Enhancing Instruction in Trigonometry: Insights from Students' Reasoning Approaches and Processes

Allan M. Canonigo

National Institute for Science and Mathematics Education Development,

University of the Philippines Diliman, Philippines

amcanonigo@up.edu.ph

Abstract: This study investigates the interplay between student reasoning and instructional strategies in trigonometry education. A qualitative case study was conducted to examine how Grade 10 students employ various reasoning approaches - deductive, inductive, abductive, analogical, and algorithmic - when solving trigonometry problems. Each approach offers unique strengths and limitations, impacting students' problem-solving strategies and comprehension of trigonometric concepts. The research also emphasizes the importance of validation, justification, and formal proving as integral steps in solidifying mathematical reasoning, transitioning students from practical experimentation to theoretical understanding. Classroom observations and teacher interviews provide insights into effective instructional strategies that leverage student reasoning. Teachers adapted methods to accommodate these diverse approaches, emphasizing scaffolding, guided inquiry, and contextualized learning experiences. Hands-on and experiential learning proved effective, fostering deeper engagement and intuitive understanding. Integration of formal mathematical reasoning ensured students grasped abstract concepts and their real-world applications. Meta-cognitive reflection and discourse enhanced students' problem-solving abilities and collaborative skills. These findings underscore the importance of flexible and inclusive mathematics instruction, equipping students with essential skills for success in mathematics and beyond.

Keywords: mathematics reasoning, deduction, abduction, justification, validation

INTRODUCTION

Developing meaningful mathematical knowledge hinges on students actively engaging with prior experiences and constructing new understanding (Romberg, 2000). This requires students to identify patterns, create models of concepts, develop symbolic representations, and devise solutions to complex problems (Battista, 1999). Trigonometry, integrating geometry, algebra, and graphical representations, offers a rich environment for fostering these skills. However, current practices often emphasize rote memorization and calculations, neglecting opportunities for deeper reasoning (Shield, 2004).

The abstract nature of trigonometry and its intricate connections between geometry and algebra pose learning challenges (Bressoud, 2010; Cavanagh, 2008; Thompson, 2008). Overcoming these hurdles necessitates a shift from purely technical instruction to a broader approach that cultivates reasoning and understanding. Investigating how students approach problem-solving in trigonometry holds the key to unlocking effective pedagogical methods.

Previous research highlights the importance of recognizing diverse student reasoning styles and tailoring instruction to cater to individual learning preferences (Thompson et al., 2007). However, a gap exists in the literature regarding the interconnectedness between student reasoning and teacher strategies specifically within trigonometry education.

The effectiveness of teaching trigonometry hinges on how teachers engage with students' solutions, responses, and reasoning (Hiebert & Stigler, 2023; Herbert & Bragg, 2021). Understanding how teachers navigate these aspects significantly impacts the quality of instruction. This study investigates this dynamic relationship between student reasoning and teacher pedagogical methods in teaching trigonometry. By examining diverse student problem-solving strategies, the research explores how teachers can leverage these solutions to foster conceptual understanding.

The study addresses the following key research questions:

1. What mathematical reasoning approaches and processes do students employ when solving and explaining solutions to trigonometry problems?
2. What instructional methods do teachers employ to accommodate and leverage students' diverse mathematical reasoning approaches and processes?

LITERATURE REVIEW

Innovative Pedagogical Approaches in Teaching Trigonometry

Traditional trigonometry instruction often relies on rote memorization, hindering students' ability to apply concepts effectively (Gillman, 1991; van Laren, 2012). To address this, innovative pedagogical approaches are emerging to foster deeper understanding and critical thinking (Quinlan, 2004). One critical challenge is overcoming common misconceptions (Gholami, 2022). Cooperative Learning Strategies (CLS) such as Cooperative Teaching and Learning (CTL) have demonstrated effectiveness in addressing this issue by creating collaborative learning environments (Asomah et al., 2023; Patterson et al., 2020; Quinlan, 2004).

Developing critical thinking and problem-solving skills is paramount in trigonometry instruction. Structured classroom discussions, hands-on activities, and real-world applications can significantly enhance student understanding (Kohlmeier & Saye, 2019; Thompson et al., 2007; Vale et al., 2019; Weber, 2005). To cater to diverse student needs, differentiated instruction is essential. By tailoring teaching strategies to individual learning styles and cognitive abilities, educators can create inclusive learning environments (Aiyub et al., 2024; Hiebert & Stigler, 2023).

Ultimately, innovative pedagogical approaches are crucial for transforming trigonometry instruction from rote memorization to a discipline that fosters deep understanding, critical thinking, and real-world application.

Reasoning Approaches in Trigonometry

The enhancement of trigonometry instruction is closely linked to understanding the various reasoning approaches that students employ. By examining different approaches to reasoning, teachers can gain critical insights into students' cognitive processes, which is essential for fostering deeper mathematical thinking and problem-solving skills (Gómez-Veiga et al. 2018).

Inductive reasoning, a key component of cognitive functioning, is crucial for students as they observe patterns and generalize from specific instances in trigonometry. This type of reasoning helps students recognize patterns and representations, which supports higher-order cognitive abilities like abstract thought and problem-solving (Bao et al., 2022; Klauer & Phye, 2008; Molnár et al., 2013). Inductive reasoning is important in discovering trigonometric identities, demonstrating that encouraging students to explore patterns significantly enhances their problem-solving abilities and deepens their conceptual understanding (Gunderson & Rosen, 2010; Hamers et al., 1998; Pellegrino & Glaser, 1984). This suggests that instructional strategies that engage students in pattern recognition and conjecture formation are particularly effective in trigonometry education.

On the other hand, deductive reasoning, which involves deriving specific conclusions from general principles, is essential for solving trigonometric problems using known identities and formulas (Sánchez et al., 2023). This reasoning process is crucial for enabling students to apply established facts to new problems, thereby generating new knowledge (Díaz Quezada & Sepúlveda Albornoz, 2023; Duval, 1991; Harel & Weber, 2020). However, research indicates that many students rely heavily on memorized formulas without fully understanding the underlying logic of trigonometric principles, which can limit their ability to apply these principles flexibly across different contexts (Siyepu, 2020). Therefore, enhancing trigonometry instruction requires a stronger emphasis on deductive reasoning, encouraging students to understand and apply logical rules rather than relying solely on memorization.

Furthermore, analogical reasoning involves drawing parallels between similar situations or problems to infer solutions or understand new concepts. This type of reasoning is essential in mathematics, where students often encounter new problems that can be understood by relating them to previously solved problems. Analogical reasoning is a powerful tool in mathematics education because it enables students to transfer knowledge from one context to another (Goswami, 1991). For instance, understanding the solution to a geometric problem can often be facilitated by recognizing its similarity to another problem that the student has already solved (Gray & Holyoak, 2021). Furthermore, analogical reasoning helps in the consolidation of knowledge, as it allows students to build on existing knowledge and apply it to new, more complex situations (Richland et al., 2007).

Moreover**,** algorithmic reasoning refers to the step-by-step application of procedures or algorithms to solve problems. This type of reasoning is pervasive in mathematics, especially in areas like algebra and calculus, where specific algorithms are applied to simplify expressions, solve equations, or compute values. Algorithmic reasoning is critical in developing procedural fluency in mathematics (Rittle-Johnson et al., 2001). While it is often seen as mechanical, algorithmic reasoning also requires a deep understanding of the underlying principles to be applied effectively. Star (2005) notes that students who develop strong algorithmic reasoning skills are better equipped to handle complex problems, as they can efficiently and accurately apply the necessary procedures. However, it is also important to balance algorithmic reasoning with other types of reasoning to ensure that students do not become overly reliant on procedures without understanding the concepts behind them.

The inclusion of inductive, deductive, abductive, analogical, and algorithmic reasoning in this study is grounded in the belief that a comprehensive understanding of problem-solving involves a multifaceted approach (Gyan et al., 2021). These reasoning modes represent distinct cognitive processes that collectively contribute to a robust problem-solving repertoire (Gyan et al., 2021; Pruner, 2023). Inductive and deductive reasoning form the cornerstone of scientific inquiry, allowing for the observation of patterns, the formulation of hypotheses, and the application of logical rules to test those hypotheses (Molnár et al., 2013). Abductive reasoning stimulates creativity and critical thinking by encouraging students to generate multiple potential explanations for observed phenomena, fostering a deeper understanding of problem complexity (Hidayah et al., 2023; Paavola, 2023; Pedemonte & Reid, 2011). Analogical reasoning facilitates knowledge transfer and problem-solving flexibility by enabling students to connect new problems with familiar situations, promoting adaptability and innovation. And algorithmic reasoning emphasizes computational thinking and systematic problem-solving, equipping students with essential skills for the digital age and preparing them to tackle complex challenges. By incorporating these diverse reasoning approaches, the study aims to provide a rich and engaging learning experience that develops students' cognitive flexibility, critical thinking, and problem-solving abilities. This holistic approach aligns with contemporary educational goals that emphasize the importance of 21st-century skills.

