# The Joining of Quaternions with Grassmann algebras: William Kingdon Clifford

*Algebra which cannot be translated into good English and sound common sense, is bad algebra.*
*-William Kingdon Clifford*

*PowerPoint presentation by Johannes Familton PhD*

# Hamilton and Quaternions

- Most of us here know the story of Hamilton and the quote on Quaternion on the plaque on Brougham (Broom) Bridge, Dublin which says:

- “Here as he walked by on the 16th of October 1843
  Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication
  $i^2 = j^2 = k^2 = i\,j\,k = -1$
  & cut it on a stone of this bridge.”

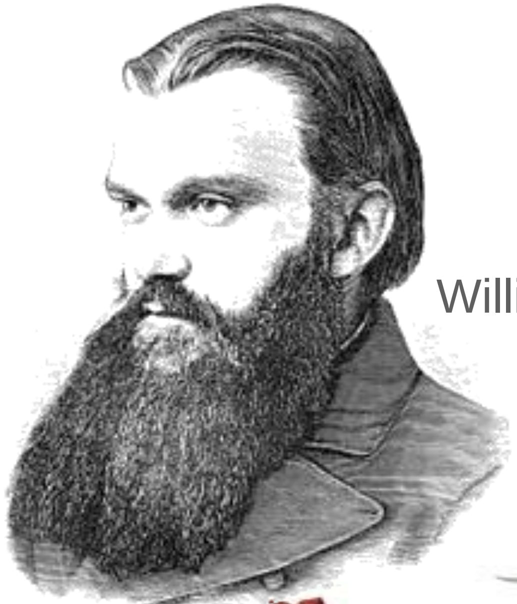
# Hamilton and Quaternions

- This was the legendary beginning of what he called “Quaternions”.
- Which is what brought us all here today:
- to learn from each other the relevance of Quaternions today, more than 150 years after they were discovered.

# Other contributors to this subject

- But some of us may not know much about
- Grassmann (Hermann Günther Grassmann; April 15, 1809 – September 26, 1877)
- Clifford (William Kingdon Clifford FRS; May 4,1845 – March 3, 1879)
- and the history of their contributions to this subject.

Hermann Günther Grassmann



William Kingdon Clifford

# Hermann Günther Grassmann

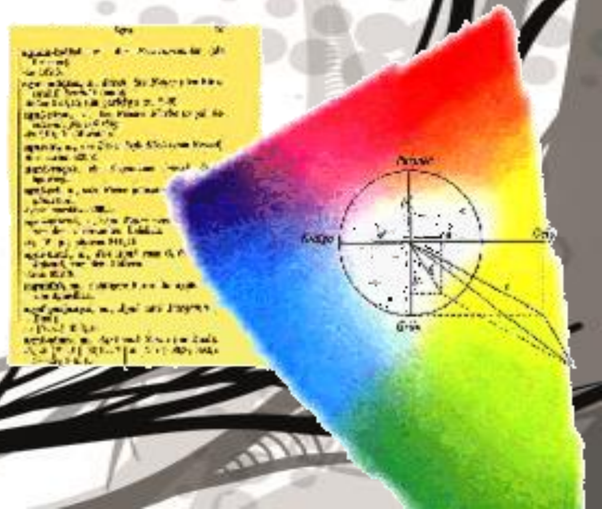
- Grassmann was a person who was truly ahead of his times.
- He anticipated Hamilton in non-commutative multiplication,
- He introduced the idea of n dimensions, which is truly a modern concept.
- Much of Grassmann’s work would conceptually look familiar to us today.

# Grassmann and Outer Product

- Grassmann’s outer products, also known as exterior products or wedge products, came before both vector and tensor analysis.
- They also overlap both these subjects.
- The wedge product is defined to be associative, anticommutative, and distributive over addition.
- Grassmann provided an algebraic setting to answer geometric questions.
- For this reason a more detailed mathematical description of Grassmann’s work needs to be discussed in order to understand what Clifford eventually did.

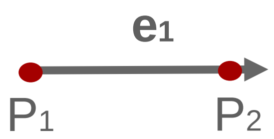
# Grassmann and Outer Products

- Grassmann used lower dimensions as building blocks for higher dimensions.
- He let a line be defined by 2 connected points, a plane by 3 connected points and so on.
- For example, a vector is usually associated with a point P or a line from 0 to P.
- What Grassmann did was discuss situations where the lines start at $P_1$ and end at $P_2$, but didn't necessarily go through 0.
- This idea allowed for greater generality.

