Editorial of the Summer 2025 Issue TR 59, Vol 17 No 3

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This editorial introduces the main and quite diverse foci of investigations in the present issue. The mathematical thinking process has the largest number of papers (see below) followed by the investigations of stage developments of different variables, and the impact of augmented reality on learning. We have two comparison papers and three investigations looking for especially effective approaches to learning.

TR 59 has a variety of often intersecting foci—some of the authors are interested in *developmental stages* within different environments. **Biro and Csikos** explore the developmental stages of reaching competence in a given profession. The four stages are made of a 2×2 matrix with corresponding entries obtained from (competent, not competent) and (conscious, unconscious) pairs. It's interesting to see the arguments establishing the sequential developmental order within the entrees. On the other hand, **Jimenez-Valderama et al** examine reading comprehension development across four levels—literal, inferential, critical, and visual—in the context of understanding the Mean Value Theorem of Calculus (MVT). The visual level, of course, is necessary because of the conceptual understanding of MVT, so that the previous three levels seem to be the levels of critical thinking development.

The Broadwell-Burch Four-stage Model of Competence Development and the Mathematics Teaching Profession by **Fanni Biró, Csaba Csíkos, (**Hungary**)**…….. page 5

Reading Comprehension of the Mean Value Theorem in Engineering Students by **Pamela Jiménez-Valderrama, Francisco Niño-Rojas, Weimar Muñoz Villate, Oscar Espinel** (Columbia)……. page 26

Two teams embraced an Augmented Reality (AR) approach. Osman et al. are interested in the impact of AR on learning isometric geometry, or transformations that preserve the distance

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between points, such as translation or rotation. Indeed, there is no doubt that AR can deepen the understanding of some mathematical concepts. **Susanto et al.** extend the use of AR into an ethno-geometric environment and are interested in the impact of that environment upon students' creative thinking, which is defined following Guilford by skills such as fluency, flexibility, elaboration and originality. This suggests that blending immersive digital experiences with culturally relevant mathematical contexts provides a promising avenue for fostering higher-order cognitive abilities in learners.

Development of an Augmented Reality-Based Learning Module on Isometric Transformation for Form 2 Students in Malaysia, **by Noorul Shuhadah Osman, Ahmad Fauzi Mohd Ayub, Nurul Nadwa Zulkifli, Jazihan Mahat** (Indonesia).....page 42

Enhancing Mathematical Creative Thinking in Ethno-Geometry Learning Using Augmented Reality Technology. **Susanto, Dafik, Arika Indah Kristiana, Arif Fatahillah, Ridho Alfarisi** (Indonesia)……..page 70

The concern for the mathematical thinking process appears in several other papers in the TR 59 collection. **Cerrado and Limjap** report on an interesting intervention in the context of Realistic Mathematics Education (RME), which uses metacognitive prompts to increase students' metacognitive activity. They find that student metacognitive activity has three components: metacognitive knowledge, metacognitive experiences, and metacognitive skills. The intervention with its specially designed performance task on quadratic functions indeed significantly increased student metacognitive awareness**. Canonigo**, on the other hand, studies reasoning approaches of students in the trigonometric domain and finds that they use several different types of reasoning: algorithmic, analytical, abductive, inductive, and deductive methods. Different teaching techniques are used in the investigation to improve student reasoning with each of the methods: scaffolding, guided inquiry, and contextualized learning experiences. He also notes that meta-cognitive reflection and discourse enhanced students' problem-solving abilities and collaborative skills. This theme of reasoning also appears in **Hassan et al**.'s paper, which investigates the relationship between working capacity of memory and the levels of mathematical anxiety as independent variables to determine their impact upon students' creative reasoning in the context of problem solving. They find out that, as one could expect, prospective mathematics teachers with high working memory capacity demonstrate flexibility and fluency (as measures of creativity) in generating new ideas. This allowed them to connect known mathematical concepts and provide logical arguments to support the truth of the ideas created. However, high and low levels of mathematics anxiety interfere with cognitive performance, causing a loss of focus in solving more complex problems and not providing arguments for the ideas created. The paper by **Fatmanissa et al.** searches for the improvement of mathematical reasoning through the development of mathematical justification of arguments. The authors used three different tasks designed for the explicit use

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of technology. They showed that addressing these three situations without technological support has significantly reduced the impact of justification on reasoning.

Enhancing Metacognitive Awareness, Skills, and Academic Performance: A Study on the Impact of Performance-Based Assessment with Metacognitive Prompts in Quadratic Functions Using the RME Approach by **Pamela Mae Cerrado, Auxencia Limjap** (Philipines) ……...…page 88

Enhancing Instruction in Trigonometry: Insights from Students' Reasoning Approaches and Processes by **Allan Canonigo** (Philippines)……page 120

The Influence of Working Memory Capacity and Mathematical Anxiety on the Creative Reasoning of Prospective Mathematics Teachers by **Buaddin Hasan, Dwi Juniati, Masriyah** Indonesia……page 154

Collaborative Problem Solving With and Without Access to Technology: Emphasis on Mathematical Justifications by **Namirah Fatmanissa, Anis Farida Jamil, Tatag Yuli Eko Siswono, Agung Lukito** (Indonesia)……..page 178

We have two comparison papers in the issue, one by **Yi Wei et al**. and the other by **Sabri and Mainali**. The first one is concerned about the PISA levels of competence of Malaysian students. Malaysia maintains a steady average level of achievement among nations but Malaysian educators want to reach higher levels. They hope to reach them by adopting Singapore's subject-based banding system to give equal opportunities and a flexible learning environment for students; continuous improvement of professional development for teachers by adopting the Singapore SFEd roadmap and the World Bank teacher development roadmap; and incorporating PISA-released items into classroom practices and the national assessment mode. They have a good model to follow in Singapore, which continuously holds the highest levels on PISA exam. The second paper, by **Sabri and Mainali** compares the teaching approaches of Turkish and American teachers in terms of representations used in their classrooms. The main difference between them is interesting: whereas Turkish teachers emphasize symbolic and life-based representations, American teachers emphasize concrete materials, visual representations, and technology. The issue of representation is also touched upon by **Meryiati et al**. who related it to the question of self-perception vis-à-vis mathematics and technological confidence. The authors found that technological self-confidence and self-perception vis-à-vis mathematics were positively related.

Challenges and Opportunities: A Comparison of PISA 2022 Mathematics Performance in Malaysia and other ASEAN Countries by **Lim Yi Wei, Darmesah Gabda, Nicholas Pang Tze Ping, Ho Chong Mun** (Malaysia)……..page 201

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The Perceptions of Multiple Representation of the US and Turkish Mathematics Teachers by **Ali Sabri IPEK, Bhesh Mainali** (Turkey, US)……page 223

The Role of Self-Perception of Mathematics and Technological Self Confidence in Predicting Mathematical Representations in Secondary Education by **Meriyati Meriyati, Suherman Suherman, Anizar Rahayu, Umi Hijriyah, Agus Jatmiko, Erwanto Erwanto** (Indonesia)……...page 247

The papers by Taraldsen, Doz et al., and Ngoveini present three different methods of impacting learning, correspondingly by gaming, general problem solving in a third grade, and by introducing a new framework for questioning students in the context of financial mathematics. *Introducing Teacher Education Students to Escape Room as a Didactic Tool in Mathematics* by **Lene Hayden Taraldsen** (Norway)……page 261

An Experimental Problem-Based Approach in Mathematics Teaching and Mathematical Problem-Solving Performance in Grade 3 Primary School Pupils by **Daniel Doz, Mara Cotič, Nastja Cotič** (Slovenia).....page 289

Optimizing Classroom Engagement in National Certificate (Vocational) Mathematics: A Framework for Questioning Techniques by **Mbazima Amos Ngoveni** (South Africa)…page 337

The issue wouldn't be complete without the Problem Corner, as always prepared by Dr. Ivan Retamoso, our Problem Corner editor……page 363

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The Broadwell-Burch Four-stage Model of Competence Development and the Mathematics Teaching Profession

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Abstract: This study aims to reveal the possible benefits of incorporating the Broadwell-Burch four-stage model of competence development into mathematics teacher training. First, a historical overview is given about the appearance of the four-stage model starting from the original sources by Broadwell and Burch arriving at today's widespread use in the medical and other professions. The case of the teaching profession and teacher education has almost been untouched in the literature; consequently, it is the right time to raise questions and dilemmas about the strength, usefulness, and applicability of the four-stage competence model. We conclude that mathematics teacher education and teachers' continuous professional development training can benefit from considering and taking note of the phases described in the four-stage competence development model. A hypothetical "reflective competence" stage seems to be especially important when describing the pedagogical content knowledge required from math educators

Keywords: mathematics teacher education research, competence development, continuous professional development, pyramid model, pedagogical content knowledge

INTRODUCTION

The Broadwell-Burch Four-stage Model of Competence Development: Historical Roots, Theoretical Rationale and Visualization

Historical emergence of the model

Several decades ago, a speculative model of skill development was published. Broadwell (1969) proposed a list of sequential developmental stages by which the development of teaching skills could be described. The four stages and their brief descriptions were as follows. (1) Unconscious Incompetent – poor teaching while being unaware of it. (2) Conscious Incompetent – poor teaching while being aware of it and seeking help. (3) Conscious Competent – starting to get results in teaching while knowing the reason for it. Finally, at the (4) Unconscious Competent level, the

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teacher is a "good teacher by nature" (Broadwell, 1969). According to Broadwell, the Unconscious Competent level applies solely to a small subset of teachers. The fact that these teachers are unaware of what makes them effective and unable to articulate their decision-making and instructional practices presents significant challenges.

The descriptions and the labels of the four stages in Broadwell's model form a combinatorically complete, 2x2 system with two dimensions and two values in both dimensions. Consciousness and unconsciousness are two possible values with respect to the self-referential or explicit knowledge the teachers may possess about themselves and their teaching practices. Nevertheless, using the term awareness would be a better way to name this dimension since, according to Hacker (1998), awareness refers to potentially reportable knowledge, and in Broadwell's model, the dimension of consciousness or unconsciousness is defined in this sense, i.e., referring to potentially available, verbalized, or potentially verbalizable knowledge. The other dimension is the success of teaching measured in some way.

The success or effectiveness of teaching has already been widely studied. One of the highly important studies (van de Grift et al., 2014) explicitly advocated a theoretical model that covers the range of observable teaching skills from simple to complex. In their conceptualization, the teaching competence – a term often used generously and sketchily in the literature – can be evaluated by means of measuring a range of teaching skills that build on each other and are classified into hierarchical levels. A further, general assumption about the development of teachers' competence is that the evaluation of teachers' professional development may be described by the principle of "sequential outcomes", i.e., thinking and doing, affecting and strengthening each other (see Thurlings and de Brok, 2017).

The Counter-intuitive Nature of the Model

One of the strengths of the sequential four-stage model was probably the surprisingly counterintuitive order of the stages. When it comes to the first two stages, it seems evident that understanding the reasons behind poor performance is advantageous. However, opinions may differ regarding the significance of awareness associated with successful completion. Why should one perceive it as a higher level of performance when individuals can no longer verbalize their decisions and lack conscious control over ongoing processes? Given the direct and immediate relevance of this question to the teaching professions, it is important to acknowledge that Broadwell (1969) employed the term skill in the broad, everyday sense: Teaching is indeed a skill.

Throughout the previous decades, the meaning of the term "skill" has evolved in many ways and directions in different content domains. In line with the aims of the current paper, we recall a definition that comes from the field of reading research. Afflerbach et al. (2008) paid off an old debt by defining the difference between skills and strategies. Although they focused on the difference between reading skills and reading strategies, the essence of the distinction can be easily

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transferred to other domains where rich systems of skills develop. Their definition was the following (Afflerbach et al., 2008, p. 368): "Reading skills are automatic actions that result in decoding and comprehension with speed, efficiency, and fluency and usually occur without awareness of the components or control involved." In other words, properly functioning reading skills are characterized by automaticity. Other skill systems at the heart of school curricula, like writing and arithmetic, can also be characterized by quick and efficient functioning without much awareness.

The sequence of levels 3 and 4 in the four-stage model becomes understandable from the idea that similar automatisms apply to fairly different skills. As Afflerbach et al. (2008) emphasized in the case of reading skills, conscious and deliberate strategies may become skills with months of practice. In the area of arithmetic skills, Csíkos (2022) presented a model about the possibly diminishing role of strategic (metacognitive) components. Effortless and automatic operation can be the key to development in many (or almost all) essential areas of human performance, as argued by Leahey and Harris (1993, p. 284), "what had formerly required conscious thought becomes intuitive, and an important question concerns what happened to the rules followed consciously by the novice".

The four-stage model proposed by Broadwell fell on fertile ground, and the Gordon Training Institution had great merit in dissemination and propagation. In addition to making the four-stage model widely known, there is a relevant content innovation attributed to Noel Burch (1970). While the original Broadwell-model was about teaching skills, the potential of generalizability made a conceptual extension of the model to learning any new skill. Later, Adams (2011) gave a summary of the generalized model studded with personal experiences, and she claimed that it was Noel Burch (1970) who developed the so-called "Learning Stages Model". In this generalized model, the labels for each stage are as follows: (1) Unconsciously unskilled, (2) Consciously unskilled, (3) Consciously skilled, and (4) Unconsciously skilled.

Consistent with the common trend observed in models that hold the promise of extensive applicability, the four-stage model has been applied in various training fields, reflecting the spirit that nothing is more practical than a good theory. The counterintuitive order of levels 3 and 4 merited particular attention. Across different domains, it has become evident that the role of awareness (consciousness, or metacognition) serves as a temporary support mechanism, akin to scaffolding. While conscious planning, monitoring, and evaluation can undoubtedly play crucial roles at certain stages of development or in response to specific tasks or contexts, it is not inherently valuable to consistently engage in conscious planning, monitoring, and evaluating for their own sake.

Indeed, the model's popularity started to rise in the 10s of this century, and in recent years a real re-recognition or renaissance of the Burch- or Broadwell-model has taken place. In scientific databases, a spectacular increase in the number of citations can be observed. It seems that the power of the model was recognized in several different fields of training and coaching. Various visu-

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alizations were developed to better convey the model's message since the original sources (either Broadwell's or Burch's) provided a purely verbal and linearly sequential structure.

Different Types of Visualizations of the Model

Various two-dimensional visualizations appeared, with the four stages allocated to different quadrants of the plane. One of the earliest examples of such visual representation in scientific literature is a quadrant model that lacked labelled axes (Kent 2010). Peel and Nolan (2015) offered a visualization featuring two axes, each with dichotomous values. Focusing on medical residents' learning, they positioned the axis of instruction and feedback horizontally (whether its intensity is high or low), and the axis of needed support vertically (high or low). Incompetence was linked to a high level of instruction and feedback, while consciousness was associated with support residents might require. The consecutive levels of competence were thus associated with four quadrants in a two-dimensional plane (see Figure 1).

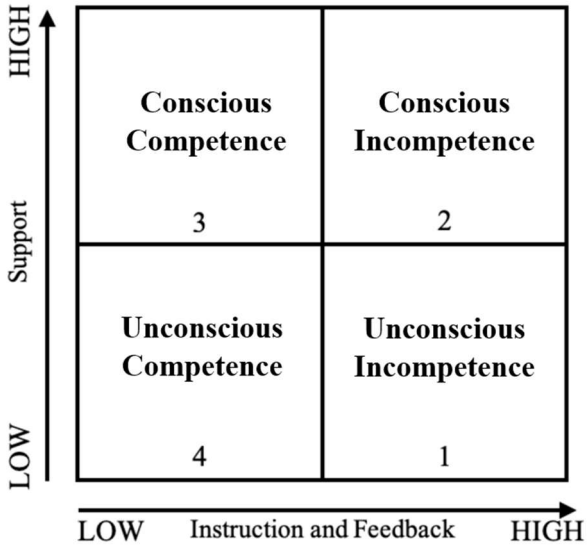


Figure 1: Two-dimensional arrangement of the model (adapted from Peel and Nolan 2015, p. 2)

Esterle and Brown (2020) suggested a two-dimensional quadrant model where confidence in one's ability is presented as a function of experience (Figure 2). With the four consecutive stages, experience grows, while confidence draws a U-shaped curve, i.e., greater confidence can be observed in the unconscious first and fourth phases.

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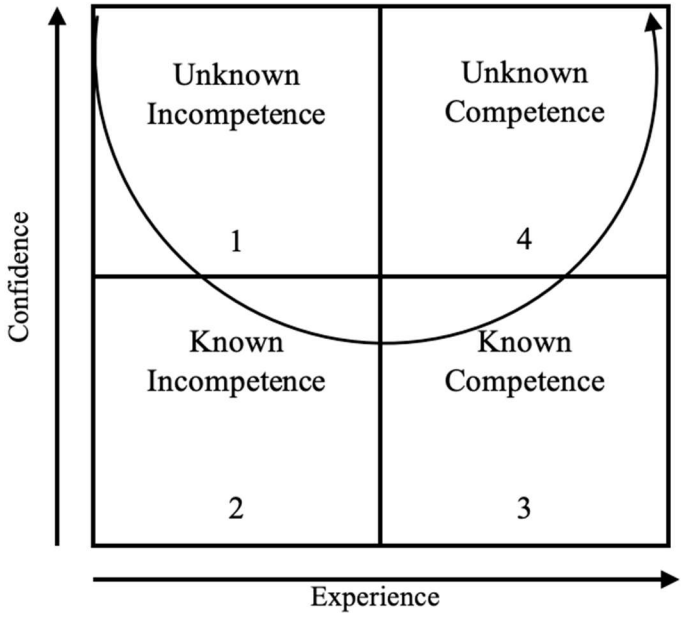


Figure 2: Two-dimensional arrangement of the stages of the model enriched by further variables (adapted from Esterle and Brown 2020, p. 1)

Interestingly, some researchers discuss the concept of cycles or wheels of competence development. They propose that the four quadrants of the plane represent consecutive stages, where the level of unconscious competence can lead back to unconscious incompetence once again (see Bhutalazi & Singh, 2019). While it might be argued that reaching a very high level of competence in a particular domain or skill system could unveil the recognition of unconscious incompetence, from an educational perspective, particularly in formal and non-formal training settings, the assumption of such self-recurring cycles might be discouraging or ambiguous. Nevertheless, Bierwolf and Frijns (2019) claim that a simple two-dimensional quadrant model may be misleading, so it is better to talk about a kind of spiral, i.e., a series of continuous learning cycles (Figure 3). Burgess et al. (2020) warn that there is a long-term potential (danger) arriving once again at the stage of unconscious incompetency, like drivers who overestimate their skills.

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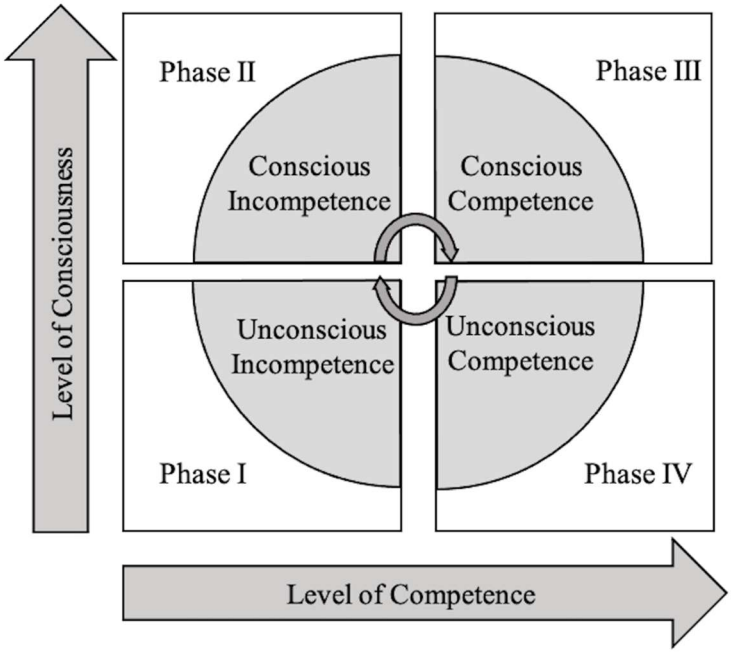


Figure 3: The recurring cycles model of the four phases (adapted from Bierwolf and Frijns 2019, p. 33)

As Neves (2022) provocatively highlighted in the context of psychotherapists, becoming aware of the unconscious incompetence can be a distressing experience when a therapist encounters unexpected information. Neves emphasizes the co-existence and multi-layered nature of conscious and unconscious competencies within the same individual. Furthermore, it is understood that different stages of competence can coexist within one person. This possible co-existence of different levels of competencies within the teaching profession poses challenges when attempting to apply the four-stage model in teacher education.

Another kind of two-dimensional representation used the horizontal line as a timeline, depicting the four stages as consecutive while incorporating additional characteristics. It is worth considering that the original four stages of the development of teaching skills were not inherently tied to specific time periods or a linear timeline. Keeley (2021) illustrated the four developmental stages using a timeline-like graph with the four stages following each other, and performance was depicted on the vertical axis (Figure 4). In this kind of visualization, the course of the performance curve resembles the logistic curve model widely used in developmental psychology.

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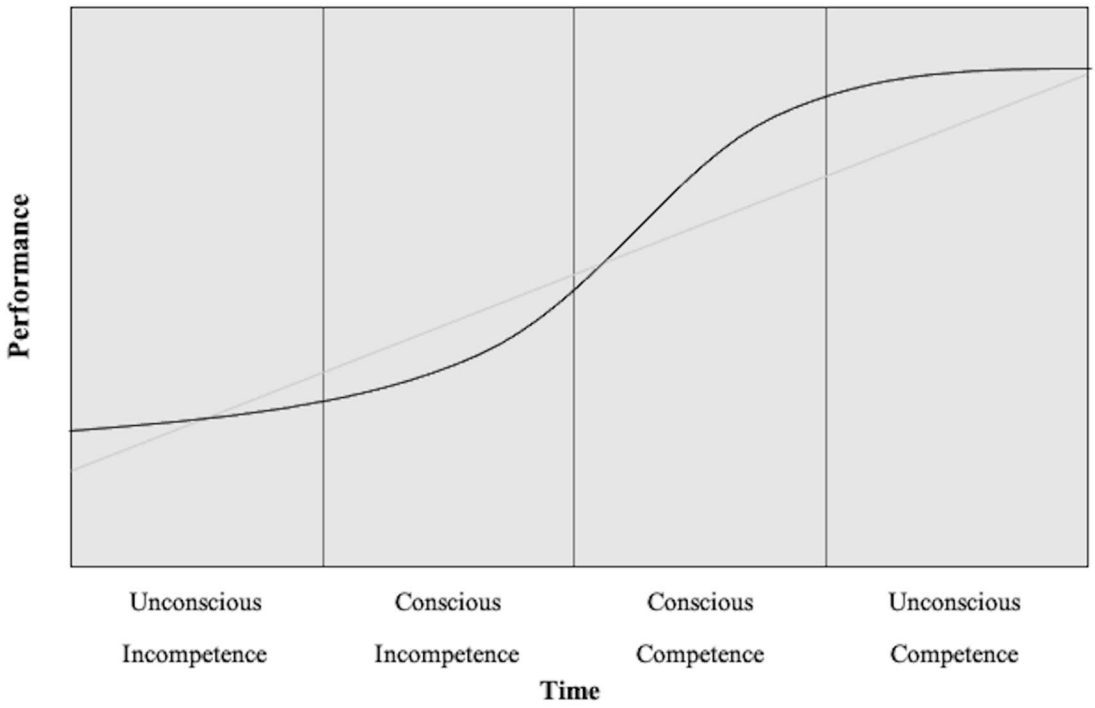


Figure 4: Timeline-based arrangement of the four levels of the model (adapted from Keeley 2021, p. 2)

The steeply rising section of the performance curve is, of course, between the second and third stages. Luczak and Kubo (2022) found neural correlates represented on the timelines for the four-stage model. Their model reassures the steep developmental phase, but not between the second and third, but between the first two stages. Indeed, considering the pedagogical relevance, we can confirm the importance of this first transition phase, which was also reassured by an empirical work on environmental awareness (Permana et al., 2020).

A pyramid-like visualization of the four-stage model appeared in Moore and Durham's (2011) work (Figure 5). In the field of dental education, they showed how the four stages are built on each other within a rectangular-shaped figure.

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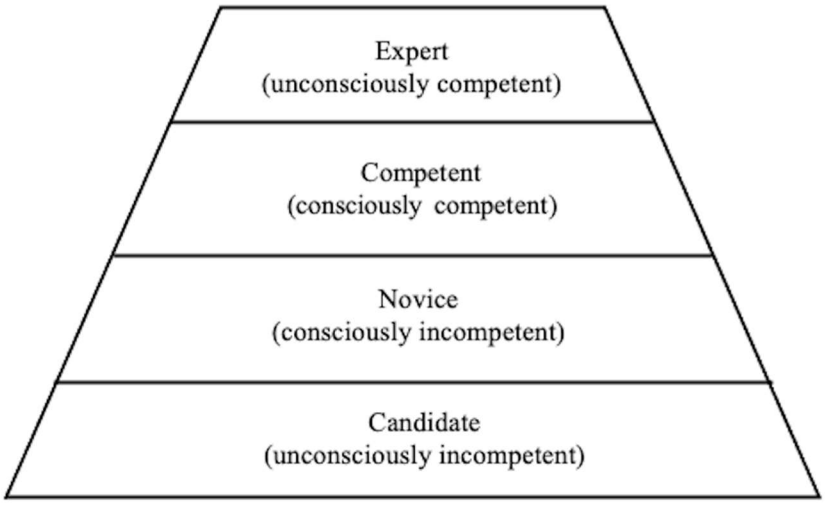


Figure 5: The first pyramid-like arrangement of the four stages labeling them with terms widely used in the cognitive science literature (adapted from Moore and Durham 2011, p. 54)

Drawing from the cognitive science literature, Moore and Durham (2011) extrapolated that the ultimate stage of unconscious competency would require approximately ten years of dedicated practice. This notion of a pivotal "10-year" period aligns with the findings of numerous expert-novice studies conducted among musicians or chess players (see, e.g., Ericsson et al., 1993; Schneider et al., 1993).

LITERATURE REVIEW

Applications in Several Disciplines: Widespread Recognition of the Broadwell-Burch Model

Applications in Psychology and the Social Sciences

The initial applications of the four-stage model encompassed coaching and adult training, and established a significant connection with university courses in teacher training and continuous professional development, particularly within the domains of medical and dental trainings and CPD programs. Curtiss and Warren (1973) succinctly captured the essence of the four-stage model in the context of adult training practices, without explicitly referencing either Broadwell or Burch. They introduced a broader term, "competence," replacing the previous concepts of teaching or other specific skills. The stages are now referred to as the typical sequence for acquiring competence. As for the need to proceed to the third to the fourth stage, they argue:

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> *With practice, you feel "conscious competence" where the skills are painfully obvious and undeveloped, and you feel embarrassed at your attempts and the lack of smooth and polished performance. (p. 89)*

Jarvis and Baloyi (2020) explicitly connected the second and third stages (conscious incompetence and conscious competence) to Vygotsky's theory of the proximal zone of development. When the student is already aware of being unsuccessful, a more experienced tutor can help achieve successful functioning while maintaining awareness. One possible description of this kind of help uses the term metacognitive scaffolding (Wright, 2018). As for the connections of the four-stage model with another important and recognized model of cognitive development, Syroyid (2021) found connections between the four stages and the Piagetian stages of general cognitive development. Therefore, the four-stage competence development model is compatible with relevant developmental theories and models, and provides the hope for effective yet understandable generalization for many areas of training development.

In search for the power of the model not only as a description of what happens in consecutive phases of skill development but as a suggestion and explanation at the same time, there were several attempts to describe how to teach, how to intervene in order to help students proceed from one level to the next. In the context of scaffolding PhD students, Elliot (2022) suggests different kinds of support along the four stages in both the formal and the hidden curriculum. The transition from the first to the second stage can be considered a period of purely scientific (academic) support. In later stages, it is the emotional, social, and psychological support tools that will play a leading role. In an action plan devised for librarians, Van Camp (2019, p. 99) emphasized that "the first stages are inherently frustrating". This is in line with what Elliot suggested about the importance of emotional support at the beginning. Another aspect of the specialty of the first transition phase (between the first and second stages) was revealed when students from liberal arts and sciences took part in an investigation. De Krom (2023) collected empirical evidence on how students express themselves in a writing assignment at the conscious incompetence stage of development. Some of them believed that a demonstration of effort such as indicating "I have no idea what you mean here" (p. 70) is required to get a better grade.

According to Wisniewska et al. (2021), the practical activities of organizational learning can be associated with the second stage of the competence development model. In other words, it is crucial to reach at least the stage of conscious incompetence to feel the need for self-study or self-reflection. Kortmann and Scholten (2020) attributed considerable explanatory power to the four-stage model in cooperative game-based learning in the sense that the importance of the second stage en route to conscious competence was also supported by empirical data.

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The Broadwell-Burch Model in the Medical Professions

The application of the four-stage competence development model extends to the field of medical education as well, as it has proven to be fertile ground and a significant area of application in the past decade. In an empirical investigation focusing on a specific technique, Asadoorian et al. (2022) revealed a potential connection between a decreasing level of confidence in ultrasonic instrumentation and participants' shift from unconscious incompetence to conscious incompetence. It becomes evident that although the experience of diminishing confidence may be regarded as a negative emotion, it is an unavoidable aspect of the developmental process.

A reasonably general model for clinical training was developed by Lane and Roberts (2022). They warn that stage 4 can be misinterpreted as the final mastery of a skill, where the learner has nothing more to learn. They also emphasize the contextualized nature of the model. Lane and Roberts raise some other concerns as well, which are instructive when the model is generalized to the field of teacher training. One is considering the possible effects of preparatory reading on the actual performance. Another issue is the measurability of performance in terms of both quantitative and qualitative indicators.

The work of Lee et al. (2017, p. 132) provides a direct and immediate source of inspiration for educational science: "Most competent endoscopists fall into the unconsciously competent category, where they can perform a procedure skillfully, but not necessarily describe or verbalize how or why they do it, making them poor instructors." Burgess et al. (2020) suggest that continuous maintenance of acquired skills through deliberate practice can be beneficial. Deliberate practice of a skill already mastered may even enable the attainment of a hypothetical fifth stage (Jensen, 2013) of competence development (Figure 6).

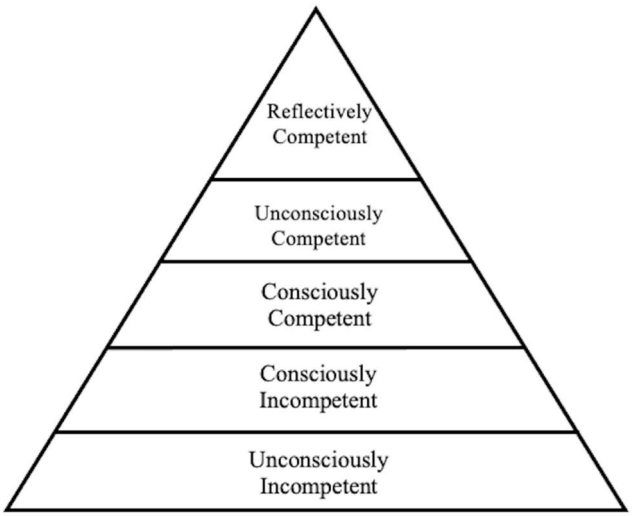


Figure 6: Pyramid-type arrangement of the five stages of competence development

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Jensen's (2013) research report for the Karolinska Institute was the first to propose a fifth stage: reflective competence. This stage has since been adopted by others to describe the "conscious competence of unconscious competence"—a phase in which unconsciously competent individuals, particularly when tasked with teaching others, must become aware of and able to articulate their previously automatic practices, effectively stepping out of "auto-pilot" mode (Jensen, 2013, p. 11). In their review of coaching as a form of continuous professional development for surgeons, Willemot et al. (2023) focused on the first three stages of the competence development model. They argue that effective coaching should support the coachee in progressing from unconscious incompetence through conscious incompetence to conscious competence. While this does not imply that the fourth stage is overlooked, the authors emphasize that, for the purposes of designing coaching interventions, the initial stages are the most relevant.

The Broadwell-Burch Model and the Teaching Profession

Differences and Similarities Between the Teaching and the Medical Professions

Continuous professional development is essential not only in medicine but also in education. Given the growing number of studies applying the competence stage model in medical contexts, it is timely to explore its potential in teacher education. While the model's relevance is clear for foundational skills across educational settings, its applicability to more complex competencies, such as teaching, may be less immediately apparent. Building on Darling-Hammond's (2006) observation that teacher training lags nearly a century behind medical education, it is worth considering how the four-stage competence model could inform the development of teaching expertise. The system of skills necessary for being an effective teacher can be labelled as teaching skills. In this sense, we return to the origins of the model, as Broadwell's (1969) original four-stage framework was developed with the teaching profession in mind. Without further operationalization or elaboration of what the "teaching skills" exactly are, we may rightly propose that this is a system of several psychological constructs involving factual knowledge, more or less automatized skills, and the capability to make decisions at the strategic (planning, monitoring and evaluation) level. From this point of view, there is a striking similarity between the medical and teaching professions.

While considering the similarities between teaching and the health professions, our attention turns to the meaning of the different developmental levels and how teacher educators can utilize this knowledge effectively. To facilitate this exploration, we will simplify and dichotomize the two aspects of the model and assume that teachers can be efficient in their work (as an overall, yes-or-no aspect), and they can be either capable or not capable of reflecting on their work (as possibly judged by experts of the field). Additionally, our focus encompasses both pre-service teachers and in-service teachers as learners. Furthermore, we aim to delve into the description of

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the hypothetical fifth stage of the model, specifically examining what reflective competence entails in the teaching profession (see Figure 6).

The Pyramid Model and the Teaching Profession

At the initial stage, pre-service teachers may be considered ineffective and they lack the ability to engage in professional reflection on their activities. However, as they progress through teacher training courses, their competence in reflecting on their own activities, as well as those of their peers, gradually increases. It is important to note that this growth in reflective competence does not automatically translate into becoming effective teachers. For university-based teacher trainers, it presents a significant yet attainable challenge to provide guidance and support to pre-service teachers as they progress to this second stage, i.e., the level of conscious incompetence.

According to popular belief and naïve hypotheses, "practice makes perfect," and the teaching profession is no exception. In this maxim, there is often a limited perspective on the process of learning from experience, as if the mere act of doing something would automatically elevate the level of competence through accumulated experience. However, the role of practice and experience is fairly different for novice teachers and for their experienced peers. Based on McIntyre's maxims, Hagger et al. (2008) identified two distinct types of reflective activities. For novice teachers, the essential tools of reflective practice involve actively reflecting and consciously considering ideas shared by their more experienced peers, as well as educational researchers. In contrast, experienced teachers engage in a different form of reflective practice, primarily centered on bringing their deeply embedded beliefs rooted in their practice to conscious awareness.

Becoming an effective and reflective teacher requires long years of deliberate practice. We use the term "deliberate practice" in line with Ericsson et al.'s (1993) formulation, which distinguished between work, play, and deliberate practice. Teacher training programs designed for continuous professional development may facilitate the process. However, an intriguing question arises: Is it genuinely possible for teachers to reach the fourth stage in the model, i.e., becoming efficient while simultaneously incapable of reflecting or verbally reporting on their teaching skills? Equally important is the exploration of how (and whether) the capability of reporting on strategic decisions may diminish or be lost.

There is another critical aspect of the model to consider before we can reasonably propose it for further consideration and as a useful starting point for empirical research. In addition to the so-called teaching skills, there exists another essential component of teaching competence consisting of domain-specific or content-related processes. This component can be identified as pedagogical content knowledge (PCK) and it necessarily contains the knowledge and acquisition of various skills that are not purely teaching skills, but different skills to be taught to the learners. A key question arises: Which stage of these skills to be taught to children should teachers reach in order to be effective in their profession?

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An Example From Basic Skills Development

Let us consider some examples from the realm of basic skills and some other systems of skills. The classic three Rs – reading, writing, arithmetic – must be thoroughly mastered by those who teach these basic skills. We would even allow the assumption that elementary teachers must reach the fourth stage in the sense that their reading, writing and calculation skills must be effortless and fluent. Other examples of more complex skills include playing an instrument or driving a vehicle. Those who undertake to teach these complex skills to others are expected to possess a reasonably high level of proficiency. A piano teacher must be able to play masterpieces, although not necessarily at the level of the top artists going on world tours or even performing daily as their livelihood. A driving instructor must be a very experienced driver, but it is unlikely that they consciously accomplish the decisions and those high amplitude movements (e.g., clearly perceptible turning of the head in a desired direction) that they demand from their students. Similarly, master chefs, synchronous interpreters, masons, and experts in most occupations exhibit high level of efficiency in their work, while they may engage in consciously planning, monitoring and evaluations of their actions at times, it is not feasible for these conscious efforts to be present throughout their daily activities. What is the key condition for a professional to be able to teach others?

According to a cynical opinion, as cited from G. B. Shaw by Shulman (1986, p. 4), “He who can, does. He who cannot, teaches.” Although one could argue that this often-mentioned phrase is not even worth talking about, the low prestige of the teaching profession is a real problem, with a significant number of teachers feeling that their work and profession are undervalued by society (OECD, 2019). In order to better understand the message of this ill-famed maxim, we can use the four-stage model. According to one possible interpretation, many professionals are more competent and more effective in a given field than an instructor might be. Car racers, piano virtuosos, or, to stick to basic skills, professional writers, world-class mathematicians, or the best stage actors are likely to be on a higher level of achievement in their very own field of expertise. However, if we would ask questions about their reflective competence, they may be less skilled in talking about the decisions they make during performance. Only some of the greatest artists and scientists are capable of reporting on the processes they are executing. In light of the four-stage model of competence development, they are indeed at level 4, with excellent efficiency, but with limited consciousness, i.e., overwhelmingly relying on quick and intuitive decisions, having probably already lost the ability to provide introspection about the ongoing processes.

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Bloom's Taxonomy and the Broadwell-Burch Model

Finally, the already powerful four-stage developmental model may be enhanced by showing its connection to Bloom's taxonomy: according to Shtaltovna (2021), the four stages not only resemble but essentially connect to the widely known taxonomy of educational objectives originally proposed by Bloom (1956). Shtaltovna provided a comparison of the revised Bloom taxonomy of educational objectives and the levels of the model, which she named the Burch/Bartlett-model. In essence, the six levels of the revised Bloom-taxonomy can be matched with the four levels of the Broadwell-Burch model (in her paper, she used the label 'Burch and Bartlett model'). The two highest levels of Bloom's taxonomy, i.e., Evaluation and Creation, can be matched with the level of unconscious competence, since the already mastered skills enable automatic and easy performance. The third and fourth levels of the revised Bloom-taxonomy, i.e., Application and Analysing (the latter is labelled Experimenting in Shtaltovna's work), represent the level of conscious competence. Finally, the first and second stages of Bloom's revised taxonomy can be matched with the stages of unconscious and conscious incompetence. Since the Bloom-taxonomy has been the basis for curriculum development for long decades in many education systems and, therefore, a basis for operationalizing learning targets, the correspondence and parallels between the two models are noteworthy but still speculative in nature.

METHOD

The Broadwell-Burch Model and the Mathematics Teaching Profession

An Example: Adding Fractions

It is time to address the applicability of the four- or five-stage model in the field of mathematics teaching. As a first step, we apply the model to the topic of adding fractions. There are at least two reasons for choosing this content. First, as emphasized by prominent scholars and the National Mathematics Advisory Panel (2008, p. 18), "The most important foundational skill not presently developed appears to be proficiency with fractions (including decimals, percents, and negative fractions)." Second, the handling of incorrect examples plays a crucial role in learning fractions (see Heemsoth & Heinze, 2013). This requires a high level of pedagogical content knowledge in order to discuss possible misconceptions while also taking advantage of the benefits of learning from errors.

<u>*Unconscious Incompetence*</u>: A well-known classical example in the literature was given by Borasi (1987). She gave the example of a mistake often made by students: $2/3 + 5/7 = 7/10$. Many children simply add the two numerators and the two denominators making the new numerator and denominator from their sums. This error is labeled a "rational error" (Ben-Zeev, 1995), since obviously there is a kind of rational and conscious strategy behind the decision of using an otherwise faulty algorithm. Therefore, unconscious incompetence in this case does not refer to the lack of any kind of consciousness, but to the lack of consciousness concerning the actual prob-

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lem to be solved. The children who use the algorithm presented above are usually not aware of why the procedure leads to bad results.

Conscious Incompetence: On the way towards mastering the algorithm of adding fractions, a next step may be that the students are aware of the above algorithm's flaw, i.e., they know that they must not automatically add the numerators and the denominators, but for some reason they are still not capable of getting the right answer. This might be due to a calculation error, or they may forget how to determine the least common multiple, and further additional problems may hinder achieving the correct result. In this stage, a student usually knows that they must do something different from simply adding the 'upper' and the 'lower' figures, but are still unable to perform well.

Conscious Competence: This is the stage we expect from good problem solvers. Students are not only expected to perform well, but we also expect them to be aware of the steps of the algorithm. Moreover, we also expect conscious and precise use of concepts like numerators, denominators, and least common multiple. Since pre-service teachers proved to be good performers in mathematics, this is also the level that we can assume as the competence level already acquired by prospective mathematics teachers.

Unconscious Competence: Imagine an upper secondary student who is capable to add fractions quite quickly and flawlessly, but unable to name the parts of the fractions, or cannot recall the concept of the least common multiple. They may perform well, but would certainly have some difficulties when trying to provide scaffolding to struggling learners. Only a part of the difficulties would come from the lack of verbally fluent use of the different concepts. These may be replaced by simpler substitution concepts (e.g., "upper figure"), but the main difficulty would come from the lack of introspection. In order to help others, we should first know well our own mental processes, and above this, the long-term accumulated pedagogical content knowledge components should be activated.

We may rightly claim for mathematics teacher training that, above the expected high level of performance (unconscious performance), a fifth level of knowledge should be built that allows for reflection on all four levels below. Of course, mathematics educators are not expected to provide introspections about every step they make. However, they should develop their knowledge to potentially report on their own thinking and actions, and on their students' assumed representations and activities. This kind of extra knowledge can be classified under the Shulmanian umbrella concept of pedagogical content knowledge. Shulman (1986, p. 9) mentions some forms of pedagogical content knowledge, e.g., the most useful forms of representations of an idea, "the most powerful analogies, illustration, examples, explanation", without which the only option left for the students is to watch and admire the master. Returning to the example of adding fractions, the teacher may use a variety of approaches based on the idea of multiple modalities, including pie charts, bar charts or special number line. One may be suitable for one student, the other for another.

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Response to G.B. Shaw's Maxim

We can reformulate Shulman's (1986) witty riposte to G. B. Shaw's degrading maxim. Shulman claimed that those who can do, and those who, in addition to this, understand, can teach. In light of the four-stage model of competence development, the hypothetical fifth stage of reflective competence may be the level that indicates a high level of pedagogical competence. Good teachers are expected to possess a high level of content knowledge, reaching even the level of being unconsciously competent in the skills they want to teach. However, in order to provide effective and individual scaffolding to the students, teachers need to regain or learn some insights about how others can be helped in reaching the next level in their proximal zone of development (see Jarvis and Bayoli, 2020). Furthermore, in line with Mason's (1998, p. 249) insights, teacher educators should work with their teacher-students in order to "extend their awareness of what students might be attending to by being explicitly aware of what they are attending to." Here, both teachers and teacher educators have a tough challenge to meet, since there are a number of math skills to be used automatically. E.g., a mental calculation task such as 9 + 6 can be solved quite quickly by most people, however, math teachers need to be aware of the "making to ten" strategy by which scaffolding can be provided to their elementary students. As Jaworski (2008) emphasized using a visual model, the math teachers' knowledge is grounded in several components including the knowledge of the learner, but, of course, additional elements of knowledge are needed beyond that. Going one step further, math teacher educators need analogue equivalents of the knowledge components a math teacher must possess. Compared to the knowledge elements expected of the student, both the teachers and the teacher trainers need additional knowledge elements that are born through conscious reflection. In other words, mathematical competence at the highest possible levels (which can be indeed called unconscious level of competence) is not enough for teaching mathematics. Consequently, we consider it necessary to supplement the pyramid model with the hypothetical fifth level, i.e., reflective competence.

CONCLUSIONS

Having reviewed the historical development of the four-stage competence development model and having discussed the potential of applications in different areas, we can now formulate suggestions for both theoretical and empirical research initiatives in the field of mathematics teacher education. One pillar of our suggestions is the promotion and urging of empirical research to test how the four- or five-stage model of competence development takes place in the mathematics teaching profession. For this, we feel the need to highlight a marked difference that exists compared to the medical professions. To do so, we need to reconsider the original Shulmanian distinctions among pedagogical knowledge, content knowledge, and pedagogical content knowledge. From a theoretical point of view, there is a chance that various competence levels are

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achieved and desired in different competencies of the teaching profession. In the medical professions, no such distinction is used, at least pedagogical knowledge in itself is not a central element of those professions. Content knowledge and even pedagogical content knowledge (as we saw from the examples taken from the applications of the four-stage model in the medical professions) have their analogous counterparts. We may rightly claim that the main novelty and challenge of the application of the competence development model lies in an application where the three Shulmanian clusters of competencies can be taken and investigated both separately and in an integrative way.

In some cases, "teachers" at the third stage of the competence development model (i.e., being consciously competent) can also be effective in their teaching. Just think about the possible effective use of peer-learning in order to raise awareness of mathematical ideas. What may pose a problem at this level is the perceived lack of professional commitment and the possibly excessive use of cognitive resources.

Professional commitment and responsibility for professional development occupy an important place on the list of teacher competencies judged to be important by upper-secondary students (Zagyváné, 2016). Excellence or at least a high level of knowledge in the discipline is an often-stated expectation (see e.g., Hamachek, 1969, p. 344), since "teachers rarely fail because of lack of knowledge. They fail more often because they are unable to communicate what they know".

The second issue that a person who wants to teach but is only at level three in the developmental model must deal with is the possible (and rather likely) cognitive overload. The teaching profession itself is fairly demanding in terms of making conscious decisions in planning, monitoring, and evaluation activities. Therefore, if the content knowledge itself demands significant resources, then teaching may cause cognitive overload. Is it possible to teach effectively and attentively the types of conditional sentences in English when they need conscious monitoring during formulating conditional sentences? How could one teach others to start smoothly from a parking lot if they always need to cite the MSM (mirror, signal, and maneuver) rule for themselves in such situations? Can one teach the strategy of "making to ten" when adding 9 + 6 unless they can use the strategy fluently and quickly, and can one mobilize their cognitive resources to scaffold learners effectively? Yes, one can teach, but beyond the excessive cognitive overload, they may be seen by others as a "blind leading the blind."

In addition to these model-related concerns, the pedagogical framework within which competence development is analyzed also deserves critical reflection. While this study has applied the Broadwell-Burch model within a primarily teacher-centered framework, we acknowledge that alternative pedagogical paradigms may involve different patterns of competence development. For instance, in constructivist classrooms, where learning is often co-constructed through student engagement and dialogic teaching, the progression through the stages may be less linear or more context-dependent. However, we believe the model's flexibility allows for meaningful adaptation in such settings as well. The emphasis on reflective competence, particularly at the fifth stage,

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aligns with the goals of student-centered pedagogy, where teachers must be capable of critically evaluating and adjusting their instructional strategies in response to diverse learner needs. Future empirical research should explore how the model functions within constructivist environments and whether the developmental sequence retains its explanatory power across varying instructional approaches.

Teachers who reach the fourth developmental stage while also acquiring additional pedagogical content knowledge are the most likely to become effective educators. Others in the fourth stage just "can and do." The issue of peer teaching may serve as a compelling example to highlight the importance of the need for separately considering the applicability of the four-stage model for the various competencies required for effective teaching, including both content-bound knowledge and teaching skills. The hypothetical fifth stage, i.e., the "reflective competence" stage will help in describing the knowledge of those who have a high level of competence in diverse (or almost all) areas of mathematics. In line with what Krauss et al. (2008) found about the connectedness of pedagogical content knowledge and content knowledge, mathematics educators with higher mathematical expertise seem to be more capable to scaffold their learners' understanding.

The idea of the Birch- or Broadwell-model of competence development may fall on fertile ground in the field of teacher training and teachers' continuous professional development. In the spirit of "nothing is more practical than a good theory" the competence levels can be a useful starting point for empirical investigations. The complexity of such empirical research is remarkable in itself, but beyond that the complexity is further increased by an additional factor. The general pedagogical knowledge (or skills) and the primarily content-bound PCK are two constructs that may and should be measured and observed simultaneously.

ACKNOWLEDGMENTS

This work was supported by the Research Programme for Public Education Development of the Hungarian Academy of Sciences.

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MATHEMATICS TEACHING RESEARCH JOURNAL 26
SUMMER 2025
Vol 17 no 3

Reading Comprehension of the Mean Value Theorem in Engineering Students

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Abstract: In this article, we analyzed the levels of reading comprehension: literal, inferential, and critical, using a diagnostic test about the understanding of the Mean Value Theorem (MVT) in engineering students of the Universidad de La Salle in the Calculus I lecture (Differential Calculus). The objectives of this article are to identify in which of these three levels of comprehension a student is in order to serve like an input to the teacher or the university to carry out an improvement plan. The other aim is to collaborate to narrow the gap between the analysis of the comprehension of mathematical theorems and comprehension of expository texts. The applied written test had seven questions created to recognize how students analyzed the hypotheses, how to elaborate graphs from the theorem assumptions and, finally, if they make an example to validate the theorem. One of the interesting results was the fact that there were no changes in the understanding of the MVT for students with some prior knowledge, contrary to what was observed with those studying it for the first time. The intervention of the teacher is important in the reading process in at least three moments: before, at the time of reading and after reading, considering motivation a primary characteristic of the process.

Keywords**:** Learning processes, Reading Comprehension, Mathematics, Calculus, the Mean Value Theorem

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INTRODUCTION

Teaching and learning differentiation of a function and some calculus applications (like rate of changes) requires special attention in university courses. Studies have shown students have problems trying to understand and relate the concepts involved in these subjects. For example, Hitt and Dufour (2021) mention some difficulties in the cognitive processes that a student often uses to understand differentiation and its relations to the concept of speed.

To find solutions, Dolores-Flores, Rivera-López and García-García (2019) explore the importance of mathematical connections with the concept of exchange rate. On the other hand, Dawkins and Epperson (2014) investigated interactions between calculus learning and problem-solving in the context of two first-semester undergraduate calculus courses.

Algebraic fluency should be promoted from the elementary levels. Thus, from these scenarios, practices that support algebraic reasoning must be prioritized, where analytical and structural processes are emphasized, which can support the meaningful learning of algebra in students (Ellis & Özgür, 2024). This would support the translation of statements (going from everyday language to mathematical language) and the development of variational thinking at higher education levels.

Talking about this subject, at different levels of higher education, fundamental mathematical skills are required to successfully face subsequent courses. Particularly in differential and integral calculus courses, given that many of the concepts and procedures studied in these areas are applied to everyday contexts, students must possess prior knowledge and certain skills that help them solve problems related to the derivative of a function and the definite or indefinite integral. This shows the importance of relating the derivative in concrete situations and problems, and ensuring that the student tends to generate a cognitive interest and foster a deeper understanding of differential calculus (Dagan, et al., 2018).

The difficulties arise in both directions (teaching and learning). Some authors state that practices have a greater impact on learning and professional development if the teacher is provided with the appropriate materials for their design and instruction (Bognar, et al., 2024). Others emphasize that the student's mood or motivational state and attitude play a fundamental role in their learning of mathematics (Geisler, et al., 2023; Schukajlow, et al., 2023).

We know about the low level of reading comprehension developed in many students, especially in the mathematical field. Certainly, several students don't have a deep understanding of definitions, theorems, ideas, and hypotheses addressed in mathematical lectures. From here arises the need to delve into this problem with a group of first semester students from La Universidad de La Salle (Colombia) and try to offer a solution.

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The Universidad de La Salle applies every semester a diagnostic test to the new students of the Engineering programs. It detects deficiencies in all levels of reading comprehension of expository texts in the mathematics field. In the same way, the problems visualized by teachers in classroom show that a good percentage of this population have troubles when reading, interpreting, and later applying the theorems that are introduced in the different calculus courses. This type of reading comprehension inconvenience hinders the correct appropriation of the basic concepts; the hypotheses raised; the theorems' applications that are essential tools in subsequent mathematics courses or even in engineering lectures. Not to mention the drawbacks they have in reading the statements of problem applications (Montero & Mahecha, 2020).

Thus, this research sought to diagnose the levels of reading comprehension of undergraduate students of engineering programs. For this, we considered the Mean Value Theorem (MVT) as the starting point of the study. Then we try to point out student appropriation, and we draw attention to student requirements; specifically, the mastery of multiple types of mathematical thinking, as well as the development of reading comprehension skills of expository texts, to fulfill this goal.

On the other hand, it is necessary to indicate this theorem is introduced in the final part of the differential calculus course, allowing most of the students to already be familiar with the calculus notation. So, the research question we try to answer is: *How to identify the level of reading comprehension of the differential calculus student on the Mean Value Theorem?*

In this research, we got the MVT statement proposed in Larson and Edwards (2016, p. 170):

Sean a y b números reales y f una función continua sobre $[a,b]$, y derivable en (a,b). Entonces existe un número c en (a,b) tal que

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

We keep the Spanish version because we used it in the applied written test. The statement of the MVT by the same author (Larson & Edwards, 2010, p. 212) in English version is:

If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a number c in (a,b) such that:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Historically, this theorem was first proved by the Italian mathematician and physicist, Giuseppe Lagrangia (also known as Joseph-Louis Lagrange), during the 18th century. His ingenuity and mathematical skills produced important contributions in almost all branches of pure mathemat-

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ics, and even helped the development of analytical mechanics and astronomy (McCormack, 2008).

THEORETICAL FRAMEWORK

The concepts of reading comprehension and levels of reading comprehension are the basis for this research.

Reading Comprehension

There is much academic research focused on reading and reading comprehension. Some studies affirm that the low levels of reading comprehension, together with the student laziness and limited parental involvement, are a mixture that necessarily leads to academic failure (Murcia & Henao, 2015). Other investigations promote strategies that enhance reading comprehension in students of all educational levels. They assumed that reading and understanding transcends the simple activity of decoding words and phrases, because these skills are associated with cognitive and metacognitive processes. Furthermore, we have to remember the influence of the social, cultural, and educational media of each student. There is also research on the significant difference between the number of "correct answers of the students when developing algorithms and the number of correct answers when solving problems involving similar algorithms" (Montero & Mahecha, 2020, p. 3), whose have less correct answers have lowest reading comprehension in mathematical sentences.

Reading is a complex activity, so "reading means, for the knowledge of the 21st century, to *be located* in the international context, and *to be* in accordance with its advances" (Durango, 2017, p. 158). For Goyes (2016), reading is an effort that occurs in the *biconditionality* of intentions between the reader and the text. This relation makes possible many meanings of the same object. However, Cairney (2002) adds a third element: the contextual factors. In this triangular approach between the *readers* who reads a text written by an *author* in their respective *particular contexts*, the reading comprehension has its genesis. It is necessary to clarify that none of these three components is more important than the other one, due to they are strongly correlated. Consequently, the reader is the one who turns these three elements and who "processes, criticizes, contrasts and values the information, who enjoys it or rejects it, who gives meaning and sense to what he reads" (Solé, 2001, p.9).

From this perspective, a text has a meaning borrowed from the author, and based on this, Solé (2001) defines *efficient reading* as the process that occurs when the meaning extracted by the

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reader matches what the author intended to communicate. So *efficient reading* is not a technique for fast reading, it means reading with purpose, understanding and strategy. Again, this emerges when, both the author and the reader use the same codes: "linguistic systems, lexicon, syntax, semantics, narrative grammar, pragmatics, etc." (Cairney, 2002, p. 31). Therefore, *efficient reading* provides a structure that facilitates the measurement of reading comprehension processes. Finally, context influences both the author and the reader. Indeed, the context is one of the most important keys in the construction of meaning. Thus, it is impossible to get a correct interpretation about an object without thinking on the contextual factors, such as culture, historical moment, and social context.

Levels of Reading Understanding

Levels of reading understanding take place in the reading process, which the three previously stated actors, text, reader, and context are involved. Undoubtedly, we reach these levels by using our previous knowledge. In this order of ideas, Strang (1965), Jenkinson (1976), and Smith (1989) describe three levels of reading comprehension: literal, inferential, and critical. However, authors such as Durango (2017) classify it as literal, inferential, and critical-intertextual. In this written, we will use the first classification. Namely:

Literal Level

At the literal level, the student must make a recognition of everything that appears explicitly in the text. So that, the reader can reconstruct the text with his own words in which the structure of the text is included.

Inferential Level

A reader who reaches the inferential level must identify relationships that are not explicitly in the text. Inferential or interpretive reading involves both logical deductions and conjectures. These conclusions could be made from certain data or using other suppositions.

Critical Level

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The critical level is considered the ideal level, since it implies a formation of own judgments of subjective character that involve identification with the characters, with the author and the context. Reading at this level has an evaluative character, which takes account of the training of the reader, as well as criteria and knowledge on the subject one read.

Visualization

For the reading comprehension of the MVT it is necessary that the student have previous knowledge such as: real numbers, intervals, Cartesian plane, slope, function, continuity, and derivative. Besides, they need to know how objects are related in the construction of a mathematical statement, so in a theorem. This relationship requires actions of representation and visualization of mathematical objects. Thus, in accordance with Bråting (2012, p. 17), "visualizations can certainly be sufficient for convincing oneself of the truth of a statement in mathematics, provided that one has sufficient knowledge of what they represent".

In this work, the articulation of the treatment of hypotheses and graphs for the reading comprehension of the MVT has a central role. The teaching of this theorem begins with mediation by the teacher where the reading and writing of the hypothesis of Rolle's Theorem is introduced and the graphic representation is visualized by each student that satisfies these hypotheses.

METHOD AND RESULTS

The population on which the research is carried out focuses on a convenient sample of 29 first-semester students of the Engineering programs of the Universidad de La Salle, to whom a questionnaire (based on the MVT) was applied. The purpose was to measure the level of comprehension and appropriation of the mathematical language that is formally used at the university lectures. We didn't use a control group because the instrument applied has a geometrical point of view employed in order to improve the understanding of the MVT. The traditional form to explain this theorem at the university selected is to introduce it by an algebraic and algorithmic way, so the students of a control group had not had the same mathematical tools.

The research was developed in three stages. The first one focused on the activation of previous knowledge, in the second one, the instrument was applied, and in the final stage the results were analyzed. For the validity of this instrument, we used a methodology with expert triangulation made by professors with more than 15 years of teaching experience, and as a result, one author

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of this article made a presentation in the CIAEM 2023[1] about reading comprehension in mathematics.

1. Activation of Prior Knowledge

The activation of the previous knowledge of the students allows to identify the capacities, and in a certain way, the attitudes that the students assume before the new information that arrives. According to Solé (2001), when the information is not understood, to a large extent, it is because there is no adequate establishment of such knowledge, or because the text is not adequately developed for its understanding. In this context, if a teacher wanted to implement this didactic proposal, he could develop it in the following way:

a) Introduce Rolle's Theorem in class, review some concepts, and generate spaces where the student can create hypotheses and ideas with the help of the visualization of graphic behaviors.

b) Recognize the level of reading comprehension through inquiry into concepts such as domain, interval, continuous function, derivative of a function, tangent line, among others.

c) Finally, each student is asked for their own graphic description of the theorem.

2. Application of the Instrument

In the second stage, a written instrument of seven questions was applied. This instrument was a questionnaire divided in the following manner: Question (1) asks the student about the previous knowledge of the MVT; question (2) asks what they hope to learn from the theorem. These two questions were taken as the basis to identify the degree of progress a student can reach after reading and trying to use the theorem without teacher measurement. Next, the theorem is stated to them and questions (3) and (4) are asked, in order to identify the vocabulary, lexicon, previous mathematical concepts and key ideas. That is, the third question allows us to recognize, through the description itself, what the student conceives of the theorem. Question (4) inquires about the level of decoding of the theorem through graphical representations. Thus, question (3) indicates whether the student reaches the *literal level* and question (4) indicates whether the *inferential level* was reached.

[1] https://xvi-ponencias.ciaem-iacme.org/index.php/xviciaem/xviciaem/paper/view/2094

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Questions (5), (6) and (7) were designed to measure the *critical level*, since they allow analyzing the connection of the themes, the validity of the information provided and the formation of judgments.

3. Results

In this case, the implementation of the exercise was carried out in a Differential Calculus course with 29 students from engineering programs. The assessment was carried out in two steps, with the answers to the questions (1) and the following scale was established:

• The student has no knowledge about the MVT

• The student recognizes some objects involved with the MVT (but not all).

• The student evidences the objects on the MVT, but does not relate them correctly

• The student recognizes the objects required for the construction of the MVT and writes them correctly.

Of the 29 students, 18 had no prior knowledge of the mean value theorem. Statements such as:

Original Statements	**Translated Statements**
“Es el punto medio de una función”	"It is the midpoint of a function"
“No sé nada sobre este teorema”	“I don't know anything about this theorem”
“Sé que tiene tres hipótesis (reglas) que se deben cumplir, sino se cumplen las tres no sería teorema del valor medio (Rolle)”	"I know that it has three hypotheses (rules) that must be fulfilled, without these, it would not be the mean value theorem (Rolle)"
“Que la pendiente de la recta tangente debe ser 0, que f(a)=f(b) y al ser la pendiente 0, es derivable”	"That the slope of the tangent line must be 0, that f(a)=f(b) and since the slope is 0, it is differentiable"

Table 1: Student interpretations

Of the 29 students, 10 were identified who recognized some objects involved, finding responses such as: "It is a continuous function, which is differentiable and has a minimum point c, in which its derivative is 0"; only one that recognized all the objects, but did not relate them correctly, an example of student writing in this case was: “It meets the requirements such as: it must be continuous, it must be differentiable, the point must be parallel. It must not be cut at any point”. And no student showed complete and correct prior knowledge.

However, after reading about the Mean Value Theorem, the answers to questions (3) were classified as *literal level* if only an adequate paraphrasing of the Mean Value Theorem was found. The

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inferential level was assigned if both the answer to question (3) as well as the graph in (4) were adequate. And finally, a *critical scope* was assigned if from question (3) to (7) the answers were adequate. It was found that 15 of the 29 failed to extract information from the reading, text such as:

• "It is the subtraction of the points [a, b] with the intention of finding the derivative of f´(c) where c would be the mean value"

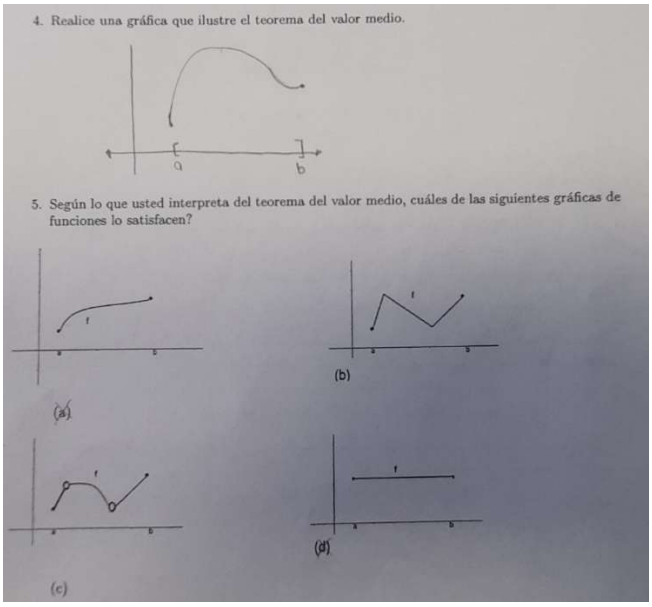
• "That a and b are mean value intervals and are differentiable and c is the variable that is in function f, but is also in a ratio of a and b"

We also found 7 students achieved *literal level*. We can read phrases like:

• "That when a and b are real numbers and there is a continuous function on it and the function is differentiable, c exists in the plan, that is, it has slope"

• "f is continuous on the interval [a, b], f is differentiable on (a, b), this implies that c lies within (a, b) such that its derivative is the subtraction of $f(b) - f(a)$ over $b - a$. It implies that the functions may be different."

We saw that 6 students reached the inferential level. The following graphic responses were obtained to questions (4) and (5):



Translate statements
4. Make a graph that illustrates the mean value theorem.
5. Based on your interpretation of the mean value theorem, which of the following function graphs satisfies it?

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Figure 1: Student responses to questions (4) and (5)

For question (4) it should be noted that the example in Figure 1 is different from the particular application of Rolle's Theorem.

And only one student reached the critical scope, this case was classified as atypical within the participating group, because he is one of the students who started from ignorance of the theorem, but the whole process led him to a good application:

> Question (1): "I know that the mean value theorem is used to find the derivative of the central or intermediate point of the interval in a function f(x) continuous and differentiable on said interval, which can be found from when the derivative gives 0, in addition to being used as the basis of Rolle's theorem".

After reading the theorem, the student indicates in Question (3):

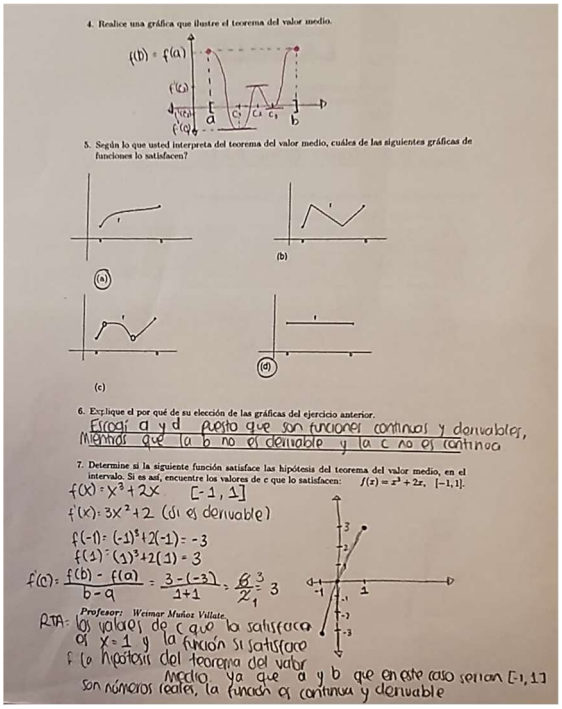
> I learned that there is an average value of a continuous function f(x) as long as the interval is composed both at its beginning and at its end by real numbers and so with these values, the formula of this theorem can be applied to find the derivative of that midpoint.

For questions with a graphic response (4) and (5), an example different from Rolle's theorem was obtained and he achieved an adequate justification that he presented in the answer to question (6), and he was the only student who applied the theorem in the proposed exercise of question (7) (Figure 2).

What is interesting in these results is to identify the changes in the appropriation of the mathematical object MVT, relating prior knowledge with that achieved after reading. In Figure 3, it can be seen the group has the most changes it's the one that had no prior knowledge about the theorem. These were the only ones to reach *inferential* and *critical level*. Those who somehow had some preconception did not pass the *literal level*.

Although the degree of activation of preconceptions, at a statistical level, did not show to be related to the level of reading comprehension (Fisher's exact test of independence, p_value of 0.1162), it can be assumed that if there is a motivation (not knowing the specific factor that generates) by extracting information from reading, while for those with some previous level it seems to be satisfied with what is already known.

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Translated Statements

4. Make a graph that illustrates the mean value theorem.
5. Based on your interpretation of the mean value theorem, which of the following function graphs satisfies it?
6. Explain why you chose graphs in the previous exercise.

Student solution:"I chose **a** and **d** since they are continuous and differentiable functions, while **b** is not differentiable and **c** is not continuous."

7. Determine whether the following function satisfies the assumptions of the mean value theorem, on the interval. If so, find the values of "c" that satisfy it: $f(x) = x^3 + 2x, [-1,1]$.

Student solution: "The values of **c** that satisfy it are x=1 and the function does satisfy the hypothesis of the mean value theorem, since **a** and **b** which in this case would be [-1,1] are real numbers, the function is continuous and differentiable"

Figure 2: Student responses that reached the critical level

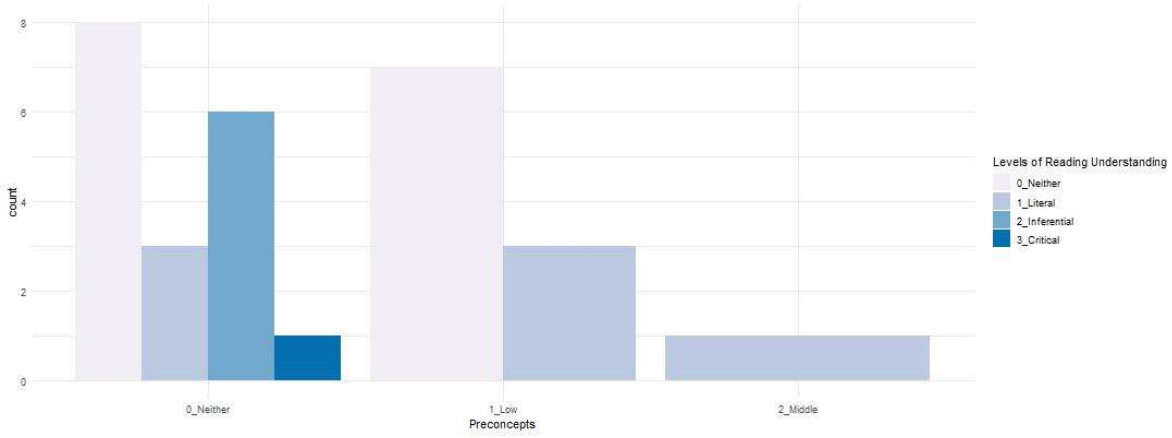


Figure 3: Distribution of students according to activation question ratio and maximum level of reading comprehension reached

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DISCUSSION

To improve the understanding of the MVT and the derivative applications of a function through its different representations (e.g., algebraic and graphic), students must develop the cognitive process to allow them to establish an appropriate relationship between concepts, hypotheses, theorems, and mathematical objects (García-García & Dolores-Flores, 2021). To achieve appropriate results in reading comprehension of theorems, the activation of prior knowledge must include the visualization of mathematical objects and the development of skills that promote mathematical connections. In this order of ideas, visualization allows the empowerment of some representations of mathematical objects and their relationship. Particularly, it will help the reading comprehension of MVT as shown in the college's texts.

In many cases the mathematical text is limited to the conceptualization of the students (de Almeida & da Silva, 2018), consequently they achieve at most a literal level of reading comprehension. To develop the other levels, it is required that the teaching dynamics tend to improve conceptual reasoning and the functions of visual presentations, since usually these presentations are limited to the illustrative part (Natsheh & Arseniy, 2014). In addition, care must be taken to ensure that the lessons planned by teachers promote cognitive growth, algorithmic mastery and reduce the cognitive load on students (Toh, 2022).

In accordance with Park (2015), visual mediations with symbolic, algebraic, and graphic notations would help students understand concepts involved in the definition of a derivative, such as limit of a function and interval. These mediations would allow the activation of prior knowledge necessary for reading comprehension.

CONCLUSIONS AND RECOMMENDATIONS

Reading comprehension becomes a mediator for the appropriation of mathematical language, without said appropriation becoming a guarantee that the student learns mathematics. In other words, a student may master mathematical language appropriately, but not understand the text, and this may be one of the reasons for the need to develop strategies and didactical tools in order to promote reading comprehension.

Reading comprehension strategies allow students to deepen and appropriate the contents of a mathematical text. In addition, they promote the analysis of the hypotheses or main ideas; the importance to verify veracity of the assumptions; formulate diagrams in order to make an alternative graphical representation to the idea that the text intends to develop; and, finally, clarify the information through particular examples.

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The teacher's intervention is crucial in the reading process in at least three moments: before (activation of prior knowledge and planning of activities), at the time of reading (implementation of activities), and after reading (evaluation of activities and appropriation of learning). In all three moments, motivation becomes an essential characteristic of the process.

When students with prior knowledge of the MVT achieve lower results, it suggests that their understanding may be hindered by a non-geometric or non-contextualized approach to the theorem. This tendency might lead them to merely grasp a superficial understanding of the MVT, lacking true ownership of its concepts. Conversely, those who approach the MVT from a geometric perspective or within a contextualized framework tend to develop a deeper comprehension of the theorem.

It's notable that a small proportion of readers reach a critical level, primarily due to inadequate preparation in developing essential cognitive processes such as note-taking and creating diagrams or graphics to enhance text comprehension (Vásquez, 2015). Therefore, ensuring the attainment of critical reading proficiency necessitates motivating students beyond mere text consumption. Instead, they should be encouraged to actively engage in reconstructing meanings through diagram creation. Education ought to discourage conformist or passive attitudes, as observed in students with prior knowledge in the experiment.

This research is the first step to improve comprehension of theorems we use to introduce in calculus classes. In order to expand the scope of this research, the next step will be to offer more didactical tools to teach others theorems, (e.g., the Fundamental Theorem of Calculus), perhaps using an historical and also geometrical point of view (Muñoz, et al., 2023). In addition, we can use those material to think about explaining in other manner the Green’s or Stokes’s Theorem.

The limitations of this study concern the small sample size and the collection of information in a single educational institution. This could make it difficult to generalize the results. Future research requires a more in-depth review of the development of reading comprehension skills at all educational levels, which are scalable to other areas of knowledge and involve other topics in mathematics. Furthermore, we suggest making longitudinal studies where measures of the impact of mathematics on the development of the inferential, literal, and critical levels could be possible.

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MATHEMATICS TEACHING RESEARCH JOURNAL 42
SUMMER 2025
Vol 17 no 3

Development of an Augmented Reality-Based Learning Module on Isometric Transformation for Form 2 Students in Malaysia

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Abstract: Augmented Reality (AR) enhances education by providing immersive and interactive learning experiences, particularly for students with low spatial skills, as well as in conceptual and procedural knowledge. In the Malaysian Form 2 mathematics curriculum, Isometric Transformation presents significant challenges. Traditional teaching methods often fall short, leading to gaps in learning. AR facilitates real-time visualization, enhancing spatial reasoning and conceptual understanding.. Simultaneously, the learning module offers structured guidance, activities, and exercises to reinforce learning outcomes. This study examined the effectiveness of an AR-based learning module in enhancing students' spatial reasoning and their conceptual and procedural knowledge. The study involved 80 Form Two students from a public high school in Putrajaya. The researchers created tests based on the level of competence that students must achieve according to the Curriculum Standard Document and Form 2 Assessment (DKSP). The difficulty index testing is conducted following the guidelines set by Brookhart and Nitko (2018). The document analysis revealed that over 80% of Form 2 students in a public high school in Putrajaya, Malaysia, struggle with questions related to Isometric Transformation, indicating the students' weaknesses in spatial visualization and conceptual and procedural understanding. The analysis also showed that traditional teaching approaches may not adequately support students' learning needs. The results emphasize the potential of AR-based learning modules to provide targeted support, enhance comprehension, and improve performance in Isometric Transformation. This study highlights the need for innovative, technology-enhanced instructional methods to tackle these educational challenges.

Keywords: Augmented Reality, Isometric Transformation, Module Development, Spatial Visualization, Conceptual and Procedural Knowledge.

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INTRODUCTION

The fast growth of technology in the 21st century has considerably impacted the teaching-learning experience, posing new challenges and broad consequences for educators. The rapid growth of technology in the education system significantly affects mathematics teaching and learning (Karim & Zoker, 2023). Students are eager to use Information and Communication Technology (ICT) in Mathematics classes, but traditional teaching methods used by teachers have kept academic performance unchanged (Mendoza-Rodríguez & Caranqui-Sánchez, 2024).

Tools like Mathematica software improved students' mathematical abstraction ability and independence in analytic geometry compared to traditional learning (Murtianto et al., 2019). Similarly, Dynamic Mathematics Software (DMS) such as GeoGebra has been particularly noted for its ability to facilitate quick calculations and assist students in abstracting complex mathematical ideas. This promotes discovery learning and enhances student engagement (Garba, 2019; Yerizon et al., 2021). Integration tools like the Ethnobra (Ethnomathematical Geogebra) learning model are practical and effective in enhancing students' geometry skills (Hamidah et al., 2024).

Technology creates immersive learning experiences through extended reality technologies, including Augmented Reality (AR), virtual reality, and mixed reality (Meccawy, 2022). These technologies provide interactive and engaging platforms for students to explore geometric shapes and spatial reasoning, making learning more exciting and effective (Hwang et al., 2021; Ipek et al., 2021).

AR is becoming more recognized as a valuable tool in mathematics education. Several studies have investigated the potential of AR to strengthen mathematics learning outcomes. Its ability to create interactive, real-world learning experiences makes it a powerful tool for teaching complex concepts (Sáez-López et al., 2020). By offering a mobile-based platform, AR allows a more engaging and practical understanding of mathematical principles (Angraini et al., 2022). For instance, AR boosts motivation and academic performance (Lam et al., 2023). Additionally, a study by Ahmad and Junaini (2020) demonstrated that AR allowed students to interact with 3D models, which enhanced their understanding of geometry.

Similarly, Fernandez-Enriquez and Delgado-Martin (2020) stated that AR positively contributed to a better interpretation of geometry, enhanced spatial visualization, and increased student motivation. Other than that, Flores-Bascuñana et al. (2020) discovered that AR helped 6th-grade students better comprehend shape concepts than the classical didactic method. In a different study, Pozo-Sánchez et al. (2021) examined the combined use of AR and the flipped classroom method, showing that both strategies contributed positively to the learning process. The flipped learning model enriches student-teacher interaction, autonomy, learning depth, classroom time utilization, and academic performance. Meanwhile, AR as a techno-educational tool boosts motivation, content accessibility, peer interaction, and problem-solving skills. Finally, Ahmad (2021) confirmed

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that AR boosts visual thinking in 10th-grade students learning geometry, compared to the traditional method. In conclusion, the findings from these studies confirm that AR is a technology that can significantly amplify specific skills and deepen students' discernment of geometry.

One key area where AR can be applied is in teaching Isometric Transformation, a crucial concept in mathematics. Isometric Transformation in mathematics preserves the distance between points in a given space. This type of transformation is essential in various fields, including signal processing, machine learning, and mathematics education (Kolouri et al., 2017). For instance, in the Malaysian Secondary School Mathematics Curriculum, Isometric Transformation is a fundamental topic (Nadarajan et al., 2023).

Isometric Transformation, which includes translations, reflections, rotations, and combinations, is complex for students to grasp. Research has shown that students often face difficulties executing and identifying these transformations, comprehending their underlying concepts, and overcoming misconceptions about their functionalities (Güven, 2012; Yanik, 2011). Students who face challenges in learning Isometric Transformation may face various obstacles in understanding the concept, subsequently affecting their performance. Various studies have explored teaching aids and instructional methods to optimize student achievement in Isometric Transformation.

Unlike conventional learning, Hamid et al. (2022) determined that non-digital games intensify engagement, motivation, and performance, especially when paired with cognitive-level questions. Similarly, Muhammad and Adnan (2023) developed *Transformasi Catur*. This chess-based game made learning more interactive, while Mohd Nasir et al. (2023) introduced the *Kit Transformasi Isometri*, which effectively improved student achievement and motivation. These studies highlight the benefits of game-based learning compared to traditional teaching methods.

However, most teaching aids developed are physical tools like colored cards and boards with storage and durability issues. Meisuri et al. (2023) suggest that technology-based teaching aids can overcome these limitations. In this context, Nadarajan et al. (2023) implemented a flipped classroom model to deepen Higher-Order Thinking Skills (HOTS), showing significant improvements in students' critical thinking and understanding compared to conventional instruction. Karatas (2022) and Mukamba & Makamure (2020) established that GeoGebra significantly improved students' understanding, problem-solving skills, and overall performance in Geometric Transformations compared to traditional methods. These studies highlight the importance of innovative teaching methods for Isometric Transformation. However, there is a gap in research on using AR to strengthen skills, specifically in Isometric Transformation.

Spatial abilities and conceptual and procedural knowledge are essential in learning Isometric Transformation. Research has indicated that students' spatial skills substantially impact their learning achievements when exposed to dynamic and static visualizations (Sudatha et al., 2018). Moreover, studies have highlighted the relationship between visual-spatial skills and mathemat-

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ics ability, emphasizing the importance of spatial visualization ability for high achievement in mathematics (Bakker et al., 2022). Conceptual and procedural knowledge are both crucial in learning Isometric Transformation. In mathematics education, highlighting conceptual and procedural knowledge is essential for optimal learning outcomes (Hussein, 2022).

Visual-Spatial

Studies show that students face considerable challenges with spatial visualization skills, essential for understanding topics such as Geometry and Isometric Transformation (Gonzalez-Campos et al., 2022). Research indicates that effective technology integration in mathematics education can significantly improve students' spatial abilities, critical thinking skills, and overall academic performance (Sevari & Falahi, 2018; Paulo et al., 2022). AR strengthens students' spatial abilities, particularly in education, with significant benefits for those with low spatial visualization skills (Papakostas et al., 2021; Hawes & Ansari, 2020; Judd & Klingberg, 2021; Danakorn Nincarean et al., 2019). By transforming abstract concepts into tangible, interactive experiences, AR heightens students' spatial abilities, making complex ideas more accessible and engaging (Sukriadi et al., 2023; Masduki et al., 2022).

Furthermore, interactive AR has been claimed to provide cognitive benefits for learning over typical 2D interfaces, notably spatial visualization (Dünser et al., 2012). Studies showed that AR-based learning environments had a relatively more significant influence on low-spatial-ability students, indicating the potential of AR to benefit this specific group of learners (Phon et al., 2015). By allowing students to interact with 3D models in real-time, AR enables them to manipulate objects and directly observe the effects of transformations like translation, reflection, and rotation. This hands-on approach makes abstract concepts more tangible and comprehensible (Küçük et al., 2016). Moreover, AR can contextualize geometric concepts in real-world settings, facilitating better comprehension and retention among students, especially those who find abstract concepts challenging (Gerup et al., 2020). Additionally, AR can contextualize geometric concepts in real-world settings, assisting students in better relating to and understanding the material.

