Collaborative Problem Solving With and Without Access to Technology: Emphasis on Mathematical Justifications

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Abstract: Many studies highlighted the importance of mathematical justification in problem-solving. This paper describes students' mathematical justifications for solving derivative problems collaboratively, especially before and after the use of technology was allowed. We asked two undergraduate students who were preservice mathematics teachers in a paired problem-solving to see how they solved a derivative problem. To better understand the role of technology, we asked them to solve the problem without technology in the first session and with technology in the second. The pair progressed from the perceptual level of justification to the symbolic example with generalization level as the technology entered the discussions. The pair used technology to validate claims and exploring conjectures, yet it did not directly contribute to how they revised inaccurate claims or offered better justifications. Further studies could utilize a task that explicitly includes instructions on technology to promote mathematical justifications during CPS.

Keywords: mathematical justification, problem-solving, derivative, technology, preservice teacher

INTRODUCTION

Mathematical justification, an activity to provide arguments to support or refute a mathematical claim, was shown to be essential in promoting good reasoning skills in problem-solving (Brodie, 2010; Cirillo et al., 2016). By providing justification for a claim, students try to find reasons for their claims and do not merely accept the ideas they receive. Several studies stated that mathematical justification determines the success of problem solving, especially in the collaborative ones (Chiu, 2008; Díez-Palomar et al., 2021). Chiu (Chiu, 2008) analyzed group problem solving of high school students and investigated the factors that most influence the success of a group in reaching a solution. This research found that justification had the greatest effect. Additionally, another study by Díez-Palomar et al. (2021) found that student interactions dominated by mathematical justification were closely related to whether or not a group's answer was correct in solving problems because students checked the validity of the claim and used it as the basis for their answer. Thus, the group's success in solving problems cannot be avoided from the existence of mathematical justification which provides space for students involved in problem solving to question why the steps they take can be done and are valid to reach the answer. By justifying claims, students would try to think why their claims are valid using their prior mathematical knowledge and persuade others to accept them (de Villiers, 1990; M. E. Staples et al., 2012).

Some challenges were faced in the mathematics classroom despite the importance of mathematical justifications in problem solving. From studies involving teachers, it was found that encouraging students to justify was difficult for teachers (Brodie, 2010; Martino & Maher, 1999). It was because students were not familiar with questioning their own claims or to criticize if an argument was mathematically acceptable without teachers' questioning to provoke them. In addition, students tended not to make or ask for justification for the claims or steps they took during the collaborative problem-solving process (Hamidy & Suryaningtyas, 2016; Nafi'an, 2020; Stylianou & Blanton, 2002). Students did not try to convince others or explain why the answer was correct but tended to only go through the process of solving problems by focusing on getting an answer rather than to convince or validate it.

Many studies suggested using technology to support students' problem-solving, especially dynamic software (NCTM, 2020). The use of such technology helps students to interact with mathematical objects and explore them concerning the problems they face. Recent studies have also demonstrated that the use of technology such as digital learning models effectively supports students in solving problems aligned with the curriculum (Effendi et al., 2024), while Dynamic Geometry Environments (DGE) assist students in abstracting mathematical ideas through visual and interactive exploration (Dintarini et al., 2024). Regarding mathematical justification, technology allows students to examine the data provided, create a temporary claim, and justify it under several possibilities (Erbas et al., 2020) or even an infinite range of possibilities (Noss et al., 2009).

In collaborative problem-solving, technologies can help promote collaboration by giving more opportunities for practical mathematics activities, like manipulating values or objects, bridging into a more abstract and complex phase (Geiger et al., 2010; Jacinto & Carreira, 2021; Olive et al., 2010). However, technology's benefits do not automatically happen once it is introduced or used. Despite being shown to help students develop mathematical thinking even for complex mathematics (Erbas et al., 2020), we still need careful analysis of what students can achieve through technology (Sinclair et al., 2009) and whether students' justification of claims happened, even without the use of technology (Hollebrands et al., 2010). Furthermore, studies exploring technology with the focus of collaborative context was still limited (Bray & Tangney, 2017). Therefore, we believe it is crucial to analyze the role of technology through the lens of mathematical justification. In this paper, we explored students' mathematical justifications while solving derivative problems collaboratively without and with access to technology. Further, we looked upon the quality of mathematical justifications and how they progressed before and after the use of technology. By addressing this issue, how students should be supported in mathematically justifying claims during their problem-solving process can be learned, especially when they use technology during the process.

LITERATURE REVIEW

Mathematical Justifications

There are two primary concerns in understanding mathematical justification. First, mathematical justification can be seen as arguing against or proving a claim or as the result of that process. This interpretation is consistent with the assertion made by Staples et al. (Staples et al., 2012) that mathematical justification is an activity or argument that uses statements or accepted mathematical reasoning to demonstrate or disprove a claim's truth. Similarly, Yackel and Cobb (Yackel & Cobb, 1996) defined mathematical justification as a disagreement or agreement regarding an acceptable mathematical explanation for a proposed mathematical approach. Second, the "acceptance" of the participants in the activity is required for mathematical justification. The foundation of this acceptance is that these individuals agree that a particular mathematical assertion or response is reasonable (K N Bieda et al., 2022). Therefore, the social context in which mathematical activities are carried out is also considered in mathematical justification (Simon & Blume, 1996; Staples & Conner, 2022; Thanheiser & Sugimoto, 2022).

Mathematical justification is both a social and a cognitive activity. In completing mathematical justification, other than involving discernment in contending to help (or disprove) a case, one expects work to persuade parties beyond himself who engage with that action (Roy et al., 2014; Sowder & Harel, 1998). There are times when mathematical justification is not always logically complete because of the involvement of the community where the activity is carried out and social considerations (Jaffe, 1997; Kilpatrick et al., 2001). For instance, in a study by Sowder and Harel (Sowder & Harel, 1998), students were asked to determine whether a rhombus would result from connecting the midpoints of the sides of an isosceles trapezoid. By drawing an isosceles trapezoid, marking the midpoints of the sides, and demonstrating that the points were connected to form a rhombus, most students justified their answer to this question. Participants in this study were attracted to persuade other participants of his mathematical cases despite the fact that what they were doing was inaccurate.

