

Teaching Calculus for Engineering Students Using Alternative Representations of Graph-formula Problems

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Abstract: This paper, comes as our response to the nowadays situation, of a gap between the skills required in academia studies and the cognitive skills with which the students come to us. We see the main reason for this gap in insufficient development of sovereign thinking and research skills, necessary in academic studies as well as in the future work of the modern engineer. This situation arises in secondary and high schools, which continue to focus on algorithmic skills (the value of which decreased in our time) and not on the development of critical thinking and understanding mathematics as a language of science and engineering. We address this challenge and develop a pedagogical strategy where we present objects from different use and different math language using technologies. We see active use of alternative representations of graph-formula problems in teaching calculus to novice engineering students as an effective way to change this situation. Our many years of experience shows that this way promotes cognitive interest, research thinking and deeper understanding of principal concepts of calculus. We have not found any literature focusing on using such an approach to graph-formula problems in calculus education. We believe that the examples presented in this paper which are chosen from our lengthy experience will demonstrate how we develop mental orientation as well as digital orientation in the study of calculus for first year engineering students. It may be also interesting to teachers of mathematics in academic studies, as well as high school teachers.

Keywords: knowledge gap, calculus teaching, graph-formula problems, different representations, pedagogical strategy, creative thinking, cognitive motivation.

INTRODUCTION

Calculus is the first mathematics course given to engineering students. The greatest importance of this course is in using it as a base for obtaining mathematical tools that are necessary for advanced academic studies. However, and just as salient, it is an important means for developing deep and creative mathematical thinking by the novice engineering student. In order to internalize the abstract mathematical concepts of Calculus students should be able to solve mathematical problems, to master algebraic technique, to have an advanced mathematical thinking strategy, and to master the mathematical language. In recent years, many researchers and lecturers around the world have pointed to a gap between the skills, required in academia studies and cognitive abilities of novice engineering students (Gueudet, 2013).

The way studies relate to the knowledge gap

For years, academic institutions have been dealing with the failure of the students in the first year of their studies in general and especially in mathematical courses (Lowe and Cook, 2003), (Yorke and Longden, 2004). With technological advances and the flow of students learning STEM professions at universities and colleges, there is a constant decline in the mathematical basic knowledge of the novice academic students (Gueudet, 2013), (Bosch, Fonseca & Gascon, 2004), (Trevor et al 1999), (EMS Committee on Education, 2011). It is noted that the decrease in the mathematical knowledge of students in the world exists also among students in departments of Mathematics, Exact Sciences and the various engineering professions.

Although high school students, in many countries, reach a high level of technical skills for solving exercises (such as finding derivatives, calculating infinite integrals and more), we still discover a lack of understanding of the concepts that the techniques are based on, for example, in the case of limits of functions and sums. Some of the reasons for this situation are the needs of students to go through many high-risk exams in high school, in addition to the fact that the mathematical concepts are abstract and complex (Holton, 2001).

Undergraduate students also have difficulties in understanding definitions and various representations of mathematical concepts in general and the concept of function in particular (Tall & Vinner, 1981), (Vinner, 1980) & (Hershkowitz, 1980). According to them, even when the student has the formal definition, he does not use it in his mathematical activity, in most cases he or she decides on the basis of concept image solely, namely on the set of all mental images associated in his mind with the name of this concept. “Mental image” refers to any type of representation: visual, symbolic form, chart, graph, etc. In other words, there is a problem among students in the first year of thought processes (such as the conceptualization, generalization, abstraction, assessment, reasoning and critical thinking). Therefore, it is important in teaching to examine the images of students regarding various mathematical concepts (such as a function, limits, continuity, derivative, integral, etc.); it is important to focus on the aspects related to the conceptual-building of concepts, and on the process of reasoning and proof. Mark and others (Mark et al, 2004) present a research framework and proposals for teaching undergraduate students. In this study, the principles of constructivism are presented (according to Piaget) and ways to cope with the problem of concept assimilation.

Another factor is the way of teaching and learning in high school that were essentially "mechanical" (algorithmic learning and memorization). In some literature this is called "rote learning" (Tami Yechiely, 2015, 2012), (Hamidreza Kashefi et al, 2012), which is defined as "ritual participation" as a Latin language learning based mainly on memory and requires more physical efforts and less mental efforts.

Heyd-Metzuyanim (2013, 2015), based on Anna Sfard, notes that this type of study can lead to a cycle of difficulties and failures, and does not lead to investigative thinking, but rather strengthens a ritual participation. Learning of this type may impair motivation to learn mathematics. The researchers (Kira J. et al, 2013) reinforce Heyd-Metzuyanim and claim that it also exists among students in academic institutions. (Yudariah bT., 1998) indicates that one of the reasons for this gap is also the technological development that led to decrease in the motivation of students to learn topics that are no longer useful in every day.

The gap is also evident that students in the academic studies need to give and to understand the deductive proofs, which did not learn enough in high school. (Mariotti et al, 2004), (Engelbrecht, 2010). Review of the literature (Hanna & de Villiers, 2008; Hemmi, 2008) indicates that books do not actually help students acquire the deductive method of proofs. The researchers ((Lithner 2000, Harel 2008) claim that the mathematics at the academy depicted as experts ' mathematics. According to the researchers, students are required to find examples, develop flexible use of different types of representations, experiment and control them at a theoretical level and more.

Many academic institutions try to overcome this gap by special preparing courses, reinforcements and extra hours, but they have not yet found perfect solutions (de Guzman, Hodgson, Robert & Villani, 1998); (Gueudet, 2008);(Thomas at al., 2012), (Gueudet, 2013), (Artigue Batanero & Kent, 2007); (Kira J., 2013).

Various representations

Representations are descriptions of mathematical concepts, operations, algorithms and problems. Researchers determined that the ability to identify and present a particular concept in different representations and the ability to move from one representation to another of that entity is an essential component of mathematical knowledge (Friedlander & Tabach, 2001). Some Researchers report that understanding of the concept in one representation does not necessarily guarantee the understanding of it in the other representation (Bakker, 2004). Therefore, in order to ensure the improvement of learning and understanding during the teaching of a particular mathematical idea there is a variety of questions to be presented and to ask the students to express the idea in other representations (Gagatsis & Shiakalli, 2004).

Our pedagogical strategy

In our attempts to find a solution to the gap mentioned above, we use the theory of constructivism in a gentle or easy way that is, to bring the learner to take responsibility for his learning until understanding. We believe that this can be done with the model built according to the following recommendations: "**to develop flexible use of different types of representations**" (Lithner, 2000; Harel, 2008) and "**teaching in a visual-intuitive approach**" (Eisenberg and Dryfus, 1991), they think that mathematical reasoning based on visualization facilitates students. In the article of (Kathleen M., 1988), emphasized a teaching method that encourages students to "**enable an active thinking about graphs by using computerized technologies.**"

Our pedagogical strategy is in this spirit, demanding to integrate the teaching at least in the following three representations: algebraic, graphic, verbal, by using scientific calculators, so we built a collection of Questions for understanding and deepening in calculus (Dagan, M., Satianov, P., 2009).

