

Editorial from Bronislaw Czarnocha

In this new issue of MTRJ, we turn to the central matter of classroom teaching-research: teaching experiments and classroom investigations of mathematical creativity of our students. We present two teaching experiments connected with teaching calculus.

The first one is a teaching experiment during which the advanced college topics of Infinite Series together with Frieze Patterns and Wallpaper Groups were taught to middle school students during a Bridge to Enter Advanced Mathematics summer program (BEAM). The authors, college faculty teaching the course, draw interesting reflections comparing the learning process of middle school students with that of college students. In particular, a successful idea of reviewing pre-requisites to the upcoming topic to be conducted by the “just-in-time” method is suggested for college teaching. The authors report success in enabling learning advanced mathematical topics in middle school; they point out however, to the behavioral difficulties of this population.

The second teaching experiment, also in Calculus, here for Engineering students addresses familiar procedural/conceptual divide in the mathematics knowledge of incoming university students. The teaching experiments addresses the divide by focusing student attention on transitions between different representations of the same concepts. Interesting problems are designed to promote student reflection and translations to different representations. The presentation contains examples of instructor/student dialogues with the help of which instructor directs student attention. Authors report an increase of understanding of mathematical concepts by the experimental group in relation to the control group.

These teaching experiments are followed by the report of our colleagues from BMCC of CUNY about the design and implementation of a new developmental course of Intermediate Algebra and Trigonometry based completely on Open Educational Resources (OER). The theme is very much “just-in-time” at CUNY which strongly promotes use of OER materials. The report shows nuts and bolts elements of the design and implementation. The course included an online platform, videos relating to the text discussed, as well as conceptual algebra games. The course was piloted by several colleagues/instructors and accepted into the mathematics department’s curriculum.

We present two book reviews, one by Małgorzata Marciniak, directing oneself to the classroom learning written by the previous dean of Education at CCNY, Dr. Posamentier who shows us his deep knowledge of the classroom cognition as well as his appreciation of subtle moments in the history of early mathematics. The second review, by Roy Berglund is of the book written by Cédric Villani, the director of the *Institut Henri Poincaré* in Paris and a recipient of the Fields Medal in 2010. There we learn about pathways of thinking of a research mathematician, its advances and traps. Being the director of the *Institut Henri Poincaré*, he closes his story with the recollection of the Aha!Moment which led him to the final proof of the theorem – very similar to Henri Poincare famous descriptions of his own Aha!Moments. That particular ending leads us to the second part of the current issue and that is mathematical creativity of our students with a special attention to the creativity of Aha!Moment.

The next three papers addressing mathematical creativity of our students refer, in different degrees to Arthur Koestler's bisociation, the theory of Aha!Moment described in his Act of Creation.

Our colleague from LaGCC, Małgorzata Marciniak, reflects upon creativity of her students, especially research students whom she mentors in mathematics research projects, and upon her own creativity in discovering new patterns in sushi making. Both he and her research student realize that creativity takes place all the time in everyday life. And that is of course the "added value" of doing research project in which one can observe his own creativity; we can transfer our newly developed awareness of creativity into other domains of life. In other words, once you see and witness creativity somewhere, you see and can witness it everywhere. Marciniak emphasizes the role of a group in facilitating student creativity.

William Baker and his colleagues delve into details of facilitating Aha!Moments of creativity in the social environment of a class. The method of investigation is accomplished by class visits of a second professor, the teacher-researcher. In these classes the teacher presents some methods of teaching which might facilitate the moments of understanding or making meaning. One can call it a mini-lesson study. The teacher-researcher observes the classroom and takes notes, which are discussed after the class. Baker adopts a particular angle in his investigations, that of "internalization" that is, in Vygotsky's understanding, "an internal reconstruction of external operation".

Bronisław Czarnocha and Hannes Stoppel present the first part of the discussion describing internal structure of the creativity occasioned by Aha!Moments. What strikes them as surprising is that the three different levels of the bisociative structure of Aha!Moments observed in mathematical classroom or in mathematicians reports correspond closely to the types of bisociative structure discovered by a new bisociative search engine BISON. That prompts them to wonder on what is really human in human creativity.

Contents

Teaching Advanced Mathematics to Middle School Students: Success, Challenges, and Applications to the College Classroom	4
<i>Rebin Muhammad and Megan Powell</i>	
Teaching Calculus for Engineering Students Using Alternative Representations of Graph-formula Problems	12
<i>Miriam Dagan, Pavel Satianov, Mina Teicher</i>	
So, You Want to Write an OER? Three Authors Share Triumphs, Pitfalls, and Options	43
<i>Kathleen Offenholley, Jenna R. Hirsch, Yevgeniy Milman</i>	
Book Review for “Numbers: Their Tales, Types and Treasures” by A.S. Posamentier and B. Thaller	49
<i>Małgorzata Marciniak</i>	
Book Review for BIRTH OF A THEOREM by Cédric Villani	51
<i>Roy Berglund</i>	
Reflections on Creativity in a Diverse College Classroom	54
<i>Małgorzata Marciniak</i>	
Creative Moments within Internalization: Mathematics Classroom	65
<i>William Baker, Anders Stachelek and Lauren Wolf</i>	
The Structure of Creativity of Aha!Moments in Mathematics (part 1)	77
<i>Bronisław Czarnocha and Hannes Stoppel</i>	

Teaching Advanced Mathematics to Middle School Students: Success, Challenges, and Applications to the College Classroom

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Abstract: In this paper, we discuss teaching in a three-week mathematics summer program for middle school students and how lessons learned from teaching in that environment can be applied to the college classroom. The program is run by the Bridge to Enter Advanced Mathematics (BEAM) program who seek to reach under-served populations and help support them on a path to college and beyond through the teaching of advanced mathematics. We detail two courses taught by the authors in Summer 2019 on Infinite Series and Frieze Patterns, documenting both the successes and challenges of teaching such material to inexperienced students. As college instructors, we discuss how teaching middle school and college students is both similar and different and ways we believe the successes in the BEAM courses can be applied in the college classroom.

INTRODUCTION

Bridge to Enter Advanced to Mathematics (BEAM) is a national program with offices in New York City, NY and Los Angeles, CA that serve to create pathways for underserved students to enter STEM fields. One of their major programs is a three-week summer camp where rising 8th graders are housed on a college campus outside of the city in which they reside and learn advanced mathematics through a series of week-long courses. Forty students are accepted to the program, with an equal number of boys and girls admitted. Students are admitted based on general math aptitude (not specific math skills) and motivation in studying mathematics. Each week, four topics courses were introduced to the students and they were asked to both try a sample problem from the course and rate their interest in the course. All instructors then met, went through the students work, and both a computer algorithm and faculty input were used to place students into one topics courses each week.

In this paper, we describe the successes and challenges of teaching upper level mathematics to such a young age-group. We compare how teaching high level material to middle school students compares to teaching college students and what we can learn from teaching in this unique environment.

The two courses we will discuss are Infinite Series, as usually seen in a Calculus 2 undergraduate course and Frieze Patterns and Wallpaper Groups that would be seen as a

part of a special topics course in an undergraduate or graduate curriculum. The courses were taught by faculty with extensive experience teaching college students and were supported by an undergraduate or graduate teaching assistant who could help with student questions, especially during small group work time. Both courses consisted of 12 students who self-selected as interested in the course. Each course ran for 9 two-hour time blocks over the course of 5 days with no homework or exams given.

INFINITE SERIES

The goal of this course was for students to understand the concept of an infinite series and develop tools for determining if a series converged or diverged. The students were introduced to infinite series with a paper cutting activity (based on 3.7 Fun with Infinite Series p. 101 in [1]). They lined up strips of paper created by cutting a square piece of paper in half, either cut consistently the same way, or rotated at a right angle between cuts. In this, students were able to see the values (length of the strip) they were adding and determined if the length of an infinite number of such strips lined up would be finite or infinite. Students were then introduced to geometric series using a bouncy ball activity [1] determining the total distance a bouncy ball bounces when dropped from a consistent height. The class then developed the geometric series formula together and worked on problems finding sums of geometric series. The divergence test was then introduced with work on intuitively finding limits at infinity based on comparison of the sizes of the numerator and denominator of a term. The harmonic series was presented as a classic example of a series that does not converge even though the corresponding sequence does converge to 0. Finally, the direct comparison and limit comparison tests were introduced, and students were given ample time to practice using the comparison tests to determine convergence and divergence of series. Students were asked to write their solutions in complete sentences using proper sigma notation and citing which test they used. Immediately preceding any work requiring specific skill sets, students were asked to complete a short worksheet on basic skills such as comparing fractions, multiplying/dividing fractions, or combining exponents in rational expressions.

Successes

Starting with a hands-on activity in cutting the paper, students were able to visually see what was being represented with an infinite series. Tactile activities are seldom used in the college classroom, and students of any age can benefit from physically interacting with the concept they are learning. Students picked up on using the formal sigma notation well and it seems, as long as properly introduced, sophisticated mathematical notation can and should be introduced as early as possible, so students become familiar with using it. Worksheets on basic skills immediately preceding a new idea allowed students to recall skills learned in the past and then apply them to the present concept. The transition between a short worksheet on

combining exponents such as x^3x^6 and $\frac{x^7}{x^{-3}}$ to geometric series allowed students to more immediately see how to manipulate an expression such as $\sum_{n=1}^{\infty} \frac{2^{n+3}}{3^{n-2}}$ into the form $\sum_{n=1}^{\infty} ar^n$.

When teaching college students, we often just assume they have all prerequisite knowledge immediately ready, even when it may have been years since they have seen a specific concept or skill. Small prerequisite worksheets immediately before a skill or concept is needed in a course may prove more valuable than a longer review of prerequisite material at the beginning of a course, where students may not see the need for specific knowledge until much later in the course.

Students were strategically put together in small groups based on ability. The groups of strong and/or focused students worked together extremely well. They made their group members convince them of their proof and why it worked and came up with creative ways to manipulate geometric series or use the comparison tests in ways that the instructor and TA even did not immediately see. The stronger groups worked well independently, allowing the instructor and TA to spend more time with groups that were more challenged by the material. Group members were given specific roles of writer, math proofreader, and writing proofreader which helped each student feel they had something specific to contribute to the group. Having specific roles in a group work is suggested in active learning classrooms [4][3], and college students may feel more engaged if they have a specific role when working in a group. By the end of the course, all students were able to apply the comparison tests correctly and write their answer in complete sentences. Near the end of the course when four groups were working independently, additional support staff were asked to come, so each group had an instructor guiding them. This was very valuable for helping each group to achieve the maximum success.

While small class sizes and additional instructional support is generally not feasible in a college setting, strategic grouping may help the instructor spend more time with students who need additional support. Finally, since all the staff and student lived together, if something was off with a student, generally the instructor could get immediate feedback from other staff members if there had been a physical, mental, emotional, or social problem that may be changing a student's behavior in the classroom. Extensive understanding of the cause of student behavior is not achievable and sometimes not appropriate in the college classroom, but acknowledging when student behavior is out of the norm may allow instructors to interact with the students in a way that considers how external stressors may affect student behavior and performance in the classroom.

Challenges

The students at BEAM are often away from home for the first time and have trouble managing themselves in a new environment. One student consistently refused to work because she was

cold in the air-conditioned classroom but did not then bring warmer clothes to subsequent classes. Other students did not sleep properly and found it challenging to stay awake during class time. This was exceptionally obvious during Week 1 when the Infinite Series course was taught. While college students do not always manage their eating, sleeping, and life habits perfectly, they are generally more able to control how physical discomfort affects their emotions in the classroom. While college students are generally motivated to focus in the classroom even when tired, hungry, cold, etc. in order to achieve the desired grade in the class, they can have underlying discomfort due to anxiety, food insecurity, financial issues, and other personal problems that can inhibit them being able to fully engage in class.

Understanding why BEAM students were uncomfortable did not always allow the instructor to immediately improve the unfocused behavior but allowed for greater understanding of how to address issues of discomfort in subsequent classes. This can be applied to a lesser extent in college classes in making an effort to understand what discomfort students are facing in an attempt to be supportive where applicable. BEAM students were not held accountable for their learning through any sort of assessment. Therefore, unless they were consistently unfocused or disruptive, there was little consequence to not paying attention in class.

BEAM students also struggled to stay engaged with the material sometimes and it was challenging to get them to do anything without consistent prompting. It was unclear if this was due to a lack of understanding, interest in the material, or simply being overwhelmed by the amount of math they were ingesting each day. Instructors in college settings often face similar challenges in not knowing if students need more instruction in order to stay focused, or are simply distracted, usually by cell phones and other technology which were not allowed in the BEAM classrooms.

One of the other main difficulties students in BEAM had was putting their ideas into writing. Students were consistently encouraged to work on the board and were excited to do so in contrast to college students who seldom want to work on the board for fear of making mistakes in front of others. Both BEAM and college students struggle to write mathematics in complete sentences, a critical skill for future study and employment. Mathematics instructors at all levels need to embed more mathematical writing into the curriculum to help develop writing skills over time.

FRIEZE PATTERNS AND WALLPAPER GROUPS

The original intention of the course was to do both one dimension and two-dimensional symmetry but due to time, only Frieze patterns were covered in depth using [6] and [2]. Students were given an understanding of the definition of rigid motion and specific definitions of rotation, reflection (horizontal and vertical), translation, glide reflection. The meaning of symmetric rigid motion and how the four types of symmetries can be combined

was discussed as a class and students then worked in small groups to determine if given patterns contained the different types of symmetries or combination of symmetries.

In the first two days of the class, students were only working on symmetry of finite shapes, using physical Islamic geometric pattern tiles. Students were asked to find the types of symmetry they had each tile had and then worked with the patterns in an organized manner. For example, one of the symmetry of one tile was D_4 student where creating Cayley table (Table 1) then asked to consider if there were types of patterns that were not on the table.

D_4	1	r	r^2	r^3	m	mr	mr^2	mr^3
1	1	r	r^2	r^3	m	mr	mr^2	mr^3
r	r	r^2	r^3	1	mr^3	m	mr	mr^2
r^2	r^2	r^3	1	r	mr^2	mr^3	m	mr
r^3	r^3	1	r	r^2	mr	mr^2	mr^3	m
m	m	mr	mr^2	mr^3	1	r	r^2	r^3
mr	mr	mr^2	mr^3	m	r^3	1	r	r^2
mr^2	mr^2	mr^3	m	mr	r^2	r^3	1	r
mr^3	mr^3	m	mr	mr^2	r	r^2	r^3	1

Table 1: The Cayley table for D_4

The rest of the week was about symmetry of infinite shapes such as Frieze Patterns. Students received a variety of Frieze patterns and the Islamic geometric pattern tiles and were asked to construct patterns themselves.

Students were then were asked to create conjectures about types of symmetries that exist in Frieze patterns such as “every Frieze pattern has translation”, “a Frieze pattern with horizontal reflection has a glide reflection.” Students then came together as a class to determine if they agree and can prove or disagree and can provide a counterexample with each of the listed conjectures. Through this process, the class identified the existence and uniqueness of the 7 groups of Frieze patterns (F1, ..., F7). Students were then given a list of one-dimensional patterns that are all Islamic Geometric Patterns such as in Figure 1 and asked to determine whether it is a Frieze pattern. After recognizing a pattern as a Frieze pattern, students identified the repeated tile of the pattern, the types of symmetry of the pattern, and classified the pattern as one of the 7 groups of Frieze patterns.

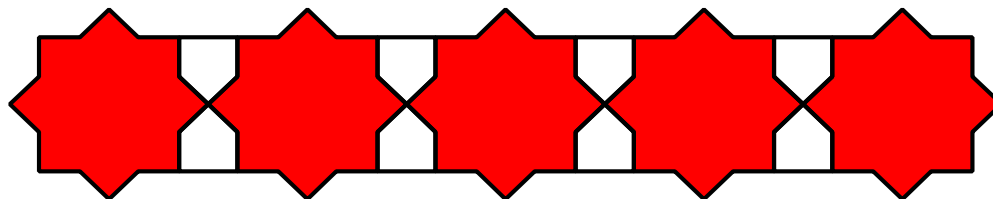


Figure1: Khatem of Sulaman

Successes

The biggest success of the course was the students' ability to make conjectures about Frieze patterns. Even those students who were not exceptionally strong or motivated were able to successfully make conjectures such as Student A "if a pattern has a reflection, then it has a glide reflection" and Student B "if a pattern has both a mirror reflection and a translation, then it has most likely a glide reflection." They were also successful in critiquing each other's conjecture such as Student A recognizing Student B's conjecture was not correct and not well written due to "most likely." This was a Week 2 course, so students had a foundation in the mostly proof-based courses that were taken in Week 1. The class was able to successfully prove the conjectures they decided were true and well-posed and showed the existence and uniqueness of the seven types of Frieze patterns. In college, students are seldom given the opportunity to make conjectures themselves. Usually in proof-based courses, they are given theorems and either given the proof or asked to prove. Allowing more time to discover conjectures may be helpful to deepen student understanding, even though it is time consuming. Even in an introductory course such as Calculus 2, if students are given the time for discovery such as the proper format for a partial fraction decomposition, the material becomes more their own than a list of formulas to remember.

Challenges

Students struggled when applying two symmetries, how to write and read the order. They often assumed that applying symmetries is commutative when it is not and had a hard time understanding symmetries are written from right to left as they are applied. This is similar to the struggle college students often have in pre-Calculus in understanding composition of functions in reading or writing it from right to left as the function is applied or applying the chain rule properly in Calculus. We suggest spending time thoroughly exploring the commutative property (where it is true and not true) in ways the students are more familiar (example addition and subtraction) before exploring new concepts that are unfamiliar.

Students assume that every operation is commutative which can cause fundamental misunderstanding of operations that are not. This struggle with the application of symmetries being non-commutative was especially apparent when students were asked to create a Cayley table. They struggled to understand how to read the rows and columns. When they confused reading the column and then row, it was unclear if they assumed that they could be read in both directions or did not understand the order in which to take the symmetry. Additionally, when filling in the Cayley table, if the students came across an element that was not one of the 8 elements of the Dihedral group D_4 , they could not readily see how their element was equivalent to an element of D_4 , especially when the element involved two different types of symmetries.

CONCLUSION

Through both structured learning opportunities and inquiry-based activities [5], middle school students at BEAM were able to learn sophisticated mathematics at a level equivalent or beyond that of many college students. The curricular freedom allowed instructors to spend additional time if needed for specific concepts without being concerned with not covering all the concepts intended. Additionally, small classes, TA support, and a controlled environment allowed students to engage with the material without worrying about other courses, homework, grades, and external stressors. Students were also in a single topic course each week for four hours a day, which allowed for much more continuity than is found in the college classroom. While much of the environment at BEAM Summer Away is not reproducible in the college classroom, we believe a few successes of the BEAM classroom may enhance college student learning. First, BEAM students were given time to physically work with items (paper cutting in Infinite Series and Islamic geometric tiles in Frieze Patterns) to help engage with a new and difficult concept. We think students of all ages can benefit from some time spent allowing them to play will help them take ownership of the new material more.

