grades and school behaviors. Additionally, a universally formatted database should be developed which will streamline the analysis process for future statistical studies. Furthermore, alternate statistical methods such as logistic regression and cluster should be considered to analyze the information collected in order to improve the model accuracy and efficiency. Boy With A Ball fully relies on the donation from sponsors. We recommend further analysis regarding fundraising methods that could clarify and improve upon the donor recruiting process. This analysis could help the organization reach additional members and increase individual donations.

Acknowledgment

The authors would like to thank the PIC Math program, the Boy With A Ball and the University of North Georgia which made it possible for the research team to work on this project.

References

[1] An Introduction to Statistical Learning, with Applications in R *by James, Witten, Hastie and Tibshirani: Springer, 2013.*

[2] Maslow's Hierarchy of Needs and Its Relation to Learning and Achievement. *Burleson, Sarah E., and Andrew C. Thoron. EDIS New Publications RSS. Agricultural Education and Communication, 25 Sept. 2014. Web. 21 Mar. 2017.*

[3] Maslow's Hierarchy of Needs. Simply Psychology. *Saul McLeod. Creative Commons, 16 Sept. 2016. Web. 25 Mar. 2017.*

[4] Maslow and Your Self Esteem Needs. *Karl Perera. More-selfesteem.com. Better Internet Bureau, n.d., 12 Jan. 2015. Web. 25 Mar. 2017*

[5] Data Science For Business-What You Need to Know About Data Mining and Data-Analytic Thinking. *Foster Provost and Tom Fawcett.*

[6] Fundamentals of Machine Learning for Predictive Data Analytics Algorithms, Wored Examples, and Case Studies. *John D Kelleher, Brian Mac Namee, and Aoife D'arcy.*

[7] Statistical Design and Analysis of Experiments. *Wiley, NY. 1989.*

[8] Probability and Statistics. *Michael J Evans and Jerey S Rosenthal.*

Textbook reviews

Małgorzata Marciniak

City University of New York, LaGuardia Community College

Long Island City, NY, USA

The textbook **"Introduction to Trigonometry"** by Terence Brenner and Daniel Maysonet begins conveniently with a pretest on the basics of trigonometry. Each pretest question is accompanied by the section number, where the reader can find the material related to that topic. Each section is accompanied by sample problems that can be assigned as homework or solved in class. A sample test is provided at the end.

The style of the book is significantly different than the style of other modern trigonometry books. Everything here is brief and straightforward, presented without excess of talking, which will be highly appreciated by students. The book contains lots of pictures and many solved examples explained in a very compacted manner. This book can easily compete with the popular Khan Academy videos in terms of explaining the topics in a shortest and simplest way.

The book begins with basic material and terminology about angles, their orientation, and measurements. Section 1.8 contains instructions for accurate use of calculator to solve equations with trigonometric functions. Few examples are solved showing the pictures of the key-strokes to explain the process of using the calculator, so the students can compare their results on each stage of work. The authors carefully point out the most common mistake, which is the incorrect MODE of the calculator, to avoid confusions in students work between degrees and radians. Section 1.9 contains numerous application problems, again accompanied by pictorial solutions. Section 1.10 with trigonometric identities contains insightful guidelines for students to work on verifying trigonometric identities.

The appendix of the book contains the tables of trigonometric functions for angles between 0 and 90 degrees. The answers to odd numbered problems and the index of mathematical terms follow the appendix.

The textbook **"Review of Intermediate Algebra and Introduction to Trigonometry"** by Terence Brenner and Dae Hong is written in a transparent and available way. In addition to covering the standard material of college algebra book, such as rational equations, radical equations, and quadratic equations, it contains a chapter on complex numbers conveniently presented before the chapter on quadratic equations.
Here are few features that distinct this textbook from others. The vertical and horizontal methods of adding rational expressions with different denominators are explained and compared in a table. The chapter on rational equations contains numerous examples of applications using the formula:

(rate of work per hour) x (time worked) = part of the job completed. Sample problem sounds as follows: If a mason can build a wall in 10 hours and an apprentice in 15 hours then how long will it take them to build it when they work together?

Chapter on radical expressions contains a careful explanation of the notation and clarifies a common confusion of the minus inside or outside of the square root sign. Chapter on quadratic equations elaborates generously on the most fundamental method of solving those equation by completing the squares. This method, however, seems to be quite challenging for students. In addition, numerous application problems from basic finances and motion in space are provided. The most impressive part of the textbook is devoted to graphing quadric curves: parabola, circle, ellipse, and hyperbola. All graphing methods are explained briefly but with all necessary details. In addition, graphing of the hyperbolas contains procedure of sketching the asymptotes.

The textbook contains the answers to all assigned problems.

A brief report from a visit at the Children's Creativity Museum in San Francisco on January 27, 2018

Małgorzata Marciniak

City University of New York, LaGuardia Community College

Long Island City, NY, USA

Abstract: *This brief report contains an overview of creative activities collected at the Children's Creativity Museum in San Francisco, CA.*

Keywords: creativity, children activities, museum, San Francisco

Introduction

Some information about the museum can be found on the web page https://creativity.org/ and in the Wikipedia article

https://en.wikipedia.org/wiki/Children%27s_Creativity_Museum.

This report contains numerous pictures with descriptions of the stations and various activities that the museum organizes on regular basis. The visit is highly recommended to children of age 2-12 and their parents!

Adults are allowed to enter the museum only if accompanied by children of age 2-12. However, an adult educator can request a brief tour which lasts about 30 minutes. Photography is allowed as long as the photos do not show children and their faces. The museum takes space of two floors and consists of several stations that address different aspects of creativity for different ages. My guide, Kelly Garrett, mentioned that according to the recent survey, children tend to spend about one hour at each station and about two and half hour at the entire museum. The museum allows re-entrance on the same day.

Personally, I would enjoy every activity that I saw at the museum. Thus, I wonder whether a similar museum could be designed for adults to revitalize their creativity?

Some stations are unfacilitated, but most contain a detailed description of what can be done and how to approach the station. However, some stations have a facilitator that provides directions and support with the supplies. Some activities are designed for younger kids, some for older but there is no formal division, and everybody can enjoy every station.

Stations

STATION 1: SAND. There are two boxes with sand. The first one indicates the level curves showing the elevation, including below the sea level. The sand has different color for the terrain and for water to make easier distinction between those two. The second box contains simulations of animals that can move along the land or only

in the water. Children design the terrain and observe the animals moving around. This station is designed for small children, but the curiosity builds up and even an adult will hang out here trying to figure out how the simulation works and how the silhouettes of the animals are projected on the sand.

After the work, participants can clean their hands and get creative with applications of the colorful fuzzy pompons.

STATION 2: BULIDING BLOCKS. Foam blocks of various shapes can be crated into various objects. This space is designed for families so that kids can build their favorite geometrical spaces for playing and hanging out. It reminded me of playing under the table, making a tent in my room just to enjoy the change of the space around me. This change has a transforming impact on the mind and can literally take an individual far away from their current location.

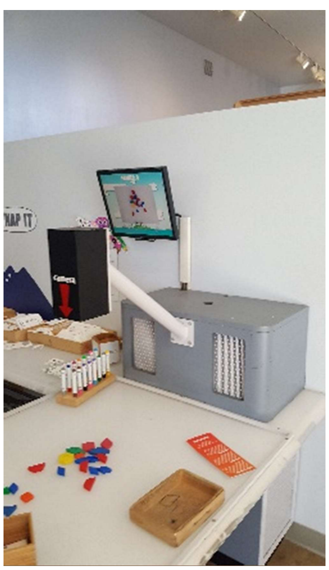
In the background under the ceiling there is a collection of flags made by children during one of the workshops. I can only imagine kids' excitement when they see their own work preserved by the museum and presented to the visitors in a form of an exposition.

STATION 3: PUPPET THEATER where students arrange their own plays and can wear dresses provided on the racks. There is a space for the audience to sit and watch the show but when I visited nobody was having rehearsals. The station seemed to be unfacilitated and the shows unscripted but there may be scripts hidden somewhere in the drawers.

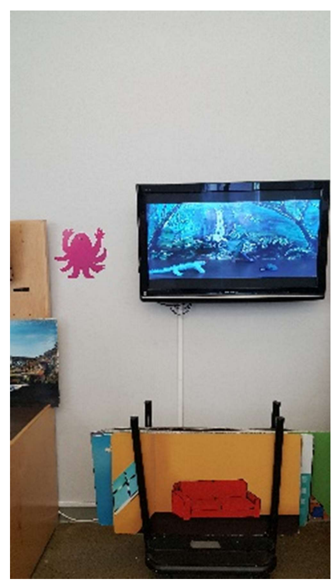
STATION 4: GEOMETRICAL ARTWORK where students write and draw, and later organize their work in geometrical expression. The exhibition is arranged on a hanging frame above the table. The entire station looks incredibly appealing even from a distance.

STATION 5: READING where parents can read with their children and teach them letters. The station is arranged so that parents and children sit around a low, round table with a shape of a tree in the middle. Build-in shelf contain books for various ages.

STATION 6: CREATIVE WRITING where students write about a theme provided seasonally by the museum staff. Their work is displayed inside the room and available for reading to other children. The theme is more mature than is STATION 4 and requires more content. The mini-exhibition on the wall contains sample writings from the visitors. This station takes the entire room, not just a small round table as in STATION 4. The horizontal shelf running around the room provides an ample space for writers. Sample questions (the property of the museum) are "What do you wish for the future?" or "What reminds you of peace?" and more. The announcement board with lights around it is a space for displaying the works of visitors.

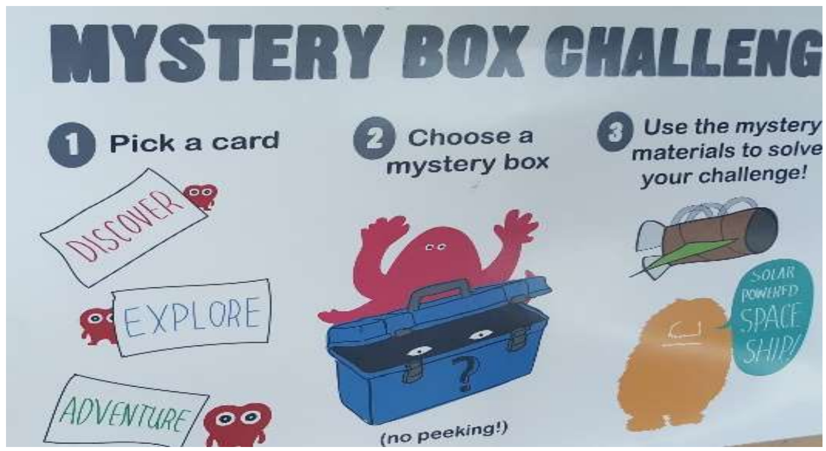


STATIONS 7 and 8: CREATIVE STORY TELLING where visitors can present their stories and record them with high quality camcorders. At those stations children are encouraged to tell their own stories with magnet shapes or with characters made in the modelling clay. The stories can be recorded and presented in a form of a video on a screen. While watching one of those kids-made videos I had an impression of a professional work. The backgrounds for the clay models is pre-made but the choices are vast. The models are made and moved by children and their parents. All created videos can be send to the authors via email.

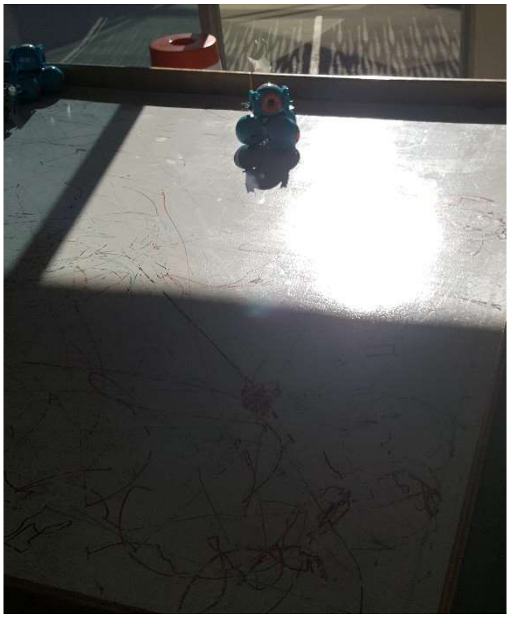


STATION 9: CLASSIC WOODEN BLOCKS that brought joy to generations of kids are here presented, for easier assembling, with magnets. The magnets are highly appreciated since the station is creatively arranged in a narrow hallway with limited space for playing. Most assembling happens on the metal wall requiring magnets to hold the assembly in place.

The wooden blocks, even if so simple, has been source of joy for generations of children across the planet. I recall hours spent on creative building that usually ended with equally creative methods of destruction.



STATION 10. MYSTERY BOX CHALLENGE offers a truly creative work for a young engineer. This assignment, oriented towards problem-solving skills, asks visitors to build something from provided materials with given tools. This station is facilitated by a museum personnel who distributes the assignments and the boxes making sure that nobody peeks what is in the box.



STATION 11: DESIGNING A ROBOT is a truly modern game that was not available to previous generations. The station is located in a large sunny room with three tables where students can design and test their robots for simple movements, drawing skills and maze drill. A facilitator offers help with learning how to program a robot and communicate with it. The tables are aligned from the most basic (simple movements) to the most advanced (running through the maze.)

STATION 12. DESIGNING THE CITY where children design their own city by coloring buildings, houses, airplanes, cars, and highways. When the drawings are ready, children scan and display their work on a large wall of the room. The projectors “shake” the buildings once the light is disturbed by for example waving the hand at the item. It looks like houses dance after someone waves at them.

Summary

The exit of the museum contains exhibitions of children’s work from workshops held at the museum. There are few examples of that.

Leaving the museum, I could not stop thinking what should a creativity museum for other ages contain?

What could spark creativity of adults who simply forgot the joy of being creative and search for some inspiring activities beyond painting and storytelling?