Enhancing trigonometry instruction requires a comprehensive approach that goes beyond traditional methods. By addressing misconceptions, fostering collaboration, promoting critical thinking, and accommodating diverse learning needs, educators can significantly improve students' understanding and application of trigonometric concepts. Furthermore, integrating insights from various reasoning approaches—inductive, deductive, abductive, analogical, and algorithmic—into instructional design can lead to more effective learning outcomes in trigonometry. As the field continues to evolve, future research should focus on examining the long-term impacts of these innovative pedagogical approaches on student achievement and their overall attitudes toward mathematics.

Mathematical Reasoning Processes

Mathematical reasoning forms the foundation of problem-solving, critical thinking, and innovation. This paper presents a structured framework for mathematical reasoning, centering on four key processes: validation, justification, informal proving, and formal proving. Each process is examined through the practical problem of determining ladder stability, providing a concrete application that illustrates the framework in action.

Validation serves as the initial step in mathematical problem-solving and is extensively discussed in literature. This process resembles experimentation and prototyping in design thinking (Brown et al., 1989) and can be likened to empirical verification in mathematics, akin to data collection in scientific inquiry. While empirical data provides essential insight, theoretical support is often needed for broader applicability and robustness.

Justification is the process of linking observations to foundational principles, a crucial step in constructing coherent arguments. This process is integral to disciplines such as philosophy, law, and mathematics (Toulmin, 2003). Polya's (1945) problem-solving heuristics underscore the importance of reflection and understanding the rationale behind solutions, reinforcing the role of justification in establishing sound reasoning.

Proving is a core activity in mathematics, involving the development of logical arguments based on axioms, definitions, and established theorems. Euclid's *Elements* remains a classic example of deductive reasoning, demonstrating the foundational approach to proofs. Contemporary research has delved into the philosophical and logical underpinnings of proof, enriching our understanding of its principles and methods (Balacheff, 1988; Hartshorne, 2000).

Formal proving represents a rigorous evolution in the practice of mathematical proof, gaining traction with the advent of computer-assisted proof systems (de Villiers, 1999). This process entails converting mathematical statements into formal language and utilizing automated tools to verify their validity with precision. Foundational work by De Bruijn (1980) and Boyer and

Moore (1979) has shaped formal verification methodologies, establishing frameworks that continue to influence modern mathematical proof systems.

Together, these four processes—validation, justification, informal proving, and formal proving—offer a comprehensive framework for understanding mathematical reasoning and applying it to concrete problems.

Theoretical Framework

This study centers on a framework for examining students' mathematical reasoning (MR) approaches and processes, developed by Jeannotte and Kieran (2017). This framework provides a comprehensive structure for analyzing MR in trigonometry education, interweaving four interconnected strands: mathematical objects, activities, processes, and concepts.

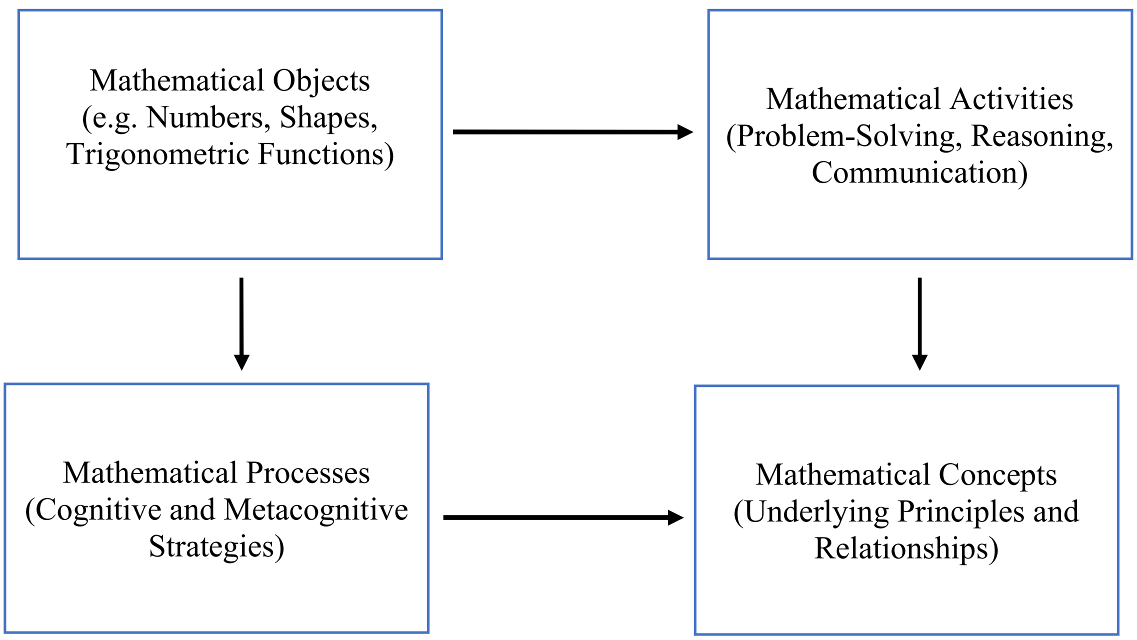

Figure 1: Mathematical Reasoning Framework based on Jeannotte and Kieran (2017).

Figure 1 shows the framework consists of two main sections which are the mathematical objects and activities, and the mathematical processes and concepts. These sections are interconnected, representing the interplay between the elements students work with and the cognitive processes they employ. Students engage in activities using mathematical objects. They utilize mathematical processes to understand and make sense of these objects and activities. These mathematical objects encompass the entities students interact with, such as numbers, shapes, and trigonometric functions in this context. Mathematical activities encompass the observable actions students undertake, including problem-solving, trigonometric reasoning, and solution articulation. In contrast, mathematical processes refer to the underlying cognitive and metacognitive strategies employed by students as they engage with mathematical tasks. Lastly, mathematical concepts explore the underlying principles and relationships students are expected to grasp within trigonometry, such as connections between angles and their trigonometric ratios. This diagram provides a visual representation of the framework, highlighting the interconnected nature of the components and their role in understanding students' mathematical reasoning in trigonometry education.

By investigating these interconnected strands, the study goes beyond simply identifying correct or incorrect answers. Instead, it examines the 'why' behind student reasoning. This comprehensive framework provides valuable insights into students' thought processes and potential areas of difficulty. This information can then be used to develop targeted instructional strategies that cater to diverse learning styles and promote a deeper understanding of trigonometry for all students.

Instructional Design

Based on the framework, a multifaceted approach to analyzing mathematical reasoning through four key components are employed:

1. *Mathematical Objects:* These are the entities students interact with, such as numbers, shapes, and, in this context, the angle in a right triangle.
2. *Mathematical Activities:* These refer to the actions students take when grappling with trigonometry concepts. The primary activity here is answering problems, focused on a real-world scenario—the "leaning ladder challenge."
3. *Mathematical Processes:* These examine the cognitive and metacognitive strategies students employ when engaging with mathematical tasks. In this activity, students apply prior knowledge and reasoning skills to explore and solve real-world problem using mathematical thinking.
4. *Mathematical Concepts:* These explore the underlying principles students are expected to learn, such as the relationship between the angle of inclination and the stability of the ladder.

Reasoning Approaches

In this study, various reasoning approaches are examined as students engage with a complex real-world problem. Each reasoning approach contributes uniquely to their understanding and supports different aspect of the problem-solving processes:

Inductive reasoning involves students observing how changes in certain conditions affect outcomes through experimentation. By analyzing patterns in these observations, they form general conclusions and make predictions about similar scenarios.

Deductive reasoning comes into play when students apply established mathematical or physical principles to reach specific conclusions. By using known theorems or formulas, they reason from general laws to determine precise outcomes in the problem context.

Abductive reasoning involves students generating plausible explanations for observed phenomena. This form of reasoning encourages creative thinking and critical thinking as they consider multiple possible causes for the behavior they observe, promoting deeper investigation and understanding..

Analogical reasoning is used when students compare the leaning ladder to other familiar objects or situations. For example, by relating the scenario to common objects like ramps or seesaws, they transfer prior knowledge to new contexts, enhancing their conceptual understanding.

Algorithmic reasoning involves breaking down into a complex situation into a series of logical, step-by-step procedures. By systematically considering relevant factors, they develop a structured approach that leads them to a solution efficiently and consistently.

Reasoning Processes

The following reasoning processes are investigated as students worked on the solution to the problem.

Validation. Students physically test various scenarios to observe outcomes directly. This hands-on experimentation provides immediate feedback and helps develop an intuitive understanding of the relationships involved.

Justification. After initial observations, students support their conclusions by connecting their observations to relevant concepts (e.g. balance, friction, or geometric relationships. This process bridges experiential learning (e.g. physical observations) with formal mathematics or underlying mathematical principles.

Proving. Students construct mathematical models using variables that influence the situation. For example, they apply algebra, geometry and trigonometry to derive equations, allowing them to demonstrate why a particular outcome occurs under defined conditions.

Formal proving. For a more rigorous approach, students use formal logic and theorem-proving tools. They define all components of the problem within a formal system and produce proofs that verify conclusions with complete precisions and generality.

