# The Perceptions of Multiple Representation of the US and Turkish Mathematics Teachers

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*Abstract: Teacher perception is one of the most important factors determining classroom teaching practices. Multiple representations are used extensively in mathematics classrooms, especially in developing conceptual understanding. The data in this study, which aims to compare Turkish and US middle school mathematics teachers' perceptions of multiple representations, were obtained from interviews with four Turkish and three US mathematics teachers with different teaching experiences. The semi-structured interviews focused on teachers' perceptions of role, usage and importance of multiple representations. The study also compares Turkish and US teacher's perceptions about the representations in teaching and learning mathematics. While Turkish and US teachers share general perceptions of multiple representations, key differences exist—especially in how representations contribute to the concretization of mathematical concepts. According to the different practices in integrating multiple representations into instructional processes across countries, Turkish teachers emphasized more symbolic and daily life-based representations, while US teachers emphasized more on concrete materials, visual representations, and the use of technology. The results of the study comparing perceptions of representation suggest that teachers have needs for professional development.*

Keywords: Multiple representations, teacher perception, cross-national perspective in mathematics

## INTRODUCTION

Cross-country comparative studies in mathematics teaching and learning can offer unique opportunities and perspectives in defining and developing mathematical thinking (Cai, 2000). In this study, US and Turkish middle school mathematics teachers' perceptions about multiple representations were compared. Both, Turkey and the United States have consistently recorded comparatively low scores on international assessments such as PISA and TIMSS, often falling short of their educational objectives. This recurring under-performance across several cycles may be attributed to multiple contributing factors, which have been explored in various studies (Miller & Fonseca, 2021)

The teaching and learning is a complex process, involves various aspects such as teachers, students, textbooks, and so on. Effective instructional strategies is one of the important factors that most likely help to improve students' mathematical achievement. For instance, implementing classroom teaching practices that focus on conceptual understanding through innovative practices can contribute to overcoming this problem. Similarly, integrating multiple representation in teaching and learning mathematics likely enhance conceptual understanding of mathematics (NCTM, 2000; Mainali, 2021). Multiple representations are often mentioned as a way to support students' conceptual understanding of mathematical concepts (Adu-Gyamfi et al., 2019; Pape & Tchoshanov, 2001; Ross & Willson, 2012). Although teacher competencies are essential for the effective and flexible use of representations in school mathematics, there is limited research exploring teachers' perceptions of the role that multiple representations play in mathematics teaching and learning. Teachers' perceptions and beliefs significantly influence the types of representations they choose to employ in their instructional practices. It is important for mathematics teachers to assess their ability to use representations effectively in their classrooms and to determine whether they receive adequate support in developing these competencies (Stein et al., 2008).

## LITERATURE REVIEW

### Multiple Representation

Representation is a crucial element in the teaching and learning of mathematics (Vergnaud, 1987). Various scholars have defined the meaning of representation in the context of teaching and learning mathematics. For example, Goldin (2003) states that representation is a configuration of signs, characters, icons, or objects that can stand for or represent something else. Similarly, Janvier (1987) suggests that representation may be a combination of expressions on paper, physical objects, and constructed arrangements of ideas in one's mind. NCTM (2000) describes representation as both a process and a product—the act of capturing a mathematical concept or relationships in some form or as a form itself. Goldin (2001) suggests that a representation is a sign or a combination of signs, characters, objects, diagrams, or graphs, and that it can exist as either an actual physical product or a mental process. Thus, representations include various external forms such as diagrams, signs, figures, characters, symbols, tables, formulas etc., to communicate and express mathematical ideas.

The importance of usage of multiple representation in teaching and learning mathematics has been growing in the last several decades. In mathematics education, it is essential for students to have early exposure to usage of various types of representations and ample practice in translating between different types of representations (Ballard, 2000). Ainsworth (2006) refers to multiple representations as a way of using the salient aspects or advantages of a representation, selecting the most appropriate representation for individual learning needs, and using the most effective strategy for problem solving. The effective use of and translations between representations in mathematics instruction is a critical factor in enhancing students' learning (Mainali, 2021).

Therefore, teachers' preferences in using multiple representations—along with how they integrate them into mathematics lessons and their ability to use them flexibly and effectively—play a vital role.

Students have different preferences for solution strategies. For example, some students prefer visual solution methods (visual learners), while others tend to use non-visual methods (non-visual learners). Regardless of students' tendencies to use specific types of representations, it is important for them to develop the skill of employing multiple types of representations when solving mathematical problems, as certain tasks are easier to solve using specific representations over others. For example, Larkin and Simon (1987) suggest that diagrammatic or graphic representations help learners recognize features more easily and make direct inferences, thereby supporting effective and efficient problem-solving. Moreover, effective mathematical problem-solving often involves reasoning with visual representations, which depends on the nature and quality of diagrams, figures, and graphs (Bremigan, 2005). Students are more likely to struggle to understand symbolic representation than verbal and pictorial representation and therefore need more time (Khairunnisak et al., 2021). Zhang (1997) argues that visual representations—such as pictures and graphs—provide access to knowledge and skills that may not be readily accessible through other means, such as verbal representation. However, some problems can be more easily expressed using verbal representations, in which case visual aids like diagrams may introduce unnecessary information, potentially distracting students during problem-solving. Thus, it is important for students to develop the ability to use different types of representations in mathematics learning. In elementary school, for instance, students might use a number line, base-ten blocks, or traditional algorithms (using signs and numeral characters) to complete an addition problem. Indeed, the use of multiple representations in mathematics instruction enhances students' conceptual understanding. Being able to apply various types of representations not only helps students arrive at correct answers but also deepens their mathematical comprehension. Additionally, understanding how to use multiple representations enables students to choose the representation that best aligns with their learning preferences. When students can select appropriate representations, mathematical concepts become more meaningful. In fact, the more mathematical tasks that require making connections between multiple representations, the more opportunities students are likely to develop for deeper mathematical understanding (Mainali & Sabri İPEK, 2025). Ultimately, developing the ability to use multiple representations can foster a growth mindset in mathematics.