MATHEMATICS TEACHING-RESEARCH JOURNAL
VOL 10, N 1
SPRING 2018

Using Algebraic Inequalities to Solve Extremum Problems

Meirav Amram[a], Miriam Dagan[b], Pavel Satianov[b], and Michael Yoshpe[a]
meiravt@sce.ac.il, dagan@sce.ac.il, pavel@sce.ac.il, IoshpeMi@gmail.com

[a] Mathematics Unit, SCE - Shamoon College of Engineering, Ashdod, Israel,
[b] Mathematics Unit, SCE - Shamoon College of Engineering, Beer-Sheva, Israel,

Abstract

The use of nonroutine tools in mathematical problem-solving is very beneficial: it encourages creativity, enables approaches that are easier and faster, and allows simplification instead of a tedious work. In this paper we present one such situation: we discuss the use of algebraic tools in the solution of extremum problems. These questions, which at first seem to have one classical solution, in fact can be solved using the approach of algebraic tools, such as Mean Inequalities. We lay out in detail the benefits of nonroutine solutions to problems, including some positive results according to the practical experience of the authors. Then we give several examples in which the usefulness of this doctrine is demonstrated, including some graphical illustrations. These materials are aimed at teachers, lecturers, and students, and are designed for them to find out about nonstandard teaching and problem-solving methods.

Keywords: algebraic inequalities; calculus learning; challenging problems; visual representation

1 Introduction

The use of nonstandard tools in problem-solving is essential in the teaching of mathematics – both for creativity and inspiration, as well as for dealing with hard problems with accessible and non-cumbersome solutions. In the following material, an explanation is presented for unique and nonroutine solutions to problems arising from analysis, with the added use of algebraic tools. We examine why the use of these techniques is necessary in high school and undergraduate studies. Then, explicit examples of problems where an algebraic solution is of use are given, to demonstrate this argument.

In most analysis courses in undergraduate studies, the approach for solving extremum problems, such as classical min-max problems, is straightforward and is accomplished by calculating the derivative, then equating it to zero and checking for appropriate solutions. This leads to a robotic way of thinking and discourages creativity. To this end, we present a different and refreshing approach for solving min-max problems, with the use of algebraic inequalities. The nonstandard methodology intrigues the student, and allows him to learn new techniques and different approaches.

The use of algebraic tools in the research of analysis is not new. Some previous works have focused upon this subject, especially on the development of general mean-inequalities and their use in order to solve problems from a variety of topics. In a book by [3], the author deals in detail with mean-inequalities and holds various useful propositions and examples. A generalization of mean-inequalities was conducted by [1]. On different approaches to algebraic inequalities, see the work of [4].

This paper is one in a series of papers that try to present unique topics that are unpopular among lecturers and teachers, and are unknown to students. These include the authors' previous work on the staircase functions ([2]), as well as the paper on the min-max functions ([5]). These functions are complicated and sometimes create teacher confusion when trying to determine how they should be taught. In these papers, as well as in the following, we offer some solutions and new insight into such problems and subjects.

One of the special and distinctive methods for emphasizing our point is the use of graphical tools. The two papers by [6, 7] describe the cognitive development in mathematics and its relations with visual imaginary; this development enables researchers and students to use graph programs for the functions. In their paper, [8] justify using such a professional program. In order to automatically present the accurate drawing of the resulted function and to visualize our results, we use several `Python`™mathematical and graphical libraries, such as `Numpy` and `Matplotlib`.

The paper is arranged as follows. In the next section, we give a general preview, in which we explain why the use of distinctive tools and examples, which hold different approaches and use creative thinking, can lead to a better teaching process. We end that section with some concrete detail about the success of the use of algebraic inequalities in calculus examinations. Then, in the following section, we give several sets of unique examples from different subjects, in order to demonstrate our principal idea.

2 Mathematics teaching with the use of special tools and special examples

In this section we provide a justification for the use of nonroutine tools and extreme examples while teaching fundamental subjects in high school and undergraduate mathematics. This methodology is relevant not only in the mathematical environment – it applies as well to other mathematics-related disciplines such as physics and engineering. After outlining the methodology, we concentrate on our specific subject, *the solution of extremum problems through the use of algebraic inequalities*. We show that a creative and refreshing approach to the solution of problems from the standpoint of analysis, with known straightforward solutions, can be beneficial to both student and teacher.

In general, the teaching of mathematical subjects is a challenging task – one must choose carefully which subjects to include, with respect to the audience and to the level of the students. Then, the teacher must pass the material in the correct order with the right amount of explanations, justifications, and examples. Finally, at the end of the teaching process, comes the examination of the knowledge acquired in the classroom: the teacher must create a challenging but fair test, to summarize the teaching procedure for the student.

According to our experience, some teachers and lecturers choose an easy solution – they take a known set of classical and banal propositions and examples, and pass them on to the students, over and over again throughout the years. The advantages of this methodology are obvious – it is easy to teach, easy to examine, and the needed material is passed on. The biggest disadvantage of this approach is that it is based on the lack of creativity and diversification: the students get a very static image of the subject, and cannot develop the innovative thinking needed for solving problems.

To this end, the use of nonstandard methods and examples is crucial for succeeding in the teaching process. It is the teacher's job to bequeath his knowledge to his students, but also to allow them to strive for different and unique approaches. The capability of looking at problems from a different point of view can be beneficial by several means: first, one can find an easier and more elegant way for solving known problems; second, when dealing with difficult problems for which there does not seem to exist a straightforward solution, creative thinking can lead to success. Further reading upon this approach can be found in [9].

Of course we do not suggest that the teaching of classical and basic subjects is unnecessary; the fundamental and core themes and tools in the various areas of mathematics must be passed on to the students. Rather, we explain why showing **other** approaches should be required of teachers and lecturers, for a variety of reasons: for a better understanding of the materials; for providing the tools that are needed in the real world of problem-solving; and for making the learning process more interesting and beneficial.

The above assertion is also true in the case of solutions for extremum problems. There is indeed the square method for solving min-max problems, in one variable as well as in several variables, and this suffices to give the students enough information to solve such problems. However, in some cases, the use of known algebraic inequalities can be helpful; the solution gets much easier, and a new creative approach is discovered.

One of the interesting algebraic tools, which gives birth to some useful algebraic inequalities, is focused on the definitions of different *means*. By a broader definition, a *mean* is a number $M(x_1, \ldots, x_n)$ corresponding to a set of numbers $x_1, \ldots, x_n$, that satisfies

$$\min(x_1, \ldots, x_n) \leq M(x_1, \ldots, x_n) \leq \max(x_1, \ldots, x_n).$$

Unfortunately, aside from the arithmetic mean, which is very popular, students do not encounter other means. Some of the lesser known of these are the *harmonic mean*:

$$\frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}, \tag{1}$$

and the *quadratic mean*:

$$\sqrt{\frac{x_1^2+\cdots+x_n^2}{n}}. \tag{2}$$

The uses of these means are more heavily emphasized in different parts in mathematics and its applications (such as calculus, geometry, physics, finances), and their recognition is essential for the young student.

After defining the basic means, we can present one of the classical algebraic inequalities; the inequality of quadratic, arithmetic, geometric, and harmonic means (QM-AM-GM-HM inequality):

$$\sqrt{\frac{x_1^2+\cdots+x_n^2}{n}} \geq \frac{x_1+\cdots+x_n}{n} \geq \sqrt[n]{x_1\cdots x_n} \geq \frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}. \tag{3}$$

This inequality, in which equality holds if and only if all terms are equal ($x_1 = \cdots = x_n$), holds a vast power in the solution of various problems. For the proof of these statements, as well as further research of the properties of means, see [3]. Some teachers and lecturers do not emphasize these methods, and thus their students lack training in the use of such useful tools. Moreover, techniques such as these can be used to simplify some min-max problems in several variables, to the level that even a high school student can solve; this will be shown below.

In the experience of the authors, the use of unique and refreshing algebraic tools in the classroom is beneficial. The student is presented with a different approach, and is somewhat intrigued by its creativity; this increases the student's will to participate and contribute to the classroom, and the student finds a beautiful method. These tools will also help him later on: in the short term, for solving similar problems, and in the long term, for use in solving general problems from analysis, in which standard solutions are too cumbersome, or even out of reach. Indeed, one of the authors has used such methods repeatedly in the teaching of the first and second calculus courses for engineers. In one such experiment, two groups of 60 students were given a first analysis course. The first one was taught the technique of solving extremum problems using algebraic inequalities (such as the AM-GM inequality), as well as other standard techniques (involving derivation). The second group was not taught the techniques involving algebraic inequalities, but only the standard methods. Then, given an exam that contained an extremum problem that can be solved using algebraic techniques, the results were as follows. From the first group, 75% of the students solved the problem correctly, out of which 85% used algebraic inequalities in their solution. In the second group, only 30% of the students solved the problem correctly.

3 Examples

After seeing the importance of the presented subject, we give several particular examples for an unusual solution of problems in analysis, with applications to other fields as well.

3.1 min-max problems for functions of one variable

In this section we examine several examples of nonstandard solution methods for min-max problems of one variable functions.

First, let $f_1(x) = x + \frac{4}{x}$, which is an odd function; we want to find the minimal value of f_1 in the domain $x > 0$. One could, using the classical way, find the derivation of f_1 and equate it to zero. Instead, using the AM-GM inequality, as in 3, we see that

$$x + \frac{4}{x} \geq 2 \cdot \sqrt{x \cdot \frac{4}{x}} = 4.$$

Equality in the above occurs when $x = \frac{4}{x}$, meaning that $x = 2$, so the minimal point of f_1 in the domain $x > 0$ is $(2, 4)$.

For an example that is a bit more complicated, consider $f_2(x) = x^2 + \frac{8}{x^3}$, in the domain $x > 0$. We want to find the minimal value of f_2 now; the previous use of the AM-GM inequality will not work, because the multiplication of the summands is not independent of x; however, writing

$$f_2(x) = 5 \cdot \frac{\frac{x^2}{3} + \frac{x^2}{3} + \frac{x^2}{3} + \frac{4}{x^3} + \frac{4}{x^3}}{5},$$

and using the AM-GM inequality with the summands as above, we get that $f_2(x) \geq 5 \cdot \sqrt[5]{\frac{16}{27}}$. Again, equality holds when all the summands are equal, that is $\frac{x^2}{3} = \frac{4}{x^3}$, or $x = \sqrt[5]{12}$, so the minimal point is $(\sqrt[5]{12}, 5 \cdot \sqrt[5]{\frac{16}{27}})$.

To summarize this set of examples, we give a generalization of the above, by letting $f_3(x) = ax^n + \frac{b}{x^m}$, with $m, n \in \mathbb{N}$, and $a, b \in \mathbb{R}$ positive. To use the AM-GM inequality, we split the two summands of f_3 info the following sums:

$$f_3(x) = \sum_1^m am^{-1}x^n + \sum_1^n bn^{-1}\frac{1}{x^m}.$$

Then, by the AM-GM inequality:

$$f_3(x) \geq (n+m) \sqrt[n+m]{(am^{-1}x^n)^m \cdot (bn^{-1}x^{-m})^n} = (n+m) \sqrt[n+m]{\frac{a^m b^n}{m^m n^n}};$$

The minimal value is achieved when all the summands are equal, i.e., when $am^{-1}x^n = bn^{-1}x^{-m}$, which is at the point

$$x = \sqrt[n+m]{\frac{bm}{an}}.$$

The reader can insert several different values m, n in the function f_3, and see that the result is as presented in the point above. In 1, we give the graphs of $f_2(x)$ and $f_4(x) = 2x^2 + \frac{1}{x^6}$, and show that the minimum is indeed achieved in the point, as above.

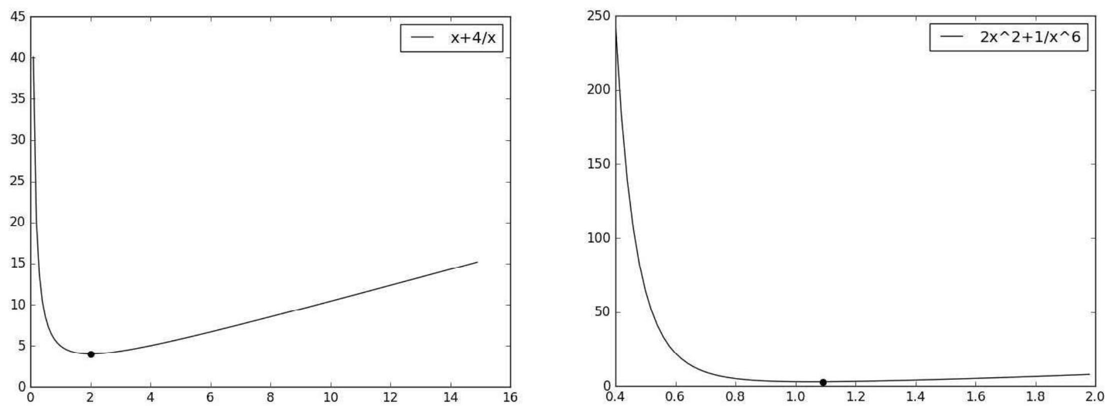


Figure 1: Graphs of the functions $f_2(x)$ and $f_4(x)$, and their minimal values

3.2 Geometric min-max problems

In this subsection we present some geometric min-max problems and their solution through the use of algebraic inequalities.

For the first example, we will find the triangle with the least possible area, which is bounded by the first quadrant and a descending line that goes through a given point $(a, b) \neq (0, 0)$.

Let m be the slope of the line ($m < 0$), which we regard as a variable. Thus, if the line is denoted by $y = mx + n$, we have that $n = b - ma$, and if $mx + n = 0$, we have $x = \frac{-n}{m} = \frac{ma-b}{m}$. Therefore, the area of the desired triangle is $S(m) = \frac{1}{2}(b - ma)\frac{ma-b}{m}$, or after a simplification:

$$S(m) = \frac{1}{2}\left(2ab - a^2m - \frac{b^2}{m}\right).$$

For $S(m)$ to be maximal, we must have that $a^2(-m) + \frac{b^2}{-m}$ is maximal. Using the AM-GM inequality:

$$a^2(-m) + \frac{b^2}{-m} \geq 2\sqrt{b^2a^2} = 2ab.$$

For an equality, we must have that the summands are equal, that is $\frac{b^2}{m} = a^2m$, so that $m = -\frac{b}{a}$ (recall that m is negative). In this situation, we get that the line is of the form $y = -\frac{b}{a}x + 2b$, and the minimal area is $S_{\min} = 2ab$. An example, with the values $a = 2, b = 4$, can be seen in 2.