Our experience shows a lack of understanding of basic concepts such as algebraic rules; relationship between formulas and graphs; elementary functions and their main properties; radian angle measurement; direct and inverse trigonometric function. Moreover, it is typical for most novice students not to be able to formulate precise definitions of main mathematical notions studied in higher school. Our pedagogical strategy is to overcome this gap based on the attitude to mathematics as a language with diverse sub-languages (verbal, analytical, graphical, numerical), and permanent and systematic using of various representations while formulating mathematical notions and solving different mathematical problems with scientific calculators if it needs (Dagan & Satianov, 2006). To internalize a mathematical concept perfectly, it is important to understand it in all its forms, that is, in all sub-languages (graphical, analytical, verbal and numerical) in order to create an accurate and effective "concept image" (Tall & Vinner, 1981). In the research literature, we did not find a reference to this type of approach.

Implement the strategy on the subject 'elementary functions and their graphs'

Our long experience shows that the permanent and systematic use of these representations, and various transitions among them in a calculus course in addition to extensive use of scientific calculators and graphic applications, yields greater achievement and a high level of understanding, all while enhancing cognitive motivation and creative thinking of the students. (Dagan, Satianov, Teicher, 2018), (Dagan, Satianov, Teicher, 2019). This approach is in accordance with the requirements of 21th century demanding profound changes in the teaching of mathematics for engineering students, (Asia at Al, 2013), (Bressoud at Al, 2016), (Kissane and Kemp, 2012), (Tall at Al, 2008). In this article, we will illustrate these ideas in the process of teaching the theme of 'elementary functions and their graphs', which is fundamental for understanding calculus.

For convenience, we will designate these basic forms of representations by means of the first letters of the appropriate words:

A – Analytical representation

V – Verbal representation

G – Graphical representation

N – Numerical representation

We will use the symbolic designations described above in order to classify problems

AG – From analytical representation to graphical representation

Example 1: Draw the graph of the formula: $y = \frac{x}{|x|}$.

GA – From graphical representation to analytical representation.

Example 2. Find a formula for the graph in Figure 1.

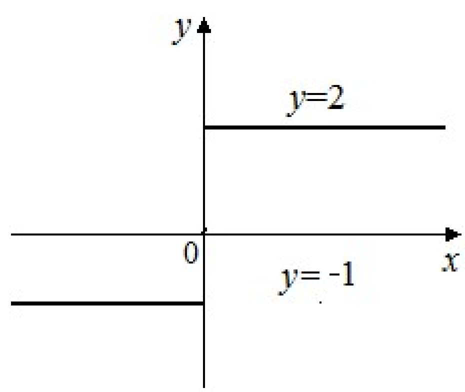

Figure 1

The GA type problems require good knowledge of graphs of elementary functions and their transformation due to the suitable formula changes.

VG – From verbal representation to graphical representation.

Example 3. Draw a curve with two points of inflection.

GV – From graphical representation to verbal representation

Example 4. Describe the main properties of a function (domain of definition, parity of function, point of discontinuity, asymptotes) given by the curve in Figure 2.

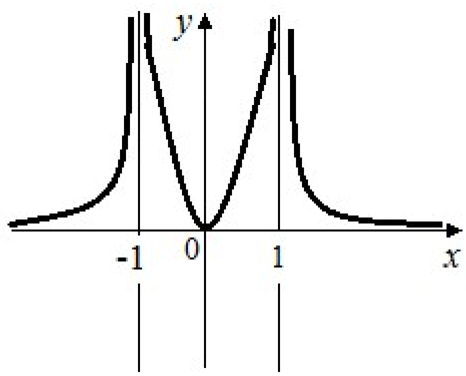

Figure 2

The VG type problems requires knowledge of the basic concepts related to elementary functions and graphical representations of these concepts.

GN – From graphical representation to numerical representation.

Example 5. Find the slope of the straight line of Figure 3.

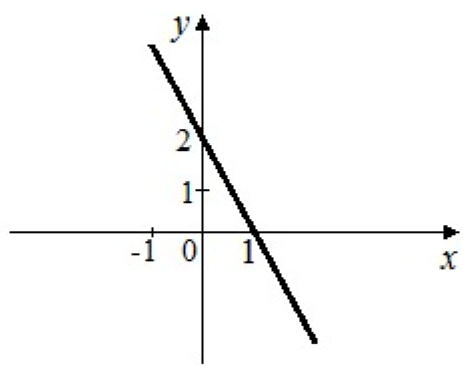

Figure 3

The GN type problems requires an acquaintance with functions concepts and suitable numerical calculations or appropriate graphical constructions to get the desired result.

NG – From numerical representation to graphical representation.

Example 6. Draw a graph of even functions that is suitable for the following table:

$$\begin{array}{llllll} x: & 0 & 1 & 2 & 3 \\ y: & 1 & 0 & 2 & 1 \end{array}$$

The NG type problems requires the use of numerical information and mention in the problem function properties for drawing the suitable curve.

The most widespread graphical problems in the studies of functions and graphs are from the AG type, connected to sketching graphs of functions given by formulas after analytical investigation. In our paper we concentrate on the opposite type of problems – constructing formulas for the functions, based on the graphical representations. Solving problems of this type requires deeper and more creative thinking. The use of GA-type problems in a calculus class provides a great opportunity to encourage different levels of thinking among the students and to develop cognitive interest in calculus studies.

We will distinguish between different types of GA problems.

Type 1. Finding the precise formula for a given graph.

Solving such problems requires acquaintance with appropriate elementary functions and understanding of suitable approaches to finding the formulas.

Example 7. Constructing the formula for the straight line in Figure 3.

Approach A. (Used by most students). Find slope m of the given straight line by numerical calculation and follow the proper formula.

$$y = y_0 + m(x - x_0); \quad m = \frac{\Delta y}{\Delta x} = \frac{2-0}{0-1} = -2;$$
$$x_0 = 1; y_0 = 0 \Rightarrow y = -2(x-1) \Rightarrow y = 2 - 2x$$

Approach B. Use the 'straight line equation in the segments':

$$\frac{x}{a} + \frac{y}{b} = 1; a = 1, b = 2 \Rightarrow \frac{x}{1} + \frac{y}{2} = 1 \Rightarrow y = 2 - 2x$$

Approach C. Construct a formula in the form $y = mx + n$ based on the data shown in Figure 3.

$$n = y(0) = 2; y(1) = m \cdot 1 + 2 = 0 \Rightarrow m = -2 \Rightarrow y = -2x + 2$$

Approach D. Use the determinant formula of the straight line passing through two given points, $(x_1, y_1); (x_2, y_2)$:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 0 \Rightarrow -2x - y + 2 = 0 \Rightarrow y = 2 - 2x$$

Example 8. Constructing the formula for the graph in Figure 4.