Second, we believe college students can be given more time to develop their own conjectures. This can be done in both lower level courses such as Calculus where students can be given an opportunity to develop conjectures related to continuous and differentiable functions and potentially developing the intermediate and mean value theorems on their own and in upper level courses such as Abstract Algebra where students can discover if symmetry group of any finite group is either Cyclic or Dihedral group.

Lastly, we found success in a discussion and short practice of a prerequisite concept or skill needed immediately before introducing a new concept. Courses such as Calculus and plagued with students of weak prerequisite knowledge and we believe short practice of the prerequisite skill immediately before the new idea will help students see the connections more readily.

While a three-week summer camp for high performing middle school students and a semester-long college class are vastly different environments for teaching and learning, we found distinct ways that understanding each can benefit teaching in the other. In the right circumstances, middle school students can understand much deeper mathematics than is usually taught in schools, and college students can greatly benefit from being allowed time for more discovery in mathematics.

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Teaching Calculus for Engineering Students Using Alternative Representations of Graph-formula Problems

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Abstract: This paper, comes as our response to the nowadays situation, of a gap between the skills required in academia studies and the cognitive skills with which the students come to us. We see the main reason for this gap in insufficient development of sovereign thinking and research skills, necessary in academic studies as well as in the future work of the modern engineer. This situation arises in secondary and high schools, which continue to focus on algorithmic skills (the value of which decreased in our time) and not on the development of critical thinking and understanding mathematics as a language of science and engineering. We address this challenge and develop a pedagogical strategy where we present objects from different use and different math language using technologies. We see active use of alternative representations of graph-formula problems in teaching calculus to novice engineering students as an effective way to change this situation. Our many years of experience shows that this way promotes cognitive interest, research thinking and deeper understanding of principal concepts of calculus. We have not found any literature focusing on using such an approach to graph-formula problems in calculus education. We believe that the examples presented in this paper which are chosen from our lengthy experience will demonstrate how we develop mental orientation as well as digital orientation in the study of calculus for first year engineering students. It may be also interesting to teachers of mathematics in academic studies, as well as high school teachers.

Keywords: knowledge gap, calculus teaching, graph-formula problems, different representations, pedagogical strategy, creative thinking, cognitive motivation.

INTRODUCTION

Calculus is the first mathematics course given to engineering students. The greatest importance of this course is in using it as a base for obtaining mathematical tools that are necessary for advanced academic studies. However, and just as salient, it is an important means for developing deep and creative mathematical thinking by the novice engineering student. In order

to internalize the abstract mathematical concepts of Calculus students should be able to solve mathematical problems, to master algebraic technique, to have an advanced mathematical thinking strategy, and to master the mathematical language. In recent years, many researchers and lecturers around the world have pointed to a gap between the skills, required in academia studies and cognitive abilities of novice engineering students (Gueudet, 2013).

The way studies relate to the knowledge gap

For years, academic institutions have been dealing with the failure of the students in the first year of their studies in general and especially in mathematical courses (Lowe and Cook, 2003), (Yorke and Longden, 2004). With technological advances and the flow of students learning STEM professions at universities and colleges, there is a constant decline in the mathematical basic knowledge of the novice academic students (Gueudet, 2013), (Bosch, Fonseca & Gascon, 2004), (Trevor et al 1999), (EMS Committee on Education, 2011). It is noted that the decrease in the mathematical knowledge of students in the world exists also among students in departments of Mathematics, Exact Sciences and the various engineering professions.

Although high school students, in many countries, reach a high level of technical skills for solving exercises (such as finding derivatives, calculating infinite integrals and more), we still discover a lack of understanding of the concepts that the techniques are based on, for example, in the case of limits of functions and sums. Some of the reasons for this situation are the needs of students to go through many high-risk exams in high school, in addition to the fact that the mathematical concepts are abstract and complex (Holton, 2001).

Undergraduate students also have difficulties in understanding definitions and various representations of mathematical concepts in general and the concept of function in particular (Tall & Vinner, 1981), (Vinner, 1980) & (Hershkowitz, 1980). According to them, even when the student has the formal definition, he does not use it in his mathematical activity, in most cases he or she decides on the basis of concept image solely, namely on the set of all mental images associated in his mind with the name of this concept. "Mental image" refers to any type of representation: visual, symbolic form, chart, graph, etc. In other words, there is a problem among students in the first year of thought processes (such as the conceptualization, generalization, abstraction, assessment, reasoning and critical thinking). Therefore, it is important in teaching to examine the images of students regarding various mathematical concepts (such as a function, limits, continuity, derivative, integral, etc.); it is important to focus on the aspects related to the conceptual-building of concepts, and on the process of reasoning and proof. Mark and others (Mark et al, 2004) present a research framework and proposals for teaching undergraduate

students. In this study, the principles of constructivism are presented (according to Piaget) and ways to cope with the problem of concept assimilation.

Another factor is the way of teaching and learning in high school that were essentially “mechanical” (algorithmic learning and memorization). In some literature this is called “rote learning” (Tami Yechiely, 2015, 2012), (Hamidreza Kashefi et al, 2012), which is defined as “ritual participation” as a Latin language learning based mainly on memory and requires more physical efforts and less mental efforts.

Heyd-Metzuyanin (2013, 2015), based on Anna Sfard, notes that this type of study can lead to a cycle of difficulties and failures, and does not lead to investigative thinking, but rather strengthens a ritual participation. Learning of this type may impair motivation to learn mathematics. The researchers (Kira J. et al, 2013) reinforce Heyd-Metzuyanin and claim that it also exists among students in academic institutions. (Yudariah bT., 1998) indicates that one of the reasons for this gap is also the technological development that led to decrease in the motivation of students to learn topics that are no longer useful in every day.

The gap is also evident that students in the academic studies need to give and to understand the deductive proofs, which did not learn enough in high school. (Mariotti et al, 2004), (Engelbrecht, 2010). Review of the literature (Hanna & de Villiers, 2008; Hemmi, 2008) indicates that books do not actually help students acquire the deductive method of proofs. The researchers ((Lithner 2000, Harel 2008) claim that the mathematics at the academy depicted as experts ' mathematics. According to the researchers, students are required to find examples, develop flexible use of different types of representations, experiment and control them at a theoretical level and more.

Many academic institutions try to overcome this gap by special preparing courses, reinforcements and extra hours, but they have not yet found perfect solutions (de Guzman, Hodgson, Robert & Villani, 1998); (Gueudet, 2008);(Thomas at al., 2012), (Gueudet, 2013), (Artigue Batanero & Kent, 2007); (Kira J., 2013).

Various representations

Representations are descriptions of mathematical concepts, operations, algorithms and problems. Researchers determined that the ability to identify and present a particular concept in different representations and the ability to move from one representation to another of that entity is an essential component of mathematical knowledge (Friedlander & Tabach, 2001). Some Researchers report that understanding of the concept in one representation does not necessarily guarantee the understanding of it in the other representation (Bakker, 2004). Therefore, in order to ensure the improvement of learning and understanding during the teaching of a particular mathematical idea

there is a variety of questions to be presented and to ask the students to express the idea in other representations (Gagatsis & Shiakalli, 2004).

Our pedagogical strategy

In our attempts to find a solution to the gap mentioned above, we use the theory of constructivism in a gentle or easy way that is, to bring the learner to take responsibility for his learning until understanding. We believe that this can be done with the model built according to the following recommendations: **“to develop flexible use of different types of representations”** (Lithner, 2000; Harel, 2008) and **“teaching in a visual-intuitive approach”** (Eisenberg and Dryfus, 1991), they think that mathematical reasoning based on visualization facilitates students. In the article of (Kathleen M., 1988), emphasized a teaching method that encourages students to **“enable an active thinking about graphs by using computerized technologies.”**

Our pedagogical strategy is in this spirit, demanding to integrate the teaching at least in the following three representations: algebraic, graphic, verbal, by using scientific calculators, so we built a collection of Questions for understanding and deepening in calculus (Dagan, M., Satianov, P., 2009).

Our experience shows a lack of understanding of basic concepts such as algebraic rules; relationship between formulas and graphs; elementary functions and their main properties; radian angle measurement; direct and inverse trigonometric function. Moreover, it is typical for most novice students not to be able to formulate precise definitions of main mathematical notions studied in higher school. Our pedagogical strategy is to overcome this gap based on the attitude to mathematics as a language with diverse sub-languages (verbal, analytical, graphical, numerical), and permanent and systematic using of various representations while formulating mathematical notions and solving different mathematical problems with scientific calculators if it needs (Dagan & Satianov, 2006). To internalize a mathematical concept perfectly, it is important to understand it in all its forms, that is, in all sub-languages (graphical, analytical, verbal and numerical) in order to create an accurate and effective “concept image” (Tall & Vinner, 1981). In the research literature, we did not find a reference to this type of approach.

Implement the strategy on the subject 'elementary functions and their graphs'

Our long experience shows that the permanent and systematic use of these representations, and various transitions among them in a calculus course in addition to extensive use of scientific calculators and graphic applications, yields greater achievement and a high level of understanding,

all while enhancing cognitive motivation and creative thinking of the students. (Dagan, Satianov, Teicher, 2018), (Dagan, Satianov, Teicher, 2019). This approach is in accordance with the requirements of 21st century demanding profound changes in the teaching of mathematics for engineering students, (Asia at Al, 2013), (Bressoud at Al, 2016), (Kissane and Kemp, 2012), (Tall at Al, 2008). In this article, we will illustrate these ideas in the process of teaching the theme of ‘elementary functions and their graphs’, which is fundamental for understanding calculus.

For convenience, we will designate these basic forms of representations by means of the first letters of the appropriate words:

A – Analytical representation

V – Verbal representation

G – Graphical representation

N – Numerical representation

We will use the symbolic designations described above in order to classify problems

AG – From analytical representation to graphical representation

Example 1: Draw the graph of the formula: $y = \frac{x}{|x|}$.

GA – From graphical representation to analytical representation.

Example 2. Find a formula for the graph in Figure 1.

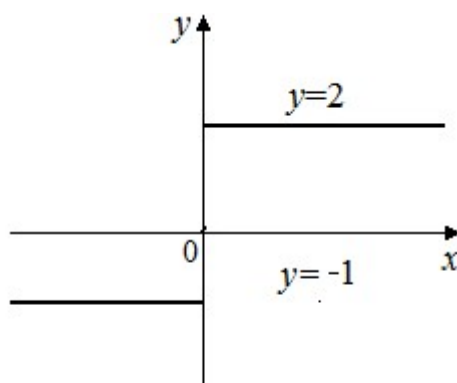


Figure 1

The GA type problems require good knowledge of graphs of elementary functions and their transformation due to the suitable formula changes.

VG – From verbal representation to graphical representation.

Example 3. Draw a curve with two points of inflection.

GV – From graphical representation to verbal representation

Example 4. Describe the main properties of a function (domain of definition, parity of function, point of discontinuity, asymptotes) given by the curve in Figure 2.

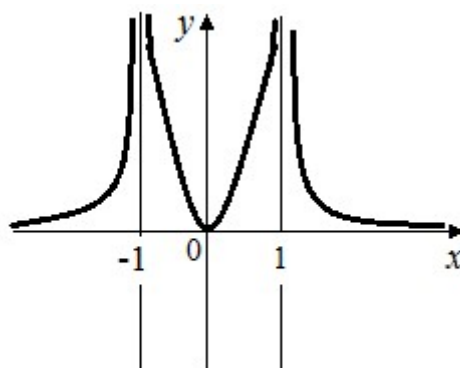


Figure 2

The VG type problems requires knowledge of the basic concepts related to elementary functions and graphical representations of these concepts.

GN – From graphical representation to numerical representation.

Example 5. Find the slope of the straight line of Figure 3.

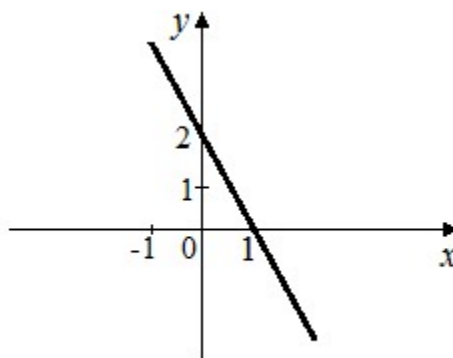


Figure 3

The GN type problems requires an acquaintance with functions concepts and suitable numerical calculations or appropriate graphical constructions to get the desired result.

NG – From numerical representation to graphical representation.

Example 6. Draw a graph of even functions that is suitable for the following table:

$$\begin{array}{cccc} x: & 0 & 1 & 2 & 3 \\ y: & 1 & 0 & 2 & 1 \end{array}$$

The NG type problems requires the use of numerical information and mention in the problem function properties for drawing the suitable curve.

The most widespread graphical problems in the studies of functions and graphs are from the AG type, connected to sketching graphs of functions given by formulas after analytical investigation. In our paper we concentrate on the opposite type of problems – constructing formulas for the functions, based on the graphical representations. Solving problems of this type requires deeper and more creative thinking. The use of GA-type problems in a calculus class provides a great opportunity to encourage different levels of thinking among the students and to develop cognitive interest in calculus studies.

We will distinguish between different types of GA problems.

Type 1. Finding the precise formula for a given graph.

Solving such problems requires acquaintance with appropriate elementary functions and understanding of suitable approaches to finding the formulas.

Example 7. Constructing the formula for the straight line in Figure 3.

Approach A. (Used by most students). Find slope m of the given straight line by numerical calculation and follow the proper formula.

$$\begin{aligned} y &= y_0 + m(x - x_0); \quad m = \frac{\Delta y}{\Delta x} = \frac{2 - 0}{0 - 1} = -2; \\ x_0 &= 1; y_0 = 0 \Rightarrow y = -2(x - 1) \Rightarrow y = 2 - 2x \end{aligned}$$

Approach B. Use the 'straight line equation in the segments':

$$\frac{x}{a} + \frac{y}{b} = 1; a = 1, b = 2 \Rightarrow \frac{x}{1} + \frac{y}{2} = 1 \Rightarrow y = 2 - 2x$$

Approach C. Construct a formula in the form $y = mx + n$ based on the data shown in Figure 3.

$$n = y(0) = 2; y(1) = m \cdot 1 + 2 = 0 \Rightarrow m = -2 \Rightarrow y = -2x + 2$$

Approach D. Use the determinant formula of the straight line passing through two given points, $(x_1, y_1); (x_2, y_2)$:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 0 \Rightarrow -2x - y + 2 = 0 \Rightarrow y = 2 - 2x$$

Example 8. Constructing the formula for the graph in Figure 4.

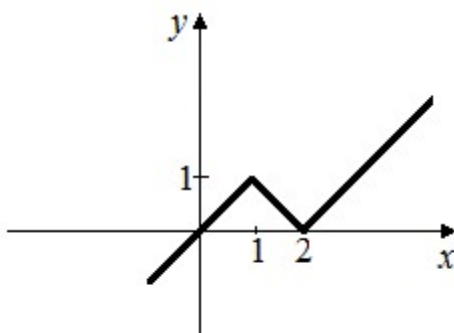


Figure 4

To find a suitable formula for the graph one needs more profound knowledge and deeper thinking and creativity (especially the first time we encounter this kind of problem).

Approach A. Use the graphs of Figure 5 and the $\min(a, b)$ elementary operation:

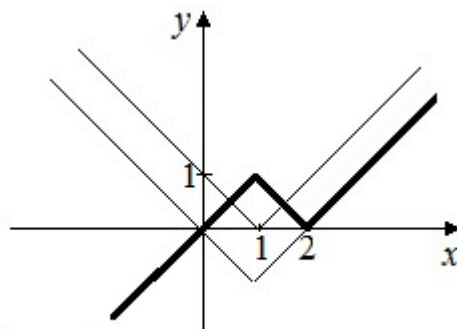


Figure 5

$$y = |x - 1| \rightarrow y = \left| |x - 1| - 1 \right| \rightarrow y = \min(x, \left| |x - 1| - 1 \right|)$$

Note that we widely use min/max functions from the very beginning of the study of calculus ([11]).

We strongly urge our students to test the constructed formula by using some graphic applications. This makes the student feel good and even be happy if he or she can find the appropriate formula. With a suitable computer program or graphic calculator, the analytical formula enables the instant drawing of the graph of the functions. This is also important for students to be acquainted with the proper software and computer input languages. If the graphics application creates a curve that does not fit the given graph it will be a sign that the student should invest a little more thought to finding a solution by correcting the previous formula or looking for another way.

In the formula $y = \min(x, \left| |x - 1| - 1 \right|)$, we used a min operation (see [11]) that sometimes does not exist in a graphic application. However, in this case the students can use the absolute value function for input to the program by means of the following formula:

$$\min(a, b) = \frac{a + b - |a - b|}{2}$$

Therefore:
$$y = \min(x, \left| |x - 1| - 1 \right|) = \frac{x + \left| |x - 1| - 1 \right| - \left| x - \left| |x - 1| - 1 \right| \right|}{2}$$

Approach B. Think about a formula for a broken line in Figure 4 in the form:

$$y = ax + b + c|x - x_1| + d|x - x_2|.$$

Here, x_1, x_2 are the turning points of the line.

In our case $x_1 = 1, x_2 = 2$ and we will subsequently get:

$$y = ax + b + c|x - 1| + d|x - 2|$$

$$\begin{cases} y(0) = b + c + 2d = 0 \\ y(1) = a + b + d = 1 \\ y(2) = 2a + b + c = 0 \\ y(3) = 3a + b + 2c + d = 1 \end{cases}$$

We now have a system of four equations with four unknowns. Here we examine the students' skill in solving such systems studied earlier in a linear algebra course, or with scientific calculators to obtain a quick solution.

From this linear system, we find: $a = 1, b = -1, c = -1, d = 1$

Therefore: $y = x - 1 - |x - 1| + |x - 2|$

It is useful to check the correctness of this formula by graphic application as well as to check it analytically by opening absolute values in the proper intervals.

Approach C. (general): The idea of obtaining such a formula for a broken line we give to our students during the first calculus course. The suitable formula for two broken points will be:

$$y = \frac{k_0 + k_2}{2}x + \frac{k_1 - k_0}{2}|x - 1| + \frac{k_2 - k_1}{2}|x - 2| + b$$

Here k_0, k_1, k_2 are the slopes of the links of a broken line (from left to right).

Using the graph in Figure 4 we will get: $k_0 = 1, k_1 = -1, k_2 = 1$ and therefore:

$$y = x - |x - 1| + |x - 2| + b$$

To find value b it suffices to substitute coordinates of one point of the graph in this equation. For example:

$$y(0) = 0 \Rightarrow 0 - 1 + 2 + b = 0 \Rightarrow b = -1$$

The final equation for a broken line in Figure 4 is:

$$y = x - |x - 1| + |x - 2| - 1$$

Note that after getting to know this formula, the creativity in the search for a formula disappears and the solution becomes a routine finding of the slopes of the links of a broken line. Therefore, we do not start from a formula in order to give students the opportunity to think independently beforehand; only then do we give a few examples explaining the idea.

Approach D. The idea of assembling the suitable formula using a number of more simple functions.