In the next example, we will find the cuboid with maximal volume that can be bounded inside the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$. For that, let (x, y, z) be the vertex of

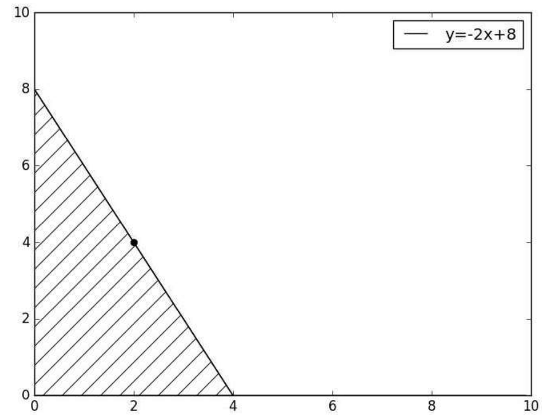


Figure 2: A triangle with minimal area, bounded by the axes and the line $y = -2x + 8$ in the first quadrate

the cuboid, with all positive parameters. Then, the length of the sides of the cuboid are $2x$, $2y$, $2z$, and thus the volume is $V(x, y, z) = 8xyz$. Now, using the AM-GM inequality on the ellipsoid formula, we get

$$1 = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} \geq 3\sqrt[3]{\frac{x^2}{4} \cdot \frac{y^2}{9} \cdot \frac{z^2}{25}},$$

and therefore

$$V(x, y, z) = xyz \leq \frac{10}{\sqrt{3}}.$$

Thus, by the maximality property of the AM-GM inequality, the maximal possible value for V is $\frac{10}{\sqrt{3}}$ which is achieved where $\frac{x^2}{4} = \frac{y^2}{9} = \frac{z^2}{25}$. Therefore, we get that $\frac{x^2}{4} = \frac{y^2}{9} = \frac{z^2}{25} = \frac{1}{3}$ so $(x, y, z) = \left(\frac{2}{\sqrt{3}}, \sqrt{3}, \frac{5}{\sqrt{3}}\right)$, so the vertices of the cuboid are $\left(\pm\frac{2}{\sqrt{3}}, \pm\sqrt{3}, \pm\frac{5}{\sqrt{3}}\right)$. The two shapes can be seen in 3.

For the last example in this subsection, we solve the following problem. From a given square paper with a side of length a, one can create a box without a cover, in the following manner: create 4 small squares of the same length x at the corners of the square, then fold them upwards into a box. For this box, we want to find the value x so that the volume is maximal. By the construction described above, the sides of this open box are $x, a - 2x, a - 2x$, so that the volume $V(x)$ is given by the formula

$$V(x) = x \cdot (a - 2x)^2.$$

Now, using the AM-GM inequality, we have that

$$\frac{2a}{3} = \frac{(a - 2x) + (a - 2x) + 4x}{3} \geq \sqrt[3]{(a - 2x)(a - 2x) \cdot 4x},$$

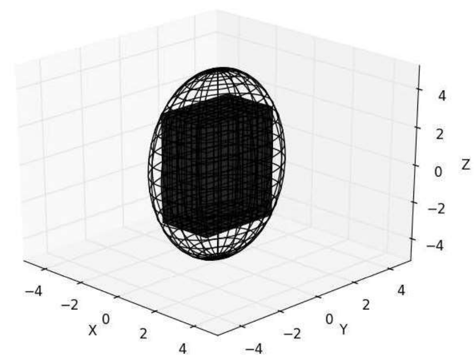


Figure 3: A cuboid bounded in a given ellipsoid, such that its volume is maximal

After a simplification we get:

$$V(x) = (a-2x)^2 \cdot x \leq \frac{2a^3}{27}.$$

Therefore, the maximal volume of the constructed box is $\frac{2a^3}{27}$, which (by the maximality of the AM-GM inequality) is achieved as the point where $a-2x = 4x$, meaning $x = \frac{a}{6}$.

3.3 min-max problems for functions of two variables subject to a condition

This subsection is devoted to solving min-max problems for functions of two variables, by again using algebraic inequalities. As opposed to the problems presented in the previous subsection, in the two-variable case, the problems are more difficult – this is due to the possibility of limiting the solution to a given domain. In such a case, the classical method for solving these problems is the *method of Lagrange multipliers*, which is a tedious procedure that can sometimes be avoided; therefore, these problems can be presented to high school students, without the use of heavy subjects from analysis.

For the first example, let $f(x,y,z) = x^2y^3z$; find its maximal value, subject to the condition $x+y+z = P$ $(P \neq 0)$.

First, we present the classical solution, using the method of Lagrange multipliers: define

$$h(x,y,z) = x^2y^3z - \lambda(x+y+z-P);$$

we want to find the zero point of the equation $\nabla h = 0$, which gives, in addition to the original constraint, the three equations:

$$2xy^3z - \lambda = 0 \tag{4}$$

$$3x^2y^2z - \lambda = 0 \tag{5}$$

$$x^2y^3 - \lambda = 0 \tag{6}$$

There are 3 generic solutions for these equations:

1. $(0, y, P - y)$ for $y \in \mathbb{R}$, in which case $\lambda = 0$;
2. $(x, 0, P - x)$ for $x \in \mathbb{R}$, in which case $\lambda = 0$;
3. $\left(\frac{P}{3}, \frac{P}{2}, \frac{P}{6}\right)$, in which case $\lambda = \frac{P^5}{3^2 \cdot 2^3}$.

Recall that we want the maximal value of f; we see that putting the first two points gives $f = 0$, where the third one gives $f = \frac{P^6}{3^2 \cdot 2^3 \cdot 6} \geq 0$, and therefore this point is the unique maximal point, with maximal value $\frac{P^6}{3^2 \cdot 2^3 \cdot 6}$.

Now, we solve the problem using algebraic tools. For that, note that we can write $P = \frac{x}{2} + \frac{x}{2} + \frac{y}{3} + \frac{y}{3} + \frac{y}{3} + z$, and then using the AM-GM inequality we get that

$$P \geq 6 \cdot \sqrt[6]{\frac{x^2y^3z}{108}};$$

rephrasing the above, we get

$$f(x, y, z) = x^2y^3z \leq \frac{P^6}{2 \cdot 6^3}.$$

By the maximality property of the AM-GM inequality, f is maximal precisely where the summands in the above are equal. Therefore the maximal point is achieved where $\frac{x}{2} = \frac{y}{3} = z$, which means that $z = \frac{p}{6}$ and so the maximal point is $\left(\frac{P}{3}, \frac{P}{2}, \frac{P}{6}\right)$, with the maximal value $\frac{P^6}{2 \cdot 6^3}$.

One notes the obvious difference between the two solutions presented above – the algebraic one is much simpler, and can be used without involving advanced tools from analysis.

Just as in the previous subsection, this example can be generalized to a more abstract scenario, for the function $x^m y^n z^k$. We leave the details for the reader; the final maximal point is

$$\left(\frac{mP}{m+n+k}, \frac{nP}{m+n+k}, \frac{kP}{m+n+k}\right) = \frac{P}{m+n+k}(m, n, k),$$

with the maximal value

$$\frac{m^m n^n k^k}{(m+n+k)^{m+n+k}} P^{m+n+k}.$$

The last example we present is quite intriguing: to find the measurements for a cuboid without a top cover, with a given surface area, such that its volume is maximal.

In order to solve this problem, let a, b, h be the measurements; we want to find the maximal value for $V(a,b,h) = abh$, with the condition $ab + 2ah + 2bh = S$, where S is given. Using the AM-GM inequality, we get

$$\frac{ab + 2ah + 2bh}{3} \geq \sqrt[3]{ab \cdot 2ah \cdot 2bh}$$

so that

$$\frac{S^3}{108} \geq a^2 b^2 h^2 = V^2(a,b,h).$$

Now, V is maximal if and only if V^2 is maximal; for V^2 to be maximal, using the maximality property of the AM-GM inequality, we must have an equality of the summands, so that $ab = 2ah = 2bh$, or $a = b = 2h$, so the desired point is $(2h, 2h, h)$, where $12h^2 = S$; that is $\left(\sqrt{\frac{S}{3}}, \sqrt{\frac{S}{3}}, \sqrt{\frac{S}{12}}\right)$.

Note that if we want the cuboid to have a top cover, the condition becomes $2ab + 2ah + 2bh = S$, and then by the same method as above, $a = b = h$, so we get a cube.

References

[1] Aldaz J. M., (2009). Selfimprovemvent of the inequality between arithmetic and geometric means. *J. Math. Inequal.*, no. 2, 213–216.

[2] Amram, M., Dagan, M., Ioshpe, M., Satianov, P., (2016). Staircase and fractional part functions. *International Journal of Mathematical Education in Science and Technology*, 47(7), 1087–1102.

[3] Bullen P. S., (1988). Handbook of Means and Their Ineqaulities. *Mathematics and its Applications*.

[4] Wang, C. L., (1979). Functional Equation Approach to inequalities. *Journal of Mathematical Analysis and Applications*, 71, 423–430.

[5] Satianov, P., Dagan, M., Amram, M., (2015). Using min and max functions in calculus teaching. *International Journal of Mathematical Education in Science and Technology*, 46, 944–951.

[6] Tall, D., (1996). Functions and Calculus. International Handbook of Mathematics Education. *Kluwer International Handbooks of Education*, 4, 289–325.

[7] Tall, D., (2000). Cognitive development in advanced mathematics using technology. *Mathematics Education Research Journal*, 12, 196–218.

[8] Tall, D., Smith, D., Piez, C., (2008). Technology and calculus. *Research on Technology and the Teaching and Learning of Mathematics*, 1, 207–258.

[9] Shneiderman, B. , Fischer, G. , Czerwinski, M., et al., (2010). Creativity Support Tools: Report From a U.S. National Science Foundation Sponsored Workshop. *International Journal of Human-Computer Interaction*, 20(2), 61–77.

Feedback from students' errors as a tool in teaching[1]

Radoslav M. Dimitrić
raddimitric at netscape.net

Department of Mathematics, CUNY-CSI

Abstract

A close look at students' written work on examinations offers a wealth of information about their performance, their knowledge of the subject, strengths, weaknesses and misconceptions, and their overall level of mathematical skills and abilities. This information can be used to better ascertain how mathematical concepts taught in class were understood by students and to suggest various approaches that could be used to improve teaching. The process of analyzing students' work, although time consuming, can be rewarding and can have positive results for both the students and the instructors. Our aim is fourfold: (a) To give a rationale for analysis of students' work, especially tests, (b) to give possible explanations of students' performance, (c) to revisit aspects of teaching some mathematical constructs, such as function and, (d) to provide possible ways of improving the teaching of concepts considered in this paper.

Keywords: Definition and conception of function, Calculus III, three-dimensional calculus, three-dimensional graphs, several variables, analytic geometry, double integral, triple integral, line integral, volume, computational errors, students' errors, lesson plans, teaching strategies, assessment of learning, assessment of teaching, diagnostic testing.

Mathematics/Education Subject Classification (MSC 2010): Primary: 97C70, 97D40, 97D70, 97I40; Secondary: 97D66, 97D80, 97I40 / (MESC 1999) Primary: B40, B50, C70, D40, D70, Secondary: D60, D80, E40, I20, I40, I60

1 Introduction

In this paper, we present the core of our analysis of one final examination in a Calculus III course for science majors (by way of answers about their majors/career goals, 42.8% of the students were chemistry or biochemistry majors, 14.3% math and math ed majors, 10.7% computer science majors, 10.7% mechanical or civil engineering, 7.1% medicine or biomedicine; remaining students were pharmacy, history, teaching and biostatistics majors, or undecided and they amounted to 14.4%. Judging by a diagnostic

[1]The first version of this paper was presented at the joint American Mathematical Society/Mathematical Association of America Meeting, Boston, January 4, 2012. The paper was presented at the 12th International Congress on Mathematics Education (ICME12), Seoul, South Korea, July 8-15, 2012.

test given at the first class meeting and made specifically for this course (the diagnostic test can be found in Dimitric (2013), pp. 28–29) these science students were not mathematically advanced, and showed considerable weakness on many points. Diagnostic test score of about 90% would indicate that the student is prepared for the course and the score below 60% requires instructor's serious discussion as to what to do about the prerequisites. In this light the mean score of 35% (median 38%, standard deviation 15.7%) of this class was rather alarming. The highest diagnostic score 63% was attained by a student entering (or already in) a biostatistics PhD program and the next highest (62%) was by a high school student who wanted to get a "head start." Most difficulties were with the diagnostic test problems no. 2,5,6,7,11,14,20. While the university has some prerequisites for this course and other higher division mathematics courses, they are not enforced, and students self-select courses according to their own opinion of their skills and knowledge.

As is often the case, there were difficulties related to the concept of functions, their properties and the operations with them. There is a considerable corpus of literature that addresses the issue of teaching and more often students' understanding of the notion of function (a sample is Breidenbach et al (1992), Dubinsky & Harel (1992), Horscovics (1982), Oehrtman et al (2008), Piaget et al (1977), Sfard (1987), Tall, D. & Vinner, S. (1981), Thomson (1994), Vinner (1989, 1992)) and a survey in Leinhardt et al (1990). The fundamental starting points in their research is in the field of psychology.

One strand of that research concentrates on students' perceptions (or subconscious metaphors) of the notion of function, dividing it into the process conception vs. action conception categories. After Dubinsky & Harel (1992, p.85), *action conception* of function involves the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time (e.g. one evaluation of an expression). A student whose function conception is limited to actions might be able to form the composition of two functions, defined by algebraic expressions, by replacing each occurrence of the variable in one expression by the other expression and then simplifying; however, the students would probably be unable to compose two functions that are defined by tables or graphs.