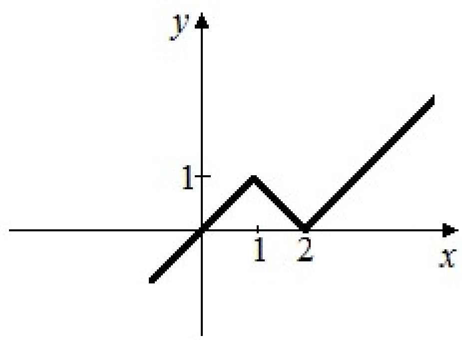

Figure 4

To find a suitable formula for the graph one needs more profound knowledge and deeper thinking and creativity (especially the first time we encounter this kind of problem).

Approach A. Use the graphs of Figure 5 and the min(a, b) elementary operation:

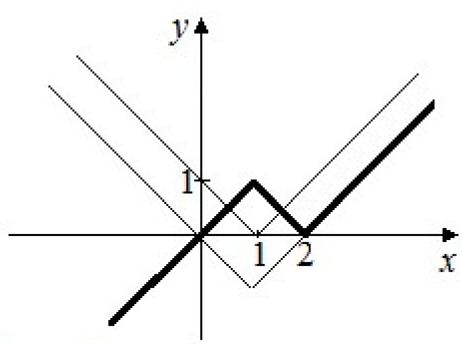

Figure 5

$$y = |x-1| \rightarrow y = ||x-1|-1| \rightarrow y = \min\left(x, ||x-1|-1|\right)$$

Note that we widely use min/max functions from the very beginning of the study of calculus ([11]).

We strongly urge our students to test the constructed formula by using some graphic applications. This makes the student feel good and even be happy if he or she can find the appropriate formula. With a suitable computer program or graphic calculator, the analytical formula enables the instant drawing of the graph of the functions. This is also important for students to be acquainted with the proper software and computer input languages. If the graphics application creates a curve that does not fit the given graph it will be a sign that the student should invest a little more thought to finding a solution by correcting the previous formula or looking for another way.

In the formula $y = \min\left(x, ||x-1|-1|\right)$, we used a min operation (see [11]) that sometimes does not exist in a graphic application. However, in this case the students can use the absolute value function for input to the program by means of the following formula:

$$\min(a,b) = \frac{a+b-|a-b|}{2}$$

Therefore: $$y = \min\left(x, ||x-1|-1|\right) = \frac{x + ||x-1|-1| - |x - ||x-1|-1||}{2}$$

Approach B. Think about a formula for a broken line in Figure 4 in the form:

$$y = ax + b + c|x - x_1| + d|x - x_2|.$$

Here, x_1, x_2 are the turning points of the line.

In our case $x_1 = 1, x_2 = 2$ and we will subsequently get:

$$y = ax + b + c|x-1| + d|x-2|$$

$$\begin{cases} y(0) = b + c + 2d = 0 \\ y(1) = a + b + d = 1 \\ y(2) = 2a + b + c = 0 \\ y(3) = 3a + b + 2c + d = 1 \end{cases}$$

We now have a system of four equations with four unknowns. Here we examine the students' skill in solving such systems studied earlier in a linear algebra course, or with scientific calculators to obtain a quick solution.

From this linear system, we find: $a = 1, b = -1, c = -1, d = 1$

Therefore: $y = x - 1 - |x-1| + |x-2|$

It is useful to check the correctness of this formula by graphic application as well as to check it analytically by opening absolute values in the proper intervals.

Approach C. (general): The idea of obtaining such a formula for a broken line we give to our students during the first calculus course. The suitable formula for two broken points will be: $y = \frac{k_0 + k_2}{2}x + \frac{k_1 - k_0}{2}|x-1| + \frac{k_2 - k_1}{2}|x-2| + b$

Here k_0, k_1, k_2 are the slopes of the links of a broken line (from left to right).

Using the graph in Figure 4 we will get: $k_0 = 1, k_1 = -1, k_2 = 1$ and therefore:

$$y = x - |x-1| + |x-2| + b$$

To find value b it suffices to substitute coordinates of one point of the graph in this equation. For example:

$$y(0) = 0 \Rightarrow 0 - 1 + 2 + b = 0 \Rightarrow b = -1$$

The final equation for a broken line in Figure 4 is:

$$y = x - |x-1| + |x-2| - 1$$

Note that after getting to know this formula, the creativity in the search for a formula disappears and the solution becomes a routine finding of the slopes of the links of a broken line. Therefore, we do not start from a formula in order to give students the opportunity to think independently beforehand; only then do we give a few examples explaining the idea.

Approach D. The idea of assembling the suitable formula using a number of more simple functions. Let us think about the functions $y = y_1(x);\ y = y_2(x);\ y = y_3(x)$ with the graphs in Figures 6-8,

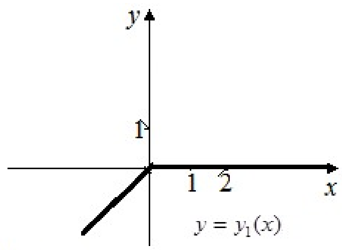

Figure 6

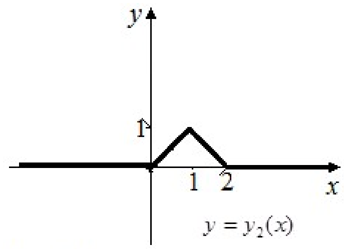

Figure 7

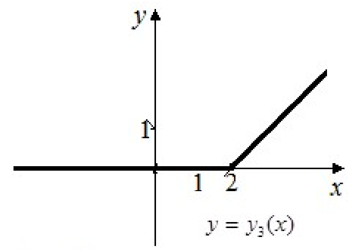

Figure 8

It is easy to construct formulas for each of them:

$$y_1 = \min(x,0); \qquad y_2 = \max(1-|x-1|,0); \qquad y_3 = \max(x-2,0)$$

It is clear that the function $y = y_1 + y_2 + y_3$ has the required graph (Figure 4).