The Leaning Ladder Challenge

Background

Proper ladder use is crucial in both occupational and domestic settings, as incorrect placement can lead to serious injuries or fatalities. The angle at which a ladder is positioned is essential for maintaining stability. A ladder set at too steep an angle risks slipping, while a too shallow angle may cause it to tip over. Safety guidelines recommend a ladder angle of approximately 75 degrees from the ground, balancing stability and reach (Simeonov et al., 2012). This angle, also known as the "4 to 1 rule," suggests that the ladder base should be one meter from the wall for every four meters of height (Schaffarczyk, 2017).

Following these guidelines is critical to prevent ladder-related accidents, with studies showing a strong link between incorrect ladder angles and increased fall risk (Smith et al., 2018; Jones & Martin, 2020). Integrating these safety principles into training helps students apply proper ladder placement in real-world situations.

Engaging Scenario

The learning experience centers on a real-world scenario—the "leaning ladder challenge." Students are presented with the following problem:
Imagine you have a 5-meter ladder that you need to lean against a wall to reach a high point safely. Setting the ladder up at the right angle is crucial for stability and safety.

Challenges to Consider

- If the ladder's base is too far from the wall, it might slide out, causing the ladder to collapse.
- If the ladder's base is too close to the wall, there's a danger of the ladder tipping over backward.

The Task

Students are tasked with determining the optimal angle for the ladder to ensure stability and safety.

Guiding Questions to Spark Inquiry

- What happens when the ladder is almost vertical (close to 90 degrees)?
- How does the ladder behave when the angle is smaller than 75 degrees?
- Can you find an angle that balances the risk of sliding and tipping over?

Collaborative Learning Through Group Work

Students work in groups to foster collaboration and knowledge sharing. This allows them to discuss prior knowledge, explore strategies, and build upon each other's ideas.

Expected Learning Outcomes

Through this activity, students are expected to achieve the following outcomes:

- *Apply Trigonometry Concepts*. Students will use their understanding of right triangles and trigonometric ratios to analyze the leaning ladder problem.
- *Develop Problem-Solving Skills*. Students will engage in a logical and systematic approach to determine the optimal angle for the ladder.
- *Refine Critical Thinking Skills*. Students will analyze the consequences of different angles and justify their reasoning through discussions.
- *Recognize Practical Applications*. The leaning ladder challenge will help students see the real-world relevance of trigonometry in ensuring stability and safety.

METHODS

To gain a deep understanding of how students reason in trigonometry and how teachers support this reasoning, a qualitative case study approach was adopted. This method was selected for its capacity to explore complex phenomena in-depth, allowing for a rich, contextualized investigation of student and teacher practices (Yin, 2014). By focusing on a specific classroom or school, this study aimed to uncover the nuanced ways students approach trigonometry problems and how teachers adapt instruction to accommodate diverse reasoning strategies.

To ensure representativeness, cross-sectional observations were conducted in three different mathematics classes within a single school. This strategic approach allowed for capturing student reasoning processes from a diverse student population and across various instructional settings. Notably, the student sample (10 students per class) was carefully chosen to encompass a range of mathematical reasoning abilities.

Participants and Participants Selection

The study involved 127 Grade 10 students, aged 16-17, who volunteered from three classes within the same school. Class 1 had 22 females and 20 males, Class 2 had 27 females and 18 males, and Class 3 had 24 females and 16 males.

To explore the students' problem-solving strategies in depth, five focus groups were formed. Each group was composed of three students, each selected from one of the three classes to ensure a representation of different mathematical reasoning approaches: inductive, deductive, analytical, algebraic, and spatial. The selection was based on their demonstrated problem-solving strategies in response to a specific problem.

This approach ensured a diverse range of perspectives in the discussions, reflecting various problem-solving methods. By including students from each class with distinct reasoning approaches, the focus group discussions provided a comprehensive view of how different mathematical strategies are applied in problem-solving.

Research Instruments

This study employed three primary research instruments to gather comprehensive data on students' reasoning approaches in trigonometry and the instructional strategies used by teachers: analysis of student work, classroom observations, and teacher interviews.

Student work analysis served as a critical instrument to evaluate and understand the mathematical reasoning processes employed by students when solving the problem. A detailed coding scheme was developed to capture the nuances of students' reasoning strategies, ensuring that the analysis accurately reflected their cognitive processes. Inter-rater reliability checks were conducted to enhance the reliability of the coding process.

Classroom observations provided a real-time perspective on teachers' instructional practices and the classroom dynamics. To mitigate observer bias, multiple observers were involved, and detailed field notes were recorded. Observers underwent training to ensure consistent data collection, and inter-observer reliability was assessed.

Teacher interviews were conducted to gain in-depth insights into teachers' instructional strategies and their perceptions of students' reasoning approaches. The interview protocol was designed to align with the study's research questions, and its clarity and structure were refined through pilot testing. Member checking was employed to ensure the accuracy of the interpretations drawn from the interviews.

Data Collection

To gain a rich understanding of student reasoning processes, the researcher employed a multifaceted approach. This involved closely observing student group work while they tackled the assigned trigonometry challenge. Particular attention was focused on groups exhibiting different reasoning approaches. This allowed the researcher to identify and track the emergence of diverse strategies students employ when grappling with the problem.

Following the group work, the researcher conducted focused follow-up discussions with selected student groups. These discussions aimed to elicit in-depth insights into students' collaborative experiences, cognitive processes, and problem-solving approaches. Participants were prompted to articulate their reasoning and justify their problem-solving strategies. Table 1 outlines the specific data collection foci.

Components	Description
Conceptual understanding	How well students grasp the underlying trigonometric concepts relevant to the challenge.
Cognitive strategies	The specific thought processes students utilize to approach the problem, such as visualizing geometric relationships, applying trigonometric formulas, or employing algebraic manipulations.
Collaboration and communication	How effectively students work together within their groups, share ideas, and build upon each other's reasoning.
Metacognitive awareness	The students' level of reflection on their own learning process, their ability to identify areas of strength and weakness, and their approaches to overcoming challenges.

Table 1: Components and Description of Data Collection

These discussions provided a crucial qualitative layer to the study, complementing the observations of group work. By triangulating the data from both sources, the researcher could gain a more comprehensive picture of how students reason in trigonometry and how these reasoning patterns relate to different instructional approaches employed by the teachers.

Data Analysis

To explore the first research question, an in-depth analysis of students' work on trigonometry problems was conducted. The student responses based on the reasoning approaches and processes they employed were categorized based on the conceptualization of these constructs. This categorization allowed for identifying patterns. Then, the students' explanations were examined to uncover the underlying reasoning processes. Through this analysis, both common and diverse reasoning strategies that students used when solving the problem were identified.

To answer the second research question, a thorough analysis of teacher practices using data from classroom observations and teacher interviews was conducted. The observations focused on instructional strategies and activities that encouraged various reasoning approaches. By analyzing this data, a range of instructional methods that teachers used to accommodate and enhance students' diverse reasoning strategies were identified. Additionally, this analysis shed light on how teachers created opportunities for students to develop and refine their reasoning skills in the classroom.

Ethical Considerations

This research was committed to upholding the highest ethical standards to ensure the well-being and rights of all participants. Adhering to core ethical principles, the study prioritized informed consent, confidentiality, and minimizing risks to participants, with a particular focus on protecting students from undue stress or harm.

Both teachers and students participated voluntarily after providing informed consent. The informed consent process clearly outlined the purpose of the study, how data would be used, and participants' right to withdraw at any time without consequence. This ensured that participants fully understood their involvement and were able to make an autonomous decision about their participation, thus respecting their rights and choices.

To safeguard participants' privacy, all data collected underwent a thorough anonymization process. Any identifiable information from student work, interview transcripts, and observation notes was removed to protect participants' identities. Maintaining confidentiality not only helped preserve privacy but also fostered trust between the researcher and participants, encouraging open and honest communication throughout the study.

The researcher implemented specific safeguards to prioritize students' mental and emotional well-being. For instance, all interactions took place in safe, familiar environments, such as classrooms, to foster a sense of security. The researcher also offered breaks during data collection to prevent fatigue and maintained a supportive, non-judgmental tone during interviews and observations. For students who appeared nervous, the researcher provided additional reassurance that there were no "right" or "wrong" answers. By respecting students' natural pacing and preferences, the study aimed to make participation feel as comfortable and stress-free as possible.

Finally, the researcher took proactive measures to monitor and mitigate any signs of stress or anxiety in students during the study. For instance, before starting data collection, the researcher conducted a rapport-building session to familiarize students with the researcher and clarify that the study was not evaluative but exploratory. The study avoided sensitive or potentially distressing topics, instead focusing on general academic and classroom experiences. Additionally, the researcher remained vigilant for any signs of discomfort; in such cases, the researcher immediately offered the option to pause, skip, or withdraw from the activity.

RESULTS

This study examines the various reasoning methods students employ when solving trigonometric problems, focusing on the Leaning Ladder Challenge as a case study.

