The Structure of Creativity of Aha!Moments in Mathematics (part 1)

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Abstract: The paper presents the classification of Aha!Moments in mathematics obtained through the analysis of collected insights of students and inventors conducted in Czarnocha and Baker (2018). It uses the definition of Aha!Moment abstracted from Koestler's bisociation theory formulated in his Act of Creation (1964). The classification scheme finds three types of Aha!Moments in the collection, Mild, Normal and Strong decided on the basis of the nature of connections made during the insight. The presentation compares this classification with the classification of connections obtained by AI based bisociative search engine BISON and finds a surprising similarity in their corresponding structures. The similarity leads to the basic question of the difference between human and computer creativity.

INTRODUCTION

Most of us are familiar with the surprisingly sudden and exceedingly pleasant moments in thinking when the issue we dwelled upon for some time suddenly and unexpectedly becomes crystal clear and we either are overwhelmed by realization of its significance or/and by the sudden boost in self-confidence. These are Aha!Moments and although several of their descriptions are in research literature (Barnes, 2000; Palatnik and Koichu, 2015; Yoon, 2012; Liljedahl, 2004), their systematic study in the practice of mathematics education has not as yet been undertaken. That job was initiated and developed into the research program by the members of the Teaching-Research Team (TRTeam) of the Bronx, who have coordinated the bisociation theory of Arthur Koestler with the classroom events (Prabhu, 2016; Stoppel and Czarnocha, 2020). The definition of Aha!Moment as bisociation extracted from Koestler (1964) work states: "*The bisociation act is the spontaneous leap of insight which connects previously unconnected matrices of experience"* (Koestler, 1964, p.45) through the discovery of a "hidden analogy". In passing let's note that Koestler's bisociation became the inspiration for the new artificial intelligence (AI) domain of computational creativity whose aim is "to model, simulate or replicate creativity with a computer" (Boden, 2004; Dubitsky et al., 2012) as well as for the design of the bisociative search engine for the two large sets of data we discuss shortly below.

Thus the creativity of Aha!Moment is the process of connecting two previously unconnected matrices of experience while building a conceptual bridge between them – the creative object. Prabhu and Czarnocha (2014) proposed bisociation as the new definition of creativity in mathematics education. Taking into account that Aha!Moments are familiar to THE population in large, the creativity of Aha!Moment can lead the process of democratization of research and practice in mathematics classrooms. The TR Team of the Bronx has analyzed, on the basis of the Koestler's definition, processes of facilitation of Aha!Moments as well as their depth of knowledge (DoK) assessment Czarnocha and Baker (2018), which is understood as the increase of understanding reached during the Aha!Moment insight. The methodology of DoK assessment allows us a glimpse into the structure of creativity of the "act of creation", that is of Aha!Moment (also called Eureka experience).

METHODOLOGY

The origins of the methodology are rooted in the teaching experiment Problem Solving in Remedial Arithmetic: Jumpstart to Reform in 2010 supported by a CUNY CIRG 7 grant (Collaborative Community College Incentive Research Grant #7). Vrunda Prabhu, a member of the TR Team of the Bronx, observed many Aha!Moments in her remedial algebra experimental classroom at Bronx CC, which she coordinated with the Koestler's theory putting first steps along the path of investigations that took us to the present multifaceted inquiry into creativity of mathematical insight. The collection of Aha!Moments which served as the data for analysis was obtained along several routes: (1) during the teaching experiment under Title V at Hostos Community College (CC) which involved peer leaders of the two experimental classrooms as student-researchers. (The full collection will be provided during the presentation.) The title V teaching experiment was conducted in two courses one section of Arithmetic/Elementary Algebra course and another section of Intermediate Algebra course; (2) through the Hunt for Aha!Moments campaign organized by Mathematics Teaching-Research Journal on line (MTRJ at hostos.cuny.edu/mtrj, a Scopus indexed journal) and (3) through the Creativity of Aha!Moment CUNY conference at Hostos CC in 2018. In addition several Aha!Moments were found in professional literature mentioned above.