Let us think about the functions $y = y_1(x); y = y_2(x); y = y_3(x)$ with the graphs in Figures 6-8,

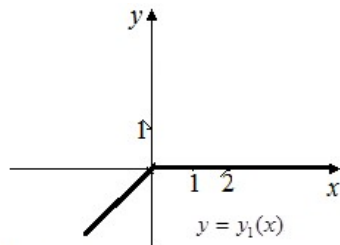


Figure 6

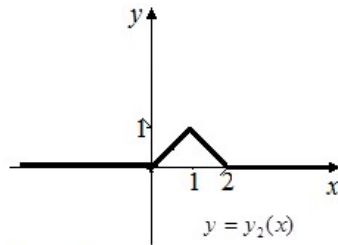


Figure 7

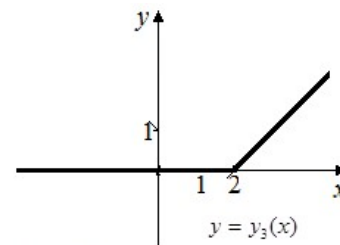


Figure 8

It is easy to construct formulas for each of them:

$$y_1 = \min(x, 0); \quad y_2 = \max(1 - |x - 1|, 0); \quad y_3 = \max(x - 2, 0)$$

It is clear that the function $y = y_1 + y_2 + y_3$ has the required graph (Figure 4).

Therefore, we will get the following formula:

$$y = \min(x, 0) + \max(1 - |x - 1|, 0) + \max(x - 2, 0).$$

In addition, if we want to use the absolute value function only, we will get:

$$y = \frac{x - |x|}{2} + \frac{1 - |x - 1| + |1 - |x - 1||}{2} + \frac{x - 2 + |x - 2|}{2}$$

After some simplifications:

$$y = \frac{1}{2} (2x - 1 - |x| - |x - 1| + |x - 2| + |1 - |x - 1||)$$

Example 9. Finding a formula for the parabola in Figure 9

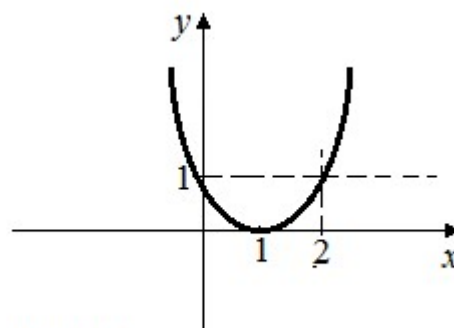


Figure 9

Solution: Because the parabola in Figure 9 touches the x -axis at point $x = 1$. its equation must be:
 $y = a(x-1)^2$.

To determine the value of coefficient a , we note that $y(0) = 1$ and so $1 = a(0-1)^2$ and therefore:
 $y = (x-1)^2 = x^2 - 2x + 1$. $a = 1$

Example 10. Finding a formula for the curve in the figure if known that it is the graph of the cubic equation $y = ax^3 + bx^2 + cx + d$.

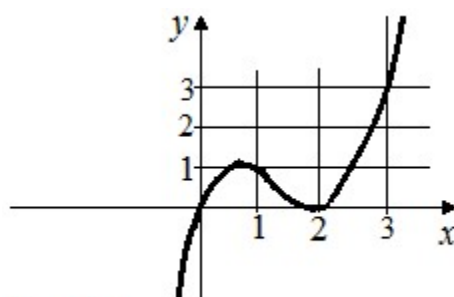


Figure 10

Solution: According to the zeros of the polynomial in Figure 10, its equation must be:

$$y = ax(x-2)^2$$

Therefore, only a single parameter a must be found and so we need only one equation and only one point substitution in this form, for example $(1,1): 1 = a \cdot 1 \cdot (-1)^2 \Rightarrow a = 1$

Thus, we obtain the required equation: $y = x(x-2)^2 = x^3 - 4x^2 + 4x$

Type 2. Constructing a formula with a graph similar that shown in the given figure.

Questions of this type require an understanding of the meaning of a graph similar to the one depicted in the given figure. Therefore, students need clarifications of what it means when we discuss the similarity of the graphs. Such questions do not usually require calculations and they are more qualitative than quantitative. The answers to such questions are often ambiguous and provide good opportunities for assessing the level of student understanding of function properties. At the same time, student answers help teachers determine the independence and creativity of their thinking.

Such questions can be divided into several subtypes in accordance with the required levels of thinking activity.

1. Recognition of the graphs of basic elementary function.

Example 11. 'What is a function whose graph is like that shown in Figure 11?'

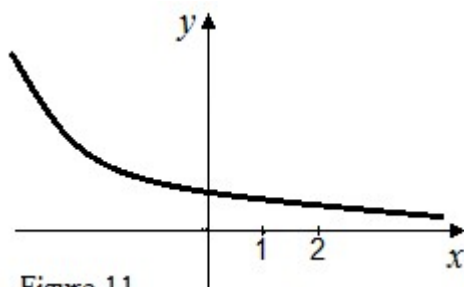


Figure 11

Possible answers can be $f(x) = (0.5)^x$ or $f(x) = a^x$ ($0 < a < 1$).

Example 12. 'What is a function whose graph is like that shown in Figure 12?'

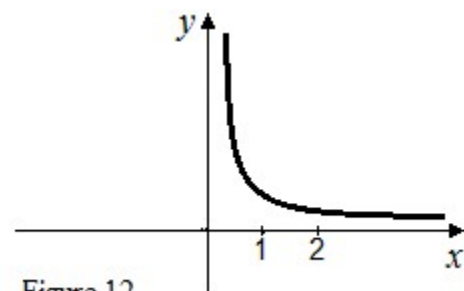


Figure 12

Possible answers can be $f(x) = \frac{1}{\sqrt{x}}$ or $f(x) = (\sqrt{x})^m$ ($m < 0$).

2. Recognition of the formula for a part of the given graph with the subsequent use of symmetry to complete the construction.

Example 13. 'What is a function whose graph is similar to that shown in Figure 13?'

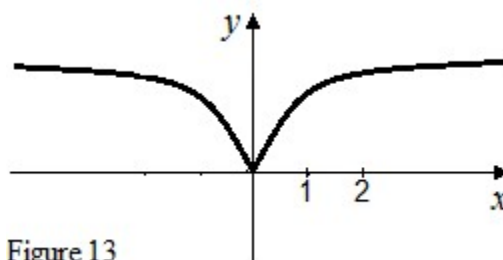


Figure 13

Here, in the right part of the curve, a student can find the graph of the function $y = \sqrt{x}$ and then use the symmetry of the curve with respect to the y -axis to complete the construction by use of the absolute value operation. The final answer is $y = \sqrt{|x|}$.

Here are some other possible answers for Figure 13:

$$y = \sqrt[3]{x^2}; \quad y = (x^2)^m \quad (0 < m < 0.5); \quad y = \ln(1 + |x|).$$

Example 14. 'What is a function whose graph is similar to that one shown in Figure 14?'

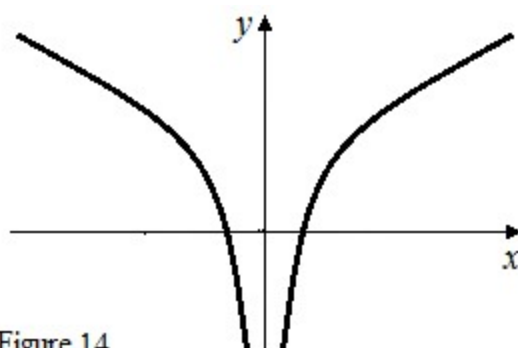


Figure 14

The possible answer may be $y = \ln|x|$.

3. Construction-required formula from the known formulas for several parts of the given graph.

Example 15. 'What is a function whose graph is like that shown in Figure 15?'

Approach A. Construction by assembly from appropriate parts

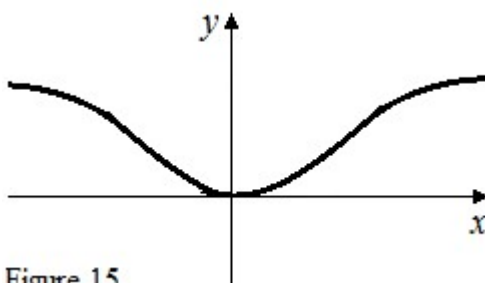


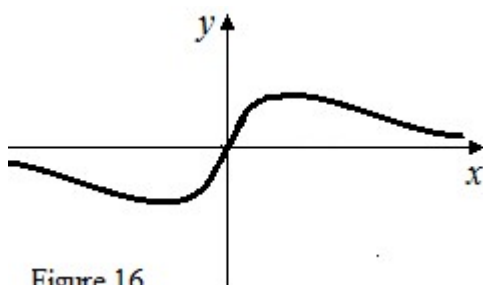
Figure 15

The graph near the origin, is like the curve $y = x^2$ and with the distance from the origin, the graph is like $y = \ln(x^2)$. Therefore, with the understanding that when x is near the origin $\ln(1 + x) \approx x$

and with even a better approximation $\ln(1+x^2) \approx x^2$, while for large values of x $\ln(1+x^2) \approx \ln(x^2) = 2\ln|x|$, the appropriate formula is: $y = \ln(1+x^2)$. In order to be sure of the correctness of our assumption we can investigate the function $y = \ln(1+x^2)$ analytically. We strongly recommend also using graphic applications for checking the constructed formula.

Approach B. Thinking about the graph of the derivative of the given function.

For the graph (Figure 15) the derivative graph looks like in Figure 16.



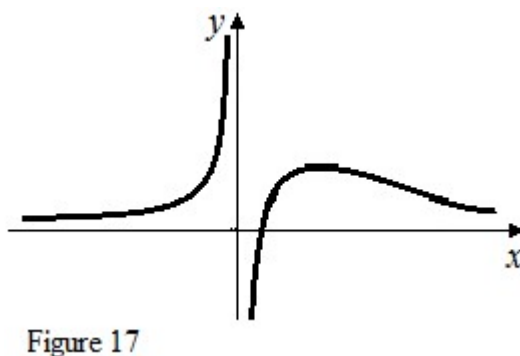
A suitable formula for this curve is: $y = \frac{x}{1+x^2}$. Hence, by integration:

$$f(x) = \int \frac{x dx}{1+x^2} = \frac{1}{2} \ln(1+x^2).$$

Note that the coefficient of 0.5 can be reduced because there are no numerical adjustments on the axes, so we will finally get: $f(x) = \ln(1+x^2)$.

4. Attention to asymptotes, zeros and signs of the function suitable to the graph.

Example 17. 'What is a function whose graph is like that shown in Figure 17?'



The appropriate formula can be $y = \frac{x-1}{x^3}$.

Example 18. 'What is a function whose graph is like that shown in Figure 18?'

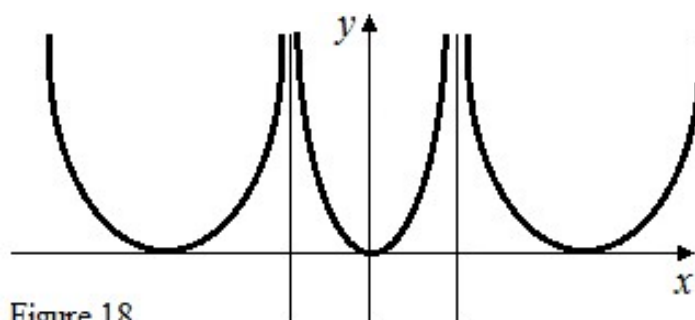


Figure 18

The appropriate formula can be $y = \frac{x^2(x^2 - 4)^2}{(x^2 - 1)^2}$.

5. The idea of parallel transfer of the graph.

Example 19. 'What is a function whose graph is like that shown in Figure 19?'

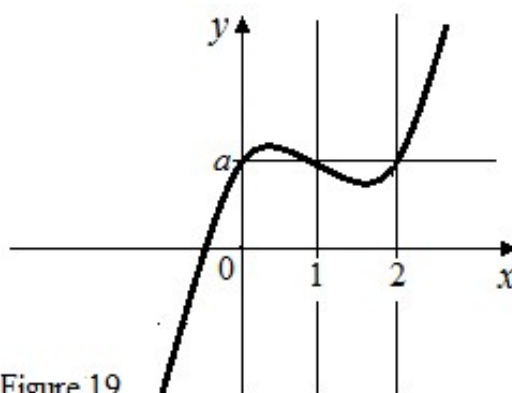


Figure 19

Let us move the graph downwards by a units. The result can be seen in Figure 20.

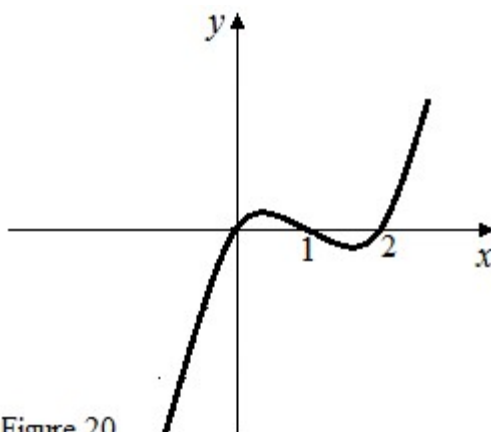


Figure 20

In Figure 20 we see 3 different roots 0, 1, 2 of a suitable function and so a formula that fits this figure may be: $y = x(x-1)(x-2)$.

Therefore, the final formula may be: $f(x) = x(x-1)(x-2) + a$.

In conclusion, we discuss the VA problem type – finding a formula for a function according to its verbal description. This type of problem is of great importance for understanding the basic concepts of function behavior. As a possible rule for finding an appropriate formula, we should think first about graphical representations that are suitable for the verbal description. Therefore, this type may be classified as providing examples of VGA problems that need to switch to a graphical representation before finding the analytical expression.

Example 20. Find a formula for a function with asymptotes $y = 0$; $y = 1$; $x = 0$.

The first step in solving such problems should be turning to the appropriate graphic representation of the textual description. It may be as shown in Figure 21:

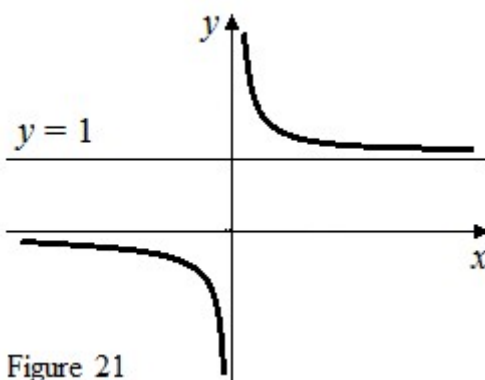
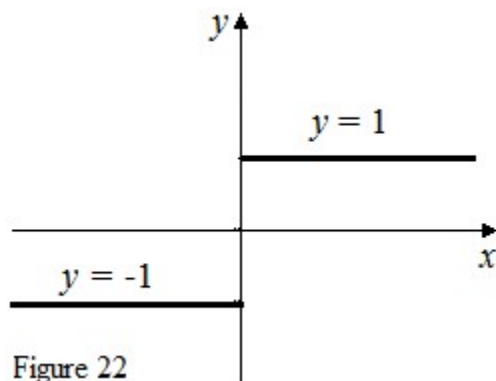
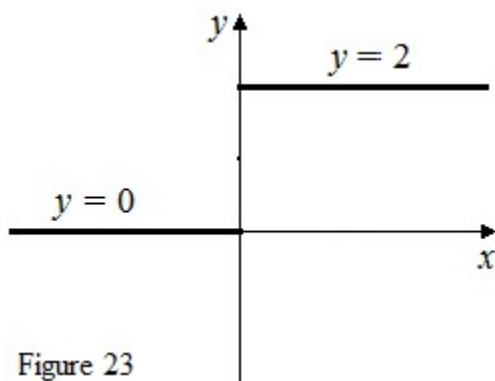


Figure 21

Let us construct first a function whose graph is horizontal asymptotes $y = 0$, $y = 1$. For this purpose, we can use a function $f_1(x) = \frac{|x|}{x}$ with a graph, as in Figure 22.



By shifting the graph in Figure 22 one unit up we will get the graph of the function $f_2(x) = f_1(x) + 1 = \frac{|x|}{x} + 1$, which is in Figure 23.



By squeezing the graph in Figure 23, we get the needed formula of the asymptotes

$$f_3(x) = \frac{1}{2} f_2(x) = \frac{|x|}{2x} + \frac{1}{2}$$

Now it is easy understand that the appropriate formula for the curve in Figure 21 may be in the form:

$$y = f_3(x) + \frac{1}{x} = \frac{|x|}{2x} + \frac{1}{2} + \frac{1}{x}$$

An optional formula for the graph in Figure 21: $y = \frac{1}{\pi} \arctg(x) + \frac{1}{2} + \frac{1}{x}$.

We recommend that students check the validity of this formula using graphical applications, as doing so always promotes a feeling of achievement when students see how to correct analytical input; the graphical application gives quick visual output to confirm the formula.

Example 21. Find a formula for a continuous elementary function with asymptotes $y = 0$; $y = x$.

Solution: First, let us draw an appropriate graph (Figure 24).

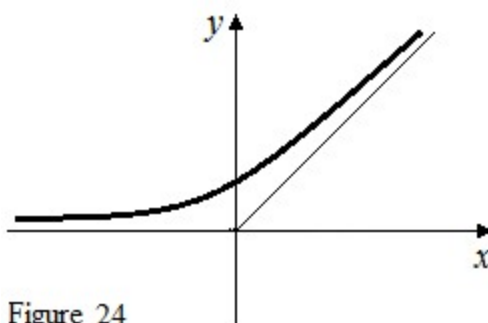


Figure 24

Now let's construct a formula for the graph in Figure 25.

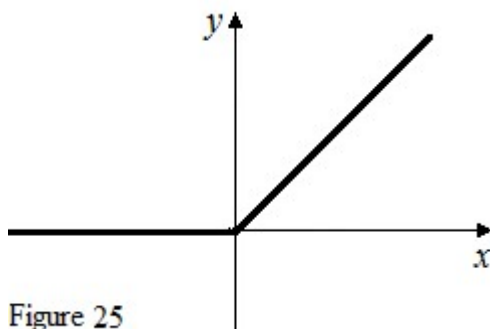


Figure 25

It is easy to verify that the suitable formula may be: $y = \frac{1}{2}(x + |x|)$.

If we now add to this function an everywhere positive expression that tends to zero at infinity, we get the required formula $y = \frac{1}{2}(x + |x|) + \frac{1}{1 + x^2}$.

Learning dialogues with students. Transition between different representations in problem solving

We want to prepare students to deal with new problems, show effective ways to approach them, and teach them not to give up when meeting new tasks. We want to prevent the approach when the student says that he has not yet encountered such problems and asks us at first to explain how to solve them and give a proper example. However, it should be noted that in the modern world there is decreasing reliance on performing algorithmic actions. For future engineers it is becoming much more important to cope with new tasks. It is therefore very important to prepare them to meet new challenges with any effective tools that are available to them. Despite the difference in engineering specialties, the ways of reasoning are much the same and the proper study of mathematics can contribute to student development.

We give some examples of our teaching dialogues with students, aimed at finding approaches to a new problem.

Dialogue 1. It is known that the curve shown in Figure 26 is a parabola. Find the equation of this curve.