Therefore, we will get the following formula:

$$y = \min(x,0) + \max(1-|x-1|,0) + \max(x-2,0)\,.$$

In addition, if we want to use the absolute value function only, we will get:

$$y = \frac{x-|x|}{2} + \frac{1-|x-1|+|1-|x-1||}{2} + \frac{x-2+|x-2|}{2}$$

After some simplifications:

$$y = \frac{1}{2}\left(2x-1-|x|-|x-1|+|x-2|+|1-|x-1||\right)$$

Example 9. Finding a formula for the parabola in Figure 9

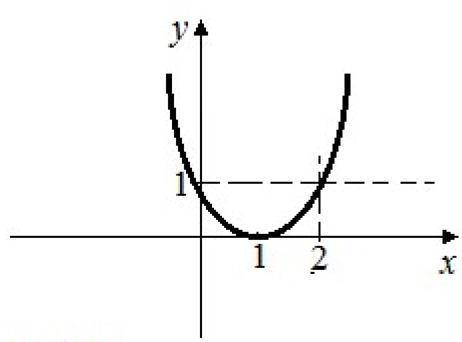

Figure 9

Solution: Because the parabola in Figure 9 touches the *x*-axis at point $x = 1$. its equation must be: $y = a(x-1)^2$.

To determine the value of coefficient *a*, we note that $y(0) = 1$ and so $1 = a(0-1)^2$ and therefore: $y = (x-1)^2 = x^2 - 2x + 1$. $a = 1$

Example 10. Finding a formula for the curve in the figure if known that it is the graph of the cubic equation $y = ax^3 + bx^2 + cx + d$.

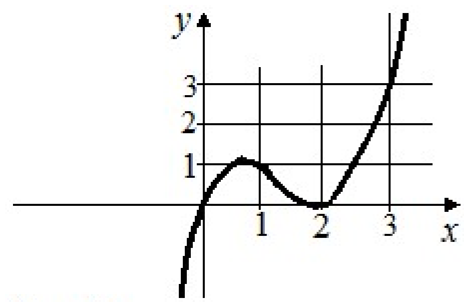

Figure 10

Solution: According to the zeros of the polynomial in Figure 10, its equation must be:

$$y = ax(x-2)^2$$

Therefore, only a single parameter *α* must be found and so we need only one equation and only one point substitution in this form, for example (1,1): $1 = a \cdot 1 \cdot (-1)^2 \Rightarrow a = 1$

Thus, we obtain the required equation: $y = x(x-2)^2 = x^3 - 4x^2 + 4x$

Type 2. <u>Constructing a formula with a graph similar that shown in the given figure</u>.

Questions of this type require an understanding of the meaning of a graph similar to the one depicted in the given figure. Therefore, students need clarifications of what it means when we discuss the similarity of the graphs. Such questions do not usually require calculations and they are more qualitative than quantitative. The answers to such questions are often ambiguous and provide good opportunities for assessing the level of student understanding of function properties. At the same time, student answers help teachers determine the independence and creativity of their thinking.

Such questions can be divided into several subtypes in accordance with the required levels of thinking activity.

1. <u>Recognition of the graphs of basic elementary function</u>.

Example 11. 'What is a function whose graph is like that shown in Figure 11?'

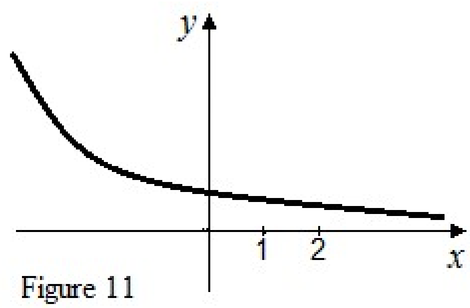

Figure 11

Possible answers can be $f(x) = (0.5)^x$ or $f(x) = a^x$ $(0 < a < 1)$.

Example 12. 'What is a function whose graph is like that shown in Figure 12?'

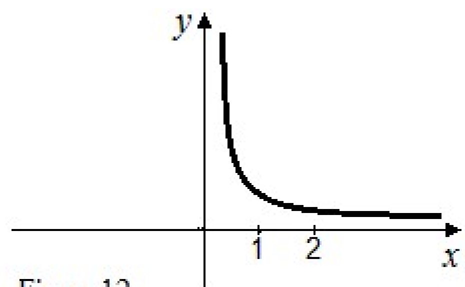

Figure 12

Possible answers can be $f(x) = \frac{1}{\sqrt{x}}$ or $f(x) = \left(\sqrt{x}\right)^m$ $(m < 0)$.

2. <u>Recognition of the formula for a part of the given graph with the subsequent use of symmetry to complete the construction</u>.

Example 13. 'What is a function whose graph is similar to that shown in Figure 13?'

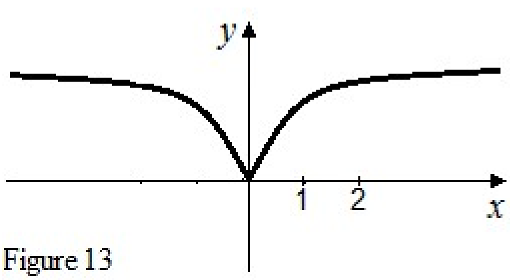

Figure 13

Here, in the right part of the curve, a student can find the graph of the function $y = \sqrt{x}$ and then use the symmetry of the curve with respect to the y-axis to complete the construction by use of the absolute value operation. The final answer is $y = \sqrt{|x|}$.

Here are some other possible answers for Figure 13:

$$y = \sqrt[3]{x^2}; \quad y = \left(x^2\right)^m \ (0 < m < 0.5); \quad y = \ln\left(1 + |x|\right).$$

Example 14. 'What is a function whose graph is similar to that one shown in Figure 14?'

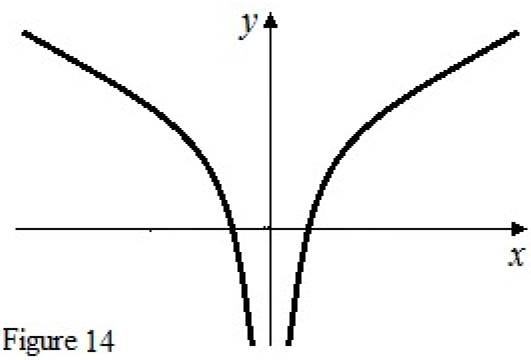

Figure 14

The possible answer may be $y = \ln|x|$.

3. Construction-required formula from the known formulas for several parts of the given graph.

Example 15. 'What is a function whose graph is like that shown in Figure 15?'