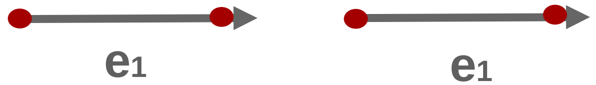


# A Pictorial Discussion of what Grassmann did

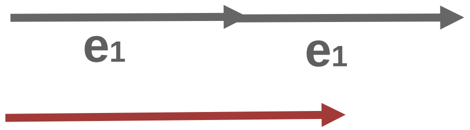
- For 1 dimension: Let $\mathbf{e}_1$ join any 2 points




- $\mathbf{e}_1$ is unchanged by parallel displacement of the 2 points.

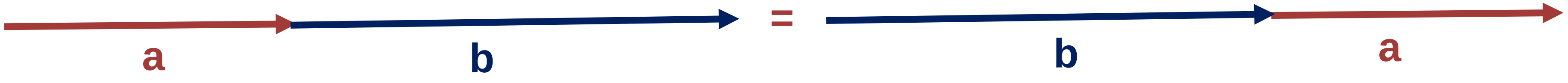



- Any vector $\mathbf{a}$ is some multiple of $\mathbf{e}_1$
- For example:




$\mathbf{a} = \frac{3}{2}\,\mathbf{e}_1$

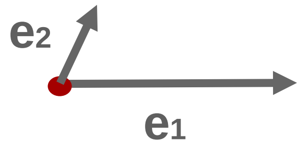
- The sum of two vectors is commutative



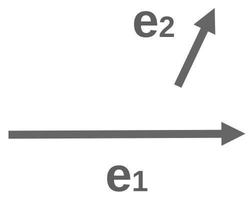

  - This is how Grassmann deals with the issue of magnitude

# A Pictorial Discussion of what Grassmann did

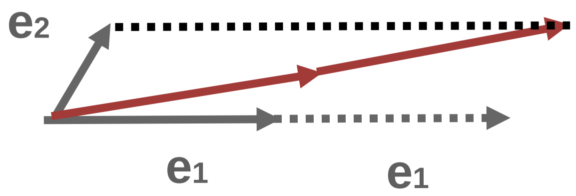
- For 2 dimensions: Let $\mathbf{e}_1$, $\mathbf{e}_2$ be any two vectors




- Shown here with a common tail
- But $\mathbf{e}_1$, $\mathbf{e}_2$ can be independently placed

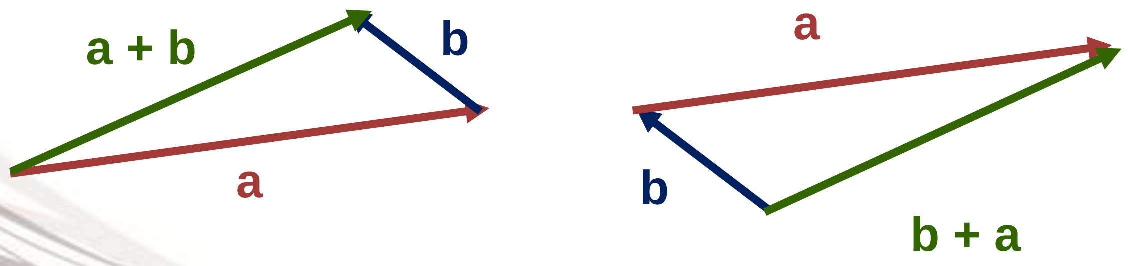



- For example **a** can be constructed as follows




$\mathbf{a} = 2\mathbf{e}_1 + \mathbf{e}_2$

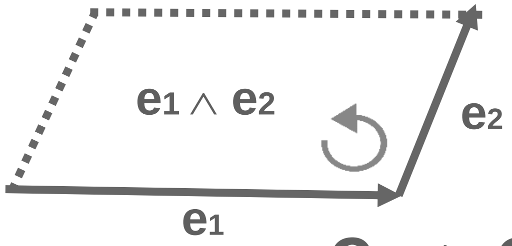
- Addition of vectors can be accomplished as follows