In contrast (ibid.), a *process conception* of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. When the subject has a process conception, he or she will be able, for example, to combine it with other processes, or even reverse it. Notions such as $1-1$ or onto become more accessible as the student's process conception strengthens.

Our own work with students certainly supports importance of perceptions and metaphors both in learning and teaching, however perhaps more fundamental issues need to be addressed prior to or simultaneously with the problem of perceptions and those involve focus on teaching and learning precise "formal" definitions of these concepts, that in turn, depend on precise, formal definitions of notions of two and higher dimensions (and representations in coordinate systems). Perhaps a notion close to this aspect is "concept definition" as discussed in Tall & Vinner (1981), Vinner (1991), Vinner & Dreyfus (1989). However these works concentrate primarily on "concept image" which is undoubtedly important, however the metaphors (such as in "concept images") cannot be made operational and stable, unless there is definite command of relevant and definite definitions these metaphors should map. By metaphor, we understand the classical (and somewhat modernized) definition of metaphor originating with Aristotle in his *On the Art of Poetry* and *Rhetoric* " carrying of expression from one object onto the other..." In short, metaphor consists of three parts: the first object, the "carrying" i.e the process of connecting the two objects, and the second object." Metaphors are different if at least one of their three parts differs from the corresponding part of another metaphor.

The present paper gives an example of analysis of students tests in a default classroom setting. Among other things, we make suggestions as a practical guide in teaching the subjects involved, but we also point out areas worthy of further education research. Notable studies exist with related themes and we mention here Selden et al (1989, 1994, 2000) where the authors test (outside of the classroom setting) self-selected sample of students (after they complete the usual calculus sequence) on routine and non-routine problems.

2 The test, its solutions and rationale

Seven concrete test problems are presented here (to be completed in 100 minutes – the time that is perhaps tight for completing all the problems). By departmental decision, graphing calculators are allowed, starting from the most elementary remedial courses to post calculus courses. Our comments accompany each individual problem with final conclusions at the end of the paper.

When making the exams, we try to adhere to some extent (and to the extent possible in classes at this level) to Bloom's taxonomy (Bloom (1956), and its revision, Anderson et al (2001)) and devise exam questions of different difficulty to test various aspects of students' knowledge. The issue of the difficulty level and all the ramifications is a complex issue. We expound on this topic in more detail in, Dimitric 2012b, while adopting here what is possibly the crudest (but still useful) graduation in Bloom's two-dimensional taxonomy array. The first two problems were meant to be of the simplest kind and students' scores on the problems show that our gage of relative difficulty was correct. This however does not mean that students' performance was appropriately good, as much as we expected or hoped for – the mean scores on the individual problems, out of maximum 15, were as follows (Problems 1 to 7, in order): 8.04, 6.39, 2.96, 3.52, 0.96, 2.22, 3.39 while the respective medians were: 8, 7, 1, 3, 0, 1, 3. Our rating of problem 5 was not the highest difficulty, for we deemed it not so time consuming, however, students' performance on this problem makes it the hardest problem. The reasons for this are manifold, the main being the fact that the continuity concept (and function in general) is an orphan in calculus education; another reason is that solving this problem required a bit of logical thinking in steps. While these problems were not out of the ordinary in comparison to problems worked out in class or assigned for homework, the problems may be more difficult than what the students have seen in previous courses and or courses taken with other instructors. We learn from Selden et al (2000) that what they term "non-routine" calculus I problems pose considerable difficulty even for students who completed the calculus sequence (including differential equations). Perhaps peculiarly, these authors report that, in place of attempting to solve the problems by calculus techniques, the students use algebra,[2] or guessing, etc., in their attempts. As it is, the problems are halting already at Bloom stage one complexity level[3] and this is the area that needs much more attention in research literature.

Since this kind of analysis is also extended to midterm exams (for any given class), the information we obtain is used to improve teaching in a number of ways.

Problem 1. Given three points in 3D space: $A(2, 0, a), B(0, 1, a)$, $C(-1-1, 0)$, where a is an arbitrary given constant.

a) Calculate the vector/cross product $\vec{AB} \times \vec{AC}$.

b) Calculate the scalar/dot product $\vec{AB} \cdot \vec{AC}$

c) For what values of a are vectors $\vec{AB}$ and $\vec{AC}$ (i) perpendicular, (ii) parallel?

d) Write an equation of the plane that contains points A,B,C.

Solution

a) First one computes the relevant vectors: $\vec{AB} = \langle -2, 1, 0\rangle = -2\mathbf{i} + \mathbf{j}$, $\vec{AC} = \langle -3, -1, -a\rangle = -3\mathbf{i} - \mathbf{j} - a\mathbf{k}$, then their vector product $\vec{AB} \times \vec{AC} = (-2\mathbf{i} + \mathbf{j}) \times (-3\mathbf{i} - \mathbf{j} - a\mathbf{k}) = -\mathbf{i}a - 2a\mathbf{j} + 5\mathbf{k} = \vec{n}$, or alternatively computing the determinant that has rows $(\mathbf{i}\,\mathbf{j}\,\mathbf{k}), (-2\,1\,0)$ and $(-3\ -1\ -a)$.

b) $\vec{AB} \cdot \vec{AC} = \langle -2, 1, 0\rangle \cdot \langle -3, -1, -a\rangle = -2(-3) + 1(-1) + 0(-a) = 5$.

c) (i) Non-zero vectors are perpendicular iff their scalar product $= 0$, which is impossible here since the scalar product is 5. Thus vectors are not perpendicular for any value of a. (ii) Non-zero vectors are parallel iff their vector product is zero. This also never happens since the vector product has the $\mathbf{k}$-component always 5, thus no value of a will force the vectors to be parallel.

d) For the equation of this plane we need one point (say A) in the plane and a vector perpendicular to the plane. But a perpendicular vector is already available, namely the vector product $= \vec{n}$. The equation is $(\vec{X} - \vec{A}) \cdot \vec{n} = 0$ or $(x-2)(-a) + (y-0)(-2a) + (z-a)5 = 0$, or $-ax - 2ay + 5z = 3a$.

[2] The issue of use of more pre-calculus in calculus was addressed in Dimitric (2001); that approach however requires a greater sophistication than is exhibited from a typical calculus student

[3] Selden et al report that their routine stage is well fulfilled by the students they test, however their testing was on the material several courses earlier than the course levels the tested students were at, at the time of testing

Normally, analytic geometry is not a part of Calculus III; rather, it is often "reviewed" in previous calculus courses, and at any rate is included in most "coffee table" calculus textbooks. Our department decided to include it in Calculus III as a "review" and rightly so, given the level of mathematical skill of students taking such courses. There is however a drawback in this kind of remediation since the topics that are left for the end of the course are sometimes, if not half of the time (depending on the class), left out for sheer lack of time. But those topics are (or should be) the core of Calculus III – Green's, Stokes' and divergence theorems. This course is fairly standard in the US – it includes topics on vector-valued functions and surfaces, partial derivatives, optimization and Lagrange multipliers, double and triple integrals and their properties and techniques of evaluation, vector fields, line integrals, surface integrals that all culminate in Green's, Stokes' and the Divergence theorems.

This problem was meant to test the elementary 3D concepts at the level of definitions and simple steps. We have used a constant a in order to prevent "button pushing" on programmable calculators. This causes some problems to some students who still think that a constant must be given as a concrete decimal number (most often an integer), not a letter, which (according to many students) is an unknown variable, by default. This kind of problem could be eliminated if there were beginning courses that would teach basic mathematical culture, including the concepts of a constant and variable; in fact such courses should be made mandatory, at least for mathematics majors.

A number of students stumble upon an objective problem, namely the fact that we represent vectors both by their end-points (that are each ordered triples) as well as by a single triple of vector's coordinates (that are the difference of those end-points). This is an objective confusion since localization and affinity of 3D space make this "gluing" possible. The instructor should thus spend some time in elaborating the concept of vector translation which produces the same vector (strictly an equivalent vector) that begins at the origin. Judging by further errors made, the instructor should also make sure that students remember that one needs to subtract the initial point from the end point of the vector to get the vector's 3D components: $\vec{AB} = \langle -2, 1, 0\rangle$ and $\vec{AC} = \langle -3, -1, -a\rangle$.

One idea to avoid this confusion is perhaps to use round brackets for ordered triples representing points and to use angle brackets to represent vectors, however we did not find that this alone can remedy the problem.

The errors even at this first step are in simple arithmetic (subtracting greater number from smaller), mishandling of minus signs and parentheses, multiplications by 0 treated as multiplications by 1, and getting vectors to be the opposite of what they should be. Arithmetic errors come somewhat from the exam pressure and to a greater extent because of addiction to calculators that result in students being unable to do even the simplest arithmetic operations in their heads. We have mentioned elsewhere [Dimitric 2003, 2009] that students did a bit better when they were not allowed calculators into the exams.

One student "finds" the vectors by dot multiplying their endpoints, which is in the domain of confounding (with the dot product). We wanted to check that students knew definitions of the cross (vector) and the dot (scalar) products and how to compute them if vectors are given in the component form. In the scalar product all that is required is component-wise multiplication and addition, but the vector product involves a bit of a twist in that students have to either use the appropriate determinant or express the vectors through the basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and use properties of cross multiplication. A majority of the students used the determinant and perhaps it may be due to the fact that we used the determinant more often than other methods in computing the product when working out exercises at the blackboard. That should then imply that including more computations with vectors spanned by the basis may be helpful.

In calculating the cross product determinant (which computes to be $-a\mathbf{i}-2a\mathbf{j}+5\mathbf{k}$), the usual arithmetic errors occur, now including calculation of the three by three determinant. Not all students remember the determinant definition of the cross product. The student that "found" vectors by multiplying their endpoints also formed a 2 by 3 "determinant" and did some inventive multiplications inside to get his incorrect result.

Not all students remember the coordinate definition (computation) of the scalar product (equal to 5 in this case). Here is how one student does it: $(-2, 1, 0)\cdot(-3, -1, -a) = (6, -1, 0)$.

Part c) was meant to check whether students learned the simple connection between the two products with vectors and orthogonality and parallelism. Some confuse the two and relate parallel to $\vec{AB} \cdot \vec{AC} = 0$ and perpendicular to $\vec{AB} \times \vec{AC} = \mathbf{0}$ (many may miss the fact that the latter 0 is a vector). Here the instructor has more work to do in devoting more time in explaining the distinctions between the two products, such as that one is a scalar and the other is a vector and spend more time on the geometric definitions of the two (in addition to their consequent computational definitions). One may add mnemonic rules to aid the students to remember which is which and how they are related to these important geometric conditions. Another student confuses perpendicularity with the condition for perpendicularity of two lines; he states: "parallel when $a = 0$. Can't be perpendicular because a cant [sic] become a negative reciprocal..." That a should be 0 for parallel or perpendicular is a frequent claim. One student computes the dot product correctly to be 5 and states correctly the condition of perpendicularity, but claims that the two vectors are always perpendicular; this could be simply a lack of concentration on the student's part? Another student "solves" this as follows: " $\vec{AB}$ and $\vec{AC}$ parallel if they point in same or opposite direction, since $\vec{AB} = \langle -2, -1, 0\rangle$ and $\vec{AC} = \langle -3, -1, -a\rangle$, $\vec{AB}$ and $\vec{AC}$ are not parallel because first component is unequal." Perhaps one needs to spend more time on the concept of parallelism by way of one vector being a scalar multiple of the other, resulting in all corresponding coordinates being proportional.

There were also students who overcomplicated, at least with some of their solutions. Calculus II or III students often take linear algebra concurrently and are only too eager to demonstrate what they are exposed to in that class, Gauss elimination being one of the favorites, since considerable attention is paid to it in linear algebra courses and it is a fairly algorithmic procedure that students always favor. Here is how one student approaches questions in part c) (decomposition into perpendicular and parallel components of a vector is also in his mind, for it may have been a part of a review session before the exam): i) $v_\perp + v_{||} = ||v||$, $a = -5$, ii) $a = 2.25$. To get this a, he does Gauss elimination in the matrix made up of rows $\vec{AB}$ and $\vec{AC}$ and gets matrix with rows $(1, -1/2, 0)$ (incorrect) and $(0, -5/2, -a)$ and then matrix with rows $(1, -1/2, 0)$ and $(0, 1, 2a/5)$ and then rows $(1, 0, 9a/10)$ and $(0, 1, 2a/5)$, then equates $9a/10 = 2a/5$ and gets $45a = 20a$ from which he derives $a = 2.25$. Ending up with an incorrect answer may be prevented, with some chance of success, by the instructor telling students to check their answers by plugging in the values they get.

Part d) was meant to give a 2D/3D analogue of the point-slope equation in the Cartesian plane. The plane is $ax + 2ay - 5z + 3a = 0$. Although students may have incorrectly found their cross product, if they wrote the correct form for the equation of the plane, they received most of the credit for this part. Some confused the plane equation with that for a line through a point. Thus, this first step of simply memorizing the plane equation is worth working on (as the lowest complexity level). Certainly, memorization comes with working out a good quantity of exercises that utilize the equation or concept to memorize; in addition our advice to students was to make an index card for a notion they were learning and write definitions and basic properties on one side and a typical example(s) using the notions on the other side of the card.

Problem 2. Find a function $f(x, y)$ that satisfies the equation $\nabla f = (y^2, x)$ or prove that such a function does not exist. (You may assume that the function has partial derivatives of any order.)

Solution

We made the assumption on the existence of all the partial derivatives in order to simplify considerations on smoothness of partial derivatives. If such an f exists, then, we have to have

$\nabla f = (\partial f/\partial x, \partial f/\partial y) = (y^2, x)$, i.e., $\partial f/\partial x = y^2$ and $\partial f/\partial y = x$,

consequently $\partial^2 f/\partial y\partial x = 2y$, $\partial^2 f/\partial x\partial y = 1$. Since we are assured that all partial derivatives exist and are therefore continuous, a Clairaut's theorem asserts that these two mixed second order partials must be equal (which does not happen here, apart from the points on a single horizontal line $2y = 1$).