### Teachers' Perceptions and Multiple Representation

Perception is defined as "a general concept or mental construct that includes beliefs, meanings, concepts, propositions, rules, mental images, and preferences" (Philipp, 2007, p. 259). The need for more research on views about multiple representations in mathematics classrooms is evident (Dreher et al., 2016). Teachers' perceptions of teaching and learning mathematics profoundly influence the processes inside and/or outside the classroom (McLeod & McLeod, 2002). This can be seen because of the strong connection between knowledge/understanding and perceptions/beliefs in mathematics. Teachers' perceptions and understandings play an important role in the design of instructional strategies (Baumert et al., 2017).

Since mathematical concepts and knowledge are not easily accessible through direct experience, representations often dominate instructional strategies (Duval, 2006). Representations are an integral component of the mathematics curriculum due to their effective role in linking students' levels of understanding with mathematical concepts. The effective use of representations depends primarily on teachers' perceptions, perspectives, and experiences with multiple representations. Rau and Matthews (2017) argue that determining when and how effectively teachers engage with different types of representations during mathematics instruction is influenced by multiple factors. Emphasizing the importance of using multiple representations in school mathematics, Bosse et al. (2011) suggest that teachers' failure to incorporate a variety of representations in the classroom directly impacts students' mathematical understanding. Therefore, it is essential for teachers to develop basic competencies in using various types of representations in their instructional strategies. Teacher competencies include selecting appropriate types of representations for effective mathematics instruction and understanding how these representations can either support or hinder student learning (Dreher et al., 2016). Indeed, teachers' perceptions of representations influence both their teaching practices and their views on the use of multiple representations—an area that still warrants further exploration (McLeod & McLeod, 2002). These perceptions are particularly important for fostering students' problem-solving abilities and conceptual understanding. However, there is still a limited number of studies that examine teachers' perceptions regarding the role of representation in the teaching and learning of mathematics.

Numerous studies have explored mathematics teachers' beliefs and perceptions, revealing both similarities and differences in how beliefs and perceptions are interpreted. Many models of teacher professional knowledge (e.g. Ball et al., 2008; Shulman, 1986) also point to the complexity of separating perceptions from beliefs in all domains in general and teaching mathematics in particular. Since the term belief is often too broad, in this study we use the more limited term perception to describe specific aspects of beliefs. Ball (1993) examined in-service and pre-service teachers, prospective and experienced teachers' knowledge, dispositions, and ways of thinking about representations in mathematics teaching in the context of fractions. Van G. et al. (2019) classify mathematics teacher knowledge models into two main categories—content knowledge and pedagogical content knowledge— where teachers' pedagogical perceptions of teaching and learning processes are not inherently considered part of these models.

Stylianou (2010) investigated middle school mathematics teachers' conceptions of representation as a process and their perspectives on the role of multiple representations. The study reported that teachers employ representations in diverse ways in their own mathematical work and tend to define the term primarily as a product used in problem solving. However, their understanding of representation as a dynamic process and as an integral part of mathematical practice appeared to be less developed. The study further noted that representations may play a more peripheral role in their instructional approaches. Similarly, Bosse et al. (2011) examined teachers' beliefs about the use of multiple representations and reported challenges related to transitioning and translating between different types of representations. Lee and Lee (2019) explored pre-service elementary teachers' perceptions of using multiple representations to solve fraction problems and found that participants lacked a clear understanding of how to use various representations effectively. Dreher et al. (2016) conducted a study on the influence of cultural differences between English and German pre-service teachers, suggesting that mathematics teachers' professional knowledge and perspectives on the use of multiple representations should be carefully considered in mathematics instruction. Expanding on this, Dreher et al. (2024) examined whether perspectives on the use of representations in mathematics classrooms differ across cultures and highlighted the challenges of applying Western views of teachers' representation practices to non-Western educational contexts.

Several studies have explored teachers' representational knowledge from various perspectives (e.g., Jones, 1997; Hitt, 1998; Knuth, 2002; Izsa & Sherin, 2003; Rosli et al., 2013; Son & Lee, 2016), while others have focused specifically on teachers' perceptions of representation (e.g., Stylianou, 2010; Bosse et al., 2011; Lee & Lee, 2019; Lee & Lee, 2023). However, relatively few studies (e.g., Cai & Wang, 2006; Dreher et al., 2016) have examined teachers' perceptions of representation through a cross-cultural lens. Therefore, this study aims to contribute to the field by shedding light on the role of representation in the teaching and learning of mathematics within an international context. The findings are expected to deepen the understanding of teachers' orientations toward multiple representations and provide valuable insights into how they utilize these tools to support students' problem-solving skills.

