Reading Comprehension of the Mean Value Theorem in Engineering Students

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Abstract: In this article, we analyzed the levels of reading comprehension: literal, inferential, and critical, using a diagnostic test about the understanding of the Mean Value Theorem (MVT) in engineering students of the Universidad de La Salle in the Calculus I lecture (Differential Calculus). The objectives of this article are to identify in which of these three levels of comprehension a student is in order to serve like an input to the teacher or the university to carry out an improvement plan. The other aim is to collaborate to narrow the gap between the analysis of the comprehension of mathematical theorems and comprehension of expository texts. The applied written test had seven questions created to recognize how students analyzed the hypotheses, how to elaborate graphs from the theorem assumptions and, finally, if they make an example to validate the theorem. One of the interesting results was the fact that there were no changes in the understanding of the MVT for students with some prior knowledge, contrary to what was observed with those studying it for the first time. The intervention of the teacher is important in the reading process in at least three moments: before, at the time of reading and after reading, considering motivation a primary characteristic of the process.

Keywords: Learning processes, Reading Comprehension, Mathematics, Calculus, the Mean Value Theorem

INTRODUCTION

Teaching and learning differentiation of a function and some calculus applications (like rate of changes) requires special attention in university courses. Studies have shown students have problems trying to understand and relate the concepts involved in these subjects. For example, Hitt and Dufour (2021) mention some difficulties in the cognitive processes that a student often uses to understand differentiation and its relations to the concept of speed.

To find solutions, Dolores-Flores, Rivera-López and García-García (2019) explore the importance of mathematical connections with the concept of exchange rate. On the other hand, Dawkins and Epperson (2014) investigated interactions between calculus learning and problem-solving in the context of two first-semester undergraduate calculus courses.

Algebraic fluency should be promoted from the elementary levels. Thus, from these scenarios, practices that support algebraic reasoning must be prioritized, where analytical and structural processes are emphasized, which can support the meaningful learning of algebra in students (Ellis & Özgür, 2024). This would support the translation of statements (going from everyday language to mathematical language) and the development of variational thinking at higher education levels.

Talking about this subject, at different levels of higher education, fundamental mathematical skills are required to successfully face subsequent courses. Particularly in differential and integral calculus courses, given that many of the concepts and procedures studied in these areas are applied to everyday contexts, students must possess prior knowledge and certain skills that help them solve problems related to the derivative of a function and the definite or indefinite integral. This shows the importance of relating the derivative in concrete situations and problems, and ensuring that the student tends to generate a cognitive interest and foster a deeper understanding of differential calculus (Dagan, et al., 2018).

The difficulties arise in both directions (teaching and learning). Some authors state that practices have a greater impact on learning and professional development if the teacher is provided with the appropriate materials for their design and instruction (Bognar, et al., 2024). Others emphasize that the student's mood or motivational state and attitude play a fundamental role in their learning of mathematics (Geisler, et al., 2023; Schukajlow, et al., 2023).

We know about the low level of reading comprehension developed in many students, especially in the mathematical field. Certainly, several students don't have a deep understanding of definitions, theorems, ideas, and hypotheses addressed in mathematical lectures. From here arises the need to delve into this problem with a group of first semester students from La Universidad de La Salle (Colombia) and try to offer a solution.

The Universidad de La Salle applies every semester a diagnostic test to the new students of the Engineering programs. It detects deficiencies in all levels of reading comprehension of expository texts in the mathematics field. In the same way, the problems visualized by teachers in classroom show that a good percentage of this population have troubles when reading, interpreting, and later applying the theorems that are introduced in the different calculus courses. This type of reading comprehension inconvenience hinders the correct appropriation of the basic concepts; the hypotheses raised; the theorems' applications that are essential tools in subsequent mathematics courses or even in engineering lectures. Not to mention the drawbacks they have in reading the statements of problem applications (Montero & Mahecha, 2020).

Thus, this research sought to diagnose the levels of reading comprehension of undergraduate students of engineering programs. For this, we considered the Mean Value Theorem (MVT) as the starting point of the study. Then we try to point out student appropriation, and we draw attention to student requirements; specifically, the mastery of multiple types of mathematical thinking, as well as the development of reading comprehension skills of expository texts, to fulfill this goal.