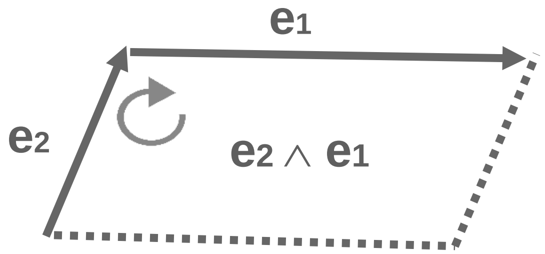



**a + b = b + a**

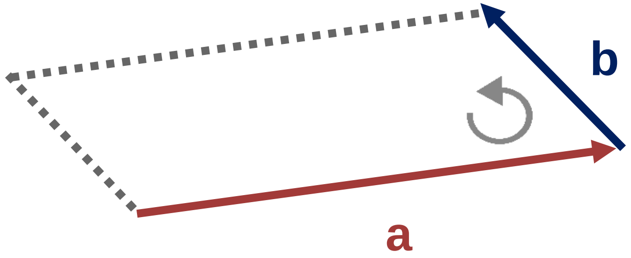
# A Pictorial Discussion of what Grassmann did

- The wedge product of two vectors is called a bivector.

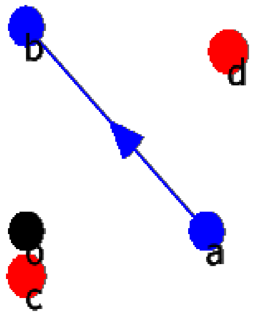



$$e_1 \wedge e_2 = -e_2 \wedge e_1$$




$$a \wedge b = A_{ab}\, e_1 \wedge e_2$$

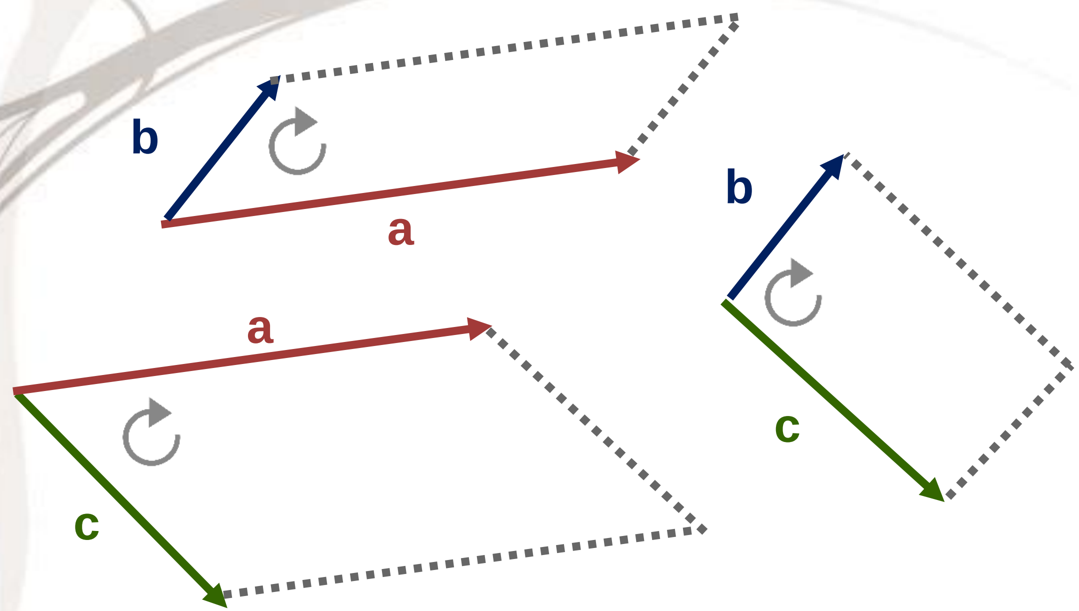



$$A_{ab} = \frac{\textit{Area of the paraelogram (made by a,b)}}{\textit{Area of the unit parallelogram constructed from } e_1, e_2}$$

- It should be noted that the same bivector can be represented by two different parallelograms provided they have the same area.