For students with low spatial abilities, AR provides a personalized learning experience where they can explore and grasp concepts at their own pace without pressure (Kase et al., 2023). Moreover, the potential of AR to offer highly realistic learning experiences suggests its applicability in helping students with low spatial-visual abilities to discern complex spatial concepts (Kamphuis et al., 2014). Research shows AR intensifies achievement and visual thinking in geometry students (Aldalalah et al., 2019), which is crucial since spatial skills directly impact mathematical performance (Qomario et al., 2022; Putri et al., 2022). AR enables hands-on inter-

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action with 3D models, making transformations more tangible, improving understanding and retention (Yanuarto et al., 2024).

Conceptual and Procedural Knowledge

In learning Isometric Transformation, conceptual knowledge plays a crucial role in facilitating the understanding and application of novel procedures. Conceptual knowledge involves interpreting and understanding the relationships between concepts, which is essential for profoundly understanding the subject (Arslan, 2010). Students' misconceptions and lack of conceptual knowledge in geometry can significantly impact their performance in examinations (Luneta, 2015). Nasir et al. (2023) stated that students had difficulty understanding the concepts of translation, reflection, and rotation. They also face challenges in comprehending the variations involved in carrying out and identifying these transformations and their combinations (Olson et al., 2020; Rollick, 2009). These challenges are compounded by traditional teaching methods that may not sufficiently address today's students' visual and interactive learning needs.

Bodensiek et al. (2019) emphasize that AR has been a focal point of educational research as it can boost conceptual understanding and facilitate effective learning. Socrates and Mufit (2022) found that integrating AR into physics learning can heighten conceptual understanding, increase interest in education, and optimize critical thinking skills. Solikhin et al. (2022) studied the impact of AR-based learning media on chemistry students' conceptual understanding, highlighting the positive effects of AR on learning outcomes. Furthermore, Putra et al. (2022) reported that AR technology received positive feedback for helping students visualize complex concepts in 3D, hence increasing their knowledge. Additionally, Rohendi and Wihardi (2020) showed that mobile-based AR can facilitate more apparent teaching of 3D geometry shapes, leading to faster comprehension among students.

Procedural knowledge in learning Isometric Transformation involves understanding how to perform specific tasks or procedures related to Isometric Transformation. Procedural knowledge consists of understanding mathematical symbols and algorithms necessary for problem-solving (Mutawah et al., 2019). The relationship between conceptual and procedural knowledge is vital for developing mathematical competencies (Schneider & Stern, 2010). Conceptual knowledge provides the cognitive control needed to execute algorithms correctly, while procedural knowledge contributes to understanding mathematical concepts (Sáenz, 2008). Both types of knowledge are essential for students' mathematical development (Lenz & Wittmann, 2020).

In mathematics education, conceptual knowledge should be taught before procedural knowledge to boost learning outcomes (Rittle-Johnson et al., 2001). Research has shown that AR facilitates contextualization and individualization of training, enabling students to practice and apply procedures in diverse contexts, ultimately leading to better mastery (Barsom et al., 2016). Fauziah

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and Sulisworo (2019) reported significant improvements in students' performance after using AR-based learning media, indicating that such technologies can effectively bridge the conceptual and procedural knowledge gap. Moreover, AR offers immediate feedback on tasks, allowing students to rectify mistakes in real time, reinforcing learning, and improving procedural knowledge (Dutta et al., 2022; Moro et al., 2021).

Learning Module

Sidek and Ariffin (2011) state that the learning module helps students master and apprehend a subject more clearly. Similarly, Nugroho et al. (2019) describe the learning module as a complete unit of learning explicitly designed for students to use either for self-learning or in the classroom. Efforts have been made to develop learning modules to strengthen understanding and foster interest in Mathematics (Marham et al., 2023; Ibrahim et al., 2020). The use of learning modules has been proven to amplify student learning outcomes across various subjects, including Mathematics (Basari & Moi, 2022), Science (Jumaat et al., 2022), Entrepreneurship (Ahmad & Siew, 2022), and History (Kaviza, 2021).

Modules can be integrated with technology, such as e-modules with computing media (Siregar & Harahap, 2020) and video-based learning modules (Kamlin & Keong, 2020). M-learning modules further elevate flexibility, enabling learning anytime and anywhere (Ismail & Kob, 2023). This integration significantly optimizes student performance through diverse technological approaches. Previous research has also found that integrating technology, specifically AR within learning modules, significantly impacts student learning outcomes more than using technology alone or a module without technological enhancements.

For instance, Nadzri et al. (2023) developed an AR-based learning module for Ruang, demonstrating that Year 4 students in the treatment group achieved higher mathematics performance than the control group using a learning module without AR integration. Similarly, Dewi and Kuswanto (2022) evaluated an AR-based physics e-module that effectively improved mathematical communication and critical thinking skills among Grade 12 students compared to the conventional teaching method. Their analysis confirmed the module's effectiveness in enhancing these abilities. In the field of chemistry, Whatoni and Sutrisno (2022) developed an AR learning module for Chemical Bonding, which significantly increased upper-secondary students' interest and motivation, with apparent differences observed between pre-and post-intervention results. Additionally, Pratamadita and Dwiningih (2022) developed an interactive e-module for Visual-Spatial Training that effectively advanced the spatial intelligence of Grade 10 students.

Their research found a significant difference between pretest and posttest results, indicating that the interactive 3D elements in the e-module positively impact students' visual-spatial intelligence. The 3D interactive elements allowed students to construct concepts and independently el-

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evate their understanding of intermolecular forces. Past studies have shown that learning modules boost students' achievement and specific skills. The effectiveness of these modules highlights their potential to optimize learning processes and educational outcomes, providing valuable insights for developing more effective learning modules in the future.

Despite the potential of AR and learning modules, their application in Malaysian education, particularly in teaching Isometric Transformation, remains limited. Most AR resources focus on other geometry or mathematics topics, leaving a gap in addressing students' challenges with visual-spatial, conceptual, and procedural understanding of Isometric Transformation. This highlights the need for a comprehensive needs analysis to develop a well-structured AR-integrated learning module specifically for Form 2 students in Malaysia.

RESEARCH OBJECTIVE

This study was conducted based on the researcher's need to obtain information about the topic, characteristics, and teaching module specifications that want to develop. Therefore, the objectives of the study are:

1. To identify students' difficulties in spatial visualization related to understanding and applying Isometric Transformation based on their test performance.
2. To analyze the conceptual and procedural knowledge gaps in students' understanding of Isometric Transformation.
3. To assess how integrating AR-based learning modules can potentially address these challenges by improving students' spatial reasoning, conceptual understanding, and procedural execution of Isometric Transformation.

RESEARCH QUESTION

The research questions for the needs analysis phase are as follows:

(a) How do the students experience spatial visualization when answering questions about the Isometric Transformation topic?
(b) How do the students demonstrate their level of conceptual and procedural knowledge of the Isometric Transformation topic while answering related questions?
(c) How does integrating an AR-based learning module improve students' spatial reasoning, as well as their conceptual and procedural knowledge of Isometric Transformation?

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METHODS

This study utilizes the method of analyzing test responses, which examines students' answer scripts on the topic of Isometric Transformations for Form 2.

Sample

This study was conducted at a public high school in Putrajaya. Eighty students from three classes were randomly selected using a simple random sampling method from a total of 231 students across seven classes. This approach ensures that the sample is representative of the entire population, allowing for the study's results to be generalized to the broader group of Form 2 students.

Procedure

Students learned about Isometric Transformation through traditional teaching methods without incorporating AR. This phase lasted for the standard duration as outlined by the school curriculum. After completing the traditional teaching phase, students were administered the achievement test on how they experienced spatial visualization in answering questions related to the Isometric Transformation topic and demonstrated their conceptual and procedural knowledge while answering related questions.

Instrument

The achievement test consisted of subjective questions adapted from items in the KSSM Mathematics textbook, focusing on the performance levels outlined in the DSKP Mathematics Form 2 curriculum for the Isometric Transformation topic. The administered test included four sections with a total of ten main questions. This achievement test comprised ten questions specifically targeting performance levels (TP) 1 to 4. This approach was chosen because these test items are designed to assess the basic and intermediate skills essential for subject mastery at this educational level. By restricting questions to TP 1 to TP 4, we ensure that all students grasp the necessary fundamentals before progressing to more complex concepts. This design guarantees that the test effectively evaluates students' knowledge of Isometric Transformation in alignment with the Mathematics Form 2 DSKP. Table 1 displays the mapping of the administered test against the curriculum specifications for the Isometric Transformation topic of Form 2.

In contrast, Table 2 shows the percentage distribution of the questions according to TP. Students are required to answer this test in 60 minutes. Each question carries a maximum of four marks.

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Category	Question	Sub Question	Performance Level (TP)
Translation, Reflection, Rotation	1		TP 1
	2		TP 2
	3	a	TP 3
		b	TP 3
	4		TP 4
Translation	1	a	TP 1
		b	TP 2
		c	TP 3
	2	a	TP 1
		b	TP 2
		c	TP 3
Reflection	1	a	TP 1
		b	TP 2
		c	TP 3
	2	a	TP 1
		b	TP 2
		c	TP 3
Rotation	1	a	TP 1
		b	TP 2
		c	TP 3
	2	a	TP 1
		b	TP 2
		c	TP 3

Table 1: Performance Test Mapping of Isometric Transformation.

Performance Level	Number	Percentage
TP 1	7	30.4
TP 2	7	30.4
TP 3	8	34.8
TP 4	1	4.4

Table 2: Percentage Distribution of Questions According to Performance Level (TP).

Difficulty Index

Difficulty index testing was conducted to verify that the constructed test items accurately assess the difficulty levels, categorized as easy, medium, or hard. Tables 3 and 4 demonstrate each

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question's difficulty index value and interpretation. Difficulty index testing for subjective items was made by referring to Brookhart and Nitko (2018) with calculations as below:

$$DifficultyIndex = \frac{(Averagemark - Minimummark)}{(Rangemark - Minimummar\ )} \quad (1)$$

Value	0.00-0.20	0.21-0.40	0.41-0.60	0.61-0.80	0.81-1.00
Interpretation	Too hard	Hard	Medium Hard	Easy	Too Easy

Table 3: Difficulty Index Value and Interpretation.

Category	Question	Sub Question	Difficulty Index	Interpretation
Transla-tion, Reflec-tion, Rota-tion	1		0.88	too easy
	2		0.08	too hard
		a	0.18	too hard
	3	b	0.06	too hard
	4		0.10	too hard
Translation	1	a	0.13	too hard
		b	0.10	too hard
		c	0.07	too hard
	2	a	0.06	too hard
		b	0.06	too hard
		c	0.04	too hard
Reflection	1	a	0.03	too hard
		b	0.03	too hard
		c	0.04	too hard
	2	a	0.03	too hard
		b	0.03	too hard
		c	0.02	too hard
Rotation	1	a	0.00	too hard
		b	0.00	too hard
		c	0.00	too hard
	2	a	0.00	too hard
		b	0.00	too hard
		c	0.00	too hard

Table 4: Difficulty Index Interpretation of Test Questions.

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Question 1 on translation, reflection, and rotation proved too easy for most students, with a difficulty index of 0.88. However, many other questions were excessively difficult, showing difficulty indices ranging from 0.00 to 0.20. The low difficulty index for numerous questions suggests that the current teaching approach may not be effective enough to help students grasp the key concepts of Isometric Transformation. Traditional teaching methods, such as lectures and textbook exercises, might not engage students sufficiently or cater to their diverse needs, limiting their ability to develop both conceptual and procedural knowledge. These methods often fail to enable students to interact with the material in a hands-on and meaningful manner. Given that Isometric Transformation requires strong spatial reasoning skills, students may struggle to visualize the concepts without appropriate tools, such as visual aids, manipulatives, or technologies like AR. Therefore, teachers should reconsider their teaching methods and may want to incorporate more interactive strategies to assist students in better understanding the topic. Additional practice and the integration of teaching aids, such as AR, could alleviate the challenges that students encounter.

Data Analysis

To ensure the validity of the achievement test, the test items were carefully reviewed and validated by an expert mathematics teacher. Feedback from the expert was incorporated into the revision of the test, enhancing its content validity. Participants in the study were selected based on specific criteria. For confidentiality, their names were replaced with labels from P1 to P80. After collecting the answer scripts, the researcher meticulously checked each one against a verified marking scheme, assessing how well students understood the Isometric Transformation.

For data analysis, the researcher organized and reviewed all scripts in line with the approved marking scheme to ensure consistent and accurate scoring. Each test item's Difficulty Index was then calculated to gauge how challenging the questions were. Consequently, all scores were recorded in Microsoft Excel, where they were analyzed to identify performance trends and patterns. This analysis revealed students' challenges and pinpointed areas for additional focus. The findings will also be used to refine and optimize the design of the AR-based learning module to be developed, ensuring it better addresses students' needs and intensifies their learning experience.

RESULT AND DISCUSSION

Document analysis methods were employed to address the research questions. Each student's achievement test answer script is reviewed and analyzed. The findings indicate that most students struggle with conceptual and procedural knowledge regarding Isometric Transformation

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across all questions. Table 5 presents the number of students categorized by the specified mark range for each question, evaluated according to the conceptual and procedural knowledge rubric.

Category	Question	Sub Question	Knowledge	Score					
				4	3	2	1	0	Not answer
Translation, Reflection, Rotation	1		Conceptual	72	0	0	4	4	0
	2		Conceptual	1	0	0	23	19	37
	3	a	Procedural	16	0	0	0	35	29
		b	Procedural	5	0	0	0	38	37
	4		Procedural	4	2	1	11	18	44
Translation	1	a	Conceptual	9	4	0	0	23	44
		b	Procedural	9	0	0	0	13	58
		c	Procedural	5	0	1	3	11	60
	2	a	Conceptual	3	3	1	0	6	67
		b	Procedural	3	2	1	0	7	67
		c	Procedural	2	1	1	2	6	68
Reflection	1	a	Conceptual	2	1	0	1	5	71
		b	Procedural	1	2	1	0	5	71
		c	Procedural	1	3	1	0	3	72
	2	a	Conceptual	1	2	0	0	2	75
		b	Procedural	0	3	0	0	2	75
		c	Procedural	0	2	0	0	2	76
Rotation	1	a	Conceptual	0	0	0	0	3	77

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	b	Procedural	0	0	0	0	1	79
	c	Procedural	0	0	0	0	1	79
	a	Conceptual	0	0	0	0	1	79
2	**b**	Procedural	0	0	0	0	1	79
	c	Procedural	0	0	0	0	1	79

Table 5: Analysis of Students' Conceptual and Procedural Knowledge Based on Test Scores.

Table 6 quantifies student performance. The first table breaks down the number of students by score ranges for questions, translation, reflection, and rotation. Consequently, it is further subdivided into conceptual and procedural knowledge categories. The data suggest that many students scored poorly, with a notable portion not answering many questions. For instance, zero scores were prevalent across various questions, especially in the procedural knowledge category. In the conceptual score analysis, most students (75%) fell within the 11-20% range, while 85% scored between 0-10% in procedural knowledge. These statistics underscore students' struggles with Isometric Transformation concepts and procedures, and the low proficiency levels suggest a need for educational intervention in these areas.

The analysis discovered that students faced more procedural problems than conceptual ones. This could be attributed to the nature of the subject matter. Isometric Transformation involves significant procedural knowledge, such as understanding translation, reflection, and rotation. These procedural demands require a deep understanding of mathematical concepts and precise execution, which can be challenging for students. Additionally, procedural knowledge often requires more practice and application than conceptual knowledge, which may contribute to the higher frequency of procedural problems among students. Mastery of procedural skills usually relies heavily on a solid understanding of the underlying concepts. Students who have not grasped the fundamental concepts may struggle to apply those procedures effectively. In education, it is crucial to ensure that students comprehend the concepts before advancing to procedural aspects, as this facilitates deeper and more enduring learning.

	Range Marks (%)					
Category	**51-60**	**41-50**	**31-40**	**21-30**	**11-20**	**0-10**
Conceptual	3	1	3	6	60	7
%	**3.75**	**1.25**	**3.75**	**7.5**	**75**	**8.75**

Table 6: Analysis of Students' Performance Based on Conceptual Knowledge Scores.

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	Range Marks (%)					
Category	**51-60**	**41-50**	**31-40**	**21-30**	**11-20**	**0-10**
Procedural	1	2	0	2	7	68
%	**1.25**	**2.5**	**0**	**2.5**	**8.75**	**85**

Table 7: Analysis of Students' Performance Based on Procedural Knowledge Scores.

Based on the conceptual and procedural scores, the study of student performance revealed that many scored low in both categories. Further analysis indicated that most students face considerable challenges, particularly with the Isometric Transformation type of rotation, as shown by the high number of students who did not answer correctly in Table 5. The number of students who answered incorrectly for rotation is significantly higher than for translation and reflection, suggesting that rotation is the most challenging Isometric Transformation type. Unlike translation (which involves straightforward shifting) or reflection (which has a clear mirror effect), rotation requires turning an object around a fixed point, which can be difficult for students to visualize mentally, especially without interactive tools. Contributing factors may include a lack of understanding of the concepts and procedures related to Isometric Transformation.

Based on the results, students face significant challenges with procedural aspects rather than conceptual understanding in Isometric Transformation. While they may grasp that a shape remains unchanged in size and orientation after a transformation, correctly executing the steps can be difficult. This difficulty contributes to the challenge of visualizing movements. Recognizing that a shape is being reflected or rotated differs from confidently placing it in the correct position. Additionally, following the step-by-step process requires careful attention to detail; even a small mistake in identifying coordinates or selecting the appropriate transformation can lead to errors. This highlights the need for targeted interventions and support in teaching and learning Isometric Transformation concepts and procedures.

In addition, it was found that students did not master other topics related to Isometric Transformation, such as Coordinates, where students could not plot coordinate points well, as shown in Figure 1.

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Question

1) Under a translation, the image of P (-2, 3) is P' (-7, -2).
   a) Describe the translation in form $\begin{pmatrix} a \\ b \end{pmatrix}$.
   b) Determine the image coordinates of Q (2, 1) under the same translation.
   c) Determine the object coordinates of R' (-1, -2) under the same translation.

Student Answer

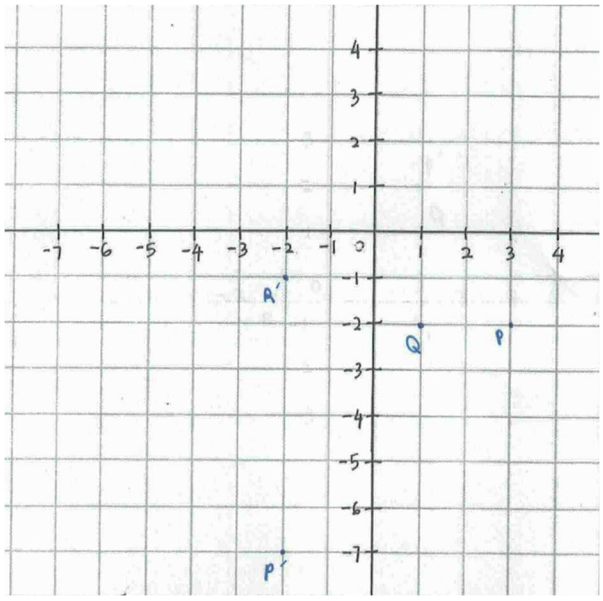


Figure 1: Example of Student Errors in Plotting Coordinate Points.

However, Figure 2 illustrates that the student has a basic knowledge of graphs. Specifically, he can plot coordinate points well but does not understand the concept of translation and reflection when he fails to perform translation correctly and determine the axis of reflection.

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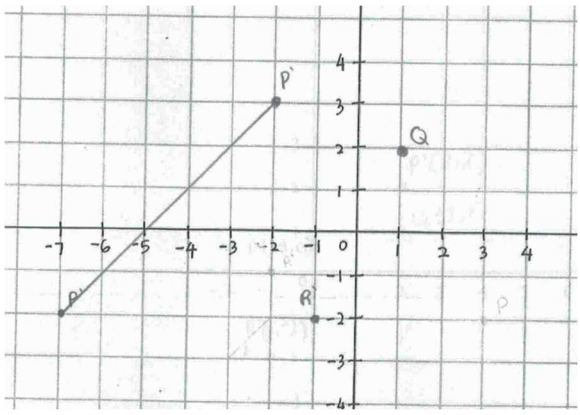
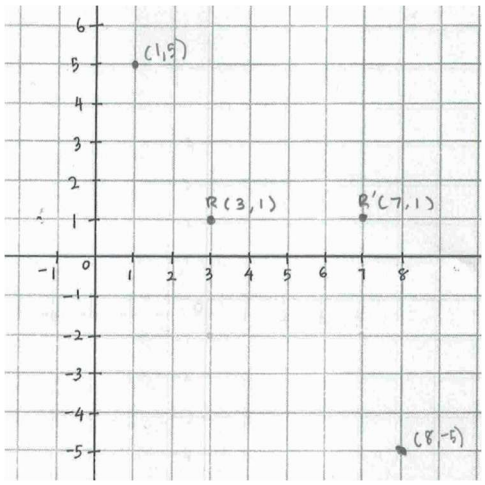
	Student 1	Student 2
Question	1) Under a translation, the image of P (-2, 3) is P′ (-7, -2). a) Describe the translation in form $\binom{a}{b}$. b) Determine the image coordinates of Q (2, 1) under the same translation. c) Determine the object coordinates of R′ (-1, -2) under the same translation.	1) R (3, 1) is mapped to R′ (7, 1) under a reflection. a) Describe the reflection by stating the axis of reflection. b) Determine the image coordinate of (8, -5) under the same reflection. c) Determine the object coordinate of (1, 5) under the same reflection.
Answer		

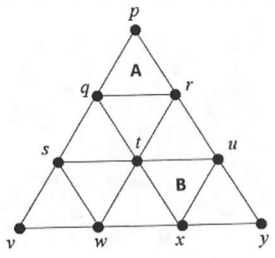
Figure 2: Student Responses Indicating Misunderstanding of Isometric Transformation.

Figure 3 illustrates the answers of students who got the highest marks among 80 students. When answering the question correctly, the first question shows the student's understanding of translation, reflection, and rotation concepts and procedures. Consequently, he can answer questions about translation and reflection well. This means that this student has good conceptual and procedural knowledge about Isometric Transformation. However, regarding the question about rotation, he did not answer, even though he was knowledgeable about it. This is likely because he could not visualize the concept of rotation, which prevented him from responding.

This demonstrates that he has a low visual-spatial ability. However, this is merely a researcher's assumption. A visual-spatial test on him could confirm this. The same observation can also be noted in the answer script of the student who received the second-highest score among all the students (Figure 4). Figures 3 and 4 illustrate the script of the student's answers.

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2) The following picture shows some matchsticks that have been arranged to form some triangles.



State the two transformations involved if triangle A wants to reach triangle B. Use the labels on the circles (p, q, r, s, t, u, v and w) to state the transformation completely.

Answer: Reflection

Transformation 1: Pantulan, q r t

Transformation 2: translasi, t u x

Translation

1

1) M (-3, -1) is mapped to M' (1, -3) under one rotation.

a) Describe the rotation by stating the direction, center, and angle of rotation.

b) Determine the image coordinates of (-2, -1) under the same rotation.

c) Determine the object coordinates of (-2, 2) under the same rotation.

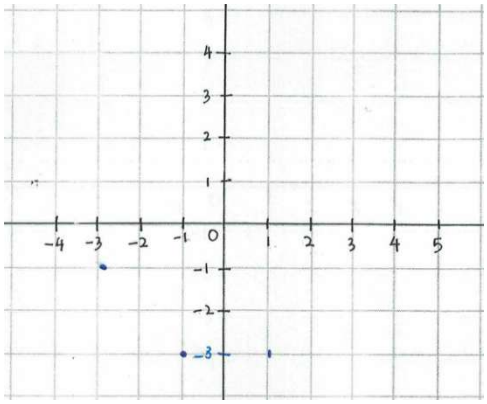
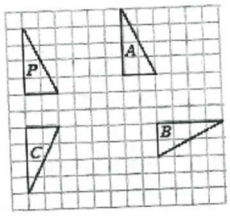


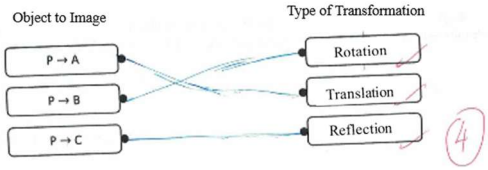
Figure 3: Answer Script of the Highest-Scoring Student in the Isometric Transformation Test.

ISOMETRIC TRANSFORMATION

1) In the diagram on the side, A, B and C are images of object P. Match their transformation types.



Answer:



4

2) If M (3, 3) is mapped to M' (3, -3) under one rotation, state the object for the followir coordinates under the same rotation.

a) Describe the rotation by stating the direction, center, and angle of rotation.

b) Determine the image coordinates of (0,0) under the same rotation.

c) Determine the object coordinates of (4, 0) under the same rotation.

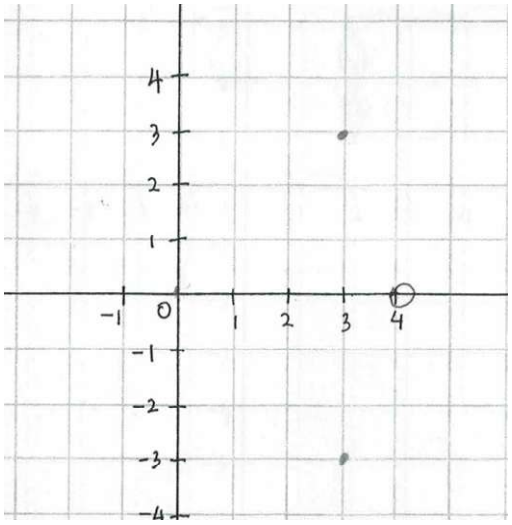


Figure 4: Answer Script of the Second Highest-Scoring Student in the Isometric Transformation Test.

The lack of responses from most students to the questions may be due to weaknesses in visualization, conceptual knowledge, and procedural knowledge. Poor conceptual understanding prevents them from fully grasping the fundamental ideas to answer the questions. At the same time,

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limited procedural knowledge makes it difficult for them to apply the correct steps to solve problems. Additionally, weak visualization skills hinder their ability to interpret spatial relationships, recognize patterns, and mentally manipulate shapes, making it challenging to understand and represent the tested concepts. This suggests that although some students understand the core ideas, they struggle to apply their knowledge in practice, particularly when following specific transformation steps. The gap between conceptual understanding, procedural execution, and visualization skills highlights students' difficulties in effectively performing transformation tasks without a well-developed spatial and conceptual foundation.

Therefore, it is necessary for the development of teaching aids that can assist students with low spatial visualization ability and be effective in improving conceptual and procedural knowledge to overcome this problem. AR has the potential to revolutionize the learning experience, particularly in the context of understanding complex mathematical concepts such as Isometric Transformation. By integrating AR technology into the learning module, students can engage in immersive and interactive experiences that deepen their ability to visualize and comprehend abstract concepts more effectively. AR bridges traditional teaching methods and student comprehension by providing real-time visualization of transformations and strengthening spatial reasoning, conceptual understanding, and procedural knowledge. Meanwhile, a structured learning module offers targeted guidance, activities, and exercises to reinforce these skills, ultimately enhancing student achievement in Isometric Transformation. Integrating AR and a structured learning module creates a dynamic and adaptive learning environment that caters to students' needs, enabling them to progress at their own pace while receiving immediate feedback. This innovative approach can potentially significantly elevate student achievement in Isometric Transformation.

CONCLUSION

This study highlights the critical need for a learning module integrating AR to enrich Form 2 students' understanding of Isometric Transformation in Malaysia. Traditional teaching methods have proven inadequate in addressing students' challenges with visualization, conceptual understanding, and procedural knowledge, leading to learning gaps. Moreover, analysis of student test performance confirmed significant struggles in these areas, reinforcing the need for AR-enhanced learning to bridge these gaps.

AR offers a transformative approach to learning transformations through real-time visualization and interactive exploration, particularly in enhancing spatial reasoning, conceptual grasp, and procedural execution. Unlike traditional methods reliant on static images, AR enables students to engage directly with 3D objects, fostering deeper comprehension of translation, reflection, and rotation. Furthermore, AR's ability to provide instant feedback supports procedural learning, helping students refine their understanding of transformation steps in real-time. In learning Iso-

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metric Transformation, students often struggle with spatial visualization and procedural execution, particularly when using traditional paper-based methods.

One of the key challenges in translation is ensuring that students understand how a shape moves without changing its orientation. In a conventional classroom, students are typically asked to plot a translated shape on graph paper by counting units. However, this process can lead to errors due to miscounting or misalignment. With AR, students can interactively select a 3D shape and apply a translation by dragging it to a new position while the app provides real-time confirmation of accuracy. Unlike manual plotting, this method reinforces conceptual understanding by allowing students to dynamically see and manipulate the transformation rather than relying on abstract visualization.

Similarly, the reflection of a shape across a line of symmetry is another area where students frequently encounter difficulties. Traditional exercises require them to manually mirror each vertex, often leading to incorrect placements and orientation errors. With AR, students can choose a quadrilateral in a virtual space, select the reflection axis, and observe how the shape flips in real-time. This allows them to instantly recognize how distances and orientations remain consistent while eliminating the guesswork of static images.

Among the most challenging transformations for students is rotation, which requires visualizing how each point moves in a circular path around a fixed point. In traditional methods, students often struggle to determine the correct quadrant, direction, and angle when performing a 90° or 180° rotation. They must rely on complex manual plotting techniques, which can be confusing, especially for those with low spatial reasoning skills. By contrast, an AR-based approach allows students to select a shape, choose a rotation point and angle, and see the transformation occur dynamically. This immediate feedback helps students correct errors, apprehend the relationship between rotation and coordinate changes, and build a more intuitive grasp of geometric transformations. Unlike static images, which only show the result, AR enables students to track the transformation as it happens, making the learning process more engaging and interactive.

A well-structured learning module is crucial in ensuring that AR integration is effective. Such a module provides clear instructional guidance and structured activities, making mathematical concepts more accessible and engaging. A strategic approach encompassing curriculum alignment, educator empowerment, and thoughtful instructional design is essential to integrate AR-based learning modules into Malaysian mathematics classrooms, particularly for topics like Isometric Transformation.

First, curriculum integration should ensure AR activities align with existing curricular goals, reinforcing key concepts without disrupting the syllabus. These modules should supplement traditional methods by visually visualizing abstract ideas, such as geometric transformations. Lessons should blend guided exploration, collaborative problem-solving, and structured practice while

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incorporating adaptive features to accommodate diverse learning paces and spatial reasoning abilities.

Second, educator readiness requires comprehensive teacher training through workshops and hands-on sessions to build technical proficiency and pedagogical confidence. Training should focus on seamless AR integration into lesson plans, real-time feedback techniques, and interactive quizzes for assessment. Establishing peer collaboration networks allows educators to share resources, refine strategies, and troubleshoot challenges, while sustained support through follow-up coaching and reflective practices ensures long-term adoption.

Third, successful classroom implementation depends on technology accessibility, ensuring schools have compatible devices such as tablets and smartphones and user-friendly AR applications. Digital or printed guides can streamline navigation, and lessons should begin with teacher-led AR demonstrations before transitioning to student-driven activities that encourage the manipulation of virtual objects to explore transformations. Peer discussions further deepen conceptual understanding and engagement.

Fourth, assessment and feedback mechanisms should combine formative tools, such as AR quizzes and real-time analytics, with summative evaluations, including student reflections and observations, to measure procedural mastery and conceptual grasp. Regular feedback from educators and learners will help refine the module design and address technical or instructional challenges.

Finally, sustainability and scalability require embedding AR training into pre-service teacher education programs to equip future educators with digital competencies. Institutional partnerships can support resource sharing and innovation. Consequently, periodic AR content and infrastructure updates, informed by emerging educational trends, will ensure continued relevance and expansion across classrooms. This cohesive strategy balances pedagogical innovation with practicality, ensuring that AR becomes an enduring and impactful component of mathematics education in Malaysia.

To successfully implement AR in the classroom, educators should consider a structured approach that strengthens engagement, deepens conceptual understanding, and ensures an effective learning experience. Align AR with learning objectives by integrating it into the curriculum to heighten understanding, especially for abstract concepts like Isometric Transformation. Choose simple, user-friendly AR tools like GeoGebra AR or Merge Cube that require minimal setup. Design interactive activities that promote problem-solving and exploration rather than passive viewing. Guide students with structured lessons using the 5E model to scaffold learning effectively. Encourage collaboration through group tasks that foster peer discussions and critical thinking. Finally, use AR-based quizzes and real-time analytics for formative assessments, providing instant feedback to track and amplify student progress.

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Future research should examine how AR-based learning modules affect students' spatial abilities over time and how they can be applied to other areas of mathematics. This study highlights the immense potential of AR, especially when paired with a well-structured learning module, to transform the way students engage with mathematics. By making learning more interactive and immersive, AR helps students grasp complex concepts and keeps them motivated and invested in learning. Hence, integrating AR into education is not just about using new technology. Still, it is about creating more prosperous, meaningful learning experiences that impact students' understanding and retention of mathematical concepts.

ACKNOWLEDGMENTS

The researcher would like to express their gratitude to all students for their participation and to schools for their permission to conduct research in schools.

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MATHEMATICS TEACHING RESEARCH JOURNAL
SUMMER 2025
Vol 17 no 3

Enhancing Mathematical Creative Thinking in Ethno-Geometry Learning Using Augmented Reality Technology

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Abstract: The purpose of this study is to develop an Android-based augmented reality learning media using Unity software to improve students' creative thinking skills in polyhedral ethno-geometry materials. The development of augmented reality learning media follows the ADDIE development model, which consists of five phases: analysis, design, development, implementation, and evaluation. The learning media developed must meet feasibility requirements such as effectiveness, practicality, and effectiveness. Data collection to measure students' creative thinking skills is currently done through written exams and documentation. We then analyzed the test data using a paired samples t-test. Based on the analysis, the results are shown to be effective and proven in improving creative thinking in mathematics. As a result of the questionnaire analysis regarding media usage, overall favorable evaluations were obtained. In summary, these results are consistent with the contribution of this study to making teaching-learning experiences effective, fun, and interactive.

Keywords: Augmented reality, mathematics creative thinking, ethno-geometry

INTRODUCTION

Learning is a process that enhances students' ability to acquire knowledge. This process involves specific steps that a teacher must take to ensure that students acquire knowledge, increase their attention and motivation in class, and become competent in this modern era. A teacher's teaching process may vary; some may focus on application, some may demonstrate concepts, while others may focus on understanding. However, the learning that takes place in the classroom depends on the teaching methods used by the teacher and the learning methods used by the students. Felder suggests that students learn through various methods such as hearing, seeing, visualizing, imagining, acting, or memorizing (Felder & Silverman, 1988). Additionally, objectivity and grammatical correctness are crucial, and any bias should be avoided. Mental visualization of concepts is crucial for understanding analogies (Gargrish et al., 2020). The success of education heavily relies on

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teachers' teaching methods, which include the use of technology. Efforts are made to improve the learning process, including implementing student-based learning and incorporating information technology into education.

Augmented reality is a form of technological advancement that can be integrated into the world of education with great potential benefits (Marrahí-Gómez & Jose Belda-Medina, 2023). Augmented reality is a technique that uses device technology to visualize graphical information adapted from an original or real part of the world (Marrahi Gomez & Belda-Medina, 2023). It is a technology that can be used to enhance users' interpretation of information about reality by overlaying digital information on the real world. Augmented Reality (AR) is becoming increasingly important in today's world. The concept of augmented reality refers to the integration of two-dimensional or three-dimensional virtual elements into the real world that are displayed or projected in real time using applications (Maulana et al., 2022). The purpose is to recreate real-life scenarios and add computer-generated virtual information, such as text, photos, 3D models, music, and videos (Chen et al., 2019).

Research by Gargrish et al., (2020) shows that the use of geometry applications provides students with a more comprehensive learning experience, allowing for better visualization and understanding of the subject matter. Mobile geometry applications have been designed to provide personalized assistance and enhance students' visualization of angles. Ibáñez et al., (2020) suggest that augmented reality technology can be an effective learning environment for students to practice basic principles of geometry. The study found that students who utilized the augmented reality-based learning environment achieved higher post-test scores compared to those who used the web-based application. One advantage of using augmented reality for learning media is that students can interact with 3D objects from multiple angles. This is not possible with web-based media. Additionally, the augmented reality-based learning environment was more effective than the web-based learning environment in schools. The study also discovered that the use of augmented reality-based learning environments increased students' learning motivation. The visualization in the use of augmented reality can facilitate the construction of shapes in the cognitive understanding process for students, particularly those with visual learning styles.

Augmented Reality

A real environment in real time augmented with virtual elements is the definition of augmented reality (Rossano et al., 2020). The purpose of using augmented reality (AR) is to add virtual elements to the data collected by the five senses (Koparan et al., 2023). These elements provide complementary information that may not be available naturally (İbili et al., 2020). Many research papers have discussed augmented reality (AR) applications in many fields, such as history, biology, physics, math, and English. However, despite the abundance of research, only a few studies have been applied in mathematics education, particularly in the area of geometry (Gargrish et al., 2021). Geometry material covers fundamental concepts such as shape, volume, and area. These concepts can be abstract and challenging to comprehend in 3 dimensions (Lainufar et al., 2020). Augmented reality technology can serve as a tool for exploration, interaction, and communication among students in

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classroom education. The visualization produced by AR is expected to fill an important role in overcoming the limitations of traditional learning methods in teaching geometry (Nazar et al., 2020). When studying geometry, AR can present realistic 3-dimensional models of real-world objects, making it easier to understand complex concepts (Widada et al., 2021). The integration of virtual and real-world elements is expected to make the geometry learning process more effective and efficient when combined with their cultural environment (Guntur et al., 2020).

Ethno-Geometry

Mathematics is the product of human thought related to concepts, procedures, and reasoning. People often unconsciously use mathematics in their daily lives. D'Ambrosio first introduced the science that refers to mathematics as ethnomathematics in 1985. It is applied by certain cultural groups, such as ethnic groups, professional groups, labor groups, and certain age groups (D'Ambrosio, 1985). Multicultural education, which connects ethnicity and culture to learning, helps students better engage with the subject matter. Accounting for different cultural approaches can foster deeper understanding and understanding of mathematics (Meng & Liu, 2022). Jayadi et al., (2022) found that a multicultural educational paradigm that includes themes such as unity in diversity, equality, cultural identity, and social justice is important for improving scientific attitudes. Suherman & Vidákovich, (2024) found that ethnicity/culture and attitudes significantly contribute to creativity in mathematics. In particular, increased ethnic diversity leads to better attitudes toward mathematics and higher levels of mathematical creative thinking. Geometry is a branch of mathematics that studies the properties of two-dimensional (2D) and three-dimensional (3D) shapes and figures (Prahmana & D'Ambrosio, 2020). Geometry plays a crucial role in elementary and middle school education Flores-Bascuñana et al., (2019), as it offers multiple visualization tools to comprehend statistical, algebraic, and arithmetic concepts (Schutera et al., 2021). Ethno-geometry is a scientific field that explores how geometry is adapted by different cultures to express their unique relationships with it. Ethno-geometry learning is based on the surrounding culture and its relevance to everyday life. This research examines the application of AR in ethno-geometry material, with the aim of enhancing students' creative thinking abilities. Recent research highlights the diverse approaches to enhancing geometrical understanding. Sari et al. (2024) investigated the role of ethnomathematics in geometry education, exploring how cultural contexts and traditional practices can inform and enrich learning. Concurrently, Zakaria et al. (2024) explored the impact of integrating geometrical design with GeoGebra, a dynamic mathematics software, on secondary students' motivation and academic performance. These studies collectively underscore the ongoing efforts to develop innovative and effective pedagogical strategies in geometry.

Mathematical Creative Thinking

Creative thinking is the ability to generate possible solutions and approaches to problems based on available information, with emphasis on both quantity and appropriateness (Yaniawati et al., 2020). Creative thinking is important in learning because improving students' creative skills has a positive impact on academic performance (Kartika et al., 2019). Educating children about creativity can in-

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crease the meaning of learning (Maskur et al., 2020). According to Regier & Savic, (2020), creative thinking is a cognitive process that involves solving problems and generating new ideas through synthesis and application. Creative thinking is considered a higher-level thinking ability (Rahmawati et al., 2019). Creative thinking in mathematics is the use of creativity to solve mathematical problems in flexible, innovative, and open-ended ways, especially by making connections between different concepts and by using imagination to develop new solutions and perspectives. It is the ability to approach and the ability to explore (Sadak et al., 2022). The Mathematics Creative Thinking (MCT) test is used to assess students' mathematical creativity (Sahliawati & Nurlaelah, 2020). Creative thinking includes several important aspects such as fluency, flexibility, elaboration, and originality (Tanjung & Nasution, 2023). Creative thinking skills encompass several aspects that help individuals generate, develop, and refine ideas effectively. These aspects include fluency, flexibility, originality, and elaboration. Each plays a unique role in fostering creative thought: *Fluency* refers to the ability to produce a large number of ideas or solutions in response to a given problem or stimulus; *Flexibility* is the ability to think about a problem from multiple perspectives or to approach it in different ways; *Originality* is the capacity to produce ideas that are novel, unique, or unexpected; and *Elaboration* involves expanding on an idea by adding details, refining it, or making it more complete and actionable. The fluency indicator measures the student's ability to provide correct solutions to the given problems. The flexibility indicator measures the ability to solve problems using different methods so that solutions are not obtained from only one point of view. The elaboration indicator measures their capacity to develop and expand upon their solutions. Meanwhile, the originality indicator measures the student's ability to find unique solutions to the given problem (Habibi et al., 2020).

This research addresses two primary questions: (1) How can valid, practical, and effective learning devices based on ethnogeometry and augmented reality be developed? (2) Does the implementation of ethnogeometry and augmented reality-based learning devices influence students' creative mathematical thinking skills? The instruments employed to assess the validity, practicality, and effectiveness of these devices include tests, questionnaires, and observations.

METHOD

Participant

In this study, 188 students were randomly selected from secondary schools in Jember, East Java, Indonesia. From a total of 7 classes, we randomly selected 6 classes, centering on Jember's class 8. In summary, researching geometry learning with AR in 8th-grade junior high school is a promising endeavor because it directly addresses common learning difficulties in geometry, aligns with the cognitive development of adolescents, and leverages the unique capabilities of AR to create an engaging, interactive, and effective learning environment for complex spatial concepts. All participants provided informed consent before participating in this study and collecting data online via Google Forms (https://bit.ly/EthnoGeometry). One of the 188 participants, there were 87 males, mean age 14.11 ± 0.68, and 101 females, mean age 14.46 ± 0.81. The mean age of all participants was 14.28 ± 0.74.

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Research Structure

This research uses the ADDIE framework (Analysis, Design, Development, Implementation, and Evaluation) which can be seen in Figure 1 below.

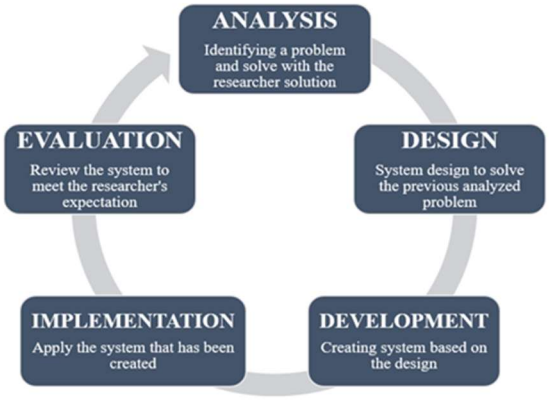


Figure 1: ADDIE model diagram

Data Collection Procedure

In this research, researchers obtained data based on the results of tests and questionnaires from 188 participating students. The test consisted of pre- and post-tests, each of which consisted of 5 questions. The purpose of the pretest was to measure students' competence before using augmented reality. Meanwhile, the posttest was used to measure the students' ability after applying augmented reality. Based on this, researchers can determine the effectiveness of augmented reality in learning geometry cube materials. In addition to the test, students also fill out a questionnaire that has been provided. Researchers also used a questionnaire that asked students about the utilization of Augmented Reality in learning, learning motivation, and student achievement. In addition, researchers took documentation in the form of student work as supporting material for data analysis. The pre-/post-test instrument can be found at (https://unej.id/PrePostTestInstrument). All instruments were used in the collection of research data.

Data Analysis

Normality tests and homogeneity tests were carried out on the pretest and posttest data as prerequisite tests for the Paired Sample Test. Data that has met the prerequisite tests is tested using the Paired Sample Test. This test is utilized to decide the distinction in the normal pre and posttest scores in the classes which apply learning using Augmented Reality. The formulation of the test hypothesis is as follows:

H_0: There is no difference between the mean pretest and posttest scores.

H_1: Average posttest scores are better than the pretest scores.

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Meanwhile, the formulation of the test criteria is as follows:

a. If significance is > 0.05, then H_0 is accepted and H_1 is rejected.

b. If significance is ≤ 0.05, then H_0 is rejected and H_1 is accepted.

Furthermore, the study conducted the N-Gain or Normalized Gain test to measure the magnitude of students' creative thinking abilities before and after treatment. The N-Gain calculation formula was used to calculate the pretest and posttest values.

$$N - Gain = \frac{S_{posttest} - S_{pretest}}{S_{max} - S_{pretest}}$$

With:

$S_{pretest}$ = pretest score

$S_{posttest}$ = posttest score

S_{max} = maximum score

RESULT

Analysis

Primary data collected came from secondary schools in Jember, East Java Province, Indonesia. In this case, the researchers created a guidebook for using this technology, because most of the students studied were still not used to using augmented reality technology. Next, the researcher taught geometry material, namely about polyhedra related to ethno-geometry. Next, students take a test at the end of the lesson to get data that helps researchers compute the effectiveness of augmented reality. The tests students will take will be administered in the classroom and each test will have an estimated time of 30 minutes. Additionally, the scoring system for the test is based on 0-100, with each question having a value of 20. In addition to the rating system, the researchers conducted a questionnaire that was filled out online via Google Form regarding whether students liked the learning experience using applications or not, and found it difficult to use augmented reality technology or not. Based on the results of the questionnaire, researchers can provide information regarding the potential for implementation or not.

Design

Figure 2 illustrates the application structure utilizing a use case diagram created with draw.io. The diagram depicts the skills of both users and administrators when utilizing the structure.

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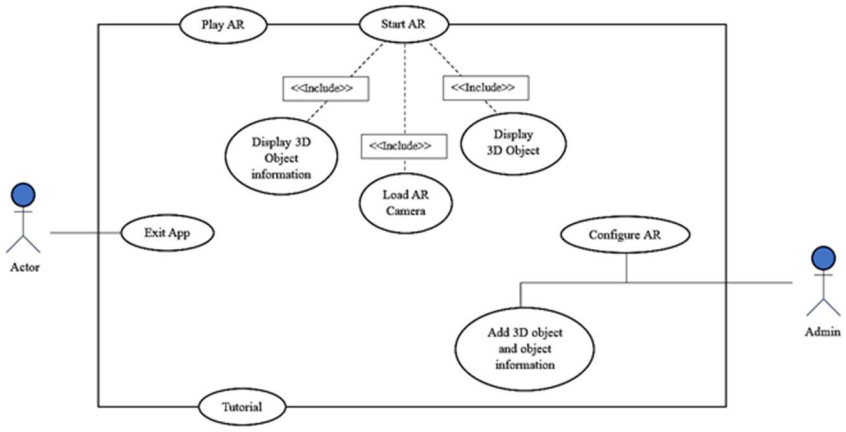


Figure 2: Use case diagram

The use case diagram illustrates the system's functional scope, outlining the interactions available to both users and administrators. Users are presented with five distinct menu options. The Competency menu displays information pertaining to media competencies. The Material menu provides access to content on surface area, volume, planar shape definitions, and augmented reality (AR) features. Within the AR menu, selecting this option activates the AR camera, allowing a 3D object relevant to a specific objective to appear when the user focuses on a designated image. The Instructional Exercise menu offers tutorials on application usage, primarily catering to experienced users. Finally, the About menu presents developer profiles and media creation references, while the Questions menu provides practice questions for students. Users can terminate the application by selecting the Exit option. In addition to these user functionalities, administrators possess elevated privileges, enabling them to configure augmented reality settings, incorporate new 3D models into the application, and perform all actions accessible to regular users, Figure 3 shows the class chart made beneath utilizing draw.io. explains class names, attribute names, and class functionality in the application. Also, all class names are connected to each other:

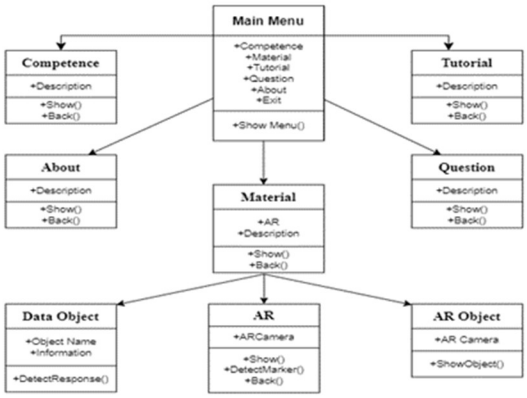


Figure 3: Class diagram

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The initial class is the main menu, which includes all available menus for user selection. Following that is the tutorial, this section describes how augmented reality can be systematically used by students. After that, the competencies and media competencies are displayed. Next, you can view your developer profile and credentials. The following question provides a practice exercise. The 'Material' section displays information on area, volume, and aspect ratio. Use the 'back' function to return to the main menu in all sections. Finally, when the user selects the AR scan option in the material menu, the AR class is triggered and can perform several functions. One of these functions is the 'display' function, which can show 3D objects and detection marks. The camera detects the target image and then returns to the camera, leading the user back to the main menu.

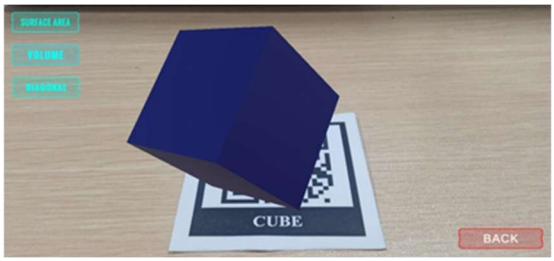
Development

The initial class is the main menu, which includes all available menus for user selection. Following that is the tutorial, this section describes how augmented reality can be systematically used by students. After that, the competencies and media competencies are displayed. Next, you can view your developer profile and credentials. The following question provides a practice exercise. The 'Material' section displays information on area, volume, and aspect ratio. Use the 'back' function to return to the main menu in all sections. Finally, when the user selects the AR scan option in the material menu, the AR class is triggered and can perform several functions. One of these functions is the 'display' function, which can show 3D objects and detection marks. The camera detects the target image and then returns to the camera, leading the user back to the main menu.



Figure 4: Target image

A prerequisite for configuring a target image in Vuforia is obtaining a valid license. This license is essential for the proper functionality of the target image within the application. Next, enter your Vuforia license into your AR camera so the camera can recognize the target image and display your assets. The asset image appears in the scene, as shown in Figure 5.



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Figure 5: 3D object

After testing the application, it can be used for education or demonstration to help elementary school teachers understand surfaces and volumes of polyhedral figures. Not only can it be used as a learning medium for junior high schools, but you can also use the application to do research and listen to explanations of specific polyhedra by junior high school teachers. Why not even junior high school students install the app on their smartphones and try it out. However, the app can only be installed on smartphones that have been updated or are running Android OS 11.0 (API level 30). The gallery menu can be seen in Figure 6 until Figure 9:

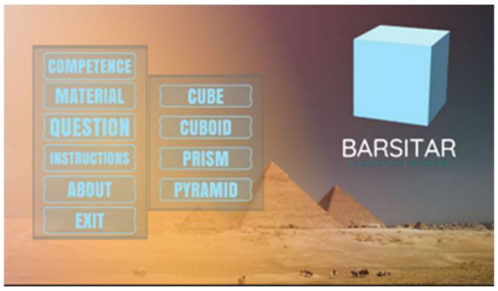


Figure 6: Main menu



Figure 7: Material menu



Figure 8: Surface area

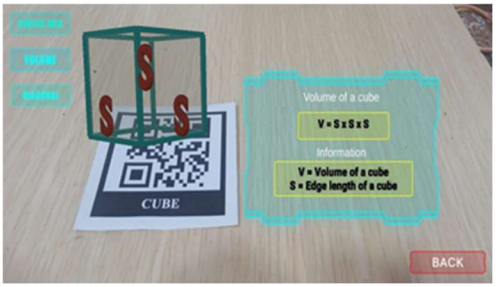


Figure 9: Volume

Implementation

During the implementation phase, the researchers conducted a baseline test at the school to evaluate students' knowledge of augmented reality. The results showed that most students had little to no understanding of the technology. Based on this finding, the researchers provided a brief introduction to the topic of augmented reality and administered a pretest to measure students' comprehension of polyhedrons. The students utilized augmented reality technology on the researcher's phone to test its effectiveness for learning. Subsequently, they completed a user experience survey to evaluate the technology's effectiveness and enjoyment. Finally, the students took a post-test to measure their understanding of polyhedrons.

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Evaluation

Following the implementation process, the researcher evaluated the results of the tests and questionnaires that the students had completed. To ascertain the efficacy of the instructional media, the researchers conducted an analysis of students' pre- and post-test scores. This comparative approach allowed for the evaluation of the media's impact on enhancing student comprehension. Supplementary data from these assessments are available at (https://unej.id/TestResult). Table 1 below was obtained by entering data into SPSS, and the results can be seen below:

	Mean	N	Std. Deviation	Std. Error Mean
Pretest	63.8617	188	8.56757	0.62485
Posttest	80.2340	188	7.41429	0.54074

Table 1: Paired sample statistics of test results

The paired sample statistics table has an average value that represents the average of the pretest and posttest. The total number of students who took part in the experiment is shown in the first column (N). In this case, the average pretest score for the 188 students is 63.8617 and the average posttest score is 80.234. Next is the paired sample correlation or Table 2 below:

	N	Correlation	Sig
Pretest & Post-test	188	0.128	0.081

Table 2: Paired samples correlation of test results

The total number of students who took part in the experiment is shown in the first column (N), and the correlation between the pre-test and post-test data is shown in the second column. The data in this instance have a positive correlation (r = 0.128). The final column represents sig. or significance in this instance, it is 0.081. Table 3 is the paired sample test table shown below: The total number of students who took part in the experiment is shown in the first column (N), and the correlation between the pre-test and post-test data is shown in the second column. The data in this instance have a positive correlation (r = 0.128). The final column represents sig. or significance in this instance, it is 0.081. Table 3 is the paired sample test table shown below:

Mean	Std. Deviation	Std. Err Mean	95% Confidence		t	df	Sig. (2-tailed)
			Lower	Upper			
-16.37234	10.59107	0.77243	-17.89614	-14.84854	-21.196	187	0.000

Table 3: Paired samples test of test results

The Paired Samples Test Table 3 has eight columns: mean, standard deviation, standard error, 95% confidence interval (lower), 95% confidence interval (upper), t, df, and Sig (both sides). The first in Table 3 average value. The difference between the pre and post-test means is -16.37234. The Sig (2-tailed) results from the average results of each test and the 5% significance level were also used in this analysis. So in this case the Sig (both sides) is 0.000. If we evaluate the results in detail, the Sig value (two-tailed) < 0.05, the meaning is to reject the null hypothesis, stating that there is no

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significant difference between each test. If the Sig (two-tailed) value is more significant than 0.05, it means accepting the null hypothesis that there is no significant difference. Based on the evaluation results, it can be seen that the Sig (2-tailed) value is lower than 0.05, consequently showing that the utilization of augmented reality media is very powerful in increasing student understanding. The next evaluation is related to the effectiveness of increasing Mathematics Creative Thinking (MCT) with indicators of fluency, flexibility, originality, and elaboration. Table 4 was obtained by entering data into SPSS, and the results are.

MCT		Mean	Std. Deviation	Std. Error Mean
Fluency	Pretest	2.38	0.487	0.036
	Posttest	4.72	0.525	0.038
Flexibility	Pretest	2.30	0.653	0.048
	Posttest	4.47	0.665	0.049
Originality	Pretest	1.84	0.750	0.055
	Posttest	4.90	0.295	0.022
Elaboration	Pretest	1.97	0.837	0.061
	Posttest	4.43	0.496	0.036

Table 4: Paired sample statistics of MCT

The paired sample statistics table has an average value that represents the average of the pretest and posttest. There is also an N column that represents the number of students who participated in the experiment. In this case, the average score on the language proficiency pretest is 2.38, and the average score on the language proficiency posttest is 4.72. The mean score for the flexibility pretest is 2.30, and the mean score for the flexibility posttest is 4.47. The mean score for the originality pretest is 1.84, and the mean score for the originality posttest is 4.90. The mean elaboration pretest score is 1.97, and the mean elaboration posttest score is 4.43. Next is the paired sample correlation or Table 5 below:

MCT		N	Correlation	Sig
Fluency	Pretest & Posttest	188	0.103	0.161
Flexibility	Pretest & Posttest	188	-0.242	0.001
Originality	Pretest & Posttest	188	-0.069	0.344
Elaboration	Pretest & Posttest	188	0.272	0.000

Table 5: Paired sample statistics of MCT

From the output above, you can see that Table 5 contains three columns. The total number of students who took part in the experiment is shown in the first column (N), and the correlation between the pre-test and post-test data is shown in the second column. The data in this instance have a positive correlation with fluency (r =0.103). The final column represents sig. or significance in this instance, it is 0.161. The data show negatively correlated with flexibility (r = -0.242), the flexibility

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of significance is 0.001. The data show negatively correlated with originality (r = -0.069). The originality of significance is 0.344. The data show positively correlated with elaboration (r = 0.272). The elaboration of significance is 0.000. Table 6 is the paired sample test table shown below:

MCT	Mean	Std. Deviation	Std. Err Mean	95% Confidence		t	df	Sig. (2-tailed)
				Lower	Upper			
Fluency	-2.340	0.679	0.050	-2.438	-2.243	-47.262	187	0.000
Flexibility	-2.165	1.039	0.076	-2.134	-2.015	-28.568	187	0.000
Originality	-3.064	0.825	0.060	-3.183	-2.945	-50.929	187	0.000
Elaboration	-2.452	0.845	0.062	-2.574	-2.330	-39.634	187	0.000

Table 6: Paired samples test of MCT

The Paired Samples Test Table or Table 6 contains the following eight values: mean, standard deviation, standard error, 95% confidence interval (lower), 95% confidence interval (upper), t, df, and Sig (two-tailed). The results are obtained by calculating the difference between the pretest and post-test means: fluency is –2.340, flexibility is -2.165, originality is -3.064, and elaboration is -2.452.

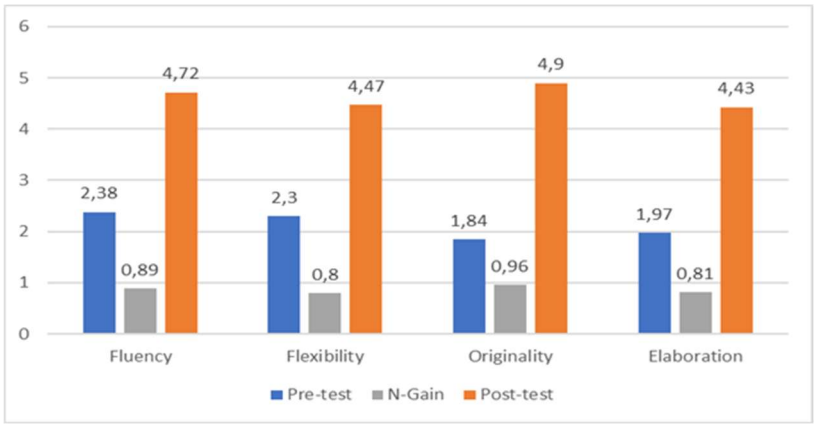


Figure 10: Average result of mathematical creative thinking

Based on Figure 10, it shows that there has been a significant increase in each indicator of mathematical creative thinking. The results of the analysis of students' creative mathematical thinking on the Fluency indicator before using learning media obtained a value of 2.38 and increased to 4.72 after using learning media, The N-Gain value was obtained at 0.89. The Flexibility indicator before using learning media obtained a value of 2.3 and increased to 4.47 after using learning media, The N-Gain value was obtained at 0.8. The Originality indicator before using learning media obtained a value of 1.84 and increased to 4.9 after using learning media, The N-Gain value was obtained at 0.96. The Elaboration indicator before using learning media obtained a value of 1.97 and increased to 4.43 after using learning media, The N-Gain value was obtained at 0.81.

To see the students' creative thinking skills, it is necessary to explore by displaying their work from the post-test results. The students' post-test results are as follows:

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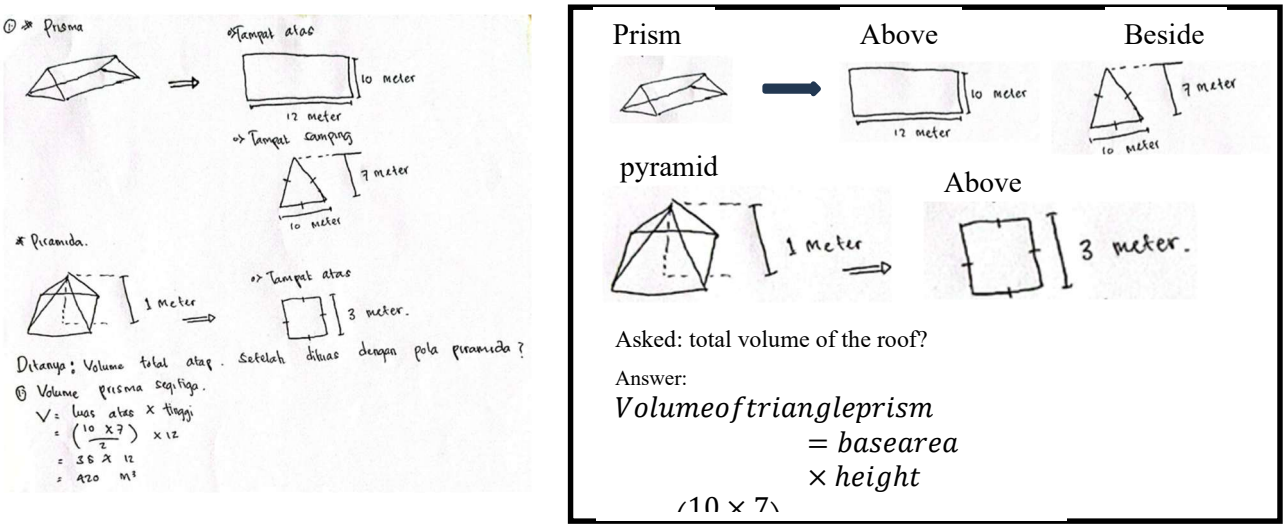


Figure 11: Example Sample snippets of student work related to creative thinking skills

Based on Figure 11, it shows that students have achieved 4 indicators of creative thinking skills, including fluency, flexibility, originality, and elaboration. Students achieve the fluency indicator by being able to draw real conditions in mathematical models in the form of geometric designs. Students meet the flexibility indicator by being able to use the volume formula of a room. The originality stage is demonstrated by students' ability to solve space volume problems from contextual problems. The elaboration stage is shown by the students' ability to answer the given problems by linking other concepts and to teach them to other students. In this way, all indicators of students' creative thinking ability can be shown in detail.

DISCUSSION

The findings indicate that integrating Augmented Reality (AR) technology within an ethno-geometric learning framework can significantly enhance students' creative thinking skills. This suggests that blending immersive digital experiences with culturally relevant mathematical contexts provides a promising avenue for fostering higher-order cognitive abilities in learners. For all aspect of creative thinking skills, namely fluency, flexibility, originality, and elaboration are improved after Augmented Reality (AR) technology with ethno-geometry learning process. It is in-line with the result given by (Thompson & Singh, 2020). The integration of Augmented Reality (AR) technology in ethno-geometry learning presents a unique blend of cultural heritage and modern technological advancements to enhance mathematical creative thinking. Augmented Reality (AR) brings an interactive dimension to learning, making abstract ethno-geometric concepts more tangible. Gonzalez & Hernandez (2023) also obtained that under AR media implementation, the interactivity can significantly increase students' engagement and motivation, since students are more likely to be intrigued by and invested in lessons that incorporate technology the find compelling. AR can visually represent 3D geometric shapes and patterns in real time, which can help students to develop a better

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spatial understanding of mathematical concepts. This visualization is particularly beneficial in ethno-geometry, where shapes and patterns are often embedded in cultural artefacts and symbols. Integrating ethno-geometry with AR technology can foster a deeper appreciation for cultural heritage. Furthermore, Kim & Choi, (2021) stated that by exploring geometric concepts through the lens of different cultures, students can connect with and appreciate the mathematical ingenuity and aesthetic beauty found in various cultural traditions. AR technology can offer personalized learning paths based on a student's progress, strengths, and areas for improvement. This adaptability can help cater to individual learning styles and paces, enhancing the overall effectiveness of the educational experience. However, the dependence on AR technology might limit access to ethno-geometry learning for students and schools with limited technological resources.

However, the digital engagement can exacerbate educational inequalities. Setting up AR technology for educational purposes can be expensive. Wright & Kanellos (2021) analysed that the cost includes not just the hardware (such as AR glasses or devices) but also the software development of educational AR applications, which can be a significant barrier for many institutions. Both educators and students may face a learning curve in effectively utilizing AR technology. Educators need to be trained to integrate AR into their teaching practices seamlessly, while students need to learn how to navigate and learn from AR applications effectively. There's a risk of overreliance on technology, where students might become too dependent on AR for learning, potentially hindering their ability to understand and apply mathematical concepts without technological aid, it was mentioned by (Sanchez & Young, 2019). Implementing ethno-geometry and augmented reality (AR) in a regular classroom can enhance student engagement, foster cultural appreciation, and provide interactive learning experiences. By merging ethno-geometry and AR, classrooms can become dynamic spaces where students learn not only math but also appreciate cultural diversity and cutting-edge technology.

CONCLUSIONS

We have done this study, and the result shows that the development of Augmented Reality (AR) technology with ethno-geometry learning process can enhance the students' creative thinking skills significantly. To develop valid, practical, and effective learning devices for ethno-geometry learning using augmented reality (AR), this study follows a systematic research and development (R&D) framework, which includes the following key phases: 1) analysis and needs assessment includes identifying the challenges in ethno-geometry learning and the potential of AR technology in enhancing mathematical creative thinking; 2) developing lesson plans, designing student worksheet, and creating AR-based learning media; 3) expert validation involving mathematics educators and technology specialists to evaluate content accuracy and AR integration; 4) gathering teacher and student feedback to evaluate usability, clarity, and engagement; 5) measuring student learning outcomes, particularly improvements in mathematical creative thinking to determine effectiveness.

The results of this study confirm that these learning devices meet the criteria of validity, practicality, and effectiveness. The validity was established through expert validation and alignment with pedagogical and technological principles. Practicality was assessed based on teacher and student responses, indicating ease of implementation and engagement. Effectiveness was demonstrated

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through improved student performance in mathematical creative thinking, as evidenced by test results and qualitative observations. Learning devices based on ethno-geometry and augmented reality that are valid, practical, and effective namely lesson plans, student worksheet, and Augmented Reality-Based Learning Media.

It also has a significant leap forward in education technology. By enhancing student engagement, improving spatial understanding, and providing a rich cultural context, AR technology has proven to be a powerful tool in the development of creative thinking skills. Since the educational practices evolving the integration of AR into learning environments ensure to unlock new possibilities for student achievement and innovation. The implication is that the research findings have significant implications for educational practice. Educators are encouraged to adopt AR technology in their teaching methodologies, particularly in subjects where abstract concepts can benefit from visual and interactive aids. Additionally, integrating cultural elements into the curriculum through ethno-geometry can provide a more holistic and engaging learning experience for students, promoting creativity alongside academic achievement. The results of this study are in the form of learning devices based on ethno-geometry and augmented reality that are valid, practical, and effective. The ability to think creatively in mathematics increased after the treatment of using learning devices based on ethno-geometry and augmented reality.

Embedding ethno-geometry learning using AR in the classroom entails several key approaches: (1) cultural exploration through augmented reality, in which students utilize AR applications to showcase 3D models of cultural artifacts; (2) interactive problem solving, where students engage in analytical tasks such as measuring angles, identifying geometric shapes, or exploring transformations using AR overlays on cultural patterns or artifacts; (3) virtual exhibits, enabling students to design cultural patterns or geometric models and present them in AR as interactive exhibitions; and (4) collaborative AR projects, where students work in groups to develop AR-enhanced presentations on cultural sites or artifacts, emphasizing their geometric properties.

ACKNOWLEDGMENTS

We gratefully acknowledge the support from University of Jember of year 2025.

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Enhancing Metacognitive Awareness, Skills, and Academic Performance: A Study on the Impact of Performance-Based Assessment with Metacognitive Prompts in Quadratic Functions Using the RME Approach

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Abstract: Assessments play a pivotal role in Mathematics education, serving as vital gauges of students' mathematical proficiency and informing teaching strategies. Despite calls for contextualization, some educators still rely on abstract teaching methods and decontextualized tasks. This study aims to address these concerns by designing and implementing a performance task on quadratic functions, following a realistic mathematics education approach, coupled with metacognitive prompts. Using mixed methods, pre-test and post-test scores from the Jr. MAI of 30 Grade 9 students, their responses to metacognitive prompts, performance task scores, and interview questionnaire responses were analyzed. Statistical analysis revealed significant improvements in students' metacognitive awareness and skills, particularly in knowledge about cognition and regulation of cognition. Strong positive correlations were observed between variables. Descriptive analysis indicated an increase in students scoring above the cut-off post-intervention. Thematic analysis of student interviews highlighted themes such as pre-task planning, progress monitoring, and post-task evaluation, underscoring enhancements in self-knowledge and mathematical performance.

Keywords: Realistic Mathematics Education, Authentic Assessments, Metacognitive Prompts, Metacognition in Mathematics

INTRODUCTION

In contemporary mathematics education, there is a growing emphasis on evaluating students' ability to apply learned concepts to real-world situations, moving beyond traditional assessments that focus solely on procedural fluency. Educators continually strive to develop effective teaching strategies and tools aimed at enhancing student comprehension and performance in mathematics. As such, performance-based assessments have emerged as effective tools in this regard, enabling educators to gauge students' conceptual understanding, problem-solving skills, and strategic thinking (Bender, 2020).

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Within the Philippine educational context, the Department of Education mandates the inclusion of performance tasks as integral components of both formative and summative assessments. These tasks are designed to assess students' mastery of competencies by requiring them to engage in meaningful, context-rich activities that mirror real-life challenges (Department of Education. 2020).These tasks are intended not only to test knowledge but to provide opportunities for students to apply mathematical concepts in context-rich, meaningful activities that reflect real-life challenges. In practice, performance tasks are often implemented at the end of instructional units to assess mastery of competencies (summative), but they can also be used throughout instruction to provide ongoing feedback and inform teaching (formative). However, the implementation of performance tasks often faces challenges, such as a lack of emphasis on students' metacognitive processes during task completion (Pranena & Pascima, 2023).

Recent studies underscore the significance of integrating metacognitive elements into performance assessments. For instance, Kadir (2022) highlights that performance assessments incorporating problem-solving and problem-posing strategies, aligned with students' metacognitive levels, can enhance mathematical achievement. Similarly, Mertasari et al. (2023) emphasized that formative performance assessments positively impact students' metacognitive abilities, fostering skills like planning, monitoring, and evaluating their learning processes.

Despite these findings, there remains a gap in effectively capturing students' metacognitive engagement during performance tasks. Addressing this, the study aims to answer the following research questions:

1. Can performance-based assessments with the metacognitive prompts in quadratic functions using the RME approach contribute to students' metacognitive awareness, skills and overall mathematical performance?
2. How do students perceive the completion of a performance-based assessment with metacognitive prompts in mathematics, and what is their perceived impact on learning outcomes?

Through this, deeper insights into how structured reflection and guided questioning, grounded in the principles of Realistic Mathematics Education (RME) can promote deeper insights into students' mathematical thinking, strengthen conceptual understanding, and improve academic performance.

LITERATURE REVIEW

Realistic Mathematics Education

Mathematics, contrary to common perceptions, emphasizes problem-solving, critical thinking, and innovation over rote memorization (Boaler, 2016). Realistic Mathematics Education (RME)

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has emerged as a successful approach to fostering motivation, self-confidence, problem-solving abilities, and reasoning skills among students (Laurens et al., 2017). By engaging students in contextual problem-solving exercises relevant to their experiences, RME aims to enhance cognitive achievement in mathematics (Bonotto, 2008).

RME is guided by six core teaching principles: activity, reality, level, intertwinement, interactivity, and guidance (Heuvel-Panhuizen and Drijvers, 2014). These principles underscore the importance of student engagement, real-life application, progressive learning, integration of mathematical concepts, social interaction, and teacher guidance.

Performance-based assessments aligned with RME should be authentic, incorporating tasks meaningful to students' abilities and values (Bland & Gareis, 2018). Such assessments provide opportunities for students to develop metacognitive skills, enabling them to effectively solve problems and apply learned concepts to real-world scenarios (Rhodes, 2019). Additionally, prompting plays a crucial role in guiding learners on when and why to employ specific problem-solving approaches (Alzahrani, 2017). In fact, Fauziah et al. (2018) conducted a study on the implementation of authentic assessment in math education, finding that it enhances students' math learning outcomes and fosters active participation from both teachers and students, both in and out of the classroom. Similarly, Winarso (2018) corroborated these findings, demonstrating that students who underwent authentic assessments showed better math learning outcomes compared to those who did not. In line with these, Sari et al. (2022) developed and validated worksheets grounded in RME, showing that students using RME-based worksheets demonstrated improved mathematical communication and engagement through contextual problem-solving activities.

Self-Assessments in Mathematics

To promote comprehensive understanding and critical thinking skills in mathematics, educators must employ varied assessment methods, including performance-based assessments (Alzahrani, 2017). These assessments challenge students to apply their knowledge and skills in practical contexts, fostering authentic learning experiences (Department of Education, 2015; Bland & Gareis, 2018 cited in Wiggins, 1993). However, challenges remain in providing timely and relevant feedback to students, necessitating innovative assessment strategies and remote monitoring systems (Delgado, 2020; Department of Education, 2020).

Many educational institutions persist in presenting mathematics as abstract and detached from real-world contexts, despite recommendations advocating for contextualization (Minaar, 2015 cited in Sandi-Urena et al., 2010). This approach limits authentic learning opportunities and deprives students of deeper learning experiences, such as reflection, crucial for their development (Andrade, 2019; Estrada, 2021). In response to these challenges, the National Council of Mathematics Supervisors (NCSM) and the National Council of Teachers of Mathematics (NCTM) recommend adapting instructional techniques and assessment practices to support student learning (Delgado, 2020). This includes fostering meaningful mathematical conversations and encouraging students to document their ideas using various tools and representations (NCTM,

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2020). Moreover, students are encouraged to engage in self-assessment to monitor their progress and tailor learning strategies to their strengths, while teachers are tasked with providing meaningful feedback to facilitate improvement (Al-Bashir et al., 2016). After all, educators must continue to explore innovative teaching methods and adapt educational practices to meet the evolving needs of students in current learning environments (Lao & Divinagracia, 2021).

Metacognition in Mathematics

Metacognition, as defined by various scholars, encompasses the processes through which students regulate their cognition, encompassing both their thoughts and behaviors (Rhodes, 2019; Chick, 2013). It comprises three main components: metacognitive knowledge, metacognitive experiences, and metacognitive skills (Laycraft, 2019). Metacognitive knowledge involves an understanding of mathematical processes and the nature of mathematics itself (Tian et al., 2018 cited in Özsoy, 2011). Metacognitive experiences, on the other hand, involve the feelings and judgments students have about learning activities, impacting both cognitive and affective processes (Efklides, 2009). Finally, metacognitive skills encompass purposeful techniques used to regulate cognition, such as planning, monitoring, and evaluating one's learning progress (Laycraft, 2019 cited in Veenman & Elshout, 1999).

These metacognitive processes have the potential to enhance students' learning experiences and academic performance in mathematics (EduCare, 2020). By enabling students to think critically and make connections between concepts, metacognition fosters a deeper understanding of mathematical concepts (Alzahrani, 2017 cited in Grant, 2014). However, traditional feedback mechanisms may fall short in supporting students' metacognitive development, necessitating innovative approaches to assessment and task design (Fogarty, 1994). With this, teachers play a crucial role in promoting metacognitive development by guiding students through the three-phase process of metacognition: planning, monitoring, and evaluating (Fogarty, 1994). By integrating metacognitive questioning techniques into lesson plans, teachers can help students develop self-awareness and problem-solving skills (Ahdhianto et al., 2020). Additionally, fostering metacognitive activities such as self-explanation and self-assessment can enhance students' academic success and overall learning experience (Setiawan & Supiandi, 2018; Lin, 2001). In fact, Ozdemir et al. (2024) found that mathematical metacognition awareness significantly contributes academic resilience.

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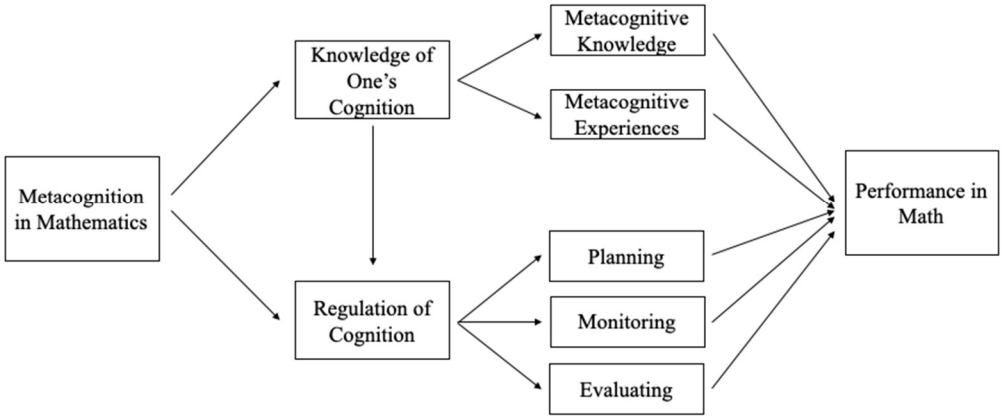


Figure 1: Theoretical Framework Adapted from Garofalo (1986)

Figure 1 shows that metacognitive knowledge and processes play a crucial role in shaping students' mathematical performance, influencing how they approach and execute cognitive tasks. Integrating a metacognitive dimension into classroom practices benefits both teachers and students by enhancing students' cognitive processing and ultimately improving their mathematical performance. This distinction in metacognition encompasses understanding one's cognitive processes and regulating them, impacting how students prepare and execute cognitive tasks, thereby affecting their mathematical performance.

Various studies emphasize the efficacy of metacognitive activities in enhancing students' metacognitive skills and performance in math. Wen-Xin et al. (2015) observed a significant improvement in students' inquiry practices, particularly in planning and analysis skills, through the use of metacognitive prompts. Additionally, Ysseldyke and Tardrew (2008) highlighted the benefits of incorporating instructional management and progress monitoring tools in ongoing math instruction, leading to improved math outcomes for students.

Overall, fostering metacognitive skills empowers students to become more self-aware learners, capable of monitoring their progress, identifying areas for improvement, and applying effective learning strategies (Chick, 2013 cited in Bransford, Brown, & Cocking). By integrating metacognitive activities into mathematics education, educators can enhance students' problem-solving abilities and promote a deeper understanding of mathematical concepts.

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METHODS

Research Design

The study used an action research design, incorporating both qualitative and quantitative data to evaluate the impact of metacognitive prompts within performance-based assessments using the RME approach on students' metacognitive awareness, skills, and academic performance. The study utilized purposive sampling to select 30 Grade 9 students. The criteria for selecting the students include: (1) active enrollment in Grade 9 during the conduct of the research; (2) consistent participation in performance-based tasks throughout the quarter; and (3) evidence of diverse learning styles and abilities based on classroom observations and formative assessments. Based on the researcher's assessment of the students, nine (9) participants were selected, representing varying proficiency levels: three below average, three average, and three above average students. The classification was based on a triangulation of their academic performance in Mathematics from the previous quarters, classroom engagement, and the quality of outputs in their performance-based tasks. Below-average students consistently scored at least one standard deviation below the class mean, average students performed within the mean range, and above-average students consistently scored above the mean. This selection ensured a representative sample to meaningfully explore metacognitive behaviors across proficiency levels.

Research Instruments

The research instruments used in the study were developed by the researcher. The first one is the performance-based assessment incorporating metacognitive prompts into students' supplementary materials, focusing on quadratic functions. The performance-based assessment task involves crafting a business proposal that integrates profit maximization principles using quadratic functions, tailored to students' personal preferences. On the other hand, the metacognitive prompts were divided into three sections: "before", "during", and "after", comprising insightful questions regarding their task experiences. Aligned with Philippines' Department of Education guidelines, these prompts served as self-monitoring tools. Additionally, specific analytic rubrics were crafted for each task, prioritizing process quality over product outcomes. Four teachers, including the Math Department coordinator, a UNIPEAT author, a head teacher and MTAP trainer, and a Math education professor, validated the performance-based assessment and rubric. To evaluate student metacognition, the Junior Metacognitive Awareness Inventory (Jr. MAI) adapted from Sperling, Howard, Miller, and Murphy in 2002 was administered, along with a parallel tool scoring students' metacognitive skills from their prompt responses. Semi-structured interview questionnaires were also designed by the researcher, validated by the same validators, to gather student experiences and perceptions on task completion with metacognitive prompts, as well as the researcher's assessment of students' metacognitive skills and Mathematics performance. Zoom was used for interviews, with recorded sessions for reference.

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Data Gathering Procedure

Participants were provided with informed consent forms before the implementation of the study. Upon consent, students completed the Jr. MAI to establish baseline data, to be compared with post-intervention responses. The researcher then conducted a lesson on Quadratic Functions, followed by the distribution of the performance-based assessment incorporated with metacognitive prompts. The task was assessed using the analytic rubric to monitor progress and offer feedback. Subsequently, students were asked to accomplish the Jr. MAI for the second time. On the other hand, the researcher evaluated students' metacognitive skills based on their prompt responses using a parallel Jr. MAI questionnaire. Comparisons were made between student scores and the researcher's scores. Additionally, participants were interviewed via Zoom at agreed-upon times, with note-taking and recording for data preservation and analysis.