On the other hand, it is necessary to indicate this theorem is introduced in the final part of the differential calculus course, allowing most of the students to already be familiar with the calculus notation. So, the research question we try to answer is: *How to identify the level of reading comprehension of the differential calculus student on the Mean Value Theorem?*

In this research, we got the MVT statement proposed in Larson and Edwards (2016, p. 170):

Sean a y b números reales y f una función continua sobre $[a, b]$, y derivable en (a, b). Entonces existe un número c en (a, b) tal que

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

We keep the Spanish version because we used it in the applied written test. The statement of the MVT by the same author (Larson & Edwards, 2010, p. 212) in English version is:

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Historically, this theorem was first proved by the Italian mathematician and physicist, Giuseppe Lagrangia (also known as Joseph-Louis Lagrange), during the 18th century. His ingenuity and mathematical skills produced important contributions in almost all branches of pure mathematics, and even helped the development of analytical mechanics and astronomy (McCormack, 2008).

THEORETICAL FRAMEWORK

The concepts of reading comprehension and levels of reading comprehension are the basis for this research.

Reading Comprehension

There is much academic research focused on reading and reading comprehension. Some studies affirm that the low levels of reading comprehension, together with the student laziness and limited parental involvement, are a mixture that necessarily leads to academic failure (Murcia & Henao, 2015). Other investigations promote strategies that enhance reading comprehension in students of all educational levels. They assumed that reading and understanding transcends the simple activity of decoding words and phrases, because these skills are associated with cognitive and metacognitive processes. Furthermore, we have to remember the influence of the social, cultural, and educational media of each student. There is also research on the significant difference between the number of "correct answers of the students when developing algorithms and the number of correct answers when solving problems involving similar algorithms" (Montero & Mahecha, 2020, p. 3), whose have less correct answers have lowest reading comprehension in mathematical sentences.

Reading is a complex activity, so "reading means, for the knowledge of the 21st century, to *be located* in the international context, and *to be* in accordance with its advances" (Durango, 2017, p. 158). For Goyes (2016), reading is an effort that occurs in the *biconditionality* of intentions between the reader and the text. This relation makes possible many meanings of the same object. However, Cairney (2002) adds a third element: the contextual factors. In this triangular approach between the *readers* who reads a text written by an *author* in their respective *particular contexts*, the reading comprehension has its genesis. It is necessary to clarify that none of these three components is more important than the other one, due to they are strongly correlated. Consequently, the reader is the one who turns these three elements and who "processes, criticizes, contrasts and values the information, who enjoys it or rejects it, who gives meaning and sense to what he reads" (Solé, 2001, p.9).

From this perspective, a text has a meaning borrowed from the author, and based on this, Solé (2001) defines *efficient reading* as the process that occurs when the meaning extracted by the reader matches what the author intended to communicate. So *efficient reading* is not a technique for fast reading, it means reading with purpose, understanding and strategy. Again, this emerges when, both the author and the reader use the same codes: "linguistic systems, lexicon, syntax, semantics, narrative grammar, pragmatics, etc." (Cairney, 2002, p. 31). Therefore, *efficient reading* provides a structure that facilitates the measurement of reading comprehension processes. Finally, context influences both the author and the reader. Indeed, the context is one of the most important keys in the construction of meaning. Thus, it is impossible to get a correct interpretation about an object without thinking on the contextual factors, such as culture, historical moment, and social context.

Levels of Reading Understanding

Levels of reading understanding take place in the reading process, which the three previously stated actors, text, reader, and context are involved. Undoubtedly, we reach these levels by using our previous knowledge. In this order of ideas, Strang (1965), Jenkinson (1976), and Smith (1989) describe three levels of reading comprehension: literal, inferential, and critical. However, authors such as Durango (2017) classify it as literal, inferential, and critical-intertextual. In this written, we will use the first classification. Namely:

Literal Level

At the literal level, the student must make a recognition of everything that appears explicitly in the text. So that, the reader can reconstruct the text with his own words in which the structure of the text is included.

Inferential Level

A reader who reaches the inferential level must identify relationships that are not explicitly in the text. Inferential or interpretive reading involves both logical deductions and conjectures. These conclusions could be made from certain data or using other suppositions.

Critical Level

The critical level is considered the ideal level, since it implies a formation of own judgments of subjective character that involve identification with the characters, with the author and the context. Reading at this level has an evaluative character, which takes account of the training of the reader, as well as criteria and knowledge on the subject one read.