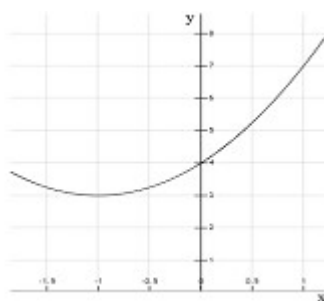


Figure 26

Student: How can I do this? I do not remember that we have solved such problems.

Lecturer: Understand first what is given and what is required of you. What does the notion of 'parabola' mean? What does the parabola equation look like?

Student: A parabola is the graph of a quadratic equation.

Lecturer: And what does a quadratic equation look like?

Student: $y = ax^2 + bx + c$

Lecturer: If so, what is unknown to you?

Student: The values of coefficients a , b , c .

Lecturer: And what is x , y in the equation $y = ax^2 + bx + c$?

Student: x is an independent variable and y depends on it.

Lecturer: For example, if $x=2$, according to this equation, what is y ?

Student: $y = a \cdot 2^2 + b \cdot 2 + c$.

Lecturer: And if you know that for $x=2$ the value of y is 4, how do you write it?

Student: $4 = a \cdot 2^2 + b \cdot 2 + c$ or $4a + 2b + c = 4$.

Lecturer: And what did you get?

Student: An equality.

Lecturer: However, you do not know the values for a, b, c , so, how would you treat this equality?

Student: As an equation, apparently.

Lecturer: And how to treat a, b, c for this equation?

Student: As unknowns of this equation.

Lecturer: And if we have 3 unknowns, how many equations are needed to find a, b, c ?

Student: Three equations.

Lecturer: Can you construct them based on the given curve?

Student: Yes, I see from Figure 1 that $y(0) = 4, y(1) = 7, y(-1) = 3$ and therefore the needed three equations are:

$$\begin{cases} a \cdot 0^2 + b \cdot 0 + c = 4 \\ a \cdot 1^2 + b \cdot 1 + c = 7 \\ a \cdot (-1)^2 + b \cdot (-1) + c = 3 \end{cases} \Leftrightarrow \begin{cases} c = 4 \\ a + b + c = 7 \\ a - b + c = 3 \end{cases}$$

Lecturer: Now that you produced a system of three equations with three unknowns, what is the next step?

Student: To solve this system. I will do it quickly. The answer is $a = 1, b = 2, c = 4$

Lecturer: What should be done now?

Student: Just to write the appropriate formula $y = x^2 + 2x + 4$.

Lecturer: How can you check the correctness of solution to ensure that there has not been an error in the calculations?

Student: We must substitute the appropriate values in the equations and ensure that they turned out to be the correct equalities:

$$\begin{cases} 4 \stackrel{?}{=} 4 \\ 1 + 2 + 4 \stackrel{?}{=} 7 \\ 1 - 2 + 4 \stackrel{?}{=} 3 \end{cases} \Rightarrow \text{all these equalities are true}$$

Lecturer: And if the figure was ‘parabola cubic’, how do you find the graph of equation $y = ax^3 + bx^2 + cx + d$?

Student: Compose the system of four equations with four unknowns and solve it.

Lecturer: OK. Maybe it is worth thinking about a proper formula that will give us the answer without composing and solving the system?

Student: It would be great. However, I have no idea how to get such a formula.

Lecturer: Let's go back to the previous task of finding a square triple by three points of its graph: $(x_1, y_1), (x_2, y_2), (x_3, y_3)$.

Lecturer: Can you construct a square triple that passes through a point (x_1, y_1) and take the value 0 for $x = x_2$ and $x = x_3$?

Student: As far as I know, according to the Bezout theorem such a triple should be divided by $x - x_2$ and by $x - x_3$ without remaining, and therefore it must be in the form: $m_1(x - x_2)(x - x_3)$

Lecturer: It's great! And how can you find the value of m_1 ?

Student: As we know $y(x_1) = y_1$ and so $m_1(x_1 - x_2)(x_1 - x_3) = y_1$. Therefore,

$$m_1 = \frac{y_1}{(x_1 - x_2)(x_1 - x_3)} \text{ and, finally } y_1(x) = y_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}.$$

Lecturer: Splendid! And if we now construct the same formulas for the rest of the values of x_2 and x_3 , like this:

$$y_2(x) = y_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}; \quad y_3(x) = y_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)},$$

can you now get the desired formula to the quadratic triple in which the graph passed through three points (x_1, y_1) , (x_2, y_2) , (x_3, y_3) ?

Student: We just need to add three received formulas: $y(x) = y_1(x) + y_2(x) + y_3(x)$ or:

$$y(x) = y_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + y_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

Lecturer: It is great! Now check this formula for the graph in Figure 26.

Student: No problem, we get:

$$y(x) = 4 \frac{(x-1)(x+1)}{(0-1)(0+1)} + 7 \frac{(x-0)(x+1)}{(1-0)(1+1)} + 3 \frac{(x-0)(x-1)}{(-1-0)(-1-1)}$$

And finally, after proper algebraic actions: $y = x^2 + 2x + 4$.

Great! I like this method!

Lecturer: This wonderful method belongs to the famous French mathematician Lagrange (1736-1813). His work has given us the polynomials that are known as ‘Lagrange polynomials’. They let us quickly get a formula for a curve that passes through any n points: (x_1, y_1) , (x_2, y_2) , ..., (x_n, y_n) , where $x_1 < x_2 < \dots < x_n$.

Lecturer: Now use this method to obtain an equation for a curve passing through four given points $(0,0)$; $(1,1)$; $(2,0)$; $(3,3)$

Student: No problem, because here $y_1=0$ and $y_3=0$, so we need two terms only in the proper

Lagrange formula: $y = 1 \cdot \frac{x(x-2)(x-3)}{1 \cdot (-1) \cdot (-2)} + 3 \cdot \frac{x(x-1)(x-2)}{3 \cdot 2 \cdot 1}$

In addition, after algebraic simplifications, we have: $y = x^3 - 4x^2 + 4x$

Lecturer: Note that there are infinitely many curves passing through these four points and the Lagrange polynomials give the simplest of suitable formulas.

Note that in our teaching approach we permanently use transitions from one presentation to another—from verbal formulation of the problem to visual (graphic) information; from graphic information to numerical; from numerical data to a symbolic presentation of it; from symbolic presentation to analytical descriptions of it by equation; and so on.

Dialogue 2

Lecturer: Give me a formula of continuous function with asymptotes $y = 2$ and $y = 0$.

Student: I have no idea how to begin.

Lecturer: Are you understand the question? Are you understand all notions mentioned in it? Can you draw a graph of a function with such properties?

Student: I don't understand how a graph can have two different horizontal asymptotes.

Lecturer: Why not? One of them is at $x \rightarrow +\infty$ and another is at $x \rightarrow -\infty$. Try to draw a suitable graph.

Student: What about the graph in Figure 27?

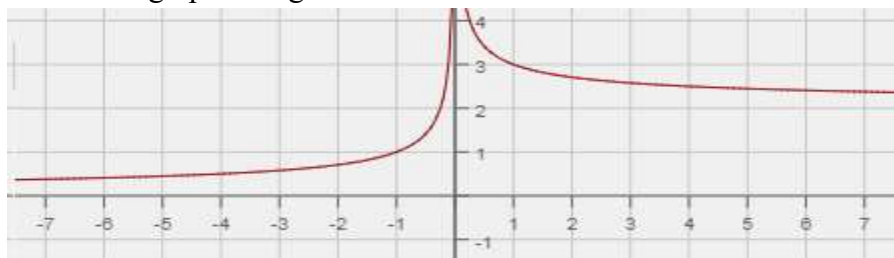


Figure 27

Lecturer: As far as horizontal asymptotes it is fit. However, what about a continuity?

Student: I let it out of sight. Here is the new option in Figure 28:

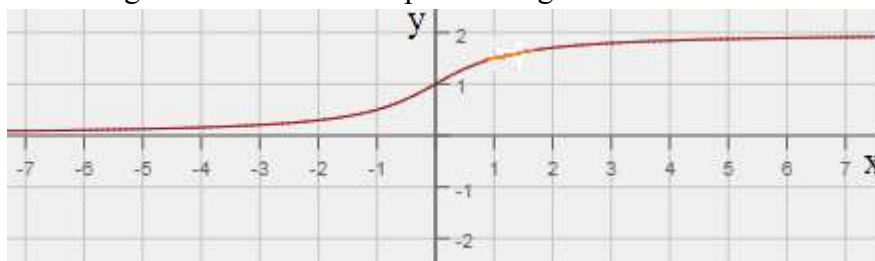


Figure 28

Lecturer: OK. Now we can see on the chart all the properties mentioned above. Does it remind you of a graph of any known functions?

Student: It looks like an arctangent graph in Figure 29:

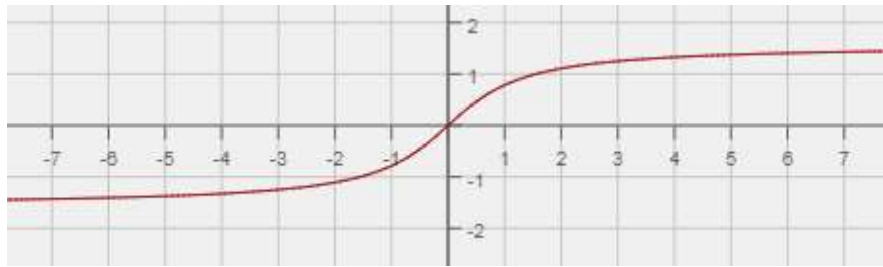


Figure 29

Lecturer: OK. But what's the difference and what needs to be done to get the required graph from what you see?

Student: An arctangent has asymptotes $y = -\frac{\pi}{2}$; $y = \frac{\pi}{2}$ and we need $y = 0$; $y = 2$.

Lecturer: What do you want to do with the formula $y = \arctan(x)$ to get the desired asymptotes?

Student: We need to change the asymptotes from $y = -\frac{\pi}{2}$; $y = \frac{\pi}{2}$ to $y = -1$; $y = 1$.

Lecturer: And how can that be done?

Student: For this, multiply the $\arctan(x)$ by $\frac{2}{\pi}$ and come to the function $y = \frac{2}{\pi} \arctan(x)$ with asymptotes $y = -1$; $y = 1$ (Figure 30).



Figure 30

Lecturer: And how would you change this function to get asymptotes $y = 0$; $y = 2$?

Student: It just needs to have 1 be added to the last function and we get the required formula:

$$y = \frac{2}{\pi} \arctan(x) + 1$$

Note that tasks of this kind require concentrated mental activity of students. Here are some important steps in solving such problems:

1. Understanding of the problem.

The student must understand all concepts mentioned in the problem, together with their graphic expression.

2. Imagination of how graphs may look.

3. Decision on which of the possible graphs is like to some known function graph.
4. Decision how to change a known formula in order to get the required graph.
5. Testing of a suggested formula by analytical investigation and by using technological tools, along with calculators and graphical applications.

Note that using graphical applications is a very positive stage for the students in the process of the fitting the formula and in strengthening the student's confidence in the final solution.

An experimental study conducted in two groups of engineering students under two approaches: Group A had 64 students; group B had 65. Group A studied according to the regular method without using 'the transition model'. Group B studied according to the model we called 'the transition between the different representations of the language of mathematics. The experiment was organized in three phases:

1. PRE-TEST

The scores received by the students in both groups examined at the end of their pre-calculus study in the preparatory program. Statistical processing carried out and t-test applied. Two hypotheses were verified: the null hypothesis and the counter hypothesis, the t-test showed no difference between the two groups at a significance level of 95%. ($P = 0.05$, $1.68 = t$, critical value 1.98).

2. MID-TEST. On the mid-term test, we gave two identical problems focused on finding an appropriate formula. In group A, the problems were answered by only 10 students; in the group B, the problems were answered by 50 students.

3. POST-TEST. In the two groups A and B mention above, students took the same semester exam at the standard level, which we have always tried to keep for years. Total exam (which contained not only graphical tasks) marks of the students in two groups were processed statistically and the t- test was applied.

The null hypothesis: There are no significant differences between the groups that are: the experimental group B and the control group A in the test results, i.e., $\bar{x}_2 - \bar{x}_1 = 0$.

Counterhypothesis: There is a significant difference between the groups: the sample group and the control group with the test results, i.e. $\bar{x}_2 - \bar{x}_1 \neq 0$.

The hypotheses tested at a significance level of 95% and the value $t = 2.513$ was obtained. The critical value of rejecting the null hypothesis is that it is 1.98. According to the results, it is possible to reject the null hypothesis and say that there is a significant gap between the experimental group B and the control group A.

Our conclusion is that in Group B the students had acquired a better understanding of mathematical concepts and their graphical representations as well as the necessary technical skills.

CONCLUDING REMARKS

The problems of mention above types we gave to students from the beginning to the end of calculus course at different levels of complexity in accordance with the studied topics. They increase the activity of students at lectures and seminars and contribute to better understanding of calculus notions while enhancing cognitive interest of the students in calculus studies. Note that this article talked about explicit function graphs. However, we are no less widely used the problems of finding equation (implicit functions) fitting to the given figure. For example, construct an equation corresponding to Figure 31.

We see this pedagogical strategy offered to students, of transition from one representation to another, and using technology, a subtle constructivist approach that will help the student create the conceptual understanding and develop effective ways to information processing, and also, it will be a tool for them to interpret and understand new topics as well.

The conceptual understanding will serve the student in building his self-confidence to help him to believe in himself to continue to process and understand the knowledge that he is previously unable to comprehend. We believe that this will lead to increasing motivation to learn new things in mathematics, it is known that motivation is a necessary condition for learning in general and for meaningful learning in particular, as stated by (Tami Yechiely, 2015), which relies on Ausubel and other researchers.

What is the formula of the figure 31?

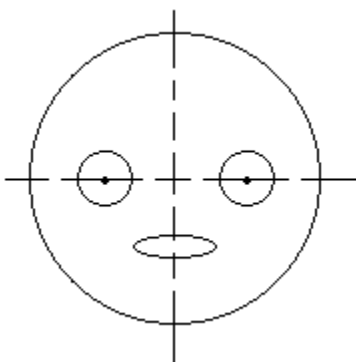


Figure 31

The possible formula:

$$(x^2 + y^2 - 16) \left((|x| - 2)^2 + y^2 - 1 \right) \left((|x| - 2)^2 + y^2 - 0.01 \right) \left(x^2 + 9(y + 2)^2 - 1 \right) = 0$$

We always tell students "Test your formula on a graphing application". Note that for checking it there is no need to write such a long formula, since most graphics applications allow drawing graphs of several formulas (four here) simultaneously.

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So, You Want to Write an OER? Three Authors Share Triumphs, Pitfalls, and Options

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Borough of Manhattan CC

Abstract: Open Educational Resources (OERs) offer a free, viable alternative to costly textbooks. The authors share their experience and advice on finding and writing online content, creating an online platform for the content, finding videos and other resources, and working with an appropriate free online homework system to match the written content. In addition, the implementation and suggestions for practitioners are discussed. At the end of the article, the bibliography contains two OERs freely available under an open commons license, one for Intermediate Algebra, the other for Mathematics for Elementary Education.

Keywords: Open educational resources, curriculum development, intermediate algebra, mathematics for elementary education, homework platform for mathematics

INTRODUCTION

Community Colleges in the United States serve the unique role of providing open access to many students seeking higher education. Community colleges educate a significantly high percentage of underrepresented students, which include low-income people, first-generation college students, and ethnic minorities (Berkner & Choy, 2008). Purchasing textbooks can present a challenge to many of these students. Open Educational Resources, or OER, offer a free, viable alternative.

OERs are educational materials that may include textbooks, modules, streaming videos, and software (Hilton, 2016). Hilton (2016), synthesized the results of sixteen studies across different disciplines at the post-secondary institutions to show that students generally achieve comparable learning outcomes when using OER vis-a-vis commercial textbooks. Two recent studies conducted at Scottsdale Community College and Mercy College showed that students enrolled in mathematics courses performed similarly on the uniform final examination whether they used a commercial textbook or OER (Fischer et al., 2013; Pawlyshyn et al., 2013). From our own experiences in the community college setting, we believe that students can do even *better* when an OER is available, because students are able to review the material and start their math homework immediately during the crucial first weeks of class, rather than waiting for financial aid or a used textbook.

The Borough of Manhattan Community College (BMCC), where all three of us teach, is one of twenty-four institutions comprising the City University of New York and serves over 23,000 students. Over 90% of the student population is comprised of minorities and groups from historically underrepresented populations. About 65% of students at BMCC receive needs-based financial assistance. Each year about 72% of BMCC's new entering students are placed into a developmental mathematics class based on their performance on the placement math proficiency test. (BMCC Fact Sheet, 2019). Our mathematics department is one of the largest departments at BMCC. Comprised of over 70 full-time and over 200 part-time faculty members, BMCC offers more than 400 sections of courses ranging from basic pre-algebra to advanced calculus. There are three levels of developmental proficiency -- prealgebra, elementary algebra and intermediate algebra -- which a student must successfully complete in order to enroll in precalculus, the first credit-bearing mathematics courses for STEM majors. We also offer many co-requisite courses, most of which at present do not have a single textbook suitable to tackle both the developmental and credit-level content.

For our OER project, we chose to focus on Intermediate Algebra with Trigonometry, a developmental course that offers no credits and meets six hours a week. On average, thirty sections of this course are offered at BMCC each semester. The development of OER materials and professional development workshops were made possible by a New York State OER Scale UP Initiative that awarded CUNY a \$4 million grant to establish new or support ongoing OER initiatives across CUNY (<https://www2.cuny.edu/libraries/open-educational-resources/nys-oer-scale-up-initiative/>). The award allowed us summer stipends, which gave us the time we needed to complete the work.

In addition to having the time for the project, the three authors of this article have, collectively, many years' experience teaching mathematics courses and developing materials for mathematics classes that informed our OER work. One of us (Offenholley), had previously created an OER for Mathematics for Elementary Education, a credit level liberal arts math class for students who wish to become PreK or elementary school teachers. We also have prior experience writing content (Hirsch, for online courses) and creating online question sets (Millman, for our Quantitative Literacy and Reasoning course for Non-STEM majors).

As the focus of this article, we will focus on the Intermediate Algebra and Trigonometry OER, our most recent project. Our work involved three main tasks: finding and writing online content, creating an online platform for the content, videos and other resources, and finding an appropriate online homework system that matched the written content. We share our experience in each of these areas, in the belief that what we learned will help others who wish to create OERs.

Written Content

For our textbook selection, we had three options. We had to decide upon (1) using an OER course that was pre-developed and published, (2) using a text that was written by our faculty, or (3) a blend of both.

For (1), a search for an appropriate textbook for this course began at OpenStax.org. OpenStax is a nonprofit educational initiative which publishes peer-reviewed, openly licensed college textbooks. These textbooks are free for electronic version and have a nominal cost at the printed level. The mathematics texts found on OpenStax represent a variety of subjects, beginning at the primary level, terminating at the higher-level undergraduate mathematics class. Faculty can download a pdf of the textbook and upload it to their own website, with their own videos and notes, or can choose to have students access the text on the OpenStax website. In addition, libretexts.org, an open textbook initiative from the University of California, is a new source for OERs.

