Creative Moments within Internalization: Mathematics Classroom

William Baker, Anders Stachelek and Lauren Wolf

Hostos Community College of the City University of New York

Abstract: Creativity in its most subjective sense within the mathematics classroom often involve an individual creating meaning of concepts presented by others during social discourse. This internalization of external concepts presented during social discourse is fundamental to the learning process. In this article we review two instructor's approaches and methods to promote creative moments within this internalization process, and student reaction.

CREATIVITY AND INTERNALIZATION

Koestler: Habit versus Original Thought

Mathematics is often viewed by students as a collection of rules to memorize and their instructors as experts in these rules. For Koestler, the essence of creativity involves a transition away from thinking about rules or 'codes' that direct one's habitual thought in any given situation, that is the ability for divergent thinking. To be more precise, such, 'thinking outside the box' is creative when it connects two previously unrelated frames of references, matrices of thought, what is referred to in mathematics education as schema. As one might expect, Koestler situates such original thought for mathematics and science within situations in which one's habitual code cannot function, to obtain the goal. "…the situation still resembles in some respect other situations encountered in the past yet contain new features or complexities which make it impossible to solve the problem by the same rules of the game which were applied to those past situations." (Koestler,1964, p.119). The blocked situation may be resolved through the discovery of a 'hidden analogy' an uncovering of what has always been there yet, was previously taken for granted. This discovery is made through selective attention on the new features and connecting this with one's existing knowledge or matrix that was previously unrelated, "It does not create something out of nothing; it uncovers, selects, re-shuffles, combines, synthesizes already existing facts, ideas, faculties, skills" (Koestler,1964, p.120)

Koestler and Mathematics Education

In that Koestler viewed the use of rules as promoting habitual semi-conscious or routine +thinking, and creative thought as occurring only when such rules are not applicable or break down it is not surprising that he decried math education's emphasis on the memorization or rules and procedures. He considered such an approach as based upon a fallacy that the mathematics and science unlike art, literature and humor is the domain of logical deductive thought. "No discovery has ever been made by 'logical deductions…" (p.264). This attitude he notes has created a language used in classrooms and textbooks that is overloaded with technical jargon. Instead Koestler considers of tantamount importance the need to emerge students in the process of guided discovery "…to re-live, to some extent the creative process." (p.265) However, Koestler spends little time elaborating on social learning situations such as the classroom, instead his focus is on creativity during untutored learning. Koestler coins the term 'bisociation' to depict the synthesis of previously unrelated frames-of-reference or matrices, (schema) that occurs during the discovery of a hidden analogy, and suggest it is the main vehicle of untutored learning (p.658). The creativity that enters into learning process that entails creating meaning in a new situation is typically referred to as accommodation. This type of learning involves restructuring or modification of existing schema when a new situation cannot be assimilated readily into existing schema i.e., what Koestler refers to as a 'blocked situation.' However, if the discovery of a hidden analogy that synthesizes previous unrelated matrices is the main vehicle of untutored learning would it not be of considerable importance during social learning situations? With this position as a premise we ask, what can an instructor do to promote such moments of little 'c' creativity within the classroom? As Sarrazy and Novotna (2013) note, you cannot teach creativity, indeed one might say the more you teach students the less opportunity for them to be creative!

In order to develop our premise that small subtle acts of creativity are inherent in the social learning process we turn to the work of Vygotsky, who had a clear focus on social learning. Two notions of Vygotsky we use to situate little 'c' creativity in the classroom are, 'internalization' and the zone of proximal development ZPD.

Vygotsky and Creativity

Singer et. al. (2017) depicts Vygotsky as a principle propagator of, "…view of creativity as one of the important mechanisms of new knowledge construction, with a distinction between creativity that leads to historical discoveries and creativity that contributes to the advancement of each student's learning." Singer et. al. (2017, p.5) While Vygotsky may well endorse creativity within the learning process he most certainly understood learning as a gradual transition between states or stages of development e.g. in the development of internalizing external signs he notes, "…it would be a great mistake…to believe that indirect operations are the result of a pure logic. They are not invented or discovered by the child in the form of a sudden insight or lightning-quick guess (the so-called 'aha' reaction). The child does not suddenly and irreducibly deduce the relationship

between the sign and the method for using it."(p.45) It is clear that Vygotsky believed that internalization, and thus development involves incremental changes or transitions yet, what is of interest to us in this paper is to what extent these incremental changes are themselves moments of little 'c' moments of creativity. First, we review what Vygotsky intends by the term 'internalization.'