Students also had a ready-made result that was used in a few exercises in class, namely that if $(F_1, F_2) = \nabla f$ is a gradient vector field (and f is a potential function) then one must have $\partial F_1/\partial y = \partial F_2/\partial x$, which would again lead to the same conclusion of no solution.

This exercise checks basic properties related to partial derivatives and also whether students know the conditions, namely that $\partial F_1/\partial y = \partial F_2/\partial x$ and that the components should have continuous partial derivatives. If these conditions are satisfied, then they should also demonstrate whether they can find such a potential function f.

However, the mathematical results on conditions and existence are not always welcome or easily adopted by science students; they prefer to "do something" – compute a number, a value of a function or such (which may be classified as an "action conception of function", Dubinsky & Harel (1992)). Thus, some students can ignore conditions and go straight to solving the system: (1) $\partial f/\partial x = y^2, \partial f/\partial y = x$ (2). After integrating the first equation with respect to x one gets $f(x, y) = y^2x + \phi(y)$, hence $\partial f/\partial y = 2yx + \phi'(y)$, which again leads to non-existence of the solution ($2y = 1$).

Indeed students' solutions (or incorrect solutions) show that they mostly attempted to find the solution by solving the system of the two pde, by finding (in)appropriate anti-derivatives. Thus students get that the function should be xy^2+C after integrating the first equation with respect to x, likewise, after integrating the second equation with respect to y, that the potential function should be equal to $xy+C$, without realizing that these C's are functions of, respectively, y and x, rather than an identical constant for both (for some the C is not there altogether). They still get the correct answer claiming that these two formulas are not the same. A couple of the students had a different "idea" namely that f is the sum (or the difference) of the two partial anti-derivatives that they obtain; $f(x, y) = xy^2 \pm xy$. This error is due most likely to shortness of time students have to separate and digest different topics they are learning at a fast pace for them. They had been exposed to various results with sums of partial derivatives, such as in the multi-variable chain rule, and they likely had had that confusion in this regard. On the other hand firm grasp of definition of a function, and its domain and codomain and integration and differentiation with respect to one variable would have prevented more blatant errors. Thus keeping in mind that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, i.e., that the domain of f is the (two-dimensional) plane would already be a good medicine.

A work due for the instructor is to emphasize repeatedly that when integrating with respect to one variable, the other variables are considered to be constants, but the "constants" that come out of such integrations are only constant with respect to the variable of integration, and, that they are in fact functions of the remaining set of variables. That is not unlike introduction of partial derivatives. Generally the transition from one dimensional to two dimensional case deserves much attention and the instructor should not spare time in making sure that students get that very basic, but important understanding. Even with one variable, the anti-derivative constant C is usually relegated to either not writing it down or to routine addition to the anti-derivative that they find, yet one should spend a bit of time on the constant as well to relate to students that an anti-derivative is in fact a family of functions that differ by constants and that C may take different values, although we use that one letter for its label (in case of discontinuities the constant may be defined piecewise). A starting point may be examples such as the following: For a (differentiable) real function of one variable x (i.e. in one dimension) $f : \mathbb{R} \longrightarrow \mathbb{R}$, assume that $df/dx = 0$. What does it say about the relationship between x and f? One should go out of one's way to prove to the students that this means that f is *independent* of x, namely that, as far as x is concerned, f is a constant. In 2 dimensions (2D), assume a (differentiable) function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is such that $\partial f/\partial x = 0$. One could use the same logical argument, namely that this will imply that f is independent of x. Students will impulsively follow the one dimensional analogy and claim that f is a constant. One should agree with this but point out the subtlety that f is a constant as far as x is concerned, but not necessarily that it is a constant with respect to the other variable y. The instructor should then produce this constant $\varphi(y)$ and say that $f(x, y) = \varphi(y)$. A concrete example such as $f(x, y) = y^2$ may be explored to show the converse that indeed the partial derivative with respect to x is 0. This is also an opportunity to explain that, unlike in the one-variable case, it is crucial to specify the variable with respect to which we are taking the derivative.

Another effort the instructor should make is to advertise conditions for these results as an essential part not only of the theory they learn, but as an essential part of their problem-solving skills.

A rare student who appealed to Clairaut's theorem forgot the crucial condition, namely the assumption on the continuity of the second partials, in order to make a conclusion about their equality. That, despite our efforts to preempt this mistake while teaching this topic, by pointing out the importance of the continuity condition; textbooks usually place an example (often in exercises section) that illustrates that continuity condition is needed.

One can also see that there are a few (luckily only a few) students who seem not to have learned yet what ∇ means. That can be seen when one student integrates the first equation with respect to y and the second with respect to x. Another error one student makes (after finding the anti-derivatives) is that he writes $f(x, y) = (xy^2, xy)$ thus confusing the roles of the players in the question. Knowing that the codomain of f is one-dimensional would have prevented that error and that, yet again would have been clear if the student wrote all three parts of the function: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Problem 3. Express partial derivatives $\partial f/\partial r, \partial f/\partial \theta$ of a function $f(x, y, z)$ in terms of $\partial f/\partial x, \partial f/\partial y, \partial f/\partial z$ where (r, θ, z) are cylindrical coordinates.

Solution

The purpose of this exercise is to test students on the routine of the chain rule in several variables (three), each of which depends on three variables. Instruction was geared towards showing that multi-variable chain rule is actually a natural extension of the one variable case and that, for that matter, the chain rule formulas

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} \tag{1}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta} \tag{2}$$

are understandable in view of the one variable chain rule. Just writing down these formulas earned the students 1/3 of the points for this problem. The students also had to know the cylindrical coordinate formulas, from which then they had to find the needed partial derivatives:

$$\frac{\partial x}{\partial r} = \cos\theta, \frac{\partial y}{\partial r} = \sin\theta, \frac{\partial z}{\partial r} = 0, \frac{\partial x}{\partial \theta} = -r\sin\theta, \frac{\partial y}{\partial \theta} = r\cos\theta, \frac{\partial z}{\partial \theta} = 0$$

(this would earn the student almost another third of the points). The students would be ready to substitute these partials into (1) and (2) to get the final answer:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta; \quad \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta.$$

A number of errors appear here. A few students in fact do not quite know the cylindrical coordinate formulas yet (Bloom's step one). Confusions are with the polar coordinates in two dimensions $x = r\cos\theta, y = r\sin\theta$ (omitting the third equation $z = z$, or writing $z = r$); this is also confused with the equation of the right circular cylinder $x^2 + y^2 = r^2$ or even $= z^2$, since the thinking is that $z = r$. At least one student confused cylindrical coordinates with spherical ones, undoubtedly attributable to the fact that, in a relatively short time, both of these coordinate sets were used in various problems and topics covered in class.

Getting incorrect derivative from cylindrical coordinates: $\partial x/\partial r = \sin\theta$ and $\partial y/\partial r = -\cos\theta$, $\partial z/\partial r = 1$ was also observed.

Some have not learned the chain rule formulas yet, even though they know the cylindrical coordinates formulas. Thus, students "invent" their own versions of what they have not yet assimilated; for instance: "$\partial f/\partial x = -r\sin\theta, \partial f/\partial y = r\cos\theta$, $\partial f/\partial z = 0$. Or $\partial f/\partial r \cdot \partial r/\partial x = \partial f/\partial x$ and $\partial f/\partial\theta \cdot \partial\theta/\partial y = \partial f/\partial y$ and $\partial f/\partial z = \partial f/\partial z$ no change. And $\partial x/\partial\theta \cdot \partial\theta/\partial r = \partial x/\partial r$ and $\partial y/\partial\theta \cdot \partial\theta/\partial r = \partial y/\partial r$."

Here instructor should spend much time in elaborating on function/variable dependencies of the various functions and variables involved, in fact making sure that students feel comfortable with functions and their compositions. The latter is especially the case in point since a good number of students are still struggling with ordinary, one-variable compositions of functions (and one variable chain rule) and compositions of functions of several variables require special attention, even if they are a very natural extension of the one-variable progenitor. One simple condition, namely that the composition $g \circ f$ of two functions exists only if *Range* $f \subseteq$ *Domain* g is entirely missing from calculus textbooks, mainly because of sole reliance on "intuitive" notions about functions (be it action, process, or "covariation"). The objective problem is in that finding the range of a function is difficult, but this is exactly where more work should be invested, in addition to introduction of a notion more relaxed than the range, namely that of codomain. The lack of understanding of these compositions is visible from the following incorrect reasoning: "Since we are expressing the partial derivatives respectively in terms of $f(x, y, z)$ to (r, θ, z) that must mean that $\partial f/\partial r = \partial f/\partial x$, $\partial f/\partial\theta = \partial f/\partial y$, $\partial f/\partial z = \partial f/\partial z$ respectively and the function $f(x, y, z) = f(r, \theta, z)$."

Certainly it is a "stopper" if students do not know formulas (1) and (2) (Bloom's level 1). Again, making a point that these formulas are an extension of the one-dimensional case would likely ease their remembering the multi-variable chain rule formulas.

An important subtlety is our notation, namely one writes $f(x, y, z)$ and, after substituting x, y, z as functions of other variables r, θ, z, we may erroneously use the same symbol and write $f(r, \theta, z)$, an infraction that could be remedied when a seasoned mathematician is doing mathematics by knowing that it is an abuse of notation. However students will not pick up on that subtlety and they will instantly conclude that $f(x, y, z) = f(r, \theta, z)$ and thus $(x, y, z) = (r, \theta, z)$. At least one student must have concluded this since he states that $\partial f/\partial r = \partial f/\partial x$ and $\partial f/\partial \theta = \partial f/\partial y$ and $\partial f/\partial z = \partial f/\partial z$, as mentioned above.

There were some students who displayed the chain rule equations, and the cylindrical coordinates but were nonetheless unable to complete the appropriate partials from the cylindrical coordinates. But then, there was at least one student who knew the cylindrical coordinates, knew how to compute the appropriate partials from them, but did not know what to do with it, since he was missing basic chain rule equations (1) and (2). One student is confounding the chain rule in this problem with the substitution formula (change of variables), thus he writes $\iiint f(r, \theta, z) dz dr d\theta$.

Here doing chain rule drills with a good number of examples seem to be the only way to learn the multi-variable chain rule.

One characteristic problem with this exercise must have been, for most students, the fact that there was no concrete formula for the function f, which made students attempt to invent it, arguably, because of the action conception of function, but even with the formula, knowledge of the (dimensions of) domain and codomain would have had helped greatly). The students were led astray also by the phonetics they had here. Thus "cylinder" and "cylindrical" induces one student to write a circular paraboloid equation $x^2 + y^2 = z$ (she likely wanted the equation of a cylinder) and then she writes $f(x, y, z) = x^2 + y^2 - z$, $f(r, \theta, z) = (r \cos\theta)^2 + (r \sin\theta)^2 - z = 2r^2 - z$ and also $\partial f/\partial x = 2x$, $\partial x/\partial r = \cos\theta$, $\partial f/\partial y = 2y$, $\partial y/\partial \theta = r(\cos\theta)$ then $\partial f/\partial r = \partial f/\partial x \cdot \partial x/\partial r = 2x \cos\theta = 2x \cos\theta$, $\partial f/\partial \theta = \partial f/\partial y \partial y/\partial \theta = 2yr \cos\theta = 2yr \cos\theta$, thus inventing the "chain rule" (notwithstanding algebraic errors). Another student writes equations of cylinders or spheres (radii $= r$) as the formula for $f(x, y, z)$ (which is not given), again confounding cylindrical coordinates with inventing a formula for f. Yet another student writes $f(x, y, z) = x^2 + y^2 + z$, $f_x = 2x, f_y = 2y, f_z = 1$ $r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}(y/x), z = z$.

Problem 4. Given points A(0,1), B(1,2) and C(3,0) (in 2D) and the curve γ that consists of the portion of the parabola $y = 1 + x^2$ between points A and B and the straight line segment connecting B and C, find the line integral

$$\int_\gamma \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$$

where $\vec{F}(x, y) = (x^2 - y, y^2 + x)$ and the curve is traversed in the direction from A to B to C. What would be the value of the integral if the orientation of the curve γ were from C to B to A?

Solution

This one is meant to test whether students have gotten basics of the line integral of a vector field. Parts of the solution(s) that were assigned partial credit were: Drawing the graph of the path and recognizing that the integral over γ is the sum of the two pieces – over the parabolic path and over the line path: $\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2}$ (2 points). Then the students should know how to parametrize the two paths, for instance: $\gamma_1 : x(t) = t, y(t) = 1 + t^2, t \in [0,1]$ and $\gamma_2 : x(t) = t, y(t) = -t + 3, t \in [1,3]$. One of course needs to know the basic technical integration formula $\int_\gamma \vec{\mathbf{F}}(x,y) \cdot d\vec{\mathbf{s}} = \int_a^b (\vec{\mathbf{F}}(x(t), y(t)) \cdot (x'(t), y'(t))dt$ and to find correct derivatives $x' = 1, y' = 2t$, for γ_1 and $x' = 1, y' = -1$ for γ_2 (3 points). Routine polynomial definite integration will then produce $\int_{\gamma_1} \vec{\mathbf{F}} d\vec{\mathbf{s}} = 2$ and $\int_{\gamma_2} \vec{\mathbf{F}} d\vec{\mathbf{s}} = 0$ (6 points), so that $\int_\gamma = 2 + 0 = 2$ (2 points). The orientation question tests whether a student knows that reversing the path orientation changes the sign of the value of the integral i.e. $\int_{\gamma^-} = -\int_{\gamma^+}$ (2 points).

Apart from the usual arithmetic and algebraic errors, some students are unable to sketch the (entire) integration path (one student attempts to draw it in 3D, undoubtedly influenced by a good number of 3D images of surfaces and curves done on other topics). Ability to draw these simple curves should be a prerequisite for taking any of the calculus courses (see here Dimitric (2012a)). Again, some additional time should be spent on functions defining curves and surfaces in different dimensions; it is highly recommended that functions are presented not only as formulas, but as ordered triples consisting of domain, codomain, and the formula (a rule of correspondence). Analysis of the dimensions of the domain and codomain would then aid the student in recognizing the context as to whether he is dealing with a surface or a curve and whether it is two, three, or more dimensions. Striving to make students switch from action to process conception of function would not help, unless students actually were well familiar with the complete functions, namely domain, assignment rules and codomain, or luckily the range.