Mathematical Reasoning Approaches Employed by Students

The research provides insights into the effectiveness and limitations of each approach in mathematical problem-solving. Table 2 presents an overview of reasoning approaches, including deductive, inductive, abductive, analogical, and algorithmic approaches, providing insights into their respective strengths and limitations in the context of trigonometric problem solving.

Reasoning Approach	Student Answer/Behavior	Explanation of Reasoning	Strengths	Limitations
Deductive Reasoning	Relying on rules and formulas (e.g., 4:1 ratio, 76-degree angle)	Applying general principles to specific situations	Provides clear guidelines, efficient	Relies on accurate initial information, may not account for all variables
Inductive Reasoning	Experimenting with different angles, observing patterns	Learning from experience and observation	Fosters hands-on learning, adaptable	Time-consuming, conclusions may be tentative
Abductive Reasoning	Making educated guesses based on observations	Generating hypotheses based on limited information	Quick decision-making, stimulates creativity	Relies on intuition, lacks concrete evidence
Analogical Reasoning	Comparing the ladder to other objects (e.g., ramp, seesaw)	Transferring knowledge from familiar situations	Facilitates understanding, promotes creativity	Depends on relevant analogies, may oversimplify
Algorithmic Reasoning	Using formulas and calculations (e.g., trigonometry)	Applying step-by-step procedures to solve problems	Precise and accurate, reliable	Requires mathematical knowledge, time-consuming

Table 2: Mathematical Reasoning Approaches in Students' Solutions to the Leaning Ladder Challenge

Deductive Reasoning

The results indicate that students were actively engaged in the deductive reasoning, demonstrating their ability to apply mathematical concepts to real-world context, collaborate effectively, and reflect on the significance of their findings. The teacher played a crucial role as a facilitator, guiding student learning and encouraging critical thinking. Below is an excerpt from a conversion among students (*S1, S2, S3, S4*) and teacher (denoted with *T*)

S1: If we need to find the safest angle for the ladder, so let's start with what we know. We have a 5-meter ladder and we want to reach a certain height.
S2: ... we know that if the ladder is too steep, it might slip. So, we need to avoid a very large angle.
S3: Also, if the angle is too small, we won't be able to reach very high.
S4: I remember our teacher saying that there's a safe ratio for ladder placement. It's something like four units out for every one unit up.
S1: That sounds right. If we use that ratio, we can calculate the distance from the wall.
S2: Yes, and then we can use trigonometry to find the angle.
S3: But we should also consider the ground. If it's slippery, we might need to adjust the angle.
S4: We can use what we know about friction to estimate how much the angle should be reduced.
S1: Okay, let's start with the 4:1 ratio and see what angle we get. Then we can adjust later.
T: That's a good start, but remember, we already know the ladder length. What else can we use?
S2: We can use the cosine function to find the angle, since we know the adjacent side (distance from the wall) and the hypotenuse (ladder length).
T: Excellent! Can you explain how you would set up the equation?

Student engagement was evident in their active participation in the problem-solving process. Students demonstrated a strong foundation in mathematical concepts by suggesting the use of the Pythagorean theorem and trigonometric functions. When faced with challenges, they actively sought clarification and guidance from the teacher. For instance, Student A's suggestion to use the Pythagorean theorem prompted the teacher to redirect their focus towards utilizing the given information more effectively.

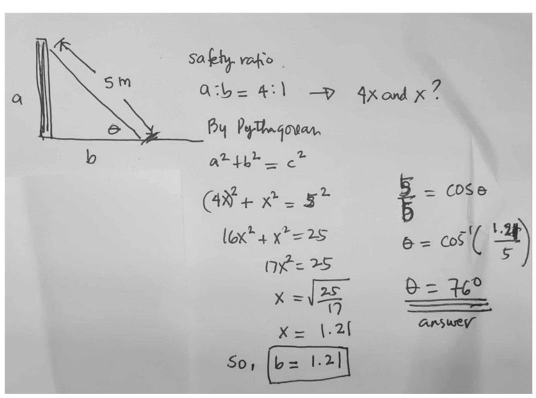

Figure 3: Sample students' calculation to find the angle

T: What does this angle 76 degrees represent in relation to the ladder and the wall?
S3: It's the angle between the ladder and the ground.
T: Correct. Now, does this angle meet the safety requirement?

The teacher's role as a facilitator was crucial in guiding students towards a deductive solution. By asking probing questions, the teacher encouraged students to think critically about the problem and justify their reasoning. For example, asking Student C to explain the meaning of the calculated angle promoted deeper understanding of the mathematical concept in relation to the real-world scenario.

Students often relied on previously learned information, such as the "4 to 1 rule" for ladder safety, demonstrating the application of deductive reasoning. However, the teacher emphasized the importance of accurate foundational knowledge. As students applied general principles to specific challenges, their understanding of the problem deepened. They recognized the influence of variables like ground conditions and load on the optimal ladder angle, indicating a nuanced grasp of deductive reasoning.

The integration of real-world constraints enhanced problem-solving, requiring students to apply mathematical concepts to practical scenarios. The use of calculators facilitated computations, allowing for a greater focus on conceptual understanding and problem-solving strategies.

Inductive Reasoning

Students began by experimenting with different ladder positions, using a miniature model of ladder, systematically collecting data on angle, distance from the wall, ladder height, stability, and reach. For each trial, they calculated the distance-to-height ratio to assess compliance with the safety standard. This process of observation and data collection laid the groundwork for inductive reasoning. An excerpt from a conversation among students (S1, S2, S3, S4(illustrates this reasoning in action:

S2: I wonder what's the best angle for this ladder. If it's too steep, it might slip.
S2: Yes, and if it's too flat, we won't reach the window.
S3: I tried leaning this miniature ladder against the wall at different angles. It seemed safer when it was a bit more flat. So we try using different angles.
S4: I saw a video online where a guy talked about how ladders should be placed. He said something about a safe distance from the wall.

S1: Let's try leaning the ladder at different angles and see what happens. Maybe we can find a pattern.

As illustrated in the following excerpt from a conversation among students (S1, S2, S3, S4) and the teacher (T), students analyzed the collected data and identified patterns among key variables. They concentrated on trials that satisfied safety criteria, observing how angle, stability, and reach were interrelated. Through these observations, they inductively concluded that an angle between 55 and 65 degrees generally offered a safe and effective solution. They understood that angles below 55 degrees might compromise reach, while angles above 65 degrees could jeopardize safety. This generalization process demonstrated the ability to infer general principles from specific observations.

S1: Our 45-degree angle is too close to the wall. It doesn't meet the 1:4 ratio.
S2: We got a 60-degree angle, but it feels a bit too steep. Let's check the ratio.
S3: Our angle is safe, but we're not reaching the window. We need to find a balance.
T: How can you adjust your ladder position to meet both the safety standard and your goal of reaching the window?
S3: We need to find an angle that's steep enough to be safe but not too steep to reach the window. So what about the safety standard that was mentioned at the start of the lesson?

Student discourse revealed a nuanced understanding of the problem's complexities. Students balanced safety concerns with practical objectives, as exemplified by the challenge of achieving desired height while maintaining stability. The teacher's role in guiding students towards a holistic perspective was crucial.

To foster inductive reasoning, controlled experimentation and rigorous data analysis were emphasized. Students systematically manipulated ladder angles and recorded observations. As one student noted, "Iterative adjustments revealed that an angle between 65 to 80 yielded optimal stability." While inductive reasoning provided valuable insights, challenges such as time constraints and uncontrolled variables emerged. Teachers stressed the importance of structured experimentation and data analysis for reliable conclusions.

Abductive Reasoning

The study highlighted the dynamic and iterative nature of abductive reasoning in mathematical problem-solving, particularly in the context of balancing safety and practicality under a 4:1 safety ratio constraint. Students generated initial hypotheses, tested them against real-world constraints, and refined their thinking based on emerging evidence. While all students ultimately arrived at viable solutions, the depth and sophistication of their reasoning varied as illustrated in the following conversation among students (*S1, S2, S3*) and the teacher (*T*). This research underscores the potential of abductive inquiry to enhance critical thinking and problem-solving skills in students.

Initial Hypotheses with Safety Consideration

> *S1: We still think a 45-degree angle is a good starting point, but we need to check if it meets the 4:1 safety ratio.*
>
> *S2: Our 60-degree angle might be too steep to meet the safety regulations. We need to adjust it.*
>
> *S3: We need to find an angle that is both safe and allows us to reach the desired height.*

The initial hypotheses reflected diverse approaches to the problem. Student A hypothesized a 45-degree angle, likely based on prior knowledge or visual estimation. Student B's suggestion of a 60-degree angle may have stemmed from considerations of stability or reach. In contrast, Student C took a more exploratory approach, aiming to find a solution that balanced both safety and practicality.

Teacher probing with safety emphasis

> *T: How can you adjust your ladder position to meet the safety regulations without compromising the ladder's reach?*
>
> *S1: We can try a steeper angle, but we need to make sure it's not too steep.*
>
> *T: Can you explain why your initial angle didn't meet the safety standard?*
>
> *S2: We focused on stability but forgot to consider the distance from the wall.*

As students engaged in experimentation and measurement, they refined their initial hypotheses. A student, after calculating the distance from the wall with a 45-degree angle, realized it did not meet the safety criteria. This led to a revised hypothesis of a steeper angle. Similarly, a measurement from another student showed that the 60-degree angle compromised safety. Through these iterative processes, students demonstrated abductive reasoning by proposing explanations for unexpected results and modifying their conjectures accordingly.