However, because the Intermediate Algebra and trigonometry class at BMCC is a 0-credit class, logarithms and trigonometry are taught from a perspective that does not include functions, so we found it difficult to find an appropriate free open-source text. Similarly, in a previous OER project for Math for Elementary Education, we were unable to find a single existing OER that had both enough mathematics and enough education content, so we had to write our own content. Since the development of those two courses, however, OpenStax has improved platforms to mix and match and modularize sections of its texts, using the CNX platform, so it is quite possible that you will be able to find the right mix for your own OER at one of these two places.

For our Intermediate Algebra and Trigonometry OER, we agreed that one author should be used to present the entire textbook to assure a pedagogy that was consistent throughout the book. Therefore, a blend of both a pre-published OER text and one that we wrote would not be acceptable. Thus, we went with option (2). An unpublished e-book for this course was available, written by a member of BMCC's mathematics department (Hirsch), and initially intended to be used for a hybrid version of the course. Permission was received to modify and edit the textbook and use the e-book version for the course that was being built. This was ultimately the text that was adopted for the Intermediate Algebra and Trigonometry OER course.

Online Platform, Videos and Games

Once we decided on our text, we wanted to have a way to present the material so that our students could easily access it, without additional cost. We also wanted to have videos available, because many of our students have a difficult time creating meaning from only reading. For our platform, we used WordPress, which is a free open-source platform and hosting service. The one issue with hosting the service with WordPress is that the students will see advertisements. Some college systems have hosting available, so that an ad-free version of WordPress can be used; within the CUNY system, the CUNY Academic Commons is beginning to be such a place (<https://commons.gc.cuny.edu>). In addition, www.pressbooks.com has a WordPress-based book-creation platform that has a free and for-pay model, both without ads. With any of those sites, students can access the content on their smartphones with adaptable pages that shrink to fit the smaller screens. (Remember that if you decide to go with an OpenStax text, you might not need a separate website.)

One of the things we liked about creating our own site was that we could use the excellent videos from www.mathispower4u.com, a website with thousands of videos for all levels of mathematics, from basic mathematics to Calculus. These videos have the advantage of being closed captioned

and ADA compliant, and there are no ads. We placed the videos separately from the text – with the text in pdf form – but in hindsight we suggest embedding the videos within the text, so that students are more likely to read *and* watch. To see the differences between the two approaches, see the two OERs links above the references section.

In addition, we were able to add links to several conceptual algebra games created through an NSF grant on which one of us (Offenholley) was the Principle Investigator. These games, like the videos and our textbook, are all available for free through Open-Commons licensing.

Homework Platform

Modern textbooks often come prepackaged with access to an online homework platform such as WebAssign, MyLab, ALEKS and HAWKES Learning (<http://www.webassign.net/>; <https://www.pearsonmylabandmastering.com/northamerica/mymathlab/>; <https://www.aleks.com/>; <http://www.hawkeslearning.com/>). These systems have become widespread particularly in developmental, introductory courses and getaway courses. The 2009 AMS survey found that 43% of bachelor's degree-granting mathematics departments were using online homework (Kehoe, 2010). Since 2009 the use of online homework platform has been required at BMCC in all pre-algebra and elementary algebra courses.

For our project, we chose MyOpenMath (MOM), an open-source homework system developed by David Lippman (<https://www.myopenmath.com/info/aboutus.php>) in 2006 at Pierce College. The first version of the platform was built on Google's OpenClass system and released as an IMathAS (Internet Mathematics Assessment System). IMathAS was one of the Learning Management Systems that was specifically designed to have capabilities of working with various algorithms used in mathematics. Initially, the system had to be installed and managed using local servers which made it more difficult to adopt. In 2011, the IMathAS evolved into MOM, which has capabilities of a cloud environment. This allows individual instructors to offer courses that use MOM without the need to locally host the system. However, the maintenance of the platform requires sponsors who offer the institutions an option to provide instructor and student support for a fee. Even though MOM does not provide direct support, it remains free and offers a large community-based support. MOM can also be integrated into Blackboard and Canvas platforms.

MOM offers a large database of automatically graded questions from basic arithmetic to calculus and beyond. MOM has various learning management features, including a fully featured gradebook. Faculty can monitor student online engagement and modify or adopt homework questions with no extensive programming knowledge. For this project, homework sets were created by selecting questions from the database that closely match examples presented in the textbook. The questions were sorted based on the number of times they were used in the past; the average time students have spent answering them, and the availability of embedded videos that demonstrated similar examples. Using already available questions in the database proved easier than modifying the already available set of questions from the template courses. One of the defining premises of the MOM platform is sharing of the resources created by the community. The

materials developed for this project are available to be accessed through MOM directly by any instructor using the system.

In developing this OER authors worked together to establish cohesiveness between the written curriculum, the online platform, and the supporting resources. The written lessons were proofread and piloted prior to wider implementation. The content of video resources was checked for accuracy and consistency with the written materials. One of the authors had a prior experience in adopting MOM in the quantitative literacy course offered at BMCC. This experience facilitated the creation of the shell for this project. In addition, this author had been granted administrative rights by David Lippman to create and manage instructor accounts on MOM. Considering the system had been successfully implemented at BMCC in other courses, it was decided that instructor support would be provided by instructors who had previously used the system.

Implementation

The policy at Borough of Manhattan Community College before adopting any new textbook for a course is to first pilot the textbook for a couple of semesters. Therefore, the faculty could facilitate constructive feedback from like-minded peers, and then to bring the text back to the department for a well-informed discussion and vote. The textbook, online homework system and videos were piloted for two semesters before the authors sought departmental permission to adopt their text.

Motivated by the zero-cost textbook, a handful of instructors piloted the text, offering suggestions about the order of the text content, editing, and pointing out glitches in the online homework system. The authors were able to polish the text, homework, and videos to a sheen before presenting them to the department for a vote.

After being presented as a possible alternative free text for our Intermediate Algebra with Trigonometry class, the department voted unanimously to offer the text as a free alternative to the already approved textbook, with the stipulation that one of us would be available for training.

Because of the funding from the New York State OER Scale UP Initiative, we were able to provide paid training to twelve instructors who wanted to adopt our OER. Two workshops were held in the summer for instructors who were teaching the class in the fall or spring of the up-coming semester. Furthermore, we were able to provide funding for the person who lead the workshops, one of the authors of this paper.

Student feedback

We received positive feedback from students who were enrolled in sections that used our OER. Students found it easy to navigate between the online textbook, homework assignments and support videos. The features provided by the online homework platform such as ability to try problems more than once, constant feedback and videos that tied to specific problems were among the common responses that we received. Students also appreciated that the materials were provided free of charge. One student wrote: *“The homework encouraged you to understand how to solve the problems and the option to have 7 chances forced me to want to get the problem. Everything [OER materials] was accessible and it was a major bonus that it was free to use”*.

Conclusion

We believe that our combined technical and teaching expertise were immensely helpful in creating our OER. We urge anyone who is considering creating an OER who does not have a technical background to team up with people who do, and anyone who is a beginning teacher to team up with people who have taught for longer. Successful implementation requires collaboration and technical expertise. Professional development is needed to introduce instructors to new online systems. Large scale implementation can require support by mathematics faculty and administration. Creating OER materials and providing professional development might require grants and/or release time. Community Colleges in the United States were founded with the premise of open access (Bragg, 2016). Developing OER materials by the experts in the field can provide high quality learning resources free to students who would otherwise not be able to afford them (Hilton, 2016). This study suggested that students benefited from OER materials.

OUR OERs

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Book Review for *“Numbers: Their Tales, Types and Treasures”* by A.S.
Posamentier and B. Thaller

Małgorzata Marciniak

LaGuardia Community College of CUNY

Title: Numbers: their tales, types, and treasures

Authors: Alfred S. Posamentier, Bernd Thaller

Publisher: Prometheus

Publication date: August 11, 2015

Paperback: 400 pages

Price: \$19.00

ISBN: 1633880303

Reviewer: Małgorzata Marciniak

2010 AMS Subject classification: general, popular, number theory, history of mathematics across the centuries, cognitive psychology,

Keywords: number theory, prime numbers, history of mathematics, Kaprekar numbers, Armstrong numbers, amicable numbers, cognitive psychology

Review:

Professional mathematicians benefit in a variety of ways from reading popular books about mathematics. Number theory specialists who become familiar with this particular book can recommend it to all students since reading requires only elementary school mathematics. Non-specialists can read this book for a pure joy of experiencing quality exposition. Certainly, while reading this volume, everybody can observe how to write about mathematics in an engaging and enriching manner. The book is written in an interesting way and contains numerous stories and a warm, human background to accompany scientific context.

The first part of the book is devoted to cognition of numbers and its developments throughout human history, and human growth from childhood to adulthood. We learn counting as children

and surprisingly as adults we do not give it much reflection. But the concept of numbers and its historical developments is tied to the neurological processes that took place in human minds. Numbers and counting are more primitive concepts than the symbols that represent them, and the words that we use to express them. Chapter 2 of the book describes psychological dimension in terms of Piaget's mathematical cognition in children and compares it to counting styles discovered among primitive Amazonian tribes. Overall the first part of the book contains insightful discussion of the growth of cognition that allows humans to count.

The second part of the book analyzes historical and linguistic developments of numbers in Babylon, Egypt, China, and India, observing that since ancient times most humans cannot perform arithmetic calculations in a perfect manner. Instead of perfecting this rather dull activity of calculations, the human mind prefers to play with numbers and discover their properties for the joy of creativity, which was first documented on a large scale in Greece. Arranging and organizing numbers into geometrical shapes leads to philosophical discoveries of prime and composite numbers based on the fact that a number which is a product of two numbers can be represented as an area of a rectangle with sides that are whole numbers. Fibonacci numbers make a returning theme in the book, for example in Pingala's three problems from Sanskrit poetry as seen by modern combinatorics.

In the last part of the book the authors encourage the reader to make their own observations and think about math problems, including open problems. Chapter 8, in particular, contains a true treasure, a sequence of ``Pythagorean curiosities`` which can engage in mathematical research everybody from students to math professors. For example, given a Pythagorean triple generate a new one using linear transformations. A student can certainly enjoy generating a Pythagorean triple, while a professor can classify all linear transformations that can do it. The last two chapters offer a brief discussion of definitions of numbers from the point of view of logicism, formalism and intuitionism. Here the authors discuss the cardinal numbers as equivalence classes of sets and the definition by Peano axioms.

To summarize, this book stands out in comparison to similar items on the shelf. Especially the first chapters that contain cognitive considerations about numbers and the last chapters with open problems in number theory. It is self-contained and, to assist with the narrative, the authors provide various tables of numbers: prime numbers, Kaprekar numbers, Armstrong numbers, amicable numbers, and more. Highly recommended to everybody!

Book Review for *BIRTH OF A THEOREM* by Cédric Villani

Roy Berglund

City Tech, Borough of Manhattan Community College of CUNY

Title: Birth of a Theorem

Author: Cédric Villani (translated from French by Malcolm DeBevoise)

Place: New York, NY

Publisher: Farrar, Straus & Giroux

Publication Date: 2016

Pages: 250

Special Features: none

Price: \$16.00

ISBN: 978-0-374-53667-1

Reviewer: Roy Berglund

By means of his own personal experience the author Cédric Villani clearly succeeds in answering key questions, such as what life is like being a mathematician, and how the work of mathematics gets done. That is what makes the reading of this book such an exciting journey. It is definitely not necessary to wade through the technical details of the interpolated mathematics; these may be ignored, as I did in reading the book.

For the author's background and credentials, I quote from the front piece of the book: "Cédric Villani is the director of the *Institut Henri Poincaré* in Paris and a professor of mathematics at the *Université de Lyon*. His work on partial differential equations and various topics in mathematical physics has been honored by a number of awards, including the Fermat Prize and the Henri Poincaré Prize. He received the Fields Medal in 2010 for results concerning Landau damping and the Boltzmann equation."

Cedric Villani begins our journey by relating a work session with his colleague Clement Mouhot on the regularity condition for the Boltzmann equation. The Boltzmann equation models the evolution of a rarefied gas made up of billions of particles that collide with one another, the statistical distribution of the positions and velocities at time t indicating the density of particles

whose position is x and whose velocity is v . The author takes us through his logical discussion and comes back to Landau damping.

The first step in mathematics research is knowing exactly what you are working on, and what you are trying to do. Speaking with various mathematicians in his group, the author garners ideas from a variety of different disciplines that he hopes will be useful in his research. Papers mentioned, although having apparently no connection with his problem, are food for thought in the search for answers.

Digression seems to be beneficial to the problem solving process, because then he digresses to a vignette on Joseph Fourier and the decomposition of vibrations. Villani remarks on the usefulness of dreams to the intellect. Sometimes dreams help to clarify the nature of the research problem: the means to the solution is often more important than the result. Recalling that other researchers have solved part of the problem, or a related problem, he realizes that a new approach with new tools will be necessary to solve his problem.

Villani describes his arrival at the Institute of Advanced Study at Princeton (IAS), and the planned work schedule. He muses about how and what it takes to win the Fields Medal. Using email and phone calls, he continues to work on his problem along with Clement in France.

Villani laments the fate of great luminaries, such as Bobby Fischer, Paul Erdős, Grigori Perelman, Alexander Grothendieck, Kurt Gödel, and John Nash as he struggles with his problem on Landau damping. Talking with a colleague at IAS about the colleague's own work helps to clarify the issues. This does not yet bring a solution to Landau damping, because the proof is not working. Also, he must decide whether to stay at IAS in Princeton versus going back to France to be director of the Institute Haute Poincaré (IHP), but ultimately decides to return to France.

Clement Mouhot is discouraged and practically wants to give up on the insurmountable problems they are confronted with in the proof and is about to email Villani, but finally he sees a way to solve the difficulties. The two mathematicians continue to work together to patch the holes in the proof, and finally put all the pieces in place to complete the proof. Exaltation is the dominant emotion.

When the theorem is presented at Princeton there are in the audience some critical physicists finding faults with the argument. This criticism stimulates Villani to work faster than normal in resolving the difficulties in the proof. Discovery in one area suggests solutions in other areas as well. Though the corrections are not finished, he goes to bed exhausted and somewhat discouraged.

However, waking early the next morning the solution immediately pops into his head. While he was unconscious during sleep, his subconscious mind continued to work on the problem. Then he sits down to work, discovers that the solution works and must be written out. Triumph! And thus it goes: between problems that crop up and the solutions, back and forth. Finally, illumination and euphoria!

As in music, mathematics is something you are struck by, enamored, until nothing else matters. Music and mathematics are known to support each other. Confronted in mathematics with the proof that remains just out of reach, happiness mixed with pain. Villani relates incidents in the life of Mikhail Gromov, who developed the theory of convex integration.

While out walking around IAS and enjoying the natural beauty of the landscape Villani encounters Vladimir Voevodsky and they discuss their respective work and the prospect of the computer checking of proofs. Villani is also reflecting on the Landau damping problem and having to say good-bye to Princeton and IAS. As often happens in thinking about mathematics, unpredictable encounters lead from one problem to another.

Villani describes his prepared address to the upcoming Congress of Mathematical Physics in Prague. But he is dominated by concern over the non-acceptance of his research in *Acta Mathematica*, yet there is consolation for recently having won the Fermat Prize. Again, he digresses to Mittag-Leffler, as founder and first editor of *Acta Mathematica*, with the recollection of Poincaré as Mittag-Leffler's favorite author, who founded dynamical systems while studying the stability of the solar system. Villani obsesses over assumed failure to publish in *Acta Mathematica*. There follows a consultation with another colleague who points out a weakness in the proof. When Villani tries to write a justification for the steps, he discovers a careless oversight, and sets about to repair it.

Receives momentous news that he has won the Fields Medal, which leads to elation. However, this must be kept a secret until the International Mathematical Union announces it. Villani describes the origin of the Fields Medal. Returning to Paris from a trip to Cairo he reflects on his entire career. At long last he receives the confirmation letter advising his award of the Fields Medal.

Attends the funeral of Paul Malliavin (Malliavin Calculus), and fondly recalls the strong early encouragement from his older mentor. Villani discusses the Poincaré Conjecture with Gregori Perelman's solution and proof, and subsequent refusal of prizes and honors.

The fevered excitement at the presentation of his Fields Medal is recalled: the grandeur, meeting distinguished friends and colleagues, partying and reveling; the participants then returning to their usual lives in universities and research centers. Villani contemplates the aftermath of fame for having received the Fields Medal. This is the process of mathematical discovery: struggle with a difficult problem, success, adoration and fame, dejection that it's over, then going on to a new problem. There are always the nagging doubts: what if the proof is wrong? At last he receives word from *Acta Mathematica* that his proof on Landau damping is accepted. A theorem is truly born at last. The process goes on unceasingly.

Reflections on Creativity in a Diverse College Classroom

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Abstract: The article reflects on challenges and joys of facilitating creativity during classroom projects. Various creative aspects of the projects are presented together with particular examples obtained by students. Diversity of the group and the interdisciplinary environment are seen as factors that give preferential treatment to the process of creation. It is emphasized that the facilitator must be well-versed in her own creative work prior to being successful in leading students through creative stages. Several examples of various appearances of creativity are furnished in the article, including these experienced on a daily basis, while fulfilling mundane activities.

INTRODUCTION

Extreme diversity of the population of students of LaGuardia Community College, a large urban college in New York City, creates a challenging, yet rewarding, environment for introducing and assessing creativity in a mathematics classroom and outside. At the same time, it is the diversity that supports the collective experience and enriches the individual learning of each student. Most students in upper level undergraduate courses major in engineering or computer science, thus creative activities were designed to address students' shortcomings and specifics of their future jobs. Fostering creativity includes brief creative assignments and 6-week long class projects, where students make an attempt to find their own topics and their own way of presenting it. This may include but is not limited to: providing historical background, inventing one's own line of inquiry, finding one's own problems, or demonstrating suitable experiments. At the beginning of the semester, students are encouraged to assess their skills of writing, speaking and mathematical exposition to create groups with complete skill sets. Since many LaGuardia students are nonnative speakers, their skills of writing and speaking require additional practice. At the same time the nature of engineering jobs often focuses on group projects, where multiple specialists in various areas can exchange their experience, which is addressed in design of in-class and outside of class creative projects.

Creativity here is understood as an internal process where an individual experiences stages of creative thought (Wallas 1926): Preparation, Incubation, Illumination, and Verification. The solved problem should be new to the individual but may be well known and published in literature. Having no doubt that working on and solving actual open problems may carry more energy and

bring more delight than working on problems already solved by others, we would still support the idea of bringing topics to the classroom that are on students' level.

CREATIVITY EXPERIENCED IN A GROUP

The stages of creativity as described by G. Wallas (1926) and the creative flow as described by M. Csikszentmihalyi (1990) treat the creative process in terms of individual experience. While visualizing a creative person, we may have in mind an image of an individual artist (or a scientist) going through creative processes in the solitary atelier, but it is often the discussions with the peers and their suggestions, appreciations, or words of criticism that are crucial in a creative process. In other words, in creativity "no man is an island entire of itself". As we recall, the most interesting research results and conclusions, revealed themselves when we were interacting with other people. Our most enlightening observations arrived when we were observing others (our students) going through creative processes. These observations prove the significance of a social experience of creativity.