Visualization

For the reading comprehension of the MVT it is necessary that the student have previous knowledge such as: real numbers, intervals, Cartesian plane, slope, function, continuity, and derivative. Besides, they need to know how objects are related in the construction of a mathematical statement, so in a theorem. This relationship requires actions of representation and visualization of mathematical objects. Thus, in accordance with Bråting (2012, p. 17), “visualizations can certainly be sufficient for convincing oneself of the truth of a statement in mathematics, provided that one has sufficient knowledge of what they represent”.

In this work, the articulation of the treatment of hypotheses and graphs for the reading comprehension of the MVT has a central role. The teaching of this theorem begins with mediation by the teacher where the reading and writing of the hypothesis of Rolle's Theorem is introduced and the graphic representation is visualized by each student that satisfies these hypotheses.

METHOD AND RESULTS

The population on which the research is carried out focuses on a convenient sample of 29 first-semester students of the Engineering programs of the Universidad de La Salle, to whom a questionnaire (based on the MVT) was applied. The purpose was to measure the level of comprehension and appropriation of the mathematical language that is formally used at the university lectures. We didn’t use a control group because the instrument applied has a geometrical point of view employed in order to improve the understanding of the MVT. The traditional form to explain this theorem at the university selected is to introduce it by an algebraic and algorithmic way, so the students of a control group had not had the same mathematical tools.

The research was developed in three stages. The first one focused on the activation of previous knowledge, in the second one, the instrument was applied, and in the final stage the results were analyzed. For the validity of this instrument, we used a methodology with expert triangulation made by professors with more than 15 years of teaching experience, and as a result, one author of this article made a presentation in the CIAEM 2023[1] about reading comprehension in mathematics.

1. Activation of Prior Knowledge

The activation of the previous knowledge of the students allows to identify the capacities, and in a certain way, the attitudes that the students assume before the new information that arrives. According to Solé (2001), when the information is not understood, to a large extent, it is because there is no adequate establishment of such knowledge, or because the text is not adequately developed for its understanding. In this context, if a teacher wanted to implement this didactic proposal, he could develop it in the following way:

a) Introduce Rolle's Theorem in class, review some concepts, and generate spaces where the student can create hypotheses and ideas with the help of the visualization of graphic behaviors.

b) Recognize the level of reading comprehension through inquiry into concepts such as domain, interval, continuous function, derivative of a function, tangent line, among others.

c) Finally, each student is asked for their own graphic description of the theorem.

2. Application of the Instrument

In the second stage, a written instrument of seven questions was applied. This instrument was a questionnaire divided in the following manner: Question (1) asks the student about the previous knowledge of the MVT; question (2) asks what they hope to learn from the theorem. These two questions were taken as the basis to identify the degree of progress a student can reach after reading and trying to use the theorem without teacher measurement. Next, the theorem is stated to them and questions (3) and (4) are asked, in order to identify the vocabulary, lexicon, previous mathematical concepts and key ideas. That is, the third question allows us to recognize, through the description itself, what the student conceives of the theorem. Question (4) inquires about the level of decoding of the theorem through graphical representations. Thus, question (3) indicates whether the student reaches the *literal level* and question (4) indicates whether the *inferential level* was reached.

[1] https://xvi-ponencias.ciaem-iacme.org/index.php/xviciaem/xviciaem/paper/view/2094

Questions (5), (6) and (7) were designed to measure the *critical level*, since they allow analyzing the connection of the themes, the validity of the information provided and the formation of judgments.

3. Results

In this case, the implementation of the exercise was carried out in a Differential Calculus course with 29 students from engineering programs. The assessment was carried out in two steps, with the answers to the questions (1) and the following scale was established:

• The student has no knowledge about the MVT

• The student recognizes some objects involved with the MVT (but not all).

• The student evidences the objects on the MVT, but does not relate them correctly

• The student recognizes the objects required for the construction of the MVT and writes them correctly.