The central issue for us has been the precise description of mathematical situation within which the insight has taken place; in order to investigate the characteristical affective impact of the insight we need to have also the description of accompanying emotional states. The classification of the depth of knowledge of each Aha!Moment has been done from the point of view of the nature of connections developed during the insight by the learner.

As implied above we are interested in the process of genesis of connections between, as well as the nature of created objects. The genesis of connections between unconnected frames of reference is one of the defining quality of Koestler's presentation of bisociation noted above. As described in Czarnocha (2012) the depth of increase of knowledge during Aha!Moment insight might be measured by the stages of the Triad of concept development of Piaget and Garcia (1989). The collection of Aha!Moments taken for the design of classification (Appendix in Czarnocha and Baker, 2020) is given by the insights observed in classes of mathematics, however, as we note below the historical Aha!Moments such as Gutenberg's (Koestler, 1964) or Einstein's (Rothenberg, 1979) neatly fit the proposed classification scheme. (The names of Aha!Moments quoted from the Appendix were established on the basis of their context and/or name of the creator. Example: Calculus Aha!Moment, Einstein Happiest Thought Aha!Moment, etc.)

We classify (1) Mild bisociation as one that involves only one conceptual connection or analogy. This includes the process in which discovering a hidden analogy involves employing elementary conceptions or patterns that are seen as relevant (Calculus Aha!Moment, Kim Aha!Moment and Physics Aha!Moment). The (2) Normal level of bisociation is the insight in which several elementary concepts are coordinated to form a functional whole. By functional whole we mean a construction of the interiorized schema, which leads to the solution to the problem or to a higher level of understanding, usually at the Inter level of the development (Fir Tree Aha!Moment and What is a Vector? Aha!Moment). We classify a (3) Strong bisociation as one that has at least two steps and/or two cycles in the progress of understanding, usually reaching Trans level of the schema development (Einstein "Happiest Thought" Aha!Moment to Factorize).

The proposed classification has been carried out on the Aha!Moments originating in mathematical classrooms; it's interesting that those of historical insights such as the Gutenberg insight (Mild) leading to printing press or that of Einstein happiest thought (Strong) leading to one of the founding principles of General Relativity Theory neatly fall into it.

BISON SEARCH ENGINE

BISON (Bisociative Networks for Creative Information Discovery) is a second generation search engine, whose bisociative methods of detecting similar patterns across different domains "promise tremendous potential for the discovery of the new insights" (Berthold, 2012;p.1). Below we point to a surprisingly clear relationship between classification of Aha!Moments insights obtained through the introductory Depth of Knowledge (DoK) described above and the classification of bisociative structures obtained by the BISON project. Berthold (2012) and Kötter & Berthold (2012) present three types of bisociative connection within Bisociative Knowledge Discovery (BKD) framework brought forth by the computer creativity approach, indicating at the same time that the typology of bisociative connections is an open field at present: (1) a single Bridging Concept, (2) Bridging Graphs and (3) Structural Similarity Bridging between two domains. This similarity is still more surprising if we note the following differences between AI definition of bisociation and that of Koestler.

Berthold (2012, p.2) asserts that informally bisociation can be seen as "(sets of) concepts that bridge two otherwise not – or only very sparsely – connected domains". Comparing this with Koestler's definition, we note the absence of the phrase "spontaneous leap of insight". Instead of "insight" of Koestler we have its product given by Berthold: the set of concepts that connects or bridges previously unconnected matrices/domains; by Berthold we have also lost the "spontaneous leap" in Koestler's definition, so that bisociative knowledge discovery employs only the mechanical, that is no spontaneous aspects inherent in connecting two different structures. This method is the basis of the AI understanding of bisociation–computers do not seem to have Aha!Moments.

