

Teaching Advanced Mathematics to Middle School Students: Success, Challenges, and Applications to the College Classroom

Rebin Muhammad¹ and Megan Powell²

¹Ohio University, Athens, OH, ²University of North Carolina, Asheville, NC

Abstract: In this paper, we discuss teaching in a three-week mathematics summer program for middle school students and how lessons learned from teaching in that environment can be applied to the college classroom. The program is run by the Bridge to Enter Advanced Mathematics (BEAM) program who seek to reach under-served populations and help support them on a path to college and beyond through the teaching of advanced mathematics. We detail two courses taught by the authors in Summer 2019 on Infinite Series and Frieze Patterns, documenting both the successes and challenges of teaching such material to inexperienced students. As college instructors, we discuss how teaching middle school and college students is both similar and different and ways we believe the successes in the BEAM courses can be applied in the college classroom.

INTRODUCTION

Bridge to Enter Advanced to Mathematics (BEAM) is a national program with offices in New York City, NY and Los Angeles, CA that serve to create pathways for underserved students to enter STEM fields. One of their major programs is a three-week summer camp where rising 8th graders are housed on a college campus outside of the city in which they reside and learn advanced mathematics through a series of week-long courses. Forty students are accepted to the program, with an equal number of boys and girls admitted. Students are admitted based on general math aptitude (not specific math skills) and motivation in studying mathematics. Each week, four topics courses were introduced to the students and they were asked to both try a sample problem from the course and rate their interest in the course. All instructors then met, went through the students work, and both a computer algorithm and faculty input were used to place students into one topics courses each week.

In this paper, we describe the successes and challenges of teaching upper level mathematics to such a young age-group. We compare how teaching high level material to middle school students compares to teaching college students and what we can learn from teaching in this unique environment.

The two courses we will discuss are Infinite Series, as usually seen in a Calculus 2 undergraduate course and Frieze Patterns and Wallpaper Groups that would be seen as a

part of a special topics course in an undergraduate or graduate curriculum. The courses were taught by faculty with extensive experience teaching college students and were supported by an undergraduate or graduate teaching assistant who could help with student questions, especially during small group work time. Both courses consisted of 12 students who self-selected as interested in the course. Each course ran for 9 two-hour time blocks over the course of 5 days with no homework or exams given.

INFINITE SERIES

The goal of this course was for students to understand the concept of an infinite series and develop tools for determining if a series converged or diverged. The students were introduced to infinite series with a paper cutting activity (based on 3.7 Fun with Infinite Series p. 101 in [1]). They lined up strips of paper created by cutting a square piece of paper in half, either cut consistently the same way, or rotated at a right angle between cuts. In this, students were able to see the values (length of the strip) they were adding and determined if the length of an infinite number of such strips lined up would be finite or infinite. Students were then introduced to geometric series using a bouncy ball activity [1] determining the total distance a bouncy ball bounces when dropped from a consistent height. The class then developed the geometric series formula together and worked on problems finding sums of geometric series. The divergence test was then introduced with work on intuitively finding limits at infinity based on comparison of the sizes of the numerator and denominator of a term. The harmonic series was presented as a classic example of a series that does not converge even though the corresponding sequence does converge to 0. Finally, the direct comparison and limit comparison tests were introduced, and students were given ample time to practice using the comparison tests to determine convergence and divergence of series. Students were asked to write their solutions in complete sentences using proper sigma notation and citing which test they used. Immediately preceding any work requiring specific skill sets, students were asked to complete a short worksheet on basic skills such as comparing fractions, multiplying/dividing fractions, or combining exponents in rational expressions.

Successes

Starting with a hands-on activity in cutting the paper, students were able to visually see what was being represented with an infinite series. Tactile activities are seldom used in the college classroom, and students of any age can benefit from physically interacting with the concept they are learning. Students picked up on using the formal sigma notation well and it seems, as long as properly introduced, sophisticated mathematical notation can and should be introduced as early as possible, so students become familiar with using it. Worksheets on basic skills immediately preceding a new idea allowed students to recall skills learned in the past and then apply them to the present concept. The transition between a short worksheet on

combining exponents such as x^3x^6 and $\frac{x^7}{x^{-3}}$ to geometric series allowed students to more immediately see how to manipulate an expression such as $\sum_{n=1}^{\infty} \frac{2^{n+3}}{3^{n-2}}$ into the form $\sum_{n=1}^{\infty} ar^n$.