Of the 29 students, 18 had no prior knowledge of the mean value theorem. Statements such as:

Original Statements	Translated Statements
“Es el punto medio de una función”	"It is the midpoint of a function"
“No sé nada sobre este teorema”	“I don't know anything about this theorem”
“Sé que tiene tres hipótesis (reglas) que se deben cumplir, sino se cumplen las tres no sería teorema del valor medio (Rolle)”	"I know that it has three hypotheses (rules) that must be fulfilled, without these, it would not be the mean value theorem (Rolle)"
“Que la pendiente de la recta tangente debe ser 0, que f(a)=f(b) y al ser la pendiente 0, es derivable”	"That the slope of the tangent line must be 0, that f(a)=f(b) and since the slope is 0, it is differentiable"

Table 1: Student interpretations

Of the 29 students, 10 were identified who recognized some objects involved, finding responses such as: "It is a continuous function, which is differentiable and has a minimum point c, in which its derivative is 0"; only one that recognized all the objects, but did not relate them correctly, an example of student writing in this case was: “It meets the requirements such as: it must be continuous, it must be differentiable, the point must be parallel. It must not be cut at any point”. And no student showed complete and correct prior knowledge.

However, after reading about the Mean Value Theorem, the answers to questions (3) were classified as *literal level* if only an adequate paraphrasing of the Mean Value Theorem was found. The *inferential level* was assigned if both the answer to question (3) as well as the graph in (4) were adequate. And finally, a *critical scope* was assigned if from question (3) to (7) the answers were adequate. It was found that 15 of the 29 failed to extract information from the reading, text such as:

• "It is the subtraction of the points [a, b] with the intention of finding the derivative of f´(c) where c would be the mean value"

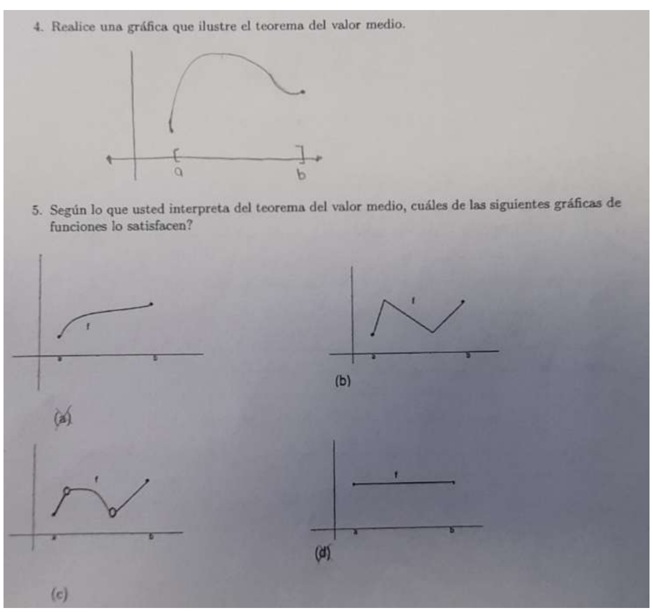
• "That a and b are mean value intervals and are differentiable and c is the variable that is in function f, but is also in a ratio of a and b"

We also found 7 students achieved *literal level*. We can read phrases like:

• "That when a and b are real numbers and there is a continuous function on it and the function is differentiable, c exists in the plan, that is, it has slope"

• "f is continuous on the interval [a, b], f is differentiable on (a, b), this implies that c lies within (a, b) such that its derivative is the subtraction of $f(b) - f(a)$ over $b - a$. It implies that the functions may be different."

We saw that 6 students reached the inferential level. The following graphic responses were obtained to questions (4) and (5):



Translate statements
4. Make a graph that illustrates the mean value theorem.
5. Based on your interpretation of the mean value theorem, which of the following function graphs satisfies it?

Figure 1: Student responses to questions (4) and (5)

For question (4) it should be noted that the example in Figure 1 is different from the particular application of Rolle's Theorem.

And only one student reached the critical scope, this case was classified as atypical within the participating group, because he is one of the students who started from ignorance of the theorem, but the whole process led him to a good application:

> Question (1): "I know that the mean value theorem is used to find the derivative of the central or intermediate point of the interval in a function f(x) continuous and differentiable on said interval, which can be found from when the derivative gives 0, in addition to being used as the basis of Rolle's theorem".