Data Analysis

The study used both quantitative and qualitative analyses to examine the gathered data. Quantitatively, several statistical methods were applied to assess learners' metacognitive awareness and skills in Mathematics. This included calculating means from pre-test and post-test scores on the Jr. MAI. A Paired Samples t-test was utilized to determine significant differences between Grade 9 students' pre-test and post-test results. Cohen's d was also employed to gauge the effect size of these differences. Additionally, the Shapiro-Wilk Test and Wilcoxon Matched-Pairs Signed-Ranks Test were conducted to assess normality in student scores. Pearson Correlation Coefficient was used to measure the association between students' Jr. MAI scores and the researcher's average scores on metacognitive skills, as assessed through the parallel Jr. MAI questionnaire. Descriptive analysis was also used to compare the number of students scoring above, equal to, or below the cut-off between previous and current performance tasks.

On the other hand, thematic analysis was used to analyze the qualitative data gathered from the interviews with the participants. This approach involved examining their responses to identify commonalities and connections within the data, ultimately leading to the creation of overarching themes. The process included coding the data to systematically organize and interpret the information, thereby highlighting key insights, patterns, and outcomes.

RESULTS

Jr. MAI Mean Scores Before and After the Intervention

The analysis of students' mean scores on the Jr. MAI revealed insightful patterns within the categories of Knowledge about Cognition and Regulation of Cognition.	**Pre-Test Mean**	**Post-Test Mean**
Knowledge about Cognition		

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• Declarative Knowledge	3.41	3.98
• Procedural Knowledge	3.37	4.07
• Conditional Knowledge	3.52	4.17
Mean	**3.43**	**4.07**
Regulation of Cognition		
• Planning	3.38	4.07
• Monitoring	3.32	3.86
• Evaluating	3.33	3.99
Mean	**3.34**	**3.97**
Overall Mean	**3.39**	**4.02**

Table 1. Jr. MAI Pre-test and Post-test Scores Before and After the Intervention

Table 1 shows the pre-test and post-test mean scores for each sub-component within these categories. Following the intervention, notable improvements were observed across all sub-components. Among the sub-components related to *Knowledge about Cognition*, *Conditional Knowledge* demonstrated the highest mean score post-intervention (4.17), indicating a substantial increase from the pre-test. Similarly, *Procedural Knowledge* and *Declarative Knowledge* exhibited notable improvements, with mean scores of 4.07 and 3.98, respectively.

In terms of *Regulation of Cognition*, the sub-component *Planning* displayed the highest mean score post-intervention (4.07), reflecting a notable increase from the pre-test. *Evaluating* and *Monitoring* also showed considerable improvements, with mean scores of 3.99 and 3.86, respectively.

Futhermore, the overall mean scores for *Knowledge about Cognition* and *Regulation of Cognition* indicate that the pre-test mean score for *Knowledge about Cognition* was 3.43, increasing to 4.07 post-intervention, while *Regulation of Cognition* increased from 3.34 to 3.97. Overall, there was a notable improvement in metacognitive awareness and skills, with the combined mean score rising from 3.39 pre-test to 4.02 post-intervention.

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Normality Test for Jr. MAI Pre-Test and Post-Test Scores of the Students

In addition to the paired sample t-test, we conducted the Shapiro-Wilk test to assess the normality of the pre-test and post-test scores for each sub-component of the Jr. MAI. For *Declarative Knowledge*, the pre-test scores showed a p-value of 0.157 and the post-test scores showed 0.029, indicating normality only in the pre-test scores. Conversely, both pre-test and post-test scores for *Procedural Knowledge* departed significantly from normality with p-values of 0.001 and 0.035, respectively. However, the pre-test and post-test scores for *Conditional Knowledge* showed no evidence of non-normality with p-values of 0.053 and 0.095, respectively.

Moving to the sub-components of *Regulation of Cognition*, both pre-test and post-test scores for *Planning and Evaluating* exhibited p-values greater than 0.05, suggesting normality. However, *Monitoring* displayed normality only in the pre-test scores ($p = 0.061$), while the post-test scores deviated from normality ($p = 0.012$).

Given the departure from normality in several sub-components, the non-parametric Wilcoxon Matched-Pairs Signed-Ranks Test for further analysis. The results show that all p-values are less than 0.05, indicating statistically significant differences between pre-test and post-test scores for these sub-categories.

Impact of the Intervention on Students' Metacognitive Awareness and Skills

The effect of implementing a performance task with metacognitive prompts in Mathematics on the metacognitive awareness and skills of Grade 9 students was rigorously assessed, focusing on their Knowledge about Cognition and Regulation of Cognition

Paired Samples Test and Effect Size		
	p-value	**Cohen's d**
PRETEST - POSTTEST	0.001	1.094

Table 2. Paired Sample t-Test of the Grade 9 Students' Overall Jr. MAI Scores and Treatment Effect of the Intervention

The paired sample t-test results in Table 2 reveal a significant difference ($p < 0.05$) between the Jr. MAI pre-test and post-test scores of the students, indicating an improvement in metacognitive awareness and skills following the intervention. This improvement was particularly notable after incorporating metacognitive prompts related to quadratic functions.

To delve deeper into the Jr. MAI scores, Cohen's d value was calculated to gauge the magnitude of the difference between the pre-test and post-test scores of Grade 9 students. It shows a Co-

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hen's d value of 1.094, indicating a very strong treatment effect. This underscores the substantial positive impact of utilizing a performance task with metacognitive prompts on the overall metacognitive awareness and skills of the students. The intervention emerged as an effective tool for enhancing students' ability to manage their thinking processes.

To further dissect the findings, an in-depth analysis was conducted on the significant differences and treatment effects of each sub-component within the two categories. Specifically, the following paired comparisons were analyzed: KACPRE–KACPOST (Knowledge About Cognition Pre-test and Post-test), DECPRE–DECPOST (Declarative Knowledge Pre-test and Post-test), PROPRE–PROPOST (Procedural Knowledge Pre-test and Post-test), and CONPRE–CONPOST (Conditional Knowledge Pre-test and Post-test).

Paired Samples Test and Effect Size		
	p-value	**Cohen's d**
KACPRE-KACPOST	0.001	1.134
DECPRE-DECPOST	0.001	0.809
PROPRE-PROPOST	0.001	0.920
CONPRE-CONPOST	0.001	0.928

Table 3. Paired Sample t-Test of Grade 9 Students' Pre-test and Post-test Scores in Knowledge about Cognition and its Sub-components and Effect Sizes of the Intervention

As shown in Table 3, significant differences ($p < 0.05$) were observed in both the Knowledge about Cognition category and its sub-components, highlighting increased metacognitive awareness post-intervention. Moreover, Cohen's d values were calculated to reveal the effect sizes of these differences.

On the other hand, it also shows the substantial effect size (Cohen's $d = 1.134$) of the category Knowledge about Cognition, particularly in Conditional Knowledge, Procedural Knowledge, and Declarative Knowledge. These findings underscore the intervention's efficacy in enhancing students' understanding and application of metacognitive strategies.

In addition, the following paired comparisons were analyzed to examine the changes within the Regulation of Cognition category: ROCPRE–ROCPOST (Regulation of Cognition Pre-test and Post-test), PLANPRE–PLANPOST (Planning Pre-test and Post-test), MONPRE–MONPOST (Monitoring Pre-test and Post-test), and EVALPRE–EVALPOST (Evaluating Pre-test and Post-test).

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Paired Samples Test and Effect Size		
	p-value	**Cohen's d**
ROCPRE-ROCPOST	0.001	0.811
PLANPRE-PLANPOST	0.001	0.847
MONPRE-MONPOST	0.005	0.559
EVALPRE-EVALPOST	0.003	0.595

Table 4. Paired Sample t-Test of Grade 9 Students' Pre-test and Post-test Scores in Regulation of Cognition and its Sub-components

Table 4 shows the significant differences ($p < 0.05$) in both the Regulation of Cognition (ROC) category and its sub-components, indicating improved metacognitive skills post-intervention. Cohen's d values were also calculated to assess the effect sizes of these differences. It also highlights a significant treatment effect (Cohen's $d = 0.811$) in the category Regulation of Cognition, particularly in Planning, Monitoring, and Evaluating. These results underscore the intervention's role in fostering students' ability to approach learning tasks strategically and evaluate their progress effectively.

The data suggest that the intervention significantly enhanced students' metacognitive skills, facilitating their ability to plan, monitor, and evaluate their learning processes. Despite varying effect sizes, the overall improvement underscores the importance of consistently engaging in metacognitive activities to achieve academic goals effectively.

Relationship between Participants' Scores and Researcher's Scores

To explore the correlation between students' average Jr. MAI scores and the researcher's scores, the Pearson correlation coefficient was utilized.

The result reveals a strong positive correlation (Pearson $r = .703$) between students' average Jr. MAI scores and the researcher's scores. This correlation is highly significant ($p < .001$), indicating that the researcher's assessment aligns closely with the students' self-reported metacognitive prompts.

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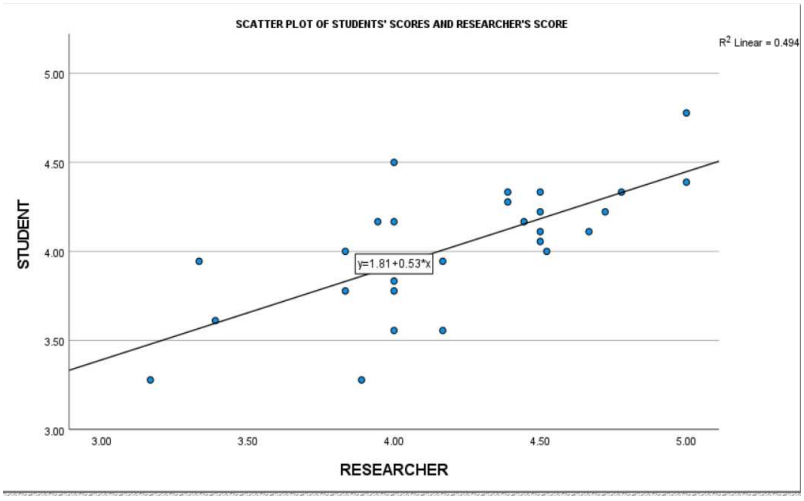


Figure 2. Scatter Plot with Regression Line of the Students' Scores and Researcher's Scores

A scatter plot, depicted in Figure 2, further illustrates this strong positive relationship, suggesting a moderate linear correlation.

To assess the intervention's impact on overall performance, descriptive analysis was conducted to determine the number of students scoring above and below the established cut-off on their performance-based assessment before and after the intervention.

	Number of Students Scored Above or Equal to the Cut-off	**Number of Students Scored Below the Cut-off**
Pre-Intervention	26	4
Post-Intervention	28	2

Table 5. Frequency Table of the Number of Students who Scored Above, Equal and Below the Cut-off

Table 5 shows the number of students who scored above, equal to, and below the established cut-off (75% passing score) on their performance-based assessment before and after the intervention. Before the intervention, 26 students achieved scores at or above the cut-off, while 4 students scored below it. Following the intervention, 28 students met or exceeded the cut-off, with only 2 students falling below it.

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Students' Perspectives on Completing a Performance-Based Assessment with Metacognitive Prompts in Mathematics

A thematic analysis was conducted to extract significant quotes from students, providing a comprehensive insight into their experiences and viewpoints regarding the metacognitive intervention. The analysis focused on two predetermined categories: students' perceptions of completing a performance-based assessment with metacognitive prompts and the perceived impact of this intervention on their learning.

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Themes	Codes		
	Below Average Group	**Average Group**	**Above Average Group**
Planning before the accomplishment of the performance task about quadratic functions	Usage of prior knowledge on quadratic equations and prior experiences Communication of mathematical ideas Utilization of math and design tools and software	Usage of prior knowledge in math and prior experiences Display of organizational techniques Communication of mathematical ideas Utilization of math and design tools and software	Usage of prior knowledge about quadratic equations and factoring Self-motivation Display of organizational techniques Communication of mathematical ideas Utilization of math and design tools and software
Monitoring progress during the accomplishment of the performance task about quadratic functions	Optimization of resources for better understanding Consideration of other sources of information	Consideration of other sources of information Usage of monitoring tools	Identification of personal strategy Situational awareness Optimization of resources for better understanding Usage of monitoring tools
Evaluating performance and output after the accomplishment of the performance task about quadratic functions	Display of fulfillment Recognition of new strategies	Display of fulfillment Recognition of new strategies	Display of fulfillment

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Improving self-knowledge in accomplishing Math tasks	Show of difficulty in math concepts Display of effort	Self-actualization Show of difficulty in math concepts Recognition of shortcomings to math tasks	Show of difficulty in math concepts Display of self-confidence towards math
Improvement of mathematical performance	Acknowledgment of self-check tasks to monitor progress	Acknowledgment of self-check tasks to monitor progress and assess feelings	Consideration of peers' mathematical improvement Recognition of the importance of the task for personal and academic growth Acknowledgment of self-check tasks to monitor progress

Table 6. Themes Emerged from the Three Groups' Significant Codes about Accomplishing the Performance-Based Assessment with Metacognitive Prompts about Quadratic Functions

First, all groups demonstrated strong planning skills, leveraging prior knowledge and seeking inspiration online to enhance their performance task on quadratic functions. For instance, one student from the average group shared, *"I looked for business ideas online to help me think of a product."* This proactive approach to planning was echoed by a student from the above-average group who said, *"Before I started, I divided the work based on what I know."* Even students in

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the below-average group showed evidence of intentional planning, as one mentioned, *"I recalled our lesson last quarter, and I used it to start making my plan."* While organizational techniques were mainly observed in the average and above-average groups, all groups effectively communicated math and business ideas, utilizing various tools and software.

Second, students showcased monitoring skills, optimizing resources such as videos and seeking external assistance to understand concepts. A student from the average group said, *"When we got confused, we asked our teacher and even watched extra videos about quadratic functions."* Even in the below-average group, persistence was observed, as one student said, *"I didn't understand it at first, but I kept watching the video until I got it."* The above-average group stood out for openly sharing their experiences and adapting to various factors during the task, showing situational awareness and personal initiative. As one reflected, *"I changed my product idea halfway because it didn't make sense mathematically."* Both average and below average students used monitoring tools as one student from the average group shared, "*I tracked which parts I already finished and what still needed checking.",* and one student from the above average group shared, *"I used the monitoring form."*

Third, students displayed evaluative skills, showing a sense of fulfillment despite setbacks. A student from the below-average group said, *"I was proud even if I didn't get all right (sic). I still tried my best."* In addition, both below-average and average groups recognized areas for improvement. One student from the average group said, *"I could have improved my design, but I think I got the math right."* Meanwhile the above-average group demonstrated confidence in their output with one stating, *"I think I did great."*

Fourth, students demonstrated self-awareness, with the below-average group showing effort to understand complex concepts as one stated, "*I knew I was having a hard time with factoring, so I focused more on that part,"* while the average group recognized potential areas for improvement as one stated, *"I realized I need to work on organizing my proposal better next time."* The above-average group also displayed confidence in their abilities, reflecting in the quality of their output. As one student expressed, *"I feel confident because I understood both the math and the business part. The checklist really helped."* They recognized the importance of monitoring progress and expressed a willingness to utilize metacognitive strategies for continuous improvement.

Lastly, all groups recognized the value of metacognitive prompts in monitoring progress and regulating emotions. The average group recognized the emotional impact of the task, with a student sharing, *"The checklist made me reflect not just on my answers but how I felt while doing the task."*The above-average group particularly appreciated the authentic task for personal and academic growth as one student shared,*"I liked this project because it felt real. It helped me grow, not just in math but also in how I plan and solve problems."* They also acknowledged its potential to enhance mathematical performance in others where students acknowledged the

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broader significance of metacognitive skills for their peers, emphasizing its role in fostering academic growth and success.

DISCUSSION

Can Performance-based Assessments with the Metacognitive Prompts in Quadratic Functions Using the RME Approach Contribute to Students' Metacognitive Awareness, Skills and Overall Mathematical Performance?

We initially tested all 30 students and obtained results that support the soundness of our method. The findings revealed a significant increase in students' metacognitive awareness, especially in *Knowledge about Cognition*, which includes declarative, procedural, and conditional knowledge. Notably, *Conditional Knowledge* showed the greatest improvement, indicating enhanced critical thinking and the ability to apply strategies appropriately in solving quadratic function tasks. *Procedural Knowledge* also showed strong gains, reflecting improved strategy application through exploration, collaboration, and problem-solving activities. Though smaller in effect, *Declarative Knowledge* still demonstrated meaningful growth, showing students' improved awareness of how they learn and process information.

The intervention also positively impacted students' *Regulation of Cognition*, particularly in planning, monitoring, and evaluating their learning processes. *Planning* had the strongest effect, likely due to guided tasks and self-check questions that helped students set goals and select effective strategies. *Monitoring* and *Evaluating* improved moderately; students reported using self-check tools to track progress and reflect on performance.

Overall, the study highlights how performance-based interventions can effectively develop metacognitive skills, enabling students to better plan, monitor, and evaluate their learning—critical abilities for academic success and lifelong learning.

How Do Students Perceive the Completion of a Performance-based Assessment with Metacognitive Prompts in Mathematics, and What is Their Perceived Impact on Learning Outcomes?

Based on predetermined criteria related to their academic standing, we then selected 9 participants for further analysis. The following results were obtained from this subgroup. Students demonstrated strong planning skills in completing the performance task on quadratic functions.

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While all groups effectively applied prior knowledge, only the below-average and average groups explicitly connected it to other subjects. Mathematical and business ideas were clearly communicated, with the average and above-average groups exhibiting better organization; the latter also showed initiative by seeking online inspiration.

In terms of monitoring, all groups managed resources well, with the above-average group notably adapting to challenges and reflecting on their experiences. Evaluation skills were evident across groups, with students identifying areas for improvement. The above-average group relied confidently on existing strategies, as reflected in their high-quality outputs. Self-awareness was also observed in all groups. The below-average group made notable efforts to understand complex concepts, the average group acknowledged areas for growth, and the above-average group exhibited confidence aligned with their performance.

Metacognitive prompts were seen as valuable in supporting progress and emotional regulation. The above-average group particularly appreciated the task's authenticity and relevance to real-world learning. Overall, the integration of metacognitive prompts in performance-based tasks effectively enhanced students' awareness, regulation, and academic performance.

CONCLUSIONS

Despite mandates to incorporate performance tasks as part of formative and summative assessments, many Philippine classrooms still rely heavily on procedural instruction that limits students' opportunities to apply mathematical concepts to real-life contexts. This results in limited opportunities for students to engage with authentic mathematical tasks, apply concepts meaningfully, or reflect deeply on their own learning processes. The findings of this study directly respond to this gap by demonstrating the effectiveness of integrating RME with metacognitive prompts in a performance task setting, particularly in teaching quadratic functions.

The improvements observed in students' metacognitive awareness—including gains in declarative, procedural, and conditional knowledge, as well as planning, monitoring, and evaluating skills—support previous research in the field, such as Ozdemir et al. (2020), which found a strong link between metacognitive awareness and academic resilience in mathematics.

Moreover, the qualitative insights from this study reinforce earlier conclusions by Winarso (2018) and Fauziah et al. (2018), who highlighted that context-rich, real-world tasks and mini-projects encourage not only cognitive growth but also emotional engagement and motivation. Similar to findings in Setiawan and Supiandi (2018), students in this study exhibited proactive behaviors such as thoughtful planning and self-monitoring, suggesting that structured support for metacognition cultivates independence and confidence in mathematical problem-solving.

By integrating metacognitive prompts into performance-based tasks, this study extends current understanding of how assessment can be both formative and reflective, moving beyond a one-

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size-fits-all approach. The intervention's differentiated effects—stronger gains in knowledge and planning, with more moderate growth in monitoring and evaluating.

The contribution of this study lies in its demonstration of a holistic, reflective assessment design that supports diverse learners in a real-world educational context. It not only validates established pedagogical frameworks but also provides practical applications relevant to local classrooms—especially where resources are limited. The study also suggests future directions for innovation in instructional design, particularly in exploring the potential of self-assessment, peer collaboration, and digital integration to further support metacognitive development. Ultimately, this research highlights that fostering metacognition through authentic, reflective performance tasks can transform mathematics classrooms—not just improving academic outcomes, but also nurturing self-directed learners prepared for lifelong learning.

ACKNOWLEDGMENTS

I extend my profound gratitude to all those who played significant roles in making this study feasible. Dr. Auxencia Limjap, my adviser, for her unwavering encouragement, support, thoughtfulness, and willingness to assist, inspire and motivate me to complete my study effectively. The evaluators contributed significantly to this study, for which I am sincerely thankful. I am grateful to the participants for their cooperation and the time they generously dedicated to completing the research instruments and openly responding to the interview questionnaires. My family's enduring love, care, and unwavering support have been invaluable throughout this journey. Lastly, I express gratitude to the Lord our God for providing me with the courage, knowledge, strength, and guidance needed to successfully complete this study.

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APPENDICES

Appendix A: Sample Output

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BUSINESS PROPOSAL

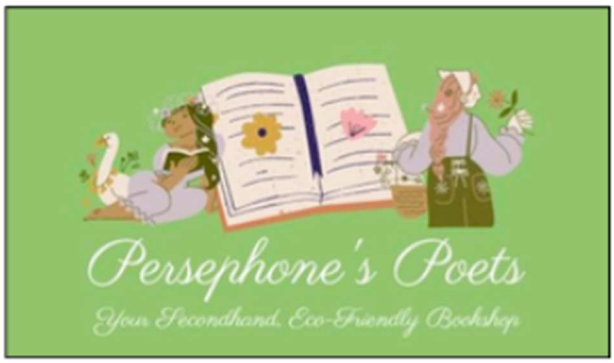
Business Name:

Persephone's Poets – Your Secondhand, Eco-Friendly Bookshop

Business Slogan:

"Let The People Read Books With Affordability & Environmental Purity!"

Business Logo



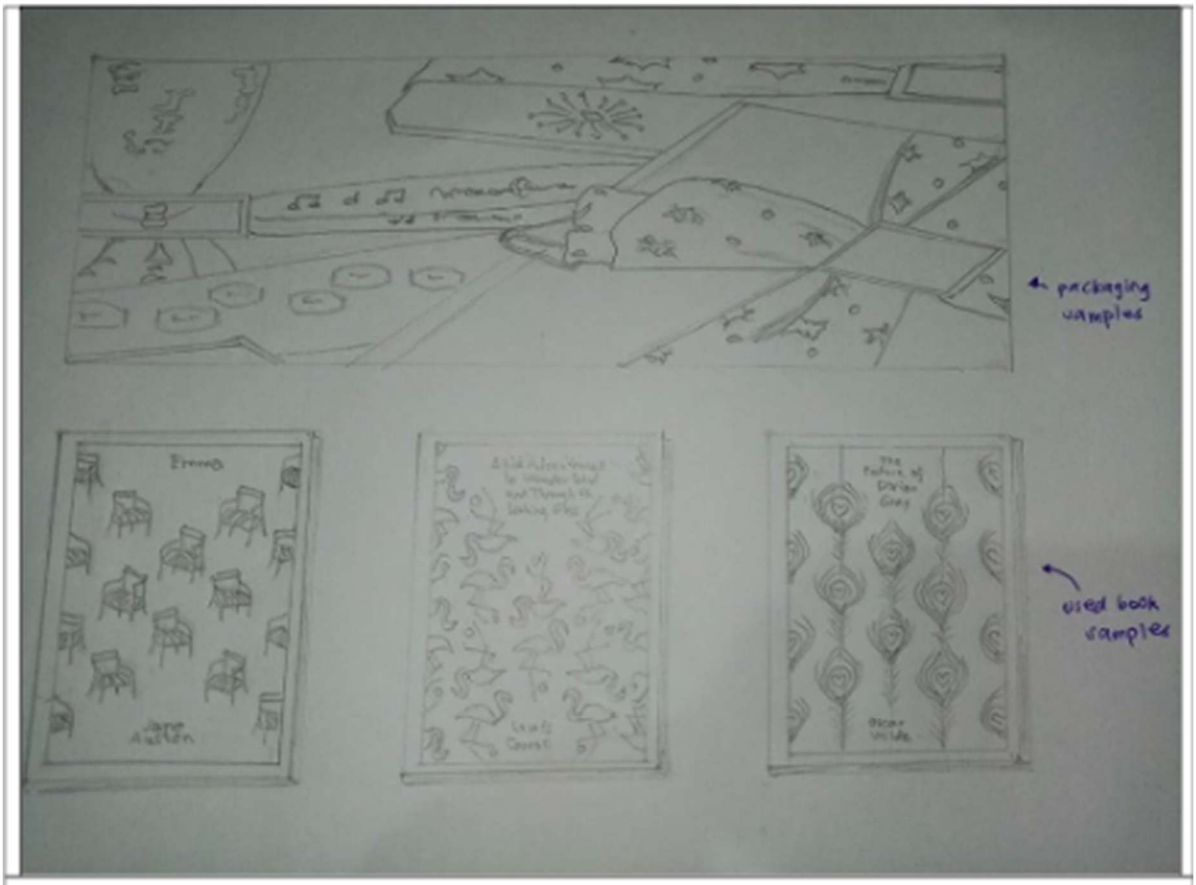
With our whimsical logo, Persephone's Poets aims to send out a warm, comforting message to our buyers that even the simplest things in life can be enjoyed and treasured. As homage to the Greek goddess Persephone, our bookshop's namesake, we chose to give an emphasis on the color green and the calm life in the countryside, hoping that the meek little books we sell would, in a special way, transport readers to cozier and happier places during tough times.

Product:

Packages that consist of three used books of satisfactory quality with genres that randomly range between literary classics, poetry, contemporaries, thrillers, etc. Every package comes with reusable, eco-friendly wrappers.

Sample of Product

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Initial Selling Price: Php 100

Sales in a Month

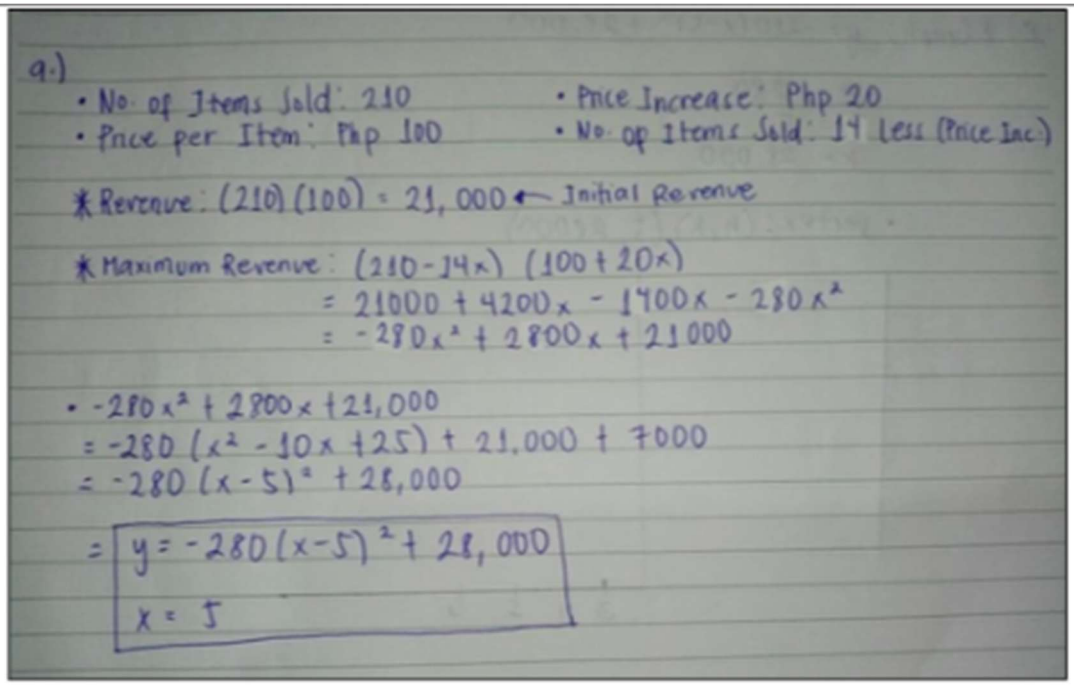
Based on the results of my given survey, a total of 210 used book packages would be sold within a month.

Increased Selling Price: Php 120

Before increasing the selling price, I've already anticipated that there would be a significant decrease in sales. Hence, for every Php 20 increase, I would sell 14 *less* used book packages.

Revenue Maximization

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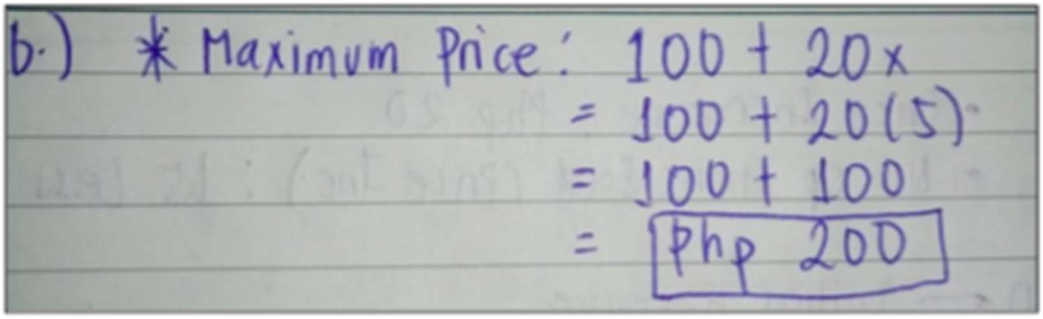

a.)
- No. of Items Sold: 210
- Price per Item: Php 100
- Price Increase: Php 20
- No. of Items Sold: 14 less (Price Inc.)

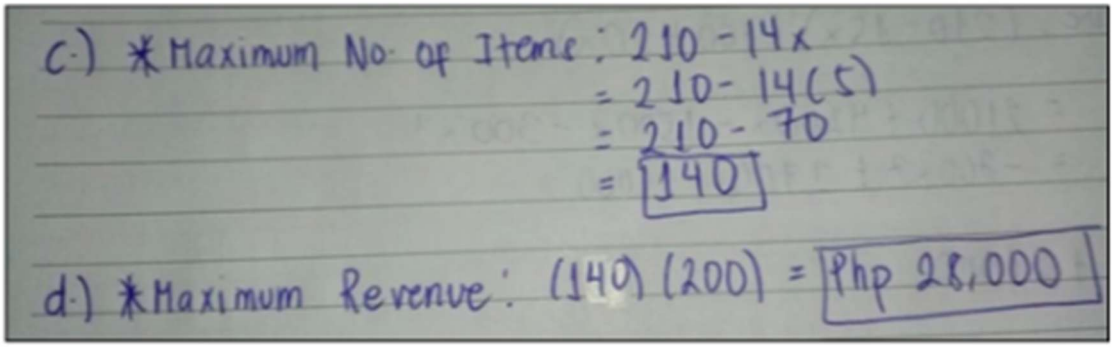
*Revenue: (210)(100) = 21,000 ← Initial Revenue

*Maximum Revenue: $(210-14x)(100+20x)$
$= 21000 + 4200x - 1400x - 280x^2$
$= -280x^2 + 2800x + 21000$

- $-280x^2 + 2800x + 21,000$
$= -280(x^2 - 10x + 25) + 21,000 + 7000$
$= -280(x-5)^2 + 28,000$

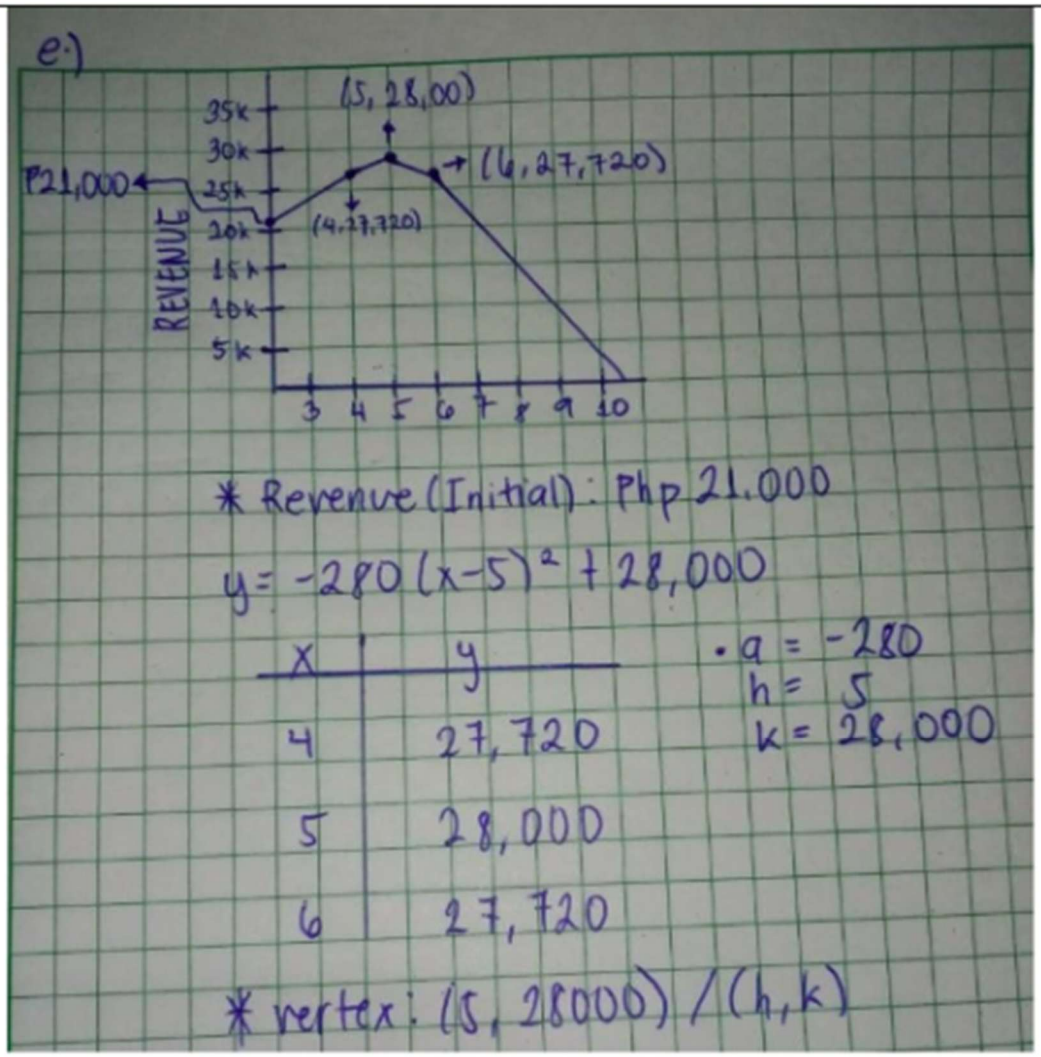
$= \boxed{y = -280(x-5)^2 + 28,000}$
$\boxed{x = 5}$


b.) *Maximum Price: $100 + 20x$
$= 100 + 20(5)$
$= 100 + 100$
$= \boxed{\text{Php } 200}$


c.) *Maximum No. of Items: $210 - 14x$
$= 210 - 14(5)$
$= 210 - 70$
$= \boxed{140}$

d.) *Maximum Revenue: $(140)(200) = \boxed{\text{Php } 28,000}$

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Final Pricing

The final price of my product is Php 200.

Marketing Strategies

While in the middle of a pandemic, I personally think that it would be best to use social media platforms like Facebook and Instagram to promote my product. Posting eye-catching photos and promising descriptions would definitely attract more buyers and put my business out in the open. Of course, when it comes to the actual selling, there are always portable online stores such as Shopee and Lazada to use that make online selling much easier and hassle-free.

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• *No, I am not struggling with the task but the solving parts are quite challenging.*
5. Should I adjust my strategies in accomplishing this task? • *I don't think adjusting my strategies is needed. They're working out fine for me.*
6. Which confusions remain and how am I going to get them clarified? • *Confusion with the "getting the vertex form" part is high at the moment. Reviewing the given materials would probably help me to get them clarified.*
7. Should I change what I planned to do? • *No, changes aren't needed since the overall plan is working like a charm.*
AFTER
1. How well did I do? • *I could confidently say that I've done well, in fact, better than I had expected.*
2. Did I get the results I expected? • *Some of the results were predicted quickly but the survey ones were quite unexpected.*
3. Is there anything I did not understand? • *I didn't understand some of the concepts in creating the graph at first but I managed to cope by eagerly reviewing the given materials about it.*
4. Do I need to go back through the task to fill in any gaps in my understanding? • *No, there aren't any gaps to fill in neither this task nor my understanding.*
5. What could I have done differently? • *There's nothing I would change since I'm highly interested and invested on the plan I've made.*
6. Can I apply the strategies I used to other problems or situations? • *Definitely, the strategies I've used are flexible routines that can be applied to almost anything.*

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SELF-CHECK. Answer the following questions before, during, and after accomplishing the performance task:

BEFORE
1. What am I supposed to do? • *I am supposed to think of a specific business plan based on my own interests that I would like to carry out professionally.*
2. What resources can I use to accomplish this task? • *Articles from online business owners themselves, blogs that give useful tips, and informative YouTube videos are the resources I can use to accomplish this task.*
3. What prior knowledge will help me with this task? • *Past activities that involved entrepreneurship and culminating from my earlier high school and elementary years were solid training grounds for me and other students alike to get a good grasp on how businesses work. Aside from that, personal stories from family members and friends who are familiar with these certain activities also gave me adequate understanding.*
4. What strategies will I use to accomplish this task? • *Finishing this as aptly as I can and following a healthy schedule are the strategies I'm planning to use to accomplish this task.*
5. What should I do first? • *Before anything else, I would consider thinking of a proper business venture first. The brand name, products I would sell, and all-around theme or message I'm trying to give the public should be my kick-starters for this task.*
6. Will I be able to accomplish this task within the given time? • *Yes, I expect to finish this task in three days.*
DURING
1. How am I doing? • *Matters are sailing through smoothly though I am a bit intimated by the parts involving equation solving.*
2. Am I being systematic in accomplishing this task? • *Yes, the tasks needed to be done for each day are being thoroughly completed.*
3. Am I maximizing the use of all the resources available? • *Yes, almost all of the resources are in good use.*
4. Am I struggling with this task? If yes, which part?

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Appendix B: Scores and Feedback

Create Your Own Business RUBRIC

Dimension	Weight	Mastered (3)	Working towards Mastery (2)	Needs Improvement (1)	Score
Defining the Situation	x3	Demonstrates the ability to construct clear and insightful business proposal with evidence of all relevant contextual factors.	Demonstrates the ability to construct a story line and problems with evidence of all relevant contextual factors and the proposal is adequately detailed.	Begins to demonstrate the ability to construct a story line and problems with evidence of most relevant contextual factors, but the proposal is superficial.	9
Identifying and monitoring appropriate Strategies	x3	Identifies multiple approaches for solving the problems that apply within a specific context.	Identifies multiple approaches for solving the problems, only some of which apply within a specific context.	Identifies only a single approach for solving the problems that does apply within a specific context.	9
Implement-ing Solution	x3	Implements the solution that addresses thoroughly and deeply multiple contextual factors of the problems.	Implements the solution in a manner that addresses multiple contextual factors of the problems in a surface manner.	Implements the solution in a manner that addresses the problems in the proposal but ignores relevant contextual factors.	9
				TOTAL	27

Teacher's Feedback:

Planning *(Defining the situation)*	All the needed information is evident and clear including the name, slogan, logo, product, and data obtained from pre- and post-surveys. Used helpful resources and knowledge in planning the business proposal Overall, the planning stage was clear and systematized.

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Monitoring *(Identifying appropriate strategies))*	Mindful thinking and decision making if strategies are needed to be adjusted took place to resolve challenges that have been experienced Done the tasks that were previously scheduled Maximized all the resources available Overall, the way that the performance task was answered reflected to the answers on the self-check.
Evaluating *(Implementing solution)*	It is clear that the plan was created with confidence and knowledge. Overall, the mathematical component of the performance task is correct and the goal was properly demonstrated.

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Appendix C. Junior Metacognitive Awareness Inventory (Jr. MAI)

Read the following sentences carefully and check the box for each answer that applies to you and the way you work on a performance task in Mathematics. Please be as honest as possible in your answers.

1 – Never 2 – Seldom 3 – Sometimes 4 – Often 5 – Always

	1	2	3	4	5
1. I am a good judge of how well I understand something.					
2. I can motivate myself to learn when I need to.					
3. I try to use strategies that have worked in the past.					
4. I know what the teacher expects me to learn.					
5. I learn best when I know something about the topic.					
6. I draw pictures or diagram to help me understand while learning.					
7. I ask myself if I learned as much as I could have once I finish a task.					
8. I ask myself if I have considered all options when solving a problem.					
9. I think about what I really need to learn before I begin a task.					
10. I ask myself questions about how well I am doing while I am learning something new.					
11. I focus on the meaning and significance of new information.					
12. I learn more when I am interested in the topic.					
13. I use my intellectual strengths to compensate for my weaknesses.					
14. I have control over how well I learn.					
15. I asked myself periodically if I am meeting my goals.					
16. I find myself using helpful learning strategies automatically.					
17. I ask myself if there was an easier way to do things after I finish a task.					
18. I set specific goals before I begin a task.					

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Enhancing Instruction in Trigonometry: Insights from Students' Reasoning Approaches and Processes

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Abstract: This study investigates the interplay between student reasoning and instructional strategies in trigonometry education. A qualitative case study was conducted to examine how Grade 10 students employ various reasoning approaches - deductive, inductive, abductive, analogical, and algorithmic - when solving trigonometry problems. Each approach offers unique strengths and limitations, impacting students' problem-solving strategies and comprehension of trigonometric concepts. The research also emphasizes the importance of validation, justification, and formal proving as integral steps in solidifying mathematical reasoning, transitioning students from practical experimentation to theoretical understanding. Classroom observations and teacher interviews provide insights into effective instructional strategies that leverage student reasoning. Teachers adapted methods to accommodate these diverse approaches, emphasizing scaffolding, guided inquiry, and contextualized learning experiences. Hands-on and experiential learning proved effective, fostering deeper engagement and intuitive understanding. Integration of formal mathematical reasoning ensured students grasped abstract concepts and their real-world applications. Meta-cognitive reflection and discourse enhanced students' problem-solving abilities and collaborative skills. These findings underscore the importance of flexible and inclusive mathematics instruction, equipping students with essential skills for success in mathematics and beyond.

Keywords: mathematics reasoning, deduction, abduction, justification, validation

INTRODUCTION

Developing meaningful mathematical knowledge hinges on students actively engaging with prior experiences and constructing new understanding (Romberg, 2000). This requires students to identify patterns, create models of concepts, develop symbolic representations, and devise solutions to complex problems (Battista, 1999). Trigonometry, integrating geometry, algebra, and graphical representations, offers a rich environment for fostering these skills. However, current

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practices often emphasize rote memorization and calculations, neglecting opportunities for deeper reasoning (Shield, 2004).

The abstract nature of trigonometry and its intricate connections between geometry and algebra pose learning challenges (Bressoud, 2010; Cavanagh, 2008; Thompson, 2008). Overcoming these hurdles necessitates a shift from purely technical instruction to a broader approach that cultivates reasoning and understanding. Investigating how students approach problem-solving in trigonometry holds the key to unlocking effective pedagogical methods.

Previous research highlights the importance of recognizing diverse student reasoning styles and tailoring instruction to cater to individual learning preferences (Thompson et al., 2007). However, a gap exists in the literature regarding the interconnectedness between student reasoning and teacher strategies specifically within trigonometry education.

The effectiveness of teaching trigonometry hinges on how teachers engage with students' solutions, responses, and reasoning (Hiebert & Stigler, 2023; Herbert & Bragg, 2021). Understanding how teachers navigate these aspects significantly impacts the quality of instruction. This study investigates this dynamic relationship between student reasoning and teacher pedagogical methods in teaching trigonometry. By examining diverse student problem-solving strategies, the research explores how teachers can leverage these solutions to foster conceptual understanding.

The study addresses the following key research questions:

1. What mathematical reasoning approaches and processes do students employ when solving and explaining solutions to trigonometry problems?
2. What instructional methods do teachers employ to accommodate and leverage students' diverse mathematical reasoning approaches and processes?

LITERATURE REVIEW

Innovative Pedagogical Approaches in Teaching Trigonometry

Traditional trigonometry instruction often relies on rote memorization, hindering students' ability to apply concepts effectively (Gillman, 1991; van Laren, 2012). To address this, innovative pedagogical approaches are emerging to foster deeper understanding and critical thinking (Quinlan, 2004). One critical challenge is overcoming common misconceptions (Gholami, 2022). Cooperative Learning Strategies (CLS) such as Cooperative Teaching and Learning (CTL) have demonstrated effectiveness in addressing this issue by creating collaborative learning environments (Asomah et al., 2023; Patterson et al., 2020; Quinlan, 2004).

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Developing critical thinking and problem-solving skills is paramount in trigonometry instruction. Structured classroom discussions, hands-on activities, and real-world applications can significantly enhance student understanding (Kohlmeier & Saye, 2019; Thompson et al., 2007; Vale et al., 2019; Weber, 2005). To cater to diverse student needs, differentiated instruction is essential. By tailoring teaching strategies to individual learning styles and cognitive abilities, educators can create inclusive learning environments (Aiyub et al., 2024; Hiebert & Stigler, 2023).

Ultimately, innovative pedagogical approaches are crucial for transforming trigonometry instruction from rote memorization to a discipline that fosters deep understanding, critical thinking, and real-world application.

Reasoning Approaches in Trigonometry

The enhancement of trigonometry instruction is closely linked to understanding the various reasoning approaches that students employ. By examining different approaches to reasoning, teachers can gain critical insights into students' cognitive processes, which is essential for fostering deeper mathematical thinking and problem-solving skills (Gómez-Veiga et al. 2018).

Inductive reasoning, a key component of cognitive functioning, is crucial for students as they observe patterns and generalize from specific instances in trigonometry. This type of reasoning helps students recognize patterns and representations, which supports higher-order cognitive abilities like abstract thought and problem-solving (Bao et al., 2022; Klauer & Phye, 2008; Molnár et al., 2013). Inductive reasoning is important in discovering trigonometric identities, demonstrating that encouraging students to explore patterns significantly enhances their problem-solving abilities and deepens their conceptual understanding (Gunderson & Rosen, 2010; Hamers et al., 1998; Pellegrino & Glaser, 1984). This suggests that instructional strategies that engage students in pattern recognition and conjecture formation are particularly effective in trigonometry education.

On the other hand, deductive reasoning, which involves deriving specific conclusions from general principles, is essential for solving trigonometric problems using known identities and formulas (Sánchez et al., 2023). This reasoning process is crucial for enabling students to apply established facts to new problems, thereby generating new knowledge (Díaz Quezada & Sepúlveda Albornoz, 2023; Duval, 1991; Harel & Weber, 2020). However, research indicates that many students rely heavily on memorized formulas without fully understanding the underlying logic of trigonometric principles, which can limit their ability to apply these principles flexibly across different contexts (Siyepu, 2020). Therefore, enhancing trigonometry instruction requires a stronger emphasis on deductive reasoning, encouraging students to understand and apply logical rules rather than relying solely on memorization.

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Furthermore, analogical reasoning involves drawing parallels between similar situations or problems to infer solutions or understand new concepts. This type of reasoning is essential in mathematics, where students often encounter new problems that can be understood by relating them to previously solved problems. Analogical reasoning is a powerful tool in mathematics education because it enables students to transfer knowledge from one context to another (Goswami, 1991). For instance, understanding the solution to a geometric problem can often be facilitated by recognizing its similarity to another problem that the student has already solved (Gray & Holyoak, 2021). Furthermore, analogical reasoning helps in the consolidation of knowledge, as it allows students to build on existing knowledge and apply it to new, more complex situations (Richland et al., 2007).

Moreover**,** algorithmic reasoning refers to the step-by-step application of procedures or algorithms to solve problems. This type of reasoning is pervasive in mathematics, especially in areas like algebra and calculus, where specific algorithms are applied to simplify expressions, solve equations, or compute values. Algorithmic reasoning is critical in developing procedural fluency in mathematics (Rittle-Johnson et al., 2001). While it is often seen as mechanical, algorithmic reasoning also requires a deep understanding of the underlying principles to be applied effectively. Star (2005) notes that students who develop strong algorithmic reasoning skills are better equipped to handle complex problems, as they can efficiently and accurately apply the necessary procedures. However, it is also important to balance algorithmic reasoning with other types of reasoning to ensure that students do not become overly reliant on procedures without understanding the concepts behind them.

The inclusion of inductive, deductive, abductive, analogical, and algorithmic reasoning in this study is grounded in the belief that a comprehensive understanding of problem-solving involves a multifaceted approach (Gyan et al., 2021). These reasoning modes represent distinct cognitive processes that collectively contribute to a robust problem-solving repertoire (Gyan et al., 2021; Pruner, 2023). Inductive and deductive reasoning form the cornerstone of scientific inquiry, allowing for the observation of patterns, the formulation of hypotheses, and the application of logical rules to test those hypotheses (Molnár et al., 2013). Abductive reasoning stimulates creativity and critical thinking by encouraging students to generate multiple potential explanations for observed phenomena, fostering a deeper understanding of problem complexity (Hidayah et al., 2023; Paavola, 2023; Pedemonte & Reid, 2011). Analogical reasoning facilitates knowledge transfer and problem-solving flexibility by enabling students to connect new problems with familiar situations, promoting adaptability and innovation. And algorithmic reasoning emphasizes computational thinking and systematic problem-solving, equipping students with essential skills for the digital age and preparing them to tackle complex challenges. By incorporating these diverse reasoning approaches, the study aims to provide a rich and engaging learning experience that develops students' cognitive flexibility, critical thinking, and problem-solving abilities. This holistic approach aligns with contemporary educational goals that emphasize the importance of 21st-century skills.

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Enhancing trigonometry instruction requires a comprehensive approach that goes beyond traditional methods. By addressing misconceptions, fostering collaboration, promoting critical thinking, and accommodating diverse learning needs, educators can significantly improve students' understanding and application of trigonometric concepts. Furthermore, integrating insights from various reasoning approaches—inductive, deductive, abductive, analogical, and algorithmic—into instructional design can lead to more effective learning outcomes in trigonometry. As the field continues to evolve, future research should focus on examining the long-term impacts of these innovative pedagogical approaches on student achievement and their overall attitudes toward mathematics.

Mathematical Reasoning Processes

Mathematical reasoning forms the foundation of problem-solving, critical thinking, and innovation. This paper presents a structured framework for mathematical reasoning, centering on four key processes: validation, justification, informal proving, and formal proving. Each process is examined through the practical problem of determining ladder stability, providing a concrete application that illustrates the framework in action.

Validation serves as the initial step in mathematical problem-solving and is extensively discussed in literature. This process resembles experimentation and prototyping in design thinking (Brown et al., 1989) and can be likened to empirical verification in mathematics, akin to data collection in scientific inquiry. While empirical data provides essential insight, theoretical support is often needed for broader applicability and robustness.

Justification is the process of linking observations to foundational principles, a crucial step in constructing coherent arguments. This process is integral to disciplines such as philosophy, law, and mathematics (Toulmin, 2003). Polya's (1945) problem-solving heuristics underscore the importance of reflection and understanding the rationale behind solutions, reinforcing the role of justification in establishing sound reasoning.

Proving is a core activity in mathematics, involving the development of logical arguments based on axioms, definitions, and established theorems. Euclid's *Elements* remains a classic example of deductive reasoning, demonstrating the foundational approach to proofs. Contemporary research has delved into the philosophical and logical underpinnings of proof, enriching our understanding of its principles and methods (Balacheff, 1988; Hartshorne, 2000).

Formal proving represents a rigorous evolution in the practice of mathematical proof, gaining traction with the advent of computer-assisted proof systems (de Villiers, 1999). This process entails converting mathematical statements into formal language and utilizing automated tools to verify their validity with precision. Foundational work by De Bruijn (1980) and Boyer and

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Moore (1979) has shaped formal verification methodologies, establishing frameworks that continue to influence modern mathematical proof systems.

Together, these four processes—validation, justification, informal proving, and formal proving—offer a comprehensive framework for understanding mathematical reasoning and applying it to concrete problems.

Theoretical Framework

This study centers on a framework for examining students' mathematical reasoning (MR) approaches and processes, developed by Jeannotte and Kieran (2017). This framework provides a comprehensive structure for analyzing MR in trigonometry education, interweaving four interconnected strands: mathematical objects, activities, processes, and concepts.

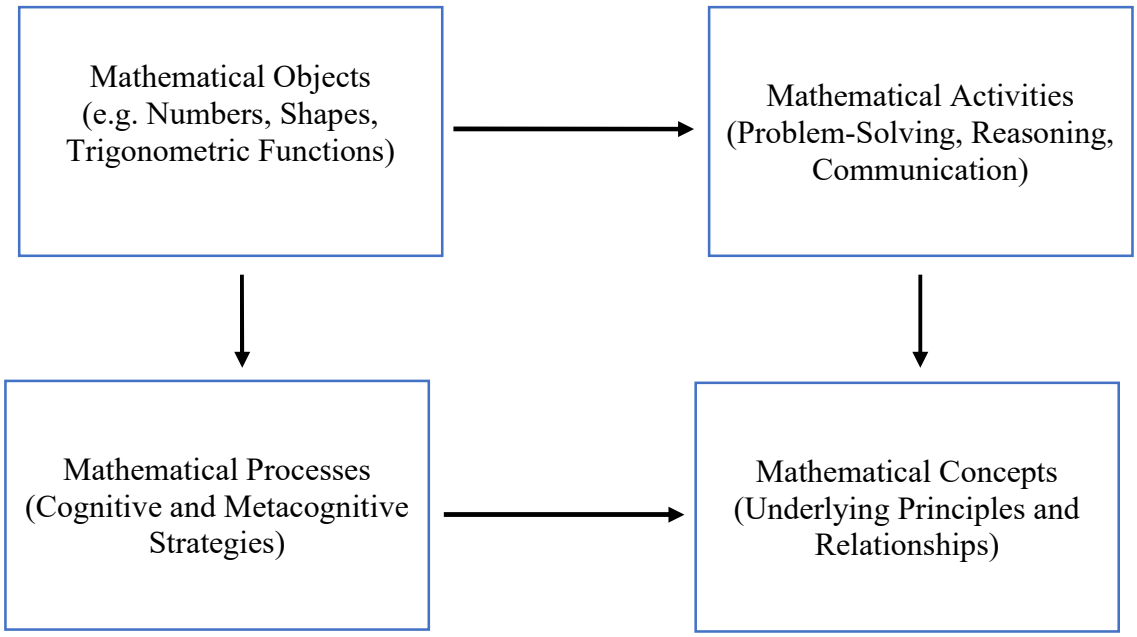


Figure 1: Mathematical Reasoning Framework based on Jeannotte and Kieran (2017).

Figure 1 shows the framework consists of two main sections which are the mathematical objects and activities, and the mathematical processes and concepts. These sections are interconnected, representing the interplay between the elements students work with and the cognitive processes they employ. Students engage in activities using mathematical objects. They utilize mathematical processes to understand and make sense of these objects and activities. These mathematical objects encompass the entities students interact with, such as numbers, shapes, and trigonometric functions in this context. Mathematical activities encompass the observable actions stu-

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dents undertake, including problem-solving, trigonometric reasoning, and solution articulation. In contrast, mathematical processes refer to the underlying cognitive and metacognitive strategies employed by students as they engage with mathematical tasks. Lastly, mathematical concepts explore the underlying principles and relationships students are expected to grasp within trigonometry, such as connections between angles and their trigonometric ratios. This diagram provides a visual representation of the framework, highlighting the interconnected nature of the components and their role in understanding students' mathematical reasoning in trigonometry education.

By investigating these interconnected strands, the study goes beyond simply identifying correct or incorrect answers. Instead, it examines the 'why' behind student reasoning. This comprehensive framework provides valuable insights into students' thought processes and potential areas of difficulty. This information can then be used to develop targeted instructional strategies that cater to diverse learning styles and promote a deeper understanding of trigonometry for all students.

Instructional Design

Based on the framework, a multifaceted approach to analyzing mathematical reasoning through four key components are employed:

1. *Mathematical Objects:* These are the entities students interact with, such as numbers, shapes, and, in this context, the angle in a right triangle.
2. *Mathematical Activities:* These refer to the actions students take when grappling with trigonometry concepts. The primary activity here is answering problems, focused on a real-world scenario—the "leaning ladder challenge."
3. *Mathematical Processes:* These examine the cognitive and metacognitive strategies students employ when engaging with mathematical tasks. In this activity, students apply prior knowledge and reasoning skills to explore and solve real-world problem using mathematical thinking.
4. *Mathematical Concepts:* These explore the underlying principles students are expected to learn, such as the relationship between the angle of inclination and the stability of the ladder.

Reasoning Approaches

In this study, various reasoning approaches are examined as students engage with a complex real-world problem. Each reasoning approach contributes uniquely to their understanding and supports different aspect of the problem-solving processes:

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Inductive reasoning involves students observing how changes in certain conditions affect outcomes through experimentation. By analyzing patterns in these observations, they form general conclusions and make predictions about similar scenarios.

Deductive reasoning comes into play when students apply established mathematical or physical principles to reach specific conclusions. By using known theorems or formulas, they reason from general laws to determine precise outcomes in the problem context.

Abductive reasoning involves students generating plausible explanations for observed phenomena. This form of reasoning encourages creative thinking and critical thinking as they consider multiple possible causes for the behavior they observe, promoting deeper investigation and understanding..

Analogical reasoning is used when students compare the leaning ladder to other familiar objects or situations. For example, by relating the scenario to common objects like ramps or seesaws, they transfer prior knowledge to new contexts, enhancing their conceptual understanding.

Algorithmic reasoning involves breaking down into a complex situation into a series of logical, step-by-step procedures. By systematically considering relevant factors, they develop a structured approach that leads them to a solution efficiently and consistently.

Reasoning Processes

The following reasoning processes are investigated as students worked on the solution to the problem.

Validation. Students physically test various scenarios to observe outcomes directly. This hands-on experimentation provides immediate feedback and helps develop an intuitive understanding of the relationships involved.

Justification. After initial observations, students support their conclusions by connecting their observations to relevant concepts (e.g. balance, friction, or geometric relationships. This process bridges experiential learning (e.g. physical observations) with formal mathematics or underlying mathematical principles.

Proving. Students construct mathematical models using variables that influence the situation. For example, they apply algebra, geometry and trigonometry to derive equations, allowing them to demonstrate why a particular outcome occurs under defined conditions.

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Formal proving. For a more rigorous approach, students use formal logic and theorem-proving tools. They define all components of the problem within a formal system and produce proofs that verify conclusions with complete precisions and generality.

The Leaning Ladder Challenge

Background

Proper ladder use is crucial in both occupational and domestic settings, as incorrect placement can lead to serious injuries or fatalities. The angle at which a ladder is positioned is essential for maintaining stability. A ladder set at too steep an angle risks slipping, while a too shallow angle may cause it to tip over. Safety guidelines recommend a ladder angle of approximately 75 degrees from the ground, balancing stability and reach (Simeonov et al., 2012). This angle, also known as the "4 to 1 rule," suggests that the ladder base should be one meter from the wall for every four meters of height (Schaffarczyk, 2017).

Following these guidelines is critical to prevent ladder-related accidents, with studies showing a strong link between incorrect ladder angles and increased fall risk (Smith et al., 2018; Jones & Martin, 2020). Integrating these safety principles into training helps students apply proper ladder placement in real-world situations.

Engaging Scenario

The learning experience centers on a real-world scenario—the "leaning ladder challenge." Students are presented with the following problem:
Imagine you have a 5-meter ladder that you need to lean against a wall to reach a high point safely. Setting the ladder up at the right angle is crucial for stability and safety.

Challenges to Consider

- If the ladder's base is too far from the wall, it might slide out, causing the ladder to collapse.
- If the ladder's base is too close to the wall, there's a danger of the ladder tipping over backward.

The Task

Students are tasked with determining the optimal angle for the ladder to ensure stability and safety.

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Guiding Questions to Spark Inquiry

- What happens when the ladder is almost vertical (close to 90 degrees)?
- How does the ladder behave when the angle is smaller than 75 degrees?
- Can you find an angle that balances the risk of sliding and tipping over?

Collaborative Learning Through Group Work

Students work in groups to foster collaboration and knowledge sharing. This allows them to discuss prior knowledge, explore strategies, and build upon each other's ideas.

Expected Learning Outcomes

Through this activity, students are expected to achieve the following outcomes:

- *Apply Trigonometry Concepts*. Students will use their understanding of right triangles and trigonometric ratios to analyze the leaning ladder problem.
- *Develop Problem-Solving Skills*. Students will engage in a logical and systematic approach to determine the optimal angle for the ladder.
- *Refine Critical Thinking Skills*. Students will analyze the consequences of different angles and justify their reasoning through discussions.
- *Recognize Practical Applications*. The leaning ladder challenge will help students see the real-world relevance of trigonometry in ensuring stability and safety.

METHODS

To gain a deep understanding of how students reason in trigonometry and how teachers support this reasoning, a qualitative case study approach was adopted. This method was selected for its capacity to explore complex phenomena in-depth, allowing for a rich, contextualized investigation of student and teacher practices (Yin, 2014). By focusing on a specific classroom or school, this study aimed to uncover the nuanced ways students approach trigonometry problems and how teachers adapt instruction to accommodate diverse reasoning strategies.

To ensure representativeness, cross-sectional observations were conducted in three different mathematics classes within a single school. This strategic approach allowed for capturing student reasoning processes from a diverse student population and across various instructional settings. Notably, the student sample (10 students per class) was carefully chosen to encompass a range of mathematical reasoning abilities.

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Participants and Participants Selection

The study involved 127 Grade 10 students, aged 16-17, who volunteered from three classes within the same school. Class 1 had 22 females and 20 males, Class 2 had 27 females and 18 males, and Class 3 had 24 females and 16 males.

To explore the students' problem-solving strategies in depth, five focus groups were formed. Each group was composed of three students, each selected from one of the three classes to ensure a representation of different mathematical reasoning approaches: inductive, deductive, analytical, algebraic, and spatial. The selection was based on their demonstrated problem-solving strategies in response to a specific problem.

This approach ensured a diverse range of perspectives in the discussions, reflecting various problem-solving methods. By including students from each class with distinct reasoning approaches, the focus group discussions provided a comprehensive view of how different mathematical strategies are applied in problem-solving.

Research Instruments

This study employed three primary research instruments to gather comprehensive data on students' reasoning approaches in trigonometry and the instructional strategies used by teachers: analysis of student work, classroom observations, and teacher interviews.

Student work analysis served as a critical instrument to evaluate and understand the mathematical reasoning processes employed by students when solving the problem. A detailed coding scheme was developed to capture the nuances of students' reasoning strategies, ensuring that the analysis accurately reflected their cognitive processes. Inter-rater reliability checks were conducted to enhance the reliability of the coding process.

Classroom observations provided a real-time perspective on teachers' instructional practices and the classroom dynamics. To mitigate observer bias, multiple observers were involved, and detailed field notes were recorded. Observers underwent training to ensure consistent data collection, and inter-observer reliability was assessed.

Teacher interviews were conducted to gain in-depth insights into teachers' instructional strategies and their perceptions of students' reasoning approaches. The interview protocol was designed to align with the study's research questions, and its clarity and structure were refined through pilot testing. Member checking was employed to ensure the accuracy of the interpretations drawn from the interviews.

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Data Collection

To gain a rich understanding of student reasoning processes, the researcher employed a multifaceted approach. This involved closely observing student group work while they tackled the assigned trigonometry challenge. Particular attention was focused on groups exhibiting different reasoning approaches. This allowed the researcher to identify and track the emergence of diverse strategies students employ when grappling with the problem.

Following the group work, the researcher conducted focused follow-up discussions with selected student groups. These discussions aimed to elicit in-depth insights into students' collaborative experiences, cognitive processes, and problem-solving approaches. Participants were prompted to articulate their reasoning and justify their problem-solving strategies. Table 1 outlines the specific data collection foci.

Components	Description
Conceptual understanding	How well students grasp the underlying trigonometric concepts relevant to the challenge.
Cognitive strategies	The specific thought processes students utilize to approach the problem, such as visualizing geometric relationships, applying trigonometric formulas, or employing algebraic manipulations.
Collaboration and communication	How effectively students work together within their groups, share ideas, and build upon each other's reasoning.
Metacognitive awareness	The students' level of reflection on their own learning process, their ability to identify areas of strength and weakness, and their approaches to overcoming challenges.

Table 1: Components and Description of Data Collection

These discussions provided a crucial qualitative layer to the study, complementing the observations of group work. By triangulating the data from both sources, the researcher could gain a more comprehensive picture of how students reason in trigonometry and how these reasoning patterns relate to different instructional approaches employed by the teachers.

Data Analysis

To explore the first research question, an in-depth analysis of students' work on trigonometry problems was conducted. The student responses based on the reasoning approaches and processes they employed were categorized based on the conceptualization of these constructs. This categorization allowed for identifying patterns. Then, the students' explanations were examined to uncover the underlying reasoning processes. Through this analysis, both common and diverse reasoning strategies that students used when solving the problem were identified.

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To answer the second research question, a thorough analysis of teacher practices using data from classroom observations and teacher interviews was conducted. The observations focused on instructional strategies and activities that encouraged various reasoning approaches. By analyzing this data, a range of instructional methods that teachers used to accommodate and enhance students' diverse reasoning strategies were identified. Additionally, this analysis shed light on how teachers created opportunities for students to develop and refine their reasoning skills in the classroom.

Ethical Considerations

This research was committed to upholding the highest ethical standards to ensure the well-being and rights of all participants. Adhering to core ethical principles, the study prioritized informed consent, confidentiality, and minimizing risks to participants, with a particular focus on protecting students from undue stress or harm.

Both teachers and students participated voluntarily after providing informed consent. The informed consent process clearly outlined the purpose of the study, how data would be used, and participants' right to withdraw at any time without consequence. This ensured that participants fully understood their involvement and were able to make an autonomous decision about their participation, thus respecting their rights and choices.

To safeguard participants' privacy, all data collected underwent a thorough anonymization process. Any identifiable information from student work, interview transcripts, and observation notes was removed to protect participants' identities. Maintaining confidentiality not only helped preserve privacy but also fostered trust between the researcher and participants, encouraging open and honest communication throughout the study.

The researcher implemented specific safeguards to prioritize students' mental and emotional well-being. For instance, all interactions took place in safe, familiar environments, such as classrooms, to foster a sense of security. The researcher also offered breaks during data collection to prevent fatigue and maintained a supportive, non-judgmental tone during interviews and observations. For students who appeared nervous, the researcher provided additional reassurance that there were no "right" or "wrong" answers. By respecting students' natural pacing and preferences, the study aimed to make participation feel as comfortable and stress-free as possible.

Finally, the researcher took proactive measures to monitor and mitigate any signs of stress or anxiety in students during the study. For instance, before starting data collection, the researcher conducted a rapport-building session to familiarize students with the researcher and clarify that the study was not evaluative but exploratory. The study avoided sensitive or potentially distressing topics, instead focusing on general academic and classroom experiences. Additionally, the

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researcher remained vigilant for any signs of discomfort; in such cases, the researcher immediately offered the option to pause, skip, or withdraw from the activity.

RESULTS

This study examines the various reasoning methods students employ when solving trigonometric problems, focusing on the Leaning Ladder Challenge as a case study.

Mathematical Reasoning Approaches Employed by Students

The research provides insights into the effectiveness and limitations of each approach in mathematical problem-solving. Table 2 presents an overview of reasoning approaches, including deductive, inductive, abductive, analogical, and algorithmic approaches, providing insights into their respective strengths and limitations in the context of trigonometric problem solving.

Reasoning Approach	Student Answer/Behavior	Explanation of Reasoning	Strengths	Limitations
Deductive Reasoning	Relying on rules and formulas (e.g., 4:1 ratio, 76-degree angle)	Applying general principles to specific situations	Provides clear guidelines, efficient	Relies on accurate initial information, may not account for all variables
Inductive Reasoning	Experimenting with different angles, observing patterns	Learning from experience and observation	Fosters hands-on learning, adaptable	Time-consuming, conclusions may be tentative
Abductive Reasoning	Making educated guesses based on observations	Generating hypotheses based on limited information	Quick decision-making, stimulates creativity	Relies on intuition, lacks concrete evidence
Analogical Reasoning	Comparing the ladder to other objects (e.g., ramp, seesaw)	Transferring knowledge from familiar situations	Facilitates understanding, promotes creativity	Depends on relevant analogies, may oversimplify
Algorithmic Reasoning	Using formulas and calculations (e.g., trigonometry)	Applying step-by-step procedures to solve problems	Precise and accurate, reliable	Requires mathematical knowledge, time-consuming

Table 2: Mathematical Reasoning Approaches in Students' Solutions to the Leaning Ladder Challenge

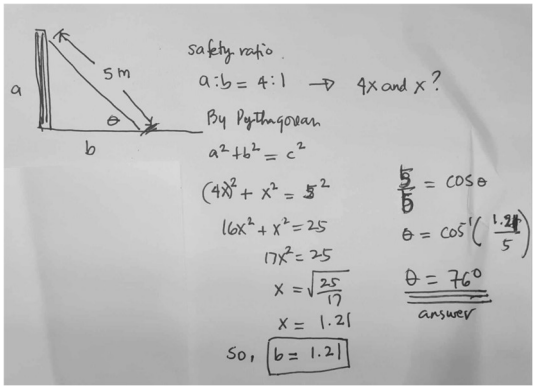
Deductive Reasoning

The results indicate that students were actively engaged in the deductive reasoning, demonstrating their ability to apply mathematical concepts to real-world context, collaborate effectively, and reflect on the significance of their findings. The teacher played a crucial role as a facilitator, guiding student learning and encouraging critical thinking. Below is an excerpt from a conversion among students (*S1, S2, S3, S4*) and teacher (denoted with *T*)

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S1: If we need to find the safest angle for the ladder, so let's start with what we know. We have a 5-meter ladder and we want to reach a certain height.
S2: ... we know that if the ladder is too steep, it might slip. So, we need to avoid a very large angle.
S3: Also, if the angle is too small, we won't be able to reach very high.
S4: I remember our teacher saying that there's a safe ratio for ladder placement. It's something like four units out for every one unit up.
S1: That sounds right. If we use that ratio, we can calculate the distance from the wall.
S2: Yes, and then we can use trigonometry to find the angle.
S3: But we should also consider the ground. If it's slippery, we might need to adjust the angle.
S4: We can use what we know about friction to estimate how much the angle should be reduced.
S1: Okay, let's start with the 4:1 ratio and see what angle we get. Then we can adjust later.
T: That's a good start, but remember, we already know the ladder length. What else can we use?
S2: We can use the cosine function to find the angle, since we know the adjacent side (distance from the wall) and the hypotenuse (ladder length).
T: Excellent! Can you explain how you would set up the equation?

Student engagement was evident in their active participation in the problem-solving process. Students demonstrated a strong foundation in mathematical concepts by suggesting the use of the Pythagorean theorem and trigonometric functions. When faced with challenges, they actively sought clarification and guidance from the teacher. For instance, Student A's suggestion to use the Pythagorean theorem prompted the teacher to redirect their focus towards utilizing the given information more effectively.



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Figure 3: Sample students' calculation to find the angle

T: What does this angle 76 degrees represent in relation to the ladder and the wall?
S3: It's the angle between the ladder and the ground.
T: Correct. Now, does this angle meet the safety requirement?

The teacher's role as a facilitator was crucial in guiding students towards a deductive solution. By asking probing questions, the teacher encouraged students to think critically about the problem and justify their reasoning. For example, asking Student C to explain the meaning of the calculated angle promoted deeper understanding of the mathematical concept in relation to the real-world scenario.

Students often relied on previously learned information, such as the "4 to 1 rule" for ladder safety, demonstrating the application of deductive reasoning. However, the teacher emphasized the importance of accurate foundational knowledge. As students applied general principles to specific challenges, their understanding of the problem deepened. They recognized the influence of variables like ground conditions and load on the optimal ladder angle, indicating a nuanced grasp of deductive reasoning.

The integration of real-world constraints enhanced problem-solving, requiring students to apply mathematical concepts to practical scenarios. The use of calculators facilitated computations, allowing for a greater focus on conceptual understanding and problem-solving strategies.

Inductive Reasoning

Students began by experimenting with different ladder positions, using a miniature model of ladder, systematically collecting data on angle, distance from the wall, ladder height, stability, and reach. For each trial, they calculated the distance-to-height ratio to assess compliance with the safety standard. This process of observation and data collection laid the groundwork for inductive reasoning. An excerpt from a conversation among students (S1, S2, S3, S4(illustrates this reasoning in action:

S2: I wonder what's the best angle for this ladder. If it's too steep, it might slip.
S2: Yes, and if it's too flat, we won't reach the window.
S3: I tried leaning this miniature ladder against the wall at different angles. It seemed safer when it was a bit more flat. So we try using different angles.
S4: I saw a video online where a guy talked about how ladders should be placed. He said something about a safe distance from the wall.

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S1: Let's try leaning the ladder at different angles and see what happens. Maybe we can find a pattern.

As illustrated in the following excerpt from a conversation among students (S1, S2, S3, S4) and the teacher (T), students analyzed the collected data and identified patterns among key variables. They concentrated on trials that satisfied safety criteria, observing how angle, stability, and reach were interrelated. Through these observations, they inductively concluded that an angle between 55 and 65 degrees generally offered a safe and effective solution. They understood that angles below 55 degrees might compromise reach, while angles above 65 degrees could jeopardize safety. This generalization process demonstrated the ability to infer general principles from specific observations.

S1: Our 45-degree angle is too close to the wall. It doesn't meet the 1:4 ratio.
S2: We got a 60-degree angle, but it feels a bit too steep. Let's check the ratio.
S3: Our angle is safe, but we're not reaching the window. We need to find a balance.
T: How can you adjust your ladder position to meet both the safety standard and your goal of reaching the window?
S3: We need to find an angle that's steep enough to be safe but not too steep to reach the window. So what about the safety standard that was mentioned at the start of the lesson?

Student discourse revealed a nuanced understanding of the problem's complexities. Students balanced safety concerns with practical objectives, as exemplified by the challenge of achieving desired height while maintaining stability. The teacher's role in guiding students towards a holistic perspective was crucial.

To foster inductive reasoning, controlled experimentation and rigorous data analysis were emphasized. Students systematically manipulated ladder angles and recorded observations. As one student noted, "Iterative adjustments revealed that an angle between 65 to 80 yielded optimal stability." While inductive reasoning provided valuable insights, challenges such as time constraints and uncontrolled variables emerged. Teachers stressed the importance of structured experimentation and data analysis for reliable conclusions.

Abductive Reasoning

The study highlighted the dynamic and iterative nature of abductive reasoning in mathematical problem-solving, particularly in the context of balancing safety and practicality under a 4:1 safety ratio constraint. Students generated initial hypotheses, tested them against real-world constraints, and refined their thinking based on emerging evidence. While all students ultimately arrived at viable solutions, the depth and sophistication of their reasoning varied as illustrated in the following conversation among students (*S1, S2, S3*) and the teacher (*T*). This research under-

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scores the potential of abductive inquiry to enhance critical thinking and problem-solving skills in students.

Initial Hypotheses with Safety Consideration

> *S1: We still think a 45-degree angle is a good starting point, but we need to check if it meets the 4:1 safety ratio.*
>
> *S2: Our 60-degree angle might be too steep to meet the safety regulations. We need to adjust it.*
>
> *S3: We need to find an angle that is both safe and allows us to reach the desired height.*

The initial hypotheses reflected diverse approaches to the problem. Student A hypothesized a 45-degree angle, likely based on prior knowledge or visual estimation. Student B's suggestion of a 60-degree angle may have stemmed from considerations of stability or reach. In contrast, Student C took a more exploratory approach, aiming to find a solution that balanced both safety and practicality.

Teacher probing with safety emphasis

> *T: How can you adjust your ladder position to meet the safety regulations without compromising the ladder's reach?*
>
> *S1: We can try a steeper angle, but we need to make sure it's not too steep.*
>
> *T: Can you explain why your initial angle didn't meet the safety standard?*
>
> *S2: We focused on stability but forgot to consider the distance from the wall.*

As students engaged in experimentation and measurement, they refined their initial hypotheses. A student, after calculating the distance from the wall with a 45-degree angle, realized it did not meet the safety criteria. This led to a revised hypothesis of a steeper angle. Similarly, a measurement from another student showed that the 60-degree angle compromised safety. Through these iterative processes, students demonstrated abductive reasoning by proposing explanations for unexpected results and modifying their conjectures accordingly.

Eventually, the groups converged on a solution that met both the safety requirements and the practical need to reach the desired height.

> *S1: We found an angle of 63 degrees that meets the safety ratio and allows us to reach the window.*
>
> *S2: After several adjustments, we also arrived at a similar angle that complies with the safety regulations.*
>
> *S3: Our initial angle was too shallow, but we were able to find a suitable angle that meets both criteria.*

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Students' approach was notably more systematic. By considering the relationship between angle, ladder length, and wall height, they were able to generate a hypothesis that closely aligned with the safety requirements. This suggests a stronger application of mathematical reasoning and problem-solving skills.

The students' approaches, while often relying on visual cues or intuitive judgments, highlighted the strengths and limitations of abductive reasoning. For instance, a student stated, "As I was adjusting the ladder, I noticed that it seemed to stand firm when the angle was somewhere around 70 degrees. I didn't have the exact data, but based on how it looked and felt, I guessed that this was probably the safest angle to use."
Abductive reasoning served as a foundation for students to explore potential solutions to the ladder challenge. However, to enhance the reliability and validity of their conclusions, students must complement their initial inferences with empirical evidence or logical reasoning. This study demonstrates how abductive inquiry, when combined with critical thinking and data analysis, can foster a deeper understanding of mathematical concepts and problem-solving strategies.

Analogical Reasoning

As illustrated in the following except from conversation among students (S1, S2, S3, S4), they identified analogous systems that shared similarities with the ladder scenario. By drawing on familiar experiences, they aimed to transfer existing knowledge to deepen their understanding and guide their problem-solving approach.

S1: This ladder problem reminds me of building a ramp.
S2: How's that?
S1: Well, a ramp needs to have more or less a correct angle to be safe and effective.
S3: And it needs to be strong enough to hold weight.
S4: So, we need to find the perfect slope for the ladder, just like a ramp.

Students demonstrated a propensity for analogical reasoning when confronted with the ladder problem. Common analogies included ramps, bridges, and trees, highlighting the transfer of knowledge from familiar systems to the novel context.

One student response exemplified this approach: "I remembered helping my dad set up a ladder when he was painting our house. He placed the ladder at an angle that looked like around 75 degrees, and it was really stable. So, I tried to mimic that same angle here, thinking it would be safe and balanced". While analogical reasoning facilitated rapid decision-making, its effectiveness depended on the appropriateness of the analogy. Students needed to critically evaluate the simi-

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larities and differences between past experiences and the current situation to avoid misconceptions.

To enhance the reliability of analogical reasoning, students were encouraged to explicitly articulate the similarities and differences between the analogical system and the ladder problem. This process facilitated more informed decision-making and reduced the likelihood of erroneous inferences.

The teacher's role in guiding students through the analogical reasoning process was crucial. By prompting students to identify key similarities and differences, the teacher helped them to transfer knowledge effectively. Analogical reasoning proved to be a valuable tool for problem-solving in the ladder challenge. By leveraging prior knowledge from familiar systems, students generated hypotheses and developed a deeper understanding of the problem. However, the effectiveness of this strategy was contingent on students' ability to critically evaluate the appropriateness of the analogy and to explicitly articulate the underlying relationships.

Algorithmic Reasoning

This study examined the application of algorithmic reasoning to determine the optimal angle for a leaning ladder, subject to safety regulations. Specifically, they developed structured, step-by-step procedures to determine a suitable ladder angle. Students like S4 and S5 identified relevant variables, applied the 4:1 safety ratio. And use trigonometric functions to calculate angles. The integration of technology further supported their work, enhancing both computational efficiency and accuracy – highlighting the valuable role of digital tools in mathematical problem-solving.

> *T: Can you outline the steps involved in solving this problem?*
> *S4: First, we need to find the distance from the wall using the safety ratio. Then, we can use the cosine function to calculate the angle.*
> *T: How can we check if this angle is practical for real-world use?*
> *S5: We can compare the height reached with the desired height.*

Teacher-student interactions played a vital role in guiding the process. By asking probing questions, the teacher encouraged students to articulate their thought process and consider the practical implications of the solution. This facilitated a deeper understanding of the problem and the algorithmic approach.

Step-by-Step Procedure

1. *Define variables:*
    - *L = ladder length (5 meters)*
    - *D = distance from the wall to the ladder base*
    - *H = height reached on the wall*

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- *A = angle between the ladder and the ground*

2. *Establish the safety constraint:*
   - *D = L / 4*
3. *Apply trigonometric function:*
   - *cos(A) = D / L*
4. *Calculate the angle:*
   - *A = arccos(D / L)*
5. *Check for feasibility:*
   - *Ensure the calculated angle provides sufficient height to reach the desired point.*

The results indicate that algorithmic reasoning is an effective strategy for well-defined problems with clear constraints. By decomposing the problem into smaller, manageable steps, students developed structured and efficient solutions. Technology integration enhanced computational efficiency, expanding problem-solving capabilities.

However, the effectiveness of algorithmic reasoning is contingent upon problem complexity and students' mathematical proficiency. While powerful for well-defined problems, it may be less suitable for complex or open-ended scenarios. A strong mathematical foundation is essential for successful implementation. Thus, balancing algorithmic approaches with other problem-solving strategies for holistic development is important.