There were a couple of students who only drew the graph of the path and did nothing else. When it comes to parametrizations the mistakes are in not parametrizing the line or the parabola piece, or not determining correct limits of the parameter in either of the two pieces. Some students however see no need for parametrizations and use the Cartesian coordinates of the two paths (and some may have the line segment Cartesian equation incorrectly). This is because those students likely confound this with the double integral exercises they have been exposed to. Problems with parametrizations should be diminished by the instructor's explanation why parametrizations are needed, as well as working out some simple but also some intricate parametrizations, while elaborating on the conditions, advantages or disadvantages of one parametrization over the other.

A number of students then use an incorrect formula to integrate this vector field, namely $\int_\gamma \vec{F} \cdot d\vec{s} = \int F(c(t)) \cdot ||c'(t)||dt$, which is a formula for the line integral of a scalar function F. Yet again, dimensional analysis here is of great help to decide the correct context for solving the problem. A list of "invented" formulas for the integral is as follows:

$\int_0^1 (x^2-y, y^2+x)ds = (\int_0^1 x^2 - yds, \int_0^1 y^2+xds)$;

$\int_0^1 \vec{F}ds + \int_1^3 \vec{F}ds$;

$\int_0^2\int_1^2 (1+x^2-y)dydx + \int_2^3\int_0^2(-x+3)dydx = \ldots = 7+17 = 24$;

$\int_{1+x^2}^0 (x^2-y, y^2+x)$, $\vec{f}(x,y) = (x^3/3 - y^2/2, y^3/3+x^2/2)$ from C to B to A.

$\int_{-x+3}^{1+x^2}(x^2-y, y^2+x)$ and stops here.

$\int_0^3 (t^2-(1+t^2), (1+t^2)^3+t)\cdot\vec{\nabla}F(x,y) =$ (this comes after a correct parametrization of the parabola path, but no parametrization of the line path, and correct determination of $\nabla F(x,y)$). The incorrect answer obtained was 71187/70.

$\int_c \vec{F}ds = \int_a^b (c(t)^2 - c(t), c(t)^2 + c(t))\cdot(c'(t))dt$.

These answers are a good indication that more work needs to be done in the area of vector field integral, certainly more examples to illustrate differences in integration of a vector field vs. scalar function.

One student has the following incorrect solution: He draws the graph fairly correctly, and seems to parametrize the portions accordingly, but instead of evaluating the integral in the two portions, he "adds" the portions of the path as follows: $C_1 = y = 1+t^2+C_2 = y = -(t-3)$, $c(t) = C_1 + C_2 = 2y = t^2-t+4$, writes $c(t) = (t^2-t+4)/2$ and $c'(t) = (2t-1)/2$ and then $F(c) = (t^2-(t^2-t+4)/2, (-(t^2-t+4)/2)^2+t)$ and $F(c)\cdot c' = (t^2-(t^2-t+4)/2, (-(t^2-t+4)/2)^2+t)\cdot(2t-1)/2$ and then $(2t-1)/2\cdot\int_{-1}^1 (t^2-(t^2-t+4)/2, ((t^2-t+-4)/2)^2+t)$.

Most students answered correctly the orientation question, but there were a couple of incorrect answers as follows:

"The value wouldn't change going the other direction since the line didn't change shape or orientation."

"The value of the integral of the orientation of curve γ went from C to B to A does not change as if it went from A to B to C."

Here it would be worthwhile to spend extra time with the one-dimensional case and why the sign of the integral changes when the orientation of the interval of integration is reversed.

There are also some semi-correct answers, as follows:

"If the orientation was from C to B to A the value of the integration would be negative" (we corrected this to "opposite"). This comes from the universal error (colloquialism) that students make, namely identification of "opposite" with "negative" which is not an innocent error, but is rather a linguistic sloppiness impeding students' understanding of concepts, and an effort should be made to correct it.[4]

Problem 5. The function $f(x,y)$ is defined as follows on the domain $[0,1]\times[0,1]$:

[4] We have noted elsewhere that this error prevents students from adopting the definition of $|x| = x$, if x is positive and $-x$, if x is negative, for $|x|$ is positive and $-x$ ("negative x") "is negative."

$$f(x,y) = \begin{cases} \frac{x}{\arctan(1/y^2)}, & \text{if} \quad y \neq 0 \\ 2x\pi, & \text{if} \quad y = 0 \end{cases}$$

Find all the points where the function is continuous (or, complementarily, where it is not continuous) in its domain.

Solution

Certainly, it would be nice to check that the function is defined in the proclaimed domain, by noting that $\arctan(1/y^2)$ never turns into 0. The first stage of the proof would be to note that for points that have non-zero y-coordinate, the first part of the definition of f is the quotient of two continuous functions. Students are expected to say that x is continuous as a polynomial and that a rational function with non-zero denominator is continuous as a composition of two continuous functions arctan and the rational function $1/y^2$. This argument so far would have earned them a good partial credit. Then they need to check continuity at points of the form $(a, 0)$ i.e. they need to check that $\lim_{(x,y)\to(a,0)} f(x,y) = f(a,0) = 2a\pi$. Students would have to find the $\lim_{(x,y)\to(a,0)} x/\arctan(1/y^2) = a/(\pi/2) = 2a/\pi$. This means, that in order to have continuity at such points, the equality $2a\pi = 2a/\pi$ must be satisfied and that happens only for $a = 0$. Hence the conclusion is that f is continuous at all points (x, y) for which $y \neq 0$, also at $(0, 0)$ and discontinuous at all other points, namely all $(a, 0)$, where $a \neq 0$.

Here students have to display some sophistication, involving the definition of a continuous function as well as the knowledge of some trigonometry/algebra, such as finding when arctan is 0, knowledge of how to do some limits, etc. We have spent some classroom time explaining that what they mostly (and erroneously) consider to be continuity (namely that the graph can be drawn without lifting the pencil off of the paper), is not very useful in functions of several variables. We have emphasized on numerous occasions that the definition of continuity of f at a point P is that $\lim_{x\to P} f(x) = f(P)$. Although this is not such an overly complicated definition students do not readily accept it, for the simple reason that it is not the instant product they like – finding limits may be an involved process and may look as a roundabout way to establishing what they like to think of as the "intuitive" concept of continuity.

Here is how one student (the academically weaker of the two high school seniors who took the class) attempts to solve the problem: She first rewrites the function in "her own" way: $\tan^{-1}(1/y^2)$ and then further "simplifies" the function into $x = \tan(1/y^2)\cdot y$, where the last y stands for the function, namely $f(x, y)$. Our experience helps us to foresee this kind of error, so we warn the students in classes that it is safer to use arctan, rather than $\tan^{-1}$ lest they mistake it for the reciprocal, rather than the inverse function. However one has to spend additional time in clearing up this notational confusion.

Almost all widely used calculus textbooks (see e.g. Rogawski (2008) or Stewart (2012)) unwittingly or purposefully use the inverse function notation, lest they appear not to be sufficiently "modern." It is as if we should be writing $square^{-1}$ to denote the square root function, just so we can show it is inverse of the basic quadratic function. The same bad habit is seen with calculator keys that almost exclusively use the inverse function notation. This notation is certainly useful in general discussions about inverse functions, but is a bad notation when names for inverse functions exist (for instance the *arc* functions such as arctan), which in their names have interpretation, if not the meaning of a particular inverse function. It has been noted many times before that good notation is golden and in this case it would result at least in students not confusing the inverse function with the reciprocal of the function.[5] To follow, the same student further draws that part of the square root function graph (in the 2D Cartesian coordinate system) that is in the unit square and claims it is the graph of $x/\arctan(1/y^2)$ – all graphs for her are in two dimensions. Then continues on to say "derivative of " $f'(x, y) =$ substitution $u = \tan^{-1}(1/y^2)$... $d2x\pi/dx = 2\pi$!! [sic]. Then claims that the function is not continuous for $x = 1$ and concludes: "The graph $x/\arctan(1/y^2)$ lies in the positive (x, y) plane and the function is discontinuous at $x = 1$." Yet again, more work must be done simply on functions of more than one variable, in what their domains and codomains are, as well as their graphs, for, even if students have better understanding of functions than the action conception, without firm grasp of domains and codomains or ranges of participating functions would not lead them to correct solution.

Another student evaluates the function at the four corners of the domain and concludes that "the function $F(x, y)$ is continuous throughout its domain." This comes from students' confusion with procedures that were used in class when finding absolute extrema where points on the boundary of the region (or at the corners of polygonal domains) were crucial. This is confirmed by another student finding the "partials" f_x, f_y (which seem to be incorrectly done "partial" anti-derivatives), which he then sets equal to zero and solves the resulting equations. Unquestionably, these confusions point to insufficient time that students had in digesting the material, be it objectively (one semester for all the advanced material) and/or subjectively – insufficient work on the students' part.

One student correctly computes the function values at the corners, then writes

$$f_x = \begin{cases} \frac{-x^2}{2\tan^{-1}(y^2)-\pi}, \\ x^2\pi \end{cases}$$

$$f_y = \begin{cases} -2\frac{1}{2\tan^{-1}(y^2)-\pi}, \\ 2x\pi y \end{cases}$$

He then solves when the parts of these "partials" are equal to 0, conveniently getting $x = 0$ as a solution to the first system and not getting anything ("false") for the solution

[5] Most students are genuinely and thoroughly surprised to learn that $\tan^{-1}$ does not denote the reciprocal of tan. It is an interesting functional equation to find all invertible functions f whose inverse equals its reciprocal.

of the first part of the second expression; however, from the second part of the second system he gets $x\pi y = x, y = 0$ and says "these points are on the boundaries." This is a confusion with the problem of finding the extrema of functions; the conclusion is that "the function is continuous on the points (1,1) and (1,0) and not continuous at (0,0) and (0,1)."

One student attempts to find the double integral, over the square, of the first defining part of the function and, after some incorrect substitutions, etc., arrives at an incorrect value of the integral. Then he states, as a matter of fact that: "It is continuous because it is defined everywhere and limits exist." Double integrals are frequent among the incorrect solutions here.

Some students know the correct definition of continuity, but cannot find the limit of the function at a general point, rather they try to find it in a corner point or two, or invent their own limit rules. Thus one states: "$2x\pi$ is not dependent on y and is a continuous function that will be well-defined within the domain" and then

$$\text{“}\lim_{y\to 0}\frac{x}{\arctan(1/y^2)} = \lim_{y\to 0} f'(x,y) = \frac{1}{\lim_{y\to 0}\operatorname{arcsec}(1/y^2)}.\text{”}$$

One student discusses continuity of the parts separately: "For $f(x,y) = 2x\pi$ this is continuous for all points work from the domain $x=$ all real numbers. For $x/\arctan(1/y^2)$ all points work except for 0 from the domain. Because only way $x/\arctan(1/y^2)$ is not continuous is if the denominator is equal to zero. Only value it will equal zero is if $y = 0$. So all points on the function $f(x,y)$ is continuous except when $y = 0$." This student did receive partial credit, but here comes again the problem of functions, in that these piecewise definitions of functions are often seen as two separate and independent functions, partly because it is more convenient to do so. Breidenbach, Dubinsky, Havks, Nichols (1992) use "piecewise defined" functions to test whether students' understanding of functions is the process one. This is something that needs to be addressed in pre-calculus or the beginning algebra courses.

Another student states: "... a function is not continuous where it is undefined," then goes on to find correctly that the denominator in the fraction part is never 0 and concludes that there are no points of discontinuity. This statement is an innocent regurgitation of classroom and textbook folklore where continuity of a function at a point is still considered even though there is no function (i.e. the function is not defined there) – the point addressed in Dimitric (2004); see furthermore, Rogawski (2008) and Stewart (2012) where the instructing examples and the exercises on "continuity" are almost exclusively such that a student finds zeros of the denominators of rational functions and pronounces that the "functions" are not continuous (since they are not defined) at those points. This is propagated as well into online homework systems such as "Webassign," etc.

Problem 6. Find the volume of the region bounded by the hyperbolic cylinders $xy = 1, xy = 9, xz = 4, xz = 36, yz = 25, yz = 49$. Do only the part in the first octant $(x, y, z \geq 0)$. [Hint: Make an appropriate change of variables and use the change of variables theorem to compute it.]

Solution

Since each of the x, y, z combinations repeat twice in a circular fashion, the substitutions seem to be imposing themselves: $u = xy, v = xz, w = yz$ with boundaries for $u|_1^9$, $v|_4^{36}$, $w|_{25}^{49}$ (*). Multiplying all the substitution equalities leads to a helpful relationship: $xyz = \sqrt{uvw}$ (**). The map $\psi(x, y, z) = (xy, xz, yz) = (u(x, y, z), v(x, y, z), w(x, y, z))$ transforms a "curvy, hyperbolic" x, y, z brick $\mathcal{V}$ into a rectangular u, v, w brick $\mathcal{V}_0$ $[1, 9] \times [4, 36] \times [25, 49]$. For ψ to be a good substitution map, we need to check that it is one-to-one. That follows from the fact that equalities $x_1y_1 = x_2y_2, x_1z_1 = x_2z_2, y_1z_1 = y_2z_2$ imply equalities $x_1 = x_2, y_1 = y_2, z_1 = z_2$. Thus existence of the inverse function $\psi^{-1} = \phi(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$ is guaranteed. In fact, solving the substitution equalities (*) gives $\phi(u, v, w) = (\sqrt{uv/w}, \sqrt{uw/v}, \sqrt{vw/u})$, which is also one-to-one. The Jacobian is then computed to be $\partial(u, v, w)/\partial(x, y, z) = -2xyz = -2\sqrt{uvw}$ ((**) was used). Both ψ, ϕ are C^1 maps, since all the partials exist and they are continuous. Thus conditions for the change of variable formula are satisfied and we find the volume as follows:

$$\iiint_{\mathcal{V}} dxdydz = \iiint_{\mathcal{V}_0} |-2\sqrt{uvw}|^{-1} dudvdw$$

$$= \frac{1}{2}\int_1^9 u^{-1/2}du \int_4^{36} v^{-1/2}dv \int_{25}^{49} w^{-1/2}dw = 64.$$

Setting up the problem and the change of variable theorem details are important and then the integral computation (with the Jacobian). We have directed students to use change of variables to check whether they have learned this very process. Equal points were allotted for substitution, for theorem conditions and for computation of the integral/Jacobian.