## THEORETICAL FRAMEWORK

Representations play a vital role in the teaching and learning of mathematics, as the use of multiple forms can greatly enrich instructional approaches and deepen students' understanding of mathematical concepts. Their impact on enhancing problem-solving abilities and fostering conceptual understanding is well-documented, with widespread agreement among scholars regarding their fundamental importance in mathematics education (Dreher et al., 2024; Duval, 2006; Greeno & Hall, 1997; Mainali, 2021; Rau & Matthews, 2017). In fact, utilizing different types of representations can enhance the understanding of the abstract nature of mathematical concepts. Goldin (2002) defines a representation from a general perspective as a configuration that can represent something else. Representations can manifest in a variety of ways, such as visuals (tables, number lines, etc.), written symbols (numbers, letter expressions, etc.), concrete materials (fraction models, decimal blocks, etc.), verbal expressions (classroom discussions, spoken language, etc.) and real-world situations (Van de Walle et al., 2004). Emphasizing both the process and product aspects of representation in mathematics, NCTM (2000) defines the process dimension of representation as the act of understanding a mathematical concept, and the product dimension as the form of representation. The process involves active engagement with mathematical ideas, often requiring flexibility and adaptability in choosing and transforming representations, while

the product is the outcome of this process, which can be used for evaluation, sharing ideas, or further exploration. Multiple representations are used extensively in understanding new concepts, problem solving and making connections between mathematical concepts (Stylianou, 2010). Ainsworth (2006) refers to multiple representations as a way to take advantage of the prominent aspects or advantages of a representation, to select the most appropriate representation for individual learning needs and to employ the most effective strategy for problem-solving. Studies on students, textbooks, curricula, etc., related to learning with multiple representations refer to the teacher's perspective and knowledge in the development of this skill in the mathematics classroom. Therefore, questions regarding teachers' knowledge and perceptions of how to use multiple representations in mathematics instruction remain central to educational research and practice.
Teachers frequently utilize multiple representations when teaching mathematical concepts and facilitating connections during the problem-solving process, regardless of the complexity of the content (Stylianou, 2010). To help students grasp abstract ideas and relate them to other mathematical concepts or real-life contexts, teachers must often generate and transform various forms of representation. As one of the key tools in mathematics instruction, representations are subject to teachers' individual perceptions and beliefs, which can vary widely. The use of representations in teaching involves several interconnected steps—including selecting the most appropriate representation for a given concept or problem, transforming one representation into another, transitioning between forms, and viewing representation as either a process or a product. Each of these decisions is shaped by the teacher's understanding and perception of representation. In this regard, mathematics teachers' knowledge and beliefs about multiple representations play a critical role in influencing their students' ability to understand and apply different representations effectively (Bharaj, 2022).

Teachers' understanding and perceptions of the role and use of representations certainly influence their instructional strategies. For instance, some teachers may believe that utilizing multiple types of representations places an additional burden on students, while others believe that incorporating different representations enhances students' mathematical understanding. Regardless of teachers' differing viewpoints regarding the role of representation, it is important to employ a variety of representations in the teaching and learning of mathematics. In fact, multiple representations can support one another due to their unique characteristics, which aid students during the problem-solving process (Dreher et al., 2024). Therefore, it is essential for teachers to understand the role of multiple representations and to apply them effectively in mathematics instruction. Similarly, it is essential to explore what teachers think and perceive about the role of representations within mathematics education. Furthermore, examining differences in teachers' perceptions across countries can provide valuable insights into the effective use of representations in mathematics instruction. In this context, the following research questions are posed:

i. How do Turkish and US middle school mathematics teachers perceive the use of representations in teaching and learning mathematics?
ii. What are the perspectives of Turkish and US middle school mathematics teachers on utilizing representations to solve mathematics problems?

iii. What are the differences in Turkish and US middle school mathematics teachers' perceptions regarding the role of representations in teaching and learning mathematics?

## METHODS

This study is a comparative qualitative case study that explores middle school math teachers' perceptions of multiple representations in two different countries: the US and Turkey.

### Study Group

The descriptive qualitative study involved semi-structured interviews with a total of seven teachers, four from Turkey (TR1, TR2, TR3, TR4) and three from the US (US1, US2, US3). All the middle school teachers that were selected voluntarily and had different educational backgrounds and levels of teaching experiences. The details about the participants are given in Table 1.

| Country | Participants | Gender | Educational Degree | Teaching experience (years) |
|---|---|---|---|---|
| | TR1 | Male | Bachelor's degree | 15 |
| **Türkiye** | TR2 | Female | Bachelor's degree | 14 |
| | TR3 | Female | Master's degree | 16 |
| | TR4 | Female | Master's degree | 8 |
| **US** | US1 | Male | Master's degree | 23 |
| | US2 | Male | Bachelor's degree | 10 |
| | US3 | Female | Master's Degree | 23 |

Table 1: In-service teacher demographics

### Research Instrument and Data Collection

Employing a qualitative research design, semi-structured interviews were conducted via Zoom. Each interview lasted approximately 15–20 minutes and was recorded for analysis purposes. The interviews were conducted in English and Turkish for the participants from the US and Turkey, respectively. The interviews conducted in Turkish were transcribed and translated into English for data analysis, as one of the researchers is a native speaker of Turkish. The semi-structured interview technique was chosen because it allows for a deeper exploration of participants' reasoning processes and knowledge structures through open-ended questions (Clement, 2000). To examine mathematics teachers' perceptions of multiple representations, a data collection tool was developed based on relevant literature. The same set of questions regarding representations was posed to participants from both countries. The data collection tool was organized into two parts. The initial section focused on teachers' perceptions of multiple representations in mathematics, particularly their views on the nature and role of these representations in the context of teaching and learning mathematics.