Approach A. Construction by assembly from appropriate parts

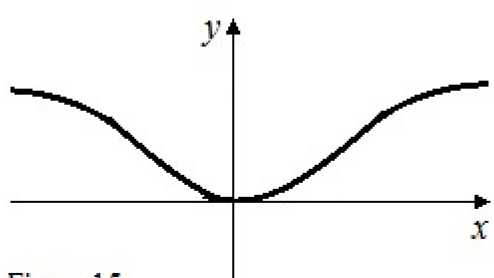

Figure 15

The graph near the origin, is like the curve $y = x^2$ and with the distance from the origin, the graph is like $y = \ln(x^2)$. Therefore, with the understanding that when x is near the origin $\ln(1 + x) \approx x$

and with even a better approximation $\ln(1+x^2) \approx x^2$, while for large values of x $\ln(1+x^2) \approx \ln(x^2) = 2\ln|x|$, the appropriate formula is: $y = \ln(1+x^2)$. In order to be sure of the correctness of our assumption we can investigate the function $y = \ln(1+x^2)$ analytically. We strongly recommend also using graphic applications for checking the constructed formula.

Approach B. Thinking about the graph of the derivative of the given function.

For the graph (Figure 15) the derivative graph looks like in Figure 16.

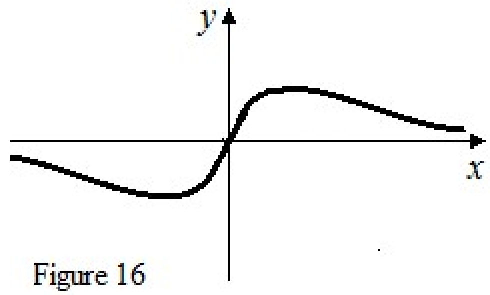

Figure 16

A suitable formula for this curve is: $y = \dfrac{x}{1+x^2}$. Hence, by integration:

$$f(x) = \int_0^x \frac{xdx}{1+x^2} = \frac{1}{2}\ln\left(1+x^2\right).$$

Note that the coefficient of 0.5 can reduced because there are no numerical adjustments on the axes, so we will finally get: $f(x) = \ln\left(1+x^2\right)$.

4. Attention to asymptotes, zeros and signs of the function suitable to the graph.

Example 17. 'What is a function whose graph is like that shown in Figure 17?'

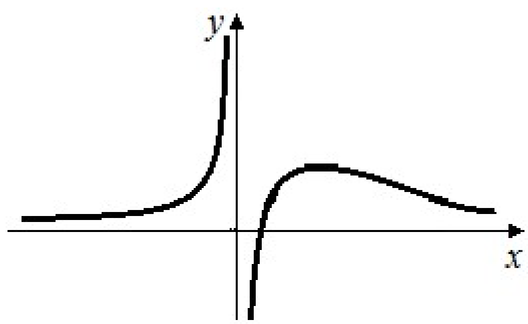

Figure 17

The appropriate formula can be $y = \frac{x-1}{x^3}$.

Example 18. 'What is a function whose graph is like that shown in Figure 18?'

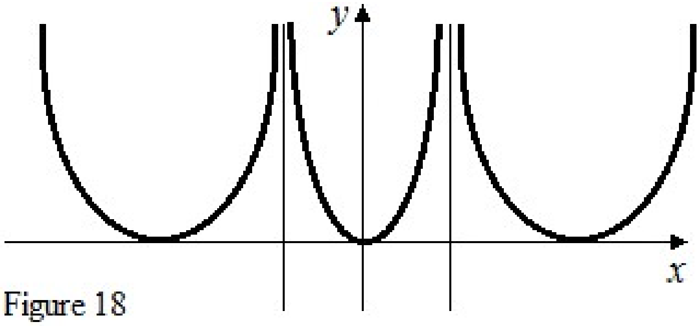

Figure 18

The appropriate formula can be $y = \frac{x^2(x^2-4)^2}{(x^2-1)^2}$.

5. The idea of parallel transfer of the graph.

Example 19. 'What is a function whose graph is like that shown in Figure 19?'

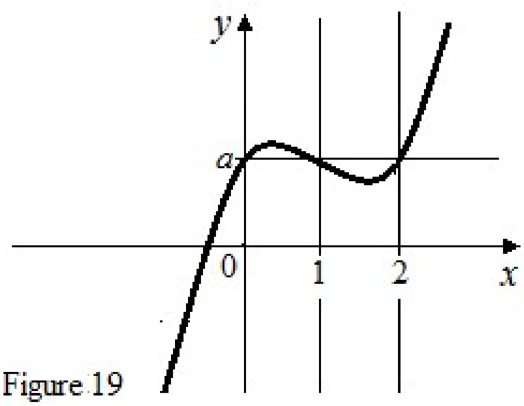

Figure 19

Let us move the graph downwards by *a* units. The result can be seen in Figure 20.

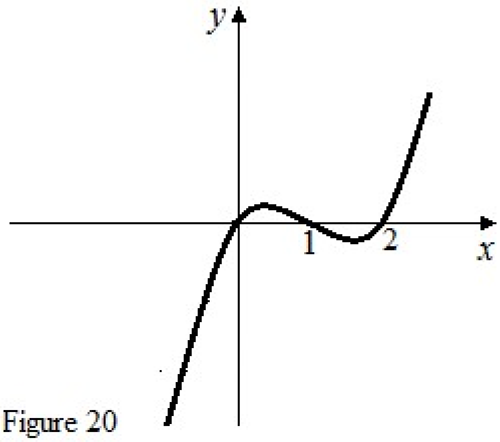

Figure 20

In Figure 20 we see 3 different roots 0, 1, 2 of a suitable function and so a formula that fits this figure may be: $y = x(x-1)(x-2)$.

Therefore, the final formula may be: $f(x) = x(x-1)(x-2) + a$.

In conclusion, we discuss the VA problem type – finding a formula for a function according to its verbal description. This type of problem is of great importance for understanding the basic concepts of function behavior. As a possible rule for finding an appropriate formula, we should think first about graphical representations that are suitable for the verbal description. Therefore, this type may be classified as providing examples of VGA problems that need to switch to a graphical representation before finding the analytical expression.

Example 20. Find a formula for a function with asymptotes $y = 0;\ y = 1;\ x = 0$.

The first step in solving such problems should be turning to the appropriate graphic representation of the textual description. It may be as shown in Figure 21:

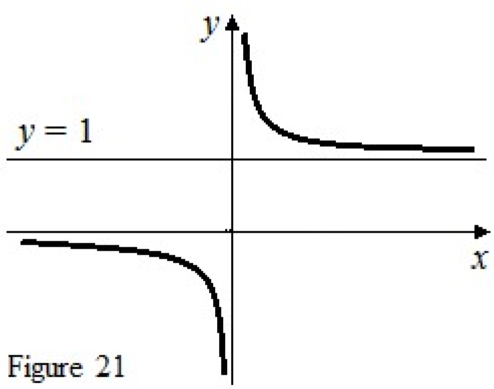

Figure 21

Let us construct first a function whose graph is horizontal asymptotes $y = 0,\ y = 1$. For this purpose, we can use a function $f_1(x) = \frac{|x|}{x}$ with a graph, as in Figure 22.