Mathematical Reasoning Processes by Students

Based on classroom observations and in-depth focus group discussions with students, this study unveiled a multifaceted spectrum of problem-solving strategies. Students exhibited a wide range of approaches, from concrete and intuitive manipulations of physical representations to more abstract and algorithmic reasoning. These findings underscore the intricate nature of mathematical cognition and highlight the necessity of instructional practices that support diverse problem-solving pathways. Table 3 shows various mathematical reasoning processes students used in approaching the problem, from validation of ladder stability to formal integration of findings into mathematical theories.

Mathematical Reasoning Processes	Description	Student Example
Validation in Determining Ladder Stability	Initial step in problem-solving; hands-on method to engage with the problem and gain immediate feedback.	Student set up the miniature ladder at different angles and observed its stability.
Justification: Adding Depth to Reasoning	Explaining the reasoning behind a chosen approach or solution; linking initial observations to broader principles or guidelines.	Student advocated for a 75-degree angle based on safety guidelines.

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Mathematical Reasoning Processes	Description	Student Example
Proving: Concrete, Theoretically Grounded Conclusions	Offering concrete, systematic conclusions based on mathematical principles; quantitative validation or refutation of initial observations.	Student employed trigonometric calculations to determine the optimal ladder angle.
Formal Proving: Integrating Findings into Existing Theories	Integrating findings into existing or new mathematical theories; elevating problem-solving to theoretically grounded inquiry.	Student formalized a mathematical theory around ladder stability.

Table 3: Mathematical Reasoning Processes in Students' Approaches to the Leaning Ladder Challenge

Validation served as the foundation for exploring ladder stability. Initially, students engaged in hands-on experimentation by adjusting ladder angles on miniature models to collect empirical data. Through this experiential learning, they gained a deeper understanding of the problem, which subsequently informed and strengthened their reasoning.

Moreover, justification enriched students' problem-solving approaches by clarifying the rationale behind their choices. Although safety guidelines provided an initial framework for decision-making, students recognized that real-world contexts often require a degree of flexibility, moving beyond strict adherence to standard protocols.

Additionally, the process of constructing mathematical proofs added a layer of mathematical rigor, essential for validating initial observations. Through trigonometric calculations, students generated concrete evidence to support their findings, thereby enhancing the credibility of their conclusions. Embedding these calculations within established mathematical frameworks demonstrated a high level of reasoning, effectively bridging empirical insight with theoretical understanding.

However, while formal proofs provide a solid foundation, a balance is necessary between theoretical rigor and practical adaptability. Although formal proofs offer reliability, real-world complexities often call for flexible, adaptable problem-solving strategies to effectively address unexpected challenges.

Instructional Strategies Employed by Teachers

This study investigated how teachers can effectively accommodate and leverage diverse student mathematical reasoning in the context of a ladder problem. Findings indicate that students employed a range of reasoning strategies, including deductive, inductive, abductive, analogical, and algorithmic approaches. Teachers responded by tailoring instruction to support these varied styles (See Table 4).

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Theme	Description of the Theme	Sample Utterances from Interviews and Probing
Hands-on and Experiential Learning	Integrating physical manipulatives and real-world contexts to enhance student engagement and understanding.	"Hands-on activities provide students with a tangible way to explore mathematical concepts." (Teacher A) "By engaging in these activities, students had the opportunity to manipulate objects, observe real-world phenomena firsthand, and develop a deeper intuitive understanding of mathematical concepts." (Teacher B)
Scaffolding and Guided Inquiry	Providing structured support and guidance to facilitate student learning and development of diverse reasoning abilities.	"By asking probing questions, teachers encouraged students to articulate their reasoning and reflect on their findings." (Teacher B)
Contextualized Learning	Grounding mathematical concepts in real-world experiences to enhance engagement and understanding.	"When students see how math is relevant to their world, they become more engaged and motivated to learn." (Teacher C "By prompting students to consider how mathematical formulas could represent relationships in a real-world scenario, teachers helped students perceive the practical utility of mathematics." (Teacher B)
Metacognitive Reflection and Discourse	Fostering student reflection and communication about their thinking processes.	"By talking through their thought processes, students can solidify their understanding, identify misconceptions, and learn from their peers." (Teacher A)
Integration of Formal Mathematical Reasoning	Balancing formal and informal reasoning to develop well-rounded mathematical thinkers.	"It's important for students to see how the math they learn in class connects to real-world problems." (Teacher E) "By fostering a balance between formal and informal reasoning, teachers can empower students to approach mathematical problems with flexibility and precision." (Teacher D)

Table 4: Strategies Employed by Teachers

Hands-on and Experiential Learning

The findings demonstrated that these instructional strategies significantly enhanced student engagement, understanding, and problem-solving skills.
Teachers effectively integrated hands-on activities, allowing students to physically manipulate objects and directly observe real-world phenomena. For example, students used miniature ladders to experiment with different angles, exploring ladder stability in a tangible way. As one teacher explained, "Hands-on activities provide students with a tangible way to explore mathematical concepts. It helps them make connections between abstract ideas and real-world applications" (Teacher A). This approach helped students develop a deeper, more intuitive understanding of mathematical concepts compared to traditional instruction.

Teacher questioning further guided student learning, encouraging them to articulate their reasoning and reflect on their findings. One teacher asked, "I noticed you tried different angles. Can

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you explain how you decided which angle was best?" (Teacher B). Such probing questions promoted critical thinking and deeper conceptual understanding.
The study supports the integration of hands-on and experiential learning in mathematics instruction, showing that these methods create a more engaging, effective, and inclusive learning environment.

Scaffolding and Guided Inquiry

Teachers used strategic prompts and feedback to facilitate student learning. For instance, when students applied deductive reasoning using the 4:1 rule, a teacher might ask, “How did you apply the 4:1 rule to determine the angle?” This question encouraged students to articulate their thought processes and deepen their understanding.

Similarly, when students engaged in inductive reasoning, teachers supported them by asking, “How can you organize your data to identify patterns or trends?” For those using abductive reasoning, teachers fostered critical thinking by prompting, “What evidence supports your hypothesis?” This approach helped students evaluate alternative explanations and justify their claims. The study emphasized that scaffolding and guided inquiry are indispensable in cultivating diverse mathematical reasoning skills, enabling students to explore, question, and construct their own knowledge.

Contextualized Learning in Mathematics

Teachers effectively used real-world examples and cultural references to bridge the gap between abstract concepts and students' lives. For instance, when a teacher asked, "How does understanding the tightrope walker help us solve the ladder problem?" (Teacher A), students were encouraged to connect familiar concepts to the mathematical challenge. This approach not only deepened students' understanding but also enhanced their appreciation of mathematics.

Furthermore, contextualized learning provided opportunities for students to apply mathematical knowledge to real-world problems. Teachers prompted students to consider how mathematical formulas represent relationships in practical scenarios, such as asking, "How can we use mathematical formulas to represent the relationships between the different parts of the problem?" (Teacher B). This helped students see the practical utility of mathematics. Contextualized learning proved to be a powerful tool for enhancing student engagement, motivation, and understanding in mathematics.

Metacognitive Reflection and Discourse

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The study found that fostering metacognitive reflection and discourse is a critical component of effective mathematics instruction. By encouraging students to think about their thinking and communicate their ideas, teachers created an environment that supported the development of higher-order thinking skills.

Teachers prompted students to articulate their reasoning processes and consider alternative perspectives, deepening their understanding and promoting collaborative learning. For instance, a teacher might ask, "Tell me about the process you went through to reach that conclusion. What patterns did you observe?" (Teacher A). Such questions encouraged students to reflect on their problem-solving processes, identify misconceptions, and learn from their peers. These findings highlight the importance of metacognitive reflection and discourse in enhancing students' mathematical problem-solving abilities and overall learning experiences.

Integration of Formal Mathematical Reasoning

A significant finding of this study was the importance of integrating formal mathematical reasoning into instruction. Students demonstrated proficiency in applying formal reasoning techniques, such as trigonometric calculations, when provided with appropriate guidance.

Teachers played a crucial role in facilitating this integration by creating contexts that bridged abstract concepts and real-world applications. For example, when students used the 4:1 ratio for deductive reasoning, teachers prompted them to connect this to formal trigonometric concepts by asking, "Remember that formula we learned about right triangles? How can we apply it here to find the missing side?" (Teacher A). This helped students see the relevance of formal reasoning in practical problem-solving.

DISCUSSION

The *Leaning Ladder Challenge* provided a valuable context for examining the variety of reasoning strategies students use to approach trigonometric problems. This study revealed that students engaged in diverse reasoning methods—deductive, inductive, abductive, analogical, and algorithmic—each offering unique strengths and limitations.

To begin with, deductive reasoning emerged as a common approach, where students applied general principles to specific cases (Jeannotte & Kieran, 2017). This structured method of problem-solving proved beneficial, although it was highly dependent on the accuracy of initial information. As Selden and Selden (1995) caution, incorrect foundational assumptions can lead to flawed conclusions, highlighting a key limitation of this approach.

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In addition, inductive reasoning, marked by experimentation and pattern recognition (Polya, 1957; Pellegrino & Glaser, 1984), allowed students to gain insights through systematic observation and data analysis. While this method promoted a deeper understanding, relying solely on inductive reasoning risked overgeneralization without the complementary support of deductive methods (Klauer & Phye, 2008; Lesh et al., 2000).

Furthermore, abductive reasoning, or the generation of hypotheses from limited information (Hidayah et al., 2023; Shodikin et al., 2021; Paavola, 2023; Pedemonte & Reid, 2011), fostered creativity and flexibility. Students using this approach often demonstrated innovative thinking by proposing rapid solutions. However, as noted by Selden and Selden (1995), abductive reasoning's intuitive nature can make it less rigorous, potentially leading to inaccuracies if not substantiated by sufficient evidence.

Moreover, analogical reasoning provided a practical problem-solving tool by drawing parallels between familiar and new situations (Gentner, 1989; Goswami, 1991). Leveraging prior knowledge, students generated creative solutions; however, the success of this approach hinged on the relevance of the analogy and the students' ability to discern meaningful similarities and differences (Vinner, 2002).

Finally, algorithmic reasoning, which involves step-by-step procedures and mathematical formulas (Zazkis & Campbell, 1996), offered precision and structure in well-defined problems. However, its effectiveness was dependent on students' mathematical proficiency, as those less fluent in mathematics faced challenges applying this reasoning method effectively.

Taken together, these findings points out the importance of integrating and validating multiple reasoning strategies in problem-solving. Effective problem-solvers demonstrated adaptability, moving flexibly between approaches to meet the specific demands of the situation—a critical skill for navigating the complexities of the Leaning Ladder Challenge.

Crucial to this problem-solving process were the stages of validation, justification, and proving, each offering distinct contributions. Validation often began with hands-on experimentation, providing immediate feedback and grounding students' understanding of the problem. For example, Student B's manipulation of a ladder to observe its stability underscored the value of experiential learning, reinforcing abstract concepts through real-world experimentation.

As students moved from observation to analysis, justification became essential for refining their reasoning. By connecting their approaches to broader principles or established guidelines, students strengthened their arguments. For instance, some students referenced to safety guidelines demonstrated how aligning observations with recognized standards added robustness to their reasoning.

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To further enhance the rigor of their solutions, students engaged in formal mathematical reasoning to derive well-supported conclusions. For example, one particular student used trigonometric calculations to determine a precise solution, demonstrating how quantitative validation can strengthen both the credibility and mathematical soundness of the problem-solving process.

In addition to these findings on student strategies, the study highlighted how teachers adapted their instructional methods to support students' diverse reasoning approaches during the Leaning Ladder Challenge. Effective strategies included hands-on and experiential learning opportunities, such as using miniature ladders to help students connect abstract mathematical concepts to real-world applications. This tangible connection fostered a deeper comprehension and retention of mathematical ideas.

Furthermore, scaffolding and guided inquiry emerged as pivotal instructional approaches. By offering structured support through prompts and feedback, teachers guided students through complex problems, helping them articulate their thought processes and deepen their understanding. This scaffolding effectively bridged the gap between students' current knowledge and desired learning outcomes (Van de Pol et al., 2020).

Equally important was the role of contextualized learning in enhancing student engagement. Teachers incorporated real-world examples and cultural references to make mathematical concepts more relatable, thereby motivating students by demonstrating the practical utility of mathematics. This approach contributed to a more inclusive and meaningful learning environment (Rubel & McCloskey, 2021; Yang et al., 2021).

Moreover, teachers promoted metacognitive reflection and discourse, encouraging students to articulate their reasoning and explore alternative perspectives. This practice fostered higher-order thinking skills and collaborative learning, ultimately enhancing students' problem-solving abilities (Schoenfeld, 2022). By balancing formal and informal reasoning approaches, teachers guided students to connect abstract concepts like trigonometry with practical applications, helping them develop into well-rounded mathematical thinkers.

CONCLUSION

The *Leaning Ladder Challenge* offered a valuable lens into the diverse reasoning strategies employed by students in tackling a trigonometric problem. Students exhibited a rich repertoire of approaches, including deductive, inductive, abductive, analogical, and algorithmic reasoning. These findings align with previous research highlighting the multifaceted nature of mathematical problem-solving (e.g., Schoenfeld, 2022).

Teachers played a pivotal role in fostering these diverse reasoning approaches. By incorporating hands-on experiences, scaffolding student thinking, and creating opportunities for metacognitive

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reflection and discourse, educators created a supportive learning environment. This aligns with research emphasizing the importance of contextualized learning (Brown et al. 1989), guided inquiry (Vygotsky, 1978), and metacognition (Flavell, 1979) in mathematics education.

The integration of formal mathematical reasoning into instruction was also crucial. While less frequently observed, its application demonstrated the potential for precise and accurate problem-solving. However, a strong foundation in mathematics is essential for effective use (Zazkis & Campbell, 1996).

Teachers play a pivotal role in fostering this flexibility by creating learning environments that prioritize experimentation, justification, and critical thinking. By accommodating and leveraging diverse reasoning approaches, teachers can promote deeper learning and enhance mathematical proficiency (Cardino & Cruz, 2020). The Leaning Ladder Challenge served as a catalyst for exploring the multifaceted nature of mathematical reasoning, equipping students with tools to navigate complex problems effectively. By understanding the nuances of student reasoning, teachers can tailor instruction to meet the needs of all learners and foster a deeper appreciation for the beauty and utility of mathematics.

Future Research Directions

Future research should investigate the intricate relationship between these reasoning modes and students' mathematical identities. Future research should delve deeper into the factors influencing the selection and application of these reasoning strategies**.** Investigating how students develop these skills over time, the impact of cultural and socioeconomic factors, and the development of assessment tools to measure reasoning proficiency are essential next steps. By understanding the nuances of student reasoning, educators can tailor instruction to optimize student learning and prepare them to become confident, adaptable problem-solvers. Moreover, longitudinal studies are necessary to ascertain the enduring impact of these instructional approaches on students' problem-solving abilities and overall mathematical disposition.

To fully realize the potential of these findings, sustained professional development is essential. Teachers require ongoing support in identifying, nurturing, and leveraging diverse reasoning strategies within their classrooms. By cultivating a culture of mathematical inquiry and exploration, educators can empower students to become proficient, critical thinkers and problem-solvers.

Limitations

The "leaning ladder challenge" offers a practical and engaging context for applying trigonometry, yet it presents several limitations. This instructional activity simplifies the complexities of real-world ladder use by excluding factors such as wind, surface traction, and the ladder’s mate-

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rial properties, which significantly impact ladder stability. Furthermore, it assumes static conditions, overlooking dynamic forces that could arise during actual ladder use. The scenario also does not consider human factors like the user's weight, experience, or the unique conditions of specific tasks, which are critical in real-life applications. While the challenge promotes ladder safety awareness, it does not cover other essential safety considerations inherent to working at heights.

Drawing on the instructional design principles of Jeannotte and Kieran (2017), this challenge seeks to foster a deeper understanding of trigonometric concepts through real-world problem-solving and collaborative learning. By grounding trigonometry in a familiar scenario, the activity aims to strengthen students' mathematical reasoning and their ability to apply theoretical concepts to practical situations.

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MATHEMATICS TEACHING RESEARCH JOURNAL
SUMMER 2025
Vol 17 no 3

The Influence of Working Memory Capacity and Mathematical Anxiety on the Creative Reasoning of Prospective Mathematics Teachers

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Abstract: This study examines the impact of working memory capacity and mathematics anxiety on the creative reasoning of prospective mathematics teachers, highlighting how these cognitive factors shape problem-solving processes. This research used a mixed-method sequence explanatory method with a sample size of 60 people for quantitative research, and four people were selected with high working memory capacity and different levels of anxiety. The quantitative phase employed multiple linear regression to assess the impact of working memory and anxiety on creative reasoning. The qualitative phase involved thematic analysis of problem-solving approaches among selected participants. To validate the consistency of the findings, the researcher uses the time triangulation method. It was found that working memory capacity and mathematics anxiety influenced the creative reasoning of prospective mathematics teachers in solving problems. Prospective mathematics teachers with high working memory capacity demonstrate flexibility and fluency in generating new ideas, can connect known mathematical concepts and provide logical arguments to support the truth of the ideas created. However, high and low levels of mathematics anxiety interfere with cognitive performance, causing a loss of focus in solving more complex problems and not providing arguments for the ideas created.

Keywords: Working memory capacity, creative reasoning, mathematics anxiety, problem solving

INTRODUCTION

A person will think and plan solutions based on mathematical concepts when facing a problem. Planning the solution leads to a process of creativity in reasoning. Planning the solution leads to a process of creativity in reasoning. Reasoning is part of mathematical thinking (T. Bergqvist & Lithner, 2005). Reasoning can be categorized as creative reasoning if the person creates their thoughts or recreates the strategies, not just remembering the procedures (Schwarz et al., 2010). Creative reasoning and logical argumentation are important parts of solving mathematical problems, leading students to build their solutions and thus benefit their learning (Olsson & Teledahl,

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2017). According to several expert opinions, creative reasoning is defined in this research as a mental activity that produces original thoughts accompanied by logical arguments and supported by the accuracy of the mathematical concepts used.

Creative reasoning plays a vital role in solving non-routine problems because a complete solution scheme is unavailable, and the problem solver must construct at least part of the reasoning (Lithner, 2008). Problems, in this case, are defined as complex tasks for the individual who solves them (Schoenfeld, 1992). This difficulty is a deadlock in thinking about creating a solution. If someone has a way to complete a particular math task, then the task becomes an exercise and no longer a problem. So far, students have tended to use memorization strategies when faced with tasks that require a lower cognitive level (T. Bergqvist & Lithner, 2005). So, they are not used to new problems that require higher thinking because they lack sufficient knowledge to solve them. In contrast, tasks emphasizing creative reasoning problems can lead to higher levels of mathematical competence (Jonsson et al., 2014). Thus, to foster creative reasoning in a person, it is necessary to get used to completing non-routine problem-oriented tasks.

The importance of understanding mathematical concepts means that creative reasoning is considered practical reasoning in solving mathematical problems. Creative reasoning is more efficient than algorithmic reasoning for completing specific tasks (Lithner (2008). However, in reality, students are rarely directed towards learning mathematics, which is oriented towards creative reasoning; they are more often taught using algorithmic reasoning in solving problems. The cognitive domain is related to problem-solving, a set of skills essential for everyday life as well as other areas, including science and technology, and the attention to individual differences and the provocations during the challenging and alternative phase stimulates the emergence of new ideas that contribute to the improvement of individuals' mathematical creativity (Shodiq et al., 2023). Problem-solving is doing non-routine tasks where the solver does not know the previously learned scheme designed to solve it (Hasan et al., 2024). So, it impacts students' reasoning development. Analysis of this gap reveals that creative reasoning is often not directly integrated into current mathematics education curricula, highlighting discrepancies between research findings and daily learning practices in universities and schools. Educators of prospective mathematics teachers should consider encouraging these prospective mathematics teachers to enable students to engage in volition-enabling prompts through problem-solving, which is a central component of mathematics teaching (Hasan et al., 2024).

Although creative reasoning research has covered many aspects, some areas are under-discussed or require further exploration. Creative reasoning research covers a lot about how the creative reasoning process occurs, how creative reasoning and imitative reasoning work, and the framework of creative reasoning itself. Based on this, further studies are needed on how cognitive and affective factors influence creative reasoning and how the reasoning process occurs.

It is also important to examine the creative reasoning abilities of prospective mathematics teachers in the affective domain because factors in this domain also influence students' mathematics

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learning. The affective domain includes students' moods and feelings (*anxiety, self-confidence, and satisfaction in completing mathematics assignments*), the atmosphere where the learning process occurs, and individual aspects of the students themselves (Hembree, 2015). Anxiety causes students to have negative perceptions of mathematics, making them more likely to feel like they cannot succeed in mathematics. Students feel unable to solve problems and come up with ideas or solutions.

Research on creative reasoning is an important research trend in education because it can provide insight into practical ways to develop creative thinking skills among students as prospective mathematics teachers. At the same time, in the curriculum and teaching methodology, this study's results can help design learning plans to support and encourage creativity. Research on creative reasoning can also help explain how the creative process occurs in individuals and groups, including analysis of factors that influence creativity, such as environment, motivation, and collaboration in problem solving.

This research aims to see the impact of creative reasoning and mathematics anxiety on creative reasoning. It is important to study creative reasoning because it is needed to develop higher thinking skills. Creative reasoning is also related to how much information or knowledge a person has in working memory. Besides that, some affective factors can hinder a person's cognitive performance.

LITERATURE REVIEW

Creative Reasoning

Creative reasoning is a thinking process that fulfills the criteria of novelty, plausibility, mathematical foundation (Lithner, 2008). This means that creative reasoning is a thinking process to create new strategies accompanied by logical arguments based on the intrinsic mathematical properties of the components involved in reasoning. According to Schwarz et al., (2010) creative reasoning is an activity that involves creating ideas, strategies, and concepts to solve problems accompanied by arguments. It refers to using a different approach to completing a task (E. Bergqvist, 2007). It can also be interpreted as concluding with new ideas and logical arguments. These arguments support the truth of what has been concluded or a new idea that has been used.

Creative Reasoning Indicators	Indicators of Creative Reasoning in Problem Solving
Novelty	Develop new ideas or strategies for solving problems.
Plausibility	Build predictive and verifiable arguments about new ideas or strategies.
Mathematical foundation	1. Use problem-solving strategies based on the relevant intrinsic properties of mathematics 2. Provide logical arguments for the intrinsic properties of

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mathematics used in solving problems

Table 1: Indicators of Creative Reasoning in Problem Solving

Based on Table 1. Lithner (2008) developed a framework for understanding creative reasoning in problem-solving, especially in the context of mathematics. The three main indicators identified are novelty, plausibility, and mathematical foundation. These indicators of creative reasoning interact with each other and contribute to effective mathematical problem-solving. In this case, novelty is defined as the novelty of an idea, either by choosing a strategy or using a strategy in solving problems. The novelty here can be shown by ideas, concepts or procedures in solving problems. Plausibility is defined as the accuracy of the arguments used to support the selection of strategies and/or implementation of strategies in solving problems. In this way, individuals explain selecting strategies and/or using strategies in solving mathematical problems. Mathematical foundation is the accuracy of selecting mathematical concepts to solve problems. Individuals select and use appropriate mathematical concepts to use in solving problems.

Problem-solving in mathematics education became a keyword in creative reasoning research in the 1980s. Research has centered on problem-solving aspects, such as the activities and competencies required for constructive problem-solving (Schoenfeld, 1992). Mathematics problem solving not only encourages students to apply their understanding but also have the potential to form a deep understanding to improve their mathematical competency (Putri et al., 2023). The accompaniment of reasonable arguments and a mathematical basis characterizes creative reasoning in problem-solving (Lithner, 2008). Its quality is typified by the novelty of ideas and reasonable reasons. Creative reasoning especially has a prominent role in non-routine problem-solving (Lithner, 2006). Non-routine math problems cannot be solved by simply applying routine steps or mathematical formulas previously studied. They do not have a unique algorithm, so they require strategic thinking (Khusna et al., 2024). In non-routine problems, solutions are not always evident or immediately apparent. Solving these problems may involve exploring various approaches, in-depth use of logic and mathematical reasoning, and creative reasoning analysis. According to Wang & Chiew (2010) problem-solving is a cognitive process of the brain that involves finding solutions to a given problem or a way to achieve a specific goal. The cognitive process is the stage in reasoning whose goal is to achieve knowledge. It consists of mental activities of remembering and manipulating information to solve problems, including mathematical problems, accompanied by logical arguments. Effective information processing is required to achieve success in problem-solving. This process is closely related to working memory capacity (WMC).

Working Memory Capacity (WMC)

Research on working memory capacity (WMC) has been conducted since the 19th century. Working memory is a theoretical construct that has come to be used in cognitive psychology to refer to the system or mechanism underlying the maintenance of task-relevant information during a cognitive task (Baddeley & Hitch, 1994). It is a contemporary term for the cognitive resources

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used to carry out mental operations and remember the results of those operations for a short period (Stillman, 1996). It is a cognitive system that temporarily stores and manipulates information in our minds. It involves three main components: verbal storage related to words and facts, visual-spatial storage related to images and objects in space, and executive control to regulate attention and mental processes.

Working memory has a very limited duration and capacity (Cowan et al., 2004). Working memory limitations as a potential barrier to effective information processing in problem-solving. Not all information that a person receives is permanently stored in working memory. During the problem solving process, a person must gather information that helps in understanding the problem at hand, engage in problem comprehension, evaluate the relevance of the information to the specific problem at hand, and process the information to develop an appropriate solution strategy. Successful problem solving depends on the effective execution of these cognitive processes, which are consistently related to working memory capacity. Working memory capacity is essential for a variety of cognitive processes, including mathematical problem solving (Wiley & Jarosz, 2012). This highlights the importance of working memory capacity in facilitating a variety of cognitive functions including creative reasoning in problem solving, thereby increasing the understanding of the cognitive mechanisms involved in performing creative reasoning tasks.

Working memory has components responsible for storing and processing information; according to Baddeley and Hitch (1994), the components in question are the central executive, the phonological loop, and the visuospatial sketchpad. The central executive components determine topics that require more attention, topics that should be ignored, and what should be done if there is a disturbance (Holyoak, 2005). The phonological loop is repeating auditory information so that the information does not fade from working memory before it is finished being (Baddeley & Hitch, 1994). The visuospatial sketchpad component maintains visual and spatial information in a limited time, for example, remembering moving objects' shape, size and direction, allowing someone to manipulate the background or scene in mental activities (Baddeley & Hitch, 1994). Next, Baddeley discovered an additional component whose task was related to interaction with long-term memory, namely the episodic buffer. Baddeley and Hitch (1994) added an episodic buffer component to the multicomponent model of working memory.

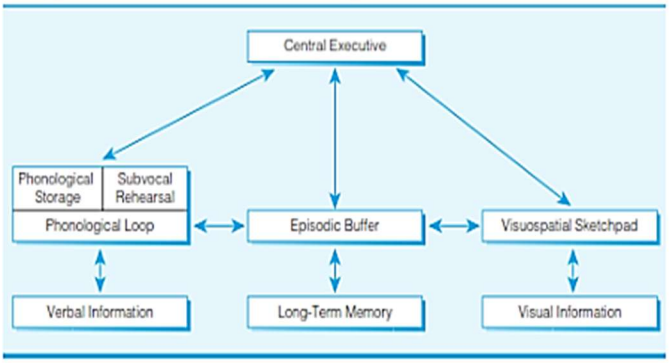


Figure 1: Multicomponent model of working memory

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Research on working memory capacity sheds light on how working memory operates, what can be stored in it, and the extent of its limitations, helping us understand human limitations in simultaneous information processing or multitasking. By recognizing the limits of working memory capacity, this research provides a basis for developing strategies to improve cognitive abilities, allowing us to optimize performance in memory-based tasks using various techniques. Knowledge of working memory capacity is also important in educational contexts, where teachers and educators find it helpful in designing learning methods that suit the human ability to store and process information effectively in short-term memory.

Working memory capacity significantly impacts various cognitive processes, including mathematical problem-solving (Wiley & Jarosz, 2012). It can be measured using working memory span tasks, where people are requested to focus on two tasks simultaneously. As tasks compete for working memory resources, people with a smaller working memory capacity will show a performance deficit on a single task, if not two, while people with a higher working memory capacity will show a smaller deficit in performance. Engle (2004) States that working memory capacity is critical in supporting cognitive abilities in understanding, reasoning, and solving problems. He also found that working memory capacity differs for every individual. Using two different groups of students. Chamandar et al. (2019) found that the group with low mathematical achievements performed significantly worse on inhibition and working memory than the group with high mathematical achievements. Differences in working memory capacity impact cognitive performance when an individual reuses the information needed to solve problems. Working memory capacity positively influences creative reasoning in mathematics by enabling individuals to manage and manipulate multiple pieces of information simultaneously (Hasan & Juniati, 2025). This capacity allows prospective mathematics teachers to think flexibly and fluently, generate diverse solutions, and develop logical arguments grounded in correct mathematical concepts. High working memory enables the use of novel ideas and supports the cognitive processes necessary for effective problem-solving and creative reasoning. According to Lerik (2016), working memory capacity affects both how many items can remain working at one point in time and the type of strategy used when someone is working on a task.

Mathematics Anxiety

Lai et al. (2015) define mathematics anxiety as a feeling of tension and anxiety that interferes with a person's ability to process numbers and solve math problems in a variety of everyday and academic situations. Mathematics anxiety can reflect anxiety that occurs in test situations as well as anxiety related to everyday life, related to math. Mathematics anxiety makes it difficult for a person to concentrate when facing math-related problems. As a result, math problems that are usually easy for him can become difficult because of excessive math anxiety. Lai et al. (2015) also said that someone who experiences math anxiety often shows lower performance in math lessons. They may feel too stressed to function well, which negatively affects their ability to solve math problems.

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Mathematics anxiety can hurt mathematical problem-solving. Not only does it cause cognitive deficits, but it also evokes a physiological reaction (Strohmaier et al., 2020). Anxiety can interfere with students' thinking and concentration when they are trying to solve math problems. Anxious students may have difficulty processing information correctly, understanding math concepts, or remembering previously learned strategies. Math anxiety causes decreased academic performance on math tasks, preventing students from achieving good results in the learning process (García-Santillán et al., 2016). These anxious students may make more mistakes, complete assignments more slowly, or have difficulty seeing important patterns and relations in problems. They may feel they are not intelligent or intellectually competent when learning mathematical concepts and skills. The effects of math anxiety are visible in the processes of calculations, operations, and procedures. Anxiety-related disorders may affect an individual's ability to understand and work with numbers (Ashcraft & Kirk, 2001). Mathematics anxiety has a significant negative impact on mathematics achievement, where the higher the level of anxiety, the lower the mathematics achievement achieved (Juniati & Budayasa, 2020a). This study shows that prospective mathematics teachers who experience high anxiety tend to have low mathematics abilities. These impacts are interrelated and can form a negative cycle where anxiety causes low achievement, low achievement further increases anxiety, and so on. Therefore, it is important to address this anxiety in order to improve mathematics achievement and reduce its negative impacts and provide emotional support so that students feel comfortable in learning and solving mathematics problems.

There has been much research related to mathematics anxiety. Wu et al. (2012), Georges et al. (2016), and Juniati and Budayasa (2020) all show a negative influence of math anxiety on math skills and math performance. Meanwhile, İbrahimoğlu (2018) showed a significant positive relationship between mathematics anxiety and learning helplessness. In addition, self-efficacy and mathematics anxiety have been found to play a role in prospective mathematics teachers' problem-solving abilities (Georges et al., 2016). Mathematics anxiety can obscure a person's cognitive ability to solve problems. This happens because of feelings of fear, doubt, and tension when solving problems. In addition, the relationship between working memory, mathematics anxiety, and mathematics learning achievement has also been extensively studied. However, research specifically examining the effect of different levels of working memory capacity and mathematics anxiety is rare.

Furthermore, a study on the creative reasoning of prospective mathematics teachers as a basis for teaching mathematics is needed. Creative reasoning is essential for prospective mathematics teachers' understanding of mathematics material and fosters students' reasoning. When entering teacher education institutions, prospective mathematics teachers hold several beliefs about mathematics, teaching and learning, the goals of teaching, and their ability to teach mathematics (Ball et al., 2005). Someone who has low math anxiety is more successful in solving geometry problems compared to someone with higher math anxiety, and someone with excessive levels of mathematics anxiety can hinder their performance in solving geometric problems (Wahyuni et

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al., 2024). These are aspects that prospective mathematics teachers must apply in mathematics teaching. This research is focused on examining in depth how a person's creative reasoning process in solving mathematical problems will vary with differences in the levels of working memory capacity and mathematics anxiety.

Theoretical Framework

Researchers use a theoretical framework for their research, which is interrelated between creative reasoning, working memory capacity, mathematics anxiety, and problem-solving.

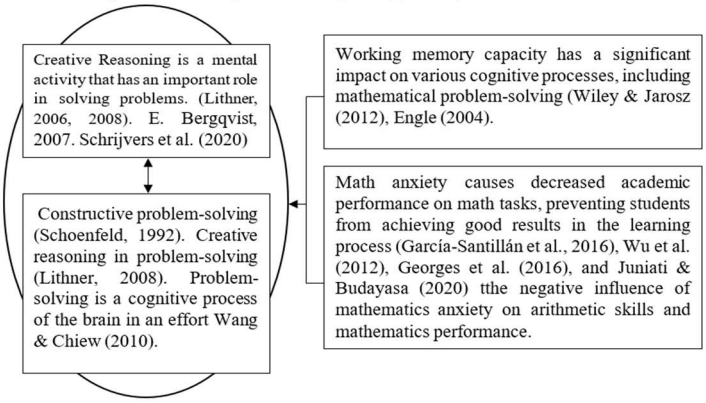


Figure 2: Theoretical Framework

Based on Figure 2. Theoretical Framework is a relationship between theories that form the basis of research and strengthen research findings. Experts have researched and studied creative reasoning as a mental activity related to the cognitive domain. This mental activity plays a role in solving mathematical problems. The problem-solving process is also inseparable from the role of working memory capacity, which shows the amount of information needed when solving a problem. On the other hand, some predictors can inhibit cognitive performance, namely the affective domain, which is related to mathematics anxiety. This theoretical basis strengthens researchers' ability to develop studies related to the creative reasoning of prospective mathematics teachers in solving problems.

METHOD

This research uses a mixed-method sequential explanatory design. Quantitative methods were used to determine the effect of working memory capacity and mathematics anxiety on creative reasoning. Qualitative methods were used to examine the creative reasoning process of prospective mathematics teachers with high working memory capacity and different levels of mathematics anxiety problem-solving.

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Sample and Data Collection

In this research, a sample was selected by cluster random sampling. The sample consisted of two clusters, one comprising second-year students and the other comprising third-year students. From each cluster, one class was randomly selected. Sixty prospective mathematics teachers were selected as subjects for this research. The sample was given the OSPAN Task as a working memory capacity and mathematics anxiety test instrument. Then, it was grouped based on working memory capacity and level of math anxiety. Working memory capacity test scores and math anxiety test scores are independent variables. In this study, several data collection instruments were used. The Operation Span Task (OSPAN Task) measured working memory capacity. It assessed a subject's ability to focus on two tasks at once. The OSPAN Task was adapted from Turner and Engle (1989) and can be seen in Juniati and Budayasa (2020). The sample was given a mathematical operation task and asked to remember numbers simultaneously. Next, the sample was given a mathematics anxiety test questionnaire to categorize the sample. The mathematics anxiety test instrument was developed and adapted from Juniati and Budayasa (2020). The mathematics anxiety test instrument consists of 15 questions about the sample's condition and feelings towards mathematics.

Participants= 60		Working Memory Capacity	
		High	Low
Mathematics Anxiety	High	14	12
	Low	14	20

Table 2: Subject Selection Process

From Table 2. It is explained that the samples were grouped based on OSPAN test scores and mathematics anxiety. The OSPAN test results were obtained by 28 prospective mathematics teachers with high working memory capacity; the remaining 32 were included in the category with low working memory capacity. Researchers only took samples of prospective mathematics teachers with high working memory capacity for further analysis of their creative reasoning because creative reasoning requires high-level thinking. Of the 28 prospective mathematics teachers with high working memory capacity, they were then given an anxiety questionnaire, and it was found that 14 prospective teachers had high mathematics anxiety and 14 prospective teachers had low mathematics anxiety.

Data on creative reasoning in problem-solving were collected using a problem-solving test instrument. This instrument contains mathematics questions on plane geometry. Problem-solving tasks are developed based on indicators of creative reasoning, namely novelty, plausibility and mathematical foundation. Before being used, the problem-solving task instrument was validated by two validators who were experts in the material and problem-solving tasks. The following are the problem-solving tasks used in this research.

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Make a non-rectangle land plan covering an area of 168m^2 with a plan of the house that will be built on the land. The house plan requirements must have three bedrooms with a minimum size of 3x3 meters, one living room with a minimum area of 20m^2, one family room with a maximum area of 25m^2, one kitchen with a minimum area of 12m^2, two bathrooms with a minimum size of minimum 2x2 meters, one car garage with a minimum area of 20m^2 and garden with a minimum area of 44m^2. Make a plan of the house. Explain your answer!

The mathematical problem-solving test developed with creative reasoning indicators is used to measure the creative reasoning performance of prospective mathematics teachers as the dependent variable. The qualitative research method involved 4 out of 60 participants as subjects. Consisting of three subjects with high working memory capacity with high mathematics anxiety and three subjects with high working memory capacity with low mathematics anxiety. Researchers used data from answers to problem-solving tasks made by the subjects, then conducted more in-depth interviews to confirm the consistency of the answers and obtain in-depth information.

Data Analysis

This study assessed working memory capacity using the OSPAN test instrument. In this test, participants were asked to remember numbers displayed for four seconds, during which participants were also asked to perform certain calculations. Upon completing the task, participants are allotted ten seconds to accurately recall and write down all the numbers they were instructed to remember. The working memory capacity score is derived by summing the number of digits recalled in the correct order, with a criterion that the percentage of correct responses for the operations must be at least 80%. The working memory capacity score ranges from 0 to 100. This study employs the quartile separation technique outlined by Conway et al. (1942). About the OSPAN test. While the instrument utilized is similar to existing tools, it features minor variations in the operations and numbers to be retained. The validity of the working memory assessment instrument is evaluated through content validity, which involves expert review to ascertain whether the instrument effectively measures its intended construct.

Additionally, the problem-solving test assesses creative reasoning by presenting participants with three non-routine questions. An assessment rubric evaluates the creativity level demonstrated in each response. The scoring for this test ranges from 0 to 80, with the final score calculated by dividing the total score obtained by the maximum possible score and multiplying by 100. Subsequently, the working memory capacity score serves as the dependent variable. In contrast, the creative reasoning score in problem-solving is treated as the independent variable, and both are analyzed using multiple linear regression analysis.

The data obtained were validated using the triangulation method by re-administering the problem-solving test at a different time to assess the consistency of the answers given by the subjects. The validated data were then analyzed. Qualitative data were analyzed in three stages: data presentation, reduction, and conclusion drawing. First, data from the participants' work and in-

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terviews were presented and grouped to consider their suitability. Then, these data were sorted and reduced, where data containing information relevant to the indicators of creative reasoning were used. Finally, conclusions were drawn based on the validity of the data obtained, which was in line with indicators of creative reasoning and the formulated research problems and objectives.

RESULTS

The research findings outline the results of both quantitative and qualitative analysis. Quantitative data was analyzed through multiple linear regression tests to investigate how working memory capacity and mathematics anxiety affect the creative reasoning abilities of prospective mathematics teachers.

Multiple R	R-Square	Adjusted R-Squared	Standard Error	Observations
0.4586	0.2103	0.1826	25.9806	60

Table 3: Summary of Multiple Linear Regression Test Results

Based on Table 3

Multiple R is 0.4586. Thus, this regression test shows a relationship between the independent variables (working memory capacity and mathematics anxiety) and the dependent variable (creative reasoning). An R^2 value of 0.21 suggests that while working memory capacity and mathematics anxiety contribute to creative reasoning, other unmeasured factors may also play a role. Future studies should explore additional cognitive and environmental influences. This shows that the influence of working memory capacity and mathematics anxiety on the creative reasoning performance of prospective mathematics teachers in solving mathematics problems is low. Meanwhile, the R-squared value of 0.1826 shows that about 18.26% of the variability in the dependent variable can be attributed to the independent variable, in the sense that around 18% of the variance in creative reasoning of prospective mathematics teachers in problem-solving is influenced by working memory capacity and mathematics anxiety.

	df	*SS*	*MS*	*F*	*Significance F*
Regression	2	10243.89	5121.944	7.588174	0.001197
Residual	57	38474.45	674.9903		
Total	59	48718.33			

Table 4: ANOVA Results on Creative Reasoning, Working Memory Capacity, and Mathematics Anxiety

Based on Table 4

Researchers used analysis of variance (ANOVA) as a statistical technique to determine whether the overall regression model is significant in explaining data variations. Table 4 shows that the multiple linear regression had 57 degrees of freedom (df) associated with residuals. This is the number of observations minus the number of predictors and one (n - k - 1), where n is the total

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observations and *k* is the number of predictors. Total degrees of freedom are equivalent to the total observations minus one (*n-1*). The regression sum of squares (SS) was 10243.89, the residual SS was 38474.45, and the total SS was 48718.33. The mean square (MS) was the result of dividing SS by the degrees of freedom (*df*) of each component. This analysis obtained an F-value of 7.5881 and a significance value of 0.0012. This value indicates the level of significance of the results of the ANOVA analysis, which shows strong statistical evidence to reject the null hypothesis (*which states that there is no relationship between the dependent and independent variables*). So, it can be concluded that there is a significant relationship between the independent and dependent variables based on the results of the ANOVA analysis Because the significance value was smaller than the alpha level (0.05). This means that working memory capacity and mathematics anxiety significantly influenced the creative reasoning of prospective mathematics teachers in solving mathematical problems. The regression model can explain a certain proportion of variance even though the previous Adjusted R-squared reflects only around 18% of the variance the predictor explains. However, a significant influence is still identified through the F test.

	Coefficients	*Std. Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	-3.48674	26.16609	-0.13325	.894462	-55.8834	48.90994
WMC (X_1)	0.6309	0.168241	3.74997	0.000416	0.294003	0.967798
Anxiety (X_2)	0.336843	0.339391	0.992493	0.325152	-0.34278	1.016461

Table 5: Coefficients of Creative Reasoning, Working Memory Capacity, and Mathematics Anxiety

Based on Table 5

Intercept Value: -3.4867. This is the predicted value of the dependent variable when all independent variables (X_1 and X_2) are equal to zero. X_1=0.6309. This means that every one-unit increase in X_1 is expected to increase the value of the dependent variable by 0.6309, assuming X_1 remains constant. X_2 value: 0.3368. This means that each one-unit increase in X_2 is expected to increase the value of the dependent variable by 0.3368, assuming X_2 remains constant. Standard Error: This is a measure of uncertainty in the coefficient estimates. Intercept Value: 26.1661. This figure shows that there is high uncertainty in the intercept estimate. X_1 value: 0.1682, a smaller standard error indicates that the estimated coefficient X_1 is more stable; the value of X_2: 0.3393, and a larger standard error of X_1 indicates greater uncertainty in the estimated coefficient X_2.

This is the t statistic used to test the significance of coefficients, which is calculated by dividing the coefficient by the standard error Intercept value: -0.1333. This value is close to zero, indicating that the intercept is not statistically significant. X_1=3.7499, which shows that the coefficient X_1 is statistically significant. X_2=0.9925. This value is also close to zero, indicating that X_2 is not statistically significant. P-value: This is the probability that the coefficient under test equals zero. Intercepts: 0.8945. This indicates that the intercept is not significant at the 0.05 level. X_1=0.0004 shows that X1 is very significant ($p < 0.05$). X_2=0.3252, which shows that X_2 is not statistically

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significant. These are the 95% confidence intervals for the coefficients. Intercept, the confidence interval (-55.8834, 48.9099) shows that cannot reject the hypothesis that the actual intercept can vary widely, including a large negative probability. The confidence interval X_1=(0.2799, 0.9289) shows that we are very confident that X_1 has a significant positive influence on the dependent variable, and the confidence interval X_2 = (-0.294, 0.3428) indicates that we cannot conclude that X_2 has a significant effect. In the regression model, Y=-3.48674+.6309X_1+.336843X_2. If X_1=0 and X_2=0, Y=-3.48674; if X had a positive value of 1, Y=-3.70379+ 6309+336843. The greater the value of X, the greater the value of Y. In other words, the higher a person's working memory capacity, the higher his or her level of reasoning.

Creative reasoning of prospective teachers with high working memory capacity and high mathematics anxiety

Subjects with high working memory capacity and high math anxiety were able to solve a problem well. The subject (Aina=Pseudonym) made an answer by carrying out calculations using the concept of the area of a flat shape to obtain an angled trapezoid as the shape of the flat shape requested in the question. Aina also gave varied answers accompanied by logical arguments to support the answers she made. After carrying out calculations using the area of a trapezoid, the area obtained is 168m^2, This means that the flat shape meets the requirements of the question, which asks to make a ground plan in the form of a flat shape, not a rectangle, with an area of 168m^2.

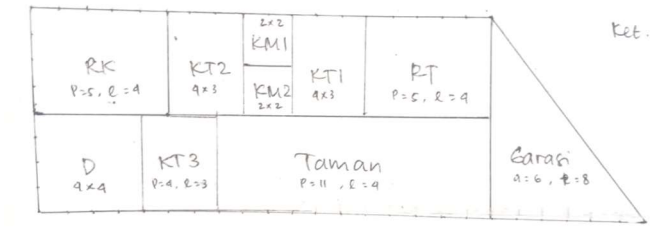


Figure 3: A Sketch by A High-Working-Memory-Capacity, High-Mathematics-Anxiety Subject

Figure 3

Aina is a category of subjects with high working memory capacity, and she has high math anxiety. Aina completed the assignment by sketching. She was confident he could complete the task because it was supported by knowledge of the area of flat shapes. Using the area formula, Aina succeeded in creating a non-rectangular shape as a trapezoid with an area of 168m^2. The trapezoid has a height of 8 meters and a base length of 24 meters, a peak length of 18 meters, and a slanted side length of 10 meters. The area of the trapezoid is obtained.
$L = (AB + DC)/2.t = (24 + 18)/2 = 168m^2$.

Interview	Question and answers	Indicator	Coding
Researcher	Where can you make this flat trapezoid shape?		

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Aina's	I used the **formula for the area of a trapezoid** so that an area of 168 m2 was obtained, thus **fulfilling the requirements**	formula for the area of a trapezoid	Mathematic Foundation (Fm)
		fulfilling the requirements	Plausibility (P)
Researcher	What was your initial idea to get the flat trapezoid shape?		
Aina's	I **made a sketch using side length calculation analysis** and **obtained a size that corresponds to an area of 168 m2**	made a sketch	Novelty (N)
		using side length calculation analysis,	Novelty (N)
		a size that corresponds to an area of 168 m2	Plausibility (P)
Researcher	Can it be made with other flat shapes?		
Aina's	Of **course, you can**, as long as we are careful in what we do	course you can	Novelty (N)

Table 6. Excerpts From Interviews With Subjects With High WMC And High Math Anxiety

Table 6

Based on Aina's work and interview (Table 6), the garage is triangular, with a base length of *6* meters and a height of 8 meters, so the area is $24m^2$, Which exceeds the minimum limit of the provisions. The park is rectangular, with a length of 11 meters and a width of 4 meters, so the area is $44m^2$, Which means it meets the minimum requirements. Next, the living room is rectangular, with a length of 5 meters and a width of 4 meters, so the area is $20m^2$. This area meets the minimum requirements. The family room is rectangular, with a length of 5 meters and a width of 4 meters.

Furthermore, the kitchen is square, with a side of 4 meters, so the area is $16m^2$, Which meets the provisions. Then, three bedrooms are rectangular with a length of 4 meters and a width of 3 meters. Each bedroom area meets the requirements. Moreover, there are two bathrooms in a square shape with a side of 2 meters, which means the area of the bathroom meets the requirements. Next, the researcher confirmed in depth with Aina regarding the answers that had been made. The following is an excerpt from the interview with Aina.

Interview	Question and answers	Indicator	Coding
Researcher	In your opinion, is your answer correct?		
Aina's	That is right, Sir. I sketched an angled "trapezoid of **the appropriate size."**	Trapezoid with the appropriate size	Mathematical Foundation (Fm)
Researcher	Try to explain this picture, especially the bathroom; where does the entrance come from?		

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Aina's	Oh no, Sir, I **did not think that far**; what I thought was as long as the area was the same as the problem.	I did not think that far	Anxiety (Ax)
Researcher	Also, where are the doors to the family room and kitchen? Can you explain		
Aina's	It meets the requirements in terms of size, but I **misplaced the position** of each room, so there are rooms that **cannot be passed through**, and the floor plan is **not structured correctly**, Sir.	It cannot be passed through	Plausibility (P)
		not structured	Anxiety (Ax)
Researcher	Why?		
Aina's	My thoughts were not there; **this problem presented a challenge**. Initially, I was asked to make a flat shape, not a rectangle, but the area had to be 1682.. This is what makes me **think higher and be less sure.**	this problem presented a challenge	Anxiety (Ax)
		Think higher and be less sure.	Anxiety (Ax)

Table 7: Excerpts From Interviews With Subjects With High Working Memory Capacity And High Math Anxiety

Based on Table 7

Aina has sketched a non-rectangular flat shape with an area of $168m^2$ in the shape of a right-angled trapezoid. Then, using mathematical concepts in the form of areas of squares and rectangles, he created the space required in the problem according to his initial knowledge of mathematical concepts. All rooms created have met the requirements. After confirmation, Aina felt that the position of the room was not suitable. The first is the bathroom; the bathrooms are positioned in a row, so he forgot to make an entrance to the bathroom. Aina confirmed that the work on the questions did not focus on the details of the room; she only paid attention to the requirements for the number and area of the room requested. Likewise, the size of the family room and kitchen is appropriate. Aina's does not provide road access to these rooms.

Creative Reasoning of Prospective Teachers With High Working Memory Capacity and Low Mathematics Anxiety

Subjects with high working memory capacity and low math anxiety were able to solve a problem. The subjects gave varied answers accompanied by logical arguments to support the answers. One such subject is (pseudonym). Indah made a house plan based on the information in the problem-solving task. Indah sketched a plot of land measuring $168m^2$ in the shape of not a rectangle but an angled trapezoid.

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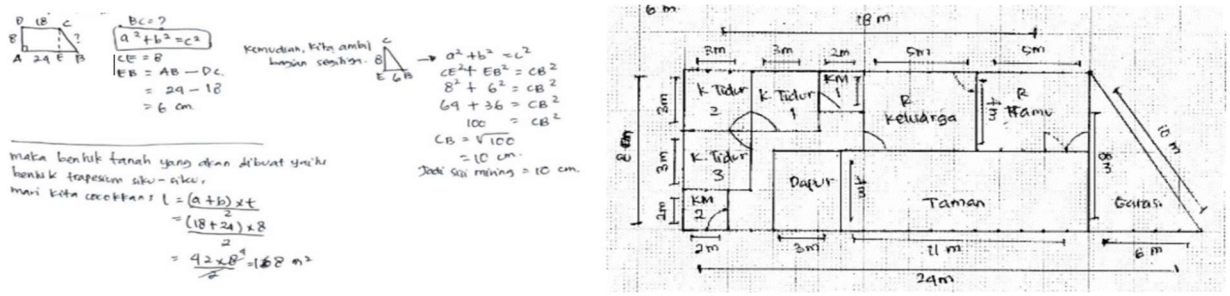


Figure 4: A Sketch by A High-Working-Memory-Capacity, High-Mathematics-Anxiety Subject

Figure 4

Indah completes assignments by making sketches. At first, he was not sure he could complete the task because he had to make a flat shape, not a rectangle, with an area of $168m^2$. After he tried many times to sketch a flat shape with an area of $168m^2$ he finally got a flat shape in the form of an angled trapezoid. The height of the trapezoid was eight meters, the base was 24 meters long, the top was 18 meters long, and the hypotenuse was 10 meters long. Indah divided the trapezoid ABCD into two parts: a rectangle AECD, with AD 8 and AE = 18 meters and correct CEB, with $EB = AB = 24 - 18 = 6$ and AC = 8. Using the Pythagorean theorem formula, the hypotenuse BC was 10. Indah calculated the area of the shape using the formula for the area of an angled trapezoid. $AB = 24, DC = 18$, and $AD = 8$; Using the formula for the area of an angled trapezoid, we obtain the area of the $trapezoid L = (AB + DC)/2.t = (24 + 18)/2.8 = 168$

Interview	Question and answers	Indicator	Coding
Researcher	How will you complete this task?		
Indah's	At first, "**I was not sure I could finish"** because the area had been determined and was not a rectangle.	I was not sure I could finish	Anxiety (Ax)
Researcher	You are making a plan in the shape of an angled trapezoid; why is that?		
Indah's	After trying to "**sketch"** with **"uncertainty"**, I finally found **a non-rectangular flannel bold 1 bold 6 bold 8, bold italic m bold 2,** namely a "**right-angled trapezoid."**	Sketch, right-angled trapezoid	Novelty (N)
		uncertainty	Anxiety (Ax)
		a non-rectangular flat shape with an area of 168	Plausibility (P)
Researcher	Apart from an angled trapezoid, are any other shapes that can be made but still meet the task requirements?		
Indah's	There could be, "**but it would take a long time to think about it."**	but it would take a long time to think about it	Anxiety (Ax)

Table 7. Excerpts From Interviews With Subjects With High Working Memory Capacity And Low Math Anxiety

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Table 7

Based on the results of Indah's work and direct confirmation through interviews with the subject. Even though you feel anxious, you are unsure you can complete the task. Indah tried several times by sketching a flat shape with an area of $168m^2$. This sketch obtains a flat figure in the form of a trapezoid with an area of $168m^2$. Indah's next step was to make part of the room according to the provisions in the question. Indah made three bedrooms in the same area, namely. $9m^2$; The room size of these rooms met the requirements in the question; then, Indah made two bathrooms the same size as requested ($4m^2$), one living room with an area of $15m^2$, family room with an area of $15m^2$, kitchen with an area of $9m^2$, garden with an area of $44m^2$ and garage with an area of $24m^2$.

Next, the researcher confirmed in depth with Indah regarding the answers that had been made. The following is an excerpt from the interview with Indah.

Interview	Question and answers	Indicator	Coding
Researcher	In your opinion, is your answer correct?		
Indah's	That is right, Sir. I sketched an angled "trapezoid of **the appropriate size."**	Trapezoid with the appropriate size	Mathematical Foundation (Fm)
Researcher	Pay attention to the position of the three bedrooms you drew!		
Indah's	Yes, Sir, it is **the right size**, but I am **confused**" about where to position it.	the right size	Mathematical Foundation (Fm)
		Confused	Anxiety (Ax)
Researcher	Pay attention to the position of the three bedrooms that you drew! Where is the door to bedroom 2?		
Indah's	I was initially confused about where to place the three bedrooms. The important thing is that they are the correct size**. It turns out that bedroom two logically does not have an entrance.**	It turns out that bedroom two logically does not have an entrance	Plausibility (P)

Table 8: Excerpts From Interviews With Subjects With High Working Memory Capacity And High Math Anxiety

Table 8

Indah has sketched a flat, non-rectangular shape with an area of $168m^2$. As an angled trapezoid, then, using mathematical concepts as an area, he created the space requested in the question according to his initial knowledge of mathematical concepts. All rooms created have met the requirements. However, after being confirmed in depth. The position of the adjacent bedrooms made Indah forget that, logically, this position was incorrect. After further confirmation, he, without realizing it, did not think that the position of the 2nd room was proper because people could not pass the 2nd room. Indah could not give a logical reason why the three rooms were

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side by side; what Indah had in mind was that the area of all the rooms did not meet the area requirements. However, in reality, the door to the second room could not be opened because there was no access.

DISCUSSION AND CONCLUSION

The regression test results explain the relationship between working memory capacity and mathematics anxiety on the creative reasoning performance of prospective mathematics teachers. The Adjusted R-squared value reflects approximately 18% of the variance as a predictor of creative reasoning. This finding is supported by the theory that working memory capacity positively affects mathematics performance. In addition, working memory capacity appears to positively affect mathematics performance in the form of basic and advanced mathematics and problem-solving skills. In contrast, mathematics anxiety hurts advanced mathematics and problem-solving skills (Juniati & Budayasa, 2020b). Cognitive independence affects basic mathematics abilities but does not affect advanced mathematics abilities. Working memory capacity positively affects mathematics performance in basic and advanced mathematics and problem-solving skills. In contrast, mathematics anxiety negatively affects advanced mathematics and problem-solving skills (Juniati & Budayasa, 2022).

The test results show that the confidence interval X_1=(0.2799, 0.9289) shows that we are very confident that X_1 has a significant favorable influence on the dependent variable. The confidence interval X_2 = (-0.294, 0.3428) indicates that we cannot conclude that X_2 has a significant effect, so it can be concluded that working memory capacity and mathematics anxiety influence a person's creative reasoning performance. Someone who has low math anxiety is more successful in solving geometry problems compared to someone with higher math anxiety, and someone with excessive levels of mathematics anxiety can hinder their performance in solving geometric problems (Wahyuni et al., 2024). From the regression test results, it can be concluded that X_1 and X_2 are independent variables that influence the dependent variable because the p-value is below 0.05. This study is consistent with previous research on the relationship of working memory to mathematics learning and achievement in children (Alloway & Alloway, 2010).

The interviews revealed the fluency and flexibility of the subject's creative reasoning. Subjects with high working memory capacity can see the novelty of their ideas from the way they sketch plans according to the requirements specified in the problem. Subjects can identify problems well and connect previously known mathematical concepts to create answers. Apart from that, the subject also provides logical arguments to support the correctness of the answers that have been made. Apart from that, the subject also uses mathematical formulas in the form of areas of triangles and quadrilaterals so that answers are made using appropriate mathematical concepts.

As their proposed new ideas demonstrate, subjects with high working memory capacity show flexibility and fluency in their creative reasoning processes. They meet the requirements for de-

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signing rooms even though the size of each room is different. This logical argumentation underlies the selection and application of new ideas in solving problems. Apart from that, they also developed a systematic and structured spatial plan. The use of the Pythagorean theorem formula in designing non-rectangular land plots shows that they use the correct mathematical basis to support their creative reasoning, and the variance they use to position the room without sacrificing the specified conditions shows the fluency of thinking and flexibility that supports it their creative reasons.

However, subjects with high math anxiety do not think about the extent to which their work is correct; their focus is only on fulfilling the requirements for the problem, so anxiety interferes with their cognitive performance, causing a deadlock in thinking, the subject is unable to provide logical reasons related to the position of the room created. So, the subject's anxiety hurt his cognitive performance. This is indicated by initial confirmation that subjects with low math anxiety are doubtful about being able to complete the task. This anxiety interferes with cognitive performance. Losing focus caused him to fail to complete the question successfully, even though he understood the meaning of the question. Upon closer inspection, it is clear that he sketched a floor plan with three bedrooms grouped. Even though the size of the room created by the subject meets the requirements, the subject placing the bedroom in an adjacent corner of the house will leave one bedroom without an entrance. The subject realized that the position of the bedroom was incorrect even though the size was correct. These results indicate that the creative reasoning process in subjects with high memory capacity can be disrupted by mathematics anxiety, both with high and low levels of mathematics anxiety. Subjects can answer systematically, identify known mathematical concepts and connect these concepts as a problem-solving strategy. Apart from that, every strategy or idea that is made is accompanied by logical arguments to support the correctness of the answers made but is less able to overcome the anxiety felt due to pressure and doubt due to limited time to solve the questions so that the answers given are less precise and the arguments given are less valid.

Subjects with high working memory capacity have much information and regularly create solutions with new ideas accompanied by reasonable arguments. Working memory capacity supports cognitive performance in understanding, reasoning, and solving problems (García-Santillán et al., 2016). However, high math anxiety due to stressful situations and limited working time interferes with concentration, resulting in less accurate answers. The level of mathematics anxiety of prospective mathematics teachers is relatively high and significantly impacts mathematics performance (Juniati & Budayasa, 2020b).

Reasoning is essential in solving mathematical problems. The cognitive system plays a massive role in this process. It performs its function of searching for appropriate information to solve problems. Information that is owned and stored in memory is recalled according to needs when faced with a problem. The amount of information stored in memory is directly proportional to the ease of responding to solving problems. Memory has different capacities in each person (Friso-Van Den Bos et al., 2013). Differences in a person's work capacity impact differences in that

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person's cognitive performance. Because the more information one has, the easier it will be for a person to solve the mathematical problems they face.

Creative reasoning as a person's cognitive domain can be disturbed due to anxiety. Anxiety makes it difficult for a person to concentrate and not focus on problems, which has an impact on cognitive activities. Something that is usually easy for someone can become difficult because of excessive anxiety. Anxiety about mathematics causes a decrease in academic achievement in mathematics tasks, thus preventing students from achieving good results in the learning process (Miller & Bichsel, 2004). Mathematics anxiety can be described as doubt, uncertainty, and a lack of courage regarding mathematical abilities. It negatively influences arithmetic skills and mathematics performance (Friso-Van Den Bos et al., 2013). As a result, the problem-solving process becomes less flexible and systematic, and mathematical concepts cannot be identified and connected to solve problems. High mathematics anxiety made it even harder for the subjects to sketch their plans systematically and organise. Mathematics anxiety can hurt mathematical problem-solving. Not only does it cause cognitive deficits, but it also evokes a physiological reaction (Palengka et al., 2019). This shows that a person's anxiety can hinder thinking. These findings are consistent with the research results (Strohmaier et al., 2020). This showed that individuals with high mathematical anxiety are less systematic and logical in providing arguments, even though they provide proof to back their proposed ideas. They formulate assumptions based on inaccurate arguments. Working memory is limited, limiting a person's ability to process information. Working memory capacity is important for many cognitive processes, including understanding, reasoning, and problem-solving. It appears to positively impact mathematics performance, including basic and advanced mathematics skills and problem-solving skills, while mathematics anxiety has a negative effect.

The results showed that working memory capacity and mathematics anxiety significantly influenced the creative reasoning performance of prospective mathematics teachers. Prospective mathematics teachers with high working memory capacity demonstrate flexibility and fluency in generating ideas, can identify and connect previously known mathematical concepts, and can provide logical arguments to support the correctness of the strategies created. However, both high and low mathematics anxiety was found to interfere with cognitive performance in solving more complex problems. Losing focus on solving problems results in unsystematic answers and an inability to provide logical arguments for the answers made.

This research provides additional information for prospective teachers and mathematics teachers on developing creative reasoning as initial knowledge when teaching mathematics to students. It can also help teachers prepare lessons to increase students' working memory capacity. Mathematics educators should incorporate structured problem-solving exercises that build working memory capacity while gradually reducing anxiety through scaffolded support and positive reinforcement.
The results of this creative reasoning research have contributed a lot to education. One of them is in designing learning strategies. Mathematics teachers can design creative reasoning-based

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mathematics learning with an innovative and flexible approach by presenting contextual problems; teachers can create mathematical problems relevant to students' daily lives. This can increase students' interest and motivation to think creatively when solving problems and providing opportunities for students to form groups and discuss, which provides space for students to collaborate to find new ideas for solving problems and improve critical thinking skills. Designing learning with the use of visual aids, manipulatives, or mathematical software to help students understand more abstract concepts. Providing opportunities for students to explore various approaches to solving problems. Ask open-ended questions that encourage students to think deeper and explore new ideas. Open-ended questions allow students to argue and explore mathematical concepts more deeply.

The results of this study indicate that working memory capacity and math anxiety contribute to creative reasoning. However, there are still other unmeasured factors that play a role in creative reasoning. Therefore, further research is needed that explores additional cognitive and environmental influences such as cultural differences in creative reasoning, creative reasoning in the context of collaborative problem solving, how positive and negative emotions affect creative reasoning, how individuals and groups adapt, learn, and grow after experiencing failure in a creative context. By exploring these aspects, creative reasoning research can become more comprehensive and relevant to societal and world developments.

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Collaborative Problem Solving With and Without Access to Technology: Emphasis on Mathematical Justifications

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Abstract: Many studies highlighted the importance of mathematical justification in problem-solving. This paper describes students' mathematical justifications for solving derivative problems collaboratively, especially before and after the use of technology was allowed. We asked two undergraduate students who were preservice mathematics teachers in a paired problem-solving to see how they solved a derivative problem. To better understand the role of technology, we asked them to solve the problem without technology in the first session and with technology in the second. The pair progressed from the perceptual level of justification to the symbolic example with generalization level as the technology entered the discussions. The pair used technology to validate claims and exploring conjectures, yet it did not directly contribute to how they revised inaccurate claims or offered better justifications. Further studies could utilize a task that explicitly includes instructions on technology to promote mathematical justifications during CPS.

Keywords: mathematical justification, problem-solving, derivative, technology, preservice teacher

INTRODUCTION

Mathematical justification, an activity to provide arguments to support or refute a mathematical claim, was shown to be essential in promoting good reasoning skills in problem-solving (Brodie, 2010; Cirillo et al., 2016). By providing justification for a claim, students try to find reasons for their claims and do not merely accept the ideas they receive. Several studies stated that mathematical justification determines the success of problem solving, especially in the collaborative ones (Chiu, 2008; Díez-Palomar et al., 2021). Chiu (Chiu, 2008) analyzed group problem solving of high school students and investigated the factors that most influence the success of a group in reaching a solution. This research found that justification had the greatest effect. Additionally, another study by Díez-Palomar et al. (2021) found that student interactions dominated by math-

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ematical justification were closely related to whether or not a group's answer was correct in solving problems because students checked the validity of the claim and used it as the basis for their answer. Thus, the group's success in solving problems cannot be avoided from the existence of mathematical justification which provides space for students involved in problem solving to question why the steps they take can be done and are valid to reach the answer. By justifying claims, students would try to think why their claims are valid using their prior mathematical knowledge and persuade others to accept them (de Villiers, 1990; M. E. Staples et al., 2012).

Some challenges were faced in the mathematics classroom despite the importance of mathematical justifications in problem solving. From studies involving teachers, it was found that encouraging students to justify was difficult for teachers (Brodie, 2010; Martino & Maher, 1999). It was because students were not familiar with questioning their own claims or to criticize if an argument was mathematically acceptable without teachers' questioning to provoke them. In addition, students tended not to make or ask for justification for the claims or steps they took during the collaborative problem-solving process (Hamidy & Suryaningtyas, 2016; Nafi'an, 2020; Stylianou & Blanton, 2002). Students did not try to convince others or explain why the answer was correct but tended to only go through the process of solving problems by focusing on getting an answer rather than to convince or validate it.

Many studies suggested using technology to support students' problem-solving, especially dynamic software (NCTM, 2020). The use of such technology helps students to interact with mathematical objects and explore them concerning the problems they face. Recent studies have also demonstrated that the use of technology such as digital learning models effectively supports students in solving problems aligned with the curriculum (Effendi et al., 2024), while Dynamic Geometry Environments (DGE) assist students in abstracting mathematical ideas through visual and interactive exploration (Dintarini et al., 2024). Regarding mathematical justification, technology allows students to examine the data provided, create a temporary claim, and justify it under several possibilities (Erbas et al., 2020) or even an infinite range of possibilities (Noss et al., 2009).

In collaborative problem-solving, technologies can help promote collaboration by giving more opportunities for practical mathematics activities, like manipulating values or objects, bridging into a more abstract and complex phase (Geiger et al., 2010; Jacinto & Carreira, 2021; Olive et al., 2010). However, technology's benefits do not automatically happen once it is introduced or used. Despite being shown to help students develop mathematical thinking even for complex mathematics (Erbas et al., 2020), we still need careful analysis of what students can achieve through technology (Sinclair et al., 2009) and whether students' justification of claims happened, even without the use of technology (Hollebrands et al., 2010). Furthermore, studies exploring technology with the focus of collaborative context was still limited (Bray & Tangney, 2017). Therefore, we believe it is crucial to analyze the role of technology through the lens of mathematical justification. In this paper, we explored students' mathematical justifications while solv-

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ing derivative problems collaboratively without and with access to technology. Further, we looked upon the quality of mathematical justifications and how they progressed before and after the use of technology. By addressing this issue, how students should be supported in mathematically justifying claims during their problem-solving process can be learned, especially when they use technology during the process.

LITERATURE REVIEW

Mathematical Justifications

There are two primary concerns in understanding mathematical justification. First, mathematical justification can be seen as arguing against or proving a claim or as the result of that process. This interpretation is consistent with the assertion made by Staples et al. (Staples et al., 2012) that mathematical justification is an activity or argument that uses statements or accepted mathematical reasoning to demonstrate or disprove a claim's truth. Similarly, Yackel and Cobb (Yackel & Cobb, 1996) defined mathematical justification as a disagreement or agreement regarding an acceptable mathematical explanation for a proposed mathematical approach. Second, the "acceptance" of the participants in the activity is required for mathematical justification. The foundation of this acceptance is that these individuals agree that a particular mathematical assertion or response is reasonable (K N Bieda et al., 2022). Therefore, the social context in which mathematical activities are carried out is also considered in mathematical justification (Simon & Blume, 1996; Staples & Conner, 2022; Thanheiser & Sugimoto, 2022).

Mathematical justification is both a social and a cognitive activity. In completing mathematical justification, other than involving discernment in contending to help (or disprove) a case, one expects work to persuade parties beyond himself who engage with that action (Roy et al., 2014; Sowder & Harel, 1998). There are times when mathematical justification is not always logically complete because of the involvement of the community where the activity is carried out and social considerations (Jaffe, 1997; Kilpatrick et al., 2001). For instance, in a study by Sowder and Harel (Sowder & Harel, 1998), students were asked to determine whether a rhombus would result from connecting the midpoints of the sides of an isosceles trapezoid. By drawing an isosceles trapezoid, marking the midpoints of the sides, and demonstrating that the points were connected to form a rhombus, most students justified their answer to this question. Participants in this study were attracted to persuade other participants of his mathematical cases despite the fact that what they were doing was inaccurate.

According to Walton (Walton, 2001), the level of confidence or the degree to which the argument is convincing (plausibility) determines which type of argument constitutes reasonable mathematical justification. In another way, the students' mathematical justifications can be of varying quality, and the most convincing justification is the one with the highest plausibility level. Several studies have identified levels of mathematical justification in this regard. Simon and Blume (Simon & Blume, 1996) classified students' mathematical justification as level 0 (without

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justification), level 1 (appeal to external authority), level 2 (empirical demonstration), level 3 (deductive justification expressed through examples), and level 4 (deductive justification independent of example). Vale et al. (Vale et al., 2017) found similar levels, while Sowder and Harel (Sowder & Harel, 1998) and Carpenter et al. (Carpenter et al., 2005) regarded them as types rather than levels but explained that one type provided a better justification than the others.

To assist us in describing the variety of mathematical justifications while solving a problem in this study, we used five levels of mathematical justification adapted from Vale et al. (Vale et al., 2017) (a brief description can be found in Table 1). Students at level 0 simply accepted a particular claim without attempting to justify its truth. At level 1, students say a claim is valid because it comes from someone other than themselves, like a teacher or a textbook. At this level, students assume that if a mathematical claim comes from a more "authoritative" source, it will be valid. For instance, a student uses a particular strategy because his teacher previously suggested it. At level 2, students provide justification based on empirical or perceptual examples. Students demonstrate how what they see supports a claim and demonstrate it to others. Students, for instance, back up the claim that a quadrilateral picture is a model of a square because the sides are the same length. He may persuade others by estimating the lengths of the sides, which are similar, using some estimating instruments. This student justifies by citing what he experiences and sees when interacting with the quadrilateral illustration.

Students begin using symbolically expressed examples at level 3, but these examples have not been generalized. Sowder and Harel (Sowder & Harel, 1998) gave an illustration of justification at this level when participants were inquired as to whether $n^2 - 79n + 1601$ is prime for any value of n. In this model, a participant supported the case by giving instances of the qualities $n^2 - 79n + 1601$ for a few values of n and presumed that the computation generally gave indivisible numbers without considering all potential values of n. At this level, the participant affirmed his cases to others by utilizing examples, and because he didn't utilize generalization, he didn't see that for $n = 80, n^2 - 79n + 1601$ is undoubtedly not a prime number. At level 4, students use definitions, theorems, or mathematical logic rules to justify their claims or make generalizations that do not rely on examples. Vale et al. (Vale et al., 2017) stated that students found this level to be the most challenging because they did not believe this justification was necessary for other students to accept their claims.

Justification Code	Description
T	Trigger of a situation for justification
C	A claim made in the situation for justification
L0	No justification (Not justifying or accepting a claim without explaining why a claim is reasonable).
L1	Appeal to authority (Stating the truth of claims using other people's authority or other sources such as teachers, books, or friends).

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L2	Empirical or perceptual demonstration (Stating the truth of claims based on the demonstration of perceptual examples, e.g., objects, pictures, or gestures).
L3	Symbolic example without generalization (Stating the truth of a claim by using some symbolic examples without generalizing those examples)
L4	Symbolic example with generalization (Stating the truth of a claim by using definitions, theorems, or rules of mathematical logic or generalizations that do not depend on a particular example)

Table 1: Coding scheme

Collaborative Problem Solving (CPS)

In this section, we try to elaborate on literatures in understanding students' collaborative problem solving. Some literature defined CPS as the ability a person has to solve problems together, while some other literature defined it as an activity to solve problems together. PISA 2015 defined CPS as an individual capacity in which two or more individuals are involved in a process to solve a problem. PISA defined CPS by stating that CPS capabilities should include the sharing of knowledge, skills, and efforts among individuals to achieve solutions. CPS competency is assessed using a 4 x 3 matrix (OECD, 2017). In the matrix, rows represent individual problem solving skills (exploring and understanding, representing and formulating, planning and implementing, monitoring and reflecting). Columns represent collaboration skills (building and maintaining shared understanding, taking appropriate action to solve problems, and building and maintaining team organization). Unlike PISA, ATC21S (Assessment for Teaching in the 21st Century) defined CPS more generally as "approaching problems responsively by working together and exchanging ideas" (Griffin & Care, 2015). In this sense, ATC21S sees CPS as a collective problem-solving activity and the competence to approach that problem in a group.

Other sources, which focused on collaborative problem solving in mathematics, positioned CPS as a situation in which a process that students undertook was investigated. The theoretical review of those studies that discussed CPS limits it as a joint problem-solving activity. However, the quality of certain activities was identified using some indicators similar to ATC21S and PISA. For example, a study by Mercier et al. (2017) incorporated the theory of perspective taking by group members to identify successful CPS interactions in the mathematics classroom. Another example was the study by Taylor and McDonald (2007), which identified students' problem-solving skills in groups using a framework by Polya similar to PISA. Rather than viewing CPS as an overall competency, these studies considered CPS as the activity in which the focus of the study occurred or was investigated. Participants in such studies were asked to solve math problems or perform certain math tasks in groups. Participants' interactions or performance were recorded and analyzed during the ongoing problem-solving process.

In the studies, a CPS was an environment or situation in which participants solve a problem or task together, and the task or problem was designed for the aim of the study. Similarly, in this

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paper, the researcher did not aim to analyze CPS as an individual ability, but as a situation or process in which mathematical justification was observed.

Theoretical Framework

In this section, we will discuss theories on mathematical justifications, collaborative problem solving, and technology, as well as how they were related to each other. Mathematical justification in Collaborative Problem Solving can be understood through the lens of social constructivism as proposed by Piaget (1970) and Vygotsky (1930). Both scholars emphasize that learning is an active process of knowledge construction, although they differ in their perspectives on the role of social interaction in individual cognitive development. Piaget (Piaget, 1970) asserts that learning is primarily an individual process driven by cognitive conflict and self-discovery, even though such conflict can arise from social interaction. In a collaborative setting, students encounter diverse perspectives that create disequilibrium (cognitive imbalance), prompting them to construct new understandings through reasoning and justification. According to Piaget, peer interaction facilitates deeper cognitive engagement as students work through contradictions and reestablish equilibrium.

On the other hand, social constructivism, as formulated by Lev Vygotsky in 1978, emphasizes the role of social interaction in individual cognitive development. Language and culture serve as fundamental frameworks through which individuals acquire knowledge. According to Vygotsky, language and culture play crucial roles in shaping intellectual development. Learning concepts are transmitted through language, interpreted, and understood through experiences and interactions within a cultural environment. Social constructivism acknowledges the social aspects of learning, emphasizing conversation, interaction with others, and the application of knowledge as essential elements for achieving learning goals (Rytilä, 2021).

In contrast to Piaget's constructivism, which views knowledge as something that students construct independently based on their experiences, Vygotsky's social constructivism posits that knowledge is developed through collaboration—whether with peers, teachers, or more knowledgeable individuals. Social constructivism is a variant of cognitive constructivism that highlights the collaborative nature of learning under the guidance of a facilitator or in cooperation with other students. Justification emerges as students engage within the Zone of Proximal Development (ZPD)—the space between what they can achieve independently and what they can achieve with the support of a more knowledgeable other. Language and discussion play a crucial role in this process, as students use verbal justification to negotiate and internalize mathematical concepts. Vygotsky (Lev Vygotsky, 1930) introduced the concept of internalization, referring to the process by which individuals assimilate knowledge, skills, values, or ways of thinking from their social environment (through interaction with others) into their own cognitive system.

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In CPS, these two perspectives complement each other. Piaget's theory explains how students refine their justifications through cognitive conflict, while Vygotsky's theory highlights how scaffolding and peer discussions can enhance the development of higher-level justifications. This study builds on these frameworks by analyzing how students construct and refine their mathematical justifications in a collaborative setting, both with and without the use of technology.

When students are involved in CPS, they will interact with students in their group. In this process, students will think about solving the problem while comparing it with what their friends think. During discussions in the CPS stages, students' statements may be questioned or asked for reasons. This question initiates what is called a situation for justification (Cobb, Wood, et al., 1992). The person who questions or asked for reasons acts as a trigger for the justification. As a response to this, students will come up with mathematical justifications that they think of in order to convince other students that their claims are correct.

When the mathematical justification put forward by the student is agreed upon by the group, a co-constructed justification is formed (Mueller, 2009; Yackel, 2004). This agreement or acceptance might be indicated by an explicit statement or implicitly through the use of the justified claim as a new data in the next discussion (Cobb et al., 2010). A similar concept was stated by Tatsis and Koleza (2008) and Yackel and Cobb (Yackel & Cobb, 1996) as mathematical justification norms. Norm, in this study, was defined as a regularity in collective activities in the classroom. Mathematical justification norms can be understood as regularities in collective activity when a mathematical method or claim needs to be supported by a reason (McClain & Cobb, 2001). In other words, mathematical justification in a CPS situation can be seen as a mathematical justification that is built together by a group or as a regularity of mathematical justification activity being observed when a group solves a problem. Mathematical justifications that are constructed together or mathematical justification norms determine whether the ongoing CPS process can continue to the next stage or needs to return to the previous stage (McClain & Cobb, 2001; Partanen & Kaasila, 2015; Roy et al., 2014; Tatsis, 2007).

The involvement of technology while students are justifying claims during CPS might bring different issues. Some studies revealed that technology improved the problem solving process. For instance, a study by Nguyen et al. (Nguyen et al., 2023) analyzed the use of GeoGebra in learning and solve problems on geometry. They found that utilizing GeoGebra helped students in problem solving, especially in questioning the process and actively participating in the exchange of ideas, activities supporting mathematical justifications. Another study by Olive et al. (Olive et al., 2010) highlighted the role of technology in being an 'external authority" that empowers students to more or new mathematical practices. For instance, they mentioned the way technology could provide visualization of an abstract concept that made this concept more approachable to students. This visualization allows students to a new mathematical practice to manipulate a supposedly abstract concept using their visual perceptions. In problem solving, such activity could help students to justify a claim using their visual perceptions and reason its relation to the intended problem. Technology also helps students in verifying claims (Erbas et al., 2020). Erbas et al.

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explored students' strategies in solving problems using the Geometer's Sketchpad. They found that the software allowed students to model the problem and manipulate the involved variables interactively to explore it. The manipulations allow students to verify the validity of claims and to construct a valid argument under a plausible justification.

Despite all the possible benefits brought by technology in helping students to justify claims, the use of technology also brought concerns. For instance, in a study by Hollebrands et al. (Hollebrands et al., 2010), students were found to rarely provide justifications while working on a problem using technology compared to when they did not use it. One of the possible reasons was that students considered the technology itself as a form of justification, which was actually not intended from the use of technology in helping students to reason and verify claims. This issue is related to the fact that technology could be treated as an 'external authority', a source of valid claim without the need to question it. As explained in the previous section, a justification based on an external authority is considered a low level of justification, as it does not involve students' own reasoning in judging the validity of a claim. Tall (1989) stated that students might lose some autonomy if they did not follow their reasoning and allow themselves to be led by technology.

In this paper, we explore how a pair of students mathematically justify claims during CPS with and without access to technology. To investigate this, we focused on identifying the quality of mathematical justification (as presented in Table 1) while they solved a problem in the absence and presence of technology. To allow for better analysis of the role of technology, we also explore the feature of technology they used and how it was related to their mathematical justification and problem solving process.

METHODS

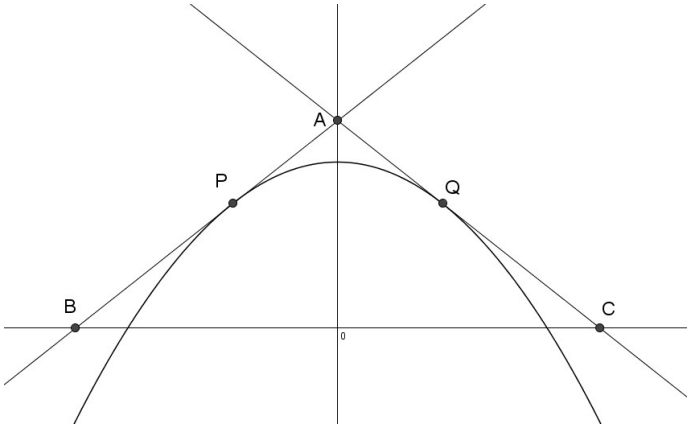
This study was part of a larger project on mathematical justification by college students while solving derivative problems collaboratively. We asked two mathematics education majors who were also preservice teachers to solve a derivative problem (Figure 1) in pair to see how they solved it. The task was designed following Chua's description of an elaboration task (Chua, 2017) as one of the task types promoting mathematical justification. Content-wise, the task was modified from Stewart (Stewart, 2010) by adding more information on the points presented in the graph. Four experts validated the task and went through phases of revisions. The validation process was to assess whether the task (1) encouraged students to elaborate their thinking upon the answer, (2) instructed students to solve the problem collaboratively, and (3) allowed various justifications along the process of finding the answer. In addition to validating the third point, the validation document included a scheme of possible mathematical justifications made by students.