The questions in the first stage were as follows:

i. What do you think "representation" means in mathematics?
ii. Do you think it is necessary and important to use more than one representation to teach mathematical concepts or problems?
iii. Do you use multiple representations while teaching in your classroom? If yes, please give an example to explain... If not, please explain why.

In the second phase of the interviews, participants were presented with three mathematical tasks and asked to solve them. The three tasks are provided in Table 2. To explore teachers' perceptions of their preferred representations for these tasks, a context was given, and participants were asked whether they considered a specific representation more suitable than others. If so, they were also asked to explain why, and whether a particular representation might be more beneficial for classroom teaching or for students. For each task, the goal was to examine what types of representations participants used while solving the problems, whether they employed more than one representation, and how they applied different solution strategies using appropriate representations. Based on their responses, participants were also asked to reflect on how these representations influenced their thinking, the advantages and disadvantages of using different representations, and whether they could generate alternative representations for the given tasks.

| |
|---|
| *2-i.* Solve the following problem for your students: |
| $1\frac{2}{3}+\frac{5}{6}=$ |
| *ii.* Solve the following problem for your students: |
| (+3) – (-4) = |
| *iii.* Solve the following problem for your students: |
| A father took his 3 children to a fair. Tickets cost twice as much for adults as for children. The father paid a total of 50 zeds for the 4 tickets. How many zeds did each child's ticket cost? (TIMSS-2007) |

Table 2: Mathematical Tasks

As shown in Table 2, the tasks focused on fractions, integers, and algebraic concepts—topics that are commonly emphasized in middle school mathematics curricula. These open-ended tasks were intentionally selected to explore participants' solution strategies and to encourage deeper conversations between the interviewer and the participants (Hunting, 1997). The goal was to examine middle school mathematics teachers' perceptions of multiple representations in both countries and to understand how they do—or do not—incorporate these representations into their classroom instruction.

**Data analysis**

The semi-structured interviews were recorded using Zoom, and the audio recordings were analyzed through thematic analysis, a method commonly employed in qualitative research (Liamputtong, 2009). Braun and Clarke's (2006) six-step approach to thematic analysis was followed, which includes: familiarizing oneself with the data, transcribing the data, generating initial codes, searching for themes, reviewing themes, defining and naming themes, and producing the report. These steps helped identify underlying codes within the data and the relationships among them, ultimately forming broader themes. Data from both phases of the study were independently coded and analyzed by each researcher. The researchers then engaged in multiple discussions to compare their analyses and finalize the themes through consensus. Initial categories were developed from the responses to the interview questions in the first section, which were informed by relevant literature to frame the teachers' perceptions of multiple representations. During this process, similar themes and sub-themes emerged among both Turkish and US mathematics teachers, although their perspectives often differed. Following the initial coding, each set of interview data was organized into broader categories using a code-mapping strategy. Saldana (2021) describes code mapping as the process of "comparing and sorting" codes into overarching themes to complete the analysis. The final code-category-theme mapping resulting from this analysis is presented in Table 3.

| Code | Sub-Theme | Main Theme |
|---|---|---|
| Expression of concepts in different forms (TR) | | |
| Explaining the mathematical idea (TR&US) | Representation as product | Representation definition |
| A tool for concretization (TR&US) | | |
| Making sense (TR&US) | | |
| Formulize (TR) | Representation as a process | |
| Expressing graphically (TR) | | |
| Establish communication (US) | | |
| Mathematical language (TR) | | |
| Expressing ideas (TR&US) | | |
| Drawing (TR&US) | External Representation | Types of Representation |
| Expressing with a table (TR) | | |
| Embodiment (TR&US) | | |
| Daily life (TR&US) | Internal Representation | |
| Expressing mathematically (TR) | | |
| Technology support (US) | | |
| Mathematical language (TR) | | |
| Making it fun (TR&US) | Positive aspects of representations | Different aspects of representations |
| Contribution to students' self-confidence (TR) | | |
| Individual Learning (US) | | |
| Students' confusion (TR&US) | Negative aspects of representations | |
| Time-taking (TR) | | |
| Preferably direct teaching strategy (TR) | | |
| Using representations in simple problems (TR) | Concept teaching | Multiple representations in the classroom |
| Visualization for problem understanding (TR&US) | | |
| Using representations in concept teaching (TR&US) | Problem solving | |
| Making connection to daily life (TR&US) | | |
| Use tangible materials (TR&US) | Mathematical skill development | |
| Concretization for abstract thinking (TR&US) | | |
| Use of mathematical language (TR&US) | | |
| Technology support (US) | | |

Table 3: Sample Code-Mapping for Math Teacher

The questions in the second phase aimed to investigate teachers' preferences regarding the use of different types of representations and to evaluate the consistency between their perceptions and actual use of representations. Each problem-solving scenario focused on identifying the primary strategies participants used to solve the tasks, their preferences for employing multiple representations, and their views on the necessity and practicality of using such representations—whether they considered them essential or time-consuming. Furthermore, the study analyzed both the similarities and differences between teachers' perceptions of representations and their actual application of these representations during problem-solving.

