

# E401: Advanced Communication Theory

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**Multi-Antenna Wireless Communications**  
Array Receivers for SIMO and MIMO

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# General Objective

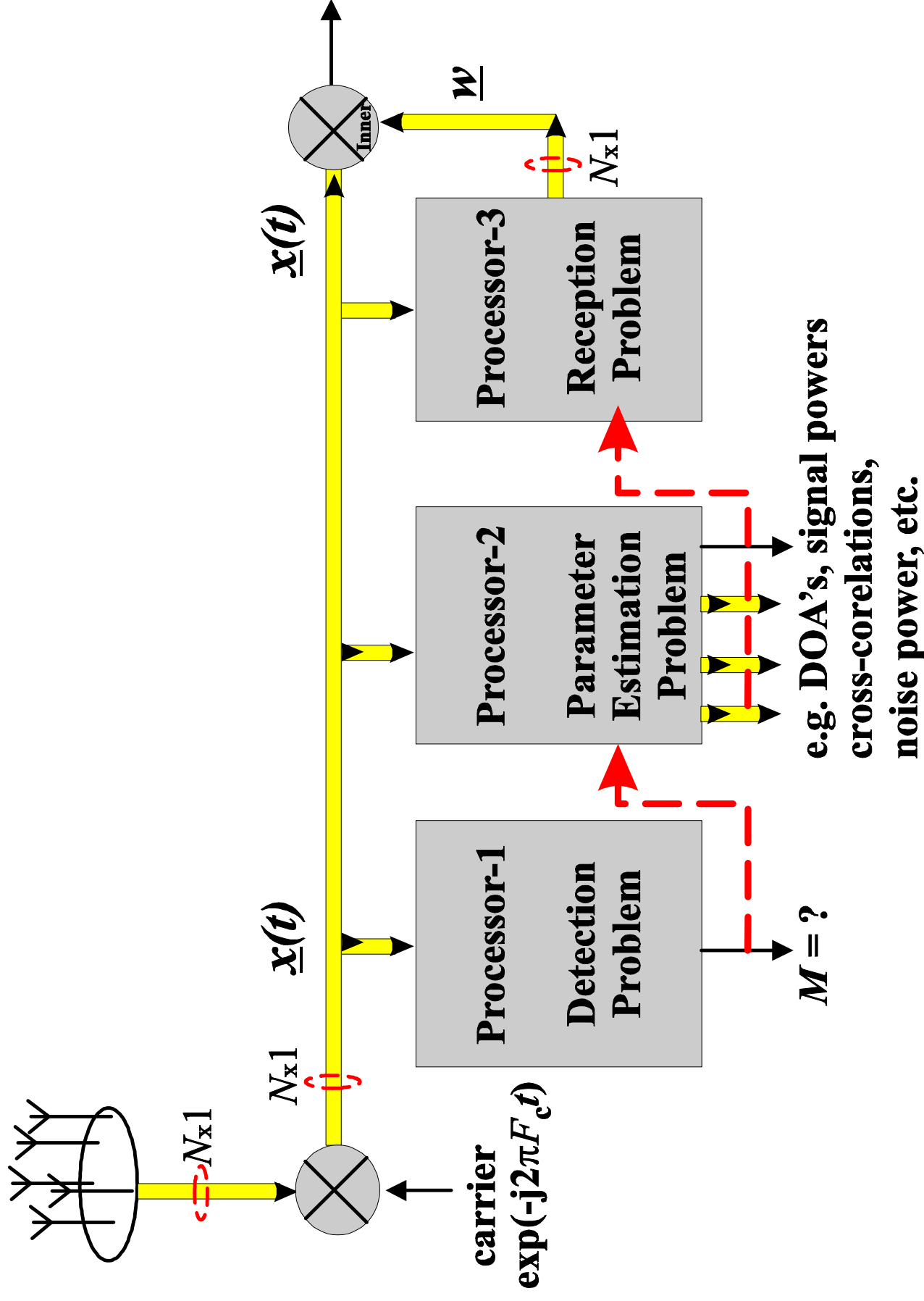
- By observing a vector-signal  $\underline{x}(t) = \underline{S}m(t) + \underline{n}(t)$  using an array system the aim is to obtain information about a signal environment.
- There are three general problems to solve.