Reflections on experiencing creativity in a group

Our personal impression is that while experiencing the creative process we may be so involved in it, that the self-reflection and the self-observation may be limited. The intensity of the creative process and assisting excitement may be so overwhelming that the person may not be able to observe the factual process of creativity and completely forget self-reflections. Thus, observing others during that process may be a crucial aspect in the growth of self-awareness of one's own creativity. That is why we encourage students in a classroom and during research meetings to work in groups and share their experience about their thinking process. In our understanding, we often learn new skills of the mind by observing others during those activities and adopt the activities that appear somehow attractive. Thus, learning the skill of creativity in a classroom may be improved by collective experience shared by groups of peers. This aspect of creative work should not involve comparing ourselves to others, but it can contain nonjudgmental comparison of the outcomes of the creative processes among the researchers. This aspect of creative work should be solely focused on building awareness and sensitivity to the signals that come from another creator. Thus, social experience of the creative process may be a natural way to grow it, since humans tend to be highly sociable creatures and enrich each other by encouragement and shared experiences.

The role of mirror neurons in experiencing creativity in a group

Mirror neurons were discovered in 1992 by Giacomo Rizzolatti and a group of researchers in Parma, Italy. They have the property of firing when one performs an action and when one observes that action being performed by others (Keysers 2010). This leads to various hypotheses about possible roles of the mirror neurons in cognitive processes.

Following the concept of "imitative" function of mirror neurons (Pineda 2009), one may inquire about their roles in the processes and flows of creativity. Claiming that mirror neurons play a role

in a process of learning supports the idea that while teaching the material, we as well teach our students some mental processes and attitudes connected to it. For example, our love of mathematics may be contagious and spread out to some individuals. Thus, we should expect that once exposed to the presence of other individuals' experiencing creative cycle and flow, our students have increased chances of experiencing such a flow on their own.

Personally, we are very supportive of the idea that observing others while they experience the creative process has a great value for the observer. That is because our interest in creativity was initiated by our own observations when working with students on creative research problems. We realized that students behaved differently and carried different attitudes when working in our office on creative problems than when they worked in a classroom on mundane assignments. We have no doubts that this curiosity which grew in us over time was detached from self-awareness and entirely rooted in observing our students. In our understanding, there is a lot to research about the role of the influence of social experiences on the creativity of individuals.

Practice of experiencing creativity in a group

The most amazing thing about experiencing creativity is that the group does not have to meet in person. The excitement can be passed across the space and time without losing enthusiasm and intensity. We all experience it while reading an interesting book written by a person who is passionate about their work and discoveries. Modern technology and its vast availability of free networking makes the sharing easier than ever before.

In our class, when introducing the assignments of creative projects, we frequently use examples of work performed by students from previous semesters. For the first time, when introducing the projects in the classroom, we used an example of students' work that came from a mentored research team. Usually, we display the slides and show a video pointing out what is valuable and significant about students' presentations. For example, recently our students in Differential Equations class found articles about designing loops of the roller coasters and prepared a quality presentation with historical background and multiple details related to various designs of the loops. Finding a topic was a form of a creative assignment since students read the guidelines and reflected on their own interests trying to make connections to the material learned in the classroom. Then they concluded with some ideas and researched them online to find artifacts that may be suitable for references.

This presentation will be shown to students next semester as an example of an excellent choice of topic. It will encourage students to search for something exciting. This is what students who prepared the roller coaster presentation wrote in their conclusions:

"After doing a significant amount of research, our group had a blast learning more about how these attractions worked, more specifically the loop aspect of the rollercoasters. It would be insane to see how these engineers create other inversions a bit more complicated than the shape of these loops. The concepts that were used in this project mainly came from physics, but it was cool to see differential equations take a role in helping determine the shape of the loop

itself. This project also was an eye-opener to show us that these engineers must work really carefully as they are responsible for the lives of people that dare to embrace the thrill.”

An attentive observer could see that one student infected others with his idea and they got excited about it just as much as he did. In one of the pictures, here presented on Figure 1, they displayed the first loop, mentioning that the French Centrifugal Railway was the first roller coaster company to introduce loops, pointing out that presenting historical aspects of the topic gave a frame to modern questions and findings.



Figure 1. Built in 1846 in Paris, France the roller coaster featured two slopes and a circular loop

EVERYDAY OCCURRENCE OF CREATIVITY

We are taught to admire great accomplishments of creativity of others, unfortunately often omitting a long process it took to arrive to the result. This leads to a misconception that creativity is reserved for the gifted and the learned for the purpose of making big scientific discoveries or great art that finds its place at a museum. But the truth is somehow on the other end of the spectrum. According to Koestler’s bisociation (Koestler 1964) it is the frequent and daily use of powerful creative flow that liberates the mind from overwhelming habits.

Reflections on everyday occurrences of creativity

We are most creative as children. However, the results of this creativity may not have many applications. Being college professors, we could claim that we are creative every day while finding new research topics or working on research projects. But it does not mean that only highly educated individuals can be creative in a valid way and with valuable topics. We can be creative on a daily basis with daily chores and mundane activities regardless of our education and employment. Moreover, this can be a source of immense joy. Recently, we received some lessons for making sushi. The idea of repeating the same procedure over and over without adding anything from ourselves seemed to be quite unbearable to us. But the idea of searching for new ingredients and testing them in sushi felt quite appealing. After placing colorful ingredients on the wrap and cutting the roll into pieces, we realized that the distribution of the shapes and colors in the resulting cuts is rather unpredictable, which encouraged numerous experiments. At the same time, a friend of

ours created a mini sushi presented in Figure 2. The joy of creating artistically appealing shapes and colors in a small bite of food was quite immense. Then reflecting on how placing ingredients on the rice affects the look and the taste of pieces of sushi after rolling and cutting was quite an entertaining activity.



Figure 2. Various patterns on cuts of sushi and miniature sushi.

Why are daily occurrences of creativity crucial for establishing the habit of creativity?

The mature mind, while rediscovering the skill of creativity, treats it as a certain novelty and studies it in various versions, particularly in casual situations (Koestler 1964). This allows the skill to grow and eventually the mind becomes fluent in being creative. Certain thrills and excitements that assist creative flow makes creativity even more attractive to the mind.

As my students pointed out in their conference proceedings presentation (Torres 2018)

‘Creativity is not something that I do during the project. I do it every day and all the time.’

It is really the everyday creativity that allows us to successfully deal with multiple daily struggles: fix a leaning shelf with remainders from a broken hook or find out how to overcome problems with the equipment that is not working properly. At the same time, allowing creativity on a daily basis makes the creative skill truly functional and available when needed for challenging research questions.

Examples of brief creative assignments

Students in my classes often go through brief creative assignments that prepare them for the final project that supposedly contains some creative aspects. These short assignments appear during lectures and students either work in groups or individually, but always share the results of their work with the class. It is important to design those assignments in a way that there are no incorrect answers, so everybody can share some results to build a feeling of accomplishment. However, it happened that students who were not used to performing creative tasks in a classroom initially had

difficulties understanding what was expected of them. This difficulty would usually diminish during the second brief creative assignment.

The first creative assignment in Calculus 3 class can be introduced during the first class meeting when the three-dimensional system of coordinates is introduced. Often students who have experience from other countries and/or other areas may prefer to sketch the 3-dimensional space with different positions of the axes and different orientation. To address this issue, we ask students to invent their own way of drawing the three-dimensional space and later, students present their work on the board. For the second assignment students plot the point $(1,2,3)$ on their system and after comparing the pictures they realize that having different systems simply interfere the communication among them and makes learning more challenging. Thus, the class arrives at the conclusion that if we have an intention of studying in one classroom and supporting each other's growth we should have a uniform way of drawing three-dimensional space. This assignment and this conclusion would not be possible without initial heterogeneity of the classroom.

Another example of a short creative assignment involves providing examples of linear differential equations and nonlinear differential equations. After introducing the definition of linear differential equations, I would provide few examples on the board and later ask students to create their own examples. Students write their results on the board and later decide whether the classification provided by the authors is accurate. There are endless possible examples and it is quite educational for the instructor to get insight into students' minds, in particular, how they process the topic and the examples provided previously. As a teacher, we learned a lot from looking at students' examples and understood that most of them created something very similar to what they already saw but some made an intellectual effort of creating something unique and funny in its own way. We were glad to observe that some students had fun playing with mathematical expressions just for a true joy of goofing around.

Creative examples of assignments invented by students

Due to limited access of examples of creative behaviors experienced by my students outside of academia, we will present samples of creative ideas and their processes that we observed when they worked on research assignments.

A student had a task of finding an assignment for herself relevant to her hobbies and/or major (electrical engineering) and learn how to use the thermal camera. Since she likes going to the gym and is interested in well-being, she wanted to research how the temperature of the body changes during exercises and how it affects the quality of the workout. After trying a few ideas, the student realized that the camera does not take proper readings through clothing, so she decided to choose running and swimming for her investigations as the sports with exposed body limbs. She asked a colleague for assistance and did several readings of him running at the gym taking measurements of the temperature of the legs and the face. Later, the student performed similar measurements for swimmers and described her results in a research report. The project had mathematical aspects,

where the student attempted to fit her observed measurements in the exponential growth or decay model. This aspect failed but she observed that the measurements fit in the logistic models. The student entirely designed all her experiments and learned from mistakes creating original research not performed before.

To provide another example of creativity we will describe the situation, where our students, while making the attempt of taking a video with a thermal camera, realized that the version of the camera that we purchased does not have such a feature. They simply took a video of the thermal camera screen with their cellphone and presented it as a thermal video. Here students combined the functions of two devices, where one of them (thermal camera) was new. They simulated the video function missing from the camera by the video function from the cellphone.

Student's self-reflection and description of the process of working on a class project

One of our students described his creative flow while working on a class assignment. His Differential Equations project was about a simple pendulum, but he wanted to show an experiment to illustrate the formula that expresses dependence between the length of the string and the period of the pendulum. He found an experiment online called a pendulum wave and decided to build a wooden frame and present the experiment in class. Here is his description of his process of searching and building the model (Delshad 2018):

"Recently, I built a pendulum wave. Pendulum wave is a structure which is based on a series of pendulums in a row, with equal distance apart from each other. Each pendulum in that series has different length, such that lengths are calculated with a precise accuracy using a formula. This difference in length of pendulums creates different time period for each and every pendulum in the series. This effect causes the pendulum wave to create different patterns like moving waves, helixes, chaotic motion and etc.

I saw someone built a pendulum wave structure online and it was mesmerizing. Now, I wanted to create my own pendulum wave. Also, I wanted to understand the laws of physics like kinetic energy, potential energy, the motion of a wave and its characteristics from this pendulum wave. First, I understood the concepts as much as I could on a piece of paper. I understood that summation of kinetic and potential energy is equal to zero. It was easy to comprehend that energy cannot be created or destroyed.

$$P.E = mgh \qquad K.E = \frac{1}{2}mv^2$$

$$P.E + K.E = 0$$

$$(1) \text{ Time period of a pendulum} = 2\pi\sqrt{\frac{l}{g}} \text{ where } l \text{ is length and } g \text{ is gravitational acceleration}$$

The process of building the pendulum wave was full of happy and challenging moments. There were a couple of stages that needed to be accomplished. I will go through all the stages one by one. First of all, I sketched the pendulum wave structure on a piece of paper. I choose to have 12 pendulums in my series. From the research, I knew that I had to choose the length of the first pendulum and Tmax for the pendulum wave. Where Tmax is the time period of the pendulum wave. Using the formula below I calculated the

length of each pendulum, and accurately noted on the piece of paper. This formula is derived from the formula above (1) with an additional constant k.

$$(2) \quad L(n) = g \left[\frac{T_{max}}{2\pi(k+n+1)} \right]^2$$

g = gravitational acceleration

T_{max} = Time period of the pendulum wave

k = constant

n = number of pendulum in the series

To find out the value of the constant k, I plugged in T_{max} as 24 seconds, g, first length = 0.254 meters, n = 1 (first pendulum) and solved for k. Whatever the k value I got, it stays the same for the entire project. So, I wrote a C++ program which helped me easily calculate the length of all the pendulums in the series. After knowing the lengths of each and every pendulum in the series, I noted all the material I needed to build a pendulum wave.

Since I knew that my pendulum weights are 0.5 cm (centimeters) wide, I kept all the pendulums 1.5 cm apart from each other so that they will not be tangled up. I think this part of the project can be understood easily because if I choose to keep the pendulums too close to each other, they would collide with each other and all pendulums get tangled up.

When I was building pendulum wave there were difficulties like setting pendulums in the series to close to each other and they would get tangled up or I would set up the pendulums in the series too far from each other and this would make it difficult to swing all pendulums at the same time. Both of these situations had to be overcome by adjusting the distance in between the pendulums. An important note here is to keep the distance from one to another pendulum the same for entire series.

One way to know if you have set up the pendulums right distance apart from each other is, the pendulum string doesn't get tangled up with another pendulum string. The right separation also depends on your hanging weight.

It is always better to have the right tools for the right job. Before starting any project, one should make sure to have all the necessary material for the project to the best of their knowledge. If something else is needed later during the project, it can be acquired, and it is fine but having all the necessary items before starting the projects will put you in a better position to build. Last but not least, when building any projects, one should keep in the mind the budget and build accordingly."

In this example the student's work was not original but he went through a creative process while figuring out what experiment he could show for his project. Then while building the frame, he had difficulties with entangled strings and had to overcome this issue. The last aspect is more of a practical problem but since the student is earning his engineering degree, this may be the kind of problem that he will be working on at work.

HETEROGENEITY

Often as teachers in the classroom we struggle and complain about the body of students being extremely diverse in terms of students' preparation, dedication and skill level. Personally, we can

relate to it and agree that homogeneity of the classroom makes lectures more aligned with students' needs. However, in the case of creative assignments the situation is slightly different. Diversity in a group offers a chance for exchanging not only the results of creative assignments but the entire process of creative thinking with its errors, reflections and corrections. Thus, while facilitating creative assignments, we encourage students to team-up based on diversity.

Creating groups based on diversity of skills

This aspect of work is closely related to experiencing creativity in a group but touches upon experiencing it within a non-homogenous group. At the beginning of the semester students receive a sheet with self-assessment questions that ask about the level of skills: math, reading, writing and public speaking. Bringing these particular categories is justified by observations and assessments from previous semesters. In our classes, students may represent extremely varied levels of math skills due to their diverse backgrounds. In addition, some students are returning to college after a long absence and may not feel comfortable with all study skills. Reading, writing and public speaking are categories motivated by the fact that a significant percent of the student body consists of non-native speakers. The range may be quite ample, containing students who just arrived from abroad and for example can read and write excellently but have serious difficulties speaking in English. There may be students who completed American high school, but English is not their first language and they do not use it on a daily basis. These students may lack some of the language skills but be strong in others. During the semester students are encouraged to observe these skills among other students and make an attempt to compose their project teams based on **complementarity of the skills**, not on friendships or homogeneity.

It has been our observation that students who have creative ideas either prepare the projects on their own or compose with ease their project teams according to their own needs. However, students who do not have creative ideas and were not chosen to complete someone's team, struggle with decisions about their topic and have difficulties delivering a quality presentation in a timely manner.

This aspect of work is still in progress with the hypothesis that the most research-efficient environment may be composed of individuals who complete each other in terms of certain aspects of the work. In the case of students, these aspects may be basic skills: reading, writing, oral presentation, and math.

Promoting interdisciplinary COLLABORATION

So far, the most interesting research in pure mathematics and in applied mathematics happened to me while working on interdisciplinary topics.

In pure mathematics we collaborated with a senior professor whose area of expertise is nonstandard analysis, and we obtained results on the intersections of that area and topology. The process of research was quite fascinating, since our knowledge of nonstandard analysis was initially

nonexistent. Thus, we followed with multiple questions trying to grasp the concept of the topic. However, our collaborators were well versed in our area, thus they did not experience the same research process.

While working in applied mathematics either with our colleagues or with students, we experienced an entirely different research environment. Our students felt free to ask questions about new aspects of mathematics that they did not understand, and we felt free to inquire about aspects of engineering. Amazingly, we all felt enormous satisfaction while teaching each other. Then we realized that the most valuable creative environment is formed by free flow of information among researchers representing various areas of expertise and various levels of insight. While working with students we found their naïve attitude particularly valuable. As we noticed, students often try to work on ideas that appear exciting. At the same time, our expertise and experience of previous unsuccessful work on such ideas, made us lean towards the direction of solvable problems. This dual approach usually placed the team within the scope of exciting problems that are workable and publishable, not only disputable in as a theory or hypothesis.

TEACHER AS AN INDIVIDUAL: FACILITATOR AND OBSERVER

How much should the teacher be involved in the creative process? In our understanding it is sufficient to show students how to start the project and then influence the work as little as possible making sure that students are enjoying the process and keep progressing on the path of understanding and growing creativity.

At the same time, we do not visualize someone without excessive experience of the creative process trying to facilitate such process for others. In our understanding working on our own research and doing creative artwork significantly improves our skills as facilitators of creative skills of others, in this case students in our classes and students who work on research projects with us. The success of research projects can be assessed based on students' motivation and presentations at the end of the project. However, we would include in the assessment the dropout rate and willingness of students to recommend working on research to other students. Since the time when we employed the ideas of facilitating creativity while mentoring students, the projects have zero dropout rate. Moreover, it is a frequent occurrence that students recommend the projects to their friends and encourage them to work together.

SUMMARY

We have had the most creative and valuable research experience while collaborating with people of different levels of expertise, and different areas of expertise, in particular, with our students. At the same time, their area of learning (various engineering majors) was rather far from ours in pure mathematics but offered some overlap in terms of general critical thinking skills and common interest. To summarize the experience that we had with creative assignments within the course

curriculum and beyond, we would say that everyone needs to find their own unique way of implementing creativity in their work and daily life.

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Creative Moments within Internalization: Mathematics Classroom

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Abstract: Creativity in its most subjective sense within the mathematics classroom often involve an individual creating meaning of concepts presented by others during social discourse. This internalization of external concepts presented during social discourse is fundamental to the learning process. In this article we review two instructor's approaches and methods to promote creative moments within this internalization process, and student reaction.