The highlight of the task is the need to use derivatives to find the slope of the tangent lines. Further, students must elaborate their steps by utilizing the graph presented in the task without dis-

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regarding its algebraic validity related to the solutions. For example, students may infer that point P is closer to point A than to point B based on the picture. However, this cannot be justified by the picture alone. They were expected to leave this as an assumption rather than valid information. The task also required students to utilize the fact that points A, B, and C are equidistant, which might be altered if students relied heavily on the perception of the distance given by the graph. The expected final answer is $P\left(\frac{-1}{2}\sqrt{3}, \frac{1}{4}\right)$ and $Q\left(\frac{1}{2}\sqrt{3}, \frac{1}{4}\right)$ to make ABC an equilateral triangle.

Given a parabola $y = 1 - x^2$ as shown below.



Lines p and q are the tangents of the parabola at P and Q. Tangent lines p and q intersect x-axis at B and C, respectively. Find points P and Q on the parabola so that triangle ABC formed by the x-axis, p, and q is equilateral. Elaborate on your steps (Note: the slope of p and q is the derivative of the parabola function at P and Q, respectively).

Figure 1: The task given in this study

The task was distributed to the pair (pseudonyms: Alan and Sarah) for two hours through an online meeting platform. They had been informed that the task was related to the derivative of functions. Both students had average mathematics ability and had been exposed to some mathematics-related technology (e.g., *GeoGebra* or *Desmos*) in college. To better understand the role of technology, we asked them to solve the problem without technology in the first session and with technology in the second. One student writes their work digitally in the session while sharing the screen with another student. In the second session, students might choose any technology they wanted and were not forced to stick to a particular technology. This strategy was offered because we wanted to reduce the factor of students' skills in utilizing a particular technology.

Students were encouraged to discuss and question their pair whenever needed. The researcher acted as an observer during the session, yet students were prompted to continue the discussions

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when they were silent for more than a minute. The session was recorded, and students' written works were collected. Semi-structured interviews were conducted to understand some critical episodes in the problem-solving process. The data sources include recordings of meetings, students' written work, and field notes. The analysis was conducted by first identifying a *situation for justification*, i.e., a situation where a claim was questioned and requested validation (Cobb, Yackel, et al., 1992). In each situation, we looked upon who made a mathematical claim and who triggered the justification of it. To further understand the quality of justification, the level of justification that appeared was also coded based on the coding scheme in Table 1.

RESULTS

We present the findings by revolving around describing three situations of justification during the CPS, i.e., situations 1 and 2 in the first session (without access to technology) and situation 3 in the second session (with access to technology). We reported students' mathematical justification levels in each session in a particular situation. The situations presented here did not happen in one period. Yet, we tried to collect related discussion episodes and elaborate on them, focusing on the progression of mathematical justification upon the corresponding claim. The pair initially decided that Sarah would share her screen and writings during the discussion.

Situation 1: a claim that $x_1 = y_1$

At the initial stage of understanding the problem, the pair started to determine the coordinates of points that might be helpful to solve the problem. By analyzing $y = 1 - x^2$, they found that the parabola intersects the x-axis at $D(-1,0)$ and $E(1,0)$ and intersects y-axis at $F(0,1)$. Sarah then continued by stating that the distances from $D, E,$and F to the origin were equal and put marks on the graph (see Figure 2). After considering the fact that $\triangle ABC$ was an equilateral triangle, Sarah inferred that if C was $(x_1, 0)$ then B was $(-x_1, 0)$. This claim was agreed upon by Alan, and he added that A would be $(0, y_1)$. The situation for justification appeared when Sarah claimed that $x_1 = y_1$ and Alan questioned it as follows (the corresponding code was put inside "[ ]").

Sarah : Look, $\triangle ABC$ is equilateral, right? So, this (puts a mark on $\overline{EC}$) would be the same length as this (puts a mark on $\overline{BD}$).
Alan : I think so.
Sarah : It means they will be the same with this (puts a mark on $\overline{AF}$), too. [C]
Alan : How do you know? [T]
Sarah : See here (pointing $\overline{OD}$), here (pointing $\overline{OF}$), and here (pointing $\overline{OE}$) are the same, right? So, this (pointing $\overline{BD}$), this (pointing $\overline{AF}$), and this (pointing $\overline{EF}$) are the same, too. [L2]
Alan : What theory says so? [T] (pause) So, you think $\triangle DEF$ is also equilateral?

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Sarah : I think so. I know it doesn't look like it (is equilateral), but, you know, the picture usually is not real. I mean, they should be the same lengths because the triangles are equilateral. Wait. So, $x_1 = y_1$ here, right? [L2]

Alan : I don't know. (pause) So, we still cannot determine the coordinates of the points (P and Q), right?

Sarah : Yes, I think so.

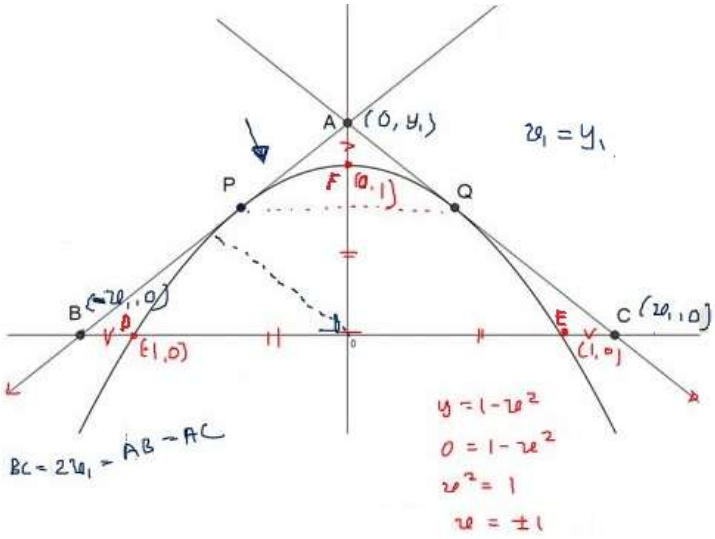


Figure 2: Pair Work in Situation 1

In this situation, Sarah used the fact that $\triangle ABC$ is equilateral and justified that $AF = BD = EC$ by saying that$\triangle DEF$ is also equilateral. Alan triggered the situation of justification by questioning this claim. Sarah at first relied on her claim on the data that $\triangle ABC$ is equilateral, then she pictured $\triangle DEF$ to be similar to $\triangle ABC$ and that led to her claim of $x_1 = y_1$. Although she explicitly stated that "picture usually is not real", she came up with " $\triangle DEF$ similar to $\triangle ABC$" based on how $\triangle DEF$ 'looked' similar to $\triangle ABC$ in the graph, not based on properties of similar triangles. Sarah used the empirical or perceptual demonstration to justify her claim (level 2), while Alan seemed unconvinced and requested a higher level of justification using theorems. However, the pair could not come up with a better justification and tentatively accepted this inaccurate claim. In summary, the claim that $x_1 = y_1$ was mainly justified by the perceived similarity of $\triangle DEF$ and $\triangle ABC$.

Situation 2: claims around the slope of tangent lines

Another situation of justification emerged during the pair's discussion about the slope of tangent lines. The pair immediately realized they could find the slope of the tangent lines from the pa-

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rabola function. They further wrote $y = 1 - x^2$ and discussed to find $m = y' = -2x$ using the derivative rule. After finding the derivative, the pair could not relate it to the two tangent lines having different slopes. This confusion led to a situation of justification where the pair tried to find other information to validate the slope. The following discussions accompanied their work in Figure 3.

Alan : *One tangent line should have a negative slope, and another is positive. But it is the same,* $-2x$. *Maybe we should try another way.*
Sarah : *Okay, but I think it should be about derivative.*
Alan : *What about stating* A, B, C *in terms of coordinate? I mean, using the facts (that* $AF = BD = EC$*).*
Sarah : *What do you mean?*
Alan : *I mean, let's say that distance from* B *to* C *is* a*, and the other two are* a*, too, right?*
Sarah : *Yes*
Alan : *So,* A *is* $(0,1 + a)$ *[C]*
Sarah : *Okay, wait (made a new page and copied the graph from the problem, wrote the coordinates for* $D, E,$ *and* F*). So, you say?*
Alan : A *is* $(0,1 + a)$*, then* C *is … [L0]*
Sarah : C *is* $(1 + a, 0)$ *(wrote the coordinate of C), right?*
Alan : *Yes*
Sarah : B *is* $(-1 + a)$*? [T]*
Alan : *No, it should be minus.* $(-1 - a)$*, it goes to the left, right? It becomes less. [L2]*
Sarah : *Oh, right. (wrote the coordinates of* B*).*
Alan : *I think we can calculate the slope from the points, right?*
Sarah : *Oh, okay. Point* A *to* C *first, ya. So,* $m = \frac{1+a-0}{0-(1+a)} = -1$ *[C]*
Alan : *But, is it true? [T]*
Sarah : *I don't know. Maybe if we see the line from* F *to* E *(further labelled as* q*), it looks parallel to the tangent. [L2] Parallel means equal slope, right?*
Alan : *Hmm, I think so.*

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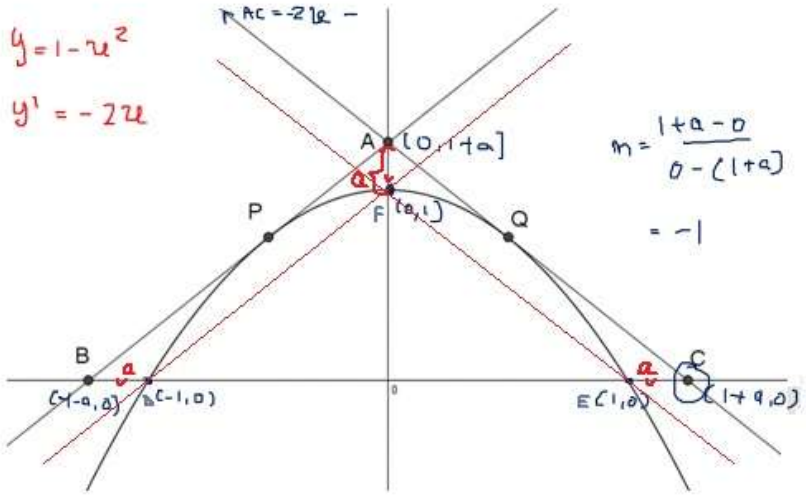


Figure *3*: Pair Work in Situation 2

The important justifications leading to the claim in Situation 2 were on the coordinates of $A, B,$ and C and the parallelism of line q and the tangent. Both justifications used perceptual demonstration (level 2), which was then triggered for better justification by Alan. The trigger resulted in the acceptance of the claim without any refined justification.

From the discussion in situations 1 and 2, it could be seen that the pair justified claims mainly using empirical or perceptual demonstration (e.g., "if it goes left, it becomes less" and "it looked parallel to the tangent"). The first claims on $A, B,$ and C coordinates were continued from Situation 1, where they inferred that the length of AF equals BD and EC. The pair accepted the justification based on perceptual and empirical demonstration as they identified the coordinates of $A, B,$ and C under a similar justification in situation 1. This accepted claim led to another claim of $m_q = -1$, which they accepted based on a perceptual demonstration that the line looked parallel to the line q passing through E and F.

Along the discussion in two situations, we see how Alan's request for better justification in Situation 1 was slowly compromised once the pair implicitly accepted the claim based on empirical or perceptual demonstration. Alan and Sarah each gave a trigger for justification of a particular claim. Yet, it was shown that Sarah's empirical or perceptual demonstration level of justification was neither refuted nor revised during the discussion. The inaccurate claim accepted by the pair in situation 1 became the underlying "agreement" in understanding new data brought into the discussion and further into accepting new claims.

The pair then continued the process by discussing that due to $m = -2x$, $m_q = -1$ and the fact that q and the tangent line have equal slope, they inferred that x as the abscissa for the coordinate of Q is $\frac{1}{2}$ and that of P is $\frac{-1}{2}$. They did not finish until they got the complete coordinates of P and

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Q due to the lack of time for the first session. The discussion was continued by allowing them to access technology in the second session.

Situation 3: claims around the coordinate of P

We discussed students' work in the second session, focusing on Situation 3, containing claims around the coordinate of P. At the beginning of the second session, we prompted the pair to access any technology they wanted to solve the problem. They were told they could do different strategies or start from any step at the first session. The pair decided to use *Desmos* to verify the abscissa of P that they found in the first session (Figure 4). Alan decided to share his screen. Alan first input the function $f(x) = 1 - x^2$. This input created the graph of $f(x)$ in the screen. As they understood that the slope of the tangent is the derivative of the function, they decided to use the *Desmos* feature (i.e., Calculus: Tangent Line) to find the slope (shown in Figure 4 as the red line $g(x)$). Alan initated to create a slider labelled as a to act as the dynamic value of x. He then dragged the slider a to change the values of $P(a, f(a))$. The following excerpt revealed the discussions using *Desmos* that their previous claim of the coordinate of P was not accurate.

Alan : *We can drag this (the slider in Desmos) to change the intersection point.*
Sarah : *Oh, okay*
Alan : *Let's try a half. Or a negative half?*
Sarah : *Okay*
Alan : *Here it is. This is the answer (Desmos window shown in* Figure 4*). So P is* $\left(\frac{-1}{2}, \frac{3}{4}\right)$ *and Q is* $\left(\frac{1}{2}, \frac{3}{4}\right)$ *because it is just positive here.*
Sarah : *Okay, the lengths are the same, ya.*
Alan : *Yes. Here (pointing to the x and y intersects of the tangent), we get the same lengths. Mm, it is around* $1\frac{1}{4}$ *both, right?*
Sarah : *Wait, no, Alan. It is not equilateral. You see, the one below (segment BC) will be twice. [T]*
Alan : *Oh, right.*
Sarah : *... and we think about the sides, right? Not the height.*
Alan : *Oh.*

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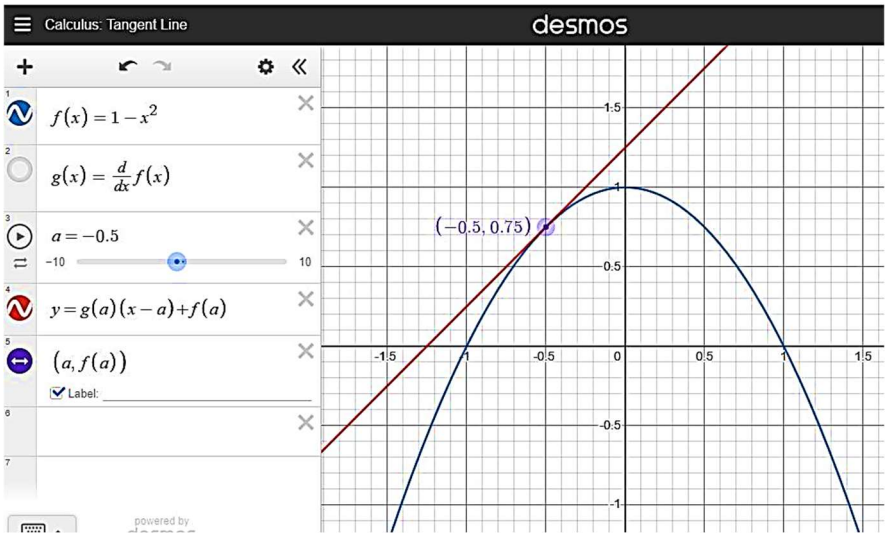
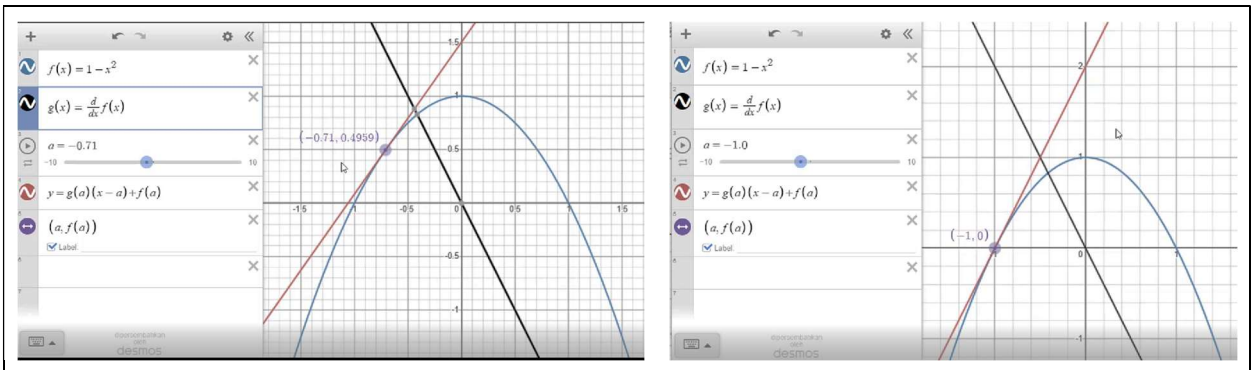


Figure 4: Initial Desmos Entry

While utilizing the slider, the pair initially slide it to a value of 0.5. This input was intended to check their previous answer of the value of x. After the graph by Desmos was shown, the pair realized that the coordinates obtained in the previous situation were incorrect. Alan then suggested to change the value of a by sliding the slider (Figure 5). They gradually changed it into smaller values, such as 0.7 and -1. After the repeated trials, the pair did not seem to find the value they wanted to create the equilateral triangle ABC by utilizing the slider alone.



Note: in these screens, they did not hide $g(x)$, so it was shown as the black line.

Figure 5: Some trials using the slider

Their repeated trials in using the slider led to different ways to analyze the data given by the problem. Sarah inferred that it might be useful to draw triangle ABO as part of the equilateral triangle ABC (Figure 6, left). By considering the angles of the triangle, they explored where P should be located using Desmos (Figure 6, right). They concluded that P should be somewhere

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close to $(-0.86,0.26)$ and thus Q would be around $(0.86,0.26)$, which is very close to the actual correct answer. The following excerpt shows how they justified this answer.

> *Sarah : I think it is true. See that the equilateral triangle (pointing to the triangle drawing) should have angles like that, right? And the graph (in Desmos) is just the same. [L3]*
> *Alan : Yeah. I think it would be true if we calculate the sides using trigonometry [L4].*
> *Sarah : I think so. It (the graph) didn't look like the question, though.*
> *Alan : Yes, it is steeper. I think I can picture the equilateral triangle in my mind (when I see the graph), now [L2]*

We saw that the justifications in situation three were refined, following the fact that their answer from the previous situations was inaccurate. The slider feature was shown to be helpful in helping them testing several values of x to satisfy the objective of the problem, despite not providing them with the answer. The pair's effort to use a drawing of an equilateral triangle as an example (level 3) to justify the equilateral triangle in the graph was shown to lead them in the right direction. The use of example was triggered by the verification made in the previous Desmos entry, showing that reliance on perceptual demonstration was insufficient. The pair, initiated by Alan, even offered a higher level of justification to use trigonometry (level 4) to justify the sides of the triangle and further validate the coordinates of P and Q, which unfortunately they did not do. They also realized that they could not base their discussion on the graph provided by the problem alone because it did not visualize the correct answer. In this situation, the justification of perceptual or empirical demonstration accompanied another justification after the pair realized its inadequacy.

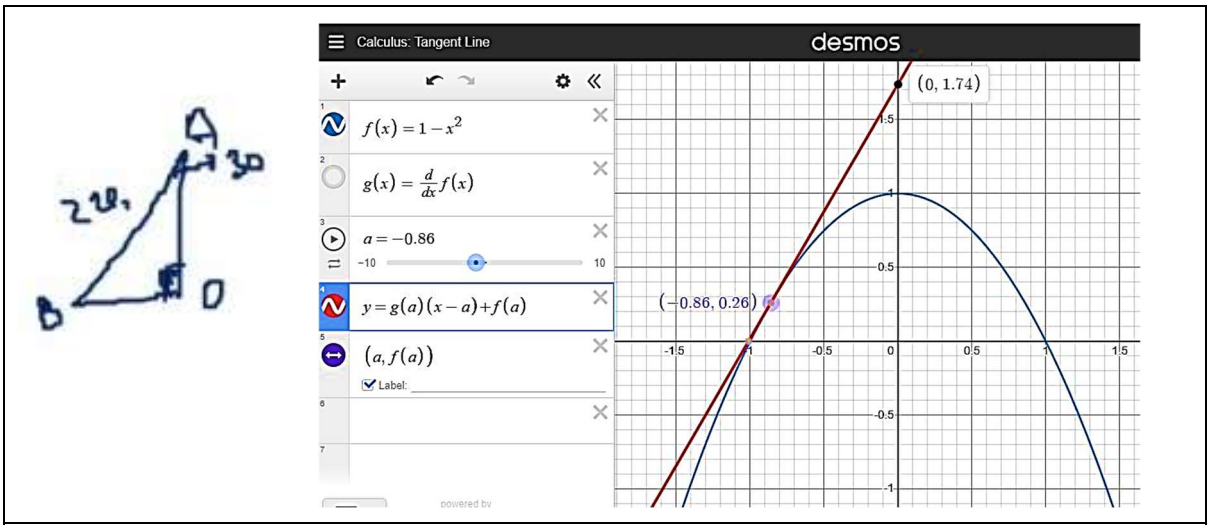


Figure *6*: Triangle drawing and Final Desmos Entry

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DISCUSSIONS AND CONCLUSIONS

The quality of mathematical justifications brought by the pair could be seen from the three situations of justification found in the study. We noticed various mathematical justifications during the first and second sessions. In the first session, we observed that claims were questioned, and perceptual or empirical demonstration type of justification was mostly offered, similar to findings by other researchers (Chazan, 1993; Sowder & Harel, 1998). We considered their reliance of perceptual or empirical demonstration was not due to the absence of trigger, a question or request of better justification, but due to the inability to fulfil such trigger. For instance, when Alan requested justification based on theories in situation 1, the pair failed to entertain such request for a higher level of justification. It led the pair to accept the claim tentatively and continue implementing strategies on inaccurate claims.

In the second session, we observed that the pair used the technology to verify their solution rather than to start working on it from the beginning. Once they realized that their previous solution could not be justified, they started to work on the initial information of the problem, such as the function of the parabola and its tangents. It was also observed that triggers for mathematical justification appeared more in the first session when the pair did not have access to technology. It raised more questions about whether the pair treated the technology itself as a trigger or justification (Hollebrands et al., 2010) so that the claims shown being valid by technology did not need further justification.

The pair used the technology to validate claims, yet it was shown that it did not directly help them improve the claims or give better justification. We believed that the offer for justification in situation 3 (when Alan wanted to use trigonometry to verify the solution) could indicate that technology promoted students to give better justification only when they understood how they came up with their claims and how their previous justification might not be enough. Nevertheless, the findings reiterated the fact that the use of technology could improve the problem solving process (Berrin et al., 2024; Nguyen et al., 2023) and accommodate students' collaboration, especially in testing their ideas (Olive et al., 2010), illustrating multiple cases to verify claims (Erbas et al., 2020) and being more responsible for their thinking (Buteau & Muller, 2006).

Another notable finding was the different use of graph or picture as justification between the first and second sessions. In situation 3, the pair seemed convinced by the graph they constructed on *Desmos* and used it as a joint justification with the triangle they made. Going back to situation 1, similar confidence did not appear on the use of a similar graph, as shown by a request for better justification. Considering this fact, we inferred what students meant by “picture” when they stated “the picture usually is not real” was not a picture constructed by a technological tool. Pictures or graphs resulting from a technological tool somehow gave them more plausibility. This finding was similarly highlighted by Zhen et al. (2016) who found that students might be unconvinced by merely a graphical perception, but they somehow believed arguments coming from a graph satis-

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fying a certain property (e.g., when it was constructed by technology). Teaching practices could benefit from this finding. Teachers might use technology as a tool to challenge students' beliefs on the truth of a claim, for example, by comparing a hand-drawn geometrical figure with the one being constructed by technology. Questions such as, "why do you think this picture seems more convincing than the one you draw in the paper?" or "Can you convince me that the technology is showing the correct picture?" could intrigue students into thinking about the reason behind their belief and at the same time allow them to critically examine outputs of technology they used.

The study allowed us to answer some questions on students' mathematical justification. The task administration to the pair without the lecturer's presence avoided the question of who would take justification as valid besides the students since the community involved was only them (Karunakaran & Levin, 2022). However, it also raised a question, as in a natural classroom setting, it was not typical to have lecturers uninvolved in any of their students' problem-solving processes. The nature of the task that could be solved without access to technology brought two concerns. First, it allows for comparing students' justification with and without technology in students' typical context, i.e., choosing their own preference, which should be taken into account if we want to explore the use of technology (Sanchez, 2020). Second, it probably diverges our observation of whether the pair's mathematical justifications were due to the task or the presence of technology itself. Students' prior exposure to the technology might contribute to how far they have solved the problem and justified the solution. It also might bring different data into the table if they decided to use a different software with different features. We also considered the possibility of even further justifications using the technology, which the pair could not make, probably due to their skill in utilizing it.

We acknowledge that the difficulty level of the task was not explicitly considered in our study, and this may have influenced the students' reliance on technology during the second session. Our focus was primarily on ensuring that students encountered no issues with the use of technology and that the task was capable of eliciting diverse mathematical justifications. However, we recognize that this could be a limitation of our research, in line with the findings of Fatmanissa et al. (2024), which indicate that the observed collaboration and justifications were influenced by an appropriate level of task difficulty. Therefore, we suggest that future research should consider designing tasks that strike a balance between difficulty and accessibility, ensuring that technology functions as a supportive tool for enhancing conceptual understanding rather than replacing students' critical thinking processes.

A study that utilizes a task that explicitly includes instructions on technology to promote mathematical justifications during CPS could be conducted in the future. Such instructions might guide students transitioning from an empirical to a more deductive justifications, for example by asking students to add supporting arguments other than relying solely on the graph to their claim of the similarity of $\triangle DEF$ and $\triangle ABC$. Such actions, called *elaborating* by Zazkis et al. (2016), encour-

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age students to not take empirical justification for granted, add supporting data, and construct more formal arguments.

This study was small in size to allow for an in-depth analysis of students' mathematical justifications, yet it is worthwhile to be developed further. An experimental study with a larger sample comparing students' mathematical justifications with and without access to technology could refine the findings of this study. The diversification of participants' mathematical or technological skill levels could also improve our understanding upon this issue. Another limitation in this study is the limited range of software tools examined due to our approach of allowing students to use a familiar software i.e., Desmos. However, this focus restricted our ability to compare the effectiveness of different software platforms in supporting mathematical justification. Future research could investigate how different software platforms might influence students problem-solving approaches and results.

ACKNOWLEDGMENTS

The authors would like to thank the School of Postgraduate Studies, State University of Surabaya, for the valuable discussions upon initial drafts of the paper, Universitas Muhammadiyah Malang, Indonesian Education Scholarship (*Beasiswa Pendidikan Indonesia*), Center for Higher Education Funding and Assessment (*PPAPT Kemdiktisaintek*), and Indonesia Endowment Funds for Education (*Lembaga Pengelola Dana Pendidikan*).

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MATHEMATICS TEACHING RESEARCH JOURNAL 201
SUMMER 2025
Vol 17 no 3

Challenges and Opportunities: A Comparison of PISA 2022 Mathematics Performance in Malaysia and other ASEAN Countries

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Abstract: Malaysia recorded the largest decline among other ASEAN countries in the 2022 PISA mathematics performance. Although the COVID-19 pandemic may have contributed to this poor mathematics performance, previous PISA trends have shown that Malaysian students were already struggling in mathematics before the pandemic. Given this pressing concern, this research aims to compare Malaysian PISA mathematics results with those of other ASEAN countries to identify the challenges Malaysia faces and discover strategies that can be adopted from others. The findings reveal that nearly 60% of Malaysian students failed to meet the baseline of mathematics proficiency levels, and only 1% reached the high-performing level. Such results reflect that a large proportion of students in Malaysia struggles with basic mathematics. This study proposes several strategies for improvement. These include adopting Singapore's subject-based banding system to give equal opportunities and a flexible learning environment for students from all levels; continuous improvement of professional development for teachers by adopting the Singapore SFEd roadmap and the World Bank teacher development roadmap; and incorporating PISA-released items into classroom practices to familiarize students with the assessment structure and problem-solving strategies. However, further investigation is required to confirm the effectiveness of the proposed implementations.

Keywords: PISA-released items, mathematics proficiency, COVID-19 pandemic, ASEAN, Malaysia, subject-based banding system, continuous professional development

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INTRODUCTION

PISA is an international programme administered by the Organisation for Economic Co-operation and Development (OECD). The main objective of PISA is to understand how well 15-year-old students perform in mathematics, science, and reading. The uniqueness of PISA mathematics assessment lies in its non-curriculum-based assessment approach, which looks at students' ability to apply their mathematics knowledge to real-life situations rather than assessing their memorisation of formulas and algorithms.

The first PISA assessment occurred in 2000. Initially, only 43 countries participated in the PISA assessment, including two ASEAN countries: Indonesia and Thailand. Since then, the assessment has taken place every three years. However, the eighth cycle of PISA, which was supposed to be conducted in 2021, was postponed to 2022 due to the disruption caused by the COVID-19 pandemic. However, it has increased to 81 countries participating in PISA 2022, including eight ASEAN countries. These ASEAN countries are Brunei Darussalam, Cambodia, Indonesia, Malaysia, Singapore, Thailand, Vietnam, and the Philippines. Notably, this was the fifth cycle of participation for Malaysia since 2009.

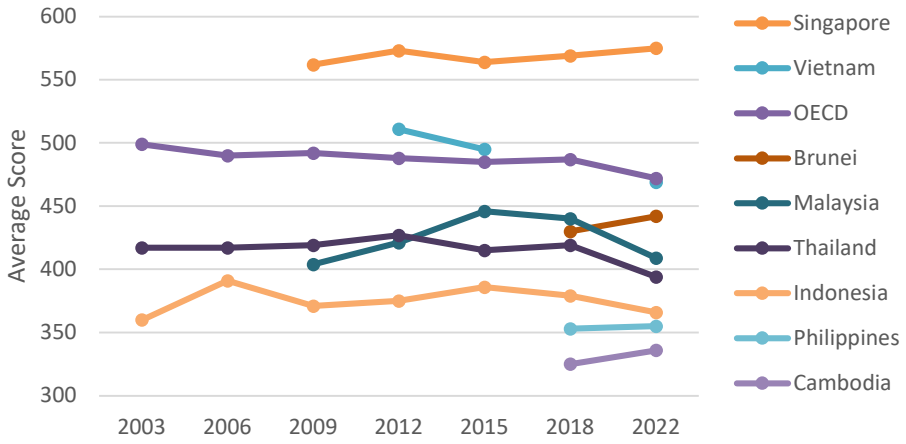


Figure 1: Trends in PISA mathematics achievement among ASEAN countries from 2000 to 2022 (Data: OECD, 2023)

Since the first participation in 2009, Malaysia's 15-year-old students have not achieved above the OECD average in mathematics. Among the participating ASEAN countries, only Singapore has always exceeded the OECD score, as depicted in Figure 1. More critically, Malaysia had the largest decline among other ASEAN countries in its most recent PISA mathematics performance. In addition to PISA, Malaysia also showed a poor performance in Trends in Mathematics and Science

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Study (TIMSS). The recent TIMSS results also showed that the average mathematics score of Malaysian Grade 8 students falls below the international intermediate-level standards (Mullis et al., 2020). Furthermore, using the measurement where 20 points equals about one year of schooling, the findings showed that Malaysian student performance was 3 years behind Vietnamese students and 8 years behind Singapore students in mathematics, a*lthough Malaysia consistently ranks among the top ASEAN countries in investing the highest percentage of its Gross Domestic Product (GDP) in education* (Lynn & Becker, 2019; Avvisati & Givord, 2023; Schleicher, 2023). Other indicators, such as the average IQ administered by Lynn & Becker in (2019), also reported that Malaysians had lower intelligence than people from Singapore, Thailand, Vietnam, and Cambodia.

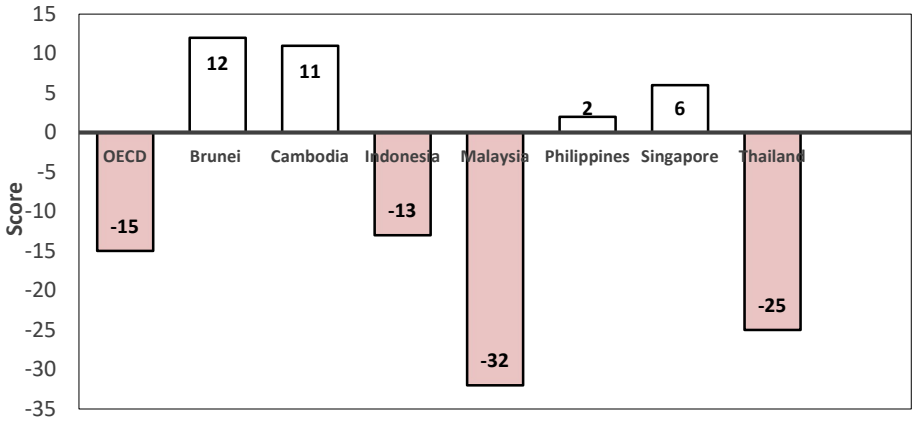


Figure 2: Trends in PISA mathematics scores among ASEAN countries from 2018 to 2022 (Data: OECD, 2023)

According to the Malaysia Education Blueprint 2013–2025, the nation aspires to rank in the top third quarter of countries in international comparative studies such as PISA and TIMSS by 2025 (Ministry Education of Malaysia, 2018). However, based on the most recent scores in the PISA and TIMSS assessments, Malaysia is still too far from realising this vision. The COVID-19 pandemic may have contributed to poor mathematics performance in Malaysia, but it is not the main factor. Based on previous findings, Malaysia already had long-standing educational issues before the COVID-19 pandemic, such as students with varying mathematics abilities in large classes, inconsistent teacher acceptance in professional development programs, and a lack of high-level cognitive mathematics problems in textbooks (Abu Hassan & Ajmain, 2022; Sianturi et al., 2021; Yaakob et al., 2020). As illustrated in Figure 2, despite the effect of the pandemic on students, it is important to note that some of the ASEAN countries improved in PISA 2022, such as Brunei, Cambodia, the Philippines, and Singapore, in their 2022 PISA math scores. Although Indonesia, Thailand, and Vietnam also experienced a decline in their mathematics scores, the decline was not

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as steep as that of Malaysia. In addition, Figure 1 shows that Malaysian students were already performing poorly before the COVID-19 pandemic began. This clearly indicates that although the pandemic has intensified the situation, long-term issues in the education system has also contributed to the decline.

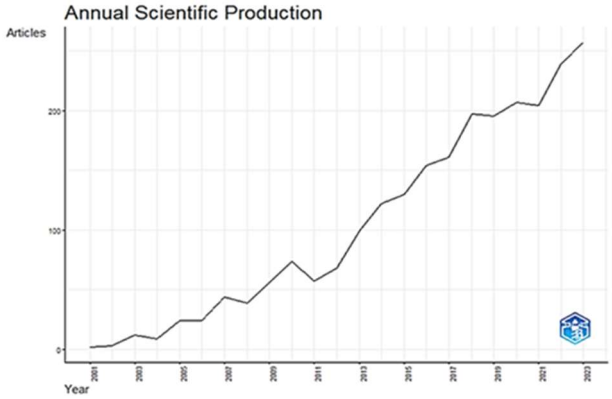


Figure 3: Annual Scientific Publication of PISA from 2001 to 2023 (Data: Scopus, 2024)

Given these pressing concerns, this research aims to compare Malaysian PISA mathematics results with those of other ASEAN countries. It also aims to identify the common issues Malaysia faces and discover the strategies we can apply from other successful ASEAN countries. Thus, this study intends to answer the following research questions: a) How did the Malaysian students perform in the PISA mathematics results compared with those of the ASEAN countries? b) What challenges do Malaysia face in achieving higher PISA mathematics proficiency? c) What are the strategies that Malaysia can apply from other successful ASEAN countries to improve PISA mathematics results? The key contribution of this study is to understand the challenges Malaysian students face in their persistent low performance in PISA. Based on comparisons with other ASEAN countries, the study seeks to understand the successful strategies they have used in their countries that Malaysia can adopt.

LITERATURE REVIEW

PISA Data for ASEAN Mathematics Improvement

Throughout the years, researchers have increasingly used PISA data to analyse the mathematics performance of countries participating in the study. Figure 3 illustrates this trend, in which a growing number of scientific articles on PISA were published annually and indexed in the Scopus database from 2001 to 2023. Various methodologies have been used to compare PISA results

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across countries. One of them is the descriptive approach. Many research papers use descriptive statistics before conducting more advanced statistical analyses (Kandeel, 2021; Kim & Kim, 2023; Rowley et al., 2019). By exploring the data with descriptive measures, researchers can better identify student performance patterns and trends across countries before moving into more in-depth investigations. For instance, Rowley et al. (2019) used descriptive analysis to compare mean achievement scores in various countries to identify patterns and differences. In addition, Rowley et al. (2019) used descriptive analysis to compare high and low-socioeconomic students to identify achievement gaps.

Beyond comparative studies, many researchers have used PISA-released items to improve their countries' educational systems. For instance, in Singapore, PISA results are not only data resources to examine the effectiveness of its educational system, but the PISA-released items are also used to revise their education assessment model (Wu et al., 2020). Apart from Singapore, Indonesia has also been active in developing PISA-like items with the aim of improving mathematics results. They emphasise connecting mathematics tasks to real-life contexts, such as cultural heritage, sports, pandemics, and other everyday life situations (e.g., Aini et al., 2023; Putri & Zulkardi, 2020; Wulandari et al., 2024). Research has indicated a significant improvement in mathematics literacy for students who practiced PISA-like tasks compared to the group of students who did not (e.g., Almarashdi & Jarrah, 2022; Efriani & Putri, 2019). Thus, Malaysia mathematics education can also benefit from the PISA-released items by ensuring that they are relevant to culture and local contexts.

It is worth mentioning that Singapore has integrated Cognitive Diagnostic Models (CDMs) with PISA items to monitor its educational assessment structure and students' cognitive abilities (Wu et al., 2020). CDMs not only give detailed diagnostic information about the students' specific strengths and weaknesses in their cognitive abilities but also enable the teachers to know exactly where the students are weak or where they require improvement (Williamson, 2023). As a result, their national mathematics assessments are always consistent with international standards. Malaysia's educational assessment can benefit from the CDM approach, but its implementation must consider teachers' and educational assessment system readiness.

In contrast, Malaysia's overall mathematics cognitive assessment is more general and lacks in-depth analysis. In 2011, Malaysia introduced school-based assessment (PBS) to reform its education policy (Ministry of Education Malaysia, 2025). PBS comprises two components: academic and non-academic. The academic component includes central (PP) and classroom-based (PBD) assessments. Central assessment is a comparable summative assessment used to evaluate students' learning outcomes at different academic levels. For instance, the Malaysian Certificate of Education (SPM) is administered to Grade 11 students in Malaysia as a national examination to

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determine their future academic paths (Ministry of Education Malaysia, 2025). Classroom-based assessment, on the other hand, consists of both formative and summative assessments (Ministry of Education Malaysia, 2025). An online integrated assessment management system is available to key in and generate a PBD report. However, compared with Singapore CDM, the analysis and assessment report lacks the in-depth details needed to diagnose the specific mathematics cognitive skills that students struggle with. Thus, there is a need to refine the cognitive assessment system and develop a more advanced and practical mathematics cognitive diagnostic assessment framework (Chin et al., 2019).

Mathematics Learning Support in ASEAN

In addition, the Ministry of Education (MOE) in Singapore has introduced ongoing projects to support children with different mathematical learning disabilities. An example of such successful projects is the Learning Support for Mathematics (LSM) program. At the beginning of primary school, children in Singapore must take a screening test to identify whether they have mastered the basic mathematics skills. For children who did not pass the screening test, it is necessary to join the LSM program (Ministry of Education Singapore, 2024). The LSM program is conducted for the selected children in four to eleven sessions weekly during regular school hours (Ministry of Education Singapore, 2024). It is conducted in small groups to ensure each child receives personalised attention while building their basic numeracy skills (Yeo & Cheng, 2021). Once these children meet specific criteria, they are discharged from the LSM program. Hence, early intervention not only reduces the number of innumerable children in the beginning stage of mathematics learning, but also supports children from low-income families who have limited support from their parents when doing homework (Ang et al., 2019).

Such interventions are also available in Malaysia. In 2010, Malaysia started the Literacy and Numeracy Screening (LINUS) program in primary schools to ensure all students acquire basic literacy and numeracy skills in their first few years of primary school (Ministry of Education Malaysia, 2018). The LINUS program was then replaced with the Primary Literacy and Numeracy (PLaN) program in 2019 (Ministry Education of Malaysia, 2023). Although both projects focus on the literacy and numeracy of primary school students, PLaN is more decentralised and school-based, which enables schools with autonomy to manage their own literacy and numeracy issues (Ministry Education of Malaysia, 2023). In addition, the Malaysia MOE introduced another intervention in 2024 to detect literacy and numeracy in primary 1 students early in 2024. It takes three months for the screening process and another three months for the intervention program (Rajaendram, 2024). As stated by the Education Director-General, Azman Adnan, this group of

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students will continue in this program if, within three months, they do not master literacy and numeracy (Rajaendram, 2024). They will only return to the original class once they have achieved the expected level of proficiency within the time frame.

Unlike the Singapore LSM program, which is integrated into the mainstream mathematics curriculum, Malaysia numeracy support is intensive. However, the students are separated from the mainstream curriculum for at least three months, which may further widen their mathematics gap and affect their social reintegration with normal class students.

Professional Development for Teachers in ASEAN

Another factor contributing to Singapore's outstanding PISA results is its special attention to the life-long learning of teachers. Under the initiative of Continual Professional Development (CPD), Singapore MOE has long offered 100 hours of fully sponsored CPD credits to all teachers annually. Teachers can use these 100 CPD hours during or after working hours, depending on their conveniences. They can update their existing knowledge and skills through attending courses, doing research projects, joining a networking group, and any other training offered in school or outside the school (Stewart, 2018).

Beginning in 2020, Singapore MOE launched the SkillsFuture for Educators (SFEd) roadmap to guide teachers in optimising and planning their 100 hours of CPD. Mrs. Chua-Lim Yen Ching, the Singapore Deputy Director-General of Education (Professional Development) in 2020, pointed out that SFEd guides teachers to identify their competency gaps and then plan for the professional development that they really need. After surveying teachers and experts about the kind of training they require to cope with the rapidly changing needs of learners, Singapore MOE focused on the six priority areas of practice that every teacher should focus on in their professional development over a five-year period. These areas include assessment literacy, differentiated instruction, special educational needs, inquiry-based learning, the use of technology for learning and character and citizenship education (Schoolbag the Education News Site, 2020). As a result of the continuous implementation of sustainable professional development for teachers, Singapore has managed to maintain its position as the top country in international student assessments, such as PISA and TIMSS.

Similarly, other ASEAN countries have also created different professional development plans for teachers. Despite this, it is not sufficient and adequate if the programs do not account for factors like teacher workload, time constraints, teacher's existing knowledge and equity, which provides individualised support and culture of growth in teachers (Fletcher-Wood, 2020; Kilag et al., 2023;

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Senawongsa et al., 2023). For instance, teachers in the Philippines reported that most professional development training was contextually irrelevant and had low applicability to practice (Bonghanoy et al., 2019). Malaysian research has also shown a mixed level of teachers' satisfaction with the professional development program compulsory for them to attend. It suggested that different teachers have different preferences and needs for professional development. For instance, Gen X (born 1964–1981) teachers were less interested in professional development that involved research activity, while Gen Y teachers (born 1982–1994) preferred to acquire information immediately and did not like top-down communications (Yaakob et al., 2020). Also, many in-service teachers in East Asia and the Pacific region have insufficient knowledge of their subjects leading to improper teaching (Afkar et al., 2023; Kurniasih & Hidayanto, 2022).

Recognising all these challenges, Afkar et al. (2023) have introduced a teacher professional development roadmap specifically for the middle- to low-income countries in East Asia and the Pacific region that face literacy and numeracy issues. The study revealed that many in-service training programs in these countries do not incorporate the four essential elements of effective teacher training programs. These elements include a focus on content knowledge, opportunities to practice what is learned with colleagues, continued support through follow-up visits focused on training content, and career incentives through promotion or increased salary (Afkar et al., 2023). By improving the quality of professional development, students' international assessment outcomes can also be improved. Again, the effectiveness of professional development for teachers depends on well-structured and thoroughly researched programs that meet teachers' specific needs. Otherwise, many teachers attend professional training programs just to fulfil the required credit hours but not to truly improve their teaching skills.

METHODS

This study used the PISA online database from 2000 to 2022. This study focuses on comparing mathematical scores across ASEAN countries using descriptive methods. The descriptive methods used in this study were mean score comparison, stacked column chart, and grouped histogram analysis. Mean scores across proficiency levels were used to compare the distribution of mathematics proficiency among ASEAN countries. The mean scores illustrate the trends over multiple PISA cycles across ASEAN countries. Stacked column charts were used to compare the percentage distribution of proficiency levels among ASEAN countries. By comparing the percentages in stacked column charts, we can easily see the proportion of students at each level for each ASEAN country. Grouped histogram analysis was used to understand the distribution shape

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and skewness of the data across ASEAN countries. By observing the distribution shape and skewness of the overall ASEAN countries, we can detect at which proficiency level they cluster.

According to the National Center for Education Statistics (2022), the PISA assessment comprises one major domain, two minor domains, and one optional domain. The major domain for the 2022 PISA cycle was mathematics literacy. The two minor domains were science literacy and reading literacy, while the optional domain was financial literacy. The mathematics literacy instrument consisted of 160 new items and 74 existing items, which took more than 6 hours for a student to answer all of them. Hence, PISA used a hybrid multistage adaptive test design (MSAT) to divide items into smaller test booklets. This helped to shorten the duration of answering the PISA items. Each student took only 60 minutes to answer 28-30 items for the mathematics literacy assessment.

The target population of this study includes the 15-year-old students from eight ASEAN countries who took part in the PISA assessment. The cohort comprised 7069 students from 199 schools in Malaysia, 6606 students from 149 schools in Singapore, 8495 students from 279 schools in Thailand, 13439 students from 410 schools in Indonesia, 7193 students from 188 schools in the Philippines, 5279 students from 183 schools in Cambodia, 5576 students from 54 schools in Brunei Darussalam, and 6068 students from 178 schools in Vietnam. The rationale for choosing these ASEAN countries, but not Myanmar and Laos, were their participation in multiple PISA cycles.

RESULTS AND DISCUSSION

Distribution of Mathematics Proficiency across ASEAN Countries

Students' performance in the PISA mathematics assessment is categorised by proficiency levels. Previously, the scale had seven proficiency levels, ranging from below Level 1 to Level 6. However, from 2022 onward, the scale has expanded to nine levels. Students are subdivided into three levels at Level 1, with 1a being the lowest, followed by 1b and 1c. Students in Level 2 are only considered to have achieved the basic level of proficiency. Levels 3 and 4 are classified as intermediate levels, while Levels 5 and 6 are grouped as advanced levels. Hence, the new adjustment in 2022 gives a more detailed view of students below the level of basic mathematics proficiency.

Figure 4 shows the distribution of students across different mathematics proficiency levels in ASEAN countries that participated in PISA 2022. Singapore has a significant proportion of students with higher proficiency levels. This indicates that most Singaporean students have strong mathematical skills. On the other hand, other ASEAN countries, such as Malaysia, Indonesia, Cambodia, Thailand, and the Philippines, share a common issue of having a higher proportion of

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students with lower proficiency levels than at any other level. This phenomenon implies that most students in these countries still struggle with basic mathematics.

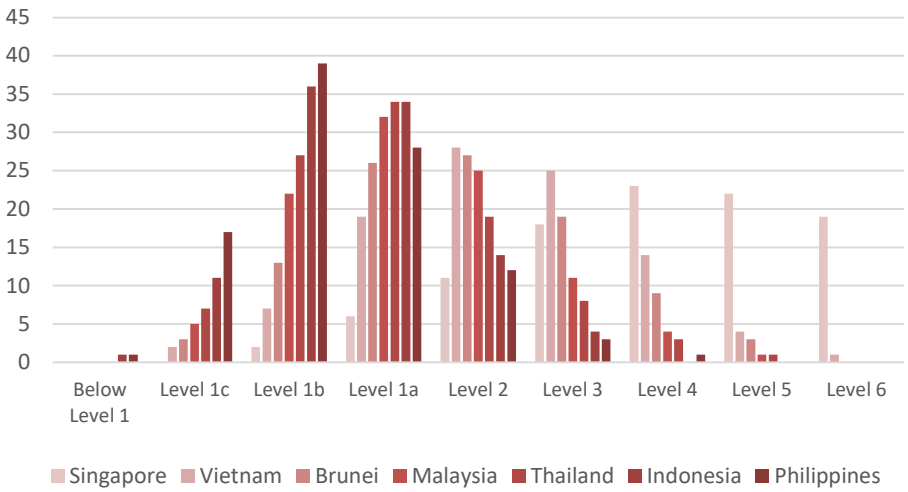
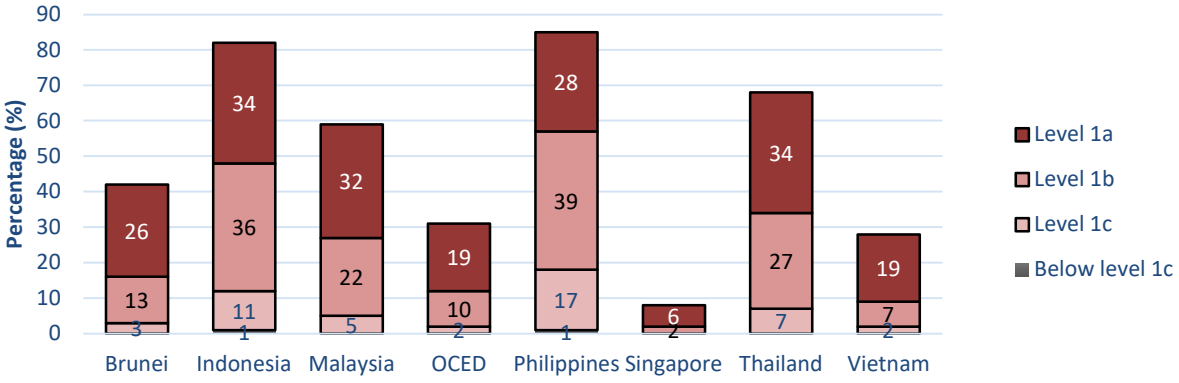


Figure 4: Mathematics proficiency levels of students across ASEAN countries in PISA 2022 (Data: OECD, 2023)

Based on the proficiency standard of OECD (2023), 15-year-old students who scored below Level 2 in the PISA mathematics assessment were regarded as low achievers. The United Nations' Sustainable Development Goals, which were launched in 2015, set a target whereby all teenagers worldwide should be able to achieve at least Level 2 in mathematics by the end of lower secondary education (United Nations, 2024). If they fail to do so, it becomes very challenging for them, whether in their further studies or competing in the workplace, both of which require some basic mathematics skills.



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Figure 5: Percentage of students below basic proficiency levels across ASEAN countries in PISA 2022 (Data: OECD, 2023)

In PISA, students who score below Level 2 typically struggle to design simple problem-solving strategies (OECD, 2023). Such problem-solving strategies involve tasks like running direct simulations with one variable, extracting key information from one or more sources that use slightly more complex modes of representation, such as two-way tables and charts or two-dimensional representations of three-dimensional objects; understanding basic functional relationships and simple ratios; and making direct interpretations of results (OECD, 2023).

To illustrate how alarming this issue is, Figure 5 provides a closer view of the percentage of students who performed below the basic proficiency levels in ASEAN countries. Most ASEAN countries, such as Malaysia, Thailand, Indonesia, and the Philippines, struggle with higher percentages of students not meeting these standards. In particular, Malaysia is in a very alarming position where 59% of students cannot even achieve minimum proficiency levels, a situation that has worsened by 17% since 2018 (OECD, 2019). Compared with Indonesia, Thailand, and the Philippines, Malaysia has the largest increase in PISA from 2018 to 2022. On the contrary, only 8% of Singapore's students and 28% of Vietnam's students perform below this proficiency level. An important point is that although Vietnam has a lower GDP (gross domestic product) per capita than Malaysia, it has 31% fewer students falling below basic proficiency.

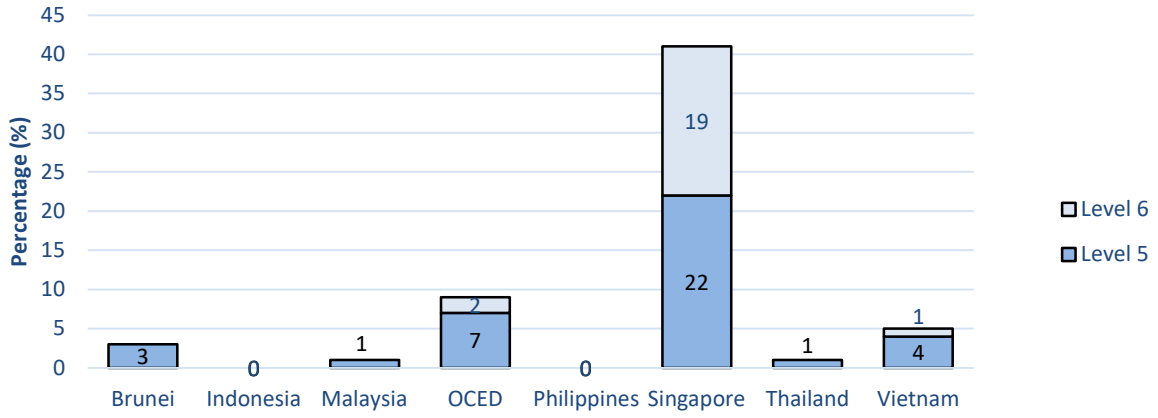


Figure 6: Percentage of students with high PISA mathematics proficiency levels across ASEAN countries in 2022 (Data: OECD, 2023)

In addition, Figure 6 shows the percentage of students in ASEAN countries that achieved high mathematics proficiency levels (Levels 5 and 6) in the PISA assessment. Singapore had 41% of its

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students at these levels. This indicates that Singapore has far more top performers (Level 5 or 6) than the international average of 9%. On the contrary, the rest of the ASEAN countries, especially Malaysia, Indonesia, Thailand, and the Philippines, scored far below the OECD average, with very few students being able to achieve a high level of proficiency. Only 1% of Malaysian students achieved Level 5, and none achieved Level 6. This implies that there could be serious issues within Malaysia's educational system that prevent the development of highly skilled students. In the long run, if these issues remain unresolved, a shortage of highly skilled workers in Malaysia will ensue. This will then force Malaysia to rely on expatriates from other countries.

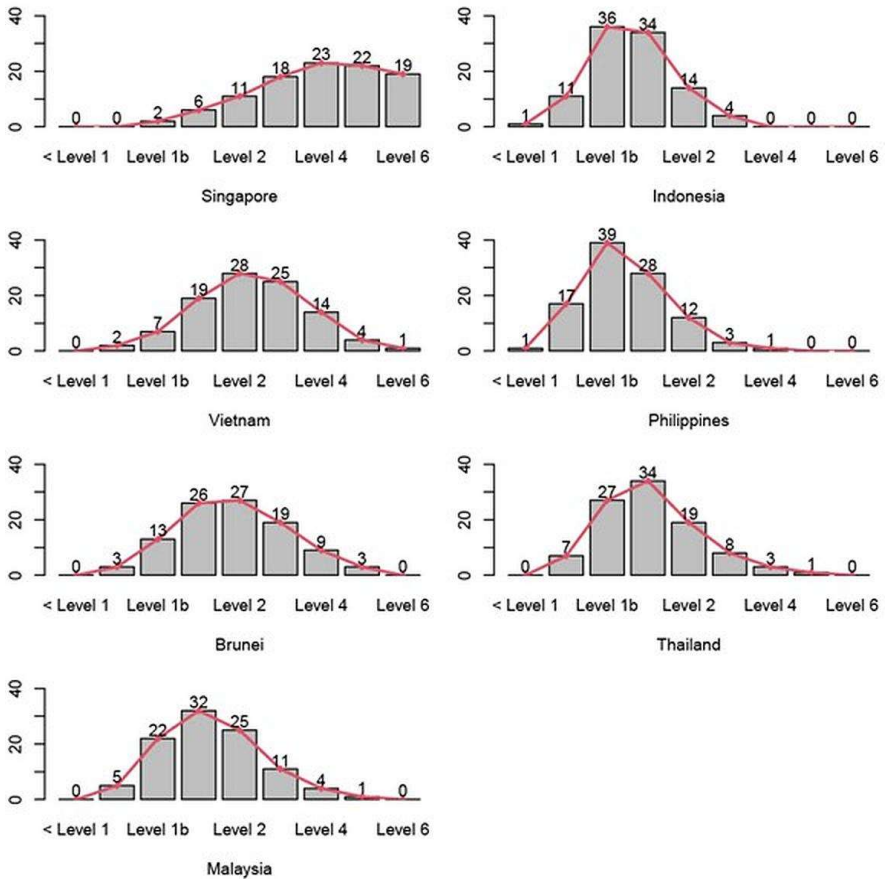


Figure 7: Distribution of PISA 2022 Mathematics Proficiency Levels in ASEAN countries. (Data: OECD, 2023)

Figure 7 shows the percentage of students at each proficiency level, ranging from 'below level 1' to 'level 6', for each of the ASEAN countries that participated in PISA 2022. In Figure 7, it can be

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observed that the distribution of students in Singapore is positively skewed. This means that most students are highly proficient in mathematics. In comparison, Vietnam and Brunei have a distribution pattern that is close to a normal curve. This indicates that most students in these two ASEAN countries have basic mathematics proficiency.

On the contrary, Figure 7 shows that other participating ASEAN countries, such as Malaysia, Indonesia, Thailand, and the Philippines, have left-skewed distributions, with most of their students below the basic proficiency level. This suggests that most students in these countries struggle in mastering basic mathematics skills. Hence, these ASEAN countries should place greater emphasis on enhancing fundemental improvements in mathematics education. As more students move up from the lowest levels, their distributions will gradually shift to the right, approaching a more normal distribution. This trend sets a more realistic goal for these ASEAN countries to accomplish in the near future.

Focusing on Malaysia in particular, Figure 7 reveals that 59% of students were below the proficiency level; 25% had basic proficiency in mathematics; 15% had intermediate proficiency; and 1% had high proficiency in the PISA 2022 mathematics assessment. Hence, other than ensuring low-performing students master basic mathematics, supporting other groups of students with different abilities and learning needs is important. This is to ensure that each students receives equal opportunities to improve their mathematical skills.

Improving Mathematics Proficiency in Malaysia: Strategies and Best Practices

Malaysia has been spending many efforts to improve its mathematics performance in international assessments. Despite not achieving the desired outcomes to rank in the top third quarter of countries in international assessment, Malaysia continues to improve its educational system. To strengthen these initiatives, Malaysia could benefit from implementing some of the best practices used in Singapore's curriculum. For instance, the study proposes a subject-based banding system for secondary schools in Singapore. In this system, all the students are grouped into subject classes, namely G1, G2, and G3, based on their Primary School Leaving Examination (PSLE) results. As shown in Figure 5, there is a diverse range of students' abilities within a classroom. With thorough research, Malaysia may consider implementing this approach. In this approach, students are free to select subject levels based on their weaknesses as well as their abilities. Aside from that, they have the freedom to adjust their subject levels based on their development during their lower secondary school years. This new system offers a more flexible learning environment for students of all levels, allowing them to improve their mathematics at their own pace.

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In addition, another key strategy to enhance Malaysia's mathematics performance at the international level is the continuous improvement of teacher professional development programs. The existing issues with the ineffectiveness of CPD in Malaysia include the lack of teachers' voice content, impractical knowledge, inadequate trainers, and insufficient time for practice (Osman et al., 2023). Malaysia can elevate its teacher professional development program by adopting effective policies from the Singapore SFEd roadmap and the World Bank teacher development roadmap. For instance, Malaysia can implement similar teacher professional development plans as those recommended in the Singapore SFEd roadmap. This roadmap strives to personalise professional development programs so that they are relevant to the needs of every teacher.

By looking at the results in Figure 7, we can see that the Malaysia mathematics proficiency level distribution was skewed to the right. The results indicate that Malaysia has more students who require foundational mathematics skills than those with higher proficiency skills.

The price and fuel consumption of four cars that Tania is considering purchasing are shown in the table below.
Fuel consumption is the number of litres of fuel needed to drive 100 kilometres. It is an estimate based on a combination of city and highway driving.

	Car A	Car B	Car C	Car D
Car Price (zeds) **Car price includes all taxes and registration fees**	8000	8700	9900	10500
Fuel Consumption (L/100km)	18.9	15.7	12.4	14.1

Some of the cells in the Cost Estimator have been filled in based on Tania's estimates.



Use the Cost Estimator to help you answer the question below.
To see how to use the Cost Estimator, click on "How to Use the Cost Estimator" above.
Based on Tania's estimates, which car would cost her the least to purchase and drive on the first year?

A. Car A
B. Car B
C. Car C
D. Car D

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Figure 8: Sample of a PISA Level 2 online question. (Source: OECD, 2023)

Hence, Malaysia policymakers can consider adopting the World Bank 2023 teacher development roadmap, which aims to reduce literacy and numeracy rates in the middle- to low-income countries in East Asia and the Pacific region. This roadmap provides Malaysia with guidelines for evidence-based teacher training programs proven to improve foundational mathematics skills in low-performing students. Through consistent improvement in teacher training programs, it is believed to improve students' mathematics performance at the international level eventually. The success of teacher training programs also depends on the teachers' commitment, not following the teacher training programs as a formality to fulfil the minimum credit hours of CPD.

As we can see from the Malaysian education blueprint for 2013–2025, Malaysia strives to elevate its mathematics and science achievement to rank within the top third quarter of countries in international comparative studies. To intensify the effectiveness of these initiatives, Malaysia should put more effort into developing and incorporating PISA-released items in classroom practices and the national assessment model. Understanding the structure of the questions set for each proficiency level of PISA will also help to better determine where students face difficulties and how to help them improve their inadequacies. Figure 8 is an example of a PISA Level 2 online question.

This question reflects a real-life situation that a student might face when making a major decision in adulthood. This is a direct question and does not require any mathematical formula because a cost estimator is provided with the question. Students only need to insert the car price and fuel consumption for each of the four cars into the provided cost estimator to determine which car has the lowest price in the first year. If they are unsure how to use the cost estimator, they can find instructions by clicking on the "How to use the Cost Estimator" link with the online question.

Although this question is simple, some students may struggle because they are unfamiliar with using the cost estimator provided. Based on Figure 5, 59% of Malaysian students cannot answer this question. This typically occurs in students who have learned mathematics, mainly through paper-based practices. Some students may find the problem challenging because they are unfamiliar with lengthy problem-solving questions. This will likely happen when their mathematics learning is mainly focused on repetitive drill questions. Hence, instead of relying on paper-based practices, teachers can include more hands-on exercises related to real-life situations in mathematics classes. For example, they can simulate the car buying process as a mathematics activity. This will allow students to role-play while discovering how to calculate fuel consumption

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and car prices. In addition, teachers can provide similar online cost estimators, and teach students how to input data and interpret it to boost their confidence in using these tools effectively.

Mara's family is moving. They can choose from two sizes of moving trucks to rent. The interior storage-compartment dimensions of the trucks are shown in the table below. All walls and the floor of the storage compartment of the trucks are rectangles.

Truck Size	Length of Floor	Width of Floor	Height
A	4 metres	2 metres	2 metres
B	6.6 metres	2.3 metres	2.3 metres

There are three different sizes of boxes available. The dimensions of these boxes are shown in the table below.

Box Size	Length	Width	Height
Small	0.4 metre	0.3 metre	0.3 metre
Medium	0.5 metre	0.5 metre	0.5 metre

Mara's family decides to rent truck A. What is the greatest number of medium boxes that could fit into truck A?

A. 320
B. 128
C. 26
D. 16

Figure 9: Sample of a PISA Level 2 online question. (Source: OECD, 2023)

The example shown in Figure 9 is also a Level 2 question. This question simulates a real-life scenario of moving house. It requires students to calculate the maximum number of medium-sized boxes Truck A can carry. Since all the boxes are the same size, students do not need to consider various box arrangements within the truck. For this question, students only need to use simple mathematical skills such as multiplication, division, and volume calculation. At the same time, they must also know how to convert between units of length. By applying these basic skills to the operation $(4\times2\times2)\div(0.5\times0.5\times0.5)$, it provides an answer of 128.

However, some students may find this question challenging if they have trouble visualising how to fit those medium-sized boxes inside the truck. This group of students still lacks a skill known as spatial visualisation, which is the ability to imagine objects in three dimensions. To help with this, teachers can use more manipulatives, such as blocks or models, when teaching volume topics. If the school is well-equipped with technology, teachers can also help students learn spatial visualisation with 3D models virtually. Improving students' spatial visualisation is essential not only for success

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in mathematics literacy but also for preparing children for successful adulthood (Gilligan-Lee et al., 2022; Sorby & Panther, 2020). However, many teachers find it challenging to effectively use spatial visualisation tools in the classroom that truly benefit students. Potential barriers such as teacher preparedness and ICT infrastructure readiness must be addressed carefully.

The prices and fuel consumption of four cars that Tania is considering purchasing are shown in the table below.
Fuel consumption is the number of liters of fuel needed to drive 100 kilometers. It is an estimate based on a combination of city and highway driving.

	Car A	Car B	Car C	Car D
Car Price (zeds) **Car price includes all taxes and registration fees**	8000	8700	9900	10500
Fuel Consumption (L/100km)	18.9	15.7	12.4	14.1

A car's resale price is the estimated price for which it can be resold at a later time. For a car that stays in excellent condition, its resale price will decrease by 5% each year.
Tania decides to buy car D and resell it after three years in excellent condition, what will be the approximate resale price of the car in zeds?

A. 1575
B. 8925
C. 9000
D. 9975

Figure 10: Sample of a PISA Level 6 question. (Source: OECD, 2023)

After exploring the basic proficiency level questions, we next examined a high proficiency level question. Example 3 is a multiple-choice question about car purchases that comes subsequently after Example 1. Based on Figure 6, only 1% of Malaysian students could answer this question. In this high-level question, students are not given mathematical formulas to solve the problem. They are required to analyse the problem and decide on a formula based on their existing knowledge. These critical thinking and problem-solving skills are necessary for students to make major decisions throughout their lives.

To solve this question, students should first decide on a process to estimate the car's value after three years. While some students may use an exponential decay formula like $10500\times(0.95)^3$, others unfamiliar with the formula can still solve this problem with an iterative process. The iterative method is carried out by subtracting 5% from the original car price (RM 10,500) to obtain RM 9,975 and repeating this calculation using the car's latest value each subsequent year. The key idea

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of the iterative method is to understand that they must always use the car's latest value for each new calculation. After three years with a depreciation of 5% annually, the car price would be approximately RM 9002.44 zeds. Hence, the closest correct answer to this question is 9000.

CONCLUSION

The main issue discussed in this study is the continuous decline in Malaysia's PISA mathematics results. Although some ASEAN countries are experiencing a decline in their PISA results, it is not as severe as the decrease in Malaysia. The percentage of Malaysian students who are below the basic proficiency level has increased from 41% to 59%. At the same time, the percentage of students who achieved high proficiency also decreased to 1%. In addition, PISA indicators also show that Malaysia is three years behind Vietnam and eight years behind Singapore in mathematics educational attainment.

From the comparative analysis of Malaysia's PISA results with those of other ASEAN countries, this study has identified a common regional issue affecting Malaysia as well as Thailand, Indonesia, and the Philippines. The results show a skewed distribution of PISA proficiency to the right in all countries. This indicates that a large proportion of students did not achieve basic proficiency levels, whereas only a small number achieved high proficiency levels.

This study proposes several strategies and best practices to improve the outcomes. First, countries with a high proportion of students below basic proficiency should put more effort into improving student's foundational skills in mathematics. At the same time, students with higher abilities should be given equal opportunities to improve their mathematical skills. This study suggests adopting Singapore's subject-based banding system for a more flexible learning environment for all students of all levels. By implementing this system, high-achieving students can learn more advanced content while low-achieving students can enhance their mathematics basic skills before moving to higher-level concepts. The results also revealed that students who scored below the basic proficiency level struggled to design simple problem-solving strategies. This highlights the need to develop and incorporate PISA-released items into classroom practices and the national assessment model. Ongoing professional teacher training is necessary to prepare teachers for the above strategies. This study suggests continuously improving teacher professional development by adopting effective policies from the Singapore SFEd roadmap and the World Bank teacher development roadmap.

Although the study used well-validated PISA data, it could provide richer information with triangulation. Future researchers could consider adding extra data sources from national

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examinations, teacher assessments, or interviews with students and teachers. They could also extend this study by conducting an in-depth analysis of the problems faced by students when solving PISA questions at different proficiency levels. In addition, the proposed strategies in this study need to be validated for their implementation and effectiveness in Malaysia's educational context.

Thus, the declining trend in Malaysia's PISA mathematics scores is a pressing concern that requires immediate action. It is only through strong collaboration among all parties that the quality of mathematics proficiency in Malaysia can be improved. If we commit ourselves to the goal of turning Malaysia into the top third of countries in international comparative studies, it is an achievable goal.

ACKNOWLEDGMENTS

The authors would like to express their gratitude to the Universiti Malaysia Sabah Postgraduate Research Grant for providing financial support for this article's production.

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MATHEMATICS TEACHING RESEARCH JOURNAL
SUMMER 2025
Vol 17 no 3

The Perceptions of Multiple Representation of the US and Turkish Mathematics Teachers

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Abstract: Teacher perception is one of the most important factors determining classroom teaching practices. Multiple representations are used extensively in mathematics classrooms, especially in developing conceptual understanding. The data in this study, which aims to compare Turkish and US middle school mathematics teachers' perceptions of multiple representations, were obtained from interviews with four Turkish and three US mathematics teachers with different teaching experiences. The semi-structured interviews focused on teachers' perceptions of role, usage and importance of multiple representations. The study also compares Turkish and US teacher's perceptions about the representations in teaching and learning mathematics. While Turkish and US teachers share general perceptions of multiple representations, key differences exist—especially in how representations contribute to the concretization of mathematical concepts. According to the different practices in integrating multiple representations into instructional processes across countries, Turkish teachers emphasized more symbolic and daily life-based representations, while US teachers emphasized more on concrete materials, visual representations, and the use of technology. The results of the study comparing perceptions of representation suggest that teachers have needs for professional development.

Keywords: Multiple representations, teacher perception, cross-national perspective in mathematics

INTRODUCTION

Cross-country comparative studies in mathematics teaching and learning can offer unique opportunities and perspectives in defining and developing mathematical thinking (Cai, 2000). In this study, US and Turkish middle school mathematics teachers' perceptions about multiple representations were compared. Both, Turkey and the United States have consistently recorded comparatively low scores on international assessments such as PISA and TIMSS, often falling short of their educational objectives. This recurring under-performance across several cycles may be attributed to multiple contributing factors, which have been explored in various studies (Miller & Fonseca, 2021)

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The teaching and learning is a complex process, involves various aspects such as teachers, students, textbooks, and so on. Effective instructional strategies is one of the important factors that most likely help to improve students' mathematical achievement. For instance, implementing classroom teaching practices that focus on conceptual understanding through innovative practices can contribute to overcoming this problem. Similarly, integrating multiple representation in teaching and learning mathematics likely enhance conceptual understanding of mathematics (NCTM, 2000; Mainali, 2021). Multiple representations are often mentioned as a way to support students' conceptual understanding of mathematical concepts (Adu-Gyamfi et al., 2019; Pape & Tchoshanov, 2001; Ross & Willson, 2012). Although teacher competencies are essential for the effective and flexible use of representations in school mathematics, there is limited research exploring teachers' perceptions of the role that multiple representations play in mathematics teaching and learning. Teachers' perceptions and beliefs significantly influence the types of representations they choose to employ in their instructional practices. It is important for mathematics teachers to assess their ability to use representations effectively in their classrooms and to determine whether they receive adequate support in developing these competencies (Stein et al., 2008).

LITERATURE REVIEW

Multiple Representation

Representation is a crucial element in the teaching and learning of mathematics (Vergnaud, 1987). Various scholars have defined the meaning of representation in the context of teaching and learning mathematics. For example, Goldin (2003) states that representation is a configuration of signs, characters, icons, or objects that can stand for or represent something else. Similarly, Janvier (1987) suggests that representation may be a combination of expressions on paper, physical objects, and constructed arrangements of ideas in one's mind. NCTM (2000) describes representation as both a process and a product—the act of capturing a mathematical concept or relationships in some form or as a form itself. Goldin (2001) suggests that a representation is a sign or a combination of signs, characters, objects, diagrams, or graphs, and that it can exist as either an actual physical product or a mental process. Thus, representations include various external forms such as diagrams, signs, figures, characters, symbols, tables, formulas etc., to communicate and express mathematical ideas.

The importance of usage of multiple representation in teaching and learning mathematics has been growing in the last several decades. In mathematics education, it is essential for students to have early exposure to usage of various types of representations and ample practice in translating between different types of representations (Ballard, 2000). Ainsworth (2006) refers to multiple representations as a way of using the salient aspects or advantages of a representation, selecting the most appropriate representation for individual learning needs, and using the most effective strategy for problem solving. The effective use of and translations between representations in mathematics instruction is a critical factor in enhancing students' learning (Mainali, 2021).

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Therefore, teachers' preferences in using multiple representations—along with how they integrate them into mathematics lessons and their ability to use them flexibly and effectively—play a vital role.

Students have different preferences for solution strategies. For example, some students prefer visual solution methods (visual learners), while others tend to use non-visual methods (non-visual learners). Regardless of students' tendencies to use specific types of representations, it is important for them to develop the skill of employing multiple types of representations when solving mathematical problems, as certain tasks are easier to solve using specific representations over others. For example, Larkin and Simon (1987) suggest that diagrammatic or graphic representations help learners recognize features more easily and make direct inferences, thereby supporting effective and efficient problem-solving. Moreover, effective mathematical problem-solving often involves reasoning with visual representations, which depends on the nature and quality of diagrams, figures, and graphs (Bremigan, 2005). Students are more likely to struggle to understand symbolic representation than verbal and pictorial representation and therefore need more time (Khairunnisak et al., 2021). Zhang (1997) argues that visual representations—such as pictures and graphs—provide access to knowledge and skills that may not be readily accessible through other means, such as verbal representation. However, some problems can be more easily expressed using verbal representations, in which case visual aids like diagrams may introduce unnecessary information, potentially distracting students during problem-solving. Thus, it is important for students to develop the ability to use different types of representations in mathematics learning. In elementary school, for instance, students might use a number line, base-ten blocks, or traditional algorithms (using signs and numeral characters) to complete an addition problem. Indeed, the use of multiple representations in mathematics instruction enhances students' conceptual understanding. Being able to apply various types of representations not only helps students arrive at correct answers but also deepens their mathematical comprehension. Additionally, understanding how to use multiple representations enables students to choose the representation that best aligns with their learning preferences. When students can select appropriate representations, mathematical concepts become more meaningful. In fact, the more mathematical tasks that require making connections between multiple representations, the more opportunities students are likely to develop for deeper mathematical understanding (Mainali & Sabri İPEK, 2025). Ultimately, developing the ability to use multiple representations can foster a growth mindset in mathematics.

Teachers' Perceptions and Multiple Representation

Perception is defined as "a general concept or mental construct that includes beliefs, meanings, concepts, propositions, rules, mental images, and preferences" (Philipp, 2007, p. 259). The need for more research on views about multiple representations in mathematics classrooms is evident (Dreher et al., 2016). Teachers' perceptions of teaching and learning mathematics profoundly influence the processes inside and/or outside the classroom (McLeod & McLeod, 2002). This can be seen because of the strong connection between knowledge/understanding and percep-

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tions/beliefs in mathematics. Teachers' perceptions and understandings play an important role in the design of instructional strategies (Baumert et al., 2017).

Since mathematical concepts and knowledge are not easily accessible through direct experience, representations often dominate instructional strategies (Duval, 2006). Representations are an integral component of the mathematics curriculum due to their effective role in linking students' levels of understanding with mathematical concepts. The effective use of representations depends primarily on teachers' perceptions, perspectives, and experiences with multiple representations. Rau and Matthews (2017) argue that determining when and how effectively teachers engage with different types of representations during mathematics instruction is influenced by multiple factors. Emphasizing the importance of using multiple representations in school mathematics, Bosse et al. (2011) suggest that teachers' failure to incorporate a variety of representations in the classroom directly impacts students' mathematical understanding. Therefore, it is essential for teachers to develop basic competencies in using various types of representations in their instructional strategies. Teacher competencies include selecting appropriate types of representations for effective mathematics instruction and understanding how these representations can either support or hinder student learning (Dreher et al., 2016). Indeed, teachers' perceptions of representations influence both their teaching practices and their views on the use of multiple representations—an area that still warrants further exploration (McLeod & McLeod, 2002). These perceptions are particularly important for fostering students' problem-solving abilities and conceptual understanding. However, there is still a limited number of studies that examine teachers' perceptions regarding the role of representation in the teaching and learning of mathematics.

Numerous studies have explored mathematics teachers' beliefs and perceptions, revealing both similarities and differences in how beliefs and perceptions are interpreted. Many models of teacher professional knowledge (e.g. Ball et al., 2008; Shulman, 1986) also point to the complexity of separating perceptions from beliefs in all domains in general and teaching mathematics in particular. Since the term belief is often too broad, in this study we use the more limited term perception to describe specific aspects of beliefs. Ball (1993) examined in-service and preservice teachers, prospective and experienced teachers' knowledge, dispositions, and ways of thinking about representations in mathematics teaching in the context of fractions. Van G. et al. (2019) classify mathematics teacher knowledge models into two main categories—content knowledge and pedagogical content knowledge— where teachers' pedagogical perceptions of teaching and learning processes are not inherently considered part of these models.

Stylianou (2010) investigated middle school mathematics teachers' conceptions of representation as a process and their perspectives on the role of multiple representations. The study reported that teachers employ representations in diverse ways in their own mathematical work and tend to define the term primarily as a product used in problem solving. However, their understanding of representation as a dynamic process and as an integral part of mathematical practice appeared to be less developed. The study further noted that representations may play a more peripheral role in their instructional approaches. Similarly, Bosse et al. (2011) examined teachers' beliefs about

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the use of multiple representations and reported challenges related to transitioning and translating between different types of representations. Lee and Lee (2019) explored pre-service elementary teachers' perceptions of using multiple representations to solve fraction problems and found that participants lacked a clear understanding of how to use various representations effectively. Dreher et al. (2016) conducted a study on the influence of cultural differences between English and German pre-service teachers, suggesting that mathematics teachers' professional knowledge and perspectives on the use of multiple representations should be carefully considered in mathematics instruction. Expanding on this, Dreher et al. (2024) examined whether perspectives on the use of representations in mathematics classrooms differ across cultures and highlighted the challenges of applying Western views of teachers' representation practices to non-Western educational contexts.

Several studies have explored teachers' representational knowledge from various perspectives (e.g., Jones, 1997; Hitt, 1998; Knuth, 2002; Izsa & Sherin, 2003; Rosli et al., 2013; Son & Lee, 2016), while others have focused specifically on teachers' perceptions of representation (e.g., Stylianou, 2010; Bosse et al., 2011; Lee & Lee, 2019; Lee & Lee, 2023). However, relatively few studies (e.g., Cai & Wang, 2006; Dreher et al., 2016) have examined teachers' perceptions of representation through a cross-cultural lens. Therefore, this study aims to contribute to the field by shedding light on the role of representation in the teaching and learning of mathematics within an international context. The findings are expected to deepen the understanding of teachers' orientations toward multiple representations and provide valuable insights into how they utilize these tools to support students' problem-solving skills.

THEORETICAL FRAMEWORK

Representations play a vital role in the teaching and learning of mathematics, as the use of multiple forms can greatly enrich instructional approaches and deepen students' understanding of mathematical concepts. Their impact on enhancing problem-solving abilities and fostering conceptual understanding is well-documented, with widespread agreement among scholars regarding their fundamental importance in mathematics education (Dreher et al., 2024; Duval, 2006; Greeno & Hall, 1997; Mainali, 2021; Rau & Matthews, 2017). In fact, utilizing different types of representations can enhance the understanding of the abstract nature of mathematical concepts. Goldin (2002) defines a representation from a general perspective as a configuration that can represent something else. Representations can manifest in a variety of ways, such as visuals (tables, number lines, etc.), written symbols (numbers, letter expressions, etc.), concrete materials (fraction models, decimal blocks, etc.), verbal expressions (classroom discussions, spoken language, etc.) and real-world situations (Van de Walle et al., 2004). Emphasizing both the process and product aspects of representation in mathematics, NCTM (2000) defines the process dimension of representation as the act of understanding a mathematical concept, and the product dimension as the form of representation. The process involves active engagement with mathematical ideas, often requiring flexibility and adaptability in choosing and transforming representations, while

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the product is the outcome of this process, which can be used for evaluation, sharing ideas, or further exploration. Multiple representations are used extensively in understanding new concepts, problem solving and making connections between mathematical concepts (Stylianou, 2010). Ainsworth (2006) refers to multiple representations as a way to take advantage of the prominent aspects or advantages of a representation, to select the most appropriate representation for individual learning needs and to employ the most effective strategy for problem-solving. Studies on students, textbooks, curricula, etc., related to learning with multiple representations refer to the teacher's perspective and knowledge in the development of this skill in the mathematics classroom. Therefore, questions regarding teachers' knowledge and perceptions of how to use multiple representations in mathematics instruction remain central to educational research and practice.