According to Walton (Walton, 2001), the level of confidence or the degree to which the argument is convincing (plausibility) determines which type of argument constitutes reasonable mathematical justification. In another way, the students' mathematical justifications can be of varying quality, and the most convincing justification is the one with the highest plausibility level. Several studies have identified levels of mathematical justification in this regard. Simon and Blume (Simon & Blume, 1996) classified students' mathematical justification as level 0 (without justification), level 1 (appeal to external authority), level 2 (empirical demonstration), level 3 (deductive justification expressed through examples), and level 4 (deductive justification independent of example). Vale et al. (Vale et al., 2017) found similar levels, while Sowder and Harel (Sowder & Harel, 1998) and Carpenter et al. (Carpenter et al., 2005) regarded them as types rather than levels but explained that one type provided a better justification than the others.

To assist us in describing the variety of mathematical justifications while solving a problem in this study, we used five levels of mathematical justification adapted from Vale et al. (Vale et al., 2017) (a brief description can be found in Table 1). Students at level 0 simply accepted a particular claim without attempting to justify its truth. At level 1, students say a claim is valid because it comes from someone other than themselves, like a teacher or a textbook. At this level, students assume that if a mathematical claim comes from a more "authoritative" source, it will be valid. For instance, a student uses a particular strategy because his teacher previously suggested it. At level 2, students provide justification based on empirical or perceptual examples. Students demonstrate how what they see supports a claim and demonstrate it to others. Students, for instance, back up the claim that a quadrilateral picture is a model of a square because the sides are the same length. He may persuade others by estimating the lengths of the sides, which are similar, using some estimating instruments. This student justifies by citing what he experiences and sees when interacting with the quadrilateral illustration.

Students begin using symbolically expressed examples at level 3, but these examples have not been generalized. Sowder and Harel (Sowder & Harel, 1998) gave an illustration of justification at this level when participants were inquired as to whether $n^2 - 79n + 1601$ is prime for any value of n. In this model, a participant supported the case by giving instances of the qualities $n^2 - 79n + 1601$ for a few values of n and presumed that the computation generally gave indivisible numbers without considering all potential values of n. At this level, the participant affirmed his cases to others by utilizing examples, and because he didn't utilize generalization, he didn't see that for $n = 80, n^2 - 79n + 1601$ is undoubtedly not a prime number. At level 4, students use definitions, theorems, or mathematical logic rules to justify their claims or make generalizations that do not rely on examples. Vale et al. (Vale et al., 2017) stated that students found this level to be the most challenging because they did not believe this justification was necessary for other students to accept their claims.

Justification Code	**Description**
T	Trigger of a situation for justification
C	A claim made in the situation for justification
L0	No justification (Not justifying or accepting a claim without explaining why a claim is reasonable).
L1	Appeal to authority (Stating the truth of claims using other people's authority or other sources such as teachers, books, or friends).

L2	Empirical or perceptual demonstration (Stating the truth of claims based on the demonstration of perceptual examples, e.g., objects, pictures, or gestures).
L3	Symbolic example without generalization (Stating the truth of a claim by using some symbolic examples without generalizing those examples)
L4	Symbolic example with generalization (Stating the truth of a claim by using definitions, theorems, or rules of mathematical logic or generalizations that do not depend on a particular example)

Table 1: Coding scheme

Collaborative Problem Solving (CPS)

In this section, we try to elaborate on literatures in understanding students' collaborative problem solving. Some literature defined CPS as the ability a person has to solve problems together, while some other literature defined it as an activity to solve problems together. PISA 2015 defined CPS as an individual capacity in which two or more individuals are involved in a process to solve a problem. PISA defined CPS by stating that CPS capabilities should include the sharing of knowledge, skills, and efforts among individuals to achieve solutions. CPS competency is assessed using a 4 x 3 matrix (OECD, 2017). In the matrix, rows represent individual problem solving skills (exploring and understanding, representing and formulating, planning and implementing, monitoring and reflecting). Columns represent collaboration skills (building and maintaining shared understanding, taking appropriate action to solve problems, and building and maintaining team organization). Unlike PISA, ATC21S (Assessment for Teaching in the 21st Century) defined CPS more generally as "approaching problems responsively by working together and exchanging ideas" (Griffin & Care, 2015). In this sense, ATC21S sees CPS as a collective problem-solving activity and the competence to approach that problem in a group.

Other sources, which focused on collaborative problem solving in mathematics, positioned CPS as a situation in which a process that students undertook was investigated. The theoretical review of those studies that discussed CPS limits it as a joint problem-solving activity. However, the quality of certain activities was identified using some indicators similar to ATC21S and PISA. For example, a study by Mercier et al. (2017) incorporated the theory of perspective taking by group members to identify successful CPS interactions in the mathematics classroom. Another example was the study by Taylor and McDonald (2007), which identified students' problem-solving skills in groups using a framework by Polya similar to PISA. Rather than viewing CPS as an overall competency, these studies considered CPS as the activity in which the focus of the study occurred or was investigated. Participants in such studies were asked to solve math problems or perform certain math tasks in groups. Participants' interactions or performance were recorded and analyzed during the ongoing problem-solving process.

In the studies, a CPS was an environment or situation in which participants solve a problem or task together, and the task or problem was designed for the aim of the study. Similarly, in this paper, the researcher did not aim to analyze CPS as an individual ability, but as a situation or process in which mathematical justification was observed.