# A Pictorial Discussion of what Grassmann did

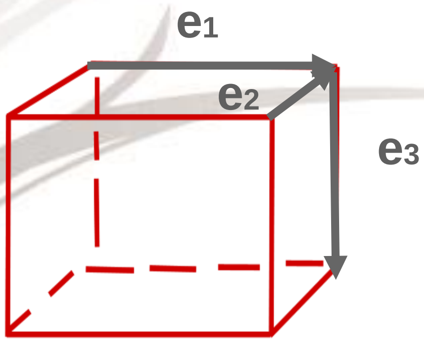
- For bivectors in 3 dimensions:




- According to Grassmann their planes can be put together in 3 dimensions in the following way

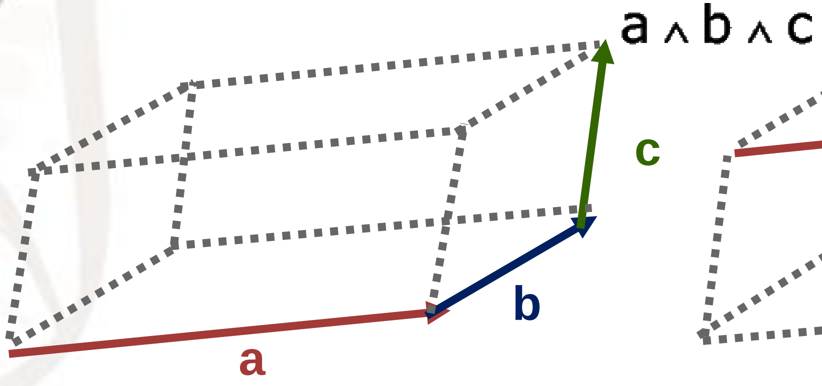
- $\mathbf{a} \wedge \mathbf{b} = \mathbf{A}_{ab}\ \mathbf{e}_1 \wedge \mathbf{e}_2$
- $\mathbf{b} \wedge \mathbf{c} = \mathbf{A}_{bc}\ \mathbf{e}_2 \wedge \mathbf{e}_3$
- $\mathbf{c} \wedge \mathbf{a} = \mathbf{A}_{ca}\ \mathbf{e}_3 \wedge \mathbf{e}_1$

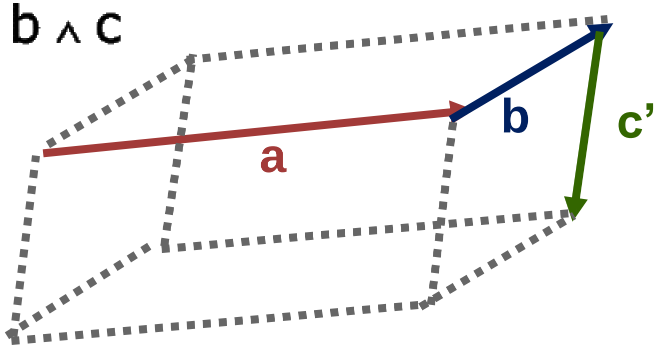
# A Pictorial Discussion of what Grassmann did




- For easy visualization we shall now take **$e_1$, $e_2$, $e_3$** to be orthonormal

- For a trivector in 3 dimensions:



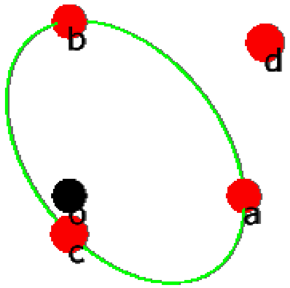




$\mathbf{a \wedge b \wedge c} =$

$\mathbf{V_{abc}\, e_1 \wedge e_2 \wedge e_3}$

$\mathbf{a \wedge b \wedge c'} =$

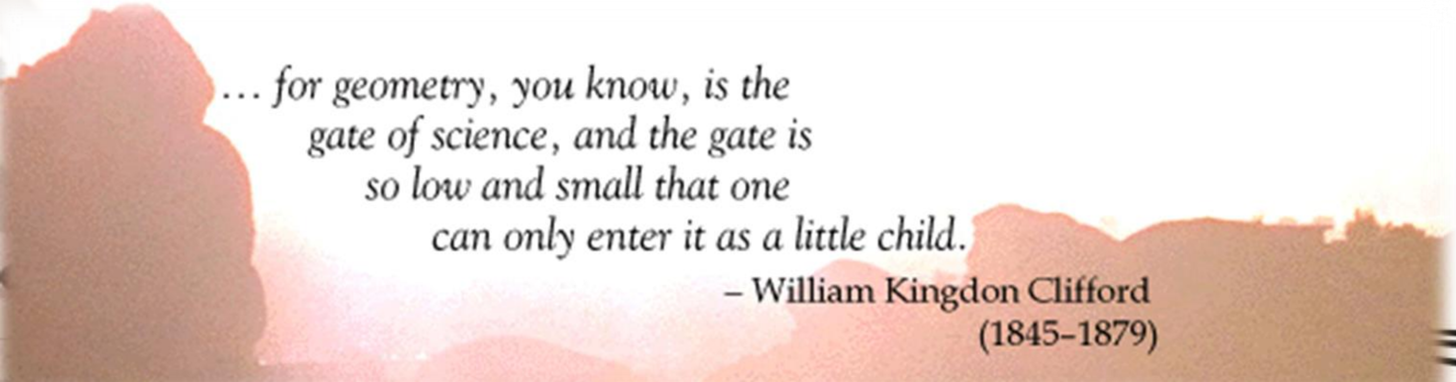
$\mathbf{-V_{abc}\, e_1 \wedge e_2 \wedge e_3}$