Teachers frequently utilize multiple representations when teaching mathematical concepts and facilitating connections during the problem-solving process, regardless of the complexity of the content (Stylianou, 2010). To help students grasp abstract ideas and relate them to other mathematical concepts or real-life contexts, teachers must often generate and transform various forms of representation. As one of the key tools in mathematics instruction, representations are subject to teachers' individual perceptions and beliefs, which can vary widely. The use of representations in teaching involves several interconnected steps—including selecting the most appropriate representation for a given concept or problem, transforming one representation into another, transitioning between forms, and viewing representation as either a process or a product. Each of these decisions is shaped by the teacher's understanding and perception of representation. In this regard, mathematics teachers' knowledge and beliefs about multiple representations play a critical role in influencing their students' ability to understand and apply different representations effectively (Bharaj, 2022).

Teachers' understanding and perceptions of the role and use of representations certainly influence their instructional strategies. For instance, some teachers may believe that utilizing multiple types of representations places an additional burden on students, while others believe that incorporating different representations enhances students' mathematical understanding. Regardless of teachers' differing viewpoints regarding the role of representation, it is important to employ a variety of representations in the teaching and learning of mathematics. In fact, multiple representations can support one another due to their unique characteristics, which aid students during the problem-solving process (Dreher et al., 2024). Therefore, it is essential for teachers to understand the role of multiple representations and to apply them effectively in mathematics instruction. Similarly, it is essential to explore what teachers think and perceive about the role of representations within mathematics education. Furthermore, examining differences in teachers' perceptions across countries can provide valuable insights into the effective use of representations in mathematics instruction. In this context, the following research questions are posed:

i. How do Turkish and US middle school mathematics teachers perceive the use of representations in teaching and learning mathematics?
ii. What are the perspectives of Turkish and US middle school mathematics teachers on utilizing representations to solve mathematics problems?

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iii. What are the differences in Turkish and US middle school mathematics teachers' perceptions regarding the role of representations in teaching and learning mathematics?

METHODS

This study is a comparative qualitative case study that explores middle school math teachers' perceptions of multiple representations in two different countries: the US and Turkey.

Study Group

The descriptive qualitative study involved semi-structured interviews with a total of seven teachers, four from Turkey (TR1, TR2, TR3, TR4) and three from the US (US1, US2, US3). All the middle school teachers that were selected voluntarily and had different educational backgrounds and levels of teaching experiences. The details about the participants are given in Table 1.

Country	Participants	Gender	Educational Degree	Teaching experience (years)
Türkiye	TR1	Male	Bachelor's degree	15
	TR2	Female	Bachelor's degree	14
	TR3	Female	Master's degree	16
	TR4	Female	Master's degree	8
US	US1	Male	Master's degree	23
	US2	Male	Bachelor's degree	10
	US3	Female	Master's Degree	23

Table 1: In-service teacher demographics

Research Instrument and Data Collection

Employing a qualitative research design, semi-structured interviews were conducted via Zoom. Each interview lasted approximately 15–20 minutes and was recorded for analysis purposes. The interviews were conducted in English and Turkish for the participants from the US and Turkey, respectively. The interviews conducted in Turkish were transcribed and translated into English for data analysis, as one of the researchers is a native speaker of Turkish. The semi-structured interview technique was chosen because it allows for a deeper exploration of participants' reasoning processes and knowledge structures through open-ended questions (Clement, 2000). To examine mathematics teachers' perceptions of multiple representations, a data collection tool was developed based on relevant literature. The same set of questions regarding representations was posed to participants from both countries. The data collection tool was organized into two parts. The initial section focused on teachers' perceptions of multiple representations in mathematics, particularly their views on the nature and role of these representations in the context of teaching and learning mathematics.

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The questions in the first stage were as follows:

i. What do you think "representation" means in mathematics?
ii. Do you think it is necessary and important to use more than one representation to teach mathematical concepts or problems?
iii. Do you use multiple representations while teaching in your classroom? If yes, please give an example to explain... If not, please explain why.

In the second phase of the interviews, participants were presented with three mathematical tasks and asked to solve them. The three tasks are provided in Table 2. To explore teachers' perceptions of their preferred representations for these tasks, a context was given, and participants were asked whether they considered a specific representation more suitable than others. If so, they were also asked to explain why, and whether a particular representation might be more beneficial for classroom teaching or for students. For each task, the goal was to examine what types of representations participants used while solving the problems, whether they employed more than one representation, and how they applied different solution strategies using appropriate representations. Based on their responses, participants were also asked to reflect on how these representations influenced their thinking, the advantages and disadvantages of using different representations, and whether they could generate alternative representations for the given tasks.

2-i. Solve the following problem for your students: $1\frac{2}{3}+\frac{5}{6}=$ *ii.* Solve the following problem for your students: (+3) – (-4) = *iii.* Solve the following problem for your students: A father took his 3 children to a fair. Tickets cost twice as much for adults as for children. The father paid a total of 50 zeds for the 4 tickets. How many zeds did each child's ticket cost? (TIMSS-2007)

Table 2: Mathematical Tasks

As shown in Table 2, the tasks focused on fractions, integers, and algebraic concepts—topics that are commonly emphasized in middle school mathematics curricula. These open-ended tasks were intentionally selected to explore participants' solution strategies and to encourage deeper conversations between the interviewer and the participants (Hunting, 1997). The goal was to examine middle school mathematics teachers' perceptions of multiple representations in both countries and to understand how they do—or do not—incorporate these representations into their classroom instruction.

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Data analysis

The semi-structured interviews were recorded using Zoom, and the audio recordings were analyzed through thematic analysis, a method commonly employed in qualitative research (Liamputtong, 2009). Braun and Clarke's (2006) six-step approach to thematic analysis was followed, which includes: familiarizing oneself with the data, transcribing the data, generating initial codes, searching for themes, reviewing themes, defining and naming themes, and producing the report. These steps helped identify underlying codes within the data and the relationships among them, ultimately forming broader themes. Data from both phases of the study were independently coded and analyzed by each researcher. The researchers then engaged in multiple discussions to compare their analyses and finalize the themes through consensus. Initial categories were developed from the responses to the interview questions in the first section, which were informed by relevant literature to frame the teachers' perceptions of multiple representations. During this process, similar themes and sub-themes emerged among both Turkish and US mathematics teachers, although their perspectives often differed. Following the initial coding, each set of interview data was organized into broader categories using a code-mapping strategy. Saldana (2021) describes code mapping as the process of "comparing and sorting" codes into overarching themes to complete the analysis. The final code-category-theme mapping resulting from this analysis is presented in Table 3.

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Code	Sub-Theme	Main Theme
Expression of concepts in different forms (TR) Explaining the mathematical idea (TR&US) A tool for concretization (TR&US) Making sense (TR&US)	Representation as product	Representation definition
Formulize (TR) Expressing graphically (TR) Establish communication (US) Mathematical language (TR)	Representation as a process	
Expressing ideas (TR&US) Drawing (TR&US) Expressing with a table (TR) Embodiment (TR&US)	External Representation	Types of Representation
Daily life (TR&US) Expressing mathematically (TR) Technology support (US)	Internal Representation	
Mathematical language (TR) Making it fun (TR&US) Contribution to students' self-confidence (TR) Individual Learning (US)	Positive aspects of representations	Different aspects of representations
Students' confusion (TR&US) Time-taking (TR)	Negative aspects of representations	
Preferably direct teaching strategy (TR) Using representations in simple problems (TR) Visualization for problem understanding (TR&US)	Concept teaching	Multiple representations in the classroom
Using representations in concept teaching (TR&US) Making connection to daily life (TR&US)	Problem solving	
Use tangible materials (TR&US) Concretization for abstract thinking (TR&US) Use of mathematical language (TR&US) Technology support (US)	Mathematical skill development	

Table 3: Sample Code-Mapping for Math Teacher

The questions in the second phase aimed to investigate teachers' preferences regarding the use of different types of representations and to evaluate the consistency between their perceptions and actual use of representations. Each problem-solving scenario focused on identifying the primary strategies participants used to solve the tasks, their preferences for employing multiple representations, and their views on the necessity and practicality of using such representations—whether they considered them essential or time-consuming. Furthermore, the study analyzed both the

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similarities and differences between teachers' perceptions of representations and their actual application of these representations during problem-solving.

In the final stage, the relevant data were thoroughly analyzed to compare the perceptions of teachers from both countries, with each consensus theme named and clearly redefined. During this process, similar codes emerged in both the Turkish and US cases, often reflecting different perspectives on the same topic. This facilitated easier comparison and validation of the data when necessary. Two researchers independently coded 25% of the dataset (one question from each stage). While the inter-coder agreement exceeded 80%, any discrepancies in the codes were revisited and discussed until a consensus was reached between the researchers.

RESULTS

Definition of Representation

The findings reveal that *concretization* emerged as a particularly salient concept in teachers' definitions of representation, as articulated by both Turkish and US participants. Furthermore, Turkish teachers expanded upon this understanding by offering interpretive rationalizations to elucidate the role and meaning of representation. For instance, one participant stated, "representation concretizes a mathematical concept in a way that is appropriate to the level of the students' understanding. I think, for example, when we present x^2 as a square, we exemplify it in a concrete way" (TR1). The data analysis suggests that participants conceptualize representation primarily as a process of concretizing mathematical ideas, while also recognizing its role in visually depicting mathematical expressions. One Turkish participant articulated this by stating, "to me, representation is a tool that should be used to facilitate students' access to mathematical content. We employ it both in everyday spoken language and in the specialized language of mathematics."

One of the Turkish participants stated that he could use representations in various ways based on his own perspective, emphasizing both the *product* and *process* dimensions of representation. As previously noted, the *process* dimension refers to capturing a mathematical concept, while the *product* dimension refers to the final form of the representation itself (NCTM, 2000). The process involves active engagement with mathematical ideas, requiring flexibility and adaptability in selecting and transforming representations, whereas the product is the outcome of this process, which can be used for evaluation, communication, or further exploration. Similar to their Turkish counterparts, the US participants also emphasized representation as a tool that aids in the concretization of mathematical concepts. For instance, one participant stated, "Using representations helps make learning math more concrete so that students can understand and visualize mathematical ideas—for example, by using manipulatives—making it a bit easier for them to access mathematical concepts" (US3).

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The data analysis suggests that representation is viewed as a product employed during the process of mathematical problem-solving. Another participant also described representation more broadly, stating as "representation is a means of expressing your own culture, point of view, or mathematical ideas" (US2). The data further revealed that US teachers, more than their Turkish counterparts, place greater emphasis on the social dimension of representation. The fact that most Turkish and US teachers focus on the process of making sense of mathematical concepts—particularly through visual or graphical forms—indicates that they perceive multiple representations as an integral part of mathematical conceptualization. However, the data also highlight the challenges teachers face in defining the concept of representation. Both Turkish and US participants frequently relied on real-life examples to explain representations. While Turkish teachers struggled to articulate a clear definition of representation, they demonstrated consistency in their understanding and showed a preference for using flexible and appropriate representations during problem-solving in the second stage of the study. For example, one Turkish participant emphasized the use of visual representations, stating: "We need to concretize abstract mathematical ideas and express them through visual representation in operational tasks—for instance, in the case of the integer operation (+3) - (-4) =?"

> I first modeled this operation using counting stamps. I began by drawing three positive stamps to represent +3. Then, I needed to subtract -4 stamps. Since I had no negative stamps available, I referred to the concept of zero pairs, which I had already introduced in the context of integer operations. To proceed, I explained that in order to subtract -4, we need to add four zero pairs (each consisting of +1 and -1) to the existing model. It is important to emphasize that adding zero pairs does not alter the value of the original expression. After adding the four zero pairs, I had four negative stamps, which allowed me to subtract -4. The result of this operation was +7. This is how I initially model the concept before introducing the formal rule for subtraction (TR3).

The data analysis suggested that participants' perceptions about representations seems to be in line with the representation as a means of concretization during their problem-solving processes. For example, one of the US participants expressed his view about representation as "I believe it is more effective to engage my students in making senses of learning math and provide hands-on learning experiences." Another participant states as follows:

> Modeling is very important when teaching math lessons, especially since I work with middle school students, and I follow a more tactile approach to make the lessons more effective. I would first try to help them reach a common denominator using manipulatives or parts of a circle-model drawing. I would also use algebra tiles or different models for integer tasks (US2).

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The data analysis suggests that a primary function of representation is to make abstract mathematical concepts more concrete, and participants from both countries share similar perspectives on defining representation.

Representation Types

Although all Turkish participants recognized the presence of different types of representations in teaching mathematics, some had difficulty elaborating on these types. Instead, they attempted to explain representations using mathematical examples. For example, one participant stated:

> Let's think about a math task where we use a number line to explain the concept. There are different types of representations we can use to represent a number line. While we mostly express it symbolically and by using the actual number line, sometimes we can use examples to explain the concept, such as a real-life example like the length of a table in the classroom (TR3).

Another Turkish participant further emphasized the use of representations through daily life contexts, symbolic forms, verbal explanations, and concrete materials. Based on the concept of integers, the participant illustrated various types of representations and stated:

> Let's think about the concept of integers. I can use the example of an elevator to teach the concept of integers. For example, I can use the elevator going up and coming down from the ground level to explore the concept of positive and negative integers. Or I can use an example of a sea submarine. When we say -5 or +3, for example, this is a verbal representation of the integers. But when we write it on the board, then we are using symbolic representation. These numbers can also be expressed concretely with materials or hands-on activities, which is more about types of visual representation (TR4).

US participants, similar to their Turkish counterparts, demonstrated a clearer understanding of the types of representation and showed confidence in using various forms of representation in their math lessons. However, they also found it challenging to explain the different types of representation in a more comprehensive way. For instance, one of the US participants stated:

> So, we can use algebra tiles to teach algebraic concepts such as solving equations. We can use other visual models, including algebra tiles. I think there are different ways to make abstract algebraic concepts more accessible to students by using appropriate representations. We should try to start with concrete materials when teaching mathematical concepts. As a result, abstract ideas can become possible and more easily accessible to students. The more we integrate different types of visual representation into our teaching, the better students understand the concept.

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> Therefore, we should definitely use more than one representation in mathematics lessons (US1).

In regard to the role of multiple representations, Turkish teachers mainly focused on symbolic representations and real-life applications, while the US teachers emphasized more on concrete materials, visual representations, and the use of technology. The various types of representations highlighted by teachers are important, as they reflect their preferences and approaches of using representations in mathematics instruction. One of the Turkish participants, who consistently emphasized the significance of symbolic representation in teaching the fractions stated that "I usually follow the traditional rule to teach the concept of fraction operations. Usually, I first convert a mixed number into an improper fraction, make the same denominators and the complete the addition/subtraction of fraction tasks (TR1)."

The data analysis suggested that using various types of representation in teaching and learning mathematics is important to provide conceptual understanding of mathematics. For instance, a US participant, who stated that he frequently used concrete materials as representations in his math lesson states as follows solving an algebraic problem:

> I would begin solving the algebraic problem using concrete objects. For example, I would give my students a roll of tickets, some monopoly money, and glue. I would encourage them to consider how many tickets they can buy with this amount of money. This hands-on approach will allow my students to engage in a more structured problem-solving process using the paper roll. Additionally, I would encourage my students to experiment with the materials to explore more about algebraic concepts, rather than focusing on correct/incorrect answers. Finally, I would gradually transition to teaching more abstract algebraic concepts (US2).

Similarly, another participant from the US stated,

> I incorporate manipulatives, tactile representations of mathematics, and computers (math software) in my math lesson. At our school, we also have access of online versions of manipulatives, and I also use them in my lesson since some of my students in my class may not feel comfortable with using technological tools (US3).

Positive (Negative) Aspects of Representation

The data analysis revealed that both Turkish and US participants primarily emphasized the positive aspects of using representations in teaching and learning mathematics. Turkish participants particularly highlighted the benefits of representations, such as making mathematical concepts more concrete, improving mathematical understanding, and helping students comprehend the unique structure of mathematical language. For instance, one of the participants stated:

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> It may sound like an emotional statement, but I would say that representations make learning mathematics more special, and they help in understanding mathematical language. Mathematics has its own language, and I think representations add more value to it. For example, we can express mathematical concepts or situations that we would normally explain in our own language by using mathematical expressions. However, we can express mathematical concepts in a very short (or long) way using mathematical representations (TR2).

Similarly, another participant emphasized the social-cognitive aspect of representations as follows:

> The use of representations makes mathematical concepts and operations easier to understand, and learning becomes more enjoyable for students. Additionally, representations help attract students' attention, as they can relate mathematics to their daily lives through the use of appropriate representations. As a result, students are likely to develop positive beliefs toward mathematics, which will help build their self-confidence in the subject (TR1). All Turkish participants, however, identified time constraints as a major drawback of using multiple representations in mathematics instruction. For instance, one teacher stated, "While modeling mathematical concepts through representations is valuable for problem-solving, incorporating representations into mathematics lessons can be time-consuming" (TR3).

Similar to Turkish teachers, US teachers also focused on the positive aspects of using representations in teaching and learning mathematics. One of the participants emphasized the role of representation in enhancing students' understanding of mathematical concepts. For example, he stated that representations can offer students alternative ways to understand mathematical concepts and to see what is going on behind the scenes (US1).

Similarly, another participant focused more on the social-cognitive aspect of representations. She stated, "Representation not only helps students feel more comfortable with their own learning style, but it also makes it easier for students to transfer mathematical concepts. Additionally, the use of appropriate representations caters to students' preferred ways of learning mathematics" (US2). US participants also highlighted a potential drawback of using representations: that they can sometimes make learning math concepts more complex. However, unlike Turkish teachers, US participants did not perceive representations as time-consuming.

Representations in the Classroom

Both Turkish and US participants frequently utilize multiple representations in their classrooms. However, there are differences in the preferences of teachers regarding when and how representations are used in mathematics instruction. Turkish teachers, in particular, tend to prefer using

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representations during the stages of concretizing concepts and understanding problems. For example, a Turkish mathematics teacher described the use of multiple representations when teaching the concept of a ray in geometry as follows:

> I use representations with the help of an interactive whiteboard to explain to students, for example, the similarities and differences between rays and lines in my geometry lessons. I encourage students to first think of a long wire, which helps them grasp the concepts of rays and lines to some extent. However, they can better comprehend these concepts in a technology-supported environment. They can further relate the idea of a ray starting from a specific point to real-life examples, such as sunlight. Later, I use symbolic representations to further explain the concepts of rays and lines during my geometry lessons (TR4).

Another Turkish teacher explained as how she used multiple representations to teach the concept of ratio:

> Let's consider the concept of ratio. While teaching the concept of ratio, I provide examples from daily life using various representations. For instance, I compare the prices of products in markets. I also integrate the concept of ratios into other subjects, such as discussing the speed of cars or the scale used on maps. These are some topics that come to mind (TR3).

Turkish teachers tend to use representations primarily to make abstract mathematical concepts more tangible and to better understand problems. The analysis also revealed that while Turkish participants favor representations for clarifying abstract concepts, they do not frequently rely on visual aids like number lines, tables, or graphs to comprehend problems. Therefore, the data suggest that the use of representations in mathematics lessons in Turkey has a limited variety and purpose. It is evident from the interviews that US teachers use concrete materials and technology-assisted applications more intensively in their mathematics classrooms. For instance, one of the US participants stated, “I'm better at using graphing calculators, so this kind of material works really well, especially for teaching algebraic concepts” (US1).

US participants stated that they often incorporate a variety of hands-on materials and manipulatives in their math lessons. They further stated that students also have the chance to apply their mathematical skills and knowledge through real-life activities. For instance, one of the participants stated that students visit the school coffee shop where they can put their math skills into practice. This allows them to engage in real-world applications, such as managing the cash register, budgeting money, or exploring ratios and proportions in the ingredients of food and drinks. The data also suggest that teachers in the US have more access to concrete materials than their counterparts in Turkey. The greater availability of resources in US classrooms allows teachers to use various forms of representation, providing more opportunities for students to learn math skills and apply mathematics in real-world scenarios. While Turkish teachers recognized the importance of using concrete materials to enhance students' understanding, they highlighted the

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challenges they face in accessing these resources. One of the Turkish participants shared the difficulty of obtaining concrete materials and hands-on activities in the following way:

> When solving a mathematics problem, teaching would be more effective if we had materials appropriately prepared ahead of time. I could use concrete materials, but if they are not available in the classroom, I would try to help my students understand the problem by drawing shapes on the board" (TR4).

Despite some constraints, both US and Turkish participants have positive views towards the role of representations in teaching and learning math lessons. The data indicated that while Turkish teachers have a positive view of using concrete materials in mathematics instruction, their access to hands-on resources is limited, making it challenging to integrate concrete representations into their lessons.

DISCUSSION

This study aims to compare Turkish and US teachers' perceptions of representations in mathematics teaching and learning and some of the results obtained in this study are noteworthy. While there is a strong emphasis on using multiple representations in teaching and learning mathematics, the effective use of representations requires teachers to have a good understanding and knowledge about representation. The results suggest that both Turkish and US participants explained about both aspects: product and process dimensions of representation. In the product dimension, teachers from both countries generally considered representation as a product used in the process of concretizing mathematical concepts, similar to the NCTM (2000) process and product dimension. While in the process dimension, Turkish teachers emphasized the mathematical aspect of representation, and US teachers focused particularly on the social aspect of representation. The discrepancy in the participants' perspectives is likely due to the different culture and context. Teachers who tried to explain the concept of representation, usually with mathematical examples, focused on the basic meaning of representation and its implications for teaching and learning mathematics. Teachers who were familiar with the concept of representation and reported using it frequently, however, had difficulties in clearly defining representation and multiple representations.

The study results suggests that teachers from both countries demonstrate a limited theoretical understanding of representation. This aligns with findings indicating that mathematics teachers are often unaware of how to effectively incorporate different modes of representation into their instructional strategies (Mainali, 2021). Additionally, there is no significant difference in how Turkish and US teachers comprehend and perceive representation, which provides important clues about the role and use of representation in teaching and learning mathematics, given the differences in teachers' definitions and perceptions of representation in mathematical problem solving. The difficulties encountered in representing multidimensional mathematical concepts

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such as numbers, ratios, and areas were similar for teachers in both countries. Both Turkish and US teachers emphasized the use of representations in mathematics courses but experienced difficulties in effectively integrating representations into their teaching practices. In general, while they highlighted the positive aspects of using representations, they were not able to use them flexibly and effectively when solving the problems presented in the interviews.

A Turkish teacher who recognized the importance of using tangible materials in whole number subtraction stated, "I use counter chips, although I am not very comfortable with them." The findings further suggested difficulties in using concrete materials as basic tools for exploring whole number operations, despite teachers being aware of the significance of representation. While Turkish teachers acknowledged the importance of concrete materials in enhancing mathematical understanding, they also mentioned having limited access to such resources. Similarly, a US teacher recognized the importance of concrete materials and expressed interest in how representations could be applied in teaching fractions. For example, one participant stated, "I honestly don't know much about using representations in fraction operations" (US1). The results also suggest that a similar situation exists for US teachers. However, it should be noted that US teachers do not face difficulties in accessing concrete materials. Overall, the results suggest that to improve teachers' representational skills and understanding in both countries, there should be a greater focus on in-service and pre-service training programs aimed at enhancing mathematics teachers' skills and knowledge related to the use of representations.

Building on the results regarding teachers limited theoretical understanding of representations, this study also highlights differences between Turkish and US teachers' perceptions of the types of representations used in teaching and learning mathematics. While Turkish teachers tended to focus on symbolic representations and daily life-based examples, US teachers placed greater emphasis on the use of concrete materials, visual representations, and technology. This pattern is consistent with Stylianou's (2010) findings, which reported that symbolic and numerical representations are more prominent than graphical and visual representations in mathematics lessons. Similarly, the current study supports Cai and Wang's (2006) results, showing that US teachers used algebraic approaches less frequently than their Chinese counterparts when representing concepts such as ratios.

Although Turkish teachers frequently emphasized the importance of incorporating real-life contexts during interviews, these contexts were seldom applied when solving the provided mathematical problems. In contrast, US classrooms appeared to integrate real-world examples more systematically, offering students opportunities, such as working in the school's canteen, to contextualize mathematical concepts. This contrast suggests that, while Turkish teachers recognize the value of real-world connections, challenges in practical implementation persist. These findings reinforce the need for professional development programs that address both the theoretical foundations and the instructional application of multiple representations in mathematics education.

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Turkish and US teachers generally shared positive and similar views on the use of multiple representations. However, they highlighted different negative aspects. Turkish teachers cited time constraints as a challenge in using representations, while US teachers felt that representations might make certain mathematical concepts more complex for students. The availability and adequacy of different types of representations likely contribute to the differing views on how various representations are used between Turkish and US teachers. Although both groups agreed that multiple representations are beneficial in school mathematics, there were differences in their preferences for using them. Turkish teachers tended to favor representations that concretize mathematical concepts and help students understand problems, but they did not often use visual representations such as number lines, tables, or graphs. The findings indicated that teachers who were aware of the importance of visual representations still did not prefer to use them frequently. Dreher et al. (2016) state that simply acknowledging the value of multiple representations is not enough to create rich learning opportunities; how these representations are used is more crucial. The following statement reflects the use of representations in mathematics lessons, with a US teacher describing this process in connection to integer operations:

> I actually had a student move and jump along a number line while teaching him to add and subtract integers. Students were required to use the number line to complete the integer operations, which helped reinforce their understanding of integer concepts. This approach allowed them to grasp that moving further to the right on the number line increases the value of the number" (US3).

The results, along with the effective use of the number line in such operations, highlight the crucial role of representations in mathematical problem-solving. Allowing students to engage with various types of representations enhances their conceptual understanding and reduces errors (Mainali, 2014). The results indicate that US teachers have better access to technological resources in their classrooms and utilize technology more extensively and effectively than their Turkish counterparts. Technological tools facilitate easier access to multiple representations and support the development of multi-representational teaching strategies in mathematics instruction (Thurm & Barzel, 2020). These results suggest that Turkish teachers, in particular, need more support in integrating technology into their mathematics courses. Technology-supported classroom environments may offer more opportunities for teachers to use multiple representations more effectively and flexibly.

LIMITATIONS, IMPLICATIONS, RECOMMENDATIONS, & CONCLUSIONS

This study was conducted with a limited number of participants from specific school districts in both countries. Moreover, it is limited to middle school teachers. As a result, the findings cannot be generalized to all in-service teachers. Nonetheless, the results offer valuable insights into both the theoretical and practical aspects of representation in mathematics teaching and learning. Teachers' knowledge and perceptions are likely to influence their preferences for using different

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representations, but further research is needed in the field of mathematics education. This study aims to contribute to the literature by comparing middle school mathematics teachers' perceptions of the use of representations in teaching mathematics. Although there are differences in the perception and use of multiple representations between the two countries—and thus two different cultural perspectives—the results are limited by the scope of the interviews.

The study has important implications for mathematics teaching and curriculum development for both countries. Research has shown that representations play a crucial role in mathematics education and are fundamental to effective teaching and learning (Goldin, 1987; Janvier, 1987; Kaput, 1987; Zazkis & Liljedahl, 2004). Therefore, it is essential for mathematics teachers to understand the role of representations and how to use them effectively during instruction. Students should be able to engage with multiple forms of representation to develop mathematical proficiency, as certain mathematical problems can be solved more efficiently using specific types of representations (Mainali, 2019). Thus, professional development programs in both countries should focus on deepening teachers' understanding of representations and their effective usage in mathematics lessons. In particular, Turkish teachers may benefit from improved technological infrastructure and additional resources to support the use of multiple representations in their classrooms. More importantly, mathematics teachers should strive to incorporate a balanced use of various representations, ensuring that students engage with different modes to develop both conceptual understanding and problem-solving skills. In fact, the greater the number of tasks or problems in textbooks that involve connecting multiple representations, the more opportunities students are likely to have for a deeper mathematical understanding (Mainali & İpek, 2025).

The findings of the study also suggest several recommendations regarding the mathematics teaching and the curriculum. In-service teacher training programs in both countries should emphasize the theoretical understanding and practical application of multiple representations in teaching and learning mathematics. Programs should incorporate strategies to help teachers effectively use representations to concretize and visualize mathematical concepts. For example, incorporating technology into teaching practices can support the use of multiple representations and improve flexibility and effectiveness in mathematics instruction. Likewise, professional development should prioritize effective methods for integrating representations in a way that does not overwhelm students or interfere with instructional time. Educators should be equipped with strategies to incorporate representations seamlessly into their daily lessons. Additionally, teachers should be encouraged to utilize diverse forms of representation, including visual, symbolic, and concrete materials, while the mathematics curriculum, including textbooks, should equally emphasize the use of various types of representations.

ACKNOWLEDGMENTS

This study was supported by The Scientific and Technological Research Council of Türkiye (TUBITAK) [grant number 1059B192300741].

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SUMMER 2025
Vol 17 no 3

The Role of Self-Perception of Mathematics and Technological Self Confidence in Predicting Mathematical Representations in Secondary Education

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Abstract: The primary objective of this research was to explore the intricate relationships and mediating among self-perception of mathematics, technological self-confidence, and mathematical representations. A cross-sectional study was conducted involving a sample of 322 students from both public and private schools in Bandar Lampung, Indonesia. The data were analyzed using structural equation modeling. The findings of this study are significant and contribute to our understanding of the roles of these variables in secondary education. Firstly, the study revealed that technological self-confidence plays a crucial role in promoting mathematical representation. Secondly, technological self-confidence was found to be positively linked with self-perception of mathematics, suggesting the interconnectedness of these factors. Lastly, self-perception of mathematics was identified as a mediating variable, but it exhibited a negative association between the relationship of technological self-confidence and mathematical representation. This study offers valuable insights into the potential keys to success in the field of mathematics education, particularly in the digital age of mathematics teaching.

Keywords: self-perception of mathematics, technological self-confidence, mathematical representations, gender, structural equation modeling.

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INTRODUCTION

In the realm of contemporary education, particularly in the context of secondary education, understanding the multifaceted interplay of self-perception and technological self-confidence concerning mathematical representations has gained prominence. The role of self-perception, which encompasses students' beliefs about their own mathematical abilities, and technological self-confidence, referring to their confidence in employing digital tools and technologies to represent mathematical concepts, is fundamental in shaping students' experiences and achievements in mathematics.

The acquisition of mathematical skills and the ability to construct mathematical representations are critical objectives within secondary education. Not only do these skills underpin academic achievement, but they also serve as vital foundations for students' analytical and problem-solving competencies that are essential in an increasingly technology-driven world.

The significance of self-perception and self-confidence in relation to mathematical achievements has been a subject of substantial research within the field of education (Kaur & Prendergast, 2022; Panaoura, 2012). Prior studies have established a clear relationship between students' self-perception of their mathematical abilities and their academic performance. A wealth of empirical evidence suggests that students who hold positive self-perceptions about their mathematical competencies tend to outperform their peers who exhibit self-doubt or negative perceptions (Dupeyrat et al., 2011; Reich & Arkin, 2006). Moreover, research has illuminated the vital role of self-confidence when it comes to adopting digital tools for mathematical learning. Students who possess higher technological self-confidence tend to engage more actively with digital resources, ultimately impacting their mathematical performance (Huang & Brainard, 2001).

However, while there is substantial research on self-perception, technological self-confidence, and mathematical achievement separately, the intersection of these constructs, specifically in the context of mathematical representations, remains an underexplored domain. This research gap points to the pressing need for an integrated investigation that considers how students' self-perception of mathematical abilities interacts with their technological self-confidence and influences their ability to create effective mathematical representations.

The research problem at the heart of this study revolves around discerning how self-perception and technological self-confidence interact and contribute to students' mathematical representations in a secondary education context. While previous studies have addressed self-perception's influence on mathematical achievement and technological self-confidence's impact on technology use, a comprehensive examination of their collective impact on the quality and effectiveness of mathematical representations is yet to be accomplished.

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By bridging this research gap, we aim to provide a more holistic perspective, shedding light on the complex web of factors influencing students' mathematical representations. This research delves into the nuanced relationships among self-perception, technological self-confidence, and mathematical representations, thereby contributing to a more comprehensive understanding of the cognitive processes involved in mathematical problem-solving and communication.

The subsequent sections of this article will delve into the pertinent theories, methodologies, empirical findings, and discussions, all in pursuit of a more comprehensive understanding of the roles that self-perception and technological self-confidence play in predicting mathematical representations in secondary education.

LITERATURE REVIEW

Technological Self-confidence and Mathematical Representations

In the realm of modern education, technological self-confidence has emerged as a pivotal factor influencing students' preparedness to effectively engage with digital tools and resources. Within the context of mathematics education, technological self-confidence holds even greater significance as it profoundly impacts students' capacity to navigate mathematical software (Roschelle et al., 2000), employ digital tools (Bolaños et al., 2023), and ultimately, enhance their ability to communicate mathematical ideas (Cretchley, 2007). This is where the concept of mathematical representations enters the equation. Mathematical representations encompass a spectrum of visual, symbolic, and verbal depictions that play a crucial role in conveying mathematical concepts and relationships (Cartwright, 2020). The intriguing intersection of technological self-confidence and mathematical representations prompts us to examine how students' confidence in technology utilization influences their proficiency in generating, interpreting, and employing mathematical representations (Hill & Uribe-Florez, 2020).

Research into the interaction between technological self-confidence and mathematical representations has unveiled intriguing connections. For example, a study by Zeldin et al. (2008). There is a positive relationship between technological self-efficacy and mathematics, science, and technology careers. Students who excel in mathematics, science, and technology fields often exhibit a high level of technology self-efficacy. Their competence and confidence in using technology to solve problems, conduct research, and engage in various technical tasks contribute to their success in these career areas (Attard et al., 2016). At the same time, a study by Ke (2014) conducted in the context of secondary education revealed that students with higher technological self-confidence tend to exhibit an increased tendency for creating visual and symbolic representations. Similarly, a survey by Magen-Nagar & Shonfeld (2018) found that students who reported higher levels of technological self-confidence were more inclined to utilize digital platforms for collaborative

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mathematical modeling, showcasing the link between self-confidence in technology and effective mathematical representations.

Furthermore, the study by Brezavšček et al. (2020) examined various factors that impact mathematics achievement in a higher education context. Although the setting differs from secondary education, their findings could still offer insights into how self-perception of mathematics and technological self-confidence can affect mathematics achievement. Their findings demonstrated that confidence with technology positively affects perceived usefulness of technology in learning mathematics. This aligns with Vygotsky's Zone of Proximal Development (ZPD) theory, suggesting that students' technological self-confidence acts as a catalyst for reaching their full potential in generating meaningful mathematical representations (Vygotsky & Cole, 1978). The existing body of research illuminates the integral connection between technological self-confidence and mathematical representations, emphasizing its pertinence within the modern educational landscape. In this paper we assume technological self-confidence effects mathematical representation.

Self-perception of Mathematics and Mathematical Representations

The self-perception of mathematics and mathematical representations refers to an individual's own assessment and beliefs about their abilities, attitudes, and understanding of mathematical concepts and their graphical or symbolic representations. In the context of education, this self-perception plays a significant role. A student's self-perception of mathematics can influence their motivation (Tully & Jacobs, 2010), learning strategies (Ben-Chaim & Zoller, 2001), and ultimately, their performance in mathematical tasks (Shen & Pedulla, 2000). If a student has a positive self-perception of their mathematical abilities, they are more likely to be motivated to learn, apply problem-solving strategies, and persist in the face of challenges. On the other hand, a negative self-perception may lead to reduced motivation and avoidance of mathematical tasks (S. Skaalvik & Skaalvik, 2004).

The finding from the literature provide some insights into the relationship between self-perception of mathematics and mathematical representations. Self-concept, which is similar to self-perception, is positively associated with mathematics achievement (OECD, 2013). Furthermore, the influence of different mathematical representations and degrees of mathematics self-efficacy on the performance of learners in pattern reasoning and their attitude towards learning mathematics has been investigated (Chen et al., 2015). We assumed that the self-perception of mathematics can lead to mathematical representation.

Self-perception of Mathematics as a Mediator Between Technological Self-confidence and Mathematical Representations

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The studies collectively underscore the intricate interplay between emotional well-being, autonomy in mathematics learning, and self-confidence in shaping students' self-perception of mathematics. These factors contribute to the broader landscape of mathematics education and can guide strategies to enhance students' mathematical self-concepts and overall competence in the subject.

Students who feel secure, relaxed, and emotionally stable while engaging in mathematics activity tend to judge their competence on the mathematics activity very highly (Zakariya, 2022). This demonstrates the strong connection between emotional states and self-perception in mathematics. Moreover, correlations exist between mathematics learning autonomy and mathematics self-concept (Bringula et al., 2021). This research indicates that students who exhibit higher levels of autonomy in their mathematics learning are more likely to develop positive self-concepts in mathematics. Autonomy allows students to take control of their learning, leading to increased confidence in their mathematical abilities. Furthermore, self-confidence in mathematics refers to how students perceive themselves in relation to their own abilities in mathematics (Aguilar, 2021). This self-confidence plays a pivotal role in shaping students' self-perception of mathematics. As students perceive themselves as capable and competent in mathematics, their self-concept in mathematics becomes more positive.

Based on the above points, it can be inferred that self-perception of mathematics, specifically mathematics self-concept and mathematics self-efficacy, can act as a mediator between technological self-confidence and mathematical representations. Students who have high levels of technological self-efficacy and mathematics self-efficacy are more likely to engage in mathematics tasks and perform better in mathematics. Additionally, students who have high levels of mathematics self-concept are more likely to perform well in mathematics. Therefore, it is important to understand how students perceive their abilities related to mathematics and technology to improve their engagement and performance in mathematics tasks.

Gender

The role of gender in the context of self-perception of mathematics and technological self-confidence in predicting mathematical representations among secondary education students is a subject of significant interest and research.

An empirical study by Xie and Liu (2023) reveals that traditional gender stereotypes have a dual impact on mathematical performance: they tend to enhance the performance of male students while undermining that of female students. Notably, the authors observe that, despite prevailing stereotypes, female students do not exhibit the classic symptoms of stereotype threat, a psychological phenomenon that typically impairs performance in male-dominated fields such as mathematics. Instead, although gender role perceptions negatively influence female students' quantitative abilities, they do not trigger the heightened anxiety commonly associated with stereotype threat.

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Moreover, the findings suggest that societal and familial beliefs about gender roles significantly shape students' engagement with mathematics.

Despite gender differences in mathematics achievement narrowing over the last decades, gender differences in self-concept in mathematics remain considerable (Mejía-Rodríguez et al., 2021; Sax, 1994). Furthermore, research has confirmed gender differences, even in secondary education, in mathematics self-concept, self-efficacy, and interest, suggesting that gender differences in mathematics motivation emerge early in life (Mejía-Rodríguez et al., 2021). Moreover, gender stereotypes are believed to play a role in the heightened math anxiety reported by female students (Justicia-Galiano et al., 2023).

Based on the above points, it can be inferred that gender plays a significant role in the self-perception of mathematics and technological self-confidence in predicting mathematical representations among secondary education students. Therefore, it is important to understand the role of gender in the self-perception of mathematics and technological self-confidence to improve students' engagement and performance in mathematics tasks.

Present Study

This study aims to investigate the role of self-perception of mathematics and technological self-confidence in predicting mathematical representations in secondary education in Indonesia. The primary focus of the research is to understand the extent to which these psychological factors influence students' ability to represent mathematical problems. According to Anderson (1949), educational psychology is defined as the science of human learning. These factors include social and psychological resources, psychological risk factors, self-efficacy, beliefs, attitudes, motivations, emotional attachments, and coping responses to life (Thomas et al., 2020). In the context of education and learning, these factors include cognitive processes such as perception, memory, reasoning, and problem-solving, which are crucial for understanding and representing mathematical problems. In Indonesia, students often face challenges in understanding and representing mathematical concepts. Therefore, it is essential to comprehend the factors that affect students' ability to comprehend and represent mathematical problems.

Research has consistently indicated that self-perception of mathematics and technological self-confidence play significant roles in shaping students' capacity to comprehend and address mathematical challenges. As established in the literature review, these psychological factors have been shown to impact students' mathematical problem-solving abilities (Brezavšček et al., 2020; S. Skaalvik & Skaalvik, 2004, 2004; Xie & Liu, 2023). However, this research focuses on the context of secondary education in Indonesia, which has unique cultural and educational characteristics. The uniqueness of this study is that it is one of the first to explore the relationship between self-perception of mathematics, technological self-confidence, and mathematical representation

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abilities in the context of secondary education in Indonesia. This research is expected to provide new insights into the factors influencing students' mathematical abilities and can serve as a basis for the development of more effective educational strategies to enhance students' mathematical representations in Indonesia.

Hypotheses

Based on the theoretical framework, the following hypotheses were formulated:

Hypothesis 1: Technological self-confidence positively affects mathematical representations.
Hypothesis 2: Self-perception of mathematics positively affects mathematical representations
Hypothesis 3: Self-perception of mathematics as expected to mediate the relationship between technological self-confidence and mathematical representations.
Hypothesis 4: There are differences between male and female among variables.

The model illustrating these hypotheses is provided in Figure 1.

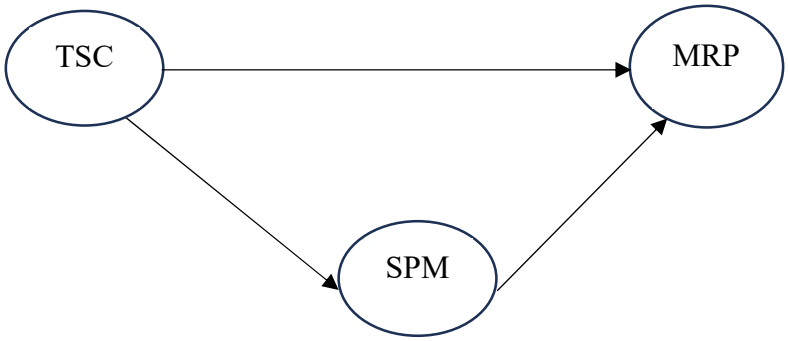


Figure 1: The model of self-perception of mathematics (SPM), technological self-confidence (TSC), and predicting mathematical representation (MRP).

METHOD

Participants

In this cross-sectional research, the participants consisted of 322 secondary education students in Bandar Lampung, Indonesia, through cluster random sampling (Gliner et al., 2016). Among the participants, 192 (59.6%) were female students with an average age of 11.27 years (*SD* = 0.46), and 130 (40.4%) were male students with an average age of 11.17 years (*SD* = 0.38). The research received approval from the Institutional Review Board of Universitas Islam Negeri Raden Intan Lampung, in compliance with the institution's ethical standards. Furthermore, all participants

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provided their informed consent prior to their involvement in the study. Further details regarding the sample characteristics can be found in Table 1.

Demographic Factor		Frequency	Percentage (%)
Gender	Female	192	59.6
	Male	130	40.4
School Type	Private	202	62.7
	Public	120	37.3
Age	11 years	248	77.1
	12 years	74	22.9
Place of Residence	City	127	39.4
	Urban	195	60.6

Table 1: The characteristic of the participants.

Instruments

In our study, students were requested to complete a questionnaire consisting of 12 items related to self-perception of mathematics and technological self-confidence. Additionally, they were presented with 5 test items assessing mathematical representation skills. Background information, including age, gender, place of residence, and school type, was also collected from the participants. Detailed information about the instruments used in the study will be provided.

Self-perception of mathematics. The self-perception of mathematics questionnaire consisted of six items that were adapted from Suherman & Vidákovich (2022a). These items were translated and validated in the Indonesian context. The questionnaire aimed to assess students' feelings and perceptions regarding their mathematical abilities. Sample items included statements such as "Math is very hard for me" and "I can tell if my answers in math make sense." Participants were asked to rate these items using a five-point Likert scale, where 5 indicated "strongly agree" and 1 indicated "strongly disagree." Cronbach's alpha was measured at 0.79, while McDonald's coefficient omega for these statements also registered a value of 0.79. We also evaluated the scale's validity and reliability.

Technological self-confidence. The technological self-confidence questionnaire in this study consisted of six items related to one's belief in their ability to successfully perform technologically sophisticated tasks. These items included statements such as "I am sure I could work with computers" and "I am sure I could learn a computer language." The questionnaire was adapted from Francis et al. (2000). Participants were asked to respond to these items using a 5-point Likert scale, where 5 represented "strongly agree" and 1 represented "strongly disagree." Cronbach's alpha this statement was measured at 0.85. Again, we also measured the validity and reliability of this scale.

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Mathematical representation. The test of mathematical representation employed in this study is designed to gauge how students interpret and express their thoughts when addressing mathematical problems. This test serves as a tool for uncovering their problem-solving skills. An example of a test question is as follows in Table 2. This particular question was adapted from (Suherman & Vidakovich, 2022). Scores are allocated according to categorized responses. Each response is then evaluated for its level and awarded one to five points accordingly. Cronbach's alpha of this instruments was 0.86 and reliability of the person and items was 0.86 and 0.85, respectively. Again, we also evaluated the instruments in this study. We provide one example of these instruments.

Questions	Picture
The pictures are part of Tapis Lampung with geometry motif. 1. Make a list of any flat shapes that you find in the Tapis Lampung motif! 2. Draw any pictures from your findings using at least one flat shape you listed in number 1. You can combine 2 or 3 or more flat shapes to create a unique image. Then name the image you have made.	

Table 2: An Example of the test.

Procedure

The initial phase involved translating the assessment instruments into the Indonesian language. The items of the tests and questionnaires were thoroughly reviewed by one expert and two mathematics teachers prior to their administration. To ensure compliance and gain permission for the research, each school principal was contacted and provided with a formal letter outlining the study's objectives. Subsequently, 7 classes were randomly chosen from five different secondary schools in Bandar Lampung. This process resulted in the participation of a total of 322 students in the current research. Data collection was carried out through the administration of paper-pencil tests.

Data Analysis

In this study, we utilized several software tools for data analysis, including MPlus version 8.3 (Muthén & Muthén, 2017), SPSS version 25 (IBM Corp, 2017), R software, and Winstep. The data analysis consisted of a multi-faceted approach. We conducted confirmatory factor analysis (CFA) using the questionnaire data to assess validity and reliability. Simultaneously, we employed the Winstep software to evaluate the validity and reliability of the test items in the context of Rasch measurement. The reliability of the data was also assessed using Cronbach's Alpha. In addition, we performed descriptive statistical analysis to elucidate the mean and standard deviation (*SD*) of

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the test items and questionnaire. Lastly, structural equation modeling (SEM) was employed to examine the hypothetical model.

We utilized various parameters to assess the model fit. Maximum likelihood was employed as an indicator for loading factors, with a threshold value of approximately 0.30. Additionally, we considered other parameters, including the Tucker-Lewis Index (TLI) and the Comparative Fit Index (CFI), Chi-square, the Root Mean Square Error of Approximation (RMSEA), and the Standardized Root Mean Squared Residual (SRMR). Following the criteria proposed by Hu & Bentler (1999), we aimed for CFI and TLI values close to or greater than 0.90, an SRMR value less than 0.80 (Hu & Bentler, 1999), RMSEA value of < 0.05 indicates a close fit, while a value < 0.08 suggests a reasonable model–data fit (Browne, 1993; Jöreskog & Sörbom, 1993; Xia & Yang, 2019). Furthermore, we also conducted Chi-square statistics to evaluate the overall model fit.

RESULTS

Validity, Reliability, and Descriptive Statistic of the Instrument.

Descriptive statistics were employed to examine the distribution of questionnaire and test responses. As indicated in Table 3, this table provides an overview of the dataset. The data reveals that the means for each variable fall within the range of 3.53 to 3.89 on a 5-point Likert scale, while the *SDs* range from 0.69 to 0.99. To assess the internal consistency or reliability of the variables, Cronbach's alpha was calculated. The Cronbach alpha values for self-perception of mathematics, technology self-confidence, and mathematical representation were approximately 0.72, 0.75, and 0.67, respectively, indicating that these variables are consistent and dependable for further analysis.

Variables	M	SD	Cronbach Alpha	McDonald Omega	Skewness	Kurtosis
Self-perception of mathematics	3.89	0.69	0.72	0.72	1.01	0.67
Technology self-confidence	3.54	0.73	0.75	0.76	0.04	0.16
Mathematical representation	3.53	0.99	0.67	0.68	0.65	-0.16

Table 3: Descriptive statistics.

Regarding data normality, Kline (2015) recommended that skewness values should not surpass |3| and kurtosis should be under |10|. In this study, skewness values varied from 0.04 to 1.01, and kurtosis ranged from -0.16 to 0.67. Table 3 presents the correlations between the elements of each variable.

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We conducted a CFA of the questionnaire to evaluate the relationships between the items and their respective latent constructs. The results of this analysis are displayed in Table 4, which provides information on the loading factors for each item in the scale. These loading factors reflect the strength and direction of the relationship between each item and the underlying construct it is intended to measure. This analysis allows us to assess the validity and reliability of the measurement model, confirming that the items effectively capture the intended constructs.

Scale	Items	Loading factor
Self-perception of mathematics	SPM1	0.55
	SPM2	0.66
	SPM3	0.56
	SPM4	0.45
	SPM5	0.48
	SPM6	0.53
Technology self-confidence	TSC1	0.46
	TSC2	0.45
	TSC3	0.45
	TSC4	0.54
	TSC5	0.49
	TSC6	0.45

Table 4: Loading factors of item questionnaire.

Table 4 reveals that the loading factor values for the self-perception of mathematics items fall within the range of 0.45 to 0.55. Similarly, for technology self-confidence, the loading factor values range from 0.45 to 0.54. These values demonstrate the strength of the relationship between each item and its respective construct. The consistency of these loading factor values across the items within each construct contributes to the reliability of the measurement model, ensuring that the items effectively measure the intended constructs.

We conducted a validity assessment of the test items using Rasch analysis. As shown in Table 5, it provides a summary of measurements by person and test items. The Cronbach alpha for assessing

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person ability was found to be 0.96, while the reliability for the test items and person ability was 0.64 and 0.71, respectively.

Object Measured	Measure [Mean (*SD*)]	Separation	Reliability	Cronbach Alpha
Person	0.03 (1.33)	1.55	0.71	0.96
Item	0.07 (0.00)	1.32	0.64	

Table 5: Summary of measured person and item test.

We also assessed the item fit using Rasch analysis, and the results are illustrated in Figure 2. It is evident that each item fits the Rasch model well, indicating the appropriateness of the items for measuring the construct.

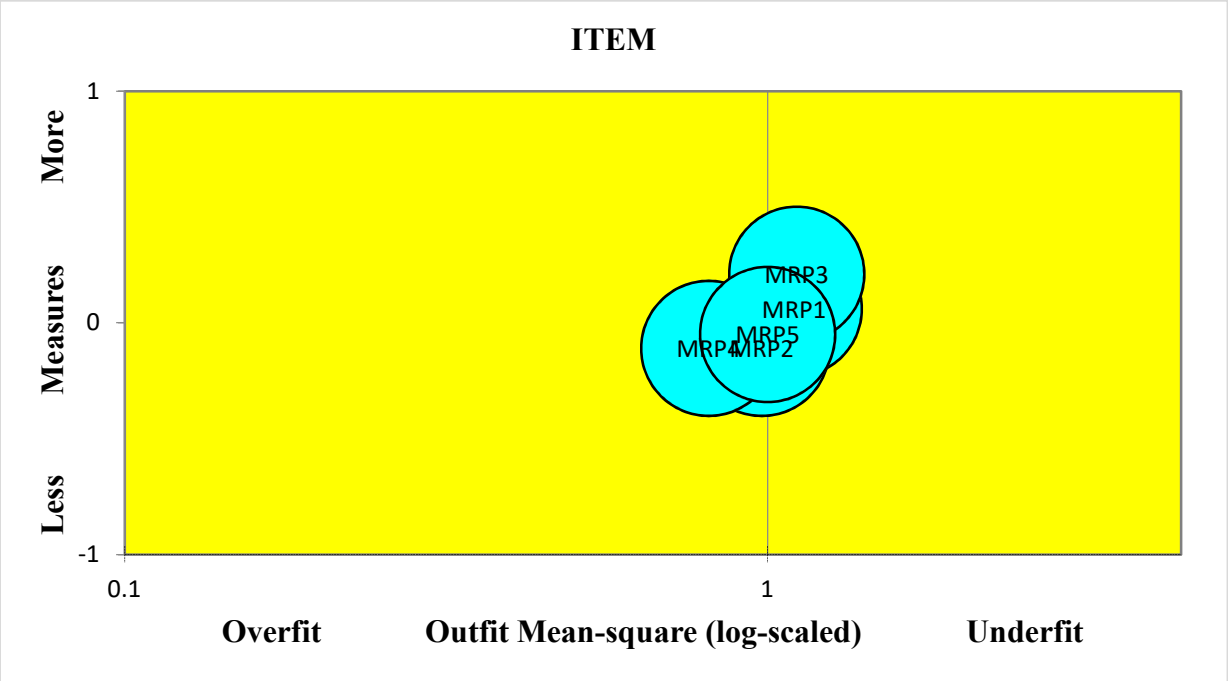


Figure 2: Bubble maps for item fit.

SEM Measurement

In our research, we proposed a hypotheses model in which self-perception of mathematics serves as a mediator between technological self-confidence and mathematical representations. Our initial

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SEM yielded the following fit indices: χ^2 (df = 116) = 411.327, p < 0.001, CFI = 0.75, TLI = 0.70, RMSEA = 0.08, SRMR = 0.05. Unfortunately, the CFI and TLI indicated a poor model fit. To improve the model fit, we made modifications (see Fig 3). The revised model demonstrated improved fit indices with χ^2 (df = 113) = 153.948, p < 0.001, CFI = 0.97, TLI = 0.96, RMSEA = 0.03, SRMR = 0.04, suggesting a much better fit to the data.

In Figure 3, we observe the SEM that provides valuable insights into the relationships between technological self-confidence, self-perception of mathematics, and mathematical representation among secondary education students. The results suggest a positive association between technological self-confidence and mathematical representation (β = 0.11, p = 0.10), although this direct relationship does not reach statistical significance. This indicates that higher levels of technological self-confidence may contribute to better mathematical representation, even though the effect is not strong. Additionally, technological self-confidence has a statistically significant positive association with self-perception of mathematics (β = 0.04, p < .05), highlighting that students with greater confidence in their technological abilities tend to have more positive self-perceptions about their mathematical skills. However, the relationship between self-perception of mathematics and mathematical representation reveals a negative association (β = -0.13, p > 0.05). This suggests that as students' self-perception of their mathematical abilities increases, their mathematical representation decreases, though the effect is not statistically significant. A noteworthy finding is that self-perception of mathematics mediates the relationship between technological self-confidence and mathematical representation with a negative association (β = -0.005, p < 0.05). This mediation suggests that the impact of technological self-confidence on mathematical representation is partially explained by students' self-perception of their mathematical abilities. The model fit for direct and indirect as also performed in table 6.

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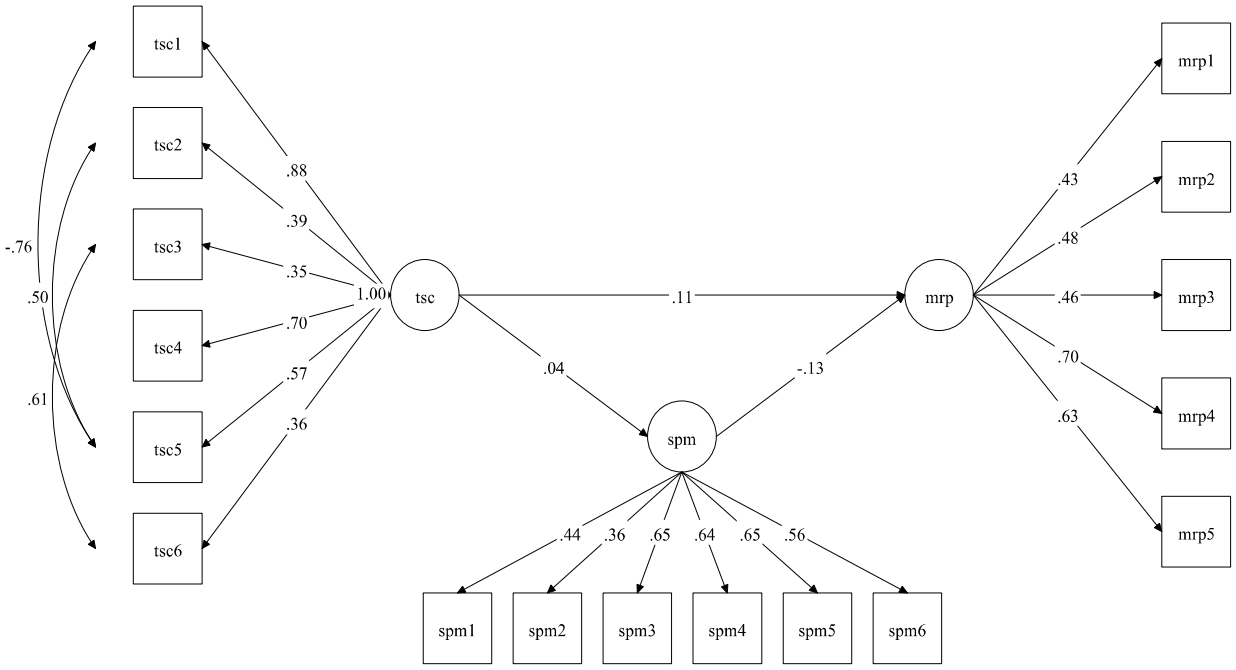


Figure 3: SEM model

Model Path	Estimate	*S.E.*	*p*	Bootstrap 95% Confidence interval	
				Lower	Upper
1. Technological self-confidence → Self-perception of mathematics	0.04	0.06	0.04	-0.07	0.14
2. Self-perception of mathematics → Mathematical representation	-0.13	0.07	0.08	-0.25	-0.03
3. Technological self-confidence → Self-perception of mathematics → Mathematical representation	-0.005	0.01	0.58	-0.03	0.00

Table 6: Path Model summary. Note: S.E. = Standard error.

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To assess the performance differences between male and female students, we analyzed the data using a violin plot. The Figure 4 violin plot visually depicts various variables, including students' mathematical representation. This type of plot, known as a violin plot, serves as an effective tool for comparing and visualizing a wide range of data. It provides insights into several key aspects of the data's distribution, such as its symmetry, the central value's location, and the dispersion of data observations. Essentially, the violin plot offers a comprehensive view of how data values are distributed, offering valuable insights into the dataset's characteristics (Potter et al., 2006). The detail of students' answer can be seen in Figure 5.

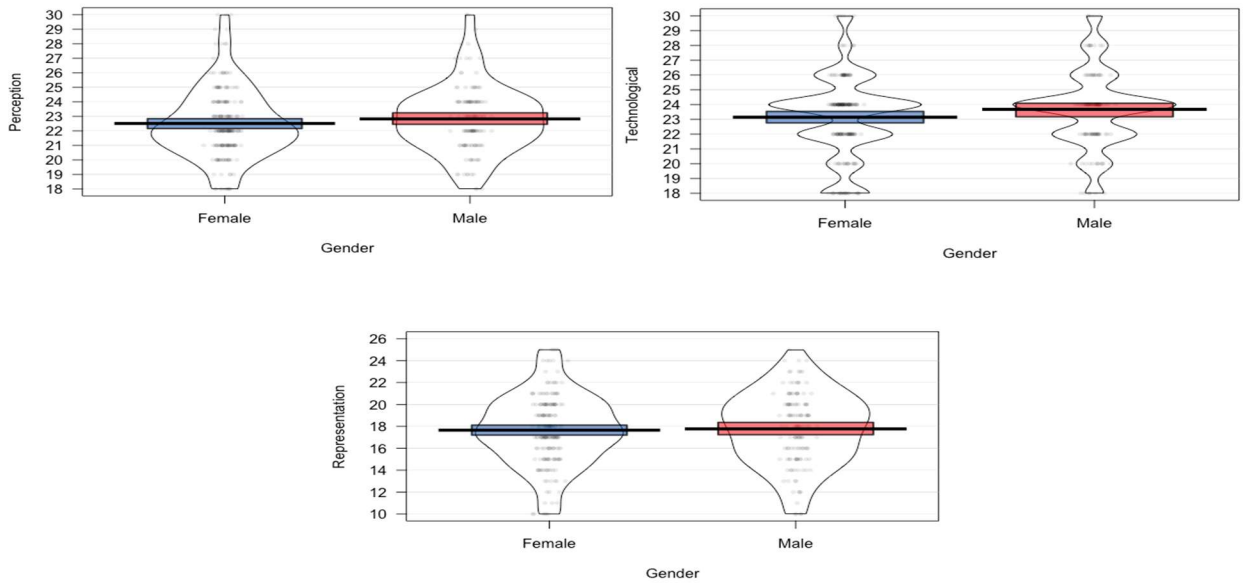
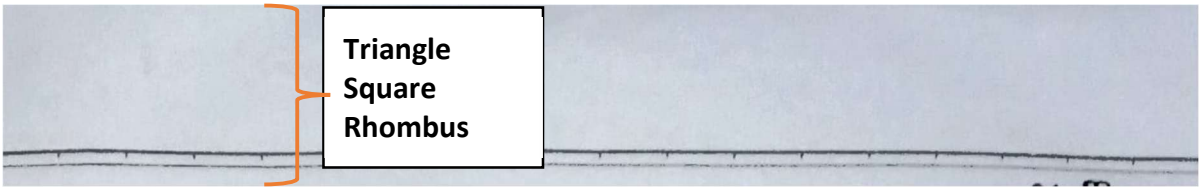


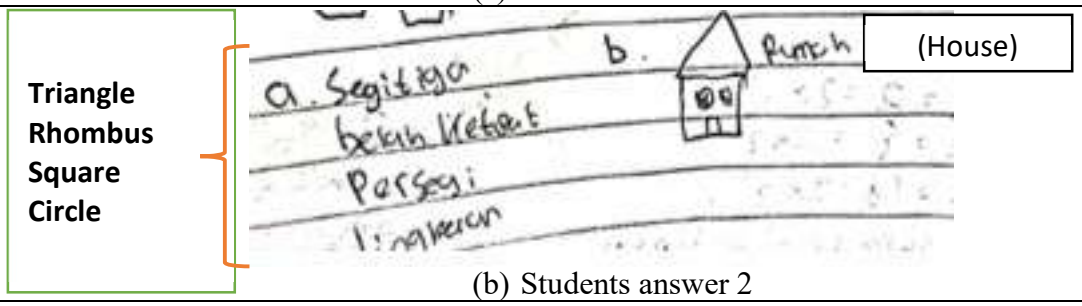
Figure 4: Violin plot of students performance across various variables.

Figure 4 displays the descriptive statistics for the variables in the study. The mean score for self-perception of mathematics was 22.64, with a standard deviation of 2.39. Technology self-confidence and mathematical representation had mean scores of 23.36 (SD = 2.77) and 17.70 (SD = 3.28), respectively. Moreover, we assessed the differences between male and female students in each variable category. The results indicated that there were no statistically significant differences between male and female students. Specifically, for mathematical representation, the t-value was -0.29, with $p > 0.05$, for self-perception of mathematics, the t-value was -1.18, with $p > 0.05$, and for technology self-confidence, the t-value was -1.95, with $p > 0.05$. The results of the statistical analysis indicate that there are no significant differences between male and female students in terms of mathematical representation, self-perception of mathematics, and technology self-confidence. In other words, both male and female students show similar performance and attitudes in these areas.

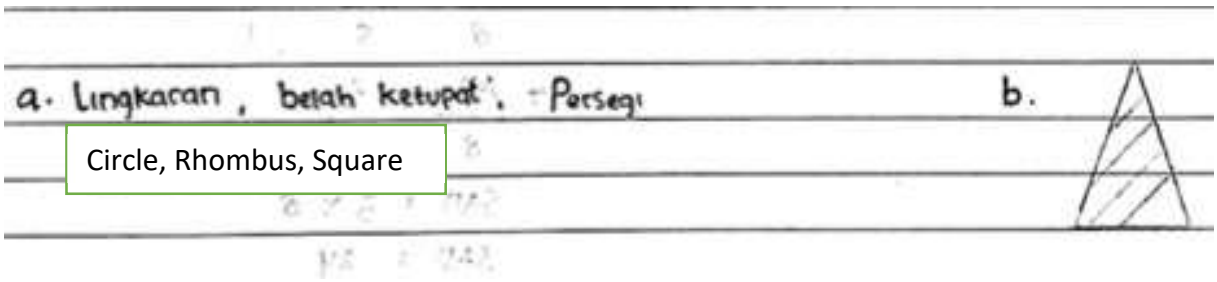
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(a) Students answer 1



(b) Students answer 2



(c) Students answer 3

Figure 5: An example of students answer.

DISCUSSION

The main goal of this research was to explore the SEM that examines the relationships between technological self-confidence, self-perception of mathematics, and mathematical representation in secondary education.

The findings of our study revealed a direct association between technological self-confidence and mathematical representation, supporting our first hypothesis. This line was supported by previous research (Attard et al., 2016; Ke, 2014). This implies that fostering technological self-confidence among students may encourage them to engage with digital educational tools and enhance their mathematical representation skills. While the statistical effect may not be particularly robust, it is noteworthy that technological self-confidence could still play a part in improving mathematical representation. Individual variations among students, such as their previous encounters with technology (Byungura et al., 2018), familiarity with particular software or tools (Frowd et al., 2011),

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and preferred learning methods (Wishart, 2005), may all influence the extent to which technological self-confidence influences their mathematical representation abilities (Chen et al., 2015; Hartsell et al., 2010). Students who have good experience and confidence with technology tools can improve their skills and represent mathematical concepts more effectively (Kunhertanti & Santosa, 2018). Therefore, it can be inferred that there may be a direct association between technological self-confidence and mathematical representation, as students who are confident in their technological skills may be better able to represent mathematical concepts effectively. By addressing these points, teachers can help students develop their technological self-confidence and improve their mathematical representation abilities, mathematical creative thinking (Suherman & Vidákovich, 2022b), promote students' skill in solving algebraic problems (Khairunnisak et al., 2021), mastery mathematics (Maskur et al., 2022), and metaphorical thinking (Farida et al., 2022) ultimately enhancing their overall mathematical literacy and problem-solving skills (Supriadi et al., 2024).

The results of this study further revealed a statistically significant positive associated between technological self-confidence and self-perception of mathematics, thereby substantiating our second hypothesis. This discovery aligns with previous research by Chen et al. (2015) and Karatas et al. (2017), which proposed similar relationships. In essence, this finding suggests that students who exhibit higher levels of confidence in their technological skills also tend to hold more positive self-perceptions regarding their mathematical abilities. According to a study by Zander et al. (2020), students who exhibit higher levels of confidence in their technological skills also tend to hold more positive self-perceptions regarding their mathematical abilities. This finding suggests that self-efficacy, or the belief in one's ability to succeed in a particular task, can be transferable across domains. Other studies have also explored the relationship between self-perceived competence in mathematics and positive affect towards mathematics on mathematics achievement (Areepattamannil & Kaur, 2012), as well as the impact of self-efficacy on problem-posing and solving skills in mathematics (Görgün & Tican, 2020). Self-confidence has been observed as the greatest non-cognitive predictor for academic achievement among other self-belief measures such as self-efficacy and self-concept (Kaur & Prendergast, 2022). Therefore, it is important to foster self-efficacy and self-confidence in students to improve their performance in mathematics. Technology's significance in education is paramount, especially considering the foundational role attributed to mathematics as a core subject (Yanuarto et al., 2023). By acknowledging and leveraging this association, educators can create an environment where students feel more confident and capable in both technology and mathematics, ultimately leading to improved academic outcomes and a more positive attitude toward math.

Our study also identified a negative association between self-perception of mathematics and mathematical representation. This research is supported by Boyer & Mailloux (2015). Although there seems to be a general trend of decreasing mathematical representation as self-perception of mathematics increases, the presence of variations among students within the sample may obscure the

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overall effect. This phenomenon suggests that while, on average, higher self-perception of mathematics is linked to decreased mathematical representation (Skaalvik & Rankin, 1994), individual differences play a crucial role in shaping this relationship. Factors such as students' prior experiences, learning styles, and problem-solving approaches can contribute to these variations. Even exploring students' perceptions of mathematics allows us to grasp how the formation of mathematical knowledge unfolds (Rosa et al., 2020). Consequently, the overall effect is not as pronounced as one might expect.

In the context of the mediation analysis, our study has shed light on the intricate relationship among technological self-confidence, self-perception of mathematics, and mathematical representation. Specifically, our findings have revealed that self-perception of mathematics acts as a mediator in the relationship between technological self-confidence and mathematical representation. Importantly, this mediation process is characterized by a negative association, aligning with the hypothesis we initially posited (Hypothesis 3). The negative association observed in this mediation warrants a deeper exploration (Phan, 2009). It can potentially be attributed to several underlying factors that influence how students perceive their mathematical abilities and their actual mathematical representation skills (Helwig et al., 2001). This can lead to errors and difficulties in problem-solving tasks that demand strong mathematical representation skills (Sari & Rosjanuardi, 2018). While students with high technological self-confidence may exhibit a positive self-perception of their mathematical abilities (Ayuso et al., 2020), this may not necessarily translate into superior mathematical representation skills (Mullins et al., 2011). Furthermore, it's conceivable that students with strong technological self-confidence may overestimate their mathematical competencies (Foster, 2016). In light of these findings, it becomes evident that addressing mathematical representation skills among students should encompass a comprehensive approach that considers both their confidence levels and actual competencies. A positive self-perception alone may not be adequate to enhance their mathematical representation abilities. Educators and policymakers should strive to bridge the gap between students' confidence and competence in mathematics, recognizing the multifaceted nature of this relationship and integration technology in the classroom (Suherman et al., 2021).

In our analysis of students' performance, we observed no significant differences between male and female students concerning mathematical representation, self-perception of mathematics, and technology self-confidence (Hypothesis 4). This indicates that both male and female students exhibit similar levels of performance in these domains (Zhan et al., 2015). In other words, gender does not appear to be a significant factor influencing students' abilities (Mejía-Rodríguez et al., 2021) and confidence related to mathematical representation (Carr et al., 2008) and technology use in the context of mathematics (Sáinz & Eccles, 2012). This finding suggests that educational efforts and interventions aimed at improving mathematical representation and technology-related skills should be designed to benefit all students regardless of their gender, as there are no apparent gender-based disparities in this regard.

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LIMITATION AND FUTURE RESEARCH

This study has made significant contributions to our understanding of the intricate connections between technological self-confidence, self-perception of mathematics, and mathematical representation in the context of secondary education. However, it is crucial to acknowledge several limitations that should be considered in future research. First, the sample used in this study consisted of secondary education students from a specific geographic region, which could limit the generalizability of the findings. Future research should endeavour to include a more diverse and representative sample that encompasses a broader spectrum of backgrounds and contexts, such as primary or tertiary education, would provide a broader perspective on these relationships.

Second, the study relied on self-report measures for variables like technological self-confidence, self-perception of mathematics, and mathematical representation. While self-reports offer valuable insights, they may be subject to response bias and may not capture the full complexity of these constructs. Combining self-report measures with objective assessments could enhance the robustness of future research. Third, the cross-sectional design of this study provided a snapshot of the relationships at a particular moment. To gain deeper insights into the dynamics of these relationships, future research should adopt longitudinal designs that track changes over time. Lastly, it's important to note that some of the associations observed in this study did not reach statistical significance. Future research should delve into the factors contributing to these non-significant relationships to uncover the underlying mechanisms at play.

CONCLUSIONS

In summary, the findings of this study revealed several key associations and mediating relationships. First, technological self-confidence was positively associated with both self-perception of mathematics and mathematical representation. However, self-perception of mathematics exhibited a negative association with mathematical representation. Furthermore, our research demonstrated that self-perception of mathematics plays a mediating role in the relationship between technological self-confidence and mathematical representation, and this mediation is characterized by a negative association. Additionally, male and female students exhibit comparable performance in these domains. These results shed light on the intricate interplay between students' confidence in technology, their perceptions of their mathematical abilities, and their mathematical representation skills, offering valuable insights for educators and policymakers seeking to enhance mathematics education, digital literacy, and digital age in secondary education.

ACKNOWLEDGMENTS

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We express our gratitude to Universitas Islam Negeri Raden Intan Lampung for their research and financial support. We also extend our thanks to the Doctoral School of Education at the University of Szeged and Universitas Persada Indonesia, for their valuable suggestions and discussions, contributing to the development of ideas in the research perspective and collaborative research.

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MATHEMATICS TEACHING RESEARCH JOURNAL
SUMMER 2025
Vol 17 no 3

Introducing Teacher Education Students to Escape Room as a Didactic Tool in Mathematics.

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Abstract: The aim of this study is to understand teacher education students' experience with an introduction to use of escape room as a didactic tool in mathematics, and to gain insight on the impact of such an introduction on teacher education students' development as mathematics teachers. Escape room is a new, game-based, didactic tool in school that offers teachers new possibilities in the mathematics classroom. Twelve teacher education students participated in the study, and data were collected through a qualitative survey, observation, and focus-group interviews. The data were analyzed through use of a framework based on game-based learning and on Dewey's perspective about what to learn and how to learn. The findings show that use of escape room in mathematics teaching was met from a traditional perspective on teaching and learning by the teacher education students, but that they showed glimpses of willingness and curiosity about its use in mathematics in school. The findings also generated hypotheses and probable conclusions that, in relation to Klafki's theory of categorical bildung, offer some understanding of possible influence on teacher education students' development of bildung as mathematics teachers, and their possible position as future change agents for mathematics teaching in school.

Keywords: Escape room, mathematics, teacher education students, bildung.

INTRODUCTION

"The aim of education is to enable individuals to continue their education" (Dewey, 1916, p. 56). The Organisation for Economic Co-operation and Development (OECD) (2019) emphasizes the future importance of citizens to be able to understand, interpret and apply knowledge and skills in various situations, and therefore that one must learn to navigate through unfamiliar contexts. Escape rooms are live-action, team-based games in which the players discover clues, solve puzzles, and accomplish tasks to complete a mission in a limited amount of time (Nicholson, 2015). A review of 70 research articles on the use of escape room in education showed that escape room

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has been adopted in teaching, both for emphasis on the participants' encountered need for application of subject area knowledge and skills, and for emphasis on necessity of cooperation and trust in co-participators' competence (Taraldsen et al., 2020).

Mathematics is a mandatory subject in primary and secondary school all over the world, and some countries are already introducing new sets of methodologies in the teaching of mathematics to meet the expectations for future schooling (European Schoolnet, 2018). In Norway, for instance, the recent revision of the national curriculum gives explicit attention to problem-solving (Udir, 2019). Polya (1981, p. 117) defines problem-solving as: "to search consciously for some action appropriate to attain a clearly conceived, but not immediately attainable, aim." According to Schoenfeld (1993) problem-solving competence is about applying mathematical concepts, methods and relations in order to solve problems. Such an action-oriented understanding of competence implies that competence does not show itself as something one has knowledge about, but as something one can put into action. The escape room idea is based on emphasis of features that provide such opportunities for the participants in an escape room event. Herein lays the novelty of use of escape room in mathematics teaching and learning. The pupil is given the opportunity to display problem-solving competence within a context that is experienced as both playful and meaningful. Within educational research we see a steady increase in the use of escape room as a didactic tool, within various contexts. Escape rooms are, for instance investigated as an active methodological approach for the purpose of teaching mathematics (Alabdulaziz, 2023; Charlo, 2020; Fuentes-Cabrera et al., 2020), or used for explicit attention to mathematical subjects, like for instance Andrews and Bagdasar's (2023) emphasis on geometry and Jiménez et al.'s (2020) focus on algebra. One also finds explicit attention to authentic scenarios related to vocational education, like for instance within professional nursing practice (Morell et al., 2020), computer security (Béguin et al., 2019), and forensic education (Ferreiro-González et al., 2019). Research on use of escape room in education is still in an early phase (Taraldsen et al., 2020), and Veldkamp et al. (2020b) conclude that the teacher's application of escape room, with their complex sets of interrelations, and the desired outcome of implementing an educational game to help achieve desired learning outcome for the pupils, so far do not align.

The study presented in this article, is concentrated around 12 Norwegian teacher education students who participate in a mathematics course on problem-solving where the students are introduced to escape room as a possible didactic tool. Borasi and Finnegan (2010) and Van der Heijden et al. (2015) highlight the potential of teachers as change agents, and Taraldsen et al. (2020) identify this potential in the teacher education student's position as well, because teacher education students may bring new ideas to the schools where they have their teaching practice as students and to the schools where they eventually will work as teachers. One of such ideas is the use of escape room as a didactic tool in mathematics. According to the theories of Dewey (e.g. Dewey, 1916, 1938), reflection on active and passive experience may transfer into learning, which means that teacher education students in practice periods, and then as qualified teachers,

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might bring new ideas, didactic tools, and possibilities into the school, based on what they learn during their teacher education. This is not just because they learn something specific about a didactic tool such as escape room, but also because, in general, they go through a significant kind of learning-based metamorphosis, from being an experienced pupil to becoming a new teacher, which may affect their beliefs about mathematics teaching and learning (e.g. Beijaard et al., 2000; Bishop, 2001; Philipp, 2007; Thompson, 1992), and therefore what priorities to make in their teaching. Beliefs seem to be in a dynamic interaction with knowledge, recognized by Bishop (2001) and Philipp (2007) as subjective knowledge. Beliefs are what one believes to be true. Subject to influence from feelings, beliefs become materialized through actions and are thereby visualized as values (Bishop, 2001). Van Zoest et al. (1994) reveal that teacher education students in general enter the teacher educational program with quite traditional, experience-based beliefs about teaching and learning priorities in school mathematics. Such beliefs may be seen in relation to experience as a pupil, for instance related to organization of mathematics lessons (Topphol, 2012) or norms in the mathematics classroom (Yackel & Rasmussen, 2002). Through their time at a teacher education program the students may be introduced to, and influenced by, approaches and opinions regarding mathematics teaching and learning that may challenge their beliefs, although this is not always the case (Amundsen & Haakstad, 2018). However, because of its well-documented novelty, the introduction to escape room, which serves as the foundation for this article, offered the participating teacher education students one such belief-influencing opportunity.

The identification of the need for attention to teachers' use of escape room (e.g. Veldkamp et al., 2020b), and the opportunity the introduction of such a didactic tool might offer regarding teacher education students' development as mathematics teachers and position as possible change agents for mathematics teaching in school, makes the objective of this study two-folded. The objectives are to understand the teacher education students' experiences of the use of escape room as a didactic tool in mathematics, and to gain insight on the impact of such an introduction on teacher education students' development as mathematics teachers. In order to approach these objectives this study intends to answer the following question: *How do teacher education students in a mathematics education course experience an introduction to use of escape room as a didactic tool?*

THEORETICAL FRAMEWORK

Game-based Learning

In educational research, facilitation for game-based learning (GBL) is recognized as an increasingly popular approach in school and teacher education (Plass et al., 2015; Moon et al., 2023). Salen and Zimmerman's (2004) definition of a game acts in this article as the foundation for

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GBL in general and for the escape room: 'a system in which players engage in an artificial conflict, defined by rules, that results in a quantifiable outcome' (p. 80). Plass et al. (2015) narrow the understanding of GBL down to a distinction between gamification and GBL. A gamification of content is, for instance, recognized by giving learners points or other types of credit for completing well-established activities (e.g. Lee & Lai, 2024). GBL replaces such an approach by redesigning the activities using artificial conflicts and rules of play to make them more engaging. The design of such activities relies on the balance between attention to content and the desire to play (e.g. Andriyani, 2023; Plass et al., 2015). On the one hand you want the participants to learn something, and on the other hand the only goal of play is for it to realize itself (Wilkinson, 2016). The context one enters with a playful purpose, together with goal and structure (Salen & Zimmerman, 2004), creates the GBL design.

According to Stolee (2021), there are four main categories of escape games: point and click, virtual reality, tabletop, and live-action escape games. Stolee (2021) claims that the live-action escape games offer the participants most tangibility and affordance conformity, which means that they best pave the way for activity, influence, reality, and affiliation. An escape room is a live-action escape game that, in short, offers the participants an artificial context where they are challenged as a group to make decisions, explore, find clues, and solve discovered problems to complete the mission within a given time frame (Nicholson & Cable, 2021). The escape room is built on the GBL design premises, where the goal of the contestants' learning through participating in the event is met by creating a play-based context in which the contestants immerse themselves to solve problems that are anchored in their curriculum.

In mathematics, the escape room setting may allow for emphasis on problem-solving in the mathematics classroom (Charlo, 2020). When applied in a school setting, the playful learning environment can be used to introduce the need for new subject content and skills (Andrews & Bagdasar, 2023; Borrego et al., 2017; Vergne et al., 2019). It can also be used to offer possibilities for repetition, deepening and transference of existing knowledge through application of well-known subject content and skills in new settings (de la Flor et al., 2020; López-Pernas et al., 2019). In addition, the participants in the escape room are made aware of the effects of their behavior on themselves and others (Lathwesen & Belova, 2021), through experienced need for communication, creativity, and cooperation (Peleg et al., 2019; Zhang et al., 2019; Veldkamp, 2022).

Because it is based on the definition of *game* (Salen & Zimmerman, 2004), being part of a learning community (Wenger, 1999) and learning through participation rather than through acquisition (Sfard, 1998) are important issues in GBL (Nicholson, 2018; Weines, 2021). The fundamental idea of GBL, however, is the opportunity to learn through the duality of experiences and challenges that the player encounters in the company of other players, and the complexity of this duality becomes visible in the wide range of reasoned attempts to associate various areas of influ-

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ence on learning and learning outcome with GBL (e.g. Bober, 2010; Christopoulos et al., 2024; Plass et al., 2015; Weines, 2021), and thereby with an escape room in a school setting. Four major areas of influence stand out because of their overarching and fundamentally established position in educational research on learning: experience, self-regulation, social interaction, and motivation.