"We call the internal reconstruction of external operation internalization. A good example of this process may be found in the development of pointing. Initially, this gesture is nothing more than an unsuccessful attempt to grasp something, a movement aimed at a certain object placed beyond his reach… When the mother comes to the child's aid and realizes his movement indicates something, the situation changes fundamentally. Pointing becomes a gesture for others. The child's unsuccessful attempt engenders a reaction not from the object he seeks but from another person. Consequently, the primary meaning of that unsuccessful grasping movement is established by others. (p.56)"

In Vygotsky's work internalization of previously external activity or signs occurs within, or at least is greatly assisted by a social environment in which another individual provides meaning to a child's activity and reflects this meaning back to them. In education this notion is often expressed as the teacher and student co-creating knowledge. Norton and D'Ambrosio (2008) characterize social constructivist pedagogy, based upon Vygotsky, as involving 'assistance' by the teacher followed with 'internalization' by students. These authors emphasize, "…internalization as the process by which functions are transferred from the social plane to create a plane of individual consciousness" (p.223). Vygotsky (1997) can be understood as viewing internalization as a transition from intuitive spontaneous concepts, based upon a child's memory of their experiences to scientific concepts, based upon an understanding of the logical structure of the concept, a transition which is accompanied by more conscious reasoning. This process occurs as a result of cooperation with the teacher, and thus modelling followed by conscious imitation which for Vygotsky involves creativity, and not mere copying or routine following of the adult's activity. "Intelligent conscious imitation comes instantly in the form of insight, not requiring repetition." (Vygotsky, 1997, p.221)

Norton and D'Ambrosio (2008) describe the role of the teacher guiding students through incubation to discover mathematics as one who poses problems within the 'upper limit' of the child's zone of proximal development (ZPD) that is, the difference between what a child can do with the assistance of an adult and what they can do on their own. In this way, instruction will promote a child's development, and the individual's creation of meaning within their range of comprehension. The focus of our discussion will be how does an instructor elicit student intuition, and subsequently challenge them to think and reason about the concepts involved, and to what extent can this process by viewed as creativity?

PEDAGOGICAL METHODOLOGY OF INSTRUCTOR STACHELEK AND WOLF

Lesson 1: Instructor Stachelek

Method 1: Work on the board from Notes

The instructor calls on students to go to the board and fill in a table, representing the different parent functions of quadratics, linear, rational and radical functions. Ask them to first name, then sketch and finally provide the domain and range of the functions.

Name	Function	Sketch	Domain & Range
	$Q(x)=x^2$		
	$F(x)=x$		
	$K(x)=\frac{1}{x}$		
	$G(x)=\sqrt{x}$		

The students willingly go to the board, they correct each other's work, especially when it comes to domain and range i.e., "No that's the range!" As they work together, they stop periodically when lost and another may fill in the rest of the work. One issue that causes confusion is whether the domain of the radical includes 0. The instructor clarifies that it does and uses this opportunity to promote the set notation for domain i.e. $[0,\infty)$.

Discussion: In this setting the students speech and gestures serves to express the need for assistance from peer's or the instructor and or to offer assistance to one another. The dialogue between peers is typically expressed in short directives or informal mathematics yet, it appears to have a lot of value in the internalization process. Serving both to communicate informal explanations for difficulties encountered and to provide emotional support. The instructor's role was to support this interaction, to make sure that the table was filled in accurately using correct mathematical expressions and notation, and to clarify issues that arose during student discussion.

Method 2: Interactive Whole Class Dialogue: Name That Function Game

Given an *x-y* coordinate table the instructor asks students to suggest random relatively small integer values for x and then fills in the corresponding value for y. Each table represents the input and output an example of one of four different types of functions given by the parent function in the first table.

The students were asked first to guess which parent function or type of function was being represented and second to write the actual function. Finally, they were asked to graph it!
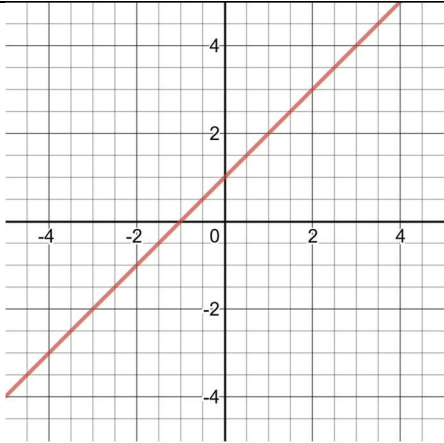
Round 1:

	Student x-value	Instructor y-value	
	2	3	
	4	5	
	7	8	The instructor asks for a negative value.
	−2	−1	

I: What is the parent function?