This process can be repeated indefinitely for n dimensions.

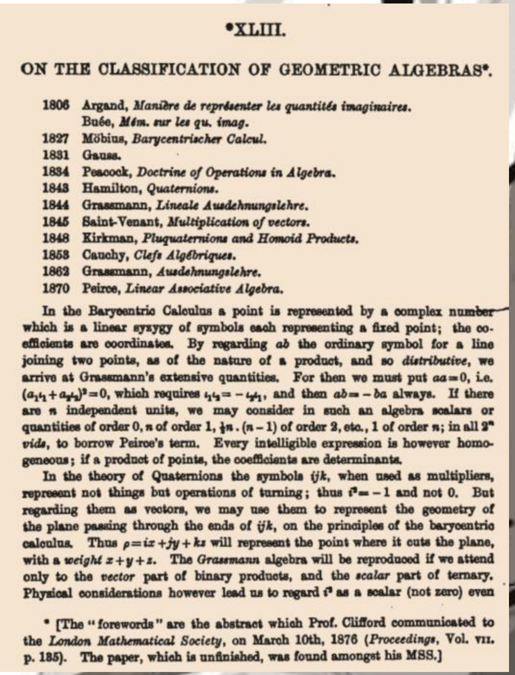
# Clifford Joins Quaternions with Grassmann algebras

- It is ironic according to D. Hestenes
- Clifford "the mathematician exhibiting the deepest understanding of Grassmann's system and advancing it in a major way, is seldom mentioned as a follower of Grassmann in historical accounts, though Clifford himself could not have been more explicit or emphatic in his claim to be following Grassmann in developing what he called 'geometric algebra'!"




# Clifford Joins Quaternions with Grassmann algebras

- In 1876 Clifford wrote, but left unfinished and unpublished, a paper called “On The Classification of Geometric Algebras”.
- In this paper Clifford succeeded in unifying Hamilton’s quaternions with Grassmann's outer product.
  - Clifford understood the deep geometric nature of the algebra Grassmann had developed.
  - He also noticed that quaternions were rotational operators that fit neatly into Grassmann’s algebras.



*XLIII.

ON THE CLASSIFICATION OF GEOMETRIC ALGEBRAS*.

1806 Argand, *Manière de représenter les quantités imaginaires.*
Buée, *Mém. sur les qu. imag.*
1827 Möbius, *Barycentrischer Calcul.*
1831 Gauss.
1834 Peacock, *Doctrine of Operations in Algebra.*
1843 Hamilton, *Quaternions.*
1844 Grassmann, *Lineale Ausdehnungslehre.*
1845 Saint-Venant, *Multiplication of vectors.*
1848 Kirkman, *Pluquaternions and Homoid Products.*
1858 Cauchy, *Clefs Algébriques.*
1862 Grassmann, *Ausdehnungslehre.*
1870 Peirce, *Linear Associative Algebra.*

In the Barycentric Calculus a point is represented by a complex number which is a linear syzygy of symbols each representing a fixed point; the co-efficients are coordinates. By regarding $ab$ the ordinary symbol for a line joining two points, as of the nature of a product, and so *distributive*, we arrive at Grassmann's extensive quantities. For then we must put $aa=0$, i.e. $(a_1\iota_1+a_2\iota_2)^2=0$, which requires $\iota_1\iota_2=-\iota_2\iota_1$, and then $ab=-ba$ always. If there are $n$ independent units, we may consider in such an algebra scalars or quantities of order 0, $n$ of order 1, $\frac{1}{2}n.(n-1)$ of order 2, etc., 1 of order $n$; in all $2^n$ *vids*, to borrow Peirce's term. Every intelligible expression is however homogeneous; if a product of points, the coefficients are determinants.