Experience, Self-regulation, Social interaction, and Motivation

Experience has been a focus of attention in educational research all the way back to Dewey's theory on learning (Dewey, 1916, 1938). For Dewey, experience includes a duality between active and passive. On the active side, the experience consists of trying out or exploring an activity, while on the passive side, the experience is the exposure to the consequences of the activity. In addition, experiences are influenced by the surroundings (one does something, and this action does something in return to the one who did something). According to Dewey, activity itself does not create learning, but learning takes place when the consequences of the activity are related to the activity. Hence, activities that are experienced to be without meaning do not facilitate learning. Such experiences are mere random impulses not transferred into learning because the passive side of experience is minimal. Dewey advocated that the value of knowledge depends on its use. Most of our experiences have phases of trial and consequence, or, in the words of Dewey, an active and a passive side. One does something, and when it fails, one does something else, and continues until something that works is found, and then that is used as a rule or starting point for new actions. One discovers and becomes aware of connections between one's actions and the consequences of them. Reflections about these actions and consequences provide learning and imply engagement and responsibility for future consequences of present actions (Dewey, 1916).

Dewey's theories about active and passive experience, and reflections about one's actions and their consequences are related to theories on *self-regulation* (Zimmerman & Schunk, 2001). Self-regulated learning includes the learner's goals, plans and self-motivational beliefs, experience of progress, and reflection on experiences in order to adapt learning strategies to adjusted goals, plans and beliefs (Zimmerman, 2000). Zimmerman (2000) divides self-regulation into a cyclic process consisting of three phases: forethought, performance or volitional control, and self-reflection. The first phase consists of the learner's conscious preparation for acting based on goals and plans for the up-coming action, influenced by the learner's self-motivational beliefs. The second phase is an action phase that contains the learner's active efforts influenced by volitional strategies, and is subject to self-observation of both personal effort and mimicking of observed skills. The third phase is a reflective, post-active process providing the learner with self-judgment and self-reaction that may influence adjustment of goals, plans and beliefs in a cycle-establishing revisiting of the first phase. In a learning community (Wenger, 1999), such as an escape room in a school setting, self-regulated learning takes place through an interactive process

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between individuals and the environment they are in, a process that is operationalized by both Zimmerman and Dewey.

The principles of continuity and interaction are core in Dewey's theory of learning and show the relation between Dewey's and Zimmerman's pragmatic point of view and the influence of *social interaction* in GBL. The principle of continuity means that the present experience is based on previous experiences, and at the same time affects subsequent experiences. The previous experiences are crucial to how a situation or problem is understood and handled, and at the same time the experiences one makes in a new situation influence and shape one's understanding, interpretation, and attempt to solve future situations. The principle of interaction relates to the influence of interplay between one's previous experiences and the surrounding environment for experiences made in the present. Hence, the social dimension emphasized by both Dewey and Zimmerman highlights the importance of social interaction in GBL. This is identified by soft skills such as cooperation, confidence, communication, and critical thinking, and within these the application and development of features such as active listening, giving and receiving constructive correction, teaching of others, leadership, and trust in others through delegation of responsibility (Morell et al., 2020). Social interaction is a paramount feature within collaborative problem-solving that may trigger critical thinking skills (e.g. Khusna et al., 2024).

According to Pintrich (2003), *motivation* is often studied from four perspectives: the choice of activity, how the individual engages with an activity, persistence, and the outcome of actions. Facilitation of learning through an escape room that motivates pupils involves all four: expectation of relevance, eagerness to play and be strategic, ability to strive and endure, and future utility. From a Deweyan perspective, the pragmatic view is that the value of knowledge depends on its use, and therefore motivation is related to future usefulness, both in its strength and stimulation. Lack of activity, lack of interdisciplinarity and lack of cooperation work to alienate and inhibit motivation, according to Dewey (1938). In GBL research, however, the self-determination theory (Deci & Ryan, 1985; Ryan & Deci, 2000a) is the prevailing motivation theory (e.g. Plass et al., 2015; Weines, 2021; Wilkinson, 2016). The self-determination theory is concerned with more than just the strength of motivation. It focuses on intrinsic or extrinsic motivation as stimulation for an activity, and the relationship between the two (Ryan & Deci, 2000b). In GBL, this relates to what motivation one has for involvement in a game experience, and how it will influence the game experience if one is intrinsically or extrinsically motivated to play and to obey the rules and structure of the game. As mentioned above, the playful purpose is one of the ingredients in a GBL design (Plass, et al., 2015; Salen & Zimmerman, 2004), whether it is intrinsically or extrinsically motivated; however, without motivation to become involved in the game, there is little chance of the game's purpose being realized. In a Deweyan perspective, the intrinsic motivation will be continuously influenced by the growth of extrinsic motivation arguments because of the increase in knowledge and skills through learning. This is also the case in a self-regulation perspective because of the increase in consciousness about one's own learning. However, one

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does not stop being possibly intrinsically motivated, as evidenced by the willingness to play for the sake of the game and social interaction. Motivation is found in the balance between experience of challenge level and skill level related to the problem one faces, which aligns with Vygotsky's zone of proximal development (Nicholson, 2018; Ouariachi & Wim, 2020) and Wenger's learning community (1999). In an educational escape room setting, the operationality of how such processes are met is found in concepts such as problem-solving, exploring, play and creativity.

Didactic and Bildung

Didacticism is theory about teaching. In the classroom, the teacher's didactic stance reveals choices the teacher has made in the planning of the teaching and actions in the actual teaching situations. It also identifies both near and distant conditions for such choices. According to Klafki, the development of a didactic stance is based on bildung (Hohr, 2011). Bildung is an individual process that is continuously influenced through the individual's informal and formal knowledge- and skill-acquisition, and becomes an expression for the ability to make independent decisions (Korsgaard & Løvlie, 2003). This means that a person's bildung refers to the dynamic relation between the person's experiences and the personal and cultural development she encounters. Consistent with Dewey's principles of continuity and interaction, Klafki and Dewey seem to share the impression of learning and environment to act as conditions for the continuous development of character that is recognized as bildung (Straum, 2018). This makes bildung a process that continuously occurs in the meeting between a person's subjective prerequisites and experiences, and surroundings that are continuously culturally disseminated (Hohr, 2011).

The teacher education student enters the teacher education program with a bildung that consists of a personal identity and beliefs about mathematics teaching that are constantly confirmed or challenged by experiences made during the program. The teacher education student's identity as a mathematics teacher and beliefs about the didactics of mathematics are operationalized and challenged through the didactic choices she makes both in her own studying and in teaching practice during the teacher education program. Potential changes and refinement of such choices are related to both education and cultural development via didactic evaluations related to pedagogical processes (Arneberg, 2008; Torjussen, 2011). This means that the teacher education student's development of bildung as a mathematics teacher emerges through active experience of connections between what to learn and how to learn in mathematics and passive experience through evaluation of consequences for her didactic stance.

Klafki (1983) identifies two main categories of the concept bildung: material bildung and formal bildung. The first involves acquisition of attitudes, knowledge and values that have emerged through culture and society. Emphasis on material bildung may be expressed through a mathe-

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matics teacher's prioritization of pupils' achievement of the national curriculum goals, as evidence of their acquisition of knowledge and skills in mathematics. Formal bildung involves developing inherent abilities and competences, and may be expressed through a mathematics teacher's prioritization of the learning environment ahead of scholar content. However, according to Klafki (1983), these two categories must not be seen as separate. This echoes Dewey's theory about learning: What to learn and how to learn are closely connected. There are no skills or methods detached from content (e.g. Toft, 2018). Klafki, as well as Dewey (Straum, 2018), suggests that such interweaving is required. On the one hand it offers an experience that opens up cultural content for the learner (here: it is possible to apply use of escape room to pave the way for learning in school mathematics). On the other hand the learner opens up for the cultural content (here: it is possible to emphasize various elements related to school mathematics through use of escape room). The result is that knowledge, skills, and experiences of a general (categorical) nature are created in the learner (Straum, 2018). The learner's awareness of this interweaving of content and form is what Klafki (1983) recognizes as categorical bildung. In this study, categorical game-based bildung emerges through the teacher education student's introduction to escape room. This includes examples that extend towards something bigger and more general (categorical) than simply what an escape room is, and the opportunities it provides for fundamental experiences that may motivate an alternative, fundamental way of thinking (here: application of a GBL design) in mathematics teaching and learning.

METHOD

Selection of Participants

In the beginning of 2021, 12 teacher education students were selected randomly (Brinkmann & Kvale, 2015), from a group of 44 students taking a 7.5 ECTS (European Credit Transfer and Accumulation System) mathematics education course on problem-solving. According to the European Commission (2015) 7.5 ECTS implies a workload of approximately 200 hours. Introduction to use of escape room in mathematics is part of this course, and therefore participation in the course was the sole selection criteria. The course was conducted in the sixth semester, and is part of the students' five-year teacher education program in Norway, where they qualify for teaching of mathematics in primary and lower secondary school. When commencing the course the 12 students had already passed 30 ECTS in mathematics education.

Description of the Introduction

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The introduction consisted of three lessons that alternated between plenary lecturing and activities. Some sessions of group work were planned as part of the course, and the 12 teacher education students were divided into three groups. They were in these groups in all organized group activity during the teaching involving escape room introduction.

The first lesson (135 minutes) was devoted to the lecturer's introduction of escape room as a concept and a phenomenon. He gave escape room a definition, elaborated on different ways of designing escape rooms and previous research on the use of escape room for educational purposes, and suggested research-based indications about how this didactic tool seemed to provide opportunities in the teaching of mathematics that related to experience, self-regulation, social interaction, and motivation in mathematics. In addition, the students tried out an escape box (Veldkamp et al., 2020a). An escape box is based on the idea of players trying to break into a box instead of trying to break out of a room (Nicholson & Cable, 2021). The escape box was used as a step on the way towards a full escape room because of the lower level of complexity that an escape box entails, when all necessary information leading to a desired code is to be found on a box instead of scattered around in a room.

In the second lesson (45 minutes) the students were challenged to try out an escape room that had a content emphasis on mathematics. In groups the students were 'trapped in a cave', as shown in figure 1 below, where they had to solve five problems in order to escape.



Figure 1: Two student groups in escape room mode, 'trapped in separate caves'

The 'caves' were shaped by using walls, chairs and desks in the classroom. In each 'cave' there were different articles and puzzles leading the students in direction of problems containing both practical and cognitive challenges, and clues on how to combine all the answers, in order to

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manage to escape the room. One of the problems was for instance written on a piece of paper hidden inside an empty bottle. In this situation the students first had to get the piece of paper out of the bottle without smashing it, and then crack the mathematical problem to be found. The problem they met was: *When writing the page numbers in a book, 3289 digits were used. How many pages were there in the book, when the first page of the book was labelled 1?*

The third lesson (90 minutes) started with a debrief session related to the experiences from the escape room tried out in the second lesson. In the debrief the students' ability to cooperate was emphasized, and strategies and solutions regarding the five problems were discussed. Towards the end of this lesson a plenary discussion about experiences with the introduction to escape room in mathematics was conducted. In this discussion the students elucidated on what possible use they found in escape room as a didactic tool in mathematics.

Collection of Data

Data were collected from the participants in the study through three stages. At the first stage they answered a small, qualitative survey. This was followed by a stage of observation. Finally semi-structured focus-group interviews were conducted.

Stage 1: Before they were introduced to escape room, the 12 students were informed about the project and their role in it, gave their consent, and then completed a small, qualitative survey. The purpose of the survey was to obtain the students' previous knowledge of escape room, expectations regarding the upcoming introduction, and their thoughts on teaching of mathematics.

Stage 2: I observed the three lessons, and filled in an observation form where it was described what happened in the situations and immediate understanding of students' active and passive experiences (Dewey, 1916, 1938). In the lessons, video recordings were used to collect data from the 12 participants during the group activities. The purpose of using observation was to obtain information about the interaction in the context being studied (Krumsvik, 2019), which on this occasion was the 12 teacher education students' meeting with an introduction to use of escape room in mathematics teaching and learning. The video recordings were watched immediately after each lesson and used to nuance the field notes. The video recordings offered opportunity to develop precise descriptions of the situation and accurate retelling of expressions the students had applied.

Stage 3: Two weeks after the introduction, the three focus-group interviews were conducted (interview A, interview B and interview C). Each student group was the same as the ones used during the introduction. The purpose of the interviews was to collect data from the participating students' sharing of opinions and discussions of their experiences about the use of escape room as a potential didactic tool in mathematics, and to use these data to investigate the meaning of what

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the participants expressed, through describing, systematizing, and interpreting. The application of focus-group interviews was chosen because of the opportunity this data collection tool provides both for the participants' sharing of individual experiences and production of data in cooperation with other participants (Brinkmann & Kvale, 2015). I moderated all three interviews, using an interview guide with a semi-structured format influenced by the theoretical framework for the study and research on escape rooms, in addition to the preliminary impressions that were made based on the previously described survey and observations. Each group was asked about their expectations associated with the use of escape room, how they encountered the introduction in the mathematics course, and what thoughts they now had about the use of escape room as a didactic tool in mathematics. By using semi-structured focus-group interviews, it was possible to capture participants' individual and joint perceptions. In addition, the focus-group setting gave participants the opportunity to communicate with other group members, and through this to discuss, enlighten and elaborate on opinions and perceptions that emerged as a desired consequence of the focus-group structure (Brinkmann & Kvale, 2015).

Analysis

The analyses in this study are based on a hermeneutical approach (e.g. Gilje & Grimen, 1993). The analytic process started with a theory-driven identification of categories related to escape room in GBL, and on Dewey's theory of learning through experience. Four categories related to learning were identified: *experience, self-regulation, social interaction,* and *motivation*, each being registered in the data material through moments of recognition given by features and aspects identified through the elaboration of each category in the theoretical framework. This elaboration is based on argumentation found in well-established educational research specifically targeting each of these four categories. A directed content analysis (Hsieh & Shannon, 2005) of the survey and observations was used to establish analytic impressions of the participants' knowledge about and experience with escape room before the interviews took place. In accordance with Hsieh and Shannon (2005) a directed content analysis can be applied to identify, validate, and extend conceptually a theoretical framework by using existing theory and prior research as key concepts for initial coding categories.

In the next round of analysis, the same four categories *experience, self-regulation, social interaction,* and *motivation* were used to analyze the transcribed data from the three focus-group interviews to further investigate how teacher education students experience an introduction to use of escape room as a didactic tool in mathematics. This round of analysis consisted of multiple close readings and interpretations of the transcripts. This was followed by attention to text segments in the transcripts, a process that provided the study with selection of quotes to be used when presenting findings. The students are identified through the following labelling: AS1, AS2, AS3, AS4, BS1, BS2, BS3, BS4, CS1, CS2, CS3, CS4 in quotes from the three focus-group inter-

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views. Utterances from the researcher is labelled R. The identification of findings also provided the opportunity to generate hypotheses and probable conclusions regarding possible impact on teacher education students' beliefs about mathematics teaching.

Trustworthiness

The trustworthiness of a qualitative research study will always be measured in relation to the validity and reliability of the study (Cresswell, 2014). In short, as a researcher, one must be aware that one's prejudices influence the research process, but still strive for transparency in the various phases of data collection and analysis. For instance, when gathering data from the observation, field notes were written, and it was made sure that they were structured in a way that allowed others to read and make sense of them. The field notes were read by the lecturer, who confirmed the registrations and impressions I had made. The dialogues from the video recordings and focus-group interviews were transcribed, and analyzed. Another researcher might make different interpretations and reach other findings because of the nature of influence from prejudices and preconceptions on interpretation of qualitative data (Gilje & Grimen, 1993). In this case, the lecturer who orchestrated all three lesson used for the introduction read through both the transcriptions from the interviews and my presentation of findings. He confirmed my interpretations and identification of quotes to support the findings. The lecturer is an associate professor in mathematics education. His credibility on this matter is founded on his broad background within mathematics education research and competence within qualitative research, his knowledge of introducing and using escape room in mathematics teaching and learning, and the fact that he was present in all three observed lessons.

The framework of the study has also been verified to strengthen the trustworthiness. In September 2020, a pilot study on the planning and data-collecting parts of the design was carried out. The survey, observation, and focus-group interview were piloted during the same course with the same lecturer but a different group of teacher education students. Minor, practical adjustments such as adding a head camera to each group during the group work were made after the pilot. In addition, the decision to use three groups, each with four students, was one of convenience because it allowed for the possibility of ruling out any insecurity related to the occasional absence of a student; data from two groups, each with three students, proved to give the study sufficient data. To sum up, it has been attempted to be as transparent as possible in the data collection and analyses, to provide replicable and valuable data that will provide trustworthy findings relevant to the research question. These findings are presented in the next section.

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FINDINGS

The imperative of the findings section is to show, with support in quotes stemming from the interviews with the students, how the introduction to the use of escape room as a didactic tool in mathematics was experienced. The attention to findings is brought forward from a participant point of view and a teacher point of view, respectively. In extension of these findings, some considerations regarding impact on the students' bildung are made.

Participant Point of View

The students described the experienced introduction as something different from what they were familiar with in mathematics (conversations translated from Norwegian):

> BS2: It contrasts traditional teaching. You might be much more active here, physically.
> BS4: Yes. Mathematics is a subject where it is easy to become accustomed with a pattern of individual solving of tasks, with offering of plenary explanations of how to solve them if there are several pupils that do not understand what to do. But here…you solve problems, and do not just sit at your desk and calculate, calculate, calculate…
> R: So it has a physical dimension…because it invites movement?
> BS4: Yes, in a way.
> BS1: But also the inclusion of cooperation in mathematics. Normally, one is left alone to work on a task at one's desk, but in the escape room a group is supposed to reach solutions together, and that contrasts pretty much what we are used to in mathematics lessons.

In their further elaboration on such an impression, a typical mathematics lesson was described as follows: first, the teacher goes through a topic, then the pupils work individually with tasks, and finally the teacher shows the pupils how to solve a selection of those tasks. This is an impression that aligns with a traditional time signature for school mathematics lessons, identified by Topphol (2012). The introduction they experienced in relation to escape room, relied on participation-based experiences with problem-solving, exploring, play, and creativity with a didactical tool that they were somewhat unfamiliar with. AS1 commented, "Well, it was something new. We were like a group…it was an entirely different way of working, and we approached it in another way than, at least I, approach a regular mathematical task." The students argued that it was essential that they got the opportunity to try out escape room themselves. This refers to the active dimension of learning through experience (Dewey, 1916, 1938), and features related to group dynamics (Lathwesen & Belova, 2021) and social interactions (Morell et al., 2020).

The students emphasized their own participation in encountering both the escape box and the escape room that had mathematical content priority. When encouraged to elaborate further on their

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experiences with the introduction, they first reflected on the organization and presentation of activities:

> R: You only had 20 minutes to work on the escape box, being the first encounter with an escape room session…
> BS1: Yes.
> R: How did you experience this encounter?
> BS1: It was fun to see how simple and down-to-earth you can present the escape room setting. Just use a box, with some tasks on it…and it instantly became apparent that we became excited and engaged.

It seemed to be a positive surprise that the escape box invited the participants so rapidly to become involved. Second, the students brought forward the problems they experienced as difficult to solve, basically because they struggled to reach solutions to them. This is an understandable attention, because of their need for solutions to all the problems, in order to escape the room, which to some extent also nourished the students' established opinion about the importance of product (here: solution):

> R: As part of the introduction lessons, [name of lecturer] also emphasized the debrief phase. What do you think about the debrief of the escape room?
> AS2: Very important.
> R: How come?
> AS2: To share both joys and frustrations…
> R: Mmm?
> AS2: Yes…in order to understand. At least on my behalf. I need to know how it is supposed to be solved, or else it may torment me for hours, days…you name it! Haha!
> R: So having the solutions to all the problems shared, so to speak…?
> AS2: Well, yes. For me it is annoying not to be enlightened about how to solve it. By having a debrief you get an impression of what the others did in the escape room as well, for instance if you split the group in pairs, and you did not catch hold of what the other pair was working on. Through a debrief I get to know what they were up to, instead of ending the escape room experience knowing only about the parts I was involved in, and thereby not being able to reflect on the holistic impression of the escape room because I do not know all the details. This has to be done in the debrief if it is not communicated while in the escape phase.
> R: Okay…so bringing attention to each of the problems met in the escape room is important in the debrief. Do you find other aspects relevant to pay attention to in a debrief?
> AS4: Yes, how to approach the room. We learned a lot about that from our first escape room try-out…What to look for, where to look, strategic use of the group. I remember that when we tried the room created for us, we did not even find all the problems. We whirled purpose-

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> lessly around, so to speak, looking neither up or down. After that we have become more synchronized and strategic in our approach.

They pointed out the importance of using problem-solving strategies, solutions to problems, sharing of choices and experiences, and self-regulation based strategy development regarding the group's choices and activity in the room. The attention to product and process culminated in the students' support to the necessity of a debrief session, in accordance with research on use of escape room as a didactic tool (Sanchez & Plumettaz-Sieber, 2019). Later on in the interview AS2 also pointed out that it is easy to forget the problems if you are not given the opportunity to reflect on them afterwards. Through these reflections, Dewey's (1916, 1938) dimension of passive experience related to approaching problems in an escape room setting comes into display.

Furthermore, with relation to frames and settings for escape rooms, the students from a participator point of view discussed that the composition of a group seems important in an escape room, and that the setting can wipe out attention to time and place. Student CS2 explained that she felt confident being a member of the group she was assigned to (in the study). While referencing the escape room her group tried she stated, "The fact that you [the other three group members] were present made me feel that I cooperated, but I did the work all by myself without knowing." Hence, her confidence also enabled this student to be completely engrossed in her individual work with one of the problems in the escape room. She indulged in the playful feature of the escape room setting.

Teacher Point of View

The students acknowledged the balanced priorities between plenary lecturing and student participation in the lessons. These holistic reflections brought forward specific attention to escape room as a possible didactic tool in mathematics, from a teacher point of view:

> BS2: I find that it is, in a way, mathematics with an aim. Traditionally pupils work with tasks, but there is no aim to reach. In the escape room you have a clear aim. If one passionately indulge the room's context, then you have to solve the tasks in order to escape the room. That makes the mathematics lesson motivating, as opposed to a regular lesson.
> R: May such an alternative have some impact on the pupils' learning in mathematics?
> BS2: Well…yes, it might. Because you can use the escape room as a teaching method, and relate it to various practical contexts and problems…and reach conclusions on site. You can choose what context you want, like building a house, or whatever…you have to solve meaningful mathematical problems in a context that makes sense in the present.

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BS2 experiences the escape room as a teaching method to possess a motivational dimension through focus on aim, instant need for resources one has as participant, and relation to a meaningful, practical context. This dimension was further exemplified through personal reflections on experiences from the specific escape room they had encountered themselves.

The students had various opinions about the necessity of presentation of a reliable setting as participants. One student (AS4) said that the story used did not catch his attention, while others (CS2 and CS4) argued that it was the playful approach given by the setting that gave meaning to the rules and motivation to solve the problems they encountered in the escape room. An example given by CS4 to underpin the latter opinion was that they were not allowed to use the internet on their mobile phones, as 'there was no coverage in the cave they were trapped in'. However, from a teacher point of view AS1 summarized the impression of possible influence of establishing a viable setting for the escape room experience:

> Well, we are grown-ups, but pretend that you are in primary school…Dim the lights, gather in a circle, have a narrative…right? Build it up…like, eeeh, once upon a time…the lights…people going back and forth…excitement about some issue that it is urgent to handle. At least I believe that the introduction to a setting has a great influence on the entering of the escape phase…at least it had for me!

Regarding the actual implementation the students mentioned the need for the escape room to be practically manageable for the teacher in the classroom, yet recognizable as an escape room for the pupils. This line of arguments was followed into the necessity of adapting the escape room to the pupils' age and maturity, but at the same time that children are used to role-playing and complying with the rules given for games. The argumentation therefore prolonged into emphasis on the importance of the possibility of learning through play, for instance through the following quote by CS2: "…I don't think they [pupils] know that play is cooperation, but it is a natural ingredient when they create a fictional world and play in it together."

The aspect of mathematical content was raised with reference to experience of the value of repetition of mathematical content:

> CS2: Yes, I also think it is a good way to use your knowledge… so that if you use it [escape room] as repetition, then you can apply the knowledge you have gained.
> R: Mmm.
> CS2: And see that you manage to use it.
> CS4: Mmm. Yes!
> CS2: That it in a way becomes beneficial.

CS2 saw a didactic possibility for use of an escape room in mathematics linked to repetition of mathematical knowledge, but then with support from CS4 extended into an argument for experi-

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ence of usefulness of application of mathematical knowledge in a new situation. To sum up the impressions made from a teacher point of view the students claimed that what one already knows may be beneficial in the present, and not just be possibly beneficial in the future. In relation to this, the students argued for the importance of a teacher orchestrated debrief when the escape room session is finished, in order to share and reflect upon the pupils' experiences in the implementation phase, and establish connections to the learning objectives that the teacher had in mind while planning and developing the escape room.

Impact on Bildung

In an extension of the above findings, it is significant that the students saw both application possibilities in their own teaching practice, and experienced glimpses of impact regarding their beliefs about mathematics teaching and learning, and thereby their development of mathematics teacher identity. One student (AS4) concluded that he saw the opportunity and value for pupils to gradually be challenged to make escape rooms for other pupils, because of the possible impact on learning through creating:

> AS4: I believe that it is more learning in developing escape rooms than it is to experience them as participants…
> AS1: Mmm.
> AS4: …when it comes to learning of mathematics. Because you actually, when making something new, you have to use your knowledge, develop tasks, create the holistic setting, pilot the tasks…does this work? Is it necessary to make adjustments? This is quite different from being in an escape room as participant.

In another interview CS4 concluded, after some contemplation, "I believe that if we had not done it [the escape room] at the teacher education … then we probably would not use it when we start working as mathematics teachers either. [Because you] have not tried it … you know." The application of escape room becomes an active choice one makes as a mathematics teacher to possibly improve one's teaching for pupils' learning of mathematics. These two quotations support the impact of active and passive experiences on learning, through two students' identification of reflection on further use (AS4) and necessity of trying oneself (CS4). All in all, the introduction to escape room provided the teacher education students with experiences related to both a hands-on participant perspective and a more reflecting teacher perspective, for instance in an argument for application of escape room in teaching of mathematics and social studies:

> AS3: On my behalf, I have decided that escape room is something that I want to use in my teaching, and especially emphasize an interdisciplinary relation.
> R: Yes?

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AS3: For instance, since I have specialized in mathematics and social studies in my teacher education, I see opportunities to intertwine knowledge-based attention to tasks and problems from these subjects with for instance priority to development of competence in cooperative work.

Such experiences may indicate an impact on the participating students' beliefs about mathematics teaching and learning, because of their willingness to change. However, the possible impact on their bildung is hidden, both for me as a researcher and for them. Hence, only hypotheses can be generated regarding bildung and the teacher education students' possible change agent influence stemming from the upcoming discussion of the findings related to the students' experience with the introduction to escape room, followed by conclusions that can only be probable.

Summary of Findings

To sum up, the findings show that traces of a traditional perspective on teaching and learning of mathematics are clear among the students, for instance through their description of a typical mathematics lesson and their stated importance of sharing the solutions or answers to the problems on the escape box and in the escape room in debrief sessions. The students compared new input regarding teaching and learning of mathematics with their established beliefs about mathematics, but revealed glimpses of willingness and curiosity to change perspective, after being introduced to use of escape room as a didactic tool in mathematics.

DISCUSSION

Through the introduction to escape room as a didactic tool in school mathematics, teacher education students were given an opportunity to experience an approach to problem-solving in mathematics that challenges the didactic of the traditional mathematics classroom (Van Zoest et al., 1994; Topphol, 2012). The above findings describe the introduction as a somewhat surprising alternative to the 12 teacher education students' expectations for the mathematics classroom. The findings, however, show that the students are open for change of perspective, after being introduced to a GBL alternative that calls for a change of beliefs about teaching and learning of mathematics from a traditional, acquisition-based individual learning perspective to a more participation-based learning community perspective (Sfard, 1998; Wenger, 1999). The students appreciated that escape room was introduced in a well-grounded manner, and that they had the opportunity to try out both an escape box and an escape room. This appreciation is shown first through clear interest and arguments regarding organizing, adaptation to pupils and emphasis on variety to positively influence motivation. Second, the students seemed to experience escape room as a means of offering pupils a collective context for repetition of mathematical knowledge and skills

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through application in a new situation while experiencing the benefits of mathematical competence and social interaction in the present.

According to Dewey (1916; 1938), learning and experience are intertwined. One makes experiences when one creates new balance through activity and consequences of the activity where there previously was imbalance. At the same time, Dewey is concerned about the value of the experiences and the direction in which the experiences orientate the learner. The intentions of, for instance, the Norwegian curriculum for compulsory schooling imply that pupils should have experiences that are not limited to cognitive knowledge and skills, but have a broader, educational perspective (Kunnskapsdepartementet, 2017). Within GBL experience, self-regulation, social interaction and motivation are key areas in such a perspective, and the findings show that the students seem aware of the opportunity to use escape room to adapt learning strategies to task requirements (Zimmerman, 2000), to emphasize group dynamics (Peleg et al., 2019; Zhang et al., 2019; Lathwesen & Belova, 2021; Veldkamp, 2022), and to generate motivation (Ryan & Deci, 2000b) regarding both subject matter content and development of 21st century skills (Andrews & Bagdasar, 2023; Kunnskapsdepartementet, 2014; Morell et al., 2020). If so, this means that the playful foundation of GBL, a socio-cultural approach to learning (Ouariachi & Wim, 2020; Plass et al., 2015; Weines, 2021), and the acknowledgment of creative processes such as problem-solving together suggest that an important objective is to go beyond subject matter content and give priority to the learning process (Charlo, 2020). To succeed in paving the way for such a learning objective the teacher must see the pupil, and not merely the subject and its content. Such a perception does indeed both depend on the teacher, and influence the teacher.

The possible change of the participating teacher education students' beliefs about mathematics teaching and learning, in this particular case, surfaces through their experience with escape room as a didactic tool that represents a change that the teacher education students seem to find interesting. The main purpose of teacher education is to educate quality teachers, that is, teachers who, in addition to scholar knowledge, know how to pave the way for pupils' learning, or apply a well-founded didactic approach. Requests for changes in the educational sector are based on societal development, educational research, technological revolutions, and education policy reforms (European Schoolnet, 2018; OECD, 2019), and therefore teacher education students have to learn how to relate to such requests for change through development of their own didactic stance, operationalized through their teaching and continuous answering of the questions of what you learn, and how you learn. One can relate to new active and passive experiences with reflection that leads to deliberate change of practice (Dewey, 1916). The opposite to adaptation and embracing of changes and development is the routine, where the possibility of change is measured in relation to what is established, and then rejected. The experienced introduction to escape room seems to offer an alternative to the traditional social and psychological norms of the mathematics classroom (Yackel & Rasmussen, 2002) through GBL's emphasis on experience, self-regulation, social interaction, and motivation.

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The learner's bildung is influenced through the meeting between object and subject, and stems from her learning based on personal judgment of the value of content, form, and relations between these. Through didactic choices for the teaching and learning environment, the teacher paves the way for experiencing such meetings. With reference to Klafki's (1983) categorical bildung, this means that if the teacher education student meets a well-founded introduction to a new didactic approach in mathematics teaching (here: use of escape room in school mathematics), and this meeting awakes interest in and awareness about the interaction between what an escape room is, how it represents learning possibilities in mathematics, and why the escape room approach is emphasized because of mathematical content one can learn, then a categorical bildung process is recognizable. When the teacher's didactic choices for this meeting are based on a balance between attention to content and the desire to play, and thereby based on the fundamental idea of GBL as an opportunity to learn through a duality of experiences and challenges encountered in the company of other players, a basis for categorical GBL bildung may be recognizable. Such a basis may provide an environment for possible change in the participating teacher education students' beliefs about mathematics teaching and learning and their identities as mathematics teachers, that is prosperous from a Klafkian and Deweyan perspective. This might support the teacher education students in becoming future change agents for the use of escape room as one of several didactic tools in mathematics at the schools where they will be working as mathematics teachers.

CONCLUSION

The findings in this study show that use of escape room in mathematics teaching was met from a traditional perspective on teaching and learning of mathematics in school. However, the findings also implied glimpses of willingness and curiosity about the idea of trying out this new, game-based, didactic tool. The introduction offered them opportunities for both active and passive experiences, and to reflect on teaching and learning of mathematics based on application of a new didactic tool. This may have made a difference when it comes to the intrinsic impact on the students' beliefs and thereby bildung as mathematics teachers. One of the students (CS4) stated that without the well-founded introduction to escape room she experienced, she would have been unlikely to try out escape room as an activity in her own teaching practice. By bringing escape rooms into mathematics teaching in school, the teacher education student may be recognized as a mathematics teaching change agent, but such an initiative will only stem from an inner conviction about the value of making such a didactic choice, related to one's didactic stance.

The students' participation in this study have made it possible to display valuable contributions to the field of mathematics education. Collection of data regarding the teacher education students' introduction to the use of escape room as a didactic tool in mathematics, has made it possible to give empirical support to theoretical frameworks that investigate what constitutes GBL

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in mathematics. Through emphasis on the pivotal role of the teacher in planning, conducting, and debriefing of an escape room experience in mathematics, this study also contributes to the growing body of knowledge about GBL in mathematics and more specifically about how escape room may work as a beneficial and meaningful part of mathematics learning in the classroom.

It is necessary to highlight that this is a qualitative study where only 12 teacher education students participated. This may open for theoretical generalization due to experiences other teachers and researchers have with use of escape room in education. Selection of other students from the group of 44 would have provided the study with other data (Brinkmann & Kvale, 2015). However, due to the equal conditions in the introduction of escape room in mathematics, I assume that such data would provide me with more or less the same foundation for analyses. Another possible limitation is the traditional perspective on mathematics teaching and learning held by the students in the study. This is not necessarily the prevailing perspective among teacher education students in mathematics neither in Norway, nor in other countries. A third limitation might be choices made for the actual introduction, for instance the design of the escape room and choice of mathematical problems. In reply to such limitations, the use of a GBL-based framework has provided the study with a methodological approach to identify and understand teacher education students' reactions and responses to an introduction to use of escape room as a didactic tool in mathematics. This offers a useful alternative for other researchers who are curious about use of escape room in mathematics, for instance in relation to frontiers such as learning outcomes, cross-sectional collaboration in both development and application of escape rooms in education, and use of digital escape rooms. I am aware that this study is limited to investigate a selection of teacher education students' experiences with an introduction to escape room in mathematics. Therefore additional research on teacher education students' use of escape room as a didactic tool in practice periods, or as new teachers in primary and lower secondary school teaching, is needed to form an impression of their position as possible change agents when applying an escape room based on the didactic questions of what you learn and how you learn.

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MATHEMATICS TEACHING RESEARCH JOURNAL
SUMMER 2025
Vol 17 no 3

An Experimental Problem-Based Approach in Mathematics Teaching and Mathematical Problem-Solving Performance in Grade 3 Primary School Pupils

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Abstract: Problem-solving is a key aspect of mathematics instruction. Although it is part of several national curricula, its implementation is still being vastly explored. The present research, which is part of a larger project, aimed to investigate the effect of using problem-oriented instruction in grade 3 of primary school on students' mathematical abilities of solving problems in arithmetic, geometry, and logic. The study used a pedagogical experiment, with a sample of 240 pupils in Grade 3 from four randomly selected Slovenian primary schools. Students were divided into two groups: 100 in the experimental group and 140 in the control group. Two tests of mathematical knowledge, constructed by the authors, were used to assess potential differences between the groups before and after the experiment took place. The results showed that the use of an experimental model of problem-based learning had a statistically significant effect on pupils' mathematics problem-solving achievement in all the domains of mathematics (i.e., arithmetic, geometry, logic & set theory). The study suggests that modern school reforms should focus on problem-based mathematics instruction as a means of achieving meaningful and lasting knowledge in students. Additional research is needed to replicate the results due to the limited number of participants in the present research.

Keywords: achievements, control group, elementary, experimental group, problems

INTRODUCTION

Mathematics education is a rapidly evolving field, as new trends and methodologies are continually being demonstrated to be more effective than previously employed ones. For instance, problem-solving is gaining more and more attention in mathematics education (Siagian et al., 2019). It provides students with opportunities to engage with mathematical concepts and ideas in meaningful and authentic contexts (English & Gainsburg, 2015; Jurdak, 2006). Through problem-

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solving, students can develop their mathematical reasoning, critical thinking, and analytical skills (Belecina & Ocampo Jr., 2018). They learn how to formulate problems, identify relevant information, and apply appropriate strategies and techniques to solve them. Moreover, problem-solving promotes active self-regulated learning (Muis, 2008), as students take responsibility for their own learning and monitor their progress and performance. By using problem-solving in mathematics education, teachers can create a student-centered (Gezim & Xhomara, 2020; Saragih & Napitupulu, 2015) and inquiry-based learning environment that fosters creativity, curiosity, and collaboration (Divrik et al., 2020). Furthermore, problem-solving can help students develop a positive attitude toward mathematics (Albay, 2019; Russo & Minas, 2020) and appreciate its relevance and applicability in various domains of life.

Teaching students to solve problems in mathematics means promoting students' learning and understanding of mathematics (Albay, 2019; Ali et al., 2010). This approach involves presenting students with problems that require them to engage in mathematical thinking and reasoning to arrive at a solution (Francisco & Maher, 2005). Problem-solving teaching encourages students to use higher-order thinking skills, such as analysis, synthesis, evaluation, and creativity, to explore and investigate mathematical concepts and relationships (Collins, 2014; Widana et al., 2018). It also promotes the development of metacognitive and affective competencies (Lai et al., 2015), as students learn how to monitor their own thinking and emotions and how to persist and persevere in the face of challenges and setbacks (Mariano-Dolesh et al., 2022). Specifically, a student's mindset is indicative of their degree of conceptual understanding in problem-solving, with those possessing a growth mindset demonstrating greater potential for successfully solving mathematical problems.

Problem-solving is particularly important in the early stages of mathematics learning (Tarim, 2009; van Bommel & Palmér, 2018). Through problem-solving, children develop their critical thinking skills and acquire mathematical knowledge that is applicable to real-life situations (Hartmann et al., 2021). Research shows that incorporating problem-solving activities in early math education can enhance children's conceptual understanding, improve their problem-solving abilities, and increase their motivation and interest in mathematics (Cheung & Kwan, 2021). Thus, it is essential to incorporate problem-solving in early math education to help children develop the necessary skills and knowledge needed for future academic success (van Bommel & Palmér, 2018).

Despite the importance of using problem-solving-based instruction, several studies have found that teachers lack competencies and knowledge about problem-solving in mathematics (Barham, 2020; Chapman, 2015). In particular, teachers' math anxiety and math self-efficacy are correlated to teachers' ability to solve problems in mathematics (Akinsola, 2008). Moreover, teachers' beliefs about mathematics and problem-solving do also affect the extent to which they use this strategy in class (Saadati et al., 2019). Teachers' attitudes toward problem-solving may also influence students' beliefs about it (Yorulmaz et al., 2021).

Considering the Slovenian context, it might be noticed that in the national mathematics curriculum for primary school (Žakelj et al., 2011), several goals regarding problem-solving are men-

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tioned. In particular, it is stated that students need to develop reasoning, generalizing, abstracting, investigating, and problem-solving. Considering, for instance, arithmetic, students should be able to solve arithmetic problems that are also related to real life. Nevertheless, the model of teaching is still traditional, i.e. behavioristic. Teachers do include elements of the national curriculum involving problem-solving, but the teaching model has not yet shifted from a behavioristic one to a problem-based approach. Therefore, it is unclear whether students exposed to a problem-based approach would be better problem solvers than those exposed to traditional lessons.

The aim of the present research is, therefore, to evaluate the efficacy of the problem-based approach with regard to the more traditional (i.e., behaviorist) model of teaching mathematics in primary school. In particular, we expect students that are exposed to a more problem-solving-oriented mathematics lesson to have higher achievements in tasks of all taxonomic levels and from all mathematics domains (i.e., arithmetic, geometry, and logic).

LITERATURE REVIEW

Mathematical problems

A mathematical problem is one in which a person attempts to overcome a problem situation by using mathematical tools (Phonopichat et al., 2014; Zhu, 2007). In the broadest sense, Magajna (2003) defines a mathematical problem as a situation that initially creates a sense of the unknown and discomfort because one is unable to explain a certain fact to oneself or is unable to achieve a desired goal, which will be resolved upon solving the problem. Problems in math can be classified according to different criteria. In general, they can be distinguished by the visibility of the solution path. Magajna (2003) distinguishes between (Kolovou et al., 2011; Mwei, 2017):

- *Routine Problems* (which are not even problems in the usual sense, but rather exercises), where the solution path is clear to the solver in advance, either because he has already been through it several times or because the path is evident from the formulation of the problem or from the context;
- *Non-Routine Problems* where the solution path is unclear and the problem is encountered for the first time. The state of the solution routine depends to a large extent on the knowledge and experience of the solver.

Problems in mathematics are also distinguished by the specificity of the goal. When the goal of the problem is well defined, we speak of a closed problem, and when the goal is not defined, we speak of an open problem, where the solver sets his own goals and tries to achieve them by investigating them (Bahar & Maker, 2015). Cotič and Valenčič Zuljan (2009) distinguish mathematical problems at the classroom level mainly according to the path and the goal and classifies them into:

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- Problems with a closed path and a closed goal;
- Problems with an open path and a closed goal;
- Open-path and open-goal problems, noting that closed-path and closed-goal problems dominate mathematics teaching at this level.

This can give students the misconception that most mathematical problems have a single path and a single solution. If we want to achieve higher levels of proficiency in students, then we need to offer them other types of problems that stimulate their thinking in finding paths and different solution strategies.

Rupnik Vec and Kompare (2006) note that most problems (that students encounter in school mathematics, e.g. arithmetic problems, and problems in science) are closed (structured). When solving closed problems in school, we usually use domain-specific strategies to solve a particular type of problem within that domain. However, in contrast to school problems, most life problems are open-ended and at least partly unstructured. Teaching, therefore, requires the design of learning situations that are characterized by similarities to life problem situations. As a rule, these problems are open-ended problems in which students go through all the stages of problem-solving, from identifying and interpreting the problem to finding, testing, and judging solutions. These types of problems can be used at all stages of the learning process: they can provide a realistic context in which to build and deepen knowledge or to build on it.

Problem-Solving in Mathematics

Mathematical problem-solving is the process of employing mathematical ideas, know-how, and techniques to resolve a mathematical issue (Schoenfeld, 1985). It entails locating the issue, comprehending it, coming up with a solution plan or strategy, putting the plan into action, and evaluating the outcome (Polya, 1945; Schoenfeld, 1987; Voskoglou, 2011). In order to foster critical thinking, reasoning, and mathematical literacy, problem-solving is a crucial component of mathematics education (Daulay & Ruhaimah, 2019; Lee, 2016). Additionally, it assists students in gaining a deeper comprehension of mathematical ideas and techniques as well as in applying what they have learned in practical settings (Carson, 2007; Stacey, 2005).

Problem-solving, therefore, emphasizes independent solving problems and overcoming the learned constraints that make it difficult to move from the initial to the final state. Thus, we solve a problem when we want to answer a question or achieve a goal, and we cannot easily recall the answer from long-term memory or we encounter obstacles to solving it.

According to Schoenfeld (1985), in order to effectively solve problems, individuals must possess and proficiently use appropriate resources (such as mathematical concepts and procedures), heuristic strategies (both general and specific), metacognitive control (the ability to oversee and

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monitor the entire problem-solving process), and appropriate beliefs (in terms of one's perspective, motivation, and confidence).

If we take a closer look at this and look at our teaching practice, we can quickly see that there are few situations in our teaching practice where students can develop these thought processes. Schoenfeld (1985) mentions several times that it is necessary to change the conception of mathematics in general from learning formulas to finding different solutions, from memorizing formulas to exploring patterns, from solving problems to formulating research questions, making connections, and solving problems. If teachers pay more attention to this, then students will have the opportunity to experience mathematics as a changing, dynamic, evolving discipline, rather than the rigid, absolute, and closed discipline that is currently imposed.

Problem-Based Learning

Problem-based learning is defined as a way to foster more creative forms of thinking, experiencing, and evaluating (Seibert, 2021), which is also the basis for problem-solving (Merritt et al., 2017; Nurlaily et al., 2019). It should therefore not be seen only from a narrow methodological perspective as a teaching method. Nevertheless, we can agree that this principle cannot be subordinated to others, because it alone cannot be used to realize all the curriculum objectives and at the same time to observe all the principles of teaching, but learning is more effective if we introduce problem-orientation and problem logic (Strmčnik, 2001). In particular, problem-based learning focuses on problems in which students can construct their knowledge (Mulyanto et al., 2018).

Strmčnik (2001) classifies problem-based learning as a teaching principle. The principle of problem-oriented teaching is applied when problem orientation is present in all phases of the learning process. On the other hand, problem-solving is a teaching method. Problem-based learning extends to the whole classroom, to all its content and process dimensions, while problem-solving has a narrower meaning and covers only part of the learning activity. Nevertheless, problem-solving is where problem-based learning innovations are most directly expressed (see Aslan, 2021).

Problem-based learning is oriented towards both exploring the problem situation and acquiring new knowledge, as well as investigating and reflecting on one's own way of learning (Downing et al., 2009). In doing so, students build on and acquire both contextual and procedural knowledge as well as metacognitive knowledge (Downing et al., 2009; Downing et al., 2011; Siagan et al., 2019). Problem-based learning is characterized by a problem situation, which can be defined by the teacher or by the teacher together with the pupils. In doing so, pupils acquire new knowledge through exploration, argumentation, verification, and taking a standpoint while

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being mentally active. In this way, problem-based learning to a large extent balances the relationship between reproductive and creative knowledge acquisition (Strmčnik, 2001).

A slightly different view of problem-based learning is offered by Wood (2003), who specifically analyses both the role of the learner and the role of the teacher, as both are involved in problem-based learning. From the learner's point of view, she mentions the important advantages of problem-based learning, which are reflected in (Wood, 2003):

- The active role of the learner in solving the problems themselves, which allows for the understanding of the knowledge acquired, while at the same time developing collaborative skills that are important for further learning;
- Facilitating cross-curricular links and knowledge transfer between subjects;
- High motivation of students to formulate and solve problems;
- Activating prior knowledge and the acquisition of related knowledge, and the development of skills (thought processes).

The effectiveness of problem-based learning can be evaluated in relation to: the teaching style or teacher guidance/orientation, the characteristics of the learners and classroom interaction, the individual learning goals of the learners, as well as the problem itself. It is therefore a rather complex teaching method with several interrelated success factors. Problem-based learning is learner-centered. It assumes that learners will explore, and link theory to practice, apply experience and knowledge to formulate the problem, and find solutions (Savery, 2006; 2015; 2019). Poorly structured problems that relate to an authentic life situation and adequate support from the teacher are crucial for the effectiveness of problem-based learning.

Previous Studies

The problem-based teaching model has been widely employed to enhance students' conceptual and problem-solving knowledge. A previous study within the same research project demonstrated the effectiveness of this model in fostering conceptual understanding in mathematics among primary school students (Doz et al., 2024). Compared to a traditional, procedural approach, problem-based instruction proved superior in developing students' ability to grasp mathematical concepts, particularly in arithmetic, logic, and geometry. These findings suggest that problem-solving skills are fundamental across all mathematical domains and that real-world problem contexts foster a deeper understanding. In particular, problem-based instruction encourages students to actively engage with problems rather than passively receive information, promoting a deeper understanding as they explore, justify, and verify their solutions. Furthermore, by incorporating real-life scenarios, this form of active learning helps students relate mathematical concepts to

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practical applications, making the learning process more meaningful and aiding conceptual retention.

Numerous studies have shown that problem-based learning significantly enhances primary school pupils' problem-solving abilities. Problem-based learning fosters critical thinking and problem-solving skills by engaging students in complex, real-world tasks (Hendriana et al., 2018). Primary school pupils exposed to problem-based learning demonstrated superior problem-solving abilities compared to those taught through traditional methods (Ali et al., 2010). Problem-based learning not only improves problem-solving performance but also promotes pupils' self-confidence in tackling mathematical challenges (Merritt et al., 2017). Therefore, integrating problem-based strategies into mathematics instruction leads to a deeper understanding of mathematical concepts, thereby enhancing problem-solving skills. These findings align with the notion that problem-based learning emphasis on active learning, critical thinking, and contextual problem solving supports the development of essential mathematical competencies. Overall, the literature consistently underscores the positive impact of problem-based learning on primary school pupils' ability to approach and solve mathematical problems.

Research conducted by Li & Tsai (2017) implemented a year-long problem-based learning intervention in a fifth-grade mathematics classroom in Taiwan. Students engaged in complex, real-world problems requiring them to apply mathematical concepts collaboratively. The study observed that students developed a deeper understanding of mathematical concepts and demonstrated improved problem-solving skills over time. Additionally, the problem-based learning approach fostered long-term knowledge retention and a positive attitude toward mathematics.

Aims of the Research

Modern school reforms around the world aim to achieve meaningful and lasting knowledge in students, and this is the reason why curricula, syllabuses, and, consequently, mathematics teaching are being updated. We decided to carry out this research in order to investigate and update the existing educational process in mathematics teaching in primary education. The research problem was focused on the design and evaluation of an experimental model of problem-based mathematics education.

The present study is part of a larger research project (Doz et al., 2024). Previous research has highlighted that a problem-based approach can help students develop better conceptual knowledge; for instance, Grade 3 pupils who followed problem-based instruction demonstrated superior performance in understanding mathematical concepts. In this study, we aimed to explore Grade 3 students' problem-solving abilities, particularly when addressing more complex problems.

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An experiment was carried out in school practice with Grade 3 pupils to determine whether the use of an experimental model of problem-based learning had a statistically significant effect on pupils' mathematics achievement. Our general research hypothesis is therefore the following:

General Hypothesis: Students receiving an experimental model of problem-based mathematics instruction will be more successful in solving mathematical problems in all mathematical domains (arithmetic and algebra, geometry with measurement, and logic and language) than students receiving classical (behavioristic) mathematics instruction.

METHODS

Participants

The study was carried out on a sample of 240 pupils in the 3rd grade in four randomly selected Slovenian primary schools from the Coastal Region, 100 pupils were included in the experimental group (57.0% females) and 140 pupils in the control group (52.1% females). All the selected primary schools were urban schools with equally solid working conditions. The socioeconomic status of students of both groups has shown that pupils are mainly from middle-class families.

In the current study, grade 3 students were selected in accordance with the objectives outlined in the National Curriculum for Mathematics (Žakelj et al., 2011), which emphasizes the importance of fostering students' problem-solving skills. Consequently, this age is well-suited for an exploration of instructional methods that hold the potential to enrich these critical skills.

The sampling procedure adhered to a systematic process aimed at ensuring the acquisition of a representative and unbiased sample. To designate the schools, a random sampling method was employed. All primary schools within the Coastal Region of Slovenia were cataloged, and a random number generator was deployed to designate four schools from this comprehensive list. Subsequently, we sought and obtained the permissions from the school principals to conduct our study. Within each of the four selected schools, all grade 3 classes were encompassed within the study. Classes from two of these schools were randomly assigned to the control group, while classes from the remaining two schools were assigned to the experimental group, based on a random coin flip to determine each school's group assignment.

Once approval was obtained from school principals and teachers, all participants and their parents were thoroughly informed about the research objectives, duration, and potential benefits. Informed parental consent was obtained. The study was conducted in full compliance with the European Code of Conduct for Research Integrity.

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Instruments

The study used a pedagogical experiment as part of the empirical research approach. Quantitatively, we analyzed the data by means of knowledge tests (two knowledge tests in mathematical problems).

Test of Knowledge

For the purposes of the research, two knowledge tests were designed (Appendix A and B). The initial and final mathematical knowledge of the experimental and control groups was assessed by means of an initial and final knowledge test. The tasks in both knowledge tests were designed taking into account Gagné's taxonomy (Cotič & Žakelj, 2004) and the mathematical content of grade 3 according to the mathematics curriculum (Žakelj et al., 2011).

The pre-test (Appendix D) consisted of 17 questions, including 10 from the domain of arithmetic, 4 concerning logic and sets, and 3 regarding geometry and measurements. The post-test (Appendix E) consisted of 22 questions, including 15 from arithmetic, 3 from logic, and 4 from geometry. The distribution of the questions among these three mathematics domains was proportional to the topics in the national curriculum for the 1st, 2nd, and 3rd years of primary schools in Slovenia (Žakelj et al., 2011).

For the pre-test, students had 1 school hour (50 minutes) to complete it; to complete the post-test, students had 2 school hours on two different days. The test was divided into two smaller tests of 11 questions each (first part: questions from 1 to 11; second part: question from 12 to 22). Most of the tasks were open-ended, but in some exercises, the pupils needed to circle one of the four given answers.

Each exercise in the pre-test was worth 1 point (Appendix D): the pupils got the point if the exercise was solved correctly, otherwise, they got 0. The post-test (Appendix E) had a variable number of points, with one point for each sub-task the exercise presented.

Regarding the taxonomic levels of the test, they were made following Gangé's taxonomy (Gagné, 1985), which is generally used to classify mathematics problems and tasks, and it is used also in the Slovenian National Assessment (Nacionalni Preizkus Znanja; RIC, 2023). Therefore, tasks regarded (1) the understanding of the concepts and definitions (conceptual knowledge), (2) the solving of simple problems (procedural knowledge), and (3) the solving of complex problems (problem-solving). In the latter, students had to decide the most suitable procedure to apply to solve the problem and needed to think about whether the tasks could be solved. For the purpose

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of the present research, problems from levels I and II are considered "elementary" problems, while problems from level III are considered "complex" problems.

The instrument we developed tests students' knowledge of (1) arithmetic (A), (2) geometry and measuring (G), and (3) logic and set theory (L). The decision to incorporate these mathematical topics is based on the fact that they correspond to the same subjects evaluated in the Slovenian National Assessment (RIC, 2023). In the test, problems concerning arithmetic were preferred, since it is the mathematical topic that pupils are the most used to in the first grades of elementary school, similarly as in the Slovenian National Assessment (RIC, 2023).

The characteristics of both knowledge tests (objectivity, reliability, and validity) were demonstrated on a pilot sample of 102 Grade 3 pupils from two randomly selected Coastal schools.

The validity of the test was determined through qualitative analysis. The contents of the test, which aimed to measure students' mathematical knowledge (and, specifically, problem-solving abilities) of grade 3 pupils, were analyzed by a group of 8 experts (3 professors of mathematics education, 2 mathematicians, and 3 primary school teachers). Based on their critical analysis of the instrument, we improved its contents and structure.

The objectivity of the instrument regarded two factors, i.e., (1) the objectivity of test-taking, and (2) the objectivity of the evaluation of pupils' answers. Regarding the first, researchers gave pupils detailed information about the test itself, thus limiting the possible researchers' influence on pupils. Concerning the second, the test was structured in such a way that there were many closed-type questions. Moreover, the tests were corrected and graded by three independent people (Hallgreen, 2012): a researcher (R) and two mathematics teachers (T1, T2), who demonstrated almost perfect inter-raters' agreement.

The reliability of the instrument was assessed using the method of parallel tests (Mueller & Knapp, 2018). The same students were tested two times; the second test was not the same test as the first, as it was a parallel version of it, i.e. a test that was composed in such a way that it tested the same contents and taxonomic levels as the first. The measure of reliability was obtained considering the correlation between the two results. The correlation is positive, high, and statistically significant ($r = 0.85$; $p < 0.001$), indicating excellent reliability of the instrument.

Moreover, we also considered the difficulty of the test and its items (Mahjabeen et al., 2017). In particular, we maintained only the test items that had the index of difficulty, i.e. the ratio between correct answers and all answers, between 10% ("difficult" questions) and 90% ("easy" questions).

Among the questions present in the test there were examples of problems:

- *Problems with More Data Than Necessary for a Solution*: When students first encounter such problems, the teacher assists them with questions and instructions (e.g., "Don't you

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think Tina's task is confusing?", "Are all the data necessary to solve the problem?", "Which data are not necessary to solve the problem? Why? Cross them out", "Which data are necessary to solve the problem? Why? Underline them"). An example of this kind of problem in the test: *Tina says: "Yesterday I went to the swimming pool. There were 8 of us: 4 girls, 3 boys, and my dog. We ate sixteen candies, one ice cream, and two lollipops there. How many sweets did we eat at the swimming pool?"*

- *Problems with Multiple Solutions*: To teach a student to apply mathematics to real-life problems, it is essential to introduce them to problems with multiple solutions. Everyday problems almost never have just one solution. Sometimes there is no solution, other times there are several different solutions, from which the best or most acceptable one must be chosen according to the situation. Thus, the "best" solution changes depending on the circumstances or the person in the specific situation. Furthermore, such problems offer the student the opportunity to realize that mathematics is not a dogmatic discipline where every situation has precisely one predetermined solution. An example of this kind of problem in the test: *Tina is looking at school supplies in the store window. She has 10 EUR. Which school supplies can she buy? Write down several solutions. What would you buy if you had 10 EUR?*
    - [notebook] □ 2.50 €
    - [scissors] □ 5 €
    - [folder] □ 4.50 €
    - [pencil] □ 3 €
- *Problems Without Enough Data for a Solution*: When solving problems with students that do not have enough data for a solution for the first time, we help them with the following questions: "Can you solve the problem? Why not?", "Which data are missing?", "Can you find the missing data?", "Where or how will you find the missing data? (If you can't find it, determine it yourself.)". An example of this kind of problem in the test: *Jan went to his grandmother's place in the countryside with his family on Sunday. The road they took to get to his grandmother's place is 80 km long. On the way back home, Jan's father chose a shorter route. How many kilometers did they travel in both directions? Can you solve the problem? If not, why not?*

Procedure

In the present research, the experimental method of pedagogical research was used. The experiment was carried out in existing primary school departments. This means that the sections were not equated to chance differences before the experiment. Pupils in the experimental group were

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exposed to a problem-based learning environment, while students from the control group continued studying in a traditional model. The experimental and control groups included female teachers, equal in terms of educational level. The experiment lasted 8 months.

In this intervention, students in the experimental group were exposed to a learning environment that promotes active learning, critical thinking, and collaborative problem-solving, potentially leading to a deeper understanding of mathematics (Doz et al., 2024). This environment included various instructional strategies and activities designed to engage students actively in the learning process. Teachers utilized (also) real-world problem scenarios to stimulate students' curiosity and encourage them to think critically about mathematical concepts. Students in the experimental group worked also in small groups to solve mathematical problems. The intervention aimed to move beyond rote memorization and procedural understanding; instead, it sought to develop student' ability to apply mathematical reasoning in various contexts, thereby deepening their overall comprehension of mathematics. During regular mathematics lessons, the teachers presented students of the experimental group with problems that were accessible to grade 3 students. These problems primarily involved arithmetic, geometry, and logic. Some problems were solved together with the teacher, who promoted critical thinking, while other problems were solved by students working in small groups cooperatively. All the problems were then reviewed by the teacher and discussed with the whole class. Each pupil had the opportunity to contribute to the solving process. Additionally, students from the experimental group received, alongside the traditional homework from the workbook, additional problems. Those problems also included ones that have multiple solutions, no solutions, or several ways to solve them and that required reasoning. Some examples of each problem solved by teachers or by students are presented in Appendix C. The exercises given to the teachers as additional material before the experiment started were developed by the researchers and printed out for the teachers, who then gave their students some photocopies of the problems. All problems had some proposed solutions that were worked out by the researchers and/or included commentaries about how to implement the exercises in their regular mathematics classes (e.g., at what stage should some problems be given, what questions might the teacher ask the students to stimulate their reasoning, etc.), which followed the directions of Polya (1945).

Meanwhile, students in the control group continued with the traditional educational model, characterized by more passive learning and teacher-centered instruction. The fact that both the experimental and control groups had female teachers with comparable educational qualifications ensured that any potential differences in outcomes were not attributed to variations in teaching expertise. The experiment's duration permitted a comprehensive assessment of the impact of the problem-based learning environment on students' performance and conceptual understanding in mathematics. This research design facilitated the exploration of potential benefits associated with this innovative pedagogical approach in comparison to the traditional teaching method.

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The initial knowledge test was administered to the control (CG) and experimental (EG) groups before the start of the experiment, and the final knowledge test was administered after the experiment under the same conditions and with the same tester. The study was carried out over a period of 8 months. The year before the start of the experiment, teachers from the EG were instructed to carry out the experiment. During these sessions, teachers acquired theoretical and practical knowledge about problem-solving, received additional material (workbooks, worksheets, etc.), and were instructed not to use a problem-oriented model of teaching during the current year. During the study itself, the EG was taught mathematics in grade 3 using the experimental model of problem-based learning, while the CG was taught mathematics in grade 3 using the (traditional) behavioral model.

Data Analysis

The data in the empirical part were analyzed using the statistical analysis software SPSS 26.0. The following statistical procedures were used:

- *Analysis of Covariance (ANCOVA)*: it was used to check for differences between the CG and EG in the final test by controlling for the pre-test;
- *Descriptive Statistics*: mean (*M*), standard deviation (*SD*), frequencies, minimum (*min*), and maximum (*max*); they were used to describe the achievements and the demographics;
- *Kolmogorov-Smirnov Test for Normality of Distribution*: it was used to determine whether the usage of parametric statistical tests was possible. All data were normally distributed;
- *t-test (Cochran-Cox approximate t-test method)* to determine the differences in knowledge of mathematical problems in mathematical content (arithmetic; geometry with measurement; logic and language) between the experimental and control group students at the beginning and at the end of the experiment.

RESULTS

Pre-test

In Table 1 we present the descriptive statistics for the pre-test. In addition, we report the results of the *t*-test, which was used to check for possible differences between the experimental and control groups. For the purposes of the research, we calculated the statistics for easier (I) and more complex problems (II). Easier problems are those that are from the second Gagné taxonomy scale, while more complex problems are those from the third Gagné taxonomy. A graphic representation of the test achievements is in Figure 1. Differences between the EG and CG might be

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found in the pre-test, in favor of the EG. In particular, students in the EG showed better performance than students in the CG in both easier (p = 0.001) and more complex (p = 0.002) problems. Regarding geometry, the EG demonstrated superior performance compared to the CG in easier problems (p = *0.*014), but there was no significant difference in more complex problems (p = 0.945). In terms of logic, there was no significant difference between the EG and the CG in easier problems (p = 0.448), but statistically significant differences were found in favor of the EG in more complex problems (p = 0.004). Overall, the results of the t-tests indicated that students in the EG initially achieved better results than students in the CG in most of the tasks.

Test	Group	*M*	*Max points*	*SD*	*min*	*max*	*t*
A I	EG	3.34	5	1.46	0	5	3.27**
	CG	2.63		1.44	0	5	
A II	EG	2.94	5	1.40	0	5	3.11**
	CG	2.30		1.35	0	5	
G I	EG	1.43	2	0.74	0	2	2.49*
	CG	1.16		0.75	0	2	
G II	EG	0.46	1	0.50	0	1	0.069
	CG	0.46		0.50	0	1	
L I	EG	0.35	2	0.55	0	2	-0.76
	CG	0.41		0.56	0	2	
L II	EG	0.62	2	0.63	0	2	2.90**
	CG	0.36		0.61	0	2	

Note. A = arithmetic; G = geometry; L = logic; EG = experimental group; CG = control group.
* $p < 0.05$; ** $p < 0.01$

Table 1: Descriptive statistics and the results of the t-test for the pre-test.

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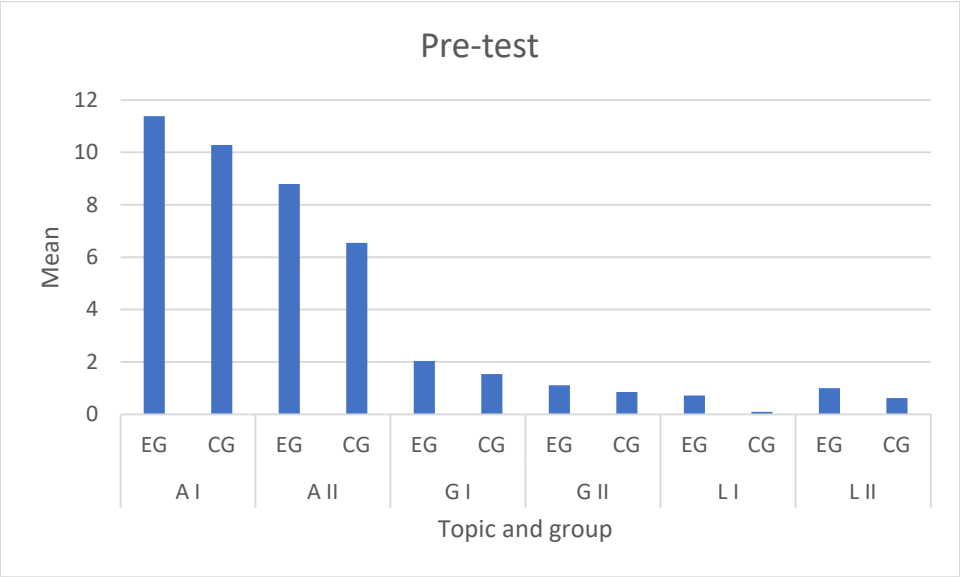


Figure 1: The means of the pre-test.

Post-test

In Table 2 we present the descriptive statistics for the post-test. In addition, we report the results of the ANCOVA test with one variable (the results of the initial tests), which was used to check for possible differences between the experimental and control group with regard to the initial test. For the purposes of the research, we calculated the statistics for easier (I) and more complex problems (II). A graphic representation of the test achievements is in Figure 2.

Test	**Group**	***Max points***	***M***	***SD***	***min***	***max***	***F***
A I	EG	13	11.38	2.00	5	13	1.17
	CG		10.29	2.99	1	13	
A II	EG	17	8.79	4.88	0	17	1.50
	CG		6.54	4.86	0	16	
G I	EG	3	2.03	0.96	0	3	6.02*
	CG		1.53	1.11	0	3	
G II	EG	2	1.10	0.94	0	2	4.01*
	CG		0.85	0.96	0	2	

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L I	EG	1	0.71	0.46	0	1	125.98***
	CG		0.09	0.29	0	1	
L II	EG	3	0.99	0.88	0	2	4.07*
	CG		0.62	0.82	0	2	

Note. A = arithmetic; G = geometry; L = logic; EG = experimental group; CG = control group; F=ANCOVA results.
* $p < 0.05$; ** $p < 0.01$; *** $p < 0.001$
Table 2: Descriptive statistics and results of the ANCOVA for the post-test.

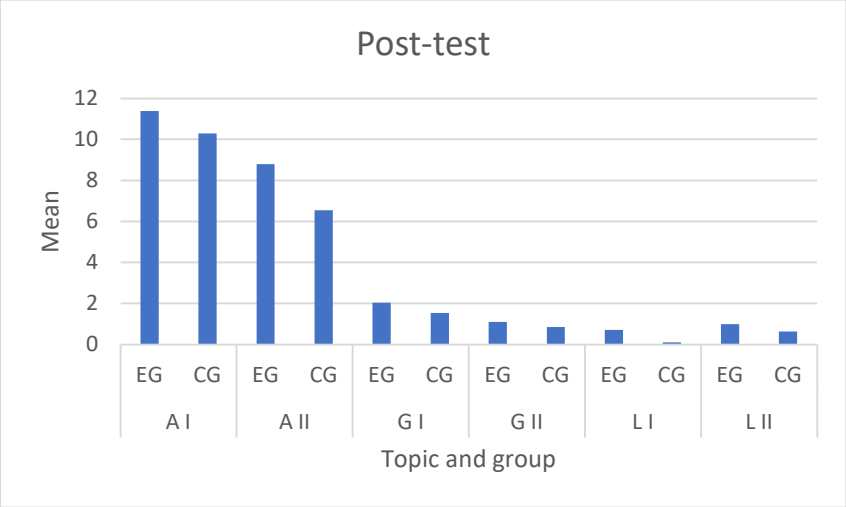


Figure 2: The means in the post-test.

From Table 2, we understand that the EG had higher achievements than the CG in solving both easier and more complex problems. The statistically higher achievements are present in the domain of geometry and logic. In particular, despite having a statistically significant advantage in the pre-test, the EG demonstrated a statistically significant difference from the CG in geometry and logic. No statistically significant differences between the EG and CG were detected in easier arithmetic problems ($p = 0.280$) and more complex arithmetic problems ($p = 0.222$), indicating that the experimental approach did not significantly change the achievements of the EG and CG when initial knowledge is considered.

On the other hand, despite having relative advantages in the pre-test, the EG showed better achievements in easier ($p = 0.015$) and more complex ($p = 0.047$) geometry problems, as well as in easier ($p < 0.001$) and more complex ($p = 0.045$) logic problems. Therefore, the EG generally performed better than the CG in two of the three mathematics topics. As noted in Table 4, the EG

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also generally outperformed the CG in arithmetic; however, considering the initial knowledge advantage of the EG over the CG, these differences might be attributed to the EG starting with a very good baseline knowledge.

Comparison Between the EG and CG

Students in both groups were good at solving simple arithmetic problems (e.g., "*Ana's mother is 31 years old. Her daughter Anna is 20 years younger. How old is Ana?*"), as these types of problems appear most frequently in our textbooks and workbooks, and are also the ones we teachers ask most often. These types of problems were solved correctly by 87.5% of EG students and 79.2% of CG students. Even simple problems in logic and geometry were solved more successfully by EG students. It was evident how narrow a range of problems was solved by CG students, as they rarely solved logic and geometry problems by measurement. In simple logic problems, pupils had to sort the elements of a given set according to two properties, using different tables (e.g. Carroll diagrams). There were 70.8% of pupils in EG were successful in these problems, while only 9% of pupils in CG were successful, even though such problems also appear in grade 3 prescribed mathematics textbook. In the EG, students solved these mathematical problems through three levels of presentation (i.e. enactive, iconic, and symbolic). The CG students solved the problem only in the workbook at the iconic or symbolic level. They did not do the concrete-experiential activity, which is, however, at the level of the first three grades, one of the obligatory steps towards the development of cognitive processes. At the same time, they did not integrate spreadsheets in a meaningful way in solving various problems, both mathematical and non-mathematical.

In simple geometry problems with measurement, 67.8% of EG pupils and 51.1% of CG pupils were successful. Although the problems were very simple, they were alien to the CG pupils, who themselves stated that they were different from the ones they solve at school. In our schools, pupils are used to solving problems in examinations that are very similar (or even identical) to those they solve in class. Since mathematics lessons do not develop different strategies for solving mathematical problems, students are then helpless in solving "different" problems.

Even with more challenging problems, the EG students were more successful. The average achievement in solving more difficult problems in arithmetic was 51.78% in EG and 38.50% in CG. In geometry problems, the average achievement was 55% in the EG and 42.70% in the CG. There was also a large difference in the knowledge of logic problems (EG achieved 49.43%, CG 31.11%). The more difficult problems included compound (guided and unguided) problems and:

- Problems that do not have sufficient data to solve;
- Problems with more data than needed for the solution;

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- Problems with multiple solutions;
- Problems in which the given data are read from a spreadsheet.

In addition, the test asked students to solve the following problems:

- To formulate a meaningful problem for a given illustration and then solve it;
- To formulate a meaningful question to the text of the problem and then solve the problem;
- Choose a mathematical problem that goes with the given calculation.

Of these problems, CG pupils in mathematics lessons solved mainly composite problems and fewer problems where they read the given data from a table and problems where they formulated a meaningful problem to a given illustration. In the EG, however, all of these problems were solved in the classroom. Interestingly, students from both groups performed the worst on the composite problems. In EG, 18.5% of students solved these problems correctly, while in CG 14.4% of students did so. Although these problems are often solved in mathematics lessons, only some students are able to decompose a compound problem into sub-problems on their own, without teacher guidance, asking intermediate questions that lead them to the solution.

In Table 3 the frequencies (in %) of how well students from EG and CG solved more challenging problems are presented.

Problems	**EG**	**CG**
Problems with more data than required	59.5	41.6
Problems with fewer data than required	82.0	63.3
Problems with multiple correct answers	20.8	10.0
Problems where data are presented in tables	80.5	71.1
Formulate a problem based on a picture	78.4	62.6
Formulate a question based on a text	54.3	37.4
Choose a mathematical problem that can be solved with a given operation	71.6	44.4

Table 3: The frequencies of correctly solved problems in EG and CG.

Table 3 shows that EG students outperformed CG students on all types of problems. Both did poorly on problems with multiple solutions (20.8% in EG and 10% in CG), as in most cases they

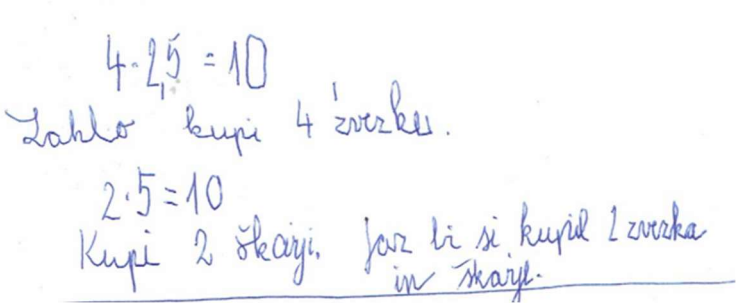
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settled for one solution, even though they were told in the task to write down all possible solutions.

Most students believe that if you find one solution, there is no need to look for others because they have "solved" the problem correctly (Figure 3). The EG teachers also found that these were the problems that caused the most problems for the pupils. Only the ablest students searched for or found more solutions even if the teacher did not guide them during the problem-solving (Figure 4).

Problem: Tina is looking at school supplies in the store window. She has 10 EUR. Which school supplies can she buy? Write down several solutions. What would you buy if you had 10 EUR?

- [notebook] □ 2.50 €
- [scissors] □ 5 €
- [folder] □ 4.50 €
- [pencil] □ 3 €



$4 \cdot 2{,}5 = 10$

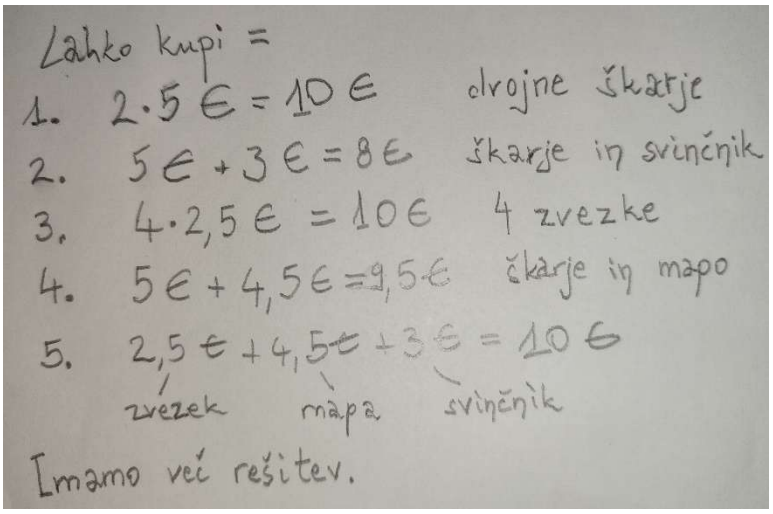
He could buy 4 notebooks.

$2 \cdot 5 = 10$

He can buy 2 scissors. I would buy 2 notebooks and scissors.

Figure 3: A boy (CG33) from the CG solving incorrectly the problem with multiple correct answers.

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He can buy =

1. $2 \cdot 5€ = 10€$ two scissors
2. $5€ + 3 = 8€$ scissors and a pencil
3. $4 \cdot 2{,}5€ = 10€$ 4 notebooks
4. $5€ + 4{,}5€ = 9{,}5€$ scissors and a folder
5. $2{,}5€ + 4{,}5€ + 3€ = 10€$ notebook, folder, pencil

We have multiple solutions

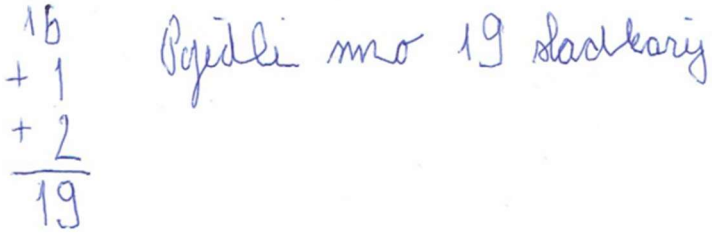
Figure 4: A boy (EG61) from the EG solving correctly the same problem with multiple correct answers.

Pupils who were in the EG in third grade are now in fourth grade. In these classes, mathematics is still taught according to the concept we developed in our experiment. The teachers of these four fourth grades have been trained in the new concept of mathematics education through continuous training and study groups. On the final test for the fourth grade, 42.2% of the students solved these types of problems correctly, which confirms the findings of some contemporary research (Ali et al., 2010; Lessani et al., 2017) that the ability to encode the components of a problem and to plan the solution of different types of problems progresses with age and with appropriate teaching.

For problems with more data than needed for the solution, it was found that 59.5% of the EG students were able to read the problem text with comprehension and were able to find the data needed for the solution (Figure 5). In CG, 41.65% of students solved the problem correctly. The incorrect solutions were mainly due to the fact that they used all the data, including unnecessary data, in the solution, as they always "had to" use all the data in the problems they encountered in class (Figure 6).

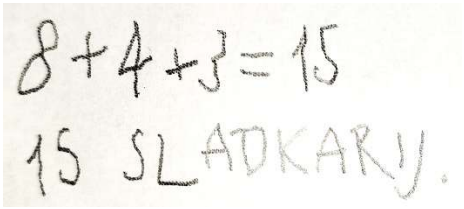
Problem: Tina says: "Yesterday I went to the swimming pool. There were 8 of us: 4 girls, 3 boys, and my dog. We ate sixteen candies, one ice cream, and two lollipops there. How many sweets did we eat at the swimming pool?

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16 We ate 19 sweets
+1
+2
___
19

Figure 5: A girl (EG85) from the EG solving correctly the problem with more data than needed.



8+4+3 = 15

15 sweets

Figure 6: A boy (CG02) from the CG solving incorrectly the same problem with more data than needed.

Students were very successful in solving problems that did not have enough data to solve, with 82% of EG students and 63.3% of CG students successfully solving this problem (Figures 7, 8, and 9). Most students stated that they liked this type of problem the most, as they found it that they could determine the value of the missing data themselves. Teachers also confirmed that pupils were very motivated to solve these types of problems, but were surprised that pupils were able to find the missing data very quickly and then assign very meaningful values to it.

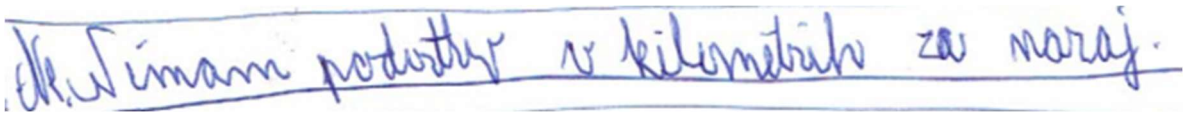
Problem: Jan went to his grandmother's place in the countryside with his family on Sunday. The road they took to get to his grandmother's place is 80 km long. On the way back home, Jan's father chose a shorter route. How many kilometers did they travel in both directions? Can you solve the problem? If not, why not?

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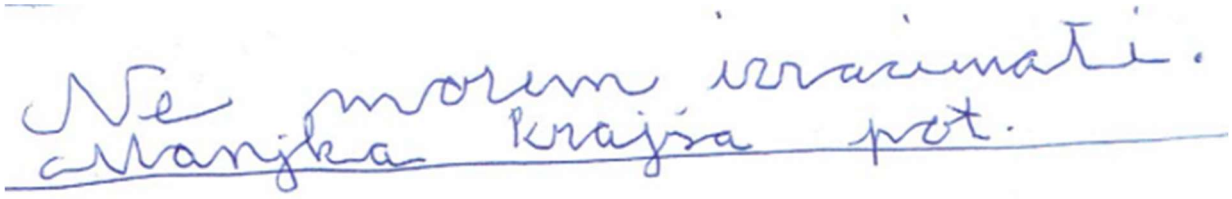
I can solve it.
The second road [grammatical error] is shorter than 80 km.
The road is long less than 160 km.

Figure 7: A girl (EG97) from the EG solving incorrectly the problem with not enough data.



No. I don't have data for the kilometers back.

Figure 8: A boy (CG09) from CG solving correctly the same problem with not enough data.



I can't calculate it.
The shorter road is missing.

Figure 9: A boy (CG74) from EG solving correctly the same problem with not enough data.

In addition to these problems, the favorite problem for students from both groups was to create a meaningful problem to go with a given illustration. Additionally, 80.5% of the EG students and 62.6% of the CG students solved this problem correctly. We also analyzed how original the students were in formulating their problems. The EG pupils formulated a wide variety of problems:

- Problems using multiplication, division, addition, and subtraction to solve them;

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- Problems in geometry;
- Problems in the data processing.

In CG, students only designed problems that they solved using multiplication and addition. At the same time, the problems were constructed exclusively according to the model in the workbook. This points to the fact that our current mathematics teaching provides little opportunity for students to solve problems in their own way and that creativity is not sufficiently encouraged. However, in the EG, the teachers developed flexible and creative thinking. The problems that the pupils constructed show that pupils at this age level are able to construct mathematical problems in a creative and meaningful way if properly encouraged by the teacher.

In the next problem, students were asked to add one or more meaningful questions to the text of the problem and then solve the problem. This problem was successfully solved by 54.3% of the EG pupils and only 37.4% of the CG pupils. The problem text was made "harder" by writing the given numerical data in words rather than numbers. Research (Moghadam et al., 2012) has shown that writing a quantity with a number or with a word has a great influence on the comprehension of the text. Therefore, the teacher needs to be aware of all the factors that influence the understanding of the problem and to guide the pupil skillfully in solving it. In the EG, teachers instructed students to underline information that is written in both number and word form.

Students in both groups did very well on problems in which they read the given data from a table. In the EG group, 80.5% of the students solved the problem correctly, and in the CG group, 71.1% of the students solved the problem correctly. In recent years, reading and writing in various forms of spreadsheets have been integrated more into classroom teaching in both mathematics and science (Isiksal & Askar, 2005). This is confirmed by the results obtained.