Theoretical Framework

In this section, we will discuss theories on mathematical justifications, collaborative problem solving, and technology, as well as how they were related to each other. Mathematical justification in Collaborative Problem Solving can be understood through the lens of social constructivism as proposed by Piaget (1970) and Vygotsky (1930). Both scholars emphasize that learning is an active process of knowledge construction, although they differ in their perspectives on the role of social interaction in individual cognitive development. Piaget (Piaget, 1970) asserts that learning is primarily an individual process driven by cognitive conflict and self-discovery, even though such conflict can arise from social interaction. In a collaborative setting, students encounter diverse perspectives that create disequilibrium (cognitive imbalance), prompting them to construct new understandings through reasoning and justification. According to Piaget, peer interaction facilitates deeper cognitive engagement as students work through contradictions and reestablish equilibrium.

On the other hand, social constructivism, as formulated by Lev Vygotsky in 1978, emphasizes the role of social interaction in individual cognitive development. Language and culture serve as fundamental frameworks through which individuals acquire knowledge. According to Vygotsky, language and culture play crucial roles in shaping intellectual development. Learning concepts are transmitted through language, interpreted, and understood through experiences and interactions within a cultural environment. Social constructivism acknowledges the social aspects of learning, emphasizing conversation, interaction with others, and the application of knowledge as essential elements for achieving learning goals (Rytilä, 2021).

In contrast to Piaget's constructivism, which views knowledge as something that students construct independently based on their experiences, Vygotsky's social constructivism posits that knowledge is developed through collaboration—whether with peers, teachers, or more knowledgeable individuals. Social constructivism is a variant of cognitive constructivism that highlights the collaborative nature of learning under the guidance of a facilitator or in cooperation with other students. Justification emerges as students engage within the Zone of Proximal Development (ZPD)—the space between what they can achieve independently and what they can achieve with the support of a more knowledgeable other. Language and discussion play a crucial role in this process, as students use verbal justification to negotiate and internalize mathematical concepts. Vygotsky (Lev Vygotsky, 1930) introduced the concept of internalization, referring to the process by which individuals assimilate knowledge, skills, values, or ways of thinking from their social environment (through interaction with others) into their own cognitive system.

In CPS, these two perspectives complement each other. Piaget's theory explains how students refine their justifications through cognitive conflict, while Vygotsky's theory highlights how scaffolding and peer discussions can enhance the development of higher-level justifications. This study builds on these frameworks by analyzing how students construct and refine their mathematical justifications in a collaborative setting, both with and without the use of technology.

When students are involved in CPS, they will interact with students in their group. In this process, students will think about solving the problem while comparing it with what their friends think. During discussions in the CPS stages, students' statements may be questioned or asked for reasons. This question initiates what is called a situation for justification (Cobb, Wood, et al., 1992). The person who questions or asked for reasons acts as a trigger for the justification. As a response to this, students will come up with mathematical justifications that they think of in order to convince other students that their claims are correct.

When the mathematical justification put forward by the student is agreed upon by the group, a co-constructed justification is formed (Mueller, 2009; Yackel, 2004). This agreement or acceptance might be indicated by an explicit statement or implicitly through the use of the justified claim as a new data in the next discussion (Cobb et al., 2010). A similar concept was stated by Tatsis and Koleza (2008) and Yackel and Cobb (Yackel & Cobb, 1996) as mathematical justification norms. Norm, in this study, was defined as a regularity in collective activities in the classroom. Mathematical justification norms can be understood as regularities in collective activity when a mathematical method or claim needs to be supported by a reason (McClain & Cobb, 2001). In other words, mathematical justification in a CPS situation can be seen as a mathematical justification that is built together by a group or as a regularity of mathematical justification activity being observed when a group solves a problem. Mathematical justifications that are constructed together or mathematical justification norms determine whether the ongoing CPS process can continue to the next stage or needs to return to the previous stage (McClain & Cobb, 2001; Partanen & Kaasila, 2015; Roy et al., 2014; Tatsis, 2007).

The involvement of technology while students are justifying claims during CPS might bring different issues. Some studies revealed that technology improved the problem solving process. For instance, a study by Nguyen et al. (Nguyen et al., 2023) analyzed the use of GeoGebra in learning and solve problems on geometry. They found that utilizing GeoGebra helped students in problem solving, especially in questioning the process and actively participating in the exchange of ideas, activities supporting mathematical justifications. Another study by Olive et al. (Olive et al., 2010) highlighted the role of technology in being an 'external authority" that empowers students to more or new mathematical practices. For instance, they mentioned the way technology could provide visualization of an abstract concept that made this concept more approachable to students. This visualization allows students to a new mathematical practice to manipulate a supposedly abstract concept using their visual perceptions. In problem solving, such activity could help students to justify a claim using their visual perceptions and reason its relation to the intended problem. Technology also helps students in verifying claims (Erbas et al., 2020). Erbas et al.

explored students' strategies in solving problems using the Geometer's Sketchpad. They found that the software allowed students to model the problem and manipulate the involved variables interactively to explore it. The manipulations allow students to verify the validity of claims and to construct a valid argument under a plausible justification.

Despite all the possible benefits brought by technology in helping students to justify claims, the use of technology also brought concerns. For instance, in a study by Hollebrands et al. (Hollebrands et al., 2010), students were found to rarely provide justifications while working on a problem using technology compared to when they did not use it. One of the possible reasons was that students considered the technology itself as a form of justification, which was actually not intended from the use of technology in helping students to reason and verify claims. This issue is related to the fact that technology could be treated as an 'external authority', a source of valid claim without the need to question it. As explained in the previous section, a justification based on an external authority is considered a low level of justification, as it does not involve students' own reasoning in judging the validity of a claim. Tall (1989) stated that students might lose some autonomy if they did not follow their reasoning and allow themselves to be led by technology.