CREATIVITY AND INTERNALIZATION

Koestler: Habit versus Original Thought

Mathematics is often viewed by students as a collection of rules to memorize and their instructors as experts in these rules. For Koestler, the essence of creativity involves a transition away from thinking about rules or 'codes' that direct one's habitual thought in any given situation, that is the ability for divergent thinking. To be more precise, such, 'thinking outside the box' is creative when it connects two previously unrelated frames of references, matrices of thought, what is referred to in mathematics education as schema. As one might expect, Koestler situates such original thought for mathematics and science within situations in which one's habitual code cannot function, to obtain the goal. "...the situation still resembles in some respect other situations encountered in the past yet contain new features or complexities which make it impossible to solve the problem by the same rules of the game which were applied to those past situations." (Koestler, 1964, p.119). The blocked situation may be resolved through the discovery of a 'hidden analogy' an uncovering of what has always been there yet, was previously taken for granted. This discovery is made through selective attention on the new features and connecting this with one's existing knowledge or matrix that was previously unrelated, "It does not create something out of nothing; it uncovers, selects, re-shuffles, combines, synthesizes already existing facts, ideas, faculties, skills" (Koestler, 1964, p.120)

Koestler and Mathematics Education

In that Koestler viewed the use of rules as promoting habitual semi-conscious or routine +thinking, and creative thought as occurring only when such rules are not applicable or break down it is not surprising that he decried math education's emphasis on the memorization of rules and procedures. He considered such an approach as based upon a fallacy that the mathematics and science unlike art, literature and humor is the domain of logical deductive thought. "No discovery has ever been made by 'logical deductions...'" (p.264). This attitude he notes has created a language used in classrooms and textbooks that is overloaded with technical jargon. Instead Koestler considers of tantamount importance the need to emerge students in the process of guided discovery "...to re-live, to some extent the creative process." (p.265) However, Koestler spends little time elaborating on social learning situations such as the classroom, instead his focus is on creativity during untutored learning. Koestler coins the term 'bisociation' to depict the synthesis of previously unrelated frames-of-reference or matrices, (schema) that occurs during the discovery of a hidden analogy, and suggest it is the main vehicle of untutored learning (p.658). The creativity that enters into learning process that entails creating meaning in a new situation is typically referred to as accommodation. This type of learning involves restructuring or modification of existing schema when a new situation cannot be assimilated readily into existing schema i.e., what Koestler refers to as a 'blocked situation.' However, if the discovery of a hidden analogy that synthesizes previous unrelated matrices is the main vehicle of untutored learning would it not be of considerable importance during social learning situations? With this position as a premise we ask, what can an instructor do to promote such moments of little 'c' creativity within the classroom? As Sarrazy and Novotna (2013) note, you cannot teach creativity, indeed one might say the more you teach students the less opportunity for them to be creative!

In order to develop our premise that small subtle acts of creativity are inherent in the social learning process we turn to the work of Vygotsky, who had a clear focus on social learning. Two notions of Vygotsky we use to situate little 'c' creativity in the classroom are, 'internalization' and the zone of proximal development ZPD.

Vygotsky and Creativity

Singer et. al. (2017) depicts Vygotsky as a principle propagator of, "...view of creativity as one of the important mechanisms of new knowledge construction, with a distinction between creativity that leads to historical discoveries and creativity that contributes to the advancement of each student's learning." Singer et. al. (2017, p.5) While Vygotsky may well endorse creativity within the learning process he most certainly understood learning as a gradual transition between states or stages of development e.g. in the development of internalizing external signs he notes, "...it would be a great mistake...to believe that indirect operations are the result of a pure logic. They are not invented or discovered by the child in the form of a sudden insight or lightning-quick guess (the so-called 'aha' reaction). The child does not suddenly and irreducibly deduce the relationship

between the sign and the method for using it.”(p.45) It is clear that Vygotsky believed that internalization, and thus development involves incremental changes or transitions yet, what is of interest to us in this paper is to what extent these incremental changes are themselves moments of little ‘c’ moments of creativity. First, we review what Vygotsky intends by the term ‘internalization.’

“We call the internal reconstruction of external operation internalization. A good example of this process may be found in the development of pointing. Initially, this gesture is nothing more than an unsuccessful attempt to grasp something, a movement aimed at a certain object placed beyond his reach... When the mother comes to the child’s aid and realizes his movement indicates something, the situation changes fundamentally. Pointing becomes a gesture for others. The child’s unsuccessful attempt engenders a reaction not from the object he seeks but from another person. Consequently, the primary meaning of that unsuccessful grasping movement is established by others. (p.56)”

In Vygotsky’s work internalization of previously external activity or signs occurs within, or at least is greatly assisted by a social environment in which another individual provides meaning to a child’s activity and reflects this meaning back to them. In education this notion is often expressed as the teacher and student co-creating knowledge. Norton and D’Ambrosio (2008) characterize social constructivist pedagogy, based upon Vygotsky, as involving ‘assistance’ by the teacher followed with ‘internalization’ by students. These authors emphasize, “...internalization as the process by which functions are transferred from the social plane to create a plane of individual consciousness” (p.223). Vygotsky (1997) can be understood as viewing internalization as a transition from intuitive spontaneous concepts, based upon a child’s memory of their experiences to scientific concepts, based upon an understanding of the logical structure of the concept, a transition which is accompanied by more conscious reasoning. This process occurs as a result of cooperation with the teacher, and thus modelling followed by conscious imitation which for Vygotsky involves creativity, and not mere copying or routine following of the adult’s activity. “Intelligent conscious imitation comes instantly in the form of insight, not requiring repetition.” (Vygotsky, 1997, p.221)

Norton and D’Ambrosio (2008) describe the role of the teacher guiding students through incubation to discover mathematics as one who poses problems within the ‘upper limit’ of the child’s zone of proximal development (ZPD) that is, the difference between what a child can do with the assistance of an adult and what they can do on their own. In this way, instruction will promote a child’s development, and the individual’s creation of meaning within their range of comprehension. The focus of our discussion will be how does an instructor elicit student intuition, and subsequently challenge them to think and reason about the concepts involved, and to what extent can this process be viewed as creativity?

PEDAGOGICAL METHODOLOGY OF INSTRUCTOR STACHELEK AND WOLF

Lesson 1: Instructor Stachelek

Method 1: Work on the board from Notes

The instructor calls on students to go to the board and fill in a table, representing the different parent functions of quadratics, linear, rational and radical functions. Ask them to first name, then sketch and finally provide the domain and range of the functions.

Name	Function	Sketch	Domain & Range
	$Q(x)=x^2$		
	$F(x)=x$		
	$K(x)=\frac{1}{x}$		
	$G(x)=\sqrt{x}$		

The students willingly go to the board, they correct each other's work, especially when it comes to domain and range i.e., "No that's the range!" As they work together, they stop periodically when lost and another may fill in the rest of the work. One issue that causes confusion is whether the domain of the radical includes 0. The instructor clarifies that it does and uses this opportunity to promote the set notation for domain i.e. $[0, \infty)$.

Discussion: In this setting the students speech and gestures serves to express the need for assistance from peer's or the instructor and or to offer assistance to one another. The dialogue between peers is typically expressed in short directives or informal mathematics yet, it appears to have a lot of value in the internalization process. Serving both to communicate informal explanations for difficulties encountered and to provide emotional support. The instructor's role was to support this interaction, to make sure that the table was filled in accurately using correct mathematical expressions and notation, and to clarify issues that arose during student discussion.

Method 2: Interactive Whole Class Dialogue: Name That Function Game

Given an x - y coordinate table the instructor asks students to suggest random relatively small integer values for x and then fills in the corresponding value for y . Each table represents the input and output an example of one of four different types of functions given by the parent function in the first table.

The students were asked first to guess which parent function or type of function was being represented and second to write the actual function. Finally, they were asked to graph it!

Round 1:

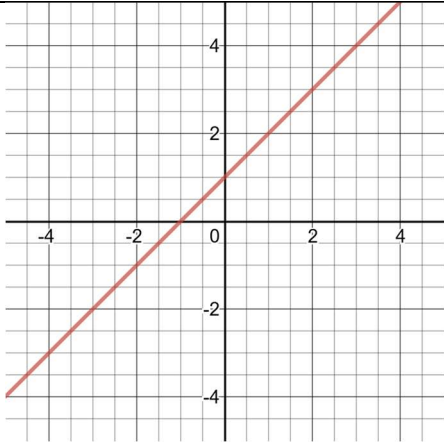
Student	Instructor	The instructor asks for a negative value.
x -value	y -value	
2	3	
4	5	
7	8	
-2	-1	

I: What is the parent function?

St1. Linear!

St2. $y=x+1$

I: $f(x) = x+2$

I: Can anyone graph it?		Student graphs the function
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Discussion: There was a preliminary round that set the stage for what was to occur later, the student readily and somewhat intuitively grasped both the parent linear function and the actual function, they also understood how to graph it.

Round 2:

	Student	Instructor	
	x -value	y -value	
	2	3	
	3	8	
	0	-1	
	1	0	
The instructor asks for a negative value.			

I: Talk amongst yourselves and discuss what type of function this is!

After pause

I: What is the parent function?

St1. Quadratic!

I: Yes, that is correct! Does that help think about it? (Think about what the actual function is)

Most students find that knowing it is a quadratic helps them connect the values with a quadratic function and begin to search for the actual function, but some are still confused.

The instructor waits for an answer...

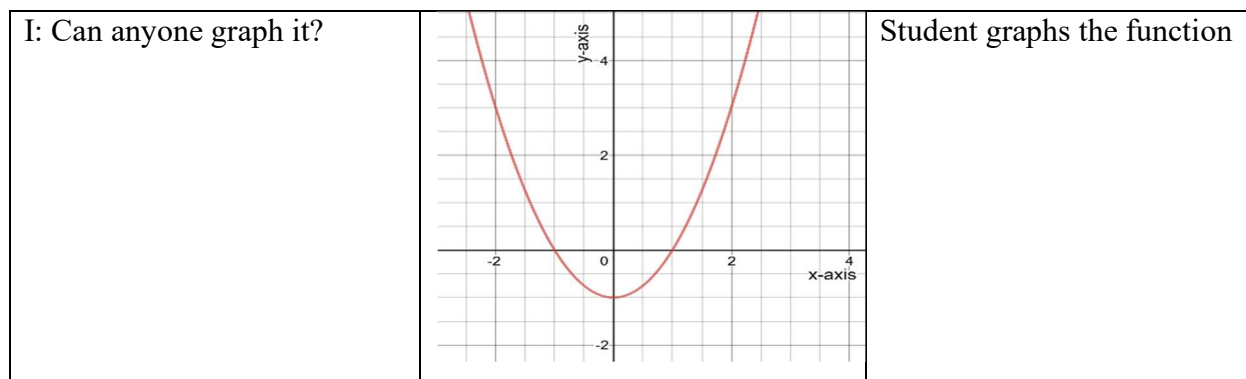
As students ponder a realization occurs

St2. Oh, now I got it!

I: What is it?

ST2: $x^2 - 1$

At this point the entire class appears to get it



Discussion:

The objective underlying this ‘Guess the function’ activity was to develop students’ conceptual relationship between tabulated data of, input-output with its associated parent function, as well as the specific translation-function that generates this data.

The student who pronounced ‘Oh now I got it!’ appears to the team (both instructor and observer) to have had a moment of insight or realization. The instructor’s pedagogical methodology to promote this moment of insight was to first introduce the intuitive-multiplicative scheme of a linear function within the context of the ‘Guess the function’ activity and then switch to the much less intuitive quadratics. In this scenario the student’s realization was a result of first, realizing that her first linear schema (matrix) was not appropriate. Then when informed by her peer that the correct matrix was quadratic, during the instructor’s pause she realized or discovered the correct additive constant to fit the data. This student certainly experienced the positive affect that Koestler values and demonstrated this by assisting other struggling students later in the period.

The effect of this pedagogical technique was to induce classical Gestalt ‘Restructuring’ as the solver needs to first put aside the initial linear schema that is intuitive and easy to apply mentally, and then begin the task of searching through the data-information presented for another schema.

Method 3: Teamwork collaboration to understand translations of parent functions

The objective was to reinforce and extend the conceptual relationship between tabulated data and graphs established during the second method to include both horizontal and vertical translations of parabolas, radical and inverse functions through collaborative work.

The class forms teams of 4-5 students each of which is given a type of function that involves either a vertical or horizontal translation. They task is to prepare a table of strategically chosen input and output values that adequately and readily represent this function. When the groups are done the give the table to the next group whose task is to determine the parent function and then complete the table of part 1, i.e., graph it and state the domain and range.

I: Please choose x-input values strategically! If you have a radical, please chose perfect squares! Also only provide input and output, do not graph it! (That is the task of the next group that does not know the function)

Student either wrote the table of data first and then the graph, ore advanced students write the function first and then the draw a graph with minimal or no use of the tabulated data. Those students who correlated the data with the graph often did not understand the curvature especially the translations of radicals. Instead they connected the dots in a linear fashion, suggesting they had not internalized the parent function graph and how their work was related to it.

Some students struggled with correct notation and what might be considered as the vertex or the minimal value of the domain and range (all graphs were in QI) of translated radical functions. The student that had the realization earlier was motivated and able to assist several of her peers with the graphing process this suggests that her realization had an affective-motivational component.

As the period draws to a close the instructor informs students that $j(x) = \sqrt{x+1}$ was the most difficult conceptually for them to graph and explains the effect of horizontal translations. The students find this a bit much to process especially so late in the period.

Lesson 2: Instructor Wolf

A more traditional format yet with distinct constructivist leanings, by which we mean a teacher led discussion within problem-solving, with a focus on active student involvement at each step of this process. The lesson begins with the Fundamental Theorem of Calculus and a subsequent review of anti-derivatives beginning with the power rule including inverse trigonometric and logarithm anti-derivatives. Most examples are review, and the students using a chart of derivatives to assist recall in this whole class problem activity.

As the review ends the instructor provides the following example to motivate the class to learn substitution involving the chain rule:

$$\int \frac{\cos \theta}{\sin^2 \theta} d\theta$$

I: We don't have the tools yet to do this one but is there something on the formula sheet you can use?

Silence

I: Let's rewrite this:

$$\int \frac{\cos \theta}{\sin^2 \theta} d\theta = \int \frac{\cos \theta}{\sin \theta} \times \frac{1}{\sin \theta} d\theta$$

$$= \int \cot \theta \times \csc \theta d\theta$$

At this point the student realize this is on the formula sheet as $-\csc \theta$

I: Ok, so how can we find the derivative of: $= \frac{2}{3}(1+x^2)^{\frac{3}{2}}$?

St1: $y' = \frac{3}{2} \frac{2}{3}(1+x^2)^{\frac{1}{2}} 2x$

I Where did this come from? (asking the rest of the class)

St2: The chain rule!

I: Do you follow this? How can we now find: $\int 2x(1+x^2)^{\frac{1}{2}} dx$?

St1: The $2x$ is the du

I: Have you ever done this before?

St1: Yes!

I: Can you be more specific about what you are doing?

St1 working with I:

$$u=1+x^2$$

$$du=2x dx$$

I: Good, (speaking for the other students in the class). It's important to know why it's du

I: (Again directing to the rest of the class). How did we do this?

St2: Asks for clarity about what

I: Just rewrite it! $\int \sqrt{1+x^2} (2x) dx =$

St2: $= \int u^{\frac{1}{2}} du$

I: Ok, what do we have?

St2: $= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$

I: Simplify! Silence...Flip that sucker!

St2:
$$= \frac{2}{3} u^{\frac{2}{3}} + C$$

I Gives wait time for student to realize they are not done

After pause

St2: Oh
$$= (1 + x^2)^{\frac{3}{2}} + C$$

I:OK, so now we have a chain-rule for anti-derivatives built upon substitution: (writes on board)

$$\int F'(g(x))g'(x)dx = F(g(x)) + C$$

I:So how can we do a problem like the one we started with, that involves trigonometric functions?
$$\int \sec x \tan x dx$$

Silence

I: Rewrites:
$$\int \frac{1}{\cos x} \times \frac{\sin x}{\cos x} dx = \int \frac{\sin x}{\cos^2 x} dx$$

I: What can we use to substitute?

St3: Ahh... $u = \cos x$

I: What does $du=$?

St3: $du = -\sin x dx$ so we have
$$-\int \frac{du}{u^2} = -\int u^{-2} du$$

I: What do we do next?

St3:
$$= -\frac{u^{-2+1}}{-2+1} + C$$
$$= -\frac{u^{-1}}{-1} + C$$

I: What do we do next? Pause ...(gives hint) the negatives are gone, let's substitute!

St3:
$$= \frac{1}{\cos x} + C$$

I: Ok, what is $1/\cos x$?
$$= \sec x + C$$

Discussion

This lecture on anti-derivatives using the chain rule can be viewed as essentially beginning and ending with an example of the type: $\int \frac{\cos \theta}{\sin^2 \theta} d\theta$. At first the instructor rewrites this example as

$\int \cot \theta \times \csc \theta d\theta$, and the students use a formula sheet to find the answer, without any understanding of the formula. After explaining substitution in the context of the chain rule, the instructor revisits this problem-type beginning with $\int \sec x \tan x dx$, The instructor first presses the students to resolve the problem, then after silence rewrites it as $\int \frac{\sin x}{\cos^2 x} dx$ and pauses until one of the students realizes how to employ the newly learned substitution technique. Specifically, what should be the value of 'u', and how does it, and the corresponding 'du' fits into the problem? The resulting realization by a student that $u = \cos x$ is a good illustration of what has been referred to as the little 'c' creativity that accompanies a typical lecture in a student centered or creative learning environment. In the follow up team discussion it was suggested that the instructor could have paused longer before re-writing the example. That being said, it was at the end of the lesson time, the student did make the correct realization, and the rest of the class appeared to readily understand.

It is interesting to note that, the lecture transitioned between the initial and final resolution of this problem type by internalization of the guiding principle or code that underlies the various examples. Specifically, the instructor first employs concrete examples, and after the students have demonstrated some degree of internalization the instructor then presents the abstract formulation of the rule. During this transition, while finding the derivative of $y = \frac{2}{3}(1 + x^2)^{\frac{3}{2}}$ when a student provides the answer, the instructor realizing he has done this before asks him to be more specific. This was done to assist the other students follow and internalize his method.

SUMMARY DISCUSSION

Koestler views creativity (discovery of a hidden analogy) as taking place when an individual searches through existing activity, tools or matrices and re-examines them in a new light that allows them to be combined with other activity, tools or matrices in a novel manner. Vygotsky understands creativity within internalization as taking place when an adult, peer or mentor reflects one's activity back to the individual with new meaning in a way that allows for creation of meaning, i.e. internalization. An important end-result for both Koestler or Vygotsky is a new code or structural understanding of the reasoning or guiding principle that underlies the novel activity.

In the first lecture by instructor S we observe a social situation in which a peer's statement allows a student to set aside her linear matrix and realize the correct quadratic matrix that is represented by the data. This information was apparently completely internalized as she then assisted other students understand the correct parent function for different data, specifically in graphing both quadratics and radicals.

In the second lecture by instructor W we observe a similar phenomenon of creativity within internalization. In this case, the initial frame of reference is the new technique of substitution previously used only to represent a polynomial within a radical-root function. The instructor introduced a trigonometric function, but its format was too complicated for the students to understand how to apply this new technique, until the instructor re-wrote the problem as a trigonometric rational. At that point a student realized that the substitution technique involved the denominator and the derivative du was the numerator of this rational trigonometric function.

In the first lecture the ‘Aha Moment’ was more distinct and the affect more noticeable. As evidenced by the students ability to assist others struggling with similar issues of extending their linear matrix to curved quadratic or root matrices. In the second lecture it was clear internalization had occurred, but the extent of independent activity was not as evidenced. That being said in the second lesson the abstract code that represents the substitution process was presented in the middle of the lesson and the students appeared to relate to it, while in the first the abstract code or rules of graphing translations appeared to come rather late to evidence student comprehension.