Eventually, the groups converged on a solution that met both the safety requirements and the practical need to reach the desired height.

> *S1: We found an angle of 63 degrees that meets the safety ratio and allows us to reach the window.*
>
> *S2: After several adjustments, we also arrived at a similar angle that complies with the safety regulations.*
>
> *S3: Our initial angle was too shallow, but we were able to find a suitable angle that meets both criteria.*

Students' approach was notably more systematic. By considering the relationship between angle, ladder length, and wall height, they were able to generate a hypothesis that closely aligned with the safety requirements. This suggests a stronger application of mathematical reasoning and problem-solving skills.

The students' approaches, while often relying on visual cues or intuitive judgments, highlighted the strengths and limitations of abductive reasoning. For instance, a student stated, "As I was adjusting the ladder, I noticed that it seemed to stand firm when the angle was somewhere around 70 degrees. I didn't have the exact data, but based on how it looked and felt, I guessed that this was probably the safest angle to use."
Abductive reasoning served as a foundation for students to explore potential solutions to the ladder challenge. However, to enhance the reliability and validity of their conclusions, students must complement their initial inferences with empirical evidence or logical reasoning. This study demonstrates how abductive inquiry, when combined with critical thinking and data analysis, can foster a deeper understanding of mathematical concepts and problem-solving strategies.

Analogical Reasoning

As illustrated in the following except from conversation among students (S1, S2, S3, S4), they identified analogous systems that shared similarities with the ladder scenario. By drawing on familiar experiences, they aimed to transfer existing knowledge to deepen their understanding and guide their problem-solving approach.

S1: This ladder problem reminds me of building a ramp.
S2: How's that?
S1: Well, a ramp needs to have more or less a correct angle to be safe and effective.
S3: And it needs to be strong enough to hold weight.
S4: So, we need to find the perfect slope for the ladder, just like a ramp.

Students demonstrated a propensity for analogical reasoning when confronted with the ladder problem. Common analogies included ramps, bridges, and trees, highlighting the transfer of knowledge from familiar systems to the novel context.

One student response exemplified this approach: "I remembered helping my dad set up a ladder when he was painting our house. He placed the ladder at an angle that looked like around 75 degrees, and it was really stable. So, I tried to mimic that same angle here, thinking it would be safe and balanced". While analogical reasoning facilitated rapid decision-making, its effectiveness depended on the appropriateness of the analogy. Students needed to critically evaluate the similarities and differences between past experiences and the current situation to avoid misconceptions.

To enhance the reliability of analogical reasoning, students were encouraged to explicitly articulate the similarities and differences between the analogical system and the ladder problem. This process facilitated more informed decision-making and reduced the likelihood of erroneous inferences.

The teacher's role in guiding students through the analogical reasoning process was crucial. By prompting students to identify key similarities and differences, the teacher helped them to transfer knowledge effectively. Analogical reasoning proved to be a valuable tool for problem-solving in the ladder challenge. By leveraging prior knowledge from familiar systems, students generated hypotheses and developed a deeper understanding of the problem. However, the effectiveness of this strategy was contingent on students' ability to critically evaluate the appropriateness of the analogy and to explicitly articulate the underlying relationships.

Algorithmic Reasoning

This study examined the application of algorithmic reasoning to determine the optimal angle for a leaning ladder, subject to safety regulations. Specifically, they developed structured, step-by-step procedures to determine a suitable ladder angle. Students like S4 and S5 identified relevant variables, applied the 4:1 safety ratio. And use trigonometric functions to calculate angles. The integration of technology further supported their work, enhancing both computational efficiency and accuracy – highlighting the valuable role of digital tools in mathematical problem-solving.

T: Can you outline the steps involved in solving this problem?
S4: First, we need to find the distance from the wall using the safety ratio. Then, we can use the cosine function to calculate the angle.
T: How can we check if this angle is practical for real-world use?
S5: We can compare the height reached with the desired height.

Teacher-student interactions played a vital role in guiding the process. By asking probing questions, the teacher encouraged students to articulate their thought process and consider the practical implications of the solution. This facilitated a deeper understanding of the problem and the algorithmic approach.

Step-by-Step Procedure

1. *Define variables:*
    - *L = ladder length (5 meters)*
    - *D = distance from the wall to the ladder base*
    - *H = height reached on the wall*

- *A = angle between the ladder and the ground*

2. *Establish the safety constraint:*
    - *D = L / 4*
3. *Apply trigonometric function:*
    - *cos(A) = D / L*
4. *Calculate the angle:*
    - *A = arccos(D / L)*
5. *Check for feasibility:*
    - *Ensure the calculated angle provides sufficient height to reach the desired point.*

The results indicate that algorithmic reasoning is an effective strategy for well-defined problems with clear constraints. By decomposing the problem into smaller, manageable steps, students developed structured and efficient solutions. Technology integration enhanced computational efficiency, expanding problem-solving capabilities.

However, the effectiveness of algorithmic reasoning is contingent upon problem complexity and students' mathematical proficiency. While powerful for well-defined problems, it may be less suitable for complex or open-ended scenarios. A strong mathematical foundation is essential for successful implementation. Thus, balancing algorithmic approaches with other problem-solving strategies for holistic development is important.

Mathematical Reasoning Processes by Students

Based on classroom observations and in-depth focus group discussions with students, this study unveiled a multifaceted spectrum of problem-solving strategies. Students exhibited a wide range of approaches, from concrete and intuitive manipulations of physical representations to more abstract and algorithmic reasoning. These findings underscore the intricate nature of mathematical cognition and highlight the necessity of instructional practices that support diverse problem-solving pathways. Table 3 shows various mathematical reasoning processes students used in approaching the problem, from validation of ladder stability to formal integration of findings into mathematical theories.

Mathematical Reasoning Processes	**Description**	**Student Example**
Validation in Determining Ladder Stability	Initial step in problem-solving; hands-on method to engage with the problem and gain immediate feedback.	Student set up the miniature ladder at different angles and observed its stability.
Justification: Adding Depth to Reasoning	Explaining the reasoning behind a chosen approach or solution; linking initial observations to broader principles or guidelines.	Student advocated for a 75-degree angle based on safety guidelines.

Mathematical Reasoning Processes	Description	Student Example
Proving: Concrete, Theoretically Grounded Conclusions	Offering concrete, systematic conclusions based on mathematical principles; quantitative validation or refutation of initial observations.	Student employed trigonometric calculations to determine the optimal ladder angle.
Formal Proving: Integrating Findings into Existing Theories	Integrating findings into existing or new mathematical theories; elevating problem-solving to theoretically grounded inquiry.	Student formalized a mathematical theory around ladder stability.

Table 3: Mathematical Reasoning Processes in Students' Approaches to the Leaning Ladder Challenge

Validation served as the foundation for exploring ladder stability. Initially, students engaged in hands-on experimentation by adjusting ladder angles on miniature models to collect empirical data. Through this experiential learning, they gained a deeper understanding of the problem, which subsequently informed and strengthened their reasoning.

Moreover, justification enriched students' problem-solving approaches by clarifying the rationale behind their choices. Although safety guidelines provided an initial framework for decision-making, students recognized that real-world contexts often require a degree of flexibility, moving beyond strict adherence to standard protocols.

Additionally, the process of constructing mathematical proofs added a layer of mathematical rigor, essential for validating initial observations. Through trigonometric calculations, students generated concrete evidence to support their findings, thereby enhancing the credibility of their conclusions. Embedding these calculations within established mathematical frameworks demonstrated a high level of reasoning, effectively bridging empirical insight with theoretical understanding.

However, while formal proofs provide a solid foundation, a balance is necessary between theoretical rigor and practical adaptability. Although formal proofs offer reliability, real-world complexities often call for flexible, adaptable problem-solving strategies to effectively address unexpected challenges.

Instructional Strategies Employed by Teachers

This study investigated how teachers can effectively accommodate and leverage diverse student mathematical reasoning in the context of a ladder problem. Findings indicate that students employed a range of reasoning strategies, including deductive, inductive, abductive, analogical, and algorithmic approaches. Teachers responded by tailoring instruction to support these varied styles (See Table 4).