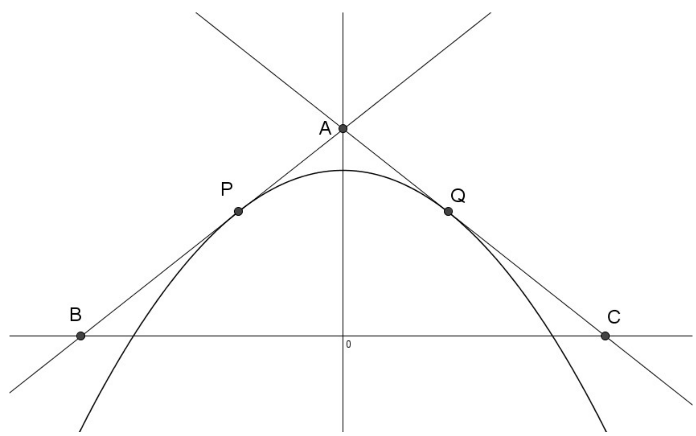
In this paper, we explore how a pair of students mathematically justify claims during CPS with and without access to technology. To investigate this, we focused on identifying the quality of mathematical justification (as presented in Table 1) while they solved a problem in the absence and presence of technology. To allow for better analysis of the role of technology, we also explore the feature of technology they used and how it was related to their mathematical justification and problem solving process.

METHODS

This study was part of a larger project on mathematical justification by college students while solving derivative problems collaboratively. We asked two mathematics education majors who were also preservice teachers to solve a derivative problem (Figure 1) in pair to see how they solved it. The task was designed following Chua's description of an elaboration task (Chua, 2017) as one of the task types promoting mathematical justification. Content-wise, the task was modified from Stewart (Stewart, 2010) by adding more information on the points presented in the graph. Four experts validated the task and went through phases of revisions. The validation process was to assess whether the task (1) encouraged students to elaborate their thinking upon the answer, (2) instructed students to solve the problem collaboratively, and (3) allowed various justifications along the process of finding the answer. In addition to validating the third point, the validation document included a scheme of possible mathematical justifications made by students.

The highlight of the task is the need to use derivatives to find the slope of the tangent lines. Further, students must elaborate their steps by utilizing the graph presented in the task without disregarding its algebraic validity related to the solutions. For example, students may infer that point P is closer to point A than to point B based on the picture. However, this cannot be justified by the picture alone. They were expected to leave this as an assumption rather than valid information. The task also required students to utilize the fact that points A, B, and C are equidistant, which might be altered if students relied heavily on the perception of the distance given by the graph. The expected final answer is $P\left(\frac{-1}{2}\sqrt{3}, \frac{1}{4}\right)$ and $Q\left(\frac{1}{2}\sqrt{3}, \frac{1}{4}\right)$ to make ABC an equilateral triangle.

Given a parabola $y = 1 - x^2$ as shown below.



Lines p and q are the tangents of the parabola at P and Q. Tangent lines p and q intersect x-axis at B and C, respectively. Find points P and Q on the parabola so that triangle ABC formed by the x-axis, p, and q is equilateral. Elaborate on your steps (Note: the slope of p and q is the derivative of the parabola function at P and Q, respectively).

Figure 1: The task given in this study

The task was distributed to the pair (pseudonyms: Alan and Sarah) for two hours through an online meeting platform. They had been informed that the task was related to the derivative of functions. Both students had average mathematics ability and had been exposed to some mathematics-related technology (e.g., *GeoGebra* or *Desmos*) in college. To better understand the role of technology, we asked them to solve the problem without technology in the first session and with technology in the second. One student writes their work digitally in the session while sharing the screen with another student. In the second session, students might choose any technology they wanted and were not forced to stick to a particular technology. This strategy was offered because we wanted to reduce the factor of students' skills in utilizing a particular technology.

Students were encouraged to discuss and question their pair whenever needed. The researcher acted as an observer during the session, yet students were prompted to continue the discussions when they were silent for more than a minute. The session was recorded, and students' written works were collected. Semi-structured interviews were conducted to understand some critical episodes in the problem-solving process. The data sources include recordings of meetings, students' written work, and field notes. The analysis was conducted by first identifying a *situation for justification*, i.e., a situation where a claim was questioned and requested validation (Cobb, Yackel, et al., 1992). In each situation, we looked upon who made a mathematical claim and who triggered the justification of it. To further understand the quality of justification, the level of justification that appeared was also coded based on the coding scheme in Table 1.

RESULTS

We present the findings by revolving around describing three situations of justification during the CPS, i.e., situations 1 and 2 in the first session (without access to technology) and situation 3 in the second session (with access to technology). We reported students' mathematical justification levels in each session in a particular situation. The situations presented here did not happen in one period. Yet, we tried to collect related discussion episodes and elaborate on them, focusing on the progression of mathematical justification upon the corresponding claim. The pair initially decided that Sarah would share her screen and writings during the discussion.

Situation 1: a claim that $x_1 = y_1$

At the initial stage of understanding the problem, the pair started to determine the coordinates of points that might be helpful to solve the problem. By analyzing $y = 1 - x^2$, they found that the parabola intersects the x-axis at $D(-1,0)$ and $E(1,0)$ and intersects y-axis at $F(0,1)$. Sarah then continued by stating that the distances from D, E,and F to the origin were equal and put marks on the graph (see Figure 2). After considering the fact that $\triangle ABC$ was an equilateral triangle, Sarah inferred that if C was $(x_1, 0)$ then B was $(-x_1, 0)$. This claim was agreed upon by Alan, and he added that A would be $(0, y_1)$. The situation for justification appeared when Sarah claimed that $x_1 = y_1$ and Alan questioned it as follows (the corresponding code was put inside "[ ]").