In the test, pupils were also given a problem where they had to find the problem that the given calculation solves among given problems. This was a twofold task: to understand the text and to have a good grasp of the basic arithmetic operations. The same numbers "appeared" in all the given problems, so the students had to read all three problems very carefully and then decide on the correct solution. In most cases, CG students chose the first problem, as only 44.4% of the students solved the problem correctly. This confirms the findings of previous studies (Bernardo, 1999; Phonapichat et al., 2014) that students do not read the whole text, but in most cases rewrite the given data and perform the computational operation that is being discussed in the mathematics lesson at the time. The EG pupils solved the given problem very well, with 71.6% of the solutions being correct. In the EG mathematics lessons, we have been putting a lot of emphasis on the first stage of problem-solving: understanding the problem.

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DISCUSSION AND CONCLUSIONS

The main purpose of our study was to test the hypothesis that EG students who received a new model of mathematics instruction, in which problem posing and problem-solving were the leading didactic-mathematical activity, would be more successful in solving all types of mathematical problems than CG students who received traditional mathematics instruction, in which the main emphasis is on training in arithmetic operations.

To check the validity of the hypothesis, a pedagogical experiment was developed. Students from the EG and CG took an initial (pre-)test to check for possible differences in the initial knowledge. Results have shown that students from EG had better achievements than students of CG in the majority of the mathematics domains. As a possible explanation for why this phenomenon occurred, teachers' experiences might be considered. For instance, the teachers from the EG were prepared for the experiment throughout the year through various forms of training (workshops, lectures, seminars, etc.) when they were teaching in the second grade. They were also given materials (method manual, task book, workbook, etc.) which they then used in the experiment when students were in the third grade. The new concept makes the problem the central content. In other words, mathematics is not taught just "for the numbers and the operations between them", but for the "problems". Thus, all fundamental mathematical concepts were built from problem situations that arose from the pupils' experience. Teachers emphasized generalized knowledge and problem-based learning, in contrast to the prevailing concept of mathematics education in this country, which is based primarily on the transfer of skills and knowledge. Instruction based on such transfer, however, "implements" knowledge and techniques immediately after the introduction of new content, regardless of whether the mental connections have been made so that the learner understands the content and can apply it. This transfer can also include the transfer of rules, principles, and methods to similar tasks without the learners understanding and being able to apply them. The scope of this transfer is relatively narrow. For example, the technique of computational operations falls within this range.

Based on all the results obtained and their analysis, we can conclude that the model of mathematics teaching implemented in the EG, in which problem posing and problem solving were the leading didactic-mathematical activity, is a success. This confirms our research hypothesis: EG will be more successful than CG in solving both simple and more complex problems in the context of arithmetic, geometry with measurement, and logic with sets.

The study suggests some important theoretical and practical implications. Firstly, the results support the effectiveness of a problem-based learning approach in mathematics teaching. This finding aligns with existing theories that emphasize the value of active learning, critical thinking, and problem-solving in developing students' mathematical skills and conceptual understanding (Chairuddin & Farman, 2019; Hu et al., 2018; Meke et al., 2018). Findings therefore suggest that when problem-posing and problem-solving become central components of mathematics instruc-

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tion, students are more successful in tackling both simple and complex mathematical problems (Darhim et al., 2020). This bolsters the theoretical foundation for the integration of problem-based learning strategies in mathematics education. Secondly, one practical implication is that educators and curriculum designers (policymakers) should consider incorporating problem-based learning approaches into the mathematics curriculum, especially for primary school students. This approach may enhance students' problem-solving abilities and overall math performance. Nevertheless, teachers should receive training and support in implementing problem-based learning effectively (Ali et al., 2010). They need strategies to create an environment that encourages problem posing and problem-solving activities in the classroom.

LIMITATIONS AND FUTURE DIRECTIONS

There are several possible limitations of our study. Firstly, the sample size is still small, so the results cannot be generalized to the whole population. Additionally, the study only focuses on students from one particular region. Future research could explore whether problem-based mathematics instruction is effective in other countries and with other curricula. This could help to determine whether the results of this study are generalizable to other contexts.

Secondly, the study only examines the impact of a specific teaching method on student performance in math, and it does not take into account other factors that may affect student performance, such as motivation (Gilbert et al., 2014), beliefs about mathematics (Mason, 2003), mathematics anxiety (Ashcraft & Ridley, 2005; Wu et al., 2012), and others.

Thirdly, the study only examines the short-term impact of the teaching method, and it is unclear whether the effects would persist over the long term. While this study showed positive results in the short term, it would be interesting to see if the effects of problem-based mathematics instruction persist over a longer period of time. Future research could follow up with the students from the experimental and control groups to see if the differences in achievement continue to hold true in later grades or even in adulthood.

This study found that teacher training was an important factor in the success of problem-based mathematics instruction. Future research could explore different types of teacher training and their effectiveness in implementing problem-based mathematics instruction. This could help to identify the most effective ways to train teachers for this type of instruction.

While this study showed that problem-based mathematics instruction was effective in improving student achievement, it would be interesting to explore whether this type of instruction also increases student engagement and motivation. Future research could measure student engagement and motivation and compare them between the experimental and control groups. Moreover, problem-based mathematics instruction could potentially be enhanced through the use of tech-

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nology (Jacinto & Carreira, 2017; Lee & Hollebrands, 2006). Future research could explore how technology can be used to support problem-based mathematics instruction and whether this enhances student achievement and engagement. Additionally, further research should explore the long-term effects and scalability of the mentioned approach in different educational settings and age groups. Investigating the transferability of our results to other cultural settings and educational systems could provide additional valuable insights.

Based on the findings and insights derived from the study, several recommendations can be made, both for educators and policymakers. Firstly, policymakers should review and potentially revise the mathematics curriculum to align it with problem-based learning principles. This should involve the development of curriculum materials, task books, and resources that facilitate problem-posing and problem-solving activities. Ensuring that the curriculum encourages critical thinking and application of mathematics in problem-solving contexts is essential. Secondly, results have shown the importance of prioritizing conceptual understanding over rote memorization and algorithmic approaches in mathematics education. Educators should therefore emphasize the application of mathematical concepts in real-world problem-solving scenarios can foster deeper comprehension among students. Also, teachers should encourage an active learning environment where students engage in critical thinking and collaborative problem-solving.

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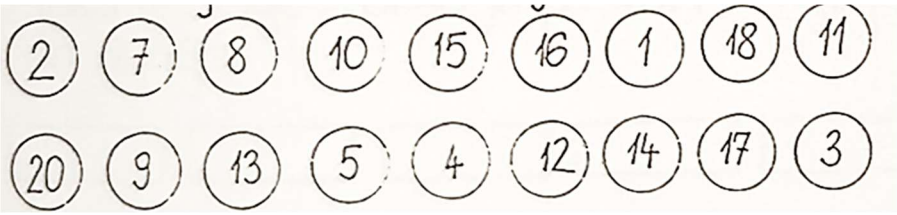
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APPENDIX
Appendix A - Pretest

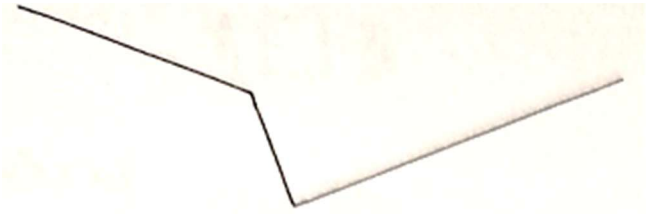
1. Complete it in such a way that the left-hand side will be equal to the right-hand side.

   $$15 - 6 = 8 + ___$$

2. Color the circles that contain numbers greater than 7 and less than 13.

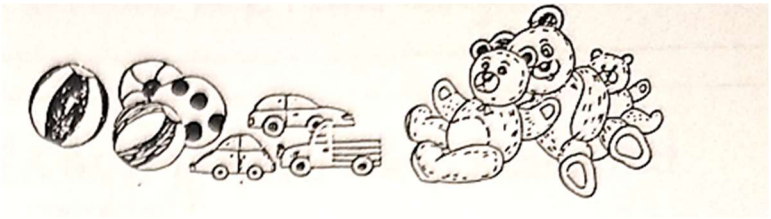


3. Tina read 8 pages of her book on the first day. On the second day, she read 6 pages more than the first day. How many pages did Tina read on the second day?
4. There were 12 cars parked in the parking lot. Then, 4 cars left, and 3 cars arrived. How many cars are in the parking lot now?
5. Tina and Rok have a total of 90 €. How much money can Tina and Rok have?
6. Maja drew the following line in her notebook:



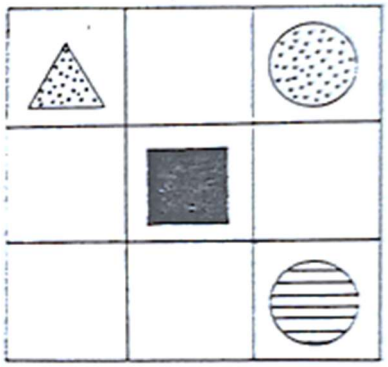
   What is the length of the drawn line?

7. Create a problem based on the following picture and solve it.



8. Look at the following table. Find the rule that was used to order the following shapes and complete the table.

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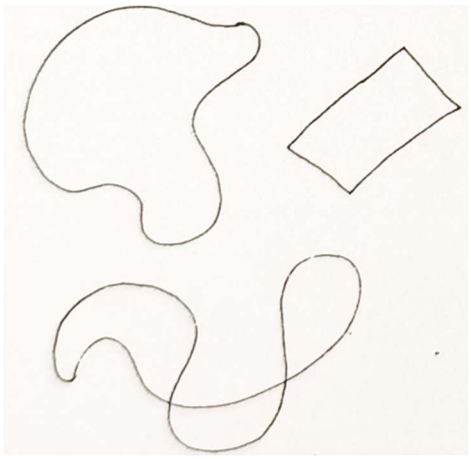
9. Maja, Jana, Nina, Tina, and Ana live in the same block of apartments. Nina is taller than Maja, Jana, and Tina, but she is not the tallest. Jana is the shortest and is right after Maja. Sort the heights of the five girls from tallest to shortest.
10. Jaka bought 2 pencils and 3 notebooks at the stationery store. Which information do you need to know how much money Jaka spent at the stationery store? Circle the correct answer:
    a. name of the stationery store;
    b. I don't need anything;
    c. price of the pencil and notebook;
    d. age of the saleswoman;
    e. only the price of the pencil.
11. Mira has   . Tine has    . Who has more money?
12. Metka is 12 years old. Her brother is 5 years old. How many years older is Metka than her brother?
13. In the orchard, they cut down 16 trees: 5 apple trees, 4 peach trees, and some plum trees. How many plum trees did they cut down?
14. Martina and Matej were drawing lines.

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Martina

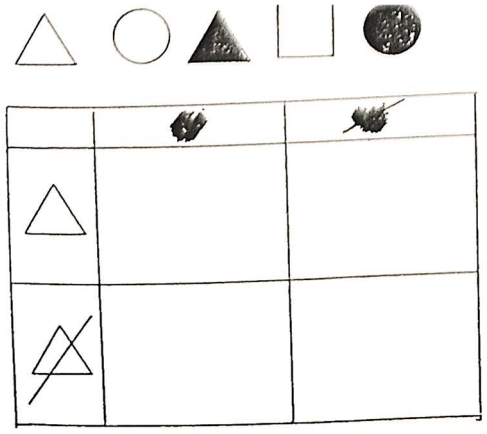


Matej



How do Martina's and Matej's lines differ?

15. Place each of the following shapes in the correct location in the table.



16. A class of 20 students is going on a trip to Piran. The bus ticket costs 2 €. On the day of the trip, Tine, Iva, and Maja fell ill, while Urška and Žiga went to an athletics competition. How many children went on the trip?
17. Which word problem matches the calculation $5 + 3 = 8$? Circle the correct answer.
    a. Tina has 8 books, and Jan has 3 books. How many books do they have together?
    b. There are 5 flowers in each bouquet. How many flowers are there in 3 bouquets?
    c. Tina is five years old. Her sister is three years old. How many years old are they together?

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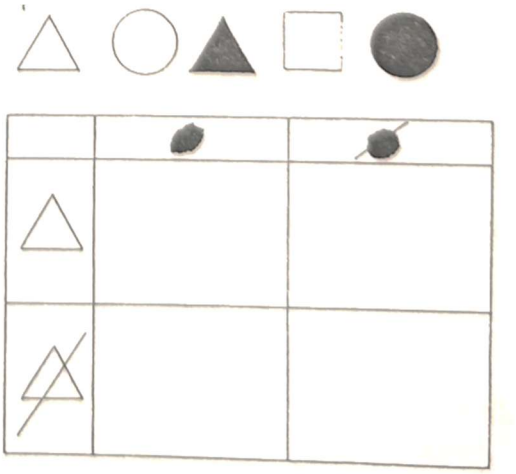
Appendix B - Post-test

1. The computation $9 \cdot 8 = 72$ is correct, Which of the following computations is also correct?
    a. $9 : 8 = 72$
    b. $72 \cdot 8 = 9$
    c. $72 : 8 = 9$
    d. $72 \cdot 9 = 8$
2. There were 136 boys and 128 girls who signed up for the competition. How many children competed?
3. The number has the following properties: it is less than 30, and you say it when you count by multiples of 3 and 8. What is this number?
4. Lena and her two friends put a bird feeder on the tree. Immediately, 4 sparrows, 8 great tits, and 9 blackbirds flew in. How many great tit legs were on the tree?
5. Matjaž has 23 blocks. Metka has 12 fewer than Matjaž, and Marko has three times as many as Metka. How many blocks do all three of them have together?
6. The price of a theater performance is 50 euros. All the seats in the theater were sold. Which of the following information do you need to determine the total price of all the tickets sold?
    a. The number of actors on stage
    b. The number of seats in the theater
    c. The age of the audience
    d. I don't need anything
7. Ana's mother is 31 years old. Her daughter Anna is 20 years younger. How old is Ana?
8. Maja has 114 sweets. She wants to divide them among her 6 friends. How many sweets do each friend get?
9. Tina is looking at school supplies in the store window. She has 10 EUR. Which school supplies can she buy? Write down several solutions. What would you buy if you had 10 EUR?
    a. [notebook] 2.50 €
    b. [scissors] 5 €
    c. [folder] 4.50 €
    d. [pencil] 3 €
10. Read the text and write a question based on it, then solve the problem. In ten desks, there are 2 students each, and in four desks, there are 4 students each.
11. Jan went to his grandmother's place in the countryside with his family on Sunday. The road they took to get to his grandmother's place is 80 km long. On the way back home,

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Jan's father chose a shorter route. How many kilometers did they travel in both directions? Can you solve the problem? If not, why not?

12. Tina says: "Yesterday I went to the swimming pool. There were 8 of us: 4 girls, 3 boys, and my dog. We ate sixteen candies, one ice cream, and two lollipops there. How many sweets did we eat at the swimming pool?"
13. Place each of the following shapes in the correct location in the table.

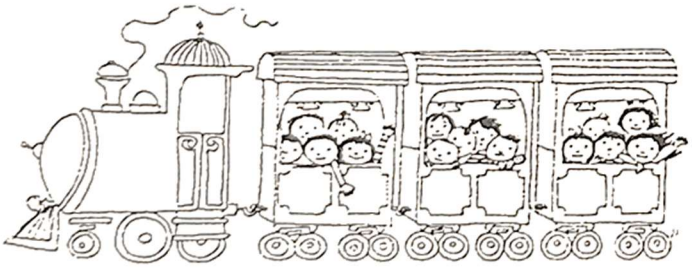


14. Rok, Jan, Ivo, Bor, and Tine are friends. Ivo is taller than Rok, Jan, and Bor, but he is not the tallest. Jan is the shortest and is just before Bor. Arrange them in order of height from shortest to tallest.
15. How much flour is there if the drawn scale is in balance?



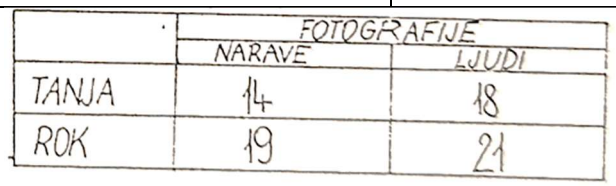
16. Mom buys 2 liters of milk every day. How many liters of milk does she buy in a month that has four weeks and 2 days?
17. Tina has 24 candies, Maja has 19. Nastja has fewer candies than Tina but more than Maja. How many candies could Nastja have?
18. Create a word problem based on the drawing and solve it.

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19. Tina and Rok are enthusiastic photographers. They particularly enjoy photographing nature and people. Look at the table to see how many photos they took in one month. [We present also the original picture of the table in the test.]

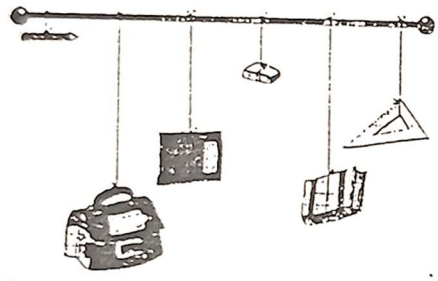
	Photos	
	Nature	People
Tanja	14	18
Rok	19	21



	FOTOGRAFIJE	
	NARAVE	LJUDI
TANJA	14	18
ROK	19	21

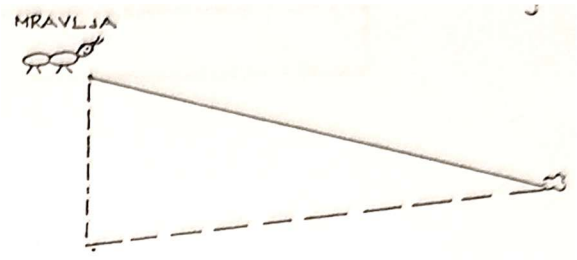
   a. How many photos did Tanja take?
   b. How many more nature photos are there?
   c. How many photos did both of them take together?

20. A painter painted 12 paintings. He displayed them on panels. He hung the same number of paintings on each panel. How many paintings could be on each panel?
21. The objects are hanging on identical elastic bands. Arrange them from the heaviest to the lightest (write the names of the objects).



22. The picture below shows two different paths from the ant to the bread crumb. How long is the longest path? [mravlja = ant]

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23. Which word problem matches the calculation $5 \cdot 3 = 15$? Circle the correct answer.
    a. Tina has 5 balls, and Jan has 3 balls. How many balls do they have together?
    b. There are 5 flowers in each bouquet. How many flowers are there in 3 bouquets?
    c. Tina is 5 years old. Her sister is 3 years old. How many years old are they together?

Appendix C – Examples of problems solved in class



A dragon with six heads has brought his sons on a walk. His sons each have half the number of heads that the father has. How many heads are there altogether (father and sons)?

Figure C1: Example of an arithmetic problem without enough data.

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The school bus has taken the children home. At the first bus stop, 12 children got off, and at the second stop, 10 children got off. How many children are still on the bus?

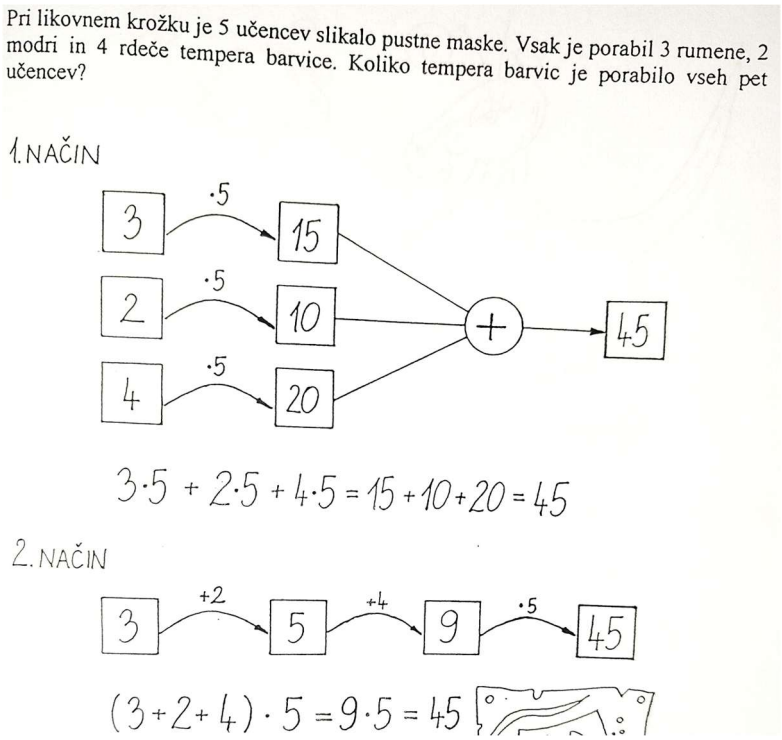
Figure C2: Example of an arithmetic problem without enough data.



Metka placed a bird feeder on the tree. Immediately, 3 sparrows, 7 tits, and 9 blackbirds arrived. How many tit legs are on the tree?

Figure C3: Example of an arithmetic problem with more data than required to solve it.

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In the art club, 5 students were painting carnival masks. Each one used 3 yellow, 2 blue, and 4 red tempera paints. How many tempera paints did all five students use in total?

Figure C4: Example of an arithmetic problem with multiple ways to solve it.

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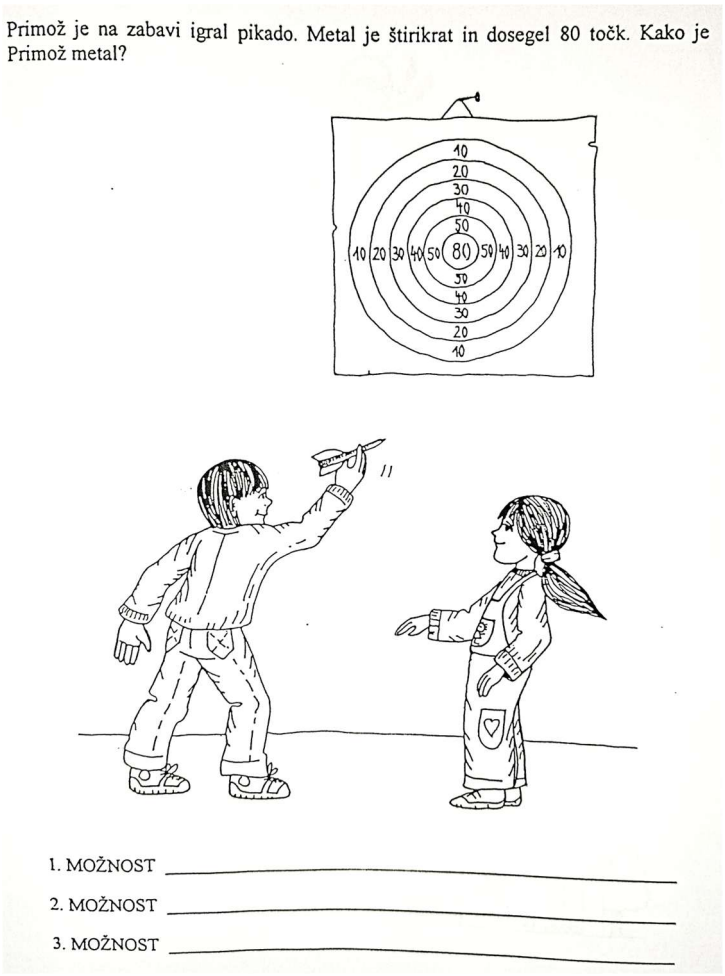


Which Martian dinosaur is inserted [i.e., it does not belong] into the following group of Martian dinosaurs?

1st solution: Martian dinosaur _____, because it has claws.
2nd solution: Martian dinosaur _____, because it has more than five legs.
3rd solution: Martian dinosaur _____, because it does not have a straight tail.

Figure C5: Example of a logic (and sets) problem with multiple ways to solve it.

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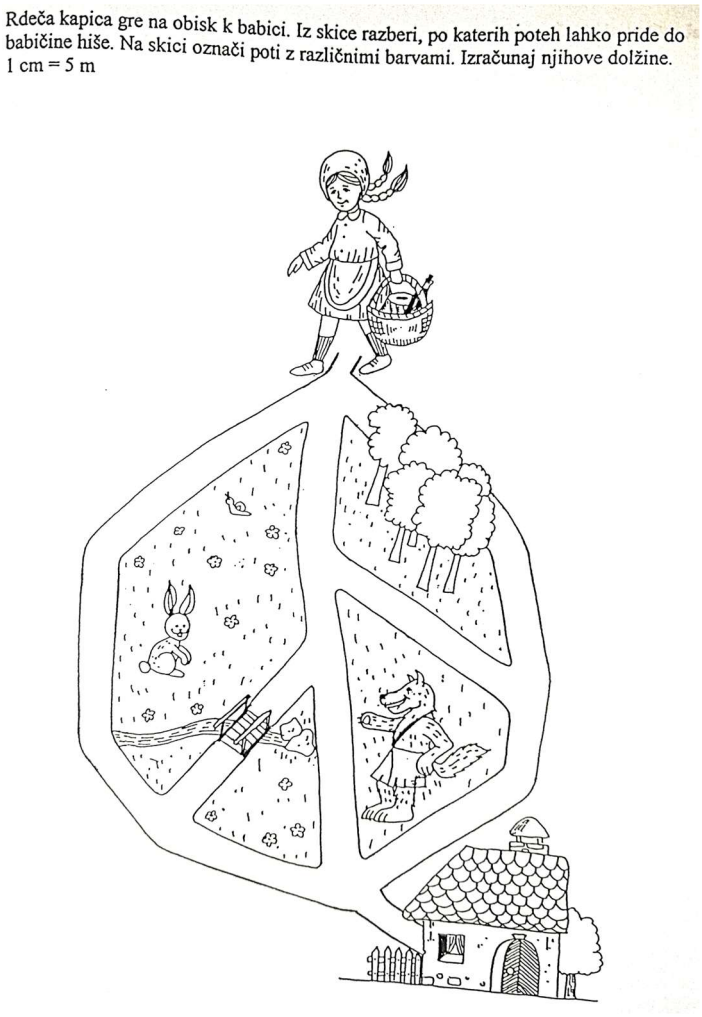


Primož played darts at the party. He threw four times and scored 80 points. How did Primož throw?

1st possibility: _____
2nd possibility: _____
3rd possibility: _____

Figure C6: Example of an arithmetic problem with multiple ways to solve it.

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Little Red Riding Hood is going to visit her grandmother. From the sketch, determine which paths she can take to get to her grandmother's house. Mark the paths on the sketch with different colors. Calculate their lengths. 1 cm = 5 m.

Figure C7: Example of a geometry (and measurement) problem with multiple ways to solve it.

Appendix D – Pre-test points, mathematics domains, and taxonomy

Exercise	Points	Mathematics domain	Taxonomy
1	1	A	I
2	1	L	I
3	1	A	I
4	1	A	I
5	1	A	II
6	1	G	I
7	1	A	II

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8	1	L	II
9	1	L	II
10	1	A	II
11	1	G	I
12	1	A	I
13	1	A	I
14	1	G	II
15	1	L	I
16	1	A	II
17	1	A	II

Note. A = arithmetic; G = geometry; L = logic; I = easier problems; II = more complex problems.

Table D: The structure of the pre-test.

Appendix E – Post-test Points, Mathematics Domains, and Taxonomy

Exercise	Points	Mathematics domain	Taxonomy
1	1	A	I
2	2	A	I
3	1	A	II
4	2	A	II
5	3	A	II
6	1	A	I
7	2	A	I
8	2	A	I
9	3	A	II
10	2	G	II
11	2	A	II
12	1	L	I
13	1	L	II
14	1	G	I
15	2	A	I
16	1	L	II
17	3	A	II
18	3	A	I
19	2	A	II
20	1	G	I
21	1	G	I
22	1	A	II

Note. A = arithmetic; G = geometry; L = logic; I = easier problems; II = more complex problems.

Table E: The structure of the post-test.

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MATHEMATICS TEACHING RESEARCH JOURNAL
SUMMER 2025
Vol 17 no 3

Optimizing Classroom Engagement in National Certificate (Vocational) Mathematics: A Framework for Questioning Techniques

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Abstract: This study develops and tests a questioning model aimed at enhancing interactive engagement in financial mathematics instruction within a single NC(V) Level 2 classroom. Grounded in formative assessment principles, the model incorporates structured and open-ended questions to foster procedural and conceptual understanding. The intervention used real-world financial scenarios like budgeting and loan repayments to make abstract concepts more relatable. Data were collected through classroom observations and a preparatory examination to evaluate student engagement and understanding. The findings indicate that consistent questioning fosters active participation, while higher-order questions promote critical thinking and real-world application. While performance improvements were noted, the primary focus was on engagement quality rather than academic outcomes. This study offers insights into how questioning strategies can enhance interactive learning in vocational education. Future research should expand the model to other subjects, gather student feedback, and integrate advanced digital tools for deeper interactivity.

Keywords: *Questioning Techniques, Interactive Engagement, Formative Assessment, Financial Mathematics*

INTRODUCTION

Globally, education systems are tasked with creating a workforce that is adaptable and equipped to meet the evolving demands of the labor market. Vocational education, particularly through Technical and Vocational Education and Training (TVET) institutions, is recognized as a crucial mechanism for bridging the gap between education and employment, providing students with the relevant skills and competencies required by industries worldwide (Venkatraman et al., 2018). Disparities between developed and developing countries regarding resources and infrastructure challenge the effectiveness of these educational frameworks. While developing countries grapple with limited resources in TVET classrooms, developed nations face the challenge of continuous-

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ly updating curricula, laboratories, and equipment to keep pace with the 5th Industrial Revolution (Al-Ismail et al., 2023).

In South Africa, TVET colleges are essential in addressing unemployment and closing the skills gap by offering industry-relevant training that prepares students for direct entry into the workforce (Ramaligela, 2022). One of the key qualifications provided by these colleges is the National Certificate (Vocational) [NC(V)], spanning three levels, each typically completed in one year. Within the NC(V) engineering program, mathematics is central to understanding other engineering disciplines and is a fundamental subject (Siddiqi, 2017).

A significant challenge that compounds these issues is the low completion rates in NC(V) qualifications, particularly in foundational subjects like mathematics. Reports indicate that students enrolled in NC(V) qualifications in 2019 had a completion rate of just 10.8% by 2021, with engineering programs showing an even lower rate of 5.7% (Khuluvhe, 2023). Moreover, Ngoveni (2018) reported that only 15.4% of NC(V) Level 2 mathematics students who enrolled in 2016 passed their final exams. These statistics highlight the need to address fundamental gaps in the program, especially in Level 2 mathematics (Financial Mathematics), which lays the groundwork for many technical skills essential in the workplace (Lago & DiPerna, 2010).

The reliance on traditional teaching methods, such as the "banking" approach where students passively receive information, further exacerbates the problem. This method limits students' engagement and fails to connect mathematical concepts to real-life applications, which is crucial for vocational training (Vimbelo & Bayaga, 2023). Teachers often fail to expose students to diverse representations of mathematical concepts, hindering deep understanding (Sehole et al., 2023). The effectiveness of mathematics instruction at TVET colleges is hampered by the reliance on traditional teaching methods and textbooks, highlighting the need for innovative teaching strategies that connect mathematical concepts to practical applications and enhance students' understanding and retention of mathematical knowledge (Adelabu & Pharamela, 2024).

Given this context, effective questioning techniques become increasingly relevant, as they can actively engage students, enhance comprehension, and foster a more interactive learning environment (Machaba & Mangwiro, 2024). Although the benefits of effective questioning in promoting student engagement and understanding are well documented across various educational settings, research on their specific application within NC(V) mathematics at TVET colleges remains limited.

This gap is particularly pronounced in financial mathematics, a foundational subject in vocational education that equips students with essential skills for navigating real-world financial situations, including budgeting, interest calculations, and loan repayments. However, the abstract nature of financial mathematics and reliance on passive teaching methods often lead to disengagement and poor conceptual understanding among students. To address this issue, this study proposes an interactive framework for questioning techniques designed to enhance student engagement and improve learning outcomes within the National Certificate (Vocational) Mathematics curriculum. By integrating structured questioning strategies that encourage critical thinking and real-world application, the study aims to bridge this gap and provide empirical insights that can

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inform and improve teaching practices in South African TVET colleges (Mahmud & Mohd Drus, 2023; Mauigoa-Tekene, 2023; Dahal, 2022; Røed et al., 2023).

The research question guiding this study is thus: "What interactive framework for questioning can be proposed to enhance engagement and learning outcomes in the National Certificate (Vocational) Mathematics curriculum at TVET colleges?"

CONCEPTUAL FRAMEWORK

The conceptual framework used in this study incorporates one of the five strategies for understanding formative assessment, as outlined in the study "Developing a Theory for Formative Assessment" (Black & Wiliam, 2009). This strategy is also recognized as a pillar of effective formative assessment in Ngoveni's (2024) research. In Ngoveni's (2024) work, two indicators were drawn from the literature and applied as benchmarks to achieve this pillar.

Pillar: Monitoring: Posing Questions That Inform Teaching

This pillar emphasizes the role of well-crafted questioning in fostering an interactive learning environment where formative assessment is integral to teaching and learning (Black & Wiliam, 2009). Adapting to NC(V) emphasizes creating responsive teaching dynamics with evaluative and instructional questions (Wiliam, 2011a).

Indicators of Monitoring: Posing Questions That Inform Teaching:

1. Engaging Students in Educational Encounters Through Consistent Questioning

Consistent questioning methods are crucial in engaging students by encouraging active participation and providing educators with immediate feedback to adjust teaching strategies (Heritage, 2007). Continuous, high-quality questioning fosters deeper comprehension and knowledge retention while promoting a dialogic and interactive learning environment (Cowie & Bell, 1999).

2. Questioning Techniques Used to Gather Evidence of Procedural and Conceptual Knowledge

Effective questioning techniques go beyond rote responses, involving open-ended and probing questions that assess students' deeper conceptual understanding and procedural knowledge (Ziyaeemehr, 2016). Such questions enhance cognitive engagement and improve learning outcomes (Mangwiro & Machaba, 2022), especially in vocational education, where the practical ap-

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plication of knowledge is critical. This indicator and consistent questioning will guide the design of the questioning model used during the intervention, ensuring it assesses and deepens procedural and conceptual knowledge (Martino & Maher, 1999; Ngoveni, 2024).

How the Framework Supports the Study and its Implication for NC(V) Mathematics

The conceptual framework supports the study's goal of developing an interactive questioning model to enhance student engagement and learning outcomes in NC(V) Mathematics at TVET colleges. Grounded in the two key indicators, the framework ensures that questioning techniques are aimed at engaging students and assessing their procedural and conceptual understanding. The intervention, designed around consistent and open-ended questions, allows for real-time formative assessment, enabling educators to adjust their teaching strategies based on immediate student feedback. By comparing the performance of students who received the intervention with those who did not, the study evaluates the effectiveness of the questioning model (Heritage, 2007).

This framework also highlights the relevance of questioning techniques in vocational education, aligning with labor market needs by focusing on the practical application of mathematical concepts (Boaler & Brodie, 2004). The findings will inform professional development programs, emphasizing questioning as a pedagogical tool for vocational educators (Fisher & Frey, 2014). The study improves teaching practices by embedding these techniques into the NC(V) curriculum, ensuring students are better prepared to apply their knowledge in real-world contexts.

LITERATURE REVIEW

This section explores the role of questioning techniques in mathematics education, their impact on student engagement and learning, challenges in implementation within the NC(V) context, and the potential of mobile learning as a complementary tool for enhancing interactive engagement.

The Role of Questioning Techniques in Mathematics Education

Questioning techniques in mathematics education are pivotal for enhancing student learning and engagement. Effective questioning assists teachers in assessing students' understanding, promotes critical thinking, and guides the learning process. However, the literature indicates that many educators do not consistently use these techniques to gather essential learning evidence, a strategy known as planned formative assessment (Ngoveni & Machaba, 2024). This approach involves the deliberate preparation and execution of tasks to measure intended learning outcomes

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(Cowie & Bell, 1999; Wiliam, 2011a). Effective questions should stimulate thinking and provide data that informs teaching, integrating skills required by the evolving labor market (Venkatraman et al., 2018; Mangwiro & Machaba, 2022). Gholami (2024) found that while traditional teaching often emphasizes routine exercises, collaborative approaches such as the Lesson Study program significantly enhance lecturers' pedagogical knowledge and improve students' problem-solving and higher-order thinking skills. This aligns with the argument that effective questioning techniques should extend beyond procedural recall to promote deep conceptual understanding and critical engagement in mathematics.

These contemporary insights highlight the importance of teacher-student interactions and peer collaboration in deepening conceptual understanding and skill development. In formative assessment, dialogue, feedback, and collaborative learning are crucial in shaping students' knowledge construction (Mercer & Littleton, 2007; Yu, 2023). Building on this, Khusna et al. (2024) emphasize that incorporating non-routine mathematical problems, particularly those requiring justification and graphical analysis, enhances students' critical thinking abilities. Their findings suggest that structured peer discussions, where students exchange ideas, assess reasoning, and refine solutions, foster deeper cognitive engagement and promote more effective problem-solving strategies.

Strategies for Effective Formative Assessment

To maximize the effectiveness of formative assessments, educators should employ various techniques to regularly evaluate student progress. These include classroom discussions, relevant questions, and tracking tasks (Fisher & Frey, 2014; Heritage, 2007). Multiple response formats, such as verbal, written, electronic, and visual, coupled with effective questioning strategies like open-ended questions and adequate wait time, can enhance student engagement and deepen their understanding (Boaler & Brodie, 2004).

Challenges in Implementing Effective Questioning Techniques

Despite the well-documented benefits of effective questioning, there remains a significant gap in developing targeted intervention strategies within the NC(V) mathematics context. The effectiveness of mathematics instruction at TVET colleges is often hampered by the reliance on traditional teaching methods and textbooks (Adelabu & Pharamela, 2024). These authors further argue that the reliance highlights the urgent need for innovative teaching strategies that connect mathematical concepts to practical applications and enhance students' understanding and retention of mathematical knowledge. Teachers often fail to expose students to diverse representations

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of mathematical concepts, limiting engagement and connecting learning to real-life applications crucial for vocational training (Vimbelo & Bayaga, 2023; Sehole et al., 2023).

Gaps in Questioning Practices in NC(V) Mathematics

The study by Ngoveni and Machaba (2024) critically evaluates the effectiveness of questioning techniques used by mathematics lecturers at South African TVET colleges. Their research reveals a predominance of lower-order questions, significantly limiting student engagement and understanding. It advocates for a shift toward higher-order questioning that fosters critical thinking and deeper learning, aligned with Bloom's Taxonomy and Vygotsky's Social Constructivism. This shift is essential for developing the cognitive abilities necessary for students to thrive in complex vocational environments.

The Need for Targeted Questioning Interventions

This gap highlights the critical necessity for more rigorous research that examines and formulates specific interventions to refine questioning practices. Such efforts are essential for developing educational strategies tailored to meet the distinct demands of NC(V) mathematics education. Undertaking this research will pave the way for more dynamic and effective educational methodologies, ensuring that questioning techniques substantially enhance student learning and engagement (Dahal, 2022; Mahmud & Mohd Drus, 2023; Mauigoa-Tekene, 2023). This study aims to develop a customized intervention for NC(V) Level 2 students who frequently face difficulties with both the conceptual and procedural aspects of mathematics. The intervention is designed to address the specific challenges these students experience and to enhance their mathematical comprehension and problem-solving abilities.

The Role of Consistent Questioning in Student Engagement

Effective questioning not only assesses knowledge but also fosters an environment of critical thinking and knowledge construction. Educators can enhance learning by defining clear objectives and systematically collecting data to monitor progress. This structured approach, as described by Cowie and Bell (1999), involves careful preparation and execution of tasks aimed at measuring intended outcomes.

Consistent questioning is crucial for engaging students, fostering dialogic interactions, and promoting active participation. Specific questioning patterns have been shown to effectively engage students in dialogic talk by encouraging early contributions, vital for developing skills necessary for the modern workforce (Hartmeyer et al., 2018). Moreover, questioning in contexts such as

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mathematics education can foster deeper reflection and understanding, emphasizing the versatility and importance of effective questioning across different fields (Di Teodoro et al., 2011). Another component with the potential to support questioning techniques while enhancing learning is digital learning.

Mobile Learning as a Support for Questioning Techniques

Integrating mobile technology in mathematics education transforms active learning by providing students immediate access to real-time information and resources, fostering a more interactive and engaging learning environment. Parry et al. (2021) highlight that mobile technology enables students to interact dynamically with content, while Pollara and Broussard (2013) emphasize its role in promoting active participation in learning. In financial mathematics, Nuphanudin et al. (2023) argue that mobile devices facilitate a hands-on approach, allowing students to apply financial concepts to real-world scenarios, thereby deepening their conceptual understanding. Similarly, Mueangpud et al. (2019) found that mobile learning applications incorporating interactive, scenario-based tasks significantly enhance students' financial literacy and decision-making skills.

Beyond enhancing engagement, mobile technology is crucial in bridging the gap between theoretical knowledge and practical application. Zhou (2023) notes that mobile devices allow students to research financial topics such as interest rates and inflation in real-time, making abstract concepts more tangible and relevant. Munoto et al. (2021) further assert that accessibility to digital tools supports knowledge retention by enabling students to apply mathematical concepts in diverse contexts. Likewise, Bónus and Korom (2022) emphasize that mobile technology encourages students to explore financial concepts through everyday experiences, reinforcing understanding and critical thinking.

Despite limited access to digital resources in some educational settings, studies indicate that technology-supported instruction can improve student performance (Mendoza-Rodriguez & Caranqui-Sánchez, 2024). Integrating mobile learning tools within structured questioning techniques can enhance student engagement, encourage deeper reasoning, and improve problem-solving skills in financial mathematics. These insights support the need for innovative teaching strategies incorporating mobile technology to facilitate meaningful, inquiry-driven learning in vocational mathematics education.

METHODS

This section discusses the research design, sample selection, data collection methods, intervention process, data analysis techniques, and ethical considerations undertaken in the study.

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Research Design

This study employed a single-case qualitative research design with a quantitative component to develop and propose an interactive framework for questioning techniques to enhance student engagement in the National Certificate (Vocational) Mathematics curriculum at Technical and Vocational Education and Training (TVET) colleges. The intervention was conducted over a three-week period with one Level 2 mathematics class, while a comparison group followed traditional instruction without the intervention. The study was situated in a real classroom environment, allowing for naturalistic observations of student engagement and performance.

Sample Selection

The study comprised seventeen students in the intervention class and thirteen in the comparison group, both from the same TVET college, ensuring curriculum and instructional consistency. The sample was selected to preserve an authentic classroom environment while maintaining a manageable class size. The intervention class, chosen for its higher enrollment, allowed for a more extensive range of engagement levels. The researcher delivered the intervention while a regular lecturer instructed the comparison group.

The comparison group was included to illustrate differences in performance between students exposed to structured questioning techniques and those following conventional instruction. The comparison was not intended for rigorous statistical performance analysis but rather for illustrative purposes, providing contextual insights into how questioning techniques may influence student participation, which may ultimately improve comprehension. Since pre-existing classroom structures were used, randomization was not applied. However, both groups had equal curriculum exposure and instructional time. The comparison group adhered to conventional teaching methods, whereas the intervention group engaged with structured questioning techniques to foster deeper engagement and conceptual understanding. This approach, widely used in pedagogical research, offers meaningful contextual insights (Creswell & Poth, 2018; Yin, 2014; Merriam & Tisdell, 2016).

Data Collection Instruments and Procedures

Data collection involved qualitative classroom observations, student assessments, and thematic coding of student interactions to ensure methodological rigor (Braun & Clarke, 2022). Observations were systematically conducted to capture student engagement with various questioning techniques and inform instructional adjustments in real-time. Field notes documented student participation, misconceptions, and engagement patterns, focusing on questioning activities and general discussions to identify emerging trends.

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A preparatory examination assessed students' understanding of financial mathematics through a 20-mark question, divided into two sections: a 10-mark component requiring the definition of five key financial concepts, and a 10-mark section involving simple and compound interest calculations. The results were analyzed to compare the effectiveness of structured questioning techniques against conventional instruction, offering insights into the possible relationship between student engagement and learning outcomes.

Intervention Process

The intervention was structured into five key phases to ensure a systematic implementation of questioning strategies and an evaluation of their impact on student engagement and understanding.

1. *Planning the Intervention*: The intervention was designed to enhance student engagement using structured and open-ended questioning techniques grounded in real-world financial scenarios. Mobile learning tools and multimedia resources were integrated to provide practical, interactive experiences.
2. *Continuous Monitoring Through Questioning*: Structured and spontaneous questions were used to assess procedural and conceptual understanding in real-time. Immediate feedback allowed for instructional adjustments, fostering deeper engagement and active participation.
3. *Implementation of Open-Ended and Wait-Time Strategies*: Open-ended questions encouraged students to reflect on financial concepts and apply them in personal finance decisions. Wait-time strategies enhanced critical thinking by allowing students to formulate well-reasoned responses.
4. *Group Activities and Collaboration*: Students worked in groups to solve practical financial problems, such as loan repayment calculations, reinforcing their understanding through peer discussions. Real-time feedback during group presentations helped clarify misconceptions and strengthened learning.
5. *Evaluation and Reflection*: The intervention's effectiveness in enhancing interactive engagement was assessed through student participation and performance trends. Results confirmed that structured questioning fostered deeper conceptual understanding and active learning in financial mathematics.

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Data Analysis Methods

Following Braun and Clarke's (2006) framework, thematic analysis was applied to field notes to systematically identify engagement patterns, student misconceptions, and the effectiveness of questioning techniques. Responses were analyzed to distinguish between procedural knowledge (calculation-based responses) and conceptual understanding (explanation-based responses). To ensure reliability, a panel of experts independently reviewed classroom observation records to maintain consistency in engagement classifications and reduce researcher bias (Miles et al., 2014; Lincoln & Guba, 1985).

A comparative analysis of preparatory examination results was conducted to assess differences in performance between Group A (intervention) and Group B (reference group). The financial mathematics question was used as the primary measure for comparison. Given the small sample size, the focus was on identifying performance trends through descriptive comparative analysis rather than applying inferential statistics. This approach aligns with established practices in small-sample educational research (Cohen et al., 2018).

Ethical Considerations

Ethical approval for this study was obtained from the relevant institutional review committee before implementation, following ethical guidelines for educational research (Cohen et al., 2018). Informed consent was collected from all participating students, ensuring they were fully aware of the study's purpose and voluntary participation. Confidentiality and anonymity were maintained, and all data was securely handled to protect student identities.

FINDINGS

Step 1: Planning the Intervention

Objective: The primary aim of this phase was to establish a strong foundation for using effective questioning strategies to continuously monitor and enhance student understanding of "Financial Mathematics," a key element of the NC(V) Level 2 curriculum. The intervention aimed to connect theoretical financial concepts to students' everyday lives by applying real-world financial examples, such as Mashonisa (informal money lending).

Selection of Topic: The researcher selected "Financial Mathematics" because it covered essential concepts such as budgeting and interest calculations, which are crucial for personal financial management. This topic aligned directly with the NC(V) Level 2 requirements and supported the broader educational goals of preparing students for real-world financial challenges. Additionally, the topic allowed the researcher to address economic challenges by exploring informal lending

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systems like Mashonisa and savings schemes like Stokvel to foster deeper engagement with real-world financial issues.

Development of a Questioning Framework: The questioning framework was designed to include structured, consistent, and open-ended questions. The structured questions were posed regularly throughout the lessons to track how well students understood the key concepts. On the other hand, the open-ended questions were intended to encourage students to think more deeply about the practical applications of financial mathematics. This included budgeting, interest calculations, and comparisons between formal and informal financial institutions like banks and Mashonisa.

Integration of Technology and Resources: To support the teaching material, the researcher used online videos to visually explain concepts such as simple and compound interest. In addition, students were encouraged to use their mobile phones to research, for example, current interest rates, inflation, and budgeting in real-time, making the learning experience more interactive and grounded in practical applications.

Resource Readiness: All resources, including videos and online tools, were pre-checked by the researcher to ensure that they functioned properly, allowing class time to focus on learning rather than dealing with technical issues. The digital tools used during the intervention were limited to the researcher's laptop and the students' mobile phones.

Incorporation of Real-Life Financial Examples: To ensure relevance, the researcher incorporated specific real-world examples, such as the following:

1. "Pinkie buys a laptop priced at R13,495. She makes a 20% deposit and finances the balance at 15% per annum simple interest. Calculate her total cost and monthly payment."
2. "Eric bought a pre-owned car for R50,000 with a five-year hire-purchase agreement at 12% per annum simple interest and an additional R350 per month for insurance. Calculate Eric's total monthly payment."
3. "Advise a friend who is new to investing about the differences between simple and compound interest. Which would you recommend for long-term financial growth, and why?"
4. "Create a detailed monthly budget based on your actual income and expenses, including your monthly allowance from parents and any bursary funds you receive."

These examples were designed to highlight the practical implications of financial decisions and prepare students for similar situations they might encounter in their personal lives.

Step 2: Implementation of Indicator 1 - Continuous Monitoring Through Questioning

Overview: This phase focused on ongoing monitoring of student progress using both structured and unstructured questioning techniques. The integration of real-world financial problems, such as informal lending and the effects of interest rates, allowed the researcher to dynamically assess student understanding.

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Classroom Engagement: The structured and unstructured questions assessed students' understanding of procedural and conceptual aspects of financial mathematics. For instance, students were asked to calculate Pinkie's monthly payment based on her 20% deposit and the 15% simple interest rate on her laptop purchase (see example below).

As part of the classroom discussion, they were also asked why Pinkie might have opted for simple interest financing and what alternative options could have been available to her. The researcher also engaged students with a question about compound interest, asking them to investigate where compound interest is applied and how it differs from simple interest in terms of long-term costs.

Additionally, questions about the Mashonisa concept were introduced, prompting students to explore why some people rely on informal lenders and the risks and benefits of such practices compared to formal institutions like banks.

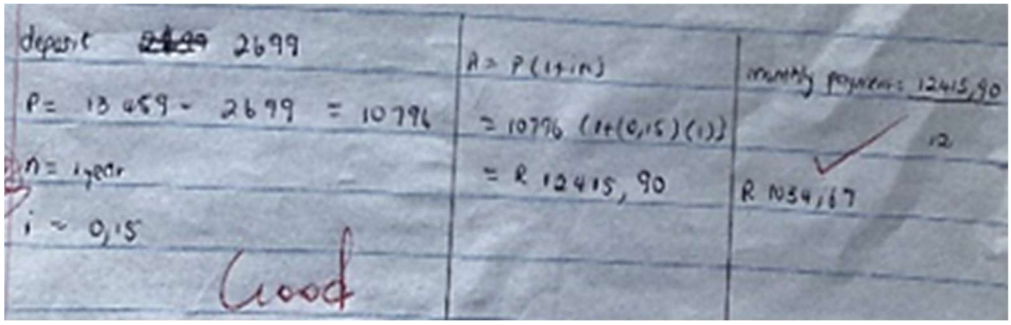


Figure 1: Student's work Example 1

Diverse Question Formats: Both verbal and written formats were used for questioning to ensure students engaged with the material in various ways. This approach encouraged students to reflect on their responses during class discussions and individual assessments.

Feedback Mechanism: The researcher utilized students' responses to provide real-time feedback, identifying areas where they struggled or excelled. This allowed for prompt intervention to address gaps in understanding, ensuring that the learning objectives were met.

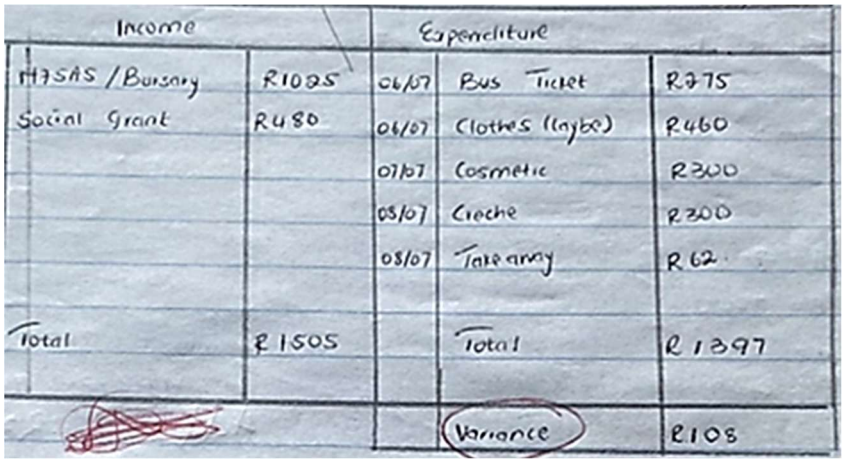
Step 3: Implementation of Indicator 2 - Use of Open-Ended and Wait-Time Strategies

Objective: This phase was designed to foster deeper student engagement through strategic questioning and personalized financial scenarios. Students were encouraged to apply financial concepts to real-life situations, such as managing a personal budget or deciding between loan options.

Design of Personalized Questions: The researcher asked students to create detailed monthly budgets based on their income and expenses (Example below). This exercise served as a foundation for understanding how to manage finances effectively. Furthermore, the activity introduced

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students to challenging financial concepts, such as variance, which refers to the difference between projected and actual outcomes in budgeting.



Income		Expenditure		
NSFAS / Bursary	R1025	06/07	Bus Ticket	R775
Social Grant	R480	06/07	Clothes (layby)	R460
		07/07	Cosmetic	R300
		08/07	Creche	R300
		08/07	Take away	R62
Total	R1505		Total	R1397
			Variance	R108

Figure 2: Student's work Example 2

Open-Ended Questions: To deepen understanding of simple interest, students were asked to investigate where simple interest is applied, why it is referred to as "simple," and what makes it different from compound interest. Similarly, they were tasked with explaining why compound interest is called "compound," where it is used, and how it impacts long-term financial growth. These questions prompted students to consider the practical implications of these types of interests in real-world financial contexts.

The researcher observed that students had expenses surpassing their income, leading to a negative balance. This allowed them to discuss variance, helping them understand that it could be positive or negative. Additionally, the occurrence of negative balances served as a teaching moment to guide students in practicing responsible spending.

Wait-Time and Reflection: The researcher incorporated sufficient wait-time after each question to encourage students to reflect on their answers and develop thoughtful responses. This approach promoted critical thinking and allowed students to engage with the material fully. For example, the researcher observed that when students were asked to explain their budgeting decisions, many initially struggled to justify their expenditure exceeding income. However, with extended wait-time they began recognizing the implications of negative balances. Additionally, when prompted to reflect on real-life financial scenarios, students gradually articulated the importance of responsible spending, linking their understanding of variance to personal financial management.

Peer Discussions: Throughout the intervention, students participated in peer discussions, sharing insights and strategies with their classmates, particularly during open-ended questioning sessions. These discussions allowed students to collaboratively reflect on financial concepts, helping to solidify their understanding through group learning and shared problem-solving approach-

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es. The researcher observed that during peer discussions, students actively challenged each other's financial decisions, particularly when debating the impact of deposit size on total interest payments, which deepened their understanding of loan repayment structures. Additionally, when comparing simple and compound interest, students engaged in collaborative problem-solving, with some referencing real-life family investments, leading to a more practical grasp of long-term financial growth strategies.

Step 4: Group Activities and Collaboration

Step 4 formalized the collaborative approach by focusing on structured group activities. In this phase, students were assigned complex financial problems that required them to work together, applying procedural and conceptual knowledge to real-world scenarios. Group work was designed to enhance deeper understanding through collective problem-solving and discussion.

Group-Based Problem Solving: Students worked in groups to solve practical financial problems, such as calculating the monthly payment for Eric's car purchase (Example 2 in Step 1). Each group calculated Eric's monthly payment, including the simple interest and monthly insurance, and discussed how the interest rate affected the overall cost. Additionally, they were asked to consider whether Eric's payment would have been significantly different if compound interest had been used instead of simple interest. This comparison helped students understand the long-term implications of different interest types.

It is important to mention that not all groups arrived at the correct answer, as demonstrated in the example below. The students received immediate feedback, including the group in the example. This group only calculated the total amount payable after five years with interest and stopped there. The real-time feedback made them aware that they still needed to divide the total amount by the number of months and include the monthly insurance payment.

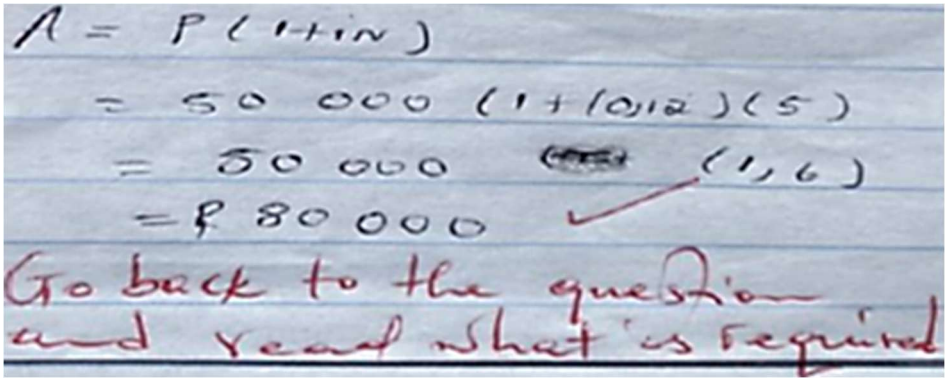


Figure 3: Student's work Example 3

In another example, students were tasked with advising a friend new to investing about the differences between simple and compound interest. The groups were required to research real-

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world applications of both types of interest and recommend which option would be more beneficial for long-term financial growth. They also explored how the frequency of compounding (annually, monthly, or daily) would affect the total amount of interest earned or paid in the long term.

Presentation and Feedback: Each group presented its findings on the whiteboard, explaining their calculations, decisions, and recommendations. The researcher provided constructive feedback, reinforcing key financial concepts and ensuring students understood the theoretical and practical aspects of interest rates.

Researcher's Observations on Group Activities and Collaboration

The researcher observed that group-based problem-solving activities encouraged students to actively engage with financial concepts, fostering peer-led learning and deeper conceptual understanding. During discussions, for example, on Eric's car purchase, students debated the impact of different interest rates on the total cost, with some initially struggling to distinguish between simple and compound interest. As they worked through the calculations, the real-time feedback process proved instrumental in correcting misconceptions, particularly for groups that overlooked dividing the total amount by the number of months or factoring in the monthly insurance cost.

Additionally, when tasked with advising a friend on investment strategies, students demonstrated varying levels of financial literacy, with some confidently explaining compounding effects. In contrast, others relied on their peers for clarification. The researcher noted that the requirement to present findings on the whiteboard prompted students to articulate their reasoning more clearly, reinforcing their understanding through peer critique and instructor feedback. Some groups conducted further research beyond the classroom discussion, highlighting the practical relevance of financial literacy in real-world decision-making.

Step 5: Evaluation and Reflection

Objective: The final phase evaluated the impact of the intervention on student learning. The focus was on assessing how well students understood financial mathematics through practical application and strategic questioning.

Evaluation Methods: The initial evaluation of this study included real-time observations, where the researcher observed students during classroom learning and adjusted the teaching approach based on their feedback. The second evaluation phase focused on analyzing student performance in the Financial Mathematics chapter covered during the intervention. This analysis compared the results of Group A (the class that received the intervention) and Group B (the class taught by the regular Lecturer), as shown in Table 1. Financial Mathematics accounted for 20 marks out of 100 in Paper 1 during the preparatory and final exams. Table 1 presents the students' perfor-

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mance in the preparatory exam, categorizing them into Group A and Group B. For example, Student 1a from Group A scored 13 out of 20, while Student 1b from Group B scored 7 out of 20.

Group A		**Group B**	
Student 1a: 13	Student 2a: 14	Student 1b: 7	Student 2b: 4
Student 3a: 16	Student 4a: 10	Student 3b: 12	Student 4b: 10
Student 5a: 8	Student 6a: 8	Student 5b: 10	Student 6b: 13
Student 7a: 14	Student 8a: 18	Student 7b: 14	Student 8b: 6
Student 9a: 12	Student 10a: 8	Student 9b: 7	Student 10b: 5
Student 11a: 6	Student 12a: 9	Student 11b: 4	Student 12b: 9
Student 13a: 8	Student 14a: 20	Student 13b: 6	
Student 15a: 10	Student 16a: 9		
Student 17a: 13			
Average (n = 17) = **11.5**		**Average** (n = 13) = **8.2**	

Table 1: Group A versus Group B in 2022

The evaluation of student performance in financial mathematics revealed a significant difference between Group A, which received an intervention, and Group B, which did not. Group A achieved an average score of 11.5 out of 20, compared to Group B's 8.2, suggesting that there could be a link between the intervention and learning outcomes. The wider range of scores in Group A (6 to 20) suggests that while many students benefited from the intervention, some struggled, showing that the approach was not uniformly effective.

The intervention included consistent questioning, personalized tasks, and group activities, all contributing to enhanced understanding and engagement. Real-time feedback was critical in helping students correct mistakes and deepen their learning. However, the lower performers in both groups indicate a need for further differentiation in teaching methods. The role of digital tools was supportive during the intervention and will be discussed next as one of the emerging themes.

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Mobile Learning and Digital Tools as a Means of Supporting Interactive Engagement

Digital tools played a complementary role in enhancing student engagement by bridging theoretical financial concepts with real-world applications. Students used mobile devices to research current interest rates and inflation trends, making financial mathematics more relevant to their daily lives. This real-time research allowed them to compare financial products, evaluate different loan repayment structures, and develop informed financial strategies, reinforcing their conceptual understanding.

To visually enhance learning, online videos were incorporated to illustrate complex financial concepts, such as simple and compound interest. These videos provided step-by-step explanations supporting procedural and conceptual knowledge, reinforcing students' ability to analyze financial decisions critically. The combination of visual learning and structured questioning encouraged deeper engagement and allowed students to reflect on financial principles beyond the classroom context.

Additionally, mobile learning supported peer discussions, where students collaboratively analyzed financial scenarios, compared their calculations, and debated financial choices. These interactions helped clarify misconceptions and reinforced knowledge through collaborative learning. Mobile learning complemented the questioning framework, allowing students to actively engage with real-world data and digital resources, fostering a more interactive and student-centered learning environment.

DISCUSSION

This study sought to explore the application of questioning techniques in enhancing interactive engagement in teaching financial mathematics for NC(V) Level 2 students. Rooted in formative assessment theories, the primary objective was to design a model that promotes deeper conceptual understanding and procedural fluency rather than simply improving performance. The findings, centered on two main indicators, 1) Engaging Students Through Consistent Questioning and 2) Questioning Techniques to Gather Evidence of Procedural and Conceptual Knowledge, offer significant insights into how these strategies foster a more dynamic and participatory learning environment.

Indicator 1: Engaging Students Through Consistent Questioning

The findings show that consistent questioning was crucial in maintaining student engagement and fostering a dialogic learning environment. Embedding questioning into daily lessons consistently challenged students to apply, reflect on, and extend their understanding of financial concepts. Black and Wiliam (1998) emphasize that consistent questioning is a vital aspect of formative assessment, offering opportunities for students to engage actively in their learning while allowing teachers to adjust instruction based on real-time feedback.

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The approach to questioning, which was grounded in real-world financial examples such as budgeting and hire-purchase agreements, aligns with Adelabu and Pharamela (2024), who argue that formative assessment should be integrated into the learning process rather than treated as a separate activity. By continually asking students to solve financial problems related to their daily lives, such as calculating loan repayments or advising on investment strategies, the questions made abstract mathematical concepts more accessible. Venkatraman et al. (2018) support this by stating that students are more likely to engage with and retain mathematical concepts when presented in practical, relatable contexts. In this study, examples like Pinkie's laptop purchase and Eric's car loan provided students with concrete frameworks to explore financial mathematics.

However, while consistent questioning effectively engaged most students, the preparatory examination results revealed that some students in Group A still performed poorly. This underscores that consistent questioning alone may not guarantee improved academic outcomes for all learners, particularly those with weaker foundational knowledge. Ngoveni and Machaba (2024) suggest that while questioning is an effective formative assessment tool, it must be coupled with instructional scaffolding for students with lower comprehension levels. Vygotsky's (1978) zone of proximal development (ZPD) theory reinforces this, suggesting that tasks or questions should challenge students within their ZPD to ensure growth without causing disengagement. Similarly, Vimbelo and Bayaga (2023) highlight the need for differentiated questioning strategies to address diverse learning needs. Future interventions should integrate scaffolding and differentiated questioning approaches to support student learning across varying ability levels.

Indicator 2: Questioning Techniques to Gather Evidence of Procedural and Conceptual Knowledge

In addition to promoting engagement, the findings show that questioning techniques effectively gathered evidence of both procedural and conceptual knowledge. Open-ended and reflective questions allowed students to articulate their understanding, apply concepts in new contexts, and engage in higher-order thinking. This aligns with Bloom's Revised Taxonomy (Anderson & Krathwohl, 2001), emphasizing the need for questions that push beyond recall, promoting analysis, evaluation, and creation.

For example, the question "Investigate where compound interest is applied and explain why it's called compound" encouraged students to perform calculations and explain the rationale behind financial mechanisms. This is consistent with Martino and Maher (1999), who advocate for questioning that promotes justification and generalization, helping students move beyond procedural fluency to develop a deeper conceptual understanding. Similarly, questions like "Advise a friend who is new to investing about the differences between simple and compound interest" required students to apply mathematical knowledge to real-life scenarios, reinforcing Hiebert and Carpenter's (1992) assertion that transferring knowledge to practical problems enhances mathematical understanding.

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While many students in Group A exhibited both procedural and conceptual mastery, some encountered difficulties with higher-order questions. To address these challenges, Ngoveni (2025) recommends pedagogical strategies that simultaneously enhance conceptual understanding and procedural fluency to strengthen mathematical proficiency in vocational education. Similarly, Ramaligela (2022) attributes performance disparities to students' struggles with higher-order thinking, often influenced by their prior knowledge and cognitive development. Mauigoa-Tekene (2023) underscores the importance of aligning questioning techniques with students' cognitive abilities to facilitate meaningful engagement. Therefore, additional scaffolding and targeted support may be required to ensure that lower-performing students fully benefit from the questioning framework.

Group A vs Group B

Although the study's primary aim was to design a model for interactive engagement, the improved performance of Group A compared to Group B suggests that questioning techniques may positively impact academic outcomes. Group A's average score of 11.5 out of 20, compared to Group B's 8.2, highlights that the questioning model, while focused on engagement, may have positively affected learning outcomes. Mueangpud et al. (2019) argue that formative assessment techniques, such as effective questioning, may lead to measurable improvements in student outcomes, even when focused on engagement rather than performance. This reinforces the value of questioning as both an engagement and assessment tool.

Mobile Learning and Digital Tools in Supporting Interactive Engagement

Although the study primarily focused on questioning techniques, the role of mobile technology in supporting interactive engagement was also noteworthy. Mobile devices allowed students to access real-time financial data, such as interest rates and inflation, providing context for the mathematical concepts taught. Zhou (2023) notes that mobile technology enhances learning by providing students immediate access to relevant information, making the learning experience more dynamic and interactive. Allowing students to research financial data independently supported the questioning model by bridging theoretical knowledge and practical application. This aligns with Nuphanudin et al. (2023), who emphasize that mobile learning encourages greater autonomy and engagement by enabling students to explore and apply concepts in real-world contexts.

Additionally, online videos were incorporated to visually illustrate complex financial concepts, such as simple and compound interest, through step-by-step explanations. Research by Dipon & Dio (2024) indicates that video-based instruction significantly improves students' performance in mathematics, with an effect size of g=1.79. This suggests that instructional videos enhance the understanding of abstract concepts and foster critical analytical skills through real-world applications. In this study, visual learning and structured questioning deepened engagement, enabling students to reflect on financial principles beyond the classroom.

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CONCLUSION

The findings of this study emphasize the effectiveness of a questioning model designed to enhance interactive engagement in NC(V) Level 2 financial mathematics classrooms. The intervention fostered a more dynamic and participatory learning environment by grounding the questions in real-world financial contexts and promoting procedural and conceptual understanding. While improved performance was a secondary outcome, it highlights the potential for questioning techniques to enhance engagement and academic success.

This study contributes a practical, evidence-based model for questioning to enhance engagement and understanding in vocational education. It extends the theoretical framework of formative assessment and questioning techniques by situating them within real-world, context-driven financial mathematics supported by digital tools. The study's findings offer immediate and scalable implications for improving interactive learning in vocational classrooms, with broader relevance for other disciplines and settings within technical education. This study expands the body of knowledge and addresses the existing gap in the literature within the TVET sector.

LIMITATIONS

This study has several limitations that may impact the interpretation and generalizability of its findings. The small sample size (n=17) and the focus on NC(V) Level 2 financial mathematics restrict broader applicability, necessitating further research across diverse student cohorts. Additionally, the three-week intervention may not have been sufficient to capture long-term learning gains, underscoring the need for extended studies. The absence of direct student feedback limited insights into engagement and perceptions. At the same time, the researcher's dual role as both educator and investigator introduced potential bias, which independent facilitators in future research could mitigate. Although many students actively participated, some required additional prompting, suggesting the need for structured scaffolding to enhance early engagement. Technological constraints also led to inconsistencies in digital engagement due to limited access to mobile learning tools. Furthermore, despite the structured questioning techniques, some students struggled with fundamental financial concepts, highlighting the need for additional instructional support.

RECOMMENDATIONS FOR FUTURE STUDIES

Future studies should incorporate qualitative feedback through surveys, interviews, or focus groups to better understand how questioning strategies influence student motivation, comprehension, and engagement. Expanding the application of the questioning model to other vocational subjects, such as science and technical trades, would provide insights into its broader effectiveness. Additionally, research involving larger and more diverse samples across multiple class-

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rooms or institutions would enhance generalizability and reveal variations in student responses across different educational settings. A longer-term study is also recommended to assess the sustained impact of the model, as extending the intervention beyond three weeks could provide valuable data on long-term engagement and retention. Furthermore, future research should explore the deeper integration of digital tools to support interactive engagement, enable real-time feedback, and facilitate adaptive learning in formative assessment practices.

ACKNOWLEDGMENTS

I thank Dr. du Plooy and Prof. Machaba for their contribution while conceptualizing the primary study and their roles as a team of experts.

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MATHEMATICS TEACHING RESEARCH JOURNAL
SUMMER 2025
Vol 17 no 3

The Problem Corner



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The purpose of ***The Problem Corner*** is to give students and instructors a chance to step out of their working "comfort zone" and solve challenging problems, whether working independently or together. Rather than solutions alone, as educators we are interested in methods, strategies, and original ideas that uncover paths to solutions. To that end, we encourage our readers to propose and solve new problems. **To submit a solution**, type it in Microsoft Word, using math type or the equation editor (PDFs are also acceptable), and email as an attachment to The Problem Corner Editor at iretamoso@bmcc.cuny.edu. Be sure to include your name, institutional affiliation, city, state, and country. Solutions must include a detailed explanation. The best solution will be published in a future issue of MTRJ, and correct solutions will be acknowledged. **To propose a problem**, follow the same formatting and submission guidelines, and include a detailed explanation of a solution.

Greetings, problem solvers!

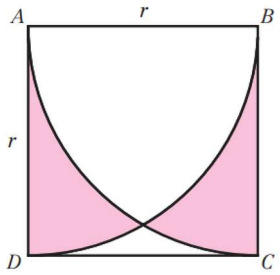
We are excited to announce that Problems 38 and 39 in *The Problem Corner* have received outstanding solutions, distinguished by their precision and creativity. Showcasing these exemplary approaches serves to inspire others and encourages a greater appreciation for mathematical thinking across the globe.

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Problem 38: "Wine Glass" Problem

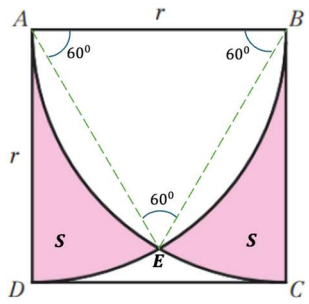
Proposed by Ivan Retamoso, BMCC, USA

The picture below shows a square $ABCD$ with each side of length $r = \sqrt{2}$ inches. Two arcs of circles are drawn with centers at points A and B. What is the area of the shaded part?



Solution 1. Submitted By Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University

In this first solution, the solver constructs an equilateral triangle, which helps create circular segments whose areas are calculated. These segments, along with 30-degree sectors and the use of symmetry, are combined to solve the problem. The solution is concise and well explained.



First, we can begin our solution by noting that the areas shaded in pink are equal. If we denote each of these areas as S, then we will need to find $2S$. Let E be the intersection point of arcs AC

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and BD, and denote the large white area above as A_1. As seen in the figure, when we draw the necessary auxiliary lines, an equilateral triangle will be formed.

$Areaofcircularsegment AE = Areaofsector ABE - A(\triangle ABE)$

$$Areaofcircularsegment AE = \frac{\pi r^2}{6} - \frac{\sqrt{3}r^2}{4}$$

$$A_1 = A(\triangle ABE) + 2(Areaofcircularsegment)$$

$$A_1 = \frac{r^2\sqrt{3}}{4} + 2\left(\frac{\pi r^2}{6} - \frac{\sqrt{3}r^2}{4}\right) = \frac{\pi r^2}{3} - \frac{\sqrt{3}r^2}{4}$$

The area of the quarter circle centered at A or B is $A_1 + S = \frac{\pi r^2}{4}$

$$S = \frac{\pi r^2}{4} - \left(\frac{\pi r^2}{3} - \frac{\sqrt{3}r^2}{4}\right) = \frac{r^2}{12}(3\sqrt{3} - \pi)$$

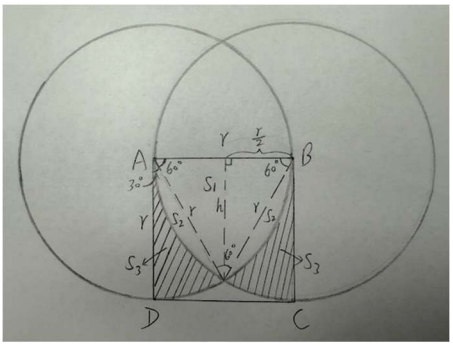
$2S = \frac{r^2}{6}(3\sqrt{3} - \pi)$ where $r = \sqrt{2}$

Hence, the area of the shaded part is $\frac{1}{3}(3\sqrt{3} - \pi)\text{in}^2$.

Solution 2. Submitted by Stephen Chen, Borough of Manhattan CC, Fuzhou, China.

In this second solution, the solver carefully explains each step, labeling all regions in the diagram, which involves two circles and an equilateral triangle. Using proportional reasoning, he determines the area of a 30-degree sector and ultimately finds a method to calculate the shaded region, which represents the final solution

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The area of the quarter of the circles shown above is $\frac{\pi r^2}{4}$

Then $S_1 + 2S_2 + S_3 = \frac{\pi}{4}r^2$……..(*)

$S_1 + S_3 = \frac{30°}{60°}\pi r^2 = \frac{\pi}{12}r^2$

$S_2 = \frac{\pi}{12}r^2 - S_3$

$h^2 + \left(\frac{r}{2}\right)^2 = r^2$ then $h = \frac{\sqrt{3}}{2}r$

$S_1 = \frac{1}{2}rh = \frac{1}{2}r\frac{\sqrt{3}}{2}r = \frac{\sqrt{3}}{4}r^2$

Substituting in (*)

$\frac{\sqrt{3}}{4}r^2 + 2\left(\frac{\pi}{12}r^2 - S_3\right) + S_3 = \frac{\pi}{4}r^2$

$\frac{\sqrt{3}}{4}r^2 + \frac{\pi}{6}r^2 - S_3 = \frac{\pi}{4}r^2$

Since $r = \sqrt{2}$

Then $S_3 = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$ and $2S_3 = \sqrt{3} - \frac{\pi}{3}$

Therefore, the area of the shaded part is $\sqrt{3} - \frac{\pi}{3}$ in^2

Problem 39: "Three Points" Problem

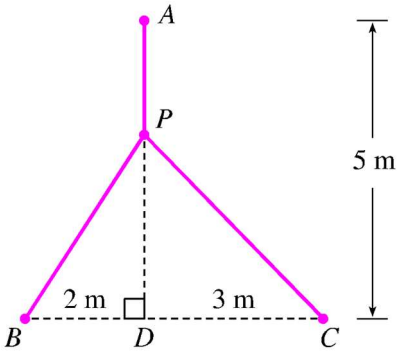
Proposed by Ivan Retamoso, BMCC, USA

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A point P must be placed somewhere along the line segment AD such that the total length L of cables connecting point P to points A, B, and C is minimized (refer to the figure).

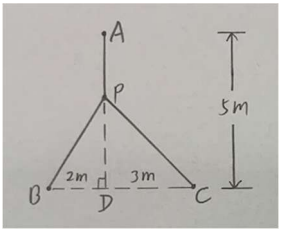
Task: Estimate the minimum value of L to *two decimal places*.

Note: Since this is an estimation problem, you are encouraged to use a graphing calculator or graphing software to approximate the solution of the equation needed to minimize L.



Solution 1. Submitted by Stephen Chen, Borough of Manhattan CC, Fuzhou, China

Our solver expresses the total length in terms of a single variable, forming a length function L that is then differentiated and minimized using the first derivative test. The critical point was identified with the aid of the DESMOS graphing calculator.



$L = PA + PB + PC$

Known $CD = 3$, $BD = 2$, $PA + PD = 5$, set $PC = x$

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$PD = \sqrt{x^2 - 9}$

$PA = 5 - PD = 5 - \sqrt{x^2 - 9}$

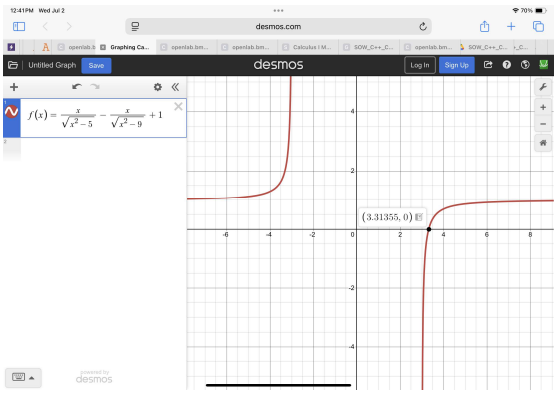
$(PB)^2 = (PD)^2 + (BD)^2 = x^2 - 9 + 2^2 = x^2 - 5$

$PB = \sqrt{x^2 - 5}$

Therefore, $L = PA + PB + PC = 5 - \sqrt{x^2 - 9} + \sqrt{x^2 - 5} + x$

Taking the derivative of L, $L' = 1 + \frac{x}{\sqrt{x^2-5}} - \frac{x}{\sqrt{x^2-9}}$

Using DESMOS graphing calculator, the graph of L' is shown below



From the graph, $L' = 0$ for $x = 3.31m$, $L' < 0$for $x < 3.31$, and $L' > 0$for $x > 3.31$.

The above shows that L achieves its minimum at $x = 3.31m$.

Therefore, the minimum value of L is given by

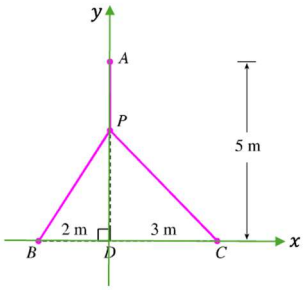
$L = (3.31) + 5 + \sqrt{(3.31)^2 - 5} - \sqrt{(3.31)^2 - 9} = 9.35m.$

Solution 2. Submitted by Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University

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This alternative solution uses a cleverly chosen xy coordinate system, chosen without loss of generality, to express the coordinates of all points in the diagram in terms of a single variable. This approach simplifies the computation of all relevant distances and leads to the construction of a distance function, which is then minimized using the derivative from calculus.

The point P will lie on the line segment AD, and the total length of the cables $L = PA + PB + PC$ is desired to be minimized.



I approach the problem geometrically by redefining the coordinate system.

To do this, we treat the line AD as the y-axis and the line BC as the x-axis. If we proceed with this approach and write the coordinates of the given points accordingly, point D can be considered as the origin. That is, $D(0,0), A(0,5), P(0,p), B(-2,0)$and $C(3,0)$. Of course, the value of p will be in the range $0 \leq p \leq 5$.

I define the function L, which is the sum of distances from the point P to points $A, B$and C.

$L(p) = PA + PB + PC = 5 - p + \sqrt{4 + p^2} + \sqrt{9 + p^2}$

To minimize this function, we take its derivative with respect to $p$and set it equal to zero.

$\frac{dL}{dp} = 0$ (solving this analytically is complex, so we use graphical methods to find the value of p that minimizes L) then $L_{min} \approx 9.35$ when $p \approx 1.41$

As a result, I used a combination of coordinate geometry, calculus (derivative to minimize distance), and graphing with Desmos to find and verify the value of $p \approx 1.41$ where $L$is minimized.

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Editor's Note: Problem 39 was also correctly solved by Hridoy Saha from Bangladesh. While his approach was similar to the ones presented, he used Newton's method at the end to solve the final equation and obtain the solution.

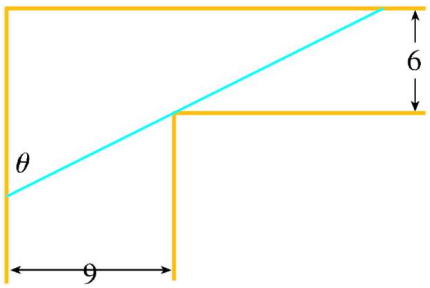
It's great to hear that you enjoyed problems 38 and 39 and that they helped you improve your mathematical skills. Now continue with the next set of questions to keep building your mastery.

Problem 40

Proposed by Ivan Retamoso, BMCC, USA

A steel pipe is being carried horizontally through a hallway that is 9 feet wide and turns at a right angle into a hallway that is 6 feet wide. What is the maximum length of pipe that can make the turn around the corner?

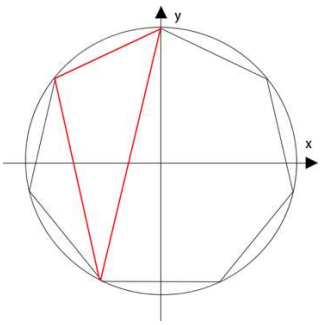
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Problem 41

Proposed by Ivan Retamoso, BMCC, USA

A regular heptagon is inscribed in a unit circle with a radius of 1 cm. Calculate the area of the red triangle.



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