In the final stage, the relevant data were thoroughly analyzed to compare the perceptions of teachers from both countries, with each consensus theme named and clearly redefined. During this process, similar codes emerged in both the Turkish and US cases, often reflecting different perspectives on the same topic. This facilitated easier comparison and validation of the data when necessary. Two researchers independently coded 25% of the dataset (one question from each stage). While the inter-coder agreement exceeded 80%, any discrepancies in the codes were revisited and discussed until a consensus was reached between the researchers.

## RESULTS

### Definition of Representation

The findings reveal that *concretization* emerged as a particularly salient concept in teachers' definitions of representation, as articulated by both Turkish and US participants. Furthermore, Turkish teachers expanded upon this understanding by offering interpretive rationalizations to elucidate the role and meaning of representation. For instance, one participant stated, "representation concretizes a mathematical concept in a way that is appropriate to the level of the students' understanding. I think, for example, when we present $x^2$ as a square, we exemplify it in a concrete way" (TR1). The data analysis suggests that participants conceptualize representation primarily as a process of concretizing mathematical ideas, while also recognizing its role in visually depicting mathematical expressions. One Turkish participant articulated this by stating, "to me, representation is a tool that should be used to facilitate students' access to mathematical content. We employ it both in everyday spoken language and in the specialized language of mathematics."

One of the Turkish participants stated that he could use representations in various ways based on his own perspective, emphasizing both the *product* and *process* dimensions of representation. As previously noted, the *process* dimension refers to capturing a mathematical concept, while the *product* dimension refers to the final form of the representation itself (NCTM, 2000). The process involves active engagement with mathematical ideas, requiring flexibility and adaptability in selecting and transforming representations, whereas the product is the outcome of this process, which can be used for evaluation, communication, or further exploration. Similar to their Turkish counterparts, the US participants also emphasized representation as a tool that aids in the concretization of mathematical concepts. For instance, one participant stated, "Using representations helps make learning math more concrete so that students can understand and visualize mathematical ideas—for example, by using manipulatives—making it a bit easier for them to access mathematical concepts" (US3).

The data analysis suggests that representation is viewed as a product employed during the process of mathematical problem-solving. Another participant also described representation more broadly, stating as "representation is a means of expressing your own culture, point of view, or mathematical ideas" (US2). The data further revealed that US teachers, more than their Turkish counterparts, place greater emphasis on the social dimension of representation. The fact that most Turkish and US teachers focus on the process of making sense of mathematical concepts—particularly through visual or graphical forms—indicates that they perceive multiple representations as an integral part of mathematical conceptualization. However, the data also highlight the challenges teachers face in defining the concept of representation. Both Turkish and US participants frequently relied on real-life examples to explain representations. While Turkish teachers struggled to articulate a clear definition of representation, they demonstrated consistency in their understanding and showed a preference for using flexible and appropriate representations during problem-solving in the second stage of the study. For example, one Turkish participant emphasized the use of visual representations, stating: "We need to concretize abstract mathematical ideas and express them through visual representation in operational tasks—for instance, in the case of the integer operation (+3) - (-4) =?"

> I first modeled this operation using counting stamps. I began by drawing three positive stamps to represent +3. Then, I needed to subtract -4 stamps. Since I had no negative stamps available, I referred to the concept of zero pairs, which I had already introduced in the context of integer operations. To proceed, I explained that in order to subtract -4, we need to add four zero pairs (each consisting of +1 and -1) to the existing model. It is important to emphasize that adding zero pairs does not alter the value of the original expression. After adding the four zero pairs, I had four negative stamps, which allowed me to subtract -4. The result of this operation was +7. This is how I initially model the concept before introducing the formal rule for subtraction (TR3).

The data analysis suggested that participants' perceptions about representations seems to be in line with the representation as a means of concretization during their problem-solving processes. For example, one of the US participants expressed his view about representation as "I believe it is more effective to engage my students in making senses of learning math and provide hands-on learning experiences." Another participant states as follows:

> Modeling is very important when teaching math lessons, especially since I work with middle school students, and I follow a more tactile approach to make the lessons more effective. I would first try to help them reach a common denominator using manipulatives or parts of a circle-model drawing. I would also use algebra tiles or different models for integer tasks (US2).

The data analysis suggests that a primary function of representation is to make abstract mathematical concepts more concrete, and participants from both countries share similar perspectives on defining representation.

**Representation Types**

Although all Turkish participants recognized the presence of different types of representations in teaching mathematics, some had difficulty elaborating on these types. Instead, they attempted to explain representations using mathematical examples. For example, one participant stated:

> Let's think about a math task where we use a number line to explain the concept. There are different types of representations we can use to represent a number line. While we mostly express it symbolically and by using the actual number line, sometimes we can use examples to explain the concept, such as a real-life example like the length of a table in the classroom (TR3).

Another Turkish participant further emphasized the use of representations through daily life contexts, symbolic forms, verbal explanations, and concrete materials. Based on the concept of integers, the participant illustrated various types of representations and stated:

> Let's think about the concept of integers. I can use the example of an elevator to teach the concept of integers. For example, I can use the elevator going up and coming down from the ground level to explore the concept of positive and negative integers. Or I can use an example of a sea submarine. When we say -5 or +3, for example, this is a verbal representation of the integers. But when we write it on the board, then we are using symbolic representation. These numbers can also be expressed concretely with materials or hands-on activities, which is more about types of visual representation (TR4).

