The Problem Corner

Ivan Retamoso, PhD, *The Problem Corner* Editor

Borough of Manhattan Community College

iretamoso@bmcc.cuny.edu

The purpose of ***The Problem Corner*** is to give students and instructors a chance to step out of their working "comfort zone" and solve challenging problems, whether working independently or together. Rather than solutions alone, as educators we are interested in methods, strategies, and original ideas that uncover paths to solutions. To that end, we encourage our readers to propose and solve new problems. **To submit a solution**, type it in Microsoft Word, using math type or the equation editor (PDFs are also acceptable), and email as an attachment to The Problem Corner Editor at iretamoso@bmcc.cuny.edu. Be sure to include your name, institutional affiliation, city, state, and country. Solutions must include a detailed explanation. The best solution will be published in a future issue of MTRJ, and correct solutions will be acknowledged. **To propose a problem**, follow the same formatting and submission guidelines, and include a detailed explanation of a solution.

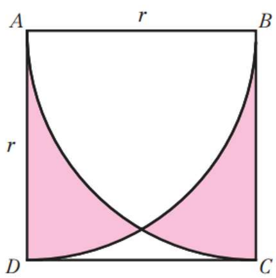
Greetings, problem solvers!

We are excited to announce that Problems 38 and 39 in *The Problem Corner* have received outstanding solutions, distinguished by their precision and creativity. Showcasing these exemplary approaches serves to inspire others and encourages a greater appreciation for mathematical thinking across the globe.

Problem 38: "Wine Glass" Problem

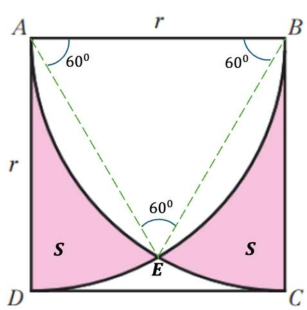
Proposed by Ivan Retamoso, BMCC, USA

The picture below shows a square $ABCD$ with each side of length $r = \sqrt{2}$ inches. Two arcs of circles are drawn with centers at points A and B. What is the area of the shaded part?



Solution 1. Submitted By Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University

In this first solution, the solver constructs an equilateral triangle, which helps create circular segments whose areas are calculated. These segments, along with 30-degree sectors and the use of symmetry, are combined to solve the problem. The solution is concise and well explained.



First, we can begin our solution by noting that the areas shaded in pink are equal. If we denote each of these areas as S, then we will need to find $2S$. Let E be the intersection point of arcs AC

and BD, and denote the large white area above as A_1. As seen in the figure, when we draw the necessary auxiliary lines, an equilateral triangle will be formed.

$$Areaofcircularsegment AE = Areaofsector ABE - A(\triangle ABE)$$

$$Areaofcircularsegment AE = \frac{\pi r^2}{6} - \frac{\sqrt{3}r^2}{4}$$

$$A_1 = A(\triangle ABE) + 2(Areaofcircularsegment)$$

$$A_1 = \frac{r^2\sqrt{3}}{4} + 2\left(\frac{\pi r^2}{6} - \frac{\sqrt{3}r^2}{4}\right) = \frac{\pi r^2}{3} - \frac{\sqrt{3}r^2}{4}$$

The area of the quarter circle centered at A or B is $A_1 + S = \frac{\pi r^2}{4}$

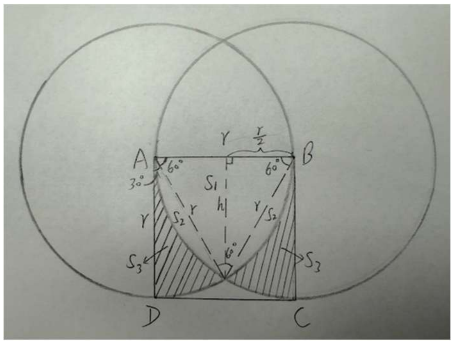
$$S = \frac{\pi r^2}{4} - \left(\frac{\pi r^2}{3} - \frac{\sqrt{3}r^2}{4}\right) = \frac{r^2}{12}\left(3\sqrt{3} - \pi\right)$$

$2S = \frac{r^2}{6}\left(3\sqrt{3} - \pi\right)$ where $r = \sqrt{2}$

Hence, the area of the shaded part is $\frac{1}{3}\left(3\sqrt{3} - \pi\right)\text{in}^2$.