When teaching college students, we often just assume they have all prerequisite knowledge immediately ready, even when it may have been years since they have seen a specific concept or skill. Small prerequisite worksheets immediately before a skill or concept is needed in a course may prove more valuable than a longer review of prerequisite material at the beginning of a course, where students may not see the need for specific knowledge until much later in the course.

Students were strategically put together in small groups based on ability. The groups of strong and/or focused students worked together extremely well. They made their group members convince them of their proof and why it worked and came up with creative ways to manipulate geometric series or use the comparison tests in ways that the instructor and TA even did not immediately see. The stronger groups worked well independently, allowing the instructor and TA to spend more time with groups that were more challenged by the material. Group members were given specific roles of writer, math proofreader, and writing proofreader which helped each student feel they had something specific to contribute to the group. Having specific roles in a group work is suggested in active learning classrooms [4][3], and college students may feel more engaged if they have a specific role when working in a group. By the end of the course, all students were able to apply the comparison tests correctly and write their answer in complete sentences. Near the end of the course when four groups were working independently, additional support staff were asked to come, so each group had an instructor guiding them. This was very valuable for helping each group to achieve the maximum success.

While small class sizes and additional instructional support is generally not feasible in a college setting, strategic grouping may help the instructor spend more time with students who need additional support. Finally, since all the staff and student lived together, if something was off with a student, generally the instructor could get immediate feedback from other staff members if there had been a physical, mental, emotional, or social problem that may be changing a student's behavior in the classroom. Extensive understanding of the cause of student behavior is not achievable and sometimes not appropriate in the college classroom, but acknowledging when student behavior is out of the norm may allow instructors to interact with the students in a way that considers how external stressors may affect student behavior and performance in the classroom.

Challenges

The students at BEAM are often away from home for the first time and have trouble managing themselves in a new environment. One student consistently refused to work because she was

cold in the air-conditioned classroom but did not then bring warmer clothes to subsequent classes. Other students did not sleep properly and found it challenging to stay awake during class time. This was exceptionally obvious during Week 1 when the Infinite Series course was taught. While college students do not always manage their eating, sleeping, and life habits perfectly, they are generally more able to control how physical discomfort affects their emotions in the classroom. While college students are generally motivated to focus in the classroom even when tired, hungry, cold, etc. in order to achieve the desired grade in the class, they can have underlying discomfort due to anxiety, food insecurity, financial issues, and other personal problems that can inhibit them being able to fully engage in class.

Understanding why BEAM students were uncomfortable did not always allow the instructor to immediately improve the unfocused behavior but allowed for greater understanding of how to address issues of discomfort in subsequent classes. This can be applied to a lesser extent in college classes in making an effort to understand what discomfort students are facing in an attempt to be supportive where applicable. BEAM students were not held accountable for their learning through any sort of assessment. Therefore, unless they were consistently unfocused or disruptive, there was little consequence to not paying attention in class.

BEAM students also struggled to stay engaged with the material sometimes and it was challenging to get them to do anything without consistent prompting. It was unclear if this was due to a lack of understanding, interest in the material, or simply being overwhelmed by the amount of math they were ingesting each day. Instructors in college settings often face similar challenges in not knowing if students need more instruction in order to stay focused, or are simply distracted, usually by cell phones and other technology which were not allowed in the BEAM classrooms.

One of the other main difficulties students in BEAM had was putting their ideas into writing. Students were consistently encouraged to work on the board and were excited to do so in contrast to college students who seldom want to work on the board for fear of making mistakes in front of others. Both BEAM and college students struggle to write mathematics in complete sentences, a critical skill for future study and employment. Mathematics instructors at all levels need to embed more mathematical writing into the curriculum to help develop writing skills over time.