Theme	Description of the Theme	Sample Utterances from Interviews and Probing
Hands-on and Experiential Learning	Integrating physical manipulatives and real-world contexts to enhance student engagement and understanding.	"Hands-on activities provide students with a tangible way to explore mathematical concepts." (Teacher A) ”By engaging in these activities, students had the opportunity to manipulate objects, observe real-world phenomena firsthand, and develop a deeper intuitive understanding of mathematical concepts." (Teacher B)
Scaffolding and Guided Inquiry	Providing structured support and guidance to facilitate student learning and development of diverse reasoning abilities.	“By asking probing questions, teachers encouraged students to articulate their reasoning and reflect on their findings." (Teacher B)
Contextualized Learning	Grounding mathematical concepts in real-world experiences to enhance engagement and understanding.	"When students see how math is relevant to their world, they become more engaged and motivated to learn." (Teacher C "By prompting students to consider how mathematical formulas could represent relationships in a real-world scenario, teachers helped students perceive the practical utility of mathematics." (Teacher B)
Metacognitive Reflection and Discourse	Fostering student reflection and communication about their thinking processes.	"By talking through their thought processes, students can solidify their understanding, identify misconceptions, and learn from their peers." (Teacher A)
Integration of Formal Mathematical Reasoning	Balancing formal and informal reasoning to develop well-rounded mathematical thinkers.	"It's important for students to see how the math they learn in class connects to real-world problems." (Teacher E) "By fostering a balance between formal and informal reasoning, teachers can empower students to approach mathematical problems with flexibility and precision." (Teacher D)

Table 4: Strategies Employed by Teachers

Hands-on and Experiential Learning

The findings demonstrated that these instructional strategies significantly enhanced student engagement, understanding, and problem-solving skills.
Teachers effectively integrated hands-on activities, allowing students to physically manipulate objects and directly observe real-world phenomena. For example, students used miniature ladders to experiment with different angles, exploring ladder stability in a tangible way. As one teacher explained, "Hands-on activities provide students with a tangible way to explore mathematical concepts. It helps them make connections between abstract ideas and real-world applications" (Teacher A). This approach helped students develop a deeper, more intuitive understanding of mathematical concepts compared to traditional instruction.

Teacher questioning further guided student learning, encouraging them to articulate their reasoning and reflect on their findings. One teacher asked, "I noticed you tried different angles. Can

you explain how you decided which angle was best?" (Teacher B). Such probing questions promoted critical thinking and deeper conceptual understanding.
The study supports the integration of hands-on and experiential learning in mathematics instruction, showing that these methods create a more engaging, effective, and inclusive learning environment.

Scaffolding and Guided Inquiry

Teachers used strategic prompts and feedback to facilitate student learning. For instance, when students applied deductive reasoning using the 4:1 rule, a teacher might ask, “How did you apply the 4:1 rule to determine the angle?” This question encouraged students to articulate their thought processes and deepen their understanding.

Similarly, when students engaged in inductive reasoning, teachers supported them by asking, “How can you organize your data to identify patterns or trends?” For those using abductive reasoning, teachers fostered critical thinking by prompting, “What evidence supports your hypothesis?” This approach helped students evaluate alternative explanations and justify their claims. The study emphasized that scaffolding and guided inquiry are indispensable in cultivating diverse mathematical reasoning skills, enabling students to explore, question, and construct their own knowledge.

Contextualized Learning in Mathematics

Teachers effectively used real-world examples and cultural references to bridge the gap between abstract concepts and students' lives. For instance, when a teacher asked, "How does understanding the tightrope walker help us solve the ladder problem?" (Teacher A), students were encouraged to connect familiar concepts to the mathematical challenge. This approach not only deepened students' understanding but also enhanced their appreciation of mathematics.

Furthermore, contextualized learning provided opportunities for students to apply mathematical knowledge to real-world problems. Teachers prompted students to consider how mathematical formulas represent relationships in practical scenarios, such as asking, "How can we use mathematical formulas to represent the relationships between the different parts of the problem?" (Teacher B). This helped students see the practical utility of mathematics. Contextualized learning proved to be a powerful tool for enhancing student engagement, motivation, and understanding in mathematics.

Metacognitive Reflection and Discourse

The study found that fostering metacognitive reflection and discourse is a critical component of effective mathematics instruction. By encouraging students to think about their thinking and communicate their ideas, teachers created an environment that supported the development of higher-order thinking skills.

Teachers prompted students to articulate their reasoning processes and consider alternative perspectives, deepening their understanding and promoting collaborative learning. For instance, a teacher might ask, "Tell me about the process you went through to reach that conclusion. What patterns did you observe?" (Teacher A). Such questions encouraged students to reflect on their problem-solving processes, identify misconceptions, and learn from their peers. These findings highlight the importance of metacognitive reflection and discourse in enhancing students' mathematical problem-solving abilities and overall learning experiences.

Integration of Formal Mathematical Reasoning

A significant finding of this study was the importance of integrating formal mathematical reasoning into instruction. Students demonstrated proficiency in applying formal reasoning techniques, such as trigonometric calculations, when provided with appropriate guidance.

Teachers played a crucial role in facilitating this integration by creating contexts that bridged abstract concepts and real-world applications. For example, when students used the 4:1 ratio for deductive reasoning, teachers prompted them to connect this to formal trigonometric concepts by asking, "Remember that formula we learned about right triangles? How can we apply it here to find the missing side?" (Teacher A). This helped students see the relevance of formal reasoning in practical problem-solving.

DISCUSSION

The *Leaning Ladder Challenge* provided a valuable context for examining the variety of reasoning strategies students use to approach trigonometric problems. This study revealed that students engaged in diverse reasoning methods—deductive, inductive, abductive, analogical, and algorithmic—each offering unique strengths and limitations.

To begin with, deductive reasoning emerged as a common approach, where students applied general principles to specific cases (Jeannotte & Kieran, 2017). This structured method of problem-solving proved beneficial, although it was highly dependent on the accuracy of initial information. As Selden and Selden (1995) caution, incorrect foundational assumptions can lead to flawed conclusions, highlighting a key limitation of this approach.

In addition, inductive reasoning, marked by experimentation and pattern recognition (Polya, 1957; Pellegrino & Glaser, 1984), allowed students to gain insights through systematic observation and data analysis. While this method promoted a deeper understanding, relying solely on inductive reasoning risked overgeneralization without the complementary support of deductive methods (Klauer & Phye, 2008; Lesh et al., 2000).

Furthermore, abductive reasoning, or the generation of hypotheses from limited information (Hidayah et al., 2023; Shodikin et al., 2021; Paavola, 2023; Pedemonte & Reid, 2011), fostered creativity and flexibility. Students using this approach often demonstrated innovative thinking by proposing rapid solutions. However, as noted by Selden and Selden (1995), abductive reasoning's intuitive nature can make it less rigorous, potentially leading to inaccuracies if not substantiated by sufficient evidence.

Moreover, analogical reasoning provided a practical problem-solving tool by drawing parallels between familiar and new situations (Gentner, 1989; Goswami, 1991). Leveraging prior knowledge, students generated creative solutions; however, the success of this approach hinged on the relevance of the analogy and the students' ability to discern meaningful similarities and differences (Vinner, 2002).

Finally, algorithmic reasoning, which involves step-by-step procedures and mathematical formulas (Zazkis & Campbell, 1996), offered precision and structure in well-defined problems. However, its effectiveness was dependent on students' mathematical proficiency, as those less fluent in mathematics faced challenges applying this reasoning method effectively.

Taken together, these findings points out the importance of integrating and validating multiple reasoning strategies in problem-solving. Effective problem-solvers demonstrated adaptability, moving flexibly between approaches to meet the specific demands of the situation—a critical skill for navigating the complexities of the Leaning Ladder Challenge.

Crucial to this problem-solving process were the stages of validation, justification, and proving, each offering distinct contributions. Validation often began with hands-on experimentation, providing immediate feedback and grounding students' understanding of the problem. For example, Student B's manipulation of a ladder to observe its stability underscored the value of experiential learning, reinforcing abstract concepts through real-world experimentation.

As students moved from observation to analysis, justification became essential for refining their reasoning. By connecting their approaches to broader principles or established guidelines, students strengthened their arguments. For instance, some students referenced to safety guidelines demonstrated how aligning observations with recognized standards added robustness to their reasoning.

To further enhance the rigor of their solutions, students engaged in formal mathematical reasoning to derive well-supported conclusions. For example, one particular student used trigonometric calculations to determine a precise solution, demonstrating how quantitative validation can strengthen both the credibility and mathematical soundness of the problem-solving process.

In addition to these findings on student strategies, the study highlighted how teachers adapted their instructional methods to support students' diverse reasoning approaches during the Leaning Ladder Challenge. Effective strategies included hands-on and experiential learning opportunities, such as using miniature ladders to help students connect abstract mathematical concepts to real-world applications. This tangible connection fostered a deeper comprehension and retention of mathematical ideas.

Furthermore, scaffolding and guided inquiry emerged as pivotal instructional approaches. By offering structured support through prompts and feedback, teachers guided students through complex problems, helping them articulate their thought processes and deepen their understanding. This scaffolding effectively bridged the gap between students' current knowledge and desired learning outcomes (Van de Pol et al., 2020).

Equally important was the role of contextualized learning in enhancing student engagement. Teachers incorporated real-world examples and cultural references to make mathematical concepts more relatable, thereby motivating students by demonstrating the practical utility of mathematics. This approach contributed to a more inclusive and meaningful learning environment (Rubel & McCloskey, 2021; Yang et al., 2021).