Sarah : Look, $\triangle ABC$ is equilateral, right? So, this (puts a mark on $\overline{EC}$) would be the same length as this (puts a mark on $\overline{BD}$).
Alan : I think so.
Sarah : It means they will be the same with this (puts a mark on $\overline{AF}$), too. [C]
Alan : How do you know? [T]
Sarah : See here (pointing $\overline{OD}$), here (pointing $\overline{OF}$), and here (pointing $\overline{OE}$) are the same, right? So, this (pointing $\overline{BD}$), this (pointing $\overline{AF}$), and this (pointing $\overline{EF}$) are the same, too. [L2]
Alan : What theory says so? [T] (pause) So, you think △DEF is also equilateral?

Sarah : I think so. I know it doesn't look like it (is equilateral), but, you know, the picture usually is not real. I mean, they should be the same lengths because the triangles are equilateral. Wait. So, $x_1 = y_1$ here, right? [L2]

Alan : I don't know. (pause) So, we still cannot determine the coordinates of the points (P and Q), right?

Sarah : Yes, I think so.

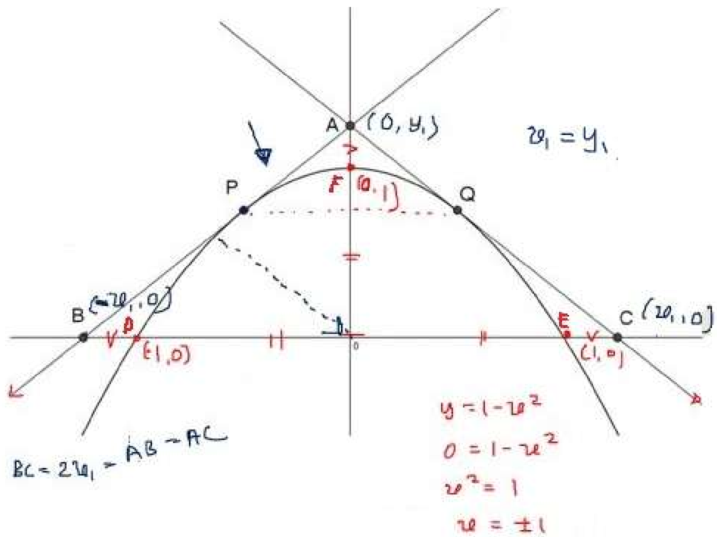


Figure 2: Pair Work in Situation 1

In this situation, Sarah used the fact that $\triangle ABC$ is equilateral and justified that $AF = BD = EC$ by saying that$\triangle DEF$ is also equilateral. Alan triggered the situation of justification by questioning this claim. Sarah at first relied on her claim on the data that $\triangle ABC$ is equilateral, then she pictured $\triangle DEF$ to be similar to $\triangle ABC$ and that led to her claim of $x_1 = y_1$. Although she explicitly stated that "picture usually is not real", she came up with " $\triangle DEF$ similar to $\triangle ABC$" based on how $\triangle DEF$ 'looked' similar to $\triangle ABC$ in the graph, not based on properties of similar triangles. Sarah used the empirical or perceptual demonstration to justify her claim (level 2), while Alan seemed unconvinced and requested a higher level of justification using theorems. However, the pair could not come up with a better justification and tentatively accepted this inaccurate claim. In summary, the claim that $x_1 = y_1$ was mainly justified by the perceived similarity of $\triangle DEF$ and $\triangle ABC$.

Situation 2: claims around the slope of tangent lines

Another situation of justification emerged during the pair's discussion about the slope of tangent lines. The pair immediately realized they could find the slope of the tangent lines from the parabola function. They further wrote $y = 1 - x^2$ and discussed to find $m = y' = -2x$ using the derivative rule. After finding the derivative, the pair could not relate it to the two tangent lines having different slopes. This confusion led to a situation of justification where the pair tried to find other information to validate the slope. The following discussions accompanied their work in Figure *3*.

Alan : One tangent line should have a negative slope, and another is positive. But it is the same, $-2x$. Maybe we should try another way.

Sarah : Okay, but I think it should be about derivative.

Alan : What about stating A, B, C in terms of coordinate? I mean, using the facts (that $AF = BD = EC$).

Sarah : What do you mean?

Alan : I mean, let's say that distance from B to C is a, and the other two are a, too, right?

Sarah : Yes

Alan : So, A is $(0,1 + a)$ [C]

Sarah : Okay, wait (made a new page and copied the graph from the problem, wrote the coordinates for D, E, and F). So, you say?

Alan : A is $(0,1 + a)$, then C is … [L0]

Sarah : C is $(1 + a, 0)$ (wrote the coordinate of C), right?

Alan : Yes

Sarah : B is $(-1 + a)$? [T]

Alan : No, it should be minus. $(-1 - a)$, it goes to the left, right? It becomes less. [L2]

Sarah : Oh, right. (wrote the coordinates of B).

Alan : I think we can calculate the slope from the points, right?

Sarah : Oh, okay. Point A to C first, ya. So, $m = \frac{1+a-0}{0-(1+a)} = -1$ [C]

Alan : But, is it true? [T]

Sarah : I don't know. Maybe if we see the line from F to E (further labelled as q), it looks parallel to the tangent. [L2] Parallel means equal slope, right?

Alan : Hmm, I think so.