US participants, similar to their Turkish counterparts, demonstrated a clearer understanding of the types of representation and showed confidence in using various forms of representation in their math lessons. However, they also found it challenging to explain the different types of representation in a more comprehensive way. For instance, one of the US participants stated:

> So, we can use algebra tiles to teach algebraic concepts such as solving equations. We can use other visual models, including algebra tiles. I think there are different ways to make abstract algebraic concepts more accessible to students by using appropriate representations. We should try to start with concrete materials when teaching mathematical concepts. As a result, abstract ideas can become possible and more easily accessible to students. The more we integrate different types of visual representation into our teaching, the better students understand the concept.

> Therefore, we should definitely use more than one representation in mathematics lessons (US1).

In regard to the role of multiple representations, Turkish teachers mainly focused on symbolic representations and real-life applications, while the US teachers emphasized more on concrete materials, visual representations, and the use of technology. The various types of representations highlighted by teachers are important, as they reflect their preferences and approaches of using representations in mathematics instruction. One of the Turkish participants, who consistently emphasized the significance of symbolic representation in teaching the fractions stated that "I usually follow the traditional rule to teach the concept of fraction operations. Usually, I first convert a mixed number into an improper fraction, make the same denominators and the complete the addition/subtraction of fraction tasks (TR1)."

The data analysis suggested that using various types of representation in teaching and learning mathematics is important to provide conceptual understanding of mathematics. For instance, a US participant, who stated that he frequently used concrete materials as representations in his math lesson states as follows solving an algebraic problem:

> I would begin solving the algebraic problem using concrete objects. For example, I would give my students a roll of tickets, some monopoly money, and glue. I would encourage them to consider how many tickets they can buy with this amount of money. This hands-on approach will allow my students to engage in a more structured problem-solving process using the paper roll. Additionally, I would encourage my students to experiment with the materials to explore more about algebraic concepts, rather than focusing on correct/incorrect answers. Finally, I would gradually transition to teaching more abstract algebraic concepts (US2).

Similarly, another participant from the US stated,

> I incorporate manipulatives, tactile representations of mathematics, and computers (math software) in my math lesson. At our school, we also have access of online versions of manipulatives, and I also use them in my lesson since some of my students in my class may not feel comfortable with using technological tools (US3).

**Positive (Negative) Aspects of Representation**

The data analysis revealed that both Turkish and US participants primarily emphasized the positive aspects of using representations in teaching and learning mathematics. Turkish participants particularly highlighted the benefits of representations, such as making mathematical concepts more concrete, improving mathematical understanding, and helping students comprehend the unique structure of mathematical language. For instance, one of the participants stated:

> It may sound like an emotional statement, but I would say that representations make learning mathematics more special, and they help in understanding mathematical language. Mathematics has its own language, and I think representations add more value to it. For example, we can express mathematical concepts or situations that we would normally explain in our own language by using mathematical expressions. However, we can express mathematical concepts in a very short (or long) way using mathematical representations (TR2).

Similarly, another participant emphasized the social-cognitive aspect of representations as follows:

> The use of representations makes mathematical concepts and operations easier to understand, and learning becomes more enjoyable for students. Additionally, representations help attract students' attention, as they can relate mathematics to their daily lives through the use of appropriate representations. As a result, students are likely to develop positive beliefs toward mathematics, which will help build their self-confidence in the subject (TR1). All Turkish participants, however, identified time constraints as a major drawback of using multiple representations in mathematics instruction. For instance, one teacher stated, "While modeling mathematical concepts through representations is valuable for problem-solving, incorporating representations into mathematics lessons can be time-consuming" (TR3).

Similar to Turkish teachers, US teachers also focused on the positive aspects of using representations in teaching and learning mathematics. One of the participants emphasized the role of representation in enhancing students' understanding of mathematical concepts. For example, he stated that representations can offer students alternative ways to understand mathematical concepts and to see what is going on behind the scenes (US1).

Similarly, another participant focused more on the social-cognitive aspect of representations. She stated, "Representation not only helps students feel more comfortable with their own learning style, but it also makes it easier for students to transfer mathematical concepts. Additionally, the use of appropriate representations caters to students' preferred ways of learning mathematics" (US2). US participants also highlighted a potential drawback of using representations: that they can sometimes make learning math concepts more complex. However, unlike Turkish teachers, US participants did not perceive representations as time-consuming.

**Representations in the Classroom**

Both Turkish and US participants frequently utilize multiple representations in their classrooms. However, there are differences in the preferences of teachers regarding when and how representations are used in mathematics instruction. Turkish teachers, in particular, tend to prefer using representations during the stages of concretizing concepts and understanding problems. For example, a Turkish mathematics teacher described the use of multiple representations when teaching the concept of a ray in geometry as follows:

> I use representations with the help of an interactive whiteboard to explain to students, for example, the similarities and differences between rays and lines in my geometry lessons. I encourage students to first think of a long wire, which helps them grasp the concepts of rays and lines to some extent. However, they can better comprehend these concepts in a technology-supported environment. They can further relate the idea of a ray starting from a specific point to real-life examples, such as sunlight. Later, I use symbolic representations to further explain the concepts of rays and lines during my geometry lessons (TR4).