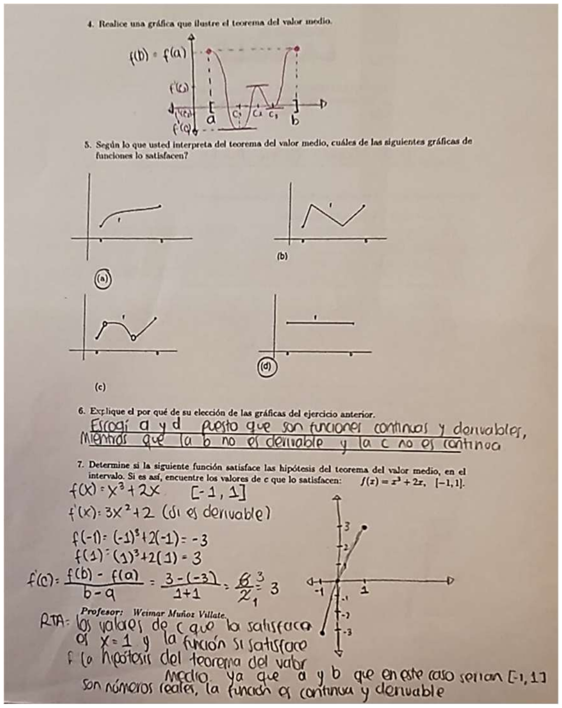
After reading the theorem, the student indicates in Question (3):

> I learned that there is an average value of a continuous function f(x) as long as the interval is composed both at its beginning and at its end by real numbers and so with these values, the formula of this theorem can be applied to find the derivative of that midpoint.

For questions with a graphic response (4) and (5), an example different from Rolle's theorem was obtained and he achieved an adequate justification that he presented in the answer to question (6), and he was the only student who applied the theorem in the proposed exercise of question (7) (Figure 2).

What is interesting in these results is to identify the changes in the appropriation of the mathematical object MVT, relating prior knowledge with that achieved after reading. In Figure 3, it can be seen the group has the most changes it's the one that had no prior knowledge about the theorem. These were the only ones to reach *inferential* and *critical level*. Those who somehow had some preconception did not pass the *literal level*.

Although the degree of activation of preconceptions, at a statistical level, did not show to be related to the level of reading comprehension (Fisher's exact test of independence, p_value of 0.1162), it can be assumed that if there is a motivation (not knowing the specific factor that generates) by extracting information from reading, while for those with some previous level it seems to be satisfied with what is already known.



Translated Statements

4. Make a graph that illustrates the mean value theorem.
5. Based on your interpretation of the mean value theorem, which of the following function graphs satisfies it?
6. Explain why you chose graphs in the previous exercise.

Student solution:"I chose **a** and **d** since they are continuous and differentiable functions, while **b** is not differentiable and **c** is not continuous."

7. Determine whether the following function satisfies the assumptions of the mean value theorem, on the interval. If so, find the values of "c" that satisfy it: $f(x) = x^3 + 2x, [-1,1]$.

Student solution: "The values of **c** that satisfy it are x=1 and the function does satisfy the hypothesis of the mean value theorem, since **a** and **b** which in this case would be [-1,1] are real numbers, the function is continuous and differentiable"

Figure 2: Student responses that reached the critical level

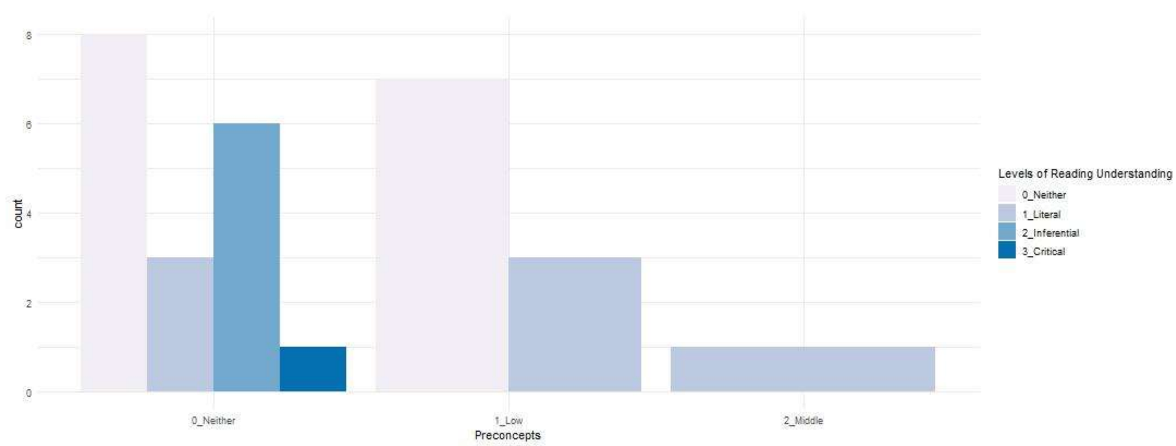


Figure 3: Distribution of students according to activation question ratio and maximum level of reading comprehension reached