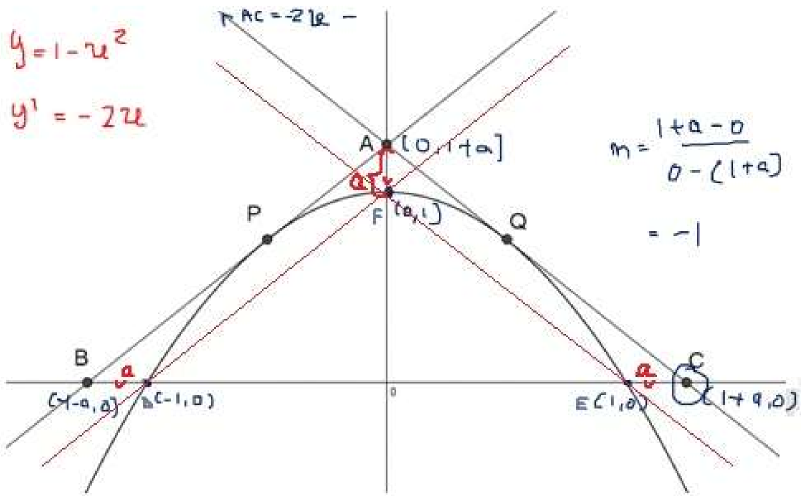


Figure *3*: Pair Work in Situation 2

The important justifications leading to the claim in Situation 2 were on the coordinates of A, B, and C and the parallelism of line q and the tangent. Both justifications used perceptual demonstration (level 2), which was then triggered for better justification by Alan. The trigger resulted in the acceptance of the claim without any refined justification.

From the discussion in situations 1 and 2, it could be seen that the pair justified claims mainly using empirical or perceptual demonstration (e.g., "if it goes left, it becomes less" and "it looked parallel to the tangent"). The first claims on A, B, and C coordinates were continued from Situation 1, where they inferred that the length of AF equals BD and EC. The pair accepted the justification based on perceptual and empirical demonstration as they identified the coordinates of A, B, and C under a similar justification in situation 1. This accepted claim led to another claim of $m_q = -1$, which they accepted based on a perceptual demonstration that the line looked parallel to the line q passing through E and F.

Along the discussion in two situations, we see how Alan's request for better justification in Situation 1 was slowly compromised once the pair implicitly accepted the claim based on empirical or perceptual demonstration. Alan and Sarah each gave a trigger for justification of a particular claim. Yet, it was shown that Sarah's empirical or perceptual demonstration level of justification was neither refuted nor revised during the discussion. The inaccurate claim accepted by the pair in situation 1 became the underlying "agreement" in understanding new data brought into the discussion and further into accepting new claims.

The pair then continued the process by discussing that due to $m = -2x$, $m_q = -1$ and the fact that q and the tangent line have equal slope, they inferred that x as the abscissa for the coordinate of Q is $\frac{1}{2}$ and that of P is $\frac{-1}{2}$. They did not finish until they got the complete coordinates of P and

Q due to the lack of time for the first session. The discussion was continued by allowing them to access technology in the second session.

Situation 3: claims around the coordinate of P

We discussed students' work in the second session, focusing on Situation 3, containing claims around the coordinate of P. At the beginning of the second session, we prompted the pair to access any technology they wanted to solve the problem. They were told they could do different strategies or start from any step at the first session. The pair decided to use *Desmos* to verify the abscissa of P that they found in the first session (Figure 4). Alan decided to share his screen. Alan first input the function $f(x) = 1 - x^2$. This input created the graph of $f(x)$ in the screen. As they understood that the slope of the tangent is the derivative of the function, they decided to use the *Desmos* feature (i.e., Calculus: Tangent Line) to find the slope (shown in Figure 4 as the red line $g(x)$). Alan initated to create a slider labelled as a to act as the dynamic value of x. He then dragged the slider a to change the values of $P(a, f(a))$. The following excerpt revealed the discussions using *Desmos* that their previous claim of the coordinate of P was not accurate.

Alan : We can drag this (the slider in Desmos) to change the intersection point.
Sarah : Oh, okay
Alan : Let's try a half. Or a negative half?
Sarah : Okay
Alan : Here it is. This is the answer (Desmos window shown in Figure 4*). So P is $\left(\frac{-1}{2}, \frac{3}{4}\right)$ and Q is $\left(\frac{1}{2}, \frac{3}{4}\right)$ because it is just positive here.*
Sarah : Okay, the lengths are the same, ya.
Alan : Yes. Here (pointing to the x and y intersects of the tangent), we get the same lengths. Mm, it is around $1\frac{1}{4}$ both, right?
Sarah : Wait, no, Alan. It is not equilateral. You see, the one below (segment BC) will be twice. [T]
Alan : Oh, right.
Sarah : ... and we think about the sides, right? Not the height.
Alan : Oh.

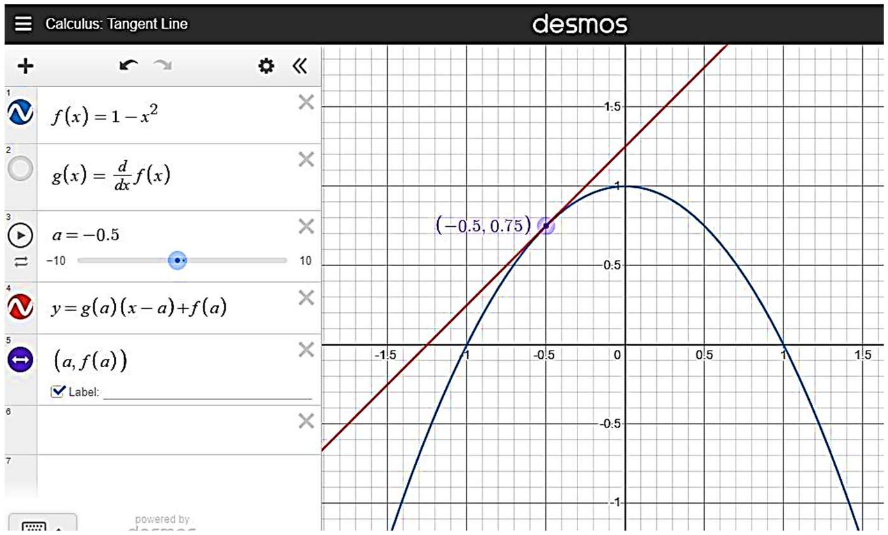
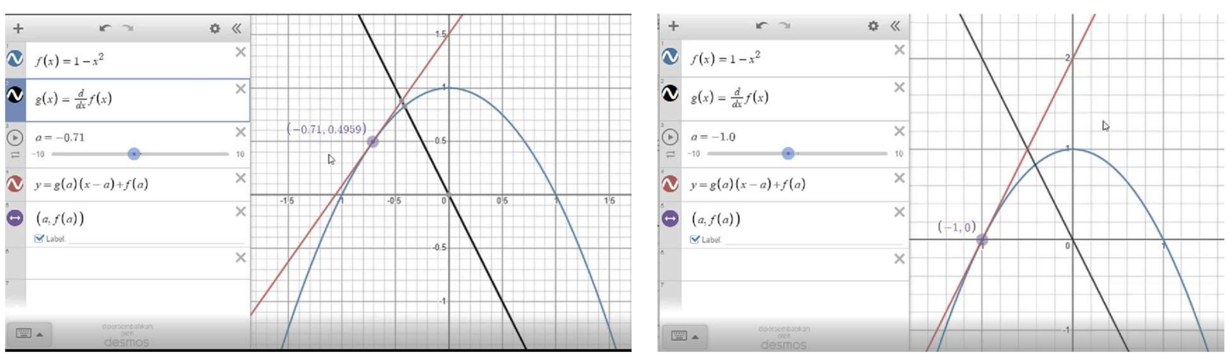