In the theory of Quaternions the symbols $ijk$, when used as multipliers, represent not things but operations of turning; thus $i^2=-1$ and not 0. But regarding them as vectors, we may use them to represent the geometry of the plane passing through the ends of $ijk$, on the principles of the barycentric calculus. Thus $\rho=ix+jy+kz$ will represent the point where it cuts the plane, with a *weight* $x+y+z$. The *Grassmann* algebra will be reproduced if we attend only to the *vector* part of binary products, and the *scalar* part of ternary. Physical considerations however lead us to regard $i^2$ as a scalar (not zero) even

* [The “forewords” are the abstract which Prof. Clifford communicated to the *London Mathematical Society*, on March 10th, 1876 (*Proceedings*, Vol. VII. p. 185). The paper, which is unfinished, was found amongst his MSS.]

# Clifford Joins Quaternions with Grassmann algebras

- Clifford’s inner product equips geometric algebra with a metric, and thereby incorporates distance and angle relationships for lines, planes, and volumes,
- The outer product gives the planes and volumes vector-like properties, for example direction.
- Clifford's algebraic system extends to higher dimensions.
- The algebraic operations have the same symbolic form as they do in 2 or 3-dimensions.
- The importance of general Clifford algebras has grown over time and has developed a life of their own.
- In this talk I will only sketch out how they connect to quaternions.

# How does Clifford do this?

- Clifford, like Grassmann, considers units in arbitrary number (n) dimensions, $e_1, e_2, \ldots, e_n$ , where $e_j\, e_k = -e_k\, e_j \ when\, j \neq k$.
  - But When $j = k$ Clifford let $e_j^2 = 1\ (or\ -1)$ where Grassmann made $e_j^2 = 0$

# How does Clifford do this?

- Clifford introduced the phrase ‘blade of order m’ to describe any wedge product $a_{j_1} \wedge a_{j_2} \wedge \ldots \wedge a_{j_m}$, of m vectors, if m ≥ 1
- A blade of order 0 is a scalar.
- By adding together blades of the same order, one can obtain more general objects which Clifford calls ‘homogeneous of order m’.
- If $m \geq 4$ these objects are not necessarily blades; $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 + \boldsymbol{e}_3 \wedge \boldsymbol{e}_4$ is not a blade, because it is not equal to any $\boldsymbol{a} \wedge \boldsymbol{b}$.
    - The vector space containing all these objects is denoted by $V_m$ .

# How does Clifford do this?

- Just as in quaternions one is allowed to add vectors to scalars, Clifford introduces the possibility of adding together objects of different order.
- This addition is commutative and associative, and generates a whole space of objects widely known as ‘clifs’.
- Clifs span all of the $V_m$’s.
  - The dimension of any $V_m$ is the binomial coef. $\binom{n}{m}$, n is the dimension of physical space.

# How does Clifford do this?

- To make the 'clifs' into an algebra, we need an associative multiplication distributive over addition.
- This is automatically provided by the definition:

$$uv = u \cdot v + u \wedge v.$$

- If $u$ and $v$ are arbitrary clifs, they can be decomposed into sums of scalars and multivectors of the form $e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_m}$; then $u \cdot v$ and $u \wedge v$ are given by the distributive law.
- The span of all the $V_m$ is now a graded algebra known as $\mathcal{Cl}$(n), in which $V_m$ contains all the objects of grade m.

The dimension of $\mathcal{Cl}$(n) as a vector space is $2^n$.

# How does Clifford do this?

- Clifford defines the product of all n units to be $\omega \equiv e_1e_2e_3 \ldots . e_n$
- Then $\omega$ commutes or anticommutes with each $e_j$ according as n odd or even.
- In the odd case ω is a true scalar; in the even case it is not, since a scalar must commute with everything.
- He then investigates $\omega^2$, finding
- $\omega^2 = e_1e_2 \ldots e_n = (1)^{n-1}e_1\, e_2 \ldots e_{n-1} = (1)^{n-1}(1)^{n-2}e_1e_2 \ldots e_{n-2} = \cdots = (-1)^s$

where $s = (n-1) + (n-2) + \cdots + 1 = \binom{n}{2} = \frac{n(n+1)}{2}$

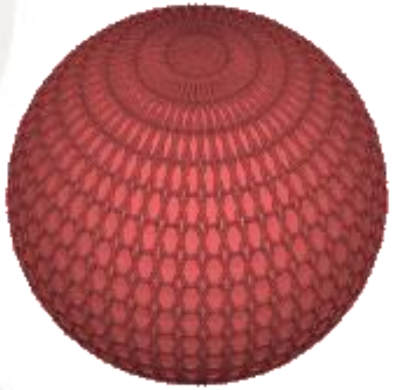
# How does Clifford do this?