Another Turkish teacher explained as how she used multiple representations to teach the concept of ratio:

> Let's consider the concept of ratio. While teaching the concept of ratio, I provide examples from daily life using various representations. For instance, I compare the prices of products in markets. I also integrate the concept of ratios into other subjects, such as discussing the speed of cars or the scale used on maps. These are some topics that come to mind (TR3).

Turkish teachers tend to use representations primarily to make abstract mathematical concepts more tangible and to better understand problems. The analysis also revealed that while Turkish participants favor representations for clarifying abstract concepts, they do not frequently rely on visual aids like number lines, tables, or graphs to comprehend problems. Therefore, the data suggest that the use of representations in mathematics lessons in Turkey has a limited variety and purpose. It is evident from the interviews that US teachers use concrete materials and technology-assisted applications more intensively in their mathematics classrooms. For instance, one of the US participants stated, “I'm better at using graphing calculators, so this kind of material works really well, especially for teaching algebraic concepts” (US1).

US participants stated that they often incorporate a variety of hands-on materials and manipulatives in their math lessons. They further stated that students also have the chance to apply their mathematical skills and knowledge through real-life activities. For instance, one of the participants stated that students visit the school coffee shop where they can put their math skills into practice. This allows them to engage in real-world applications, such as managing the cash register, budgeting money, or exploring ratios and proportions in the ingredients of food and drinks. The data also suggest that teachers in the US have more access to concrete materials than their counterparts in Turkey. The greater availability of resources in US classrooms allows teachers to use various forms of representation, providing more opportunities for students to learn math skills and apply mathematics in real-world scenarios. While Turkish teachers recognized the importance of using concrete materials to enhance students' understanding, they highlighted the challenges they face in accessing these resources. One of the Turkish participants shared the difficulty of obtaining concrete materials and hands-on activities in the following way:

> When solving a mathematics problem, teaching would be more effective if we had materials appropriately prepared ahead of time. I could use concrete materials, but if they are not available in the classroom, I would try to help my students understand the problem by drawing shapes on the board" (TR4).

Despite some constraints, both US and Turkish participants have positive views towards the role of representations in teaching and learning math lessons. The data indicated that while Turkish teachers have a positive view of using concrete materials in mathematics instruction, their access to hands-on resources is limited, making it challenging to integrate concrete representations into their lessons.

## DISCUSSION

This study aims to compare Turkish and US teachers' perceptions of representations in mathematics teaching and learning and some of the results obtained in this study are noteworthy. While there is a strong emphasis on using multiple representations in teaching and learning mathematics, the effective use of representations requires teachers to have a good understanding and knowledge about representation. The results suggest that both Turkish and US participants explained about both aspects: product and process dimensions of representation. In the product dimension, teachers from both countries generally considered representation as a product used in the process of concretizing mathematical concepts, similar to the NCTM (2000) process and product dimension. While in the process dimension, Turkish teachers emphasized the mathematical aspect of representation, and US teachers focused particularly on the social aspect of representation. The discrepancy in the participants' perspectives is likely due to the different culture and context. Teachers who tried to explain the concept of representation, usually with mathematical examples, focused on the basic meaning of representation and its implications for teaching and learning mathematics. Teachers who were familiar with the concept of representation and reported using it frequently, however, had difficulties in clearly defining representation and multiple representations.

The study results suggests that teachers from both countries demonstrate a limited theoretical understanding of representation. This aligns with findings indicating that mathematics teachers are often unaware of how to effectively incorporate different modes of representation into their instructional strategies (Mainali, 2021). Additionally, there is no significant difference in how Turkish and US teachers comprehend and perceive representation, which provides important clues about the role and use of representation in teaching and learning mathematics, given the differences in teachers' definitions and perceptions of representation in mathematical problem solving. The difficulties encountered in representing multidimensional mathematical concepts such as numbers, ratios, and areas were similar for teachers in both countries. Both Turkish and US teachers emphasized the use of representations in mathematics courses but experienced difficulties in effectively integrating representations into their teaching practices. In general, while they highlighted the positive aspects of using representations, they were not able to use them flexibly and effectively when solving the problems presented in the interviews.

A Turkish teacher who recognized the importance of using tangible materials in whole number subtraction stated, "I use counter chips, although I am not very comfortable with them." The findings further suggested difficulties in using concrete materials as basic tools for exploring whole number operations, despite teachers being aware of the significance of representation. While Turkish teachers acknowledged the importance of concrete materials in enhancing mathematical understanding, they also mentioned having limited access to such resources. Similarly, a US teacher recognized the importance of concrete materials and expressed interest in how representations could be applied in teaching fractions. For example, one participant stated, "I honestly don't know much about using representations in fraction operations" (US1). The results also suggest that a similar situation exists for US teachers. However, it should be noted that US teachers do not face difficulties in accessing concrete materials. Overall, the results suggest that to improve teachers' representational skills and understanding in both countries, there should be a greater focus on in-service and pre-service training programs aimed at enhancing mathematics teachers' skills and knowledge related to the use of representations.