COMPARISON OF TWO STRUCTURES

Type Mild of DoK. Calculus Aha!Moment single concept connection

During my Calculus 1, the teacher gave us an example to solve: $\lim_{X\to 0} \frac{\sqrt{1+X}}{X} - \frac{\sqrt{1-X}}{X} =$

1 I verify if the limit is defined when X approaching to 0. It is not.

2 I asked myself "how can I do and find a way for this limit can be define?"

3 I remember in my previews class that when the teacher gave us a rational fraction to solve, he said that we must eliminate the radical in the denominator by multiplication with the conjugate. But for this equation we don't have radical in the denominator but in the numerator.

4 I'm a little bit struggling. What can I do?

5 I was looking at the limit and said to myself why not apply the same rule for the fraction when we have the radical in the denominator.

We see here that the student doesn't see the unifying power of the method 'multiplying by a conjugate', so that different positions of the similar expression break the situation in students' mind into two different cases (see figure 1). The insight in line 5 "why not to apply the same rule" indicates the beginning of connecting the two domains, search for the limit of the function with rationalization of fractions by a single concept of algebraic conjugation.

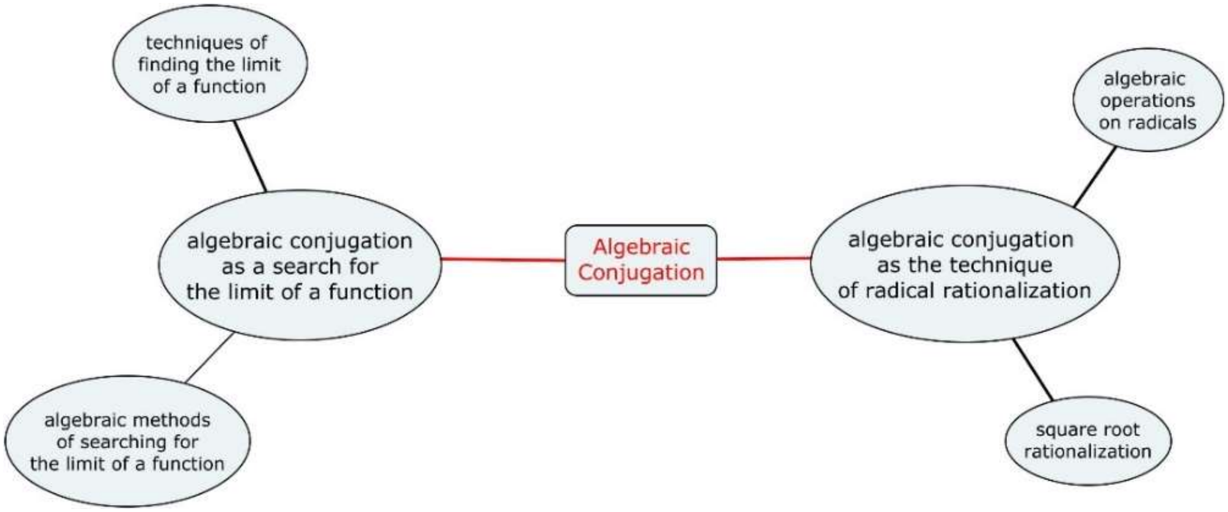

Figure 1. Algebraic conjugation as the single bridging concept (Stoppel & Czarnocha, 2020, p. 20)

The type Mild corresponds to the structure discovered by BISON search engine called the bridging by a single concept connection, see figure 2.