Solution 2. Submitted by Stephen Chen, Borough of Manhattan CC, Fuzhou, China.

In this second solution, the solver carefully explains each step, labeling all regions in the diagram, which involves two circles and an equilateral triangle. Using proportional reasoning, he determines the area of a 30-degree sector and ultimately finds a method to calculate the shaded region, which represents the final solution



The area of the quarter of the circles shown above is $\frac{\pi r^2}{4}$

Then $S_1 + 2S_2 + S_3 = \frac{\pi}{4}r^2$……………………………..(*)

$$S_1 + S_3 = \frac{30°}{60°}\pi r^2 = \frac{\pi}{12}r^2$$

$$S_2 = \frac{\pi}{12}r^2 - S_3$$

$h^2 + \left(\frac{r}{2}\right)^2 = r^2$ then $h = \frac{\sqrt{3}}{2}r$

$$S_1 = \frac{1}{2}rh = \frac{1}{2}r\frac{\sqrt{3}}{2}r = \frac{\sqrt{3}}{4}r^2$$

Substituting in (*)

$$\frac{\sqrt{3}}{4}r^2 + 2\left(\frac{\pi}{12}r^2 - S_3\right) + S_3 = \frac{\pi}{4}r^2$$

$$\frac{\sqrt{3}}{4}r^2 + \frac{\pi}{6}r^2 - S_3 = \frac{\pi}{4}r^2$$

Since $r = \sqrt{2}$

Then $S_3 = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$ and $2S_3 = \sqrt{3} - \frac{\pi}{3}$

Therefore, the area of the shaded part is $\sqrt{3} - \frac{\pi}{3}$ in^2

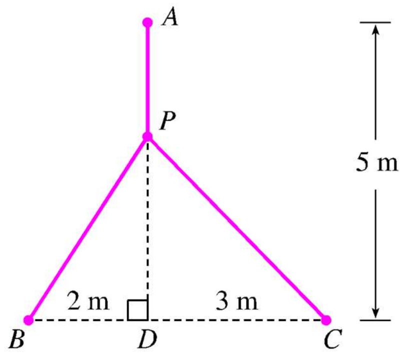
Problem 39: "Three Points" Problem

Proposed by Ivan Retamoso, BMCC, USA

A point P must be placed somewhere along the line segment AD such that the total length L of cables connecting point P to points A, B, and C is minimized (refer to the figure).

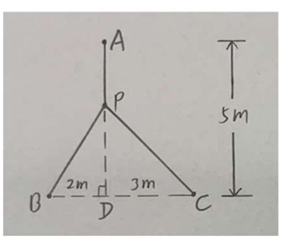
Task: Estimate the minimum value of L to <u>*two decimal places*</u>.

Note: Since this is an estimation problem, you are encouraged to use a graphing calculator or graphing software to approximate the solution of the equation needed to minimize L.



Solution 1. Submitted by Stephen Chen, Borough of Manhattan CC, Fuzhou, China

Our solver expresses the total length in terms of a single variable, forming a length function L that is then differentiated and minimized using the first derivative test. The critical point was identified with the aid of the DESMOS graphing calculator.



$L = PA + PB + PC$

Known $CD = 3$, $BD = 2$, $PA + PD = 5$, set $PC = x$

$PD = \sqrt{x^2 - 9}$

$PA = 5 - PD = 5 - \sqrt{x^2 - 9}$

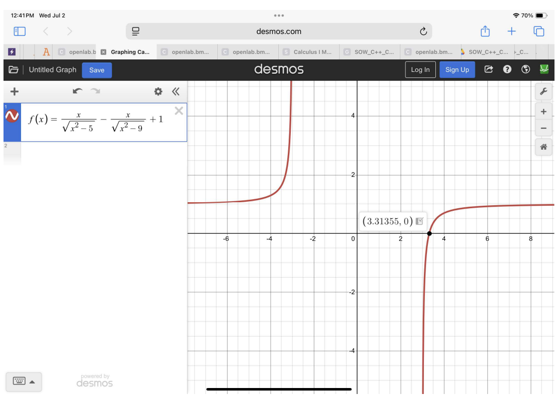
$(PB)^2 = (PD)^2 + (BD)^2 = x^2 - 9 + 2^2 = x^2 - 5$