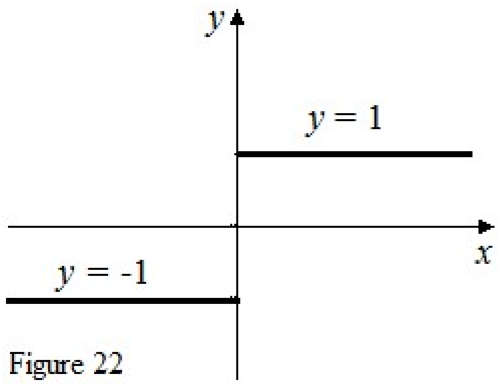

Figure 22

By shifting the graph in Figure 22 one unit up we will get the graph of the function $f_2(x) = f_1(x) + 1 = \frac{|x|}{x} + 1$, which is in Figure 23.

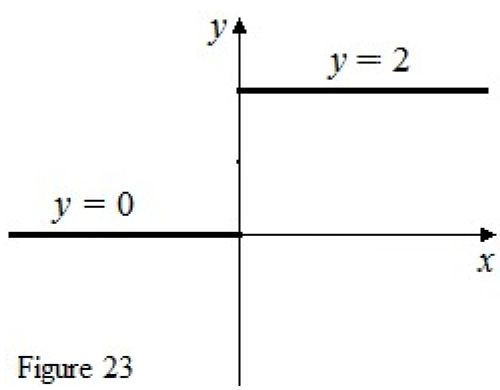

Figure 23

By squeezing the graph in Figure 23, we get the needed formula of the asymptotes

$$f_3(x) = \frac{1}{2} f_2(x) = \frac{|x|}{2x} + \frac{1}{2}$$

Now it is easy understand that the appropriate formula for the curve in Figure 21 may be in the form:

$$y = f_3(x) + \frac{1}{x} = \frac{|x|}{2x} + \frac{1}{2} + \frac{1}{x}$$

An optional formula for the graph in Figure 21: $y = \frac{1}{\pi} arctg(x) + \frac{1}{2} + \frac{1}{x}$.

We recommend that students check the validity of this formula using graphical applications, as doing so always promotes a feeling of achievement when students see how to correct analytical input; the graphical application gives quick visual output to confirm the formula.

Example 21. Find a formula for a continuous elementary function with asymptotes $y = 0$; $y = x$.

Solution: First, let us draw an appropriate graph (Figure 24).

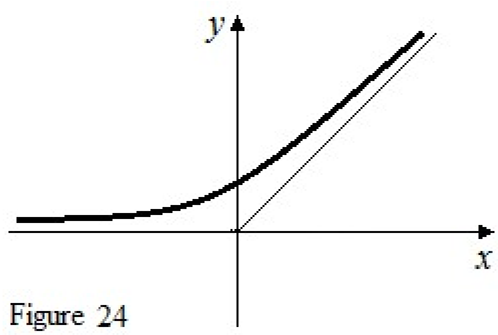

Figure 24

Now let's construct a formula for the graph in Figure 25.

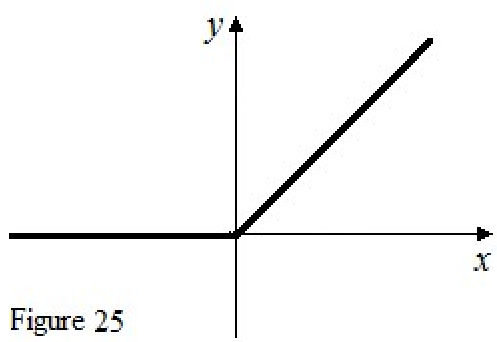

Figure 25

It is easy to verify that the suitable formula may be: $y = \frac{1}{2}(x + |x|)$.

If we now add to this function an everywhere positive expression that tends to zero at infinity, we get the required formula $y = \frac{1}{2}(x + |x|) + \frac{1}{1 + x^2}$.

<u>Learning dialogues with students. Transition between different representations in problem solving</u>

We want to prepare students to deal with new problems, show effective ways to approach them, and teach them not to give up when meeting new tasks. We want to prevent the approach when the student says that he has not yet encountered such problems and asks us at first to explain how to solve them and give a proper example. However, it should be noted that in the modern world there is decreasing reliance on performing algorithmic actions. For future engineers it is becoming much more important to cope with new tasks. It is therefore very important to prepare them to meet new challenges with any effective tools that are available to them. Despite the difference in engineering specialties, the ways of reasoning are much the same and the proper study of mathematics can contribute to student development.

We give some examples of our teaching dialogues with students, aimed at finding approaches to a new problem.

Dialogue 1. It is known that the curve shown in Figure 26 is a parabola. Find the equation of this curve.

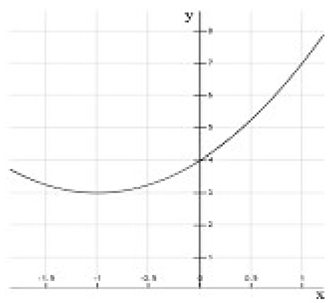

Figure 26

Student: How can I do this? I do not remember that we have solved such problems.

Lecturer: Understand first what is given and what is required of you. What does the notion of 'parabola' mean? What does the parabola equation look like?

Student: A parabola is the graph of a quadratic equation.

Lecturer: And what does a quadratic equation look like?

Student: $y = ax^2 + bx + c$

Lecturer: If so, what is unknown to you?

Student: The values of coefficients a, b, c.

Lecturer: And what is x, y in the equation $y = ax^2 + bx + c$?

Student: x is an independent variable and y depends on it.

Lecturer: For example, if $x=2$, according to this equation, what is y?

Student: $y=a\cdot 2^2+b\cdot 2+c$.

Lecturer: And if you know that for $x=2$ the value of y is 4, how do you write it?

Student: $4=a\cdot 2^2+b\cdot 2+c$ or $4a+2b+c=4$.

Lecturer: And what did you get?

Student: An equality.

Lecturer: However, you do not know the values for a, b, c, so, how would you treat this equality?

Student: As an equation, apparently.

Lecturer: And how to treat a, b, c for this equation?

Student: As unknowns of this equation.

Lecturer: And if we have 3 unknowns, how many equations are needed to find a, b, c?

Student: Three equations.

Lecturer: Can you construct them based on the given curve?

Student: Yes, I see from Figure 1 that $y(0)=4, y(1)=7, y(-1)=3$ and therefore the needed three equations are:

$$\begin{cases} a\cdot 0^2+b\cdot 0+c=4 \\ a\cdot 1^2+b\cdot 1+c=7 \\ a\cdot(-1)^2+b\cdot(-1)+c=3 \end{cases} \Leftrightarrow \begin{cases} c=4 \\ a+b+c=7 \\ a-b+c=3 \end{cases}$$

Lecturer: Now that you produced a system of three equations with three unknowns, what is the next step?