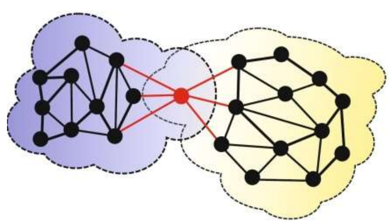

Figure 2. Bridging by a single concept (Kötter & Berthold, 2012, p. 45)

Type Normal of DoK. Aha!Moment Fir Tree

> I had a tremendous Aha! Moment. I just realized that the formula I got from the patterns [n(n+1)] was a factorized expression and if I multiply it "n(n+1)= units square" I would have something like an algebraic expression exactly a trinomial expression that can be factorized as well, and it equals real numbers for example: $n(n+1)=12$ $n^2+n=12$ $n^2+n-12=0$ $n^2+n-20=0$ $n^2+n-56=0$ n^2+n-90 $(n+4)(n-3)=0$ $(n+5)(n-4)=0$ $(n+8)(n-7)=0$ $(n+10)(n-9)$ And when it comes to $n^2+n-274$ it cannot be factorized.

The student makes the connection between the algebraic expression she found and one of the processes of solving quadratic equation, by factorization of the quadratic trinomial. Factorization is the common domain joining the other two.

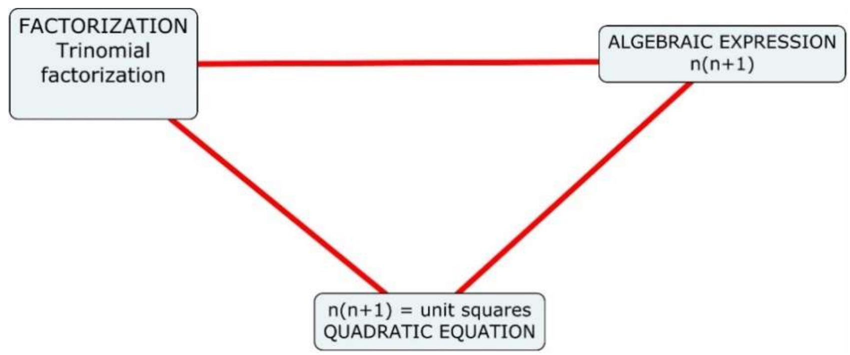

Figure 3. Bridging graph of three algebraic domains (Stoppel & Czarnocha, 2020, p. 22)

A bridging graph is given by connections between subgraphs of different domains. Connections might be given by direct connections between wedges of different graphs, see figure 4 (b).

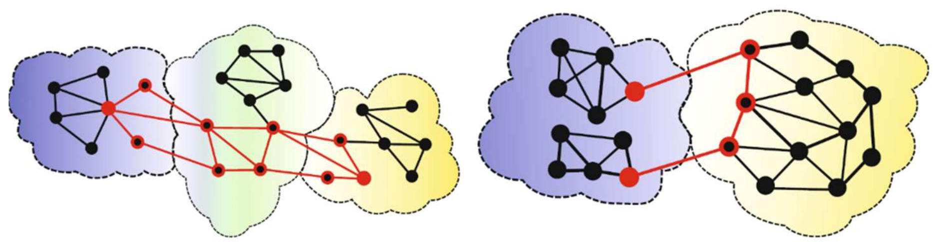

Figure 4. Bridging graphs by concepts (Berthold, 2012, p. 4)

The bridging might also be given by a relatively dense graph in connection with a network-bases representation as represented in figure 4 (a) (Berthold, 2012). Both Kotter and Berthold (2012) and Dubitsky et al (2012) suggest that Archimedes' Eureka moment can be understood through such a bridging graph connection of type (b) in figure 4.

Type Strong of DoK. The "Happiest Thought" of Einstein Aha!Moment.

> Just as in the case where an electric field is produced by electromagnetic induction, the gravitational field similarly has only a relative existence. *Thus, for an observer in free fall from the roof of a house there exists, during his fall no gravitational field*-at least not in his immediate vicinity. If the observer releases any objects, they will remain, relative to him, in a state of rest, or in a state of uniform motion, independent of their particular chemical and physical nature. (In this consideration one must naturally neglect air resistance.) The observer is therefore justified in calling its state "rest". [Einstein's italics]

The analogy is "hidden" within the phrase "Just as in the case…" To explicate the hidden analogy, we need to make a precise comparison of the components of the phenomenon of the electromagnetic induction with the phenomenon surrounding the thought experiment concerning the gravitational field. Figure 5 is helpful in grasping the hidden analogy.