Building on the results regarding teachers limited theoretical understanding of representations, this study also highlights differences between Turkish and US teachers' perceptions of the types of representations used in teaching and learning mathematics. While Turkish teachers tended to focus on symbolic representations and daily life-based examples, US teachers placed greater emphasis on the use of concrete materials, visual representations, and technology. This pattern is consistent with Stylianou's (2010) findings, which reported that symbolic and numerical representations are more prominent than graphical and visual representations in mathematics lessons. Similarly, the current study supports Cai and Wang's (2006) results, showing that US teachers used algebraic approaches less frequently than their Chinese counterparts when representing concepts such as ratios.

Although Turkish teachers frequently emphasized the importance of incorporating real-life contexts during interviews, these contexts were seldom applied when solving the provided mathematical problems. In contrast, US classrooms appeared to integrate real-world examples more systematically, offering students opportunities, such as working in the school's canteen, to contextualize mathematical concepts. This contrast suggests that, while Turkish teachers recognize the value of real-world connections, challenges in practical implementation persist. These findings reinforce the need for professional development programs that address both the theoretical foundations and the instructional application of multiple representations in mathematics education.

Turkish and US teachers generally shared positive and similar views on the use of multiple representations. However, they highlighted different negative aspects. Turkish teachers cited time constraints as a challenge in using representations, while US teachers felt that representations might make certain mathematical concepts more complex for students. The availability and adequacy of different types of representations likely contribute to the differing views on how various representations are used between Turkish and US teachers. Although both groups agreed that multiple representations are beneficial in school mathematics, there were differences in their preferences for using them. Turkish teachers tended to favor representations that concretize mathematical concepts and help students understand problems, but they did not often use visual representations such as number lines, tables, or graphs. The findings indicated that teachers who were aware of the importance of visual representations still did not prefer to use them frequently. Dreher et al. (2016) state that simply acknowledging the value of multiple representations is not enough to create rich learning opportunities; how these representations are used is more crucial. The following statement reflects the use of representations in mathematics lessons, with a US teacher describing this process in connection to integer operations:

> I actually had a student move and jump along a number line while teaching him to add and subtract integers. Students were required to use the number line to complete the integer operations, which helped reinforce their understanding of integer concepts. This approach allowed them to grasp that moving further to the right on the number line increases the value of the number" (US3).

The results, along with the effective use of the number line in such operations, highlight the crucial role of representations in mathematical problem-solving. Allowing students to engage with various types of representations enhances their conceptual understanding and reduces errors (Mainali, 2014). The results indicate that US teachers have better access to technological resources in their classrooms and utilize technology more extensively and effectively than their Turkish counterparts. Technological tools facilitate easier access to multiple representations and support the development of multi-representational teaching strategies in mathematics instruction (Thurm & Barzel, 2020). These results suggest that Turkish teachers, in particular, need more support in integrating technology into their mathematics courses. Technology-supported classroom environments may offer more opportunities for teachers to use multiple representations more effectively and flexibly.

## LIMITATIONS, IMPLICATIONS, RECOMMENDATIONS, & CONCLUSIONS

This study was conducted with a limited number of participants from specific school districts in both countries. Moreover, it is limited to middle school teachers. As a result, the findings cannot be generalized to all in-service teachers. Nonetheless, the results offer valuable insights into both the theoretical and practical aspects of representation in mathematics teaching and learning. Teachers' knowledge and perceptions are likely to influence their preferences for using different representations, but further research is needed in the field of mathematics education. This study aims to contribute to the literature by comparing middle school mathematics teachers' perceptions of the use of representations in teaching mathematics. Although there are differences in the perception and use of multiple representations between the two countries—and thus two different cultural perspectives—the results are limited by the scope of the interviews.

The study has important implications for mathematics teaching and curriculum development for both countries. Research has shown that representations play a crucial role in mathematics education and are fundamental to effective teaching and learning (Goldin, 1987; Janvier, 1987; Kaput, 1987; Zazkis & Liljedahl, 2004). Therefore, it is essential for mathematics teachers to understand the role of representations and how to use them effectively during instruction. Students should be able to engage with multiple forms of representation to develop mathematical proficiency, as certain mathematical problems can be solved more efficiently using specific types of representations (Mainali, 2019). Thus, professional development programs in both countries should focus on deepening teachers' understanding of representations and their effective usage in mathematics lessons. In particular, Turkish teachers may benefit from improved technological infrastructure and additional resources to support the use of multiple representations in their classrooms. More importantly, mathematics teachers should strive to incorporate a balanced use of various representations, ensuring that students engage with different modes to develop both conceptual understanding and problem-solving skills. In fact, the greater the number of tasks or problems in textbooks that involve connecting multiple representations, the more opportunities students are likely to have for a deeper mathematical understanding (Mainali & İpek, 2025).

The findings of the study also suggest several recommendations regarding the mathematics teaching and the curriculum. In-service teacher training programs in both countries should emphasize the theoretical understanding and practical application of multiple representations in teaching and learning mathematics. Programs should incorporate strategies to help teachers effectively use representations to concretize and visualize mathematical concepts. For example, incorporating technology into teaching practices can support the use of multiple representations and improve flexibility and effectiveness in mathematics instruction. Likewise, professional development should prioritize effective methods for integrating representations in a way that does not overwhelm students or interfere with instructional time. Educators should be equipped with strategies to incorporate representations seamlessly into their daily lessons. Additionally, teachers should be encouraged to utilize diverse forms of representation, including visual, symbolic, and concrete materials, while the mathematics curriculum, including textbooks, should equally emphasize the use of various types of representations.

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