Student: To solve this system. I will do it quickly. The answer is $a=1, b=2, c=4$

Lecturer: What should be done now?

Student: Just to write the appropriate formula $y=x^2+2x+4$.

Lecturer: How can you check the correctness of solution to ensure that there has not been an error in the calculations?

Student: We must substitute the appropriate values in the equations and ensure that they turned out to be the correct equalities:

$$\begin{cases} 4 \overset{?}{=} 4 \\ 1+2+4 \overset{?}{=} 7 \\ 1-2+4 \overset{?}{=} 3 \end{cases} \Rightarrow \text{ all these equalities are true}$$

Lecturer: And if the figure was ‘parabola cubic’, how do you find the graph of equation $y = ax^3 + bx^2 + cx + d$?

Student: Compose the system of four equations with four unknowns and solve it.

Lecturer: OK. Maybe it is worth thinking about a proper formula that will give us the answer without composing and solving the system?

Student: It would be great. However, I have no idea how to get such a formula.

Lecturer: Let's go back to the previous task of finding a square triple by three points of its graph: (x_1, y_1), (x_2, y_2), (x_3, y_3).

Lecturer: Can you construct a square triple that passes through a point (x_1, y_1) and take the value 0 for $x = x_2$ *and* $x = x_3$?

Student: As far as I know, according to the Bezout theorem such a triple should be divided by $x - x_2$ and by $x - x_2$ without remaining, and therefore it must be in the form: $m_1(x - x_2)(x - x_3)$

Lecturer: It's great! And how can you find the value of m_1?

Student: As we know $y(x_1) = y_1$ and so $m_1(x_1 - x_2)(x_1 - x_3) = y_1$. Therefore,

$$m_1 = \frac{y_1}{(x_1 - x_2)(x_1 - x_3)} \text{ and, finally } y_1(x) = y_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}.$$

Lecturer: Splendid! And if we now construct the same formulas for the rest of the values of x_2 *and* x_3, like this:

$$y_2(x) = y_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}; \quad y_3(x) = y_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)},$$

can you now get the desired formula to the quadratic triple in which the graph passed through three points (x_1, y_1), (x_2, y_2), (x_3, y_3)?

Student: We just need to add three received formulas: $y(x) = y_1(x) + y_2(x) + y_3(x)$ or:

$$y(x) = y_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + y_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

Lecturer: It is great! Now check this formula for the graph in Figure 26.

Student: No problem, we get:

$$y(x) = 4\frac{(x-1)(x+1)}{(0-1)(0+1)} + 7\frac{(x-0)(x+1)}{(1-0)(1+1)} + 3\frac{(x-0)(x-1)}{(-1-0)(-1-1)}$$

And finally, after proper algebraic actions: $y = x^2 + 2x + 4$.

Great! I like this method!

Lecturer: This wonderful method belongs to the famous French mathematician Lagrange (1736-1813). His work has given us the polynomials that are known as 'Lagrange polynomials'. They let us quickly get a formula for a curve that passes through any ***n*** points: (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$, where $x_1 < x_2 < \ldots < x_n$.

Lecturer: Now use this method to obtain an equation for a curve passing through four given points (0,0); (1,1); (2,0); (3,3)

Student: No problem, because here $y_1 = 0$ and $y_3 = 0$, so we need two terms only in the proper Lagrange formula: $y = 1 \cdot \frac{x(x-2)(x-3)}{1\cdot(-1)\cdot(-2)} + 3 \cdot \frac{x(x-1)(x-2)}{3 \cdot 2 \cdot 1}$

In addition, after algebraic simplifications, we have: $y = x^3 - 4x^2 + 4x$

Lecturer: Note that there are infinitely many curves passing through these four points and the Lagrange polynomials give the simplest of suitable formulas.

Note that in our teaching approach we permanently use transitions from one presentation to another—from verbal formulation of the problem to visual (graphic) information; from graphic information to numerical; from numerical data to a symbolic presentation of it; from symbolic presentation to analytical descriptions of it by equation; and so on.

Dialogue 2

Lecturer: Give me a formula of continuous function with asymptotes $y = 2$ and $y = 0$.

Student: I have no idea how to begin.

Lecturer: Are you understand the question? Are you understand all notions mentioned in it? Can you draw a graph of a function with such properties?

Student: I don't understand how a graph can have two different horizontal asymptotes.

Lecturer: Why not? One of them is at $x \to +\infty$ and another is at $x \to -\infty$.Try to draw a suitable graph.

Student: What about the graph in Figure 27?

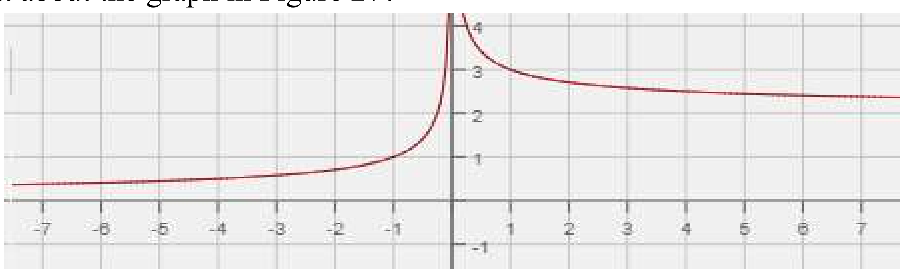

Figure 27

Lecturer: As far as horizontal asymptotes it is fit. However, what about a continuity?

Student: I let it out of sight. Here is the new option in Figure 28:

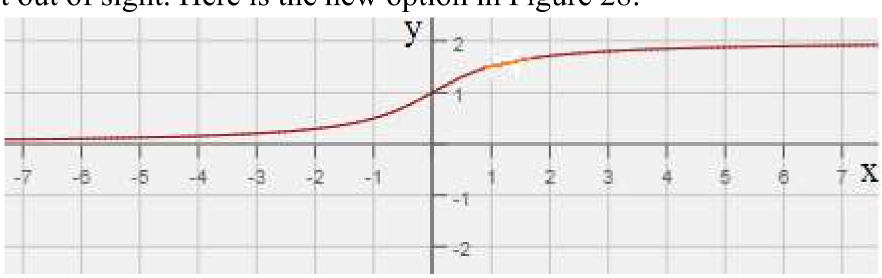

Figure 28

Lecturer: OK. Now we can see on the chart all the properties mentioned above. Does it remind you of a graph of any known functions?