There are a number of tripping points here, from routine to more subtle ones. Not knowing/remembering how to compute the Jacobian (a fairly intuitive formula) was a rare miss among the routine ones. The change of variables formula (a substitution formula) is computed by

$$\iiint_{\mathcal{V}} f(x, y, z)dxdydz =$$

$$\iint_{\mathcal{V}_0} f(x(u, v, w), y(u, v, w), z(u, v, w))|Jac(\phi)|dudvdw$$

where $\phi : \mathcal{V}_0 \longrightarrow \mathcal{V}$ is a C^1 and one-to-one map on the interior of $\mathcal{V}_0$ and f is a continuous integrand; here the Jacobian $Jac(\phi) = \partial(x, y, z)/\partial(u, v, w)$ (see e.g. [Rogawski, 2008]). This is somewhat a convoluted formula, for it seems to work "backwards" in that we do not know ϕ, rather we know an easier $\phi^{-1} = \psi$ that is usually suggested by the given shapes of regions of integration. The instructor should spend time to justify the conditions and show that the formula does not work if conditions are not satisfied. He must further spend time in explaining the built in deformation of volume and thus justify the use of the Jacobian multiplier. The shortcuts students make are often utilitarian in nature and one of them is in not bothering with the Jacobian. Thus one student says: "If Jacobian was not needed then $\iiint dV = \int_1^9 \int_4^{36} \int_{25}^{49} dV = \int_1^9 \int_4^{36} \int_{25}^{49} dzdydx$, and so on, and gets the volume 6144." Another student proceeds to do the same "formula" without even mentioning the Jacobian.

A technicality should also be explained, namely that the Jacobians of inverse functions are reciprocal to each other; some students had missed this fact. Emphasis on this however should be made already in the first calculus courses where derivatives of inverse functions are discussed.

One of the students who missed including the Jacobian in the formula (but had the correct substitution) went further in simply calculating the volume of the straight box $\mathcal{V}_0$ by multiplying its dimensions (which she found correctly from the substitution).

Another student seems to know the general procedure and he correctly chooses the substitution $u = xy, v = xz, w = yz$, but when he tries to solve for x, y, z he is not successful. He puts a big " ?" and writes "no idea. If We knew the variables, We can figure it out, but we can't come up with the variables." Thus solving these algebraic systems is important and it is almost never done in lower division algebra classes, or is mentioned only as an afterthought.

One student begins with $\int_{25}^{49} \int_4^{36} \int_1^9 r dr d\theta dz$ and proceeds with $r^2|_1^9$ etc., to get something like 30720 at the end. Here, there is confounding with specialized procedures and a number of exercises done on the topic of special substitutions, such as cylindrical and spherical coordinates (with their own Jacobians); subtleties are missing though. This again means more exercise time should be devoted both in class and by the students so as to definitely discern the confusing notions and make them clearer in students' minds.

One problem (not widespread) was not quite knowing the integral formula for the volume and it is tempting for some students to put xyz or uvw as the integrand in computing the volume ("length times width times height").

There was one "silent solution" (namely, no explanation, just a formula or two) by a student who writes:

$$\int_1^3 \int_5^7 \int_2^6 z dz dy dx$$

and goes on to integrate this convenient integral with separated variables to get 64 as the solution. One can attribute this solution to a sharp talent to see through promptly, but, unfortunately this was not the case here. It is not an infrequent luck at numerology that almost with no exception accompanies problems that have suspiciously nice, round numbers (that, in this case also seem to be squares too). One can try to justify this solution by a substitution $u^2 = xy, w^2 = xz, v^2 = yz$; however since the one-to-one requirement is needed, the substitution is $u = \sqrt{xy}, w = \sqrt{xz}, v = \sqrt{yz}$, to get the correct limits for u, v, w. When the Jacobian is found: $\partial(u, v, w)/\partial(x, y, z) = 1/4$ and its reciprocal is 4, which will then give the correct result after evaluating

$$\int_1^3 \int_5^7 \int_2^6 |Jac| \; dwdvdu$$

None of this is to be found in the student's solution who had a lucky confounding event with a formula for finding volume under a surface z (which would be a double integral). Two more students start taking square roots of the given numbers, for the temptation is too great not to do that when perfect squares of integers are given.

It would help greatly teaching (and learning) the integration method of substitution in several variables, if this method were thoroughly taught in the case of one variable (usually in calculus I). The practice of applying the method of substitution in one variable almost exclusively omits checking that the substitutions are of the right kind (namely that they are C^1 and one-to-one, etc.). Also, when doing substitutions, students retain the old limits since they want to go back to the old variable, once they eventually solve the indefinite integral via the substituting variable. Thus an opportunity is missed to introduce (implicitly) the "Jacobian" and all the other features that would become rather useful when the case of several variables is taken up. We cannot overemphasize this point.

Problem 7. Find the volume of the smaller region bounded by the sphere $x^2 + y^2 + z^2 = a^2$ (a is a given constant) and the plane $z = b$, where constant b is such that $a > b > 0$, using either cylindrical or spherical coordinates.

Solution

Setting up the volume integral earned one third of the assigned points: For cylindrical coordinates we have $x = r\cos\theta, y = r\sin\theta, z = z$ and then $V = \iiint_{\mathcal{V}} dxdydz = \iiint_{\mathcal{V}_0} rdrd\theta dz$ and for the spherical coordinates $x = \rho\cos\theta\sin\phi, y = \rho\sin\theta\sin\phi, z = \rho\cos\phi$ one has $V = \iiint_{\mathcal{V}_0} \rho^2 \sin\phi d\rho d\theta d\phi$. These are substitution formulas with Jacobians already incorporated for these specific substitutions. Textbooks usually discuss them separately and our lecture time is also partly devoted to the same. If students cannot remember these formulas they have to use the general substitution procedures and compute the Jacobians, but they certainly have to know the spherical and cylindrical substitutions.

The next stage is to determine $\mathcal{V}_0$, namely to determine integration limits for the new variables. For this, well-drawn geometric diagrams are useful, and simple appeal to the Pythagoras' theorem will give limits for cylindrical coordinates: $0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{a^2 - b^2}$ and $b \le z \le \sqrt{a^2 - r^2}$. For spherical coordinates, the limits are as follows: $0 \le \theta \le 2\pi$, $0 \le \phi \le \arccos(b/a)$, and $a \le \rho \le b/\cos\phi$. Determining these limits was worth 1/3 of total points and so was the final part of actually computing the integrals (and thus the volume): In cylindrical form

$$V = \int_0^{2\pi} d\theta \int_0^{\sqrt{a^2-b^2}} rdr \int_b^{\sqrt{a^2-r^2}} dz =$$

$$= 2\pi \int_0^{\sqrt{a^2-b^2}} rdr(\sqrt{a^2 - r^2} - b) =$$

$$= 2\pi \left(\int_0^{\sqrt{a^2-b^2}} r\sqrt{a^2-r^2}dr - b\int_0^{\sqrt{a^2-b^2}} rdr \right).$$

The first integral in parentheses is done by substitution $t = a^2 - r^2$ to get the volume $V = \pi(b^3 + 2a^3 - 3ba^2)/3$. Similarly, in spherical coordinates one has:

$$\int_0^{2\pi} d\theta \int_0^{\arccos(b/a)} d\phi \int_{b/\cos\phi}^{a} \rho^2 \sin\phi\, d\rho =$$

$$= 2\pi \int_0^{\arccos(b/a)} \left((a^3/3)\sin\phi - (b^3 \sin\phi)/(3\cos^3\phi)\right) d\phi$$

The second part of the last integral is done by substitution $t = \cos\phi$ and the volume again is $V = \pi(a-b)^2(2a+b)/3$.

This exercise could be worked out using elementary calculations, as long as the student knows formulas for volumes of balls, cones, etc. The essence of the computation is the method of cross sections, which is often taught in Calculus I courses. In the context of Calculus III, the problem is meant to check whether students have the skills to work out the volume via triple integrals, using specific substitutions, namely cylindrical or spherical coordinates. We specified these two methods perhaps to aid students in deciding how to start to solve, in view of the fact that either coordinate substitution would require about the same amount of work to find the volume. One student thought that one method was more advantageous than the other, so he was pondering which one to choose before he proceeded. Thus he thinks aloud: "Region is bounded by a sphere so we will use spherical coordinates. The Jacobian or multiplier in spherical coordinates is $\rho^2 \sin\rho$ [sic]" and then he lists spherical coordinates. But then crosses it all out and continues: "Actually since we are bounded by $z = b$, we will use cylindrical coordinates" and lists polar coordinates for x, y, but not z and writes that the Jacobian $= r$. Unfortunately he does not find the limits of integration correctly.

Here the variety of errors occur at every potential trouble spot. The instructor should spend more time in elaborating on the simple volume formula $V = \iiint_{\mathcal{V}} dxdydz$ and point out that the integrand is 1, because the volume is already encoded in the region of integration $\mathcal{V}$ whose volume is being calculated. Here students use all sorts of expressions for the integrand in this volume formula, most frequently $x^2 + y^2 + z^2$ or $\sqrt{a^2 - x^2 - y^2}$. Other "integrands" are invented by substituting either cylindrical or spherical coordinates into the sphere equation so that the integrands turn out to be $r^2 + z^2$, $\sqrt{a^2 - r^2}$, ρ^2 or $\rho^2 - a^2$. These errors come partly from students confounding procedures they used to find areas under curves (integrand is the function formula), or volumes under surfaces (integrand is the surface formula with the double integral as a device). Thus these should be contrasted with the volume formula when starting work with triple integrals.

With a number of errors along the way, students struggle through, coming up with some answers that are more often than not polynomials in a, b (including numerical constants). However, these polynomials range from being non-homogeneous, to being homogeneous of degrees 2, 4, 5 and so on. Teaching students basics of dimensional analysis should be done early on in (pre)algebra and introductory science courses, so that students are comfortable with various dimensions or units for that matter. It would then go a long way in recognizing problems with their "solutions" simply based on the incorrect dimension they come up with for, in this case the volume.

Determining limits of integration is definitely a hurdle and while no problem is found (with a couple of exceptions) in finding limits $0 \leq \theta \leq 2\pi$, since most examples done in the textbook and classroom have those exact limits, determining limits for other variables is hard for many, not the least because students fail to make a good geometric drawing, but also because they are not yet good at understanding that limits of one variable need not be constant, i.e. they may depend on (an)other variable(s). Thus, in students' papers, r is found to have limits from 0 to a, or b to $a - r$, or 0 to $a - b$ and z has limits 0 to a, or 0 to b, or b to $a - r$ or 0 to $\sqrt{a^2 - r}$, b to $a^2 + b^2$, or $a - b$ to a; ϕ varies from a to b, 0 to π, 0 to 2π.

The cursory role of 3D geometry in US school system is well-known. Even the best of our students competing at math Olympiads need to be given extra preparation on this aspect of geometry. With that in mind, an instructor has to spend some time in three dimensional calculus, drawing 3D pictures and manipulating them, through finding relationships, using similarities of triangles, and all. Computer aid in 3D imaging is pretty, but perhaps not very helpful, for students still would not be able to draw figures and subfigures, find the limits and relationships among various elements in their 3D figures, unless they made repeated attempts at drawing them by hand. With slight exaggeration, one should require simple art classes as a requirement for taking a tad more advanced math class.

One student has a definite idea to solve the problem as a double integral, namely to play with volumes under the spherical cap and the flat cap $z = b$, all using cylindrical coordinates. But she says "$x^2 + y^2 + z^2 = a^2$ projection onto x-y plane $= x^2 + y^2 = a^2$," rather than projecting the circle at level $z = b$.

3 A brief lesson plan

One of the many uses of our analysis is in making lesson plans. Let us assume that the lesson's topic is continuity, one of the most important and most neglected topics in calculus teaching practice. We have feedback here attached to Problem 5 and we can incorporate it into the lesson plan as follows:

Make sure that, by this time, students are well-familiar with the exact notion of function, as an ordered triple – domain, codomain and the assignment rule(s). This should be done at early pre-calculus, but in any case it is a must to review/teach at the beginning of the course. Give students some "trick questions" such as ask them to decide which of the following are functions and which are not: $f : A \longrightarrow B$, a) $f(x) = \sqrt{x}$, $A = \mathbb{R}$, $B = \mathbb{R}$, b) $f(x) = \sqrt{x}$, $A = \mathbb{R}^+$, $B = \mathbb{R}$, c) $f(x) = -\sqrt{x}$, $A = \mathbb{R}^+$, $B = \mathbb{R}^+$, d) $f(x) = -\sqrt{x}$, $A = \mathbb{R}^+$, $B = \mathbb{R}^-$, e) $f(x) = \sqrt{x}$, $A = \mathbb{R}^+$, $B = \mathbb{R}^-$. If complex numbers are part of the course, tailor this accordingly.

By this time enough exercises must have been done for students to feel comfortable with the notion of a limit and the routine of finding various limits, as well as basic operations with limits. This should already be done in the first calculus course. Test them on this with the following limits:

$$\lim_{x\to 0}\frac{\sin x}{x}, \lim_{x\to 0}\frac{\cos x}{x}, \lim_{x\to\infty}\frac{\sin x}{x}.$$

It is likely that the students will indiscriminately invoke "L'Hospital's rule;" remind them what conditions must be satisfied with this rule.