Figure 4: Initial Desmos Entry

While utilizing the slider, the pair initially slide it to a value of 0.5. This input was intended to check their previous answer of the value of x. After the graph by Desmos was shown, the pair realized that the coordinates obtained in the previous situation were incorrect. Alan then suggested to change the value of a by sliding the slider (Figure 5). They gradually changed it into smaller values, such as 0.7 and -1. After the repeated trials, the pair did not seem to find the value they wanted to create the equilateral triangle ABC by utilizing the slider alone.



Note: in these screens, they did not hide $g(x)$, so it was shown as the black line.

Figure 5: Some trials using the slider

Their repeated trials in using the slider led to different ways to analyze the data given by the problem. Sarah inferred that it might be useful to draw triangle ABO as part of the equilateral triangle ABC (Figure *6*, left). By considering the angles of the triangle, they explored where P should be located using Desmos (Figure *6*, right). They concluded that P should be somewhere close to $(-0.86, 0.26)$ and thus Q would be around $(0.86, 0.26)$, which is very close to the actual correct answer. The following excerpt shows how they justified this answer.

Sarah : I think it is true. See that the equilateral triangle (pointing to the triangle drawing) should have angles like that, right? And the graph (in Desmos) is just the same. [L3]

Alan : Yeah. I think it would be true if we calculate the sides using trigonometry [L4].

Sarah : I think so. It (the graph) didn't look like the question, though.

Alan : Yes, it is steeper. I think I can picture the equilateral triangle in my mind (when I see the graph), now [L2]

We saw that the justifications in situation three were refined, following the fact that their answer from the previous situations was inaccurate. The slider feature was shown to be helpful in helping them testing several values of x to satisfy the objective of the problem, despite not providing them with the answer. The pair's effort to use a drawing of an equilateral triangle as an example (level 3) to justify the equilateral triangle in the graph was shown to lead them in the right direction. The use of example was triggered by the verification made in the previous Desmos entry, showing that reliance on perceptual demonstration was insufficient. The pair, initiated by Alan, even offered a higher level of justification to use trigonometry (level 4) to justify the sides of the triangle and further validate the coordinates of P and Q, which unfortunately they did not do. They also realized that they could not base their discussion on the graph provided by the problem alone because it did not visualize the correct answer. In this situation, the justification of perceptual or empirical demonstration accompanied another justification after the pair realized its inadequacy.