Moreover, teachers promoted metacognitive reflection and discourse, encouraging students to articulate their reasoning and explore alternative perspectives. This practice fostered higher-order thinking skills and collaborative learning, ultimately enhancing students' problem-solving abilities (Schoenfeld, 2022). By balancing formal and informal reasoning approaches, teachers guided students to connect abstract concepts like trigonometry with practical applications, helping them develop into well-rounded mathematical thinkers.

CONCLUSION

The *Leaning Ladder Challenge* offered a valuable lens into the diverse reasoning strategies employed by students in tackling a trigonometric problem. Students exhibited a rich repertoire of approaches, including deductive, inductive, abductive, analogical, and algorithmic reasoning. These findings align with previous research highlighting the multifaceted nature of mathematical problem-solving (e.g., Schoenfeld, 2022).

Teachers played a pivotal role in fostering these diverse reasoning approaches. By incorporating hands-on experiences, scaffolding student thinking, and creating opportunities for metacognitive reflection and discourse, educators created a supportive learning environment. This aligns with research emphasizing the importance of contextualized learning (Brown et al. 1989), guided inquiry (Vygotsky, 1978), and metacognition (Flavell, 1979) in mathematics education.

The integration of formal mathematical reasoning into instruction was also crucial. While less frequently observed, its application demonstrated the potential for precise and accurate problem-solving. However, a strong foundation in mathematics is essential for effective use (Zazkis & Campbell, 1996).

Teachers play a pivotal role in fostering this flexibility by creating learning environments that prioritize experimentation, justification, and critical thinking. By accommodating and leveraging diverse reasoning approaches, teachers can promote deeper learning and enhance mathematical proficiency (Cardino & Cruz, 2020). The Leaning Ladder Challenge served as a catalyst for exploring the multifaceted nature of mathematical reasoning, equipping students with tools to navigate complex problems effectively. By understanding the nuances of student reasoning, teachers can tailor instruction to meet the needs of all learners and foster a deeper appreciation for the beauty and utility of mathematics.

Future Research Directions

Future research should investigate the intricate relationship between these reasoning modes and students' mathematical identities. Future research should delve deeper into the factors influencing the selection and application of these reasoning strategies**.** Investigating how students develop these skills over time, the impact of cultural and socioeconomic factors, and the development of assessment tools to measure reasoning proficiency are essential next steps. By understanding the nuances of student reasoning, educators can tailor instruction to optimize student learning and prepare them to become confident, adaptable problem-solvers. Moreover, longitudinal studies are necessary to ascertain the enduring impact of these instructional approaches on students' problem-solving abilities and overall mathematical disposition.

To fully realize the potential of these findings, sustained professional development is essential. Teachers require ongoing support in identifying, nurturing, and leveraging diverse reasoning strategies within their classrooms. By cultivating a culture of mathematical inquiry and exploration, educators can empower students to become proficient, critical thinkers and problem-solvers.

Limitations

The "leaning ladder challenge" offers a practical and engaging context for applying trigonometry, yet it presents several limitations. This instructional activity simplifies the complexities of real-world ladder use by excluding factors such as wind, surface traction, and the ladder's material properties, which significantly impact ladder stability. Furthermore, it assumes static conditions, overlooking dynamic forces that could arise during actual ladder use. The scenario also does not consider human factors like the user's weight, experience, or the unique conditions of specific tasks, which are critical in real-life applications. While the challenge promotes ladder safety awareness, it does not cover other essential safety considerations inherent to working at heights.

Drawing on the instructional design principles of Jeannotte and Kieran (2017), this challenge seeks to foster a deeper understanding of trigonometric concepts through real-world problem-solving and collaborative learning. By grounding trigonometry in a familiar scenario, the activity aims to strengthen students' mathematical reasoning and their ability to apply theoretical concepts to practical situations.

REFERENCES

[1] Aiyub, A., Suryadi, D., Fatimah, S., Kusnandi, K., & Abidin, Z. (2024) Investigation of Students' Mathematical Thinking Processes in Solving Non-routine Number Pattern Problems: A Hermeneutics Phenomenological Study. *Mathematics Teaching Research Journal, 16* (1). https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2024/04/3.pdf

[2] Asomah, R. K., Agyei, D. D., Ntow, F. D., & Benning, I. (2023). Hypothetical approach to the teaching of trigonometric concepts using cooperative learning. *Education Research International*. https://doi.org/10.1155/2023/2051776

[3] Balacheff, N. (1988). A study of students' proving processes at the junior high school level. In *Second UCSMP international conference on mathematics education*. NCTM. https://hal.science/hal-01652045v2

[4] Bao, L., Koenig, K., Xiao, Y., Fritchman, J., Zhou, S., & Chen, C. (2022). Theoretical model and quantitative assessment of scientific thinking and reasoning. *Physical Review Physics Education Research*, *18*(1), 010115. https://doi.org/10.1103/PhysRevPhysEducRes.18.010115

[5] Battista, M. T. (1999). Fifth graders' enumeration of cubes in 3D arrays: Conceptual progress in an inquiry-based classroom. *Journal for Research in Mathematics Education, 30*, 417-448. https://doi.org/10.2307/749708

[6] Bressoud, D. M. (2010). Historical reflections on teaching trigonometry. *Mathematics Teacher, 104*(2), 106-112. https://doi.org/10.5951/MT.104.2.0106

[7] Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *1989*, (1), 32-42.

[8] Calavia, M. B., Blanco, T., & Casas, R. (2021). Fostering creativity as a problem-solving competence through design: Think-Create-Learn, a tool for teachers. *Thinking skills and creativity*, *39*, 100761. https://doi.org/10.1016/j.tsc.2020.100761

[9] Cardino Jr, J. M., & Cruz, R. A. O. D. (2020). Understanding of learning styles and teaching strategies towards improving the teaching and learning of mathematics. *LUMAT: International Journal on Math, Science and Technology Education*, *8*(1), 19-43. https://doi.org/10.31129/LUMAT.8.1.1348

[10] Cavanagh, M. (2008). Trigonometry from a different angle. *The Australian Mathematics Teacher, 64*(1), 25-30. https://search.informit.org/doi/10.3316/informit.137728644563645

[11] de Villiers, M. (1999). The role and function of proof with Sketchpad. In M. deVilliers (Ed.), *Rethinking proof with Sketchpad* (pp. 3–10). Key Curriculum. https://carmamaths.org/resources/jon/Preprints/Papers/JMB%20solo/ICMI%2019/Related/mdv%20proof.pdf

[12] Díaz Quezada, V., & Sepúlveda Albornoz, M. (2023). Validation of an Instrument to Assess Deductive Reasoning in Solving Types of Problems. *International Journal of Engineering Pedagogy*, (7). https://doi.org/10.3991/ijep.v13i7.40153

[13] Duval, R. (1991). Structure du raisonnement déductif et apprentissage de la démonstration [Structure of deductive reasoning and learning of proof]. *Educational Studies in Mathematics, 22*(3), 233–261.

[14] Gentner, D. (1989). The mechanisms of analogical learning. In S. Vosniadou & A. Ortony (Eds.), *Similarity and analogical reasoning* (pp. 199–241). London: Cambridge University Press.

[15] Gholami, H. (2022). Improving the performance of mathematics teachers through preparing a research lesson on the topic of maximum and minimum values of a trigonometric function. *Mathematics Teaching Research Journal, 14*(2), 26-43. ERIC (EJ135059). https://eric.ed.gov/?id=EJ1350529.

[16] Gillman, L. (1991). π and the limit of (sin α)/α. *The American Mathematical Monthly, 98*(4), 346-349. https://doi.org/10.1080/00029890.1991.12000766

[17] Gómez-Veiga, I., Vila Chaves, J. O., Duque, G., & García Madruga, J. A. (2018). A new look to a classic issue: Reasoning and academic achievement at secondary school. *Frontiers in psychology, 9*, 400. https://doi.org/10.3389/fpsyg.2018.00400

[18] Goswami, U. (1991). Analogical reasoning: What develops? A review of research and theory. *Child development, 62*(1), 1-22. https://doi.org/10.1111/j.1467-8624.1991.tb01511.x

[19] Gray, M. E., & Holyoak, K. J. (2021). Teaching by analogy: From theory to practice. *Mind, Brain, and Education*, *15*(3), 250-263.

[20] Gunderson, D. S., & Rosen, K. H. (2010). *Handbook of mathematical induction*. Boca Raton: CRC Press LLC.

[21] Gyan, R. K., Ayiku, F., Atteh, E., & Adams, A. K. (2021). The effect of constructivism on students' performance in solving mathematical problems under trigonometry. *Asian Journal of Education and Social Studies*, *19*(2), 1-18. https://doi.org/10.9734/AJESS/2021/v19i230458

[22] Hamers, J. H., de Koning, E., & Sijtsma, K. (1998). Inductive reasoning in third grade: Intervention promises and constraints. *Contemporary Educational Psychology, 23*(2), 132-148. https://doi.org/10.1006/ceps.1998.0966

[23] Harel, G., & Weber, K. (2020). Deductive Reasoning in Mathematics Education. In: Lerman, S. (eds) Encyclopedia of Mathematics Education. Springer, Cham. https://doi.org/10.1007/978-3-030-15789-0_43

[24] Herbert, S., & Bragg, L. A. (2021). Factors in a professional learning program to support a teacher's growth in mathematical reasoning and its pedagogy. *Mathematics education research journal*, *33*(3), 409-433. https://doi.org/10.1007/s13394-020-00310-5

[25] Hidayah, I. N., Irawati, S., Agung, M., & Sa'dijah, C. (2023). Creative Conjecture: Abductive Reasoning to Generate Some Ideas in Algebra. *Mathematics Teaching Research Journal*, *15*(1), 108-126.