St1. Linear!

St2. $y=x+1$

I: $f(x) = x+2$

I: Can anyone graph it?		Student graphs the function

Discussion: The was a preliminary round that set the stage for what was to occur later, the student readily and somewhat intuitively grasped both the parent linear function and the actual function, they also understood how to graph it.

Round 2:

	Student x-value	Instructor y-value	
	2	3	
	3	8	
	0	−1	The instructor asks for a negative value.
	1	0	

I: Talk amongst yourselves and discuss what type of function this is!

After pause

I: What is the parent function?

St1. Quadratic!

I: Yes, that is correct! Does that help think about it? (Think about what the actual function is)

Most students find that knowing it is a quadratic helps them connect the values with a quadratic function and begin to search for the actual function, but some are still confused.
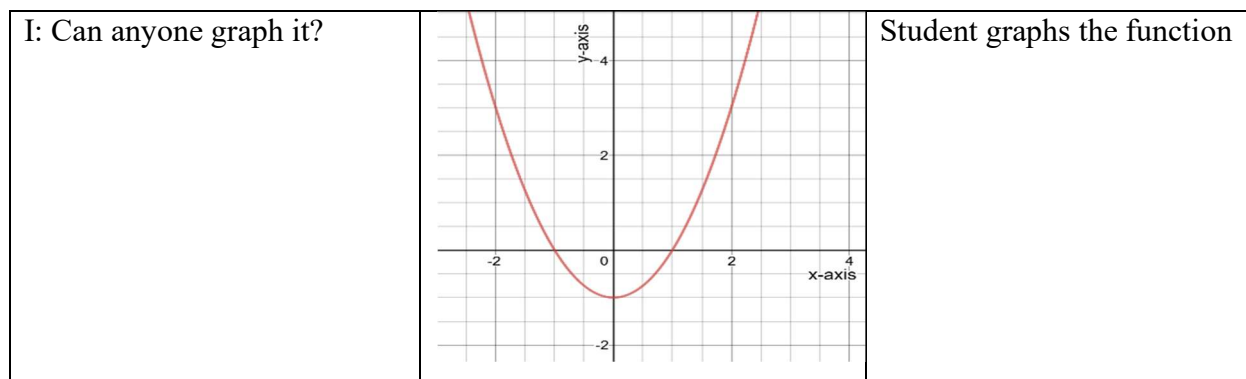
The instructor waits for an answer…

As students ponder a realization occurs

St2. Oh, now I got it!

I: What is it?

ST2: $x^2 - 1$

At this point the entire class appears to get it

I: Can anyone graph it?		Student graphs the function

Discussion:

The objective underlying this 'Guess the function' activity was to develop students' conceptual relationship between tabulated data of, input-output with its associated parent function, as well as the specific translation-function that generates this data.

The student who pronounced 'Oh now I got it!' appears to the team (both instructor and observer) to have had a moment of insight or realization. The instructor's pedagogical methodology to promote this moment of insight was to first introduce the intuitive-multiplicative scheme of a linear function within the context of the 'Guess the function' activity and then switch to the much less intuitive quadratics. In this scenario the student's realization was a result of first, realizing that her first linear schema (matrix) was not appropriate. Then when informed by her peer that the correct matrix was quadratic, during the instructor's pause she realized or discovered the correct additive constant to fit the data. This student certainly experienced the positive affect that Koestler values and demonstrated this by assisting other struggling students later in the period.

The effect of this pedagogical technique was to induce classical Gestalt 'Restructuring' as the solver needs to first put aside the initial linear schema that is intuitive and easy to apply mentally, and then begin the task of searching through the data-information presented for another schema.

Method 3: Teamwork collaboration to understand translations of parent functions

The objective was to reinforce and extend the conceptual relationship between tabulated data and graphs established during the second method to include both horizontal and vertical translations of parabolas, radical and inverse functions through collaborative work.

The class forms teams of 4-5 students each of which is given a type of function that involves either a vertical or horizontal translation. They task is to prepare a table of strategically chosen input and output values that adequately and readily represent this function. When the groups are done the give the table to the next group whose task is to determine the parent function and then complete the table of part 1, i.e., graph it and state the domain and range.