FRIEZE PATTERNS AND WALLPAPER GROUPS

The original intention of the course was to do both one dimension and two-dimensional symmetry but due to time, only Frieze patterns were covered in depth using [6] and [2]. Students were given an understanding of the definition of rigid motion and specific definitions of rotation, reflection (horizontal and vertical), translation, glide reflection. The meaning of symmetric rigid motion and how the four types of symmetries can be combined

was discussed as a class and students then worked in small groups to determine if given patterns contained the different types of symmetries or combination of symmetries.

In the first two days of the class, students were only working on symmetry of finite shapes, using physical Islamic geometric pattern tiles. Students were asked to find the types of symmetry they had each tile had and then worked with the patterns in an organized manner. For example, one of the symmetry of one tile was D_4 student where creating Cayley table (Table 1) then asked to consider if there were types of patterns that were not on the table.

D_4	1	r	r^2	r^3	m	mr	mr^2	mr^3
1	1	r	r^2	r^3	m	mr	mr^2	mr^3
r	r	r^2	r^3	1	mr^3	m	mr	mr^2
r^2	r^2	r^3	1	r	mr^2	mr^3	m	mr
r^3	r^3	1	r	r^2	mr	mr^2	mr^3	m
m	m	mr	mr^2	mr^3	1	r	r^2	r^3
mr	mr	mr^2	mr^3	m	r^3	1	r	r^2
mr^2	mr^2	mr^3	m	mr	r^2	r^3	1	r
mr^3	mr^3	m	mr	mr^2	r	r^2	r^3	1

Table 1: The Cayley table for D_4

The rest of the week was about symmetry of infinite shapes such as Frieze Patterns. Students received a variety of Frieze patterns and the Islamic geometric pattern tiles and were asked to construct patterns themselves.

Students were then were asked to create conjectures about types of symmetries that exist in Frieze patterns such as “every Frieze pattern has translation”, “a Frieze pattern with horizontal reflection has a glide reflection.” Students then came together as a class to determine if they agree and can prove or disagree and can provide a counterexample with each of the listed conjectures. Through this process, the class identified the existence and uniqueness of the 7 groups of Frieze patterns (F1, ..., F7). Students were then given a list of one-dimensional patterns that are all Islamic Geometric Patterns such as in Figure 1 and asked to determine whether it is a Frieze pattern. After recognizing a pattern as a Frieze pattern, students identified the repeated tile of the pattern, the types of symmetry of the pattern, and classified the pattern as one of the 7 groups of Frieze patterns.

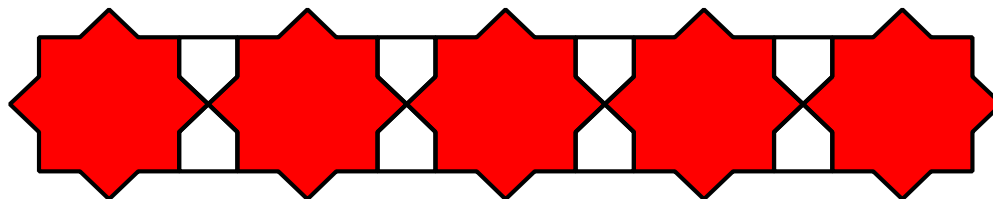


Figure1: Khatem of Sulaman

Successes

The biggest success of the course was the students' ability to make conjectures about Frieze patterns. Even those students who were not exceptionally strong or motivated were able to successfully make conjectures such as Student A "if a pattern has a reflection, then it has a glide reflection" and Student B "if a pattern has both a mirror reflection and a translation, then it has most likely a glide reflection." They were also successful in critiquing each other's conjecture such as Student A recognizing Student B's conjecture was not correct and not well written due to "most likely." This was a Week 2 course, so students had a foundation in the mostly proof-based courses that were taken in Week 1. The class was able to successfully prove the conjectures they decided were true and well-posed and showed the existence and uniqueness of the seven types of Frieze patterns. In college, students are seldom given the opportunity to make conjectures themselves. Usually in proof-based courses, they are given theorems and either given the proof or asked to prove. Allowing more time to discover conjectures may be helpful to deepen student understanding, even though it is time consuming. Even in an introductory course such as Calculus 2, if students are given the time for discovery such as the proper format for a partial fraction decomposition, the material becomes more their own than a list of formulas to remember.