This discussion highlights several issues for instructors the more peer-peer interaction provides greater opportunity for meaningful creative insights though such increased participation, yet it may limit the amount of the time spent on structural formulations that express the underlying code of the examples done in class. Conversely a more structured teacher controlled lecture while often insuring sufficient time to develop the underlying code provides less opportunity for distinct moments of student creativity and demonstration of independent internalized activity.

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The Structure of Creativity of Aha!Moments in Mathematics (part 1)

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Abstract: The paper presents the classification of Aha!Moments in mathematics obtained through the analysis of collected insights of students and inventors conducted in Czarnocha and Baker (2018). It uses the definition of Aha!Moment abstracted from Koestler's bisociation theory formulated in his Act of Creation (1964). The classification scheme finds three types of Aha!Moments in the collection, Mild, Normal and Strong decided on the basis of the nature of connections made during the insight. The presentation compares this classification with the classification of connections obtained by AI based bisociative search engine BISON and finds a surprising similarity in their corresponding structures. The similarity leads to the basic question of the difference between human and computer creativity.

INTRODUCTION

Most of us are familiar with the surprisingly sudden and exceedingly pleasant moments in thinking when the issue we dwelled upon for some time suddenly and unexpectedly becomes crystal clear and we either are overwhelmed by realization of its significance or/and by the sudden boost in self-confidence. These are Aha!Moments and although several of their descriptions are in research literature (Barnes, 2000; Palatnik and Koichu, 2015; Yoon, 2012; Liljedahl, 2004), their systematic study in the practice of mathematics education has not as yet been undertaken. That job was initiated and developed into the research program by the members of the Teaching-Research Team (TRTeam) of the Bronx, who have coordinated the bisociation theory of Arthur Koestler with the classroom events (Prabhu, 2016; Stoppel and Czarnocha, 2020). The definition of Aha!Moment as bisociation extracted from Koestler (1964) work states: “*The bisociation act is the spontaneous leap of insight which connects previously unconnected matrices of experience*” (Koestler, 1964, p.45) through the discovery of a “hidden analogy”. In passing let's note that Koestler's bisociation became the inspiration for the new artificial intelligence (AI) domain of computational creativity whose aim is “to model, simulate or replicate creativity with a computer” (Boden, 2004; Dubitsky et al., 2012) as well as for the design of the bisociative search engine for the two large sets of data we discuss shortly below.

Thus the creativity of Aha!Moment is the process of connecting two previously unconnected matrices of experience while building a conceptual bridge between them – the creative object. Prabhu and Czarnocha (2014) proposed bisociation as the new definition of creativity in mathematics education. Taking into account that Aha!Moments are familiar to THE population in large, the creativity of Aha!Moment can lead the process of democratization of research and practice in mathematics classrooms. The TR Team of the Bronx has analyzed, on the basis of the Koestler's definition, processes of facilitation of Aha!Moments as well as their depth of knowledge (DoK) assessment Czarnocha and Baker (2018), which is understood as the increase of understanding reached during the Aha!Moment insight. The methodology of DoK assessment allows us a glimpse into the structure of creativity of the “act of creation”, that is of Aha!Moment (also called Eureka experience).

METHODOLOGY

The origins of the methodology are rooted in the teaching experiment Problem Solving in Remedial Arithmetic: Jumpstart to Reform in 2010 supported by a CUNY CIRG 7 grant (Collaborative Community College Incentive Research Grant #7). Vrunda Prabhu, a member of the TR Team of the Bronx, observed many Aha!Moments in her remedial algebra experimental classroom at Bronx CC, which she coordinated with the Koestler's theory putting first steps along the path of investigations that took us to the present multifaceted inquiry into creativity of mathematical insight. The collection of Aha!Moments which served as the data for analysis was obtained along several routes: (1) during the teaching experiment under Title V at Hostos Community College (CC) which involved peer leaders of the two experimental classrooms as student-researchers. (The full collection will be provided during the presentation.) The title V teaching experiment was conducted in two courses one section of Arithmetic/Elementary Algebra course and another section of Intermediate Algebra course; (2) through the Hunt for Aha!Moments campaign organized by Mathematics Teaching-Research Journal on line (MTRJ at hostos.cuny.edu/mtrj, a Scopus indexed journal) and (3) through the Creativity of Aha!Moment CUNY conference at Hostos CC in 2018. In addition several Aha!Moments were found in professional literature mentioned above.

The central issue for us has been the precise description of mathematical situation within which the insight has taken place; in order to investigate the characteristic affective impact of the insight we need to have also the description of accompanying emotional states. The classification of the depth of knowledge of each Aha!Moment has been done from the point of view of the nature of connections developed during the insight by the learner.

As implied above we are interested in the process of genesis of connections between, as well as the nature of created objects. The genesis of connections between unconnected frames of reference is one of the defining quality of Koestler's presentation of bisociation noted above. As described

in Czarnocha (2012) the depth of increase of knowledge during Aha!Moment insight might be measured by the stages of the Triad of concept development of Piaget and Garcia (1989). The collection of Aha!Moments taken for the design of classification (Appendix in Czarnocha and Baker, 2020) is given by the insights observed in classes of mathematics, however, as we note below the historical Aha!Moments such as Gutenberg's (Koestler, 1964) or Einstein's (Rothenberg, 1979) neatly fit the proposed classification scheme. (The names of Aha!Moments quoted from the Appendix were established on the basis of their context and/or name of the creator. Example: Calculus Aha!Moment, Einstein Happiest Thought Aha!Moment, etc.)

We classify (1) Mild bisociation as one that involves only one conceptual connection or analogy. This includes the process in which discovering a hidden analogy involves employing elementary conceptions or patterns that are seen as relevant (Calculus Aha!Moment, Kim Aha!Moment and Physics Aha!Moment). The (2) Normal level of bisociation is the insight in which several elementary concepts are coordinated to form a functional whole. By functional whole we mean a construction of the interiorized schema, which leads to the solution to the problem or to a higher level of understanding, usually at the Inter level of the development (Fir Tree Aha!Moment and What is a Vector? Aha!Moment). We classify a (3) Strong bisociation as one that has at least two steps and/or two cycles in the progress of understanding, usually reaching Trans level of the schema development (Einstein "Happiest Thought" Aha!Moment to Factorize).

The proposed classification has been carried out on the Aha!Moments originating in mathematical classrooms; it's interesting that those of historical insights such as the Gutenberg insight (Mild) leading to printing press or that of Einstein happiest thought (Strong) leading to one of the founding principles of General Relativity Theory neatly fall into it.

BISON SEARCH ENGINE

BISON (Bisociative Networks for Creative Information Discovery) is a second generation search engine, whose bisociative methods of detecting similar patterns across different domains "promise tremendous potential for the discovery of the new insights" (Berthold, 2012;p.1). Below we point to a surprisingly clear relationship between classification of Aha!Moments insights obtained through the introductory Depth of Knowledge (DoK) described above and the classification of bisociative structures obtained by the BISON project. Berthold (2012) and Kötter & Berthold (2012) present three types of bisociative connection within Bisociative Knowledge Discovery (BKD) framework brought forth by the computer creativity approach, indicating at the same time that the typology of bisociative connections is an open field at present: (1) a single Bridging Concept, (2) Bridging Graphs and (3) Structural Similarity Bridging between two domains. This similarity is still more surprising if we note the following differences between AI definition of bisociation and that of Koestler.

Berthold (2012, p.2) asserts that informally bisociation can be seen as “(sets of) concepts that bridge two otherwise not – or only very sparsely – connected domains”. Comparing this with Koestler’s definition, we note the absence of the phrase “spontaneous leap of insight”. Instead of “insight” of Koestler we have its product given by Berthold: the set of concepts that connects or bridges previously unconnected matrices/domains; by Berthold we have also lost the “spontaneous leap” in Koestler’s definition, so that bisociative knowledge discovery employs only the mechanical, that is no spontaneous aspects inherent in connecting two different structures. This method is the basis of the AI understanding of bisociation—computers do not seem to have Aha!Moments.

COMPARISON OF TWO STRUCTURES

Type Mild of DoK. Calculus Aha!Moment single concept connection

During my Calculus 1, the teacher gave us an example to solve: $\lim_{X \rightarrow 0} \frac{\sqrt{1+X}}{X} - \frac{\sqrt{1-X}}{X} =$

- 1 I verify if the limit is defined when X approaching to 0. It is not.
- 2 I asked myself “how can I do and find a way for this limit can be define?”
- 3 I remember in my previews class that when the teacher gave us a rational fraction to solve, he said that we must eliminate the radical in the denominator by multiplication with the conjugate. But for this equation we don’t have radical in the denominator but in the numerator.
- 4 I’m a little bit struggling. What can I do?
- 5 I was looking at the limit and said to myself why not apply the same rule for the fraction when we have the radical in the denominator.

We see here that the student doesn’t see the unifying power of the method ‘multiplying by a conjugate’, so that different positions of the similar expression break the situation in students’ mind into two different cases (see figure 1). The insight in line 5 “why not to apply the same rule” indicates the beginning of connecting the two domains, search for the limit of the function with rationalization of fractions by a single concept of algebraic conjugation.

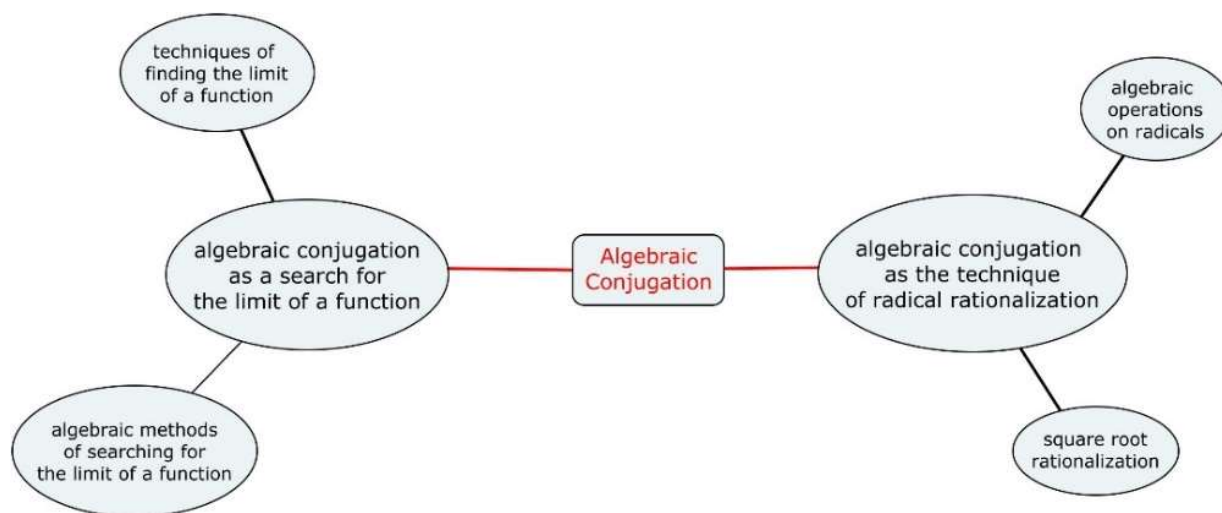


Figure 1. Algebraic conjugation as the single bridging concept (Stoppel & Czarnocha, 2020, p. 20)

The type Mild corresponds to the structure discovered by BISON search engine called the bridging by a single concept connection, see figure 2.

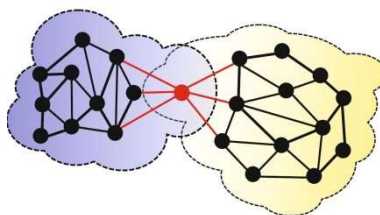


Figure 2. Bridging by a single concept (Kötter & Berthold, 2012, p. 45)

Type Normal of DoK. Aha!Moment Fir Tree

I had a tremendous Aha! Moment. I just realized that the formula I got from the patterns $[n(n+1)]$ was a factorized expression and if I multiply it " $n(n+1)=$ units square" I would have something like an algebraic expression exactly a trinomial expression that can be factorized as well, and it equals real numbers for example: $n(n+1)=12$ $n^2+n=12$ $n^2+n-12=0$ $n^2+n-20=0$ $n^2+n-56=0$ n^2+n-90 $(n+4)(n-3)=0$ $(n+5)(n-4)=0$ $(n+8)(n-7)=0$ $(n+10)(n-9)$ And when it comes to $n^2+n-274$ it cannot be factorized.

The student makes the connection between the algebraic expression she found and one of the processes of solving quadratic equation, by factorization of the quadratic trinomial. Factorization is the common domain joining the other two.

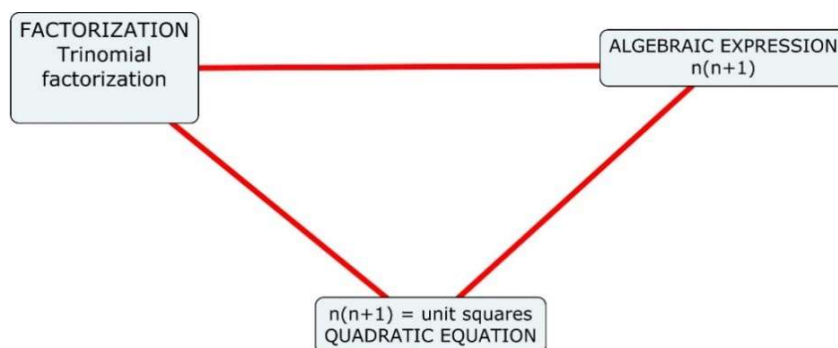


Figure 3. Bridging graph of three algebraic domains (Stoppel & Czarnocha, 2020, p. 22)

A bridging graph is given by connections between subgraphs of different domains. Connections might be given by direct connections between wedges of different graphs, see figure 4 (b).

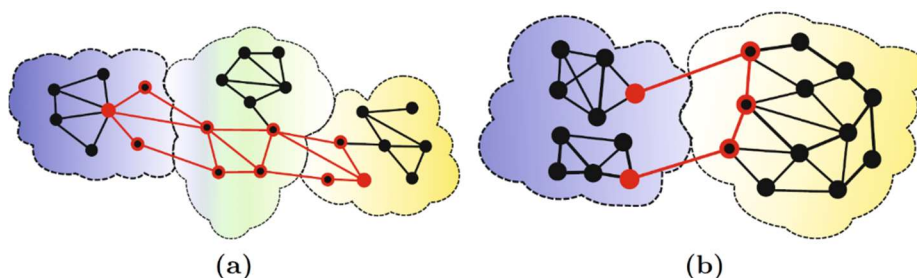


Figure 4. Bridging graphs by concepts (Berthold, 2012, p. 4)

The bridging might also be given by a relatively dense graph in connection with a network-bases representation as represented in figure 4 (a) (Berthold, 2012). Both Kotter and Berthold (2012) and Dubitsky et al (2012) suggest that Archimedes' Eureka moment can be understood through such a bridging graph connection of type (b) in figure 4.

Type Strong of DoK. The “Happiest Thought” of Einstein Aha!Moment.

Just as in the case where an electric field is produced by electromagnetic induction, the gravitational field similarly has only a relative existence. *Thus, for an observer in free fall from the roof of a house there exists, during his fall no gravitational field*-at least not in his immediate vicinity. If the observer releases any objects, they will remain, relative to him, in a state of rest, or in a state of uniform motion, independent of their particular chemical and physical nature. (In this consideration one must naturally neglect air resistance.) The observer is therefore justified in calling its state “rest”. [Einstein’s italics]

The analogy is “hidden” within the phrase “Just as in the case...” To explicate the hidden analogy, we need to make a precise comparison of the components of the phenomenon of the

electromagnetic induction with the phenomenon surrounding the thought experiment concerning the gravitational field. Figure 5 is helpful in grasping the hidden analogy.

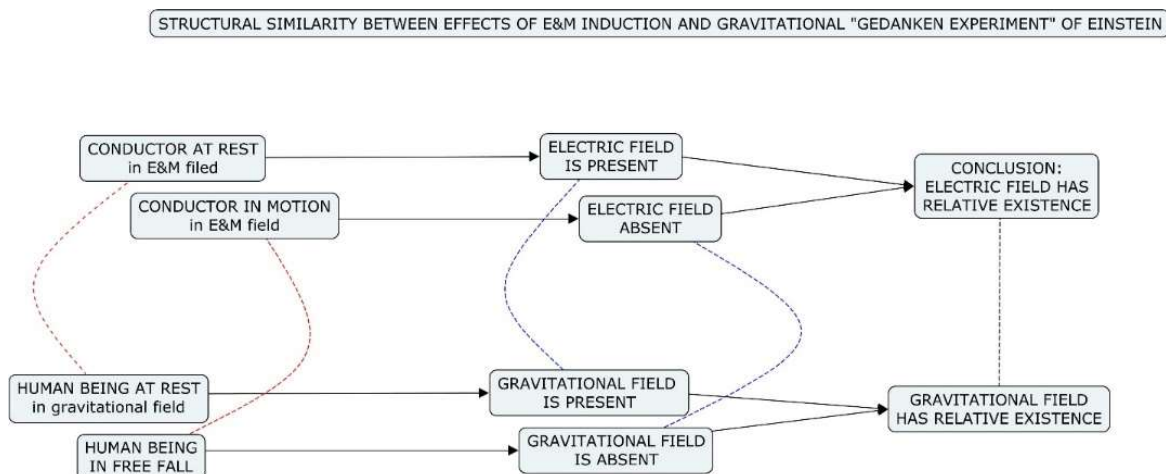


Figure 5. Structural similarity of electromagnetism and general relativity (Stoppel & Czarnocha, 2020, p. 26)

The Strong type of Aha!Moment of DoK intersects with the BISON type of Bridging by Structural similarity – the most complex graph the BISON team found.

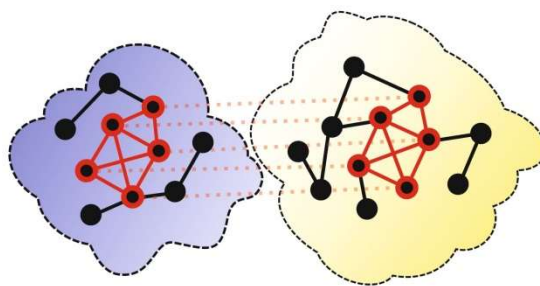


Figure 6. Bridging graphs by structural similarity (Berthold, 2012, p. 5)

CONCLUSIONS

We find it surprising that the cognitive structure of connections reached during Aha!Moments experienced by humans is essentially similar to the connections found by the bisociative search engine designed and programmed by the AI methods. We noted that the definition of bisociation used by Koestler and its definition used in the construction of the search engine, while similar in terms of role of connections between the concepts, differ significantly in the processes leading to the formation of the conceptual network. What does it mean? We conjecture therefore that cognitive nature of the created product is not the essential aspect of human creativity; the essential aspect of human bisociative creativity is in the creative process rather than in the creative product. That observation may significantly change the aim of the research in and practice of classroom

mathematical creativity from the analysis of creative products to the investigations of the nature of the creative process.

Part 2 of this presentation will be published in the next issue of MTRJ; it will compare the two different methods which led to the corresponding results presented here.

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