Student: It looks like an arctangent graph in Figure 29:

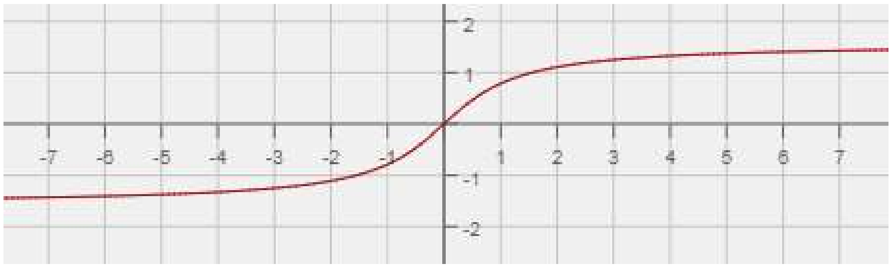

Figure 29

Lecturer: OK. But what's the difference and what needs to be done to get the required graph from what you see?

Student: An arctangent has asymptotes $y=-\frac{\pi}{2};\ y=\frac{\pi}{2}$ and we need $y=0;\ y=2$.

Lecturer: What do you want to do with the formula $y=\arctan(x)$ to get the desired asymptotes?

Student: We need to change the asymptotes from $y=-\frac{\pi}{2};\ y=\frac{\pi}{2}$ to $y=-1;\ y=1$.

Lecturer: And how can that be done?

Student: For this, multiply the arctan(x) by $\frac{2}{\pi}$ and come to the function $y=\frac{2}{\pi}\arctan(x)$ with asymptotes $y=-1;\ y=1$ (Figure 30).

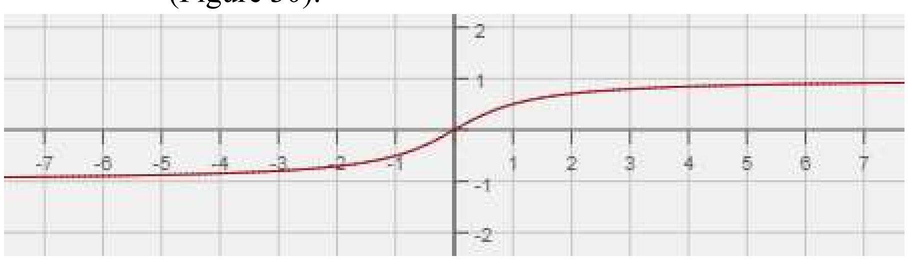

Figure 30

Lecturer: And how would you change this function to get asymptotes $y=0;\ y=2$?

Student: It just needs to have 1 be added to the last function and we get the required formula:

$$y=\frac{2}{\pi}\arctan(x)+1$$

Note that tasks of this kind require concentrated mental activity of students. Here are some important steps in solving such problems:

1. Understanding of the problem.
   The student must understand all concepts mentioned in the problem, together with their graphic expression.
2. Imagination of how graphs may look.

3. Decision on which of the possible graphs is like to some known function graph.
4. Decision how to change a known formula in order to get the required graph.
5. Testing of a suggested formula by analytical investigation and by using technological tools, along with calculators and graphical applications.

Note that using graphical applications is a very positive stage for the students in the process of the fitting the formula and in strengthening the student's confidence in the final solution.
An experimental study conducted in two groups of engineering students under two approaches: Group A had 64 students; group B had 65. Group A studied according to the regular method without using ‘the transition model’. Group B studied according to the model we called 'the transition between the different representations of the language of mathematics. The experiment was organized in three phases:

1. PRE-TEST
The scores received by the students in both groups examined at the end of their pre-calculus study in the preparatory program. Statistical processing carried out and t-test applied. Two hypotheses were verified: the null hypothesis and the counter hypothesis, the t-test showed no difference between the two groups at a significance level of 95%. (P = 0.05, 1.68 = t, critical value 1.98).

2. MID-TEST. On the mid-term test, we gave two identical problems focused on finding an appropriate formula. In group A, the problems were answered by only 10 students; in the group B, the problems were answered by 50 students.

3. POST-TEST. In the two groups A and B mention above, students took the same semester exam at the standard level, which we have always tried to keep for years. Total exam (which contained not only graphical tasks) marks of the students in two groups were processed statistically and the t- test was applied.
The null hypothesis: There are no significant differences between the groups that are: the experimental group B and the control group A in the test results, i.e., $\bar{x}_2 - \bar{x}_1 = 0$.
Counterhypothesis: There is a significant difference between the groups: the sample group and the control group with the test results, i.e. $\bar{x}_2 - \bar{x}_1 \neq 0$.
The hypotheses tested at a significance level of 95% and the value t = 2.513 was obtained. The critical value of rejecting the null hypothesis is that it is 1.98. According to the results, it is possible to reject the null hypothesis and say that there is a significant gap between the experimental group B and the control group A.
Our conclusion is that in Group B the students had acquired a better understanding of mathematical concepts and their graphical representations as well as the necessary technical skills.

CONCLUDING REMARKS

The problems of mention above types we gave to students from the beginning to the end of calculus course at different levels of complexity in accordance with the studied topics. They increase the activity of students at lectures and seminars and contribute to better understanding of calculus notions while enhancing cognitive interest of the students in calculus studies. Note that this article talked about explicit function graphs. However, we are no less widely used the problems of finding equation (implicit functions) fitting to the given figure. For example, construct an equation corresponding to Figure 31.

We see this pedagogical strategy offered to students, of transition from one representation to another, and using technology, a subtle constructivist approach that will help the student create the conceptual understanding and develop effective ways to information processing, and also, it will be a tool for them to interpret and understand new topics as well.

The conceptual understanding will serve the student in building his self-confidence to help him to believe in himself to continue to process and understand the knowledge that he is previously unable to comprehend. We believe that this will lead to increasing motivation to learn new things in mathematics, it is known that motivation is a necessary condition for learning in general and for meaningful learning in particular, as stated by (Tami Yechiely, 2015), which relies on Ausubel and other researchers.

What is the formula of the figure 31?

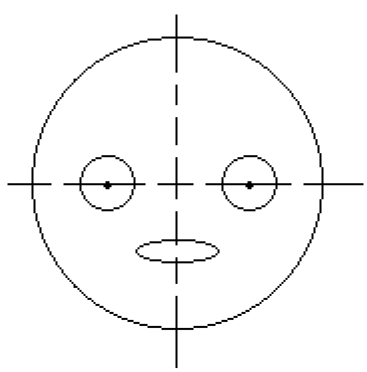

Figure 31

The possible formula:

$$\left(x^2+y^2-16\right)\left(\left(|x|-2\right)^2+y^2-1\right)\left(\left(|x|-2\right)^2+y^2-0.01\right)\left(x^2+9\left(y+2\right)^2-1\right)=0$$

We always tell students "Test your formula on a graphing application". Note that for checking it there is no need to write such a long formula, since most graphics applications allow drawing graphs of several formulas (four here) simultaneously.

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