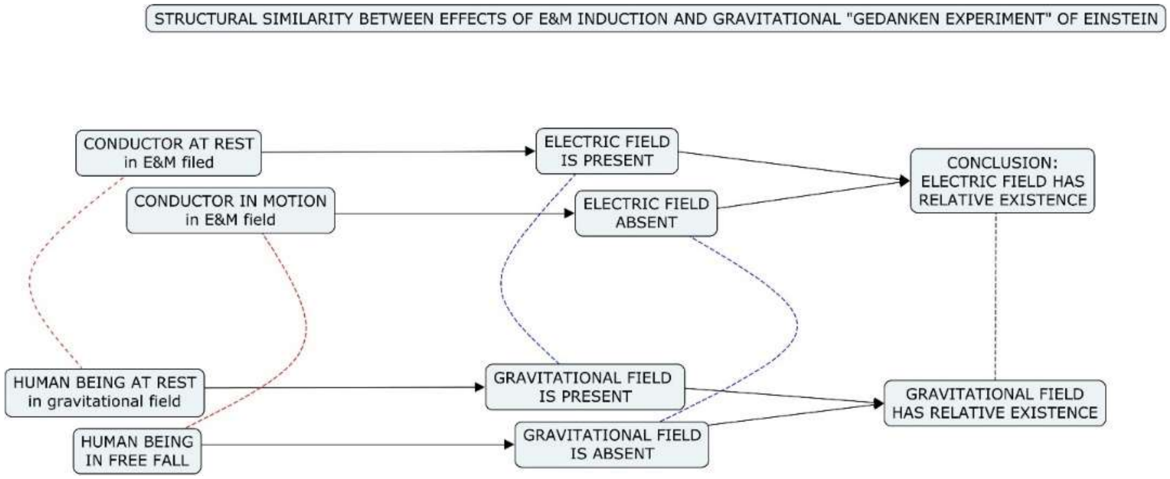

Figure 5. Structural similarity of electromagnetism and general relativity (Stoppel & Czarnocha, 2020, p. 26)

The Strong type of Aha!Moment of DoK intersects with the BISON type of Bridging by Structural similarity – the most complex graph the BISON team found.

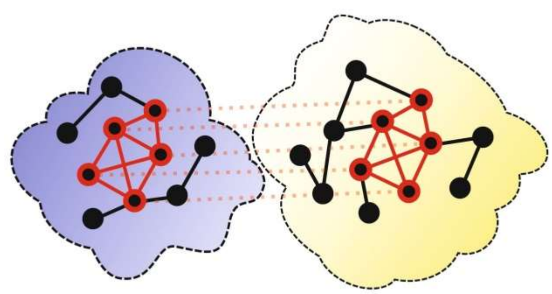

Figure 6. Bridging graphs by structural similarity (Berthold, 2012, p. 5)

CONCLUSIONS

We find it surprising that the cognitive structure of connections reached during Aha!Moments experienced by humans is essentially similar to the connections found by the bisociative search engine designed and programmed by the AI methods. We noted that the definition of bisociation used by Koestler and its definition used in the construction of the search engine, while similar in terms of role of connections between the concepts, differ significantly in the processes leading to the formation of the conceptual network. What does it mean? We conjecture therefore that cognitive nature of the created product is not the essential aspect of human creativity; the essential aspect of human bisociative creativity is in the creative process rather than in the creative product. That observation may significantly change the aim of the research in and practice of classroom mathematical creativity from the analysis of creative products to the investigations of the nature of the creative process.

Part 2 of this presentation will be published in the next issue of MTRJ; it will compare the two different methods which led to the corresponding results presented here.

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