Challenges

Students struggled when applying two symmetries, how to write and read the order. They often assumed that applying symmetries is commutative when it is not and had a hard time understanding symmetries are written from right to left as they are applied. This is similar to the struggle college students often have in pre-Calculus in understanding composition of functions in reading or writing it from right to left as the function is applied or applying the chain rule properly in Calculus. We suggest spending time thoroughly exploring the commutative property (where it is true and not true) in ways the students are more familiar (example addition and subtraction) before exploring new concepts that are unfamiliar.

Students assume that every operation is commutative which can cause fundamental misunderstanding of operations that are not. This struggle with the application of symmetries being non-commutative was especially apparent when students were asked to create a Cayley table. They struggled to understand how to read the rows and columns. When they confused reading the column and then row, it was unclear if they assumed that they could be read in both directions or did not understand the order in which to take the symmetry. Additionally, when filling in the Cayley table, if the students came across an element that was not one of the 8 elements of the Dihedral group D_4 , they could not readily see how their element was equivalent to an element of D_4 , especially when the element involved two different types of symmetries.

CONCLUSION

Through both structured learning opportunities and inquiry-based activities [5], middle school students at BEAM were able to learn sophisticated mathematics at a level equivalent or beyond that of many college students. The curricular freedom allowed instructors to spend additional time if needed for specific concepts without being concerned with not covering all the concepts intended. Additionally, small classes, TA support, and a controlled environment allowed students to engage with the material without worrying about other courses, homework, grades, and external stressors. Students were also in a single topic course each week for four hours a day, which allowed for much more continuity than is found in the college classroom. While much of the environment at BEAM Summer Away is not reproducible in the college classroom, we believe a few successes of the BEAM classroom may enhance college student learning. First, BEAM students were given time to physically work with items (paper cutting in Infinite Series and Islamic geometric tiles in Frieze Patterns) to help engage with a new and difficult concept. We think students of all ages can benefit from some time spent allowing them to play will help them take ownership of the new material more.

Second, we believe college students can be given more time to develop their own conjectures. This can be done in both lower level courses such as Calculus where students can be given an opportunity to develop conjectures related to continuous and differentiable functions and potentially developing the intermediate and mean value theorems on their own and in upper level courses such as Abstract Algebra where students can discover if symmetry group of any finite group is either Cyclic or Dihedral group.

Lastly, we found success in a discussion and short practice of a prerequisite concept or skill needed immediately before introducing a new concept. Courses such as Calculus and plagued with students of weak prerequisite knowledge and we believe short practice of the prerequisite skill immediately before the new idea will help students see the connections more readily.

While a three-week summer camp for high performing middle school students and a semester-long college class are vastly different environments for teaching and learning, we found distinct ways that understanding each can benefit teaching in the other. In the right circumstances, middle school students can understand much deeper mathematics than is usually taught in schools, and college students can greatly benefit from being allowed time for more discovery in mathematics.

References

- [1] Julie Barnes and Jessica M. Libertini. *Tactile learning activities in mathematics: a recipe book for the undergraduate classroom*. 2018. isbn: 9781470443511.
- [2] David W. Farmer. *Groups and symmetry: a guide to discovering mathematics*. Mathematical world: v. 5. American Mathematical Society, 1996. isbn: 0821804502.
- [3] Scott Freeman et al. “Active learning increases student performance in science, engineering, and mathematics”. In: *Proceedings of the National Academy of Sciences of the United States of America* 111.23 (June 2014), pp. 8410–8415. issn: 10916490. doi: 10.1073/pnas.1319030111.
- [4] Jeffrey S. Rosenthal. “Active Learning Strategies in Advanced Mathematics Classes”. In: *Studies in Higher Education* 20.2 (Jan. 1995), pp. 223–228. issn: 1470174X. doi: 10.1080/03075079512331381723.
- [5] Laursen Sandra L. et al. “Benefits for Women and Men of Inquiry-Based Learning in College Mathematics: A Multi-Institution Study.” In: *Journal for Research in Mathematics Education* 45.4 (2014), p. 406. issn: 00218251.
- [6] Belcastro Sarah-Marie and Hull Thomas C. “Classifying Frieze Patterns without Using Groups.” In: *The College Mathematics Journal* 33.2 (2002), p. 93. issn: 07468342.