- Now suppose that u and v belong to $V_m$ and $V_{m'}$ respectively.
- When we multiply them, we get two terms, $u \cdot v$ and $u \wedge v$ belonging respectively to $V_{(m+m'-2)}$ and to $V_{(m+m')}$
- These are either both even or both odd.
- In particular, if m and m' are both even then uv will be of even order.
- Therefore the units of even order: $1, e_j e_k, e_j e_k e_l e_m, \ldots$ where $j \neq k \neq l \neq m$, form a closed subalgebra.
- Clifford called this the 'even subalgebra'.

# How does Clifford do this?

- In the case where s is odd, ω can be considered as an imaginary unit. However, it is not an imaginary scalar unless n is also odd. (For example, n=3 or 7, see table below)
- Thus for n=3, Clifford is able to reduce the eight-dimensional algebra over the reals to a four-dimensional algebra (the quaternions) over the complex field.

| $n$ | $\omega = scalar?$ | $s$ | $\boldsymbol{\omega^2 = (-1)^s}$ | $\omega$ |
|---|---|---|---|---|
| 1 | yes | 0 | 1 | Real |
| 2 | no | 1 | -1 | Imaginary |
| 3 | yes | 3 | -1 | Imaginary |
| 4 | no | 6 | 1 | Real |
| 5 | yes | 10 | 1 | Real |
| 6 | no | 15 | -1 | Imaginary |
| 7 | yes | 21 | -1 | Imaginary |



# Clifford 3D Geometric Algebra

- For example let $n = 3$ and $e_j^2 = +1 \Rightarrow j = 1,2,3 \Rightarrow \omega^2 = -1$ and this commutes with all the elements of the algebra.
- Thus we can consider $\omega \equiv (\sqrt{-1})$.
- The even subalgebra has basis $\{1; e_2e_3; e_1e_3; e_1e_2\}$
- It gives the quaternions:
- $i \equiv e_2e_3, j \equiv e_1e_3, k \equiv e_1e_2$ in Hamilton's notation.
- Thus a pure quaternion is actually a bivector in the context of Clifford algebras.

# Clifford 4D Geometric Algebra

- From table: in 4 dimensions s = 3 + 2 + 1 = 6 is even. Therefore ω is real. But ω is not a scalar because n is even. This makes ω anticommute with vectors (members of $V_1$).
- But ω does commute with all members of $V_0$, $V_2$, and $V_4$. These span the even subalgebra, which contains the quaternions.
- The whole algebra in 4D is spanned by $2^4$ =16 generators, where half of these are of even order.

# Clifford 4D Geometric Algebra

- The even subalgebra is spanned by the following 8 generators:
- $V_0$ has 1 generator which we call (1).
- $V_2$ has 6 generators $(e_2e_3)$, $(e_3e_1)$, $(e_1e_2)$, $(e_0e_1)$, $(e_0e_2)$, $(e_0e_3)$.
- $V_4$ has 1 generator $(e_0e_1e_2e_3) = \omega$.

- To show that every member of the even subalgebra can be written as $(q) + (q')\omega$, where q and q' are quaternions.
- We see that (q) is just $q_0$ (1) + $q_1$ (i) + $q_2$ (j) + $q_3$ (k), where $q_0$, $q_1$, etc. are any (real) numbers. This is just what we can get from $V_0$ and the first three generators of $V_2$.

# Clifford 4D Geometric Algebra

- To deal with the q' part, let's work backwards. We are looking for (q')ω, which can be expressed as