[26] Hiebert, J., & Stigler, J. W. (2023). Creating practical theories of teaching. In *Theorizing teaching: Current status and open issues* (pp. 23-56). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-031-25613-4_2

[27] Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics, 96*, 1-16. https://doi.org/10.1007/s10649-017-9761-8

[28] Klauer, K. J., & Phye, G. D. (2008). Inductive reasoning: A training approach. *Review of educational research, 78*(1), 85-123. https://doi.org/10.3102/0034654307313402

[29] Kohlmeier, J., & Saye, J. (2019). Examining the relationship between teachers' discussion facilitation and their students' reasoning. *Theory & Research in Social Education, 47*(2), 176-204. https://doi.org/10.1080/00933104.2018.1486765

[30] Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. Kelly & R. Lesh (Eds.). *Handbook of research design in mathematics and science education* (pp. 591–646). Mahwah, NJ: Lawrence Erlbaum Associates, Inc. eBook ISBN 9781410602725

[31] Molnár, G., Greiff, S., & Csapó, B. (2013). Inductive reasoning, domain specific and complex problem solving: Relations and development. Thinking skills and Creativity, 9, 35-45. https://doi.org/10.1016/j.tsc.2013.03.002

[32] Paavola, S. (2023). Abduction as a Logic of Discovery. In *Handbook of Abductive Cognition* (pp. 43-60). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-031-10135-9_9

[33] Patterson, C. L., Parrott, A., & Belnap, J. (June 2020). Strategies for assessing mathematical knowledge for teaching in mathematics content courses. In A. Appova, R. M. Welder, and Z. Feldman, (Eds.), Supporting Mathematics Teacher Educators' Knowledge and Practices for Teaching Content to Prospective (Grades K-8) Teachers. Special Issue: The Mathematics Enthusiast, ISSN 1551-3440, vol. 17, nos. 2 & 3, pp. 807–842. ScholarWorks: University of Montana. https://scholarworks.umt.edu/tme

[34] Pedemonte, B., & Reid, D. (2011). The role of abduction in proving processes. *Educational Studies in Mathematics, 76*, 281-303. https://doi.org/10.1007/s10649-010-9275-0

[35] Pellegrino, J. W., & Glaser, R. (1984). Analyzing aptitudes for learning: Inductive reasoning. https://eric.ed.gov/?id=ED243781

[36] Polya, G. (1945). How to solve it: A new aspect of mathematical method. Princeton University Press. National Council of Teachers of Mathematics (NCTM). (2000). Principles and standards for school mathematics. NCTM.

[37] Pruner, M. J. (2023). *Complexity and collaborative problem solving*. https://summit.sfu.ca/item/36587

[38] Quinlan, C. (2004). Sparking interest in trigonometry. *The Australian Mathematics Teacher, 60*(3), 17-20. https://files.eric.ed.gov/fulltext/EJ717841.pdf

[39] Richland, L. E., Zur, O., & Holyoak, K. J. (2007). Cognitive supports for analogies in the mathematics classroom. *Science*, 316(5828), 1128–1129. https://doi.org/10.1126/science.1142103

[40] Rittle-Johnson, B., Siegler, RS & Alibali. (2001). Developing conceptual understanding and procedural skills in mathematics: An iterative process. *Journal of Educational Psychology 93*(2), 346-362.

[41] Romberg, T. (2000). Changing the teaching and learning of mathematics. *Australian Mathematics Teacher, 56*(4), 6-9. https://search.informit.org/doi/epdf/10.3316/aeipt.106031

[42] Rubel, L. H., & McCloskey, A. V. (2021). Contextualization of mathematics: which and whose world?. *Educational Studies in Mathematics*, *107*(2), 383-404. https://doi.org/10.1007/s10649-021-10041-4

[43] Sánchez, M. A., Gómez, C. P., del Rincón, T. O., & Musonda, T. M. (2023). Designing a Theoretical Proposal Using Problem–Based Learning to Improve Learning of Trigonometric Ratios among Grade Eleven Students. *American Journal of Educational Research*, *11*(2), 53-78. https://doi.org/10.12691/education-11-2-5

[44] Schaffarczyk, K. (2017). *Factors contributing to non-occupational falls from ladders in men 50 years and over*(Doctoral dissertation, UNSW Sydney). https://doi.org/10.26190/unsworks/20191

[45] Schoenfeld, A. H. (2022). Why are learning and teaching mathematics so difficult?. In *Handbook of cognitive mathematics* (pp. 1-35). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-030-44982-7_10-1

[46] Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational studies in mathematics*, *29*(2), 123-151. https://doi.org/10.1007/BF01274210

[47] Shield, M. (2004). Formal Definitions in Mathematics. *The Australian Mathematics Teacher, 60*(4), 25-28. http://eprints.qut.edu.au/view/person/Shield,Malcom.html

[48] Siyepu, S. W. (2015). Analysis of errors in derivatives of trigonometric functions. *International journal of STEM education*, *2*, 1-16. https://doi.org/10.1186/s40594-015-0029-5

[49] Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge university press.

[50] Sfard, A. (2012). Introduction: Developing mathematical discourse—Some insights from communicational research. *International Journal of Educational Research, 51*, 1-9. https://doi.org/10.1016/j.ijer.2011.12.013

[51] Shodikin, A., Purwanto, P., Subanji, S., & Sudirman, S. (2021). Students' thinking process when using abductive reasoning in problem solving. *Acta Scientiae*, *23*(2), 58-87. https://doi.org/10.17648/acta.scientiae.6026

[52] Simeonov, P., Hsiao, H., Kim, I. J., Powers, J. R., & Kau, T. Y. (2012). Factors affecting extension ladder angular positioning. *Human factors*, *54*(3), 334-345. https://doi.org/10.1177/001872081244580

[53] Star, J. R. (2005). Reconceptualizing Procedural Knowledge. *Journal for Research in Mathematics Education*, *36*(5), 404–411.

[54] Thompson, P. W. (2008). The design of tasks in support of teacher's development of coherent mathematical meanings. *Journal of Mathematics Teacher Education, 10*, 415-432. http://www.pat-thompson.net/PDFversions/2008ConceptualAnalysis.pdf.

[55] Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education, 10*, 415-432. https://doi.org/10.1007/s10857-007-9054-8

[56] Toulmin, S. E. (2003). *The uses of argument*. Cambridge university press.

[57] Vale, C., Widjaja, W., Doig, B., & Groves, S. (2019). Anticipating students' reasoning and planning prompts in structured problem-solving lessons. *Mathematics Education Research Journal*, *31*, 1-25. https://doi.org/10.1007/s13394-018-0239-5

[58] van de Pol, J., van Loon, M., van Gog, T., Braumann, S., & de Bruin, A. (2020). Mapping and drawing to improve students' and teachers' monitoring and regulation of students' learning from text: Current findings and future directions. *Educational Psychology Review*, *32*(4), 951-977. https://doi.org/10.1007/s10648-020-09560-y

[59] van Laren, L. (2012). Mathematics teachers at the intersection of policy interventions: What practising teachers offer as challenges in teaching trigonometry. *Education as Change*, *16*(2), 203-215. https://doi.org/10.1080/16823206.2012.745726

[60] Vinner, S. (2002). The Role of Definitions in the Teaching and Learning of Mathematics. In: Tall, D. (eds) Advanced Mathematical Thinking. Mathematics Education Library, vol 11. Springer, Dordrecht. https://doi.org/10.1007/0-306-47203-1_5

[61] Weber, K. (2005). Students' understanding of trigonometric functions. *Mathematics Education Research Journal*, *17*(3), 91-112. https://doi.org/10.1007/BF03217423

[62] Yang, Z., Yang, X., Wang, K., Zhang, Y., Pei, G., & Xu, B. (2021). The emergence of mathematical understanding: Connecting to the closest superordinate and convertible concepts. *Frontiers in Psychology*, *12*, 525493. https://doi.org/10.3389/fpsyg.2021.525493

[63] Yin, R. K. (2014). *Case study research: Design and methods*. Sage.

[64] Yu, J., Kreijkes, P., & Salmela-Aro, K. (2022). Students' growth mindset: Relation to teacher beliefs, teaching practices, and school climate. *Learning and Instruction*, *80*, 101616. https://doi.org/10.1016/j.learninstruc.2022.101616

[65] Zazkis, R., & Campbell, S. (1996). Prime decomposition: Understanding uniqueness. *The Journal of Mathematical Behavior*, *15*(2), 207-218. https://doi.org/10.1016/S0732-3123(96)90017-6