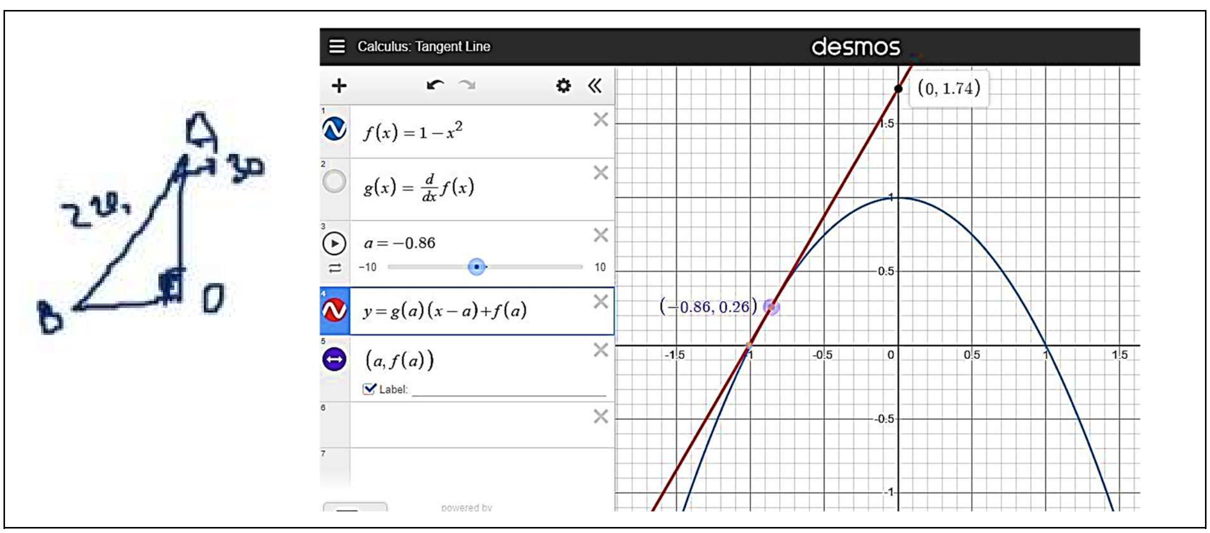


Figure *6*: Triangle drawing and Final Desmos Entry

DISCUSSIONS AND CONCLUSIONS

The quality of mathematical justifications brought by the pair could be seen from the three situations of justification found in the study. We noticed various mathematical justifications during the first and second sessions. In the first session, we observed that claims were questioned, and perceptual or empirical demonstration type of justification was mostly offered, similar to findings by other researchers (Chazan, 1993; Sowder & Harel, 1998). We considered their reliance of perceptual or empirical demonstration was not due to the absence of trigger, a question or request of better justification, but due to the inability to fulfil such trigger. For instance, when Alan requested justification based on theories in situation 1, the pair failed to entertain such request for a higher level of justification. It led the pair to accept the claim tentatively and continue implementing strategies on inaccurate claims.

In the second session, we observed that the pair used the technology to verify their solution rather than to start working on it from the beginning. Once they realized that their previous solution could not be justified, they started to work on the initial information of the problem, such as the function of the parabola and its tangents. It was also observed that triggers for mathematical justification appeared more in the first session when the pair did not have access to technology. It raised more questions about whether the pair treated the technology itself as a trigger or justification (Hollebrands et al., 2010) so that the claims shown being valid by technology did not need further justification.

The pair used the technology to validate claims, yet it was shown that it did not directly help them improve the claims or give better justification. We believed that the offer for justification in situation 3 (when Alan wanted to use trigonometry to verify the solution) could indicate that technology promoted students to give better justification only when they understood how they came up with their claims and how their previous justification might not be enough. Nevertheless, the findings reiterated the fact that the use of technology could improve the problem solving process (Berrin et al., 2024; Nguyen et al., 2023) and accommodate students' collaboration, especially in testing their ideas (Olive et al., 2010), illustrating multiple cases to verify claims (Erbas et al., 2020) and being more responsible for their thinking (Buteau & Muller, 2006).

Another notable finding was the different use of graph or picture as justification between the first and second sessions. In situation 3, the pair seemed convinced by the graph they constructed on *Desmos* and used it as a joint justification with the triangle they made. Going back to situation 1, similar confidence did not appear on the use of a similar graph, as shown by a request for better justification. Considering this fact, we inferred what students meant by "picture" when they stated "the picture usually is not real" was not a picture constructed by a technological tool. Pictures or graphs resulting from a technological tool somehow gave them more plausibility. This finding was similarly highlighted by Zhen et al. (2016) who found that students might be unconvinced by merely a graphical perception, but they somehow believed arguments coming from a graph satisfying a certain property (e.g., when it was constructed by technology). Teaching practices could benefit from this finding. Teachers might use technology as a tool to challenge students' beliefs on the truth of a claim, for example, by comparing a hand-drawn geometrical figure with the one being constructed by technology. Questions such as, "why do you think this picture seems more convincing than the one you draw in the paper?" or "Can you convince me that the technology is showing the correct picture?" could intrigue students into thinking about the reason behind their belief and at the same time allow them to critically examine outputs of technology they used.

The study allowed us to answer some questions on students' mathematical justification. The task administration to the pair without the lecturer's presence avoided the question of who would take justification as valid besides the students since the community involved was only them (Karunakaran & Levin, 2022). However, it also raised a question, as in a natural classroom setting, it was not typical to have lecturers uninvolved in any of their students' problem-solving processes. The nature of the task that could be solved without access to technology brought two concerns. First, it allows for comparing students' justification with and without technology in students' typical context, i.e., choosing their own preference, which should be taken into account if we want to explore the use of technology (Sanchez, 2020). Second, it probably diverges our observation of whether the pair's mathematical justifications were due to the task or the presence of technology itself. Students' prior exposure to the technology might contribute to how far they have solved the problem and justified the solution. It also might bring different data into the table if they decided to use a different software with different features. We also considered the possibility of even further justifications using the technology, which the pair could not make, probably due to their skill in utilizing it.

We acknowledge that the difficulty level of the task was not explicitly considered in our study, and this may have influenced the students' reliance on technology during the second session. Our focus was primarily on ensuring that students encountered no issues with the use of technology and that the task was capable of eliciting diverse mathematical justifications. However, we recognize that this could be a limitation of our research, in line with the findings of Fatmanissa et al. (2024), which indicate that the observed collaboration and justifications were influenced by an appropriate level of task difficulty. Therefore, we suggest that future research should consider designing tasks that strike a balance between difficulty and accessibility, ensuring that technology functions as a supportive tool for enhancing conceptual understanding rather than replacing students' critical thinking processes.

A study that utilizes a task that explicitly includes instructions on technology to promote mathematical justifications during CPS could be conducted in the future. Such instructions might guide students transitioning from an empirical to a more deductive justifications, for example by asking students to add supporting arguments other than relying solely on the graph to their claim of the similarity of $\triangle DEF$ and $\triangle ABC$. Such actions, called *elaborating* by Zazkis et al. (2016), encourage students to not take empirical justification for granted, add supporting data, and construct more formal arguments.

This study was small in size to allow for an in-depth analysis of students' mathematical justifications, yet it is worthwhile to be developed further. An experimental study with a larger sample comparing students' mathematical justifications with and without access to technology could refine the findings of this study. The diversification of participants' mathematical or technological skill levels could also improve our understanding upon this issue. Another limitation in this study is the limited range of software tools examined due to our approach of allowing students to use a familiar software i.e., Desmos. However, this focus restricted our ability to compare the effectiveness of different software platforms in supporting mathematical justification. Future research could investigate how different software platforms might influence students problem-solving approaches and results.

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