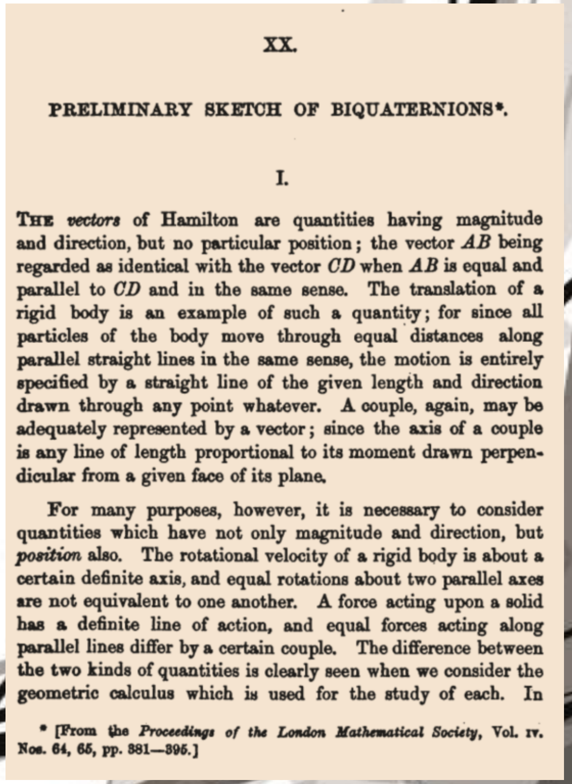
$$(q')\omega = q'_0\,\omega + q'_1\,(i)\omega + q'_2\,(j)\omega + q'_3\,(k)\omega\ .$$

- But $(i)\omega = (e_2e_3)(e_0e_1e_2e_3) = -(e_0e_1)$, $(j)\omega = -(e_0e_2)$, and $(k)\omega = -(e_0e_3)$.
- So $(q')\omega = q'_0\,(e_0e_1e_2e_3) - q'_1\,(e_0e_1) - q'_2(e_0e_2) - q'_3(e_0e_3)$.
- This is just what we can get from $V_4$ and the last three generators of $V_2$. (The signs of $q_1$', etc. are arbitrary.)

- Any expression of the form q + q'ω where q and q' is a quaternion this is what Clifford called 'biquaternions'.

# Clifford Joins Quaternions with Grassmann algebras

- Clifford had first introduced his notion of ‘biquaternions’ in "Preliminary Sketch of Biquaternions” in 1873 to the London mathematical Society.
- This was essentially the result of his synthesis of Grassmann’s and Hamilton’s ideas.
- His deeper development of these ideas were discovered later and found amongst his unpublished, unfinished papers after his death in 1879.

XX.

PRELIMINARY SKETCH OF BIQUATERNIONS*.

I.

THE *vectors* of Hamilton are quantities having magnitude and direction, but no particular position; the vector *AB* being regarded as identical with the vector *CD* when *AB* is equal and parallel to *CD* and in the same sense. The translation of a rigid body is an example of such a quantity; for since all particles of the body move through equal distances along parallel straight lines in the same sense, the motion is entirely specified by a straight line of the given length and direction drawn through any point whatever. A couple, again, may be adequately represented by a vector; since the axis of a couple is any line of length proportional to its moment drawn perpendicular from a given face of its plane.

For many purposes, however, it is necessary to consider quantities which have not only magnitude and direction, but *position* also. The rotational velocity of a rigid body is about a certain definite axis, and equal rotations about two parallel axes are not equivalent to one another. A force acting upon a solid has a definite line of action, and equal forces acting along parallel lines differ by a certain couple. The difference between the two kinds of quantities is clearly seen when we consider the geometric calculus which is used for the study of each. In

* [From the *Proceedings of the London Mathematical Society*, Vol. IV. Nos. 64, 65, pp. 381—395.]

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Pics (Not by the author of this powerpoint or Wikipedia)

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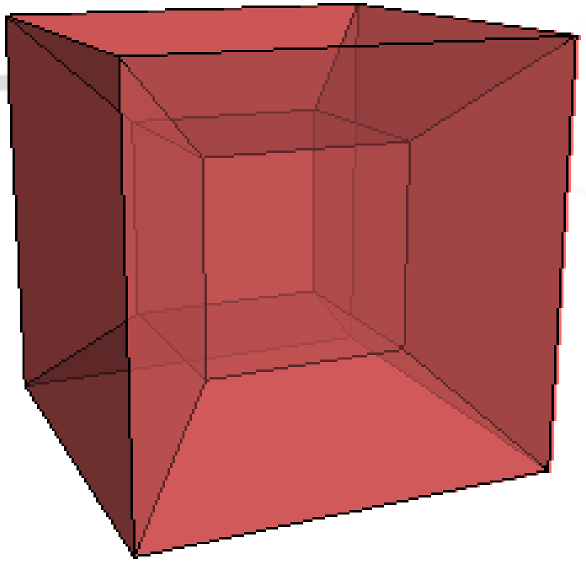
# Clifford



# Thank You

The Big Red Algebra