I: Please choose x-input values strategically! If you have a radical, please chose perfect squares! Also only provide input and output, do not graph it! (That is the task of the next group that does not know the function)

Student either wrote the table of data first and then the graph, ore advanced students write the function first and then the draw a graph with minimal or no use of the tabulated data. Those students who correlated the data with the graph often did not understand the curvature especially the translations of radicals. Instead they connected the dots in a linear fashion, suggesting they had not internalized the parent function graph and how their work was related to it.

Some students struggled with correct notation and what might be considered as the vertex or the minimal value of the domain and range (all graphs were in QI) of translated radical functions. The student that had the realization earlier was motivated and able to assist several of her peers with the graphing process this suggests that her realization had an affective-motivational component.

As the period draws to a close the instructor informs students that $j(x) = \sqrt{x + 1}$ was the most difficult conceptually for them to graph and explains the effect of horizontal translations. The students find this a bit much to process especially so late in the period.

Lesson 2: Instructor Wolf

A more traditional format yet with distinct constructivist leanings, by which we mean a teacher led discussion within problem-solving, with a focus on active student involvement at each step of this process. The lesson begins with the Fundamental Theorem of Calculus and a subsequent review of anti-derivatives beginning with the power rule including inverse trigonometric and logarithm anti-derivatives. Most examples are review, and the students using a chart of derivatives to assist recall in this whole class problem activity.

As the review ends the instructor provides the following example to motivate the class to learn substitution involving the chain rule:

$$\int \frac{\cos \theta}{\sin^2 \theta} d\theta$$

I: We don't have the tools yet to do this one but is there something on the formula sheet you can use?

Silence

I: Let's rewrite this:

$$\int \frac{\cos \theta}{\sin^2 \theta} d\theta = \int \frac{\cos\theta}{\sin\theta} \times \frac{1}{\sin\theta} d\theta$$

$$= \int \cot\theta \times \csc\theta \, d\theta$$

At this point the student realize this is on the formula sheet as $-\csc\theta$

I: Ok, so how can we find the derivative of: $= \frac{2}{3}(1 + x^2)^{\frac{3}{2}}$?

St1: $$y' = \frac{3}{2}\frac{2}{3}(1 + x^2)^{\frac{1}{2}}2x$$

I Where did this come from? (asking the rest of the class)

St2: The chain rule!

I: Do you follow this? How can we now find: $\int 2x(1 + x^2)^{\frac{1}{2}} \, dx$?

St1: The *2x* is the *du*

I: Have you ever done this before?

St1: Yes!

I: Can you be more specific about what you are doing?

St1 working with I:

$$u=1+x^2$$

$$du=2x \ dx$$

I: Good, (speaking for the other students in the class). It's important to know why it's *du*

I: (Again directing to the rest of the class). How did we do this?

St2: Asks for clarity about what

I: Just rewrite it! $\int \sqrt{1 + x^2} \, (2x)dx =$

St2: $$= \int u^{\frac{1}{2}}du$$

I: Ok, what do we have?

St2: $$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$

I: Simplify! Silence…Flip that sucker!

St2: $$= \frac{2}{3}u^{\frac{2}{3}} + C$$

I Gives wait time for student to realize they are not done

After pause

St2: Oh $$= (1 + x^2)^{\frac{3}{2}} + C$$

I:OK, so now we have a chain-rule for anti-derivatives built upon substitution: (writes on board)

$$\int F'\big(g(x)\big)g'(x)dx = F(g(x)) + C$$

I:So how can we do a problem like the one we started with, that involves trigonometric functions?
$$\int \sec x \tan x \, dx$$

Silence

I: Rewrites: $$\int \frac{1}{\cos x} \times \frac{\sin x}{\cos x} dx = \int \frac{\sin x}{\cos^2 x} dx$$

I: What can we use to substitute?

St3: Ahh… $u = \cos x$

I: What does du=?

St3: $du = -\sin x \, dx$ so we have $$-\int \frac{du}{u^2} = -\int u^{-2} du$$

I: What do we do next?

St3: $$= -\frac{u^{-2+1}}{-2+1} + C$$

$$= -\frac{u^{-1}}{-1} + C$$

I: What do we do next? Pause …(gives hint) the negatives are gone, let's substitute!