Now, use some limits, in higher dimensions, preferably those you will use to check whether functions are continuous or not. First some that separate variables:

$$\lim_{(x,y)\to(0,\pi/2)}\frac{\sin x\cos y}{xy},$$

And some that are not obviously variable separating $\lim_{(x,y)\to(a,b)}[xy]$ (the integer part function). Discussion of cases when ab is and is not an integer may be instructive.

One should also do some piecewise defined function examples, for instance the example used in Problem no. 5. Or the following one:

$$f(x,y)=\begin{cases}\frac{e^x-1}{x}\frac{\sin y}{y}, & \text{if } x\neq 0 \text{ and } y\neq 0,\\ \frac{\sin y}{y}, & \text{if } x=0 \text{ and } y\neq 0,\\ \frac{e^x-1}{x}, & \text{if } x\neq 0 \text{ and } y=0,\\ 1, & \text{if } x=0 \text{ and } y=0.\end{cases}$$

Define continuity of function f at a point $P\in A$ of the domain of the function $f: A\longrightarrow B$ if $\lim_{x\to P} f(x)=f(P)$. Emphasize that a question whether a "function" is continuous at a point outside of its domain is meaningless, since there is no function at such a point. Define what is meant to say that f is discontinuous at a point. Define (global) continuity of f on all of its domain.

Now do first some simple examples of one variable, then pick up the aforementioned examples from limit exercises and add new ones. Make sure the examples are graduated, from simple to more complicated.

If possible, visualize the functions and their limits, either by hand, or using graphing software.

4 Conclusions

The problems with understanding functions exist from early mathematics education and are dragged along throughout, including higher level or even graduate level courses. We believe that the foundation for these problems begin before functions are introduced, at the level of basic notions such as sets and ordered pairs (or triples, etc).

Graphing of functions is introduced simultaneously with the Cartesian plane, without much ado about abstract formation of the Cartesian product of (two) sets. Yet, it took many generations in human development before the idea of Cartesian product matured. Not only should considerable time be devoted to "playing" with coordinates, but a physical drawings[6] of coordinates (and at least simple objects) in two and three dimensional coordinate systems should be practiced thoroughly until the students feel this as their "second nature." Once this ground work is made for the dimensions higher than one, one can introduce a name for a subset of a Cartesian product $A \times B$ – call a subset $\rho \subseteq A \times B$ a *relation from* A *to* B, that can be denoted by $\rho : A \longrightarrow B$.[7] Some simple operations and drawings with relations should be introduced as a practice.

Then, time comes for a definition of a function $f : A \longrightarrow B$ as a relation with properties that, for every $a \in A$ there is only one $b \in B$ with $(a, b) \in f$ or $b = f(a)$. A should be called the *domain* of f and B the *codomain* and the *range* would be introduced as the set of all $b \in B$ such that $(a, b) \in f$, for all $a \in A$. An important point should be made, namely that the function consists of these three parts $f : A \longrightarrow B$ and that each of them as important as any other in their own right and that two functions $f : A \longrightarrow B$ and $g : C \longrightarrow D$ are equal if and only if all three components are equal: $A = C, B = D, f = g$. A good number of exercises that vary some or all parts of a function and discussion of resulting differences should be made until students adopt this formalism to a high degree.

Then the stage is set for metaphors. A dynamic metaphor that we use is that of a processor (a "grinding machine") as sketched in the following diagram.

One can represent this as making of bread sticks (placed in codomain C) after dough in domain A is processed by pasta machine f.[8] Here, it is very important to impart to students that a function $f : A \longrightarrow B$ is the whole assembly line consisting of A, f and B, rather than just rely on a vague hope that the students would see functions either as action or process conception. Importance of domain may be imparted for instance by replacing domain with dough with domain with iron bars that cannot be processed by the processor, or codomain that is a capillary dish that cannot accommodate the processed dough... Varying all the parts of the function would show a student that each part is in fact equally important as a constituent of what is meant by a function. Composition may be explained in a similar manner, for a packing machine $g : C \longrightarrow D$ may be the follow up "factory" that will work, only if the final product of f is in the domain of g; thus g may be able to process sticks only of certain length range, not of other dimensions, yet again underlining importance of domain and codomain (or

[6] We like slightly exaggerating and saying that "what does not go through your hand does not get to your head..."

[7] Our own experience and feedback from several educational systems show that even very young students are not intimidated by this abstraction level, which may run contrary to prejudices in existence regarding mathematics education.

[8] One can vary the description dependent on the fashionable language of the day and place – veggie burgers in California or a sausage machine in Texas...

range) – the diagram has some sticks separated from the other in B, those in domain C of g and those that are not in C.

There are other metaphors one can use, but a very important point to remember is that we cannot have correct metaphors if we do not give students correct definitions of notions we teach them and insist that they adopt these definitions, at least at the memorization level (first step in Bloom's taxonomy). Otherwise we would end up in finding metaphors, of metaphors, thus giving students an impression that the notions are negotiable moving targets. While mathematical formalism is not equivalent to what mathematics is, it is nonetheless an essential constituent of mathematics. It is most crucially visible in at least one practical development, namely computerization of human knowledge and experience which could not take place without strict mathematical formalism. Development of ideas is mostly a meandering process, not infrequently through bouncing off of the extreme views or practices. Thus an increased formalization of mathematics in the 20th century led to sometimes indiscriminate transplantation of formalism into teaching practice, which in turn led to overreaction in other direction, in denial of all the formalism where the notions of function and its properties was degraded to pre-Eulerian levels of understanding of what a function is. In a pivotal exposition of the pre-1930's history of the development of function notion, Luzin (1935; see also Yushkevich (1966, 1977)) highlights subtleties of the function concept inclusive with the then developing views on classification of functions, intuitionistic developments, etc. Bourbaki formalized the function concept in 1939 and then category theory brought about a great generality and the accompanying simplification of the concept. Adoption of these developments in school curriculum has been checkered at best and it would be an interesting topic of research to look into the reasons why function concept is not adopted and properly taught in schools (and colleges).

If this huge leap of students remembering correct definition of function gets achieved, other constructs with functions would become much easier, one of them being definition of continuity.

There are some drawbacks in analyzing or publishing an analysis of this kind. For instance, it is a fact that analyzing tests in detail is very time consuming, and even tedious. In addition, there are instances where this powerful educational tool may be misused: Fearing negative feedback from students or administration alike, some instructors would attempt to avoid potential trouble spots (as found through analyses of this kind) by considerably diluting the level of content of the courses both in the classroom as well as in levels of examinations.

As with every powerful weapon, dangers of its misuse always exist. Still, the benefits here outweigh the dangers. Based on the work students have shown on tests one can extract a good amount of useful information that can be utilized as follows (beyond the obvious and customary assessment value of tests):

1. To find out about students' specific difficulties in learning specific topics. In this case we learn about problems students are likely to have with analytic geometry, linguistic and phonetic hurdles in adopting concepts, difficulties with "higher"

dimensions, differentiation and integration hurdles, (mis)understanding of the concept of function, and continuity, especially when mapping one higher dimension to another, in graphical visualization and (in)ability to use it intelligently. In (lack of) algebraic skills and ability to perform simple mental calculations. In problems in doing integration, drawing graphs in 2D, dimensional analysis, etc.

2 To learn about points of emphasis and amount of repetition the instructor should devote to specific topics.

3. To help instructors devise appropriate lesson plans, that would preempt usual difficulties, misconceptions and obstacles that exist before and during instruction on any specific topic.

4. To get a specific content for making diagnostic tests for any particular class.

5. To learn about students' gaps in their knowledge in material they should have learned by the time they come to the specific subject (prerequisites).

6. To get very concrete snapshots of progression of teaching and learning, in fairly real time (since midterm exams are part of the analysis).

5 Topics for further exploration

a) Compare performance of students in mathematics classes dependent on whether they had previously taken or not a drawing class, with perspective drawing in particular. A default expectation is that the students who took such a class would perform considerably better in multi-dimensional calculus or in other courses where more than one dimension is involved. A worthy cause would also be to quantify the difference in math abilities of the two groups.

b) Compare performance of students in mathematics classes dependent on whether they had previously taken or not a mathematics culture class where considerable time is devoted to issues of coordinate systems (dimensions, ordered tuplets, relations and so on), variables, constants, sets and operations with them. Our belief is that students who took such a class would be considerably ahead of those who did not take that kind of class.

c) Develop a calculus complex (consisting of three segments, for instance) in a way that earlier segments set up the stage for easier sailing into the subsequent segments, by way of generality of introducing concepts and by thoughtful process of choosing expandable metaphors.

References

[1] Anderson, L.W. & Krathwohl, D.R. & Airasian, P.W. & Cruikshank, K.A. & Mayer, R.E. & Pintrich, P.R. & Raths, J. & Wittrock, M.C. (Eds.) (2001): A

taxonomy for learning, teaching, and assessing. A revision of Bloom's taxonomy of educational objectives; Addison Wesley Longman, Inc.

[2] Bloom, B.S. (Ed.) (1956). Taxonomy of Educational Objectives: The Classification of Educational Goals, pp.201–207; Susan Fauer Company, Inc.

[3] Breindenbach, Daniel & Dubinsky, Ed & Hawks, Julie & Nichols, Devilyna (1992): Development of the process conception of function. *Educational studies in mathematics* **23**, 247–285.

[4] Dimitric, R.M. (2001): Using less calculus in teaching calculus. A historical approach. *Mathematics Magazine* **74**, No.3 (June), 201-211.

[5] Dimitric, R.M. (2003): Components of Successful Education, *Tenth International Congress of Mathematics Education (ICME10)*, TSG 27, 2004; *The Teaching of Mathematics*, **6**(2003), No.2, 69–80 7–13.

[6] Dimitric, R.M. (2004): Is $1/x$ continuous at 0?, arXiv: 1210.2939v1;mathGM]

[7] Dimitric, R.M. (2009): Student placement in calculus courses. A case study. Diagnostic testing: canned vs. home-made. A case study. *Eleventh International Congress of Mathematics Education (ICME11)*, TSG , 2009; *The Teaching of Mathematics*, **12**(2009), No. 2, 83- 102.

[8] Dimitric, R.M. (2012a): How to make a course specific diagnostic test in the context of Bloom's taxonomy. In preparation.

[9] Dimitric, R.M. (2012b): On complexity distance and its applications in education. In preparation.

[10] Dimitric, R.M. (2013): Feedback from students' tests as a tool in teaching. A case study. arXiv:1305.5173v1 [math.HO] 21 May 2013.

[11] Dubinski, E. & Schoenfeld, A.H. & Kaput, J.J. (Eds) (1994): *Research in Collegiate Mathematic Education, I.* Providence, RI: American Mathematical Society.

[12] Dubinski, E. & Harel, G. (1992): The nature of the process conception of function. In G. Harel & E. Dubinsky (Eds.). The concept of function: Aspects of epistemology and pedagogy. *MAA Notes* **25**, 85–106.

[13] Horscovics, N. (1982): Problems related to the understanding of functions, in *Proceedings of the Workshop on Functions*, Organized by the Foundation for Curriculum Development, Enschede.

[14] Leinhardt, G. & Zaslavsky, O. & Stein, M. (1990): Functions, graphs and graphing: Tasks, learning and teaching, *Review of Educational Research* **60**(1), 1-64.

[15] Luzin, N.N. (1935): Function, The Great Soviet Encyclopaedia **1959**, 314-334 Translated by Abe Shenitzer: Function: Part I *Amer. Math Monthly* **105**, No. 1 (Jan. 1998), 59–67 and No. 3 (Mar. 1998), 263–270.

[16] Oehrtman, Michael & Carlson, Marylin & Thomson, Patrick W. (2008): Foundational reasoning abilities that promote coherence in students' function understanding, In: Making the Connection: Research and Teaching in Undergraduate Mathematics, SIGMA and RUME, *MAA Notes*, Vol. 75, 27–41.

[17] Piaget, J. & Grize, J.-B. & Szeminska, A. & Bang, V. (1977): *Epistemology and Psychology of Functions* (J. Castellanos and V. Anderson, trans.), Reidel, Dordrecht.

[18] Rogawski, J. (2008): *Calculus. Early Transcendentals*, 3rd printing. New York: W.H. Freeman and Company.

[19] Selden, J. & Mason, A. & Selden, A. (1989): Can average calculus students solve nonroutine problems? *Journal of Mathematical Behavior* **8**, 45–50.

[20] Selden, J. & Selden, A. & Mason, A. (1994): Even good calculus students can't solve nonroutine problems. IN: *Research issues in undergraduate mathematics learning* (Vol. 33, pp.19-26). Washington, D.C.: Mathematical Association of America.

[21] Selden, Annie & Selden, John & Hauk, Shandy & Mason, Alice(2010): Why Can't Calculus Students Access their Knowledge to Solve Nonroutine Problems? *CBMS Issues in Mathematics Education*, **8**, 128–153.

[22] Stewart, J. (2012): *Single Variable Calculus. Early Transcendentals.* 7th ed. Australia: Brooks/Cole, Cengage Learning.

[23] Sfard, A. (1987): Two conceptions of mathematical notions: Operational and structural, in J.C. Bergeron, N. Herscovics and C. Kieran (eds.), *Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education, III*, Université, Montreal, pp 162-169.

[24] Tall, D. & Vinner, S. (1981): Concept images and concept definitions in mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics* **12**, 151–169.

[25] Thomson, Patrick W. (1994): Students, functions, and the undergradute curriculum. In Dubinsky et al (Issues in Mathematics Education Vol 4, pp 21-24).

[26] Vinner, S. (1989): Images and definition for the concept of function, *Journal for Research in Mathematics Education* **20**, 356–366.

[27] Vinner, S. (1992): The function concept as a prototype for problems in mathematics learning, in Dubinsky & Harel (1992), 195–214.

[28] Vinner, S. & Dreyfus, T. (1989): Images and definitions for the concept of function, *Journal of Research in Mathematics Education* **20**, 356–366.

[29] Yushkevich, A.P. (1966): The concept of function up to the middle of the nineteenth century, *Istor.-Mat. Issled.*, **17**, 123–150.

[30] Yushkevich, A.P. (1977): The concept of function up to the middle of the nineteenth century, *Arch. History of Exact Sci.*, **16** 37–85.