DISCUSSION

To improve the understanding of the MVT and the derivative applications of a function through its different representations (e.g., algebraic and graphic), students must develop the cognitive process to allow them to establish an appropriate relationship between concepts, hypotheses, theorems, and mathematical objects (García-García & Dolores-Flores, 2021). To achieve appropriate results in reading comprehension of theorems, the activation of prior knowledge must include the visualization of mathematical objects and the development of skills that promote mathematical connections. In this order of ideas, visualization allows the empowerment of some representations of mathematical objects and their relationship. Particularly, it will help the reading comprehension of MVT as shown in the college's texts.

In many cases the mathematical text is limited to the conceptualization of the students (de Almeida & da Silva, 2018), consequently they achieve at most a literal level of reading comprehension. To develop the other levels, it is required that the teaching dynamics tend to improve conceptual reasoning and the functions of visual presentations, since usually these presentations are limited to the illustrative part (Natsheh & Arseniy, 2014). In addition, care must be taken to ensure that the lessons planned by teachers promote cognitive growth, algorithmic mastery and reduce the cognitive load on students (Toh, 2022).

In accordance with Park (2015), visual mediations with symbolic, algebraic, and graphic notations would help students understand concepts involved in the definition of a derivative, such as limit of a function and interval. These mediations would allow the activation of prior knowledge necessary for reading comprehension.

CONCLUSIONS AND RECOMMENDATIONS

Reading comprehension becomes a mediator for the appropriation of mathematical language, without said appropriation becoming a guarantee that the student learns mathematics. In other words, a student may master mathematical language appropriately, but not understand the text, and this may be one of the reasons for the need to develop strategies and didactical tools in order to promote reading comprehension.

Reading comprehension strategies allow students to deepen and appropriate the contents of a mathematical text. In addition, they promote the analysis of the hypotheses or main ideas; the importance to verify veracity of the assumptions; formulate diagrams in order to make an alternative graphical representation to the idea that the text intends to develop; and, finally, clarify the information through particular examples.

The teacher's intervention is crucial in the reading process in at least three moments: before (activation of prior knowledge and planning of activities), at the time of reading (implementation of activities), and after reading (evaluation of activities and appropriation of learning). In all three moments, motivation becomes an essential characteristic of the process.

When students with prior knowledge of the MVT achieve lower results, it suggests that their understanding may be hindered by a non-geometric or non-contextualized approach to the theorem. This tendency might lead them to merely grasp a superficial understanding of the MVT, lacking true ownership of its concepts. Conversely, those who approach the MVT from a geometric perspective or within a contextualized framework tend to develop a deeper comprehension of the theorem.

It's notable that a small proportion of readers reach a critical level, primarily due to inadequate preparation in developing essential cognitive processes such as note-taking and creating diagrams or graphics to enhance text comprehension (Vásquez, 2015). Therefore, ensuring the attainment of critical reading proficiency necessitates motivating students beyond mere text consumption. Instead, they should be encouraged to actively engage in reconstructing meanings through diagram creation. Education ought to discourage conformist or passive attitudes, as observed in students with prior knowledge in the experiment.

This research is the first step to improve comprehension of theorems we use to introduce in calculus classes. In order to expand the scope of this research, the next step will be to offer more didactical tools to teach others theorems, (e.g., the Fundamental Theorem of Calculus), perhaps using an historical and also geometrical point of view (Muñoz, et al., 2023). In addition, we can use those material to think about explaining in other manner the Green's or Stokes's Theorem.

The limitations of this study concern the small sample size and the collection of information in a single educational institution. This could make it difficult to generalize the results. Future research requires a more in-depth review of the development of reading comprehension skills at all educational levels, which are scalable to other areas of knowledge and involve other topics in mathematics. Furthermore, we suggest making longitudinal studies where measures of the impact of mathematics on the development of the inferential, literal, and critical levels could be possible.

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