St3: $$= \frac{1}{\cos x} + C$$

I: Ok, what is 1/$\cos x$? $$= \sec x + C$$

Discussion

This lecture on anti-derivatives using the chain rule can be viewed as essentially beginning and ending with an example of the type: $\int \frac{\cos\theta}{\sin^2\theta} d\theta$. At first the instructor rewrites this example as

$\int \cot\theta \times \csc\theta \, d\theta$, and the students use a formula sheet to find the answer, without any understanding of the formula. After explaining substitution in the context of the chain rule, the instructor revisits this problem-type beginning with $\int \sec x \tan x \, dx$, The instructor first presses the students to resolve the problem, then after silence rewrites it as $\int \frac{\sin x}{\cos^2 x} dx$ and pauses until one of the students realizes how to employ the newly learned substitution technique. Specifically, what should be the value of 'u', and how does it, and the corresponding 'du' fits into the problem? The resulting realization by a student that $u=\cos x$ is a good illustration of what has been referred to as the little 'c' creativity that accompanies a typical lecture in a student centered or creative learning environment. In the follow up team discussion it was suggested that the instructor could have paused longer before re-writing the example. That being said, it was at the end of the lesson time, the student did make the correct realization, and the rest of the class appeared to readily understand.

It is interesting to note that, the lecture transitioned between the initial and final resolution of this problem type by internalization of the guiding principle or code that underlies the various examples. Specifically, the instructor first employs concrete examples, and after the students have demonstrated some degree of internalization the instructor then presents the abstract formulation of the rule. During this transition, while finding the derivative of $y = \frac{2}{3}(1 + x^2)^{\frac{3}{2}}$ when a student provides the answer, the instructor realizing he has done this before asks him to be more specific. This was done to assist the other students follow and internalize his method.

SUMMARY DISCUSSION

Koestler views creativity (discovery of a hidden analogy) as taking place when an individual searches through existing activity, tools or matrices and re-examines them in a new light that allows them to be combined with other activity, tools or matrices in a novel manner. Vygotsky understands creativity within internalization as taking place when an adult, peer or mentor reflects one's activity back to the individual with new meaning in a way that allows for creation of meaning, i.e. internalization. An important end-result for both Koestler or Vygotsky is a new code or structural understanding of the reasoning or guiding principle that underlies the novel activity.

In the first lecture by instructor S we observe a social situation in which a peer's statement allows a student to set aside her linear matrix and realize the correct quadratic matrix that is represented by the data. This information was apparently completely internalized as she then assisted other students understand the correct parent function for different data, specifically in graphing both quadratics and radicals.

In the second lecture by instructor W we observe a similar phenomenon of creativity within internalization. In this case, the initial frame of reference is the new technique of substitution previously used only to represent a polynomial within a radical-root function. The instructor introduced a trigonometric function, but its format was too complicated for the students to understand how to apply this new technique, until the instructor re-wrote the problem as a trigonometric rational. At that point a student realized that the substitution technique involved the denominator and the derivative du was the numerator of this rational trigonometric function.

In the first lecture the 'Aha Moment' was more distinct and the affect more noticeable. As evidenced by the students ability to assist others struggling with similar issues of extending their linear matrix to curved quadratic or root matrices. In the second lecture it was clear internalization had occurred, but the extent of independent activity was not as evidenced. That being said in the second lesson the abstract code that represents the substitution process was presented in the middle of the lesson and the students appeared to relate to it, while in the first the abstract code or rules of graphing translations appeared to come rather late to evidence student comprehension.

This discussion highlights several issues for instructors the more peer-peer interaction provides greater opportunity for meaningful creative insights though such increased participation, yet it may limit the amount of the time spent on structural formulations that express the underlying code of the examples done in class. Conversely a more structured teacher controlled lecture while often insuring sufficient time to develop the underlying code provides less opportunity for distinct moments of student creativity and demonstration of independent internalized activity.

REFERENCES

John-Steiner, V., Souberman, E., Cole, M., & Scribner, S. (1978). LS Vygotsky–Mind in Society: The development of higher psychological processes. London: Harvard University Press

Koestler, A. (1964), *The Act of Creation*, New York, N.Y., Macmillan

Norton, A., & D'Ambrosio, B. S. (2008). ZPC and ZPD: Zones of teaching and learning. *Journal for Research in Mathematics Education*, 220-246.

Sarrazy, B., & Novotná, J. (2013). Didactical contract and responsiveness to didactical contract: A theoretical framework for enquiry into students' creativity in mathematics. *ZDM*, *45*(2), 281-293.

Singer, F. M., Sheffield, L. J., & Leikin, R. (2017). Advancements in research on creativity and giftedness in mathematics education: introduction to the special issue. *ZDM*, *49*(1), 5-12.

Vygotsky, L. S. (1997). *Thought and language*, (10th Ed.) Alex Kozulin (Ed.) MIT Press