$PB = \sqrt{x^2 - 5}$

Therefore, $L = PA + PB + PC = 5 - \sqrt{x^2 - 9} + \sqrt{x^2 - 5} + x$

Taking the derivative of L, $L' = 1 + \frac{x}{\sqrt{x^2-5}} - \frac{x}{\sqrt{x^2-9}}$

Using DESMOS graphing calculator, the graph of L' is shown below



From the graph, $L' = 0$ for $x = 3.31m$, $L' < 0$for $x < 3.31$, and $L' > 0$for $x > 3.31$.

The above shows that L achieves its minimum at $x = 3.31m$.

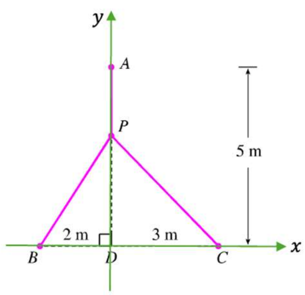
Therefore, the minimum value of L is given by

$L = (3.31) + 5 + \sqrt{(3.31)^2 - 5} - \sqrt{(3.31)^2 - 9} = 9.35m.$

Solution 2. Submitted by Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University

This alternative solution uses a cleverly chosen xy coordinate system, chosen without loss of generality, to express the coordinates of all points in the diagram in terms of a single variable. This approach simplifies the computation of all relevant distances and leads to the construction of a distance function, which is then minimized using the derivative from calculus.

The point P will lie on the line segment AD, and the total length of the cables $L = PA + PB + PC$ is desired to be minimized.



I approach the problem geometrically by redefining the coordinate system.

To do this, we treat the line AD as the y-axis and the line BC as the x-axis. If we proceed with this approach and write the coordinates of the given points accordingly, point D can be considered as the origin. That is, $D(0,0), A(0,5), P(0,p), B(-2,0)$and $C(3,0)$. Of course, the value of p will be in the range $0 \leq p \leq 5$.

I define the function L, which is the sum of distances from the point P to points $A, B$and C.

$L(p) = PA + PB + PC = 5 - p + \sqrt{4 + p^2} + \sqrt{9 + p^2}$

To minimize this function, we take its derivative with respect to $p$and set it equal to zero.

$\frac{dL}{dp} = 0$ (solving this analytically is complex, so we use graphical methods to find the value of p that minimizes L) then $L_{min} \approx 9.35$ when $p \approx 1.41$

As a result, I used a combination of coordinate geometry, calculus (derivative to minimize distance), and graphing with Desmos to find and verify the value of $p \approx 1.41$ where $L$is minimized.

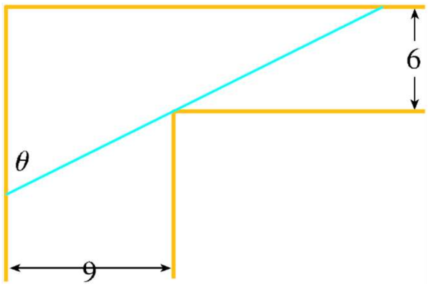
Editor's Note: Problem 39 was also correctly solved by Hridoy Saha from Bangladesh. While his approach was similar to the ones presented, he used Newton's method at the end to solve the final equation and obtain the solution.

It's great to hear that you enjoyed problems 38 and 39 and that they helped you improve your mathematical skills. Now continue with the next set of questions to keep building your mastery.

Problem 40

Proposed by Ivan Retamoso, BMCC, USA

A steel pipe is being carried horizontally through a hallway that is 9 feet wide and turns at a right angle into a hallway that is 6 feet wide. What is the maximum length of pipe that can make the turn around the corner?



Problem 41

Proposed by Ivan Retamoso, BMCC, USA

A regular heptagon is inscribed in a unit circle with a radius of 1 cm. Calculate the area of the red triangle.