1. The Detection problem:  $M = ?$   
(i.e. to detect the presence of  $M$  **co-channel** emitting sources)

2. The Estimation problem:  
to estimate various signal and channel parameters  
e.g. DOAs  $= ? \forall i; P_{m_i} = \varepsilon \{ m_i^2(t) \} = ? \forall i; P_n = \sigma_n^2 = ?;$   
 $\rho_{ij} = \varepsilon \{ m_i(t) m_j^*(t) \} = ? \forall i, j$ , with  $i \neq j$   
polarization parameters, fading coefficients, signal spread.

3. The Reception problem:  
to receive one signal (desired signal) and suppress the remaining  $M - 1$   
as unwanted cochannel interference

- These problems are highlighted in the following block structure:



# General Problem Formulation $M < N$

- Consider an observed  $(N \times 1)$  complex signal-vector  $\underline{x}(t)$  that is modelled as follows

$$\underline{x}(t) \triangleq \underbrace{S(\underline{p})}_{N \times M} \cdot \underbrace{\underline{m}(t)}_{M \times 1} + \underbrace{\underline{n}(t)}_{N \times 1} \quad (1)$$

Note that by observing  $\underline{x}(t)$ , its 2nd order statistics become known, i.e. the covariance matrix  $\mathbb{R}_{xx}$  is known, where

$$\mathbb{R}_{xx} = \mathcal{E}\{\underline{x}(t) \cdot \underline{x}(t)^H\} \quad (2)$$

- Estimate  $M, p_1, p_2, \dots, p_M, \mathbb{R}_{mm}, \sigma_n^2$ , etc.

$$S \triangleq S(\underline{p}) = [S(p_1), S(p_2), \dots, S(p_M)] - (\text{unknown})$$

where  $\underline{m}(t)$  : message signal-vector - (**unknown**)

$\mathbb{R}_{mm}$  : 2nd order statistics of  $\underline{m}(t)$  - (**unknown**)

$\underline{n}(t)$  : AWGN vector - (power  $\sigma_n^2$  **unknown**)

$\underline{p}$  = the vector of generic (**unknown**) parameters  $p_1, p_2, \dots, p_M$

$N$  = **known** (this is a system parameter)

$M$  = **unknown** (this is a channel parameter - number of signals)

with  $M < N$  (later this condition will be removed)

# Array Covariance Matrix

## Theoretical Covariance Matrix

- If the  $(N \times 1)$  vector-signal  $\underline{x}(t) = \underline{S}\underline{m}(t) + \underline{n}(t)$  is observed over infinite observation interval then its 2nd order statistics can be calculated. These are given by the theoretical covariance matrix  $\mathbb{R}_{\underline{x}\underline{x}}$  which is an  $(N \times N)$  complex matrix - always Hermitian. That is,

$$\mathbb{R}_{\underline{x}\underline{x}} \triangleq \mathcal{E} \left\{ \underline{x}(t)\underline{x}(t)^H \right\} \quad (3)$$

$$= \begin{bmatrix} \mathcal{E} \{ x_1(t)x_1(t)^* \}, & \mathcal{E} \{ x_1(t)x_2(t)^* \}, & \dots, & \mathcal{E} \{ x_1(t)x_N(t)^* \} \\ \mathcal{E} \{ x_2(t)x_1(t)^* \}, & \mathcal{E} \{ x_2(t)x_2(t)^* \}, & \dots, & \mathcal{E} \{ x_2(t)x_N(t)^* \} \\ \dots, & \dots, & \dots, & \dots \\ \mathcal{E} \{ x_N(t)x_1(t)^* \}, & \mathcal{E} \{ x_N(t)x_2(t)^* \}, & \dots, & \mathcal{E} \{ x_N(t)x_N(t)^* \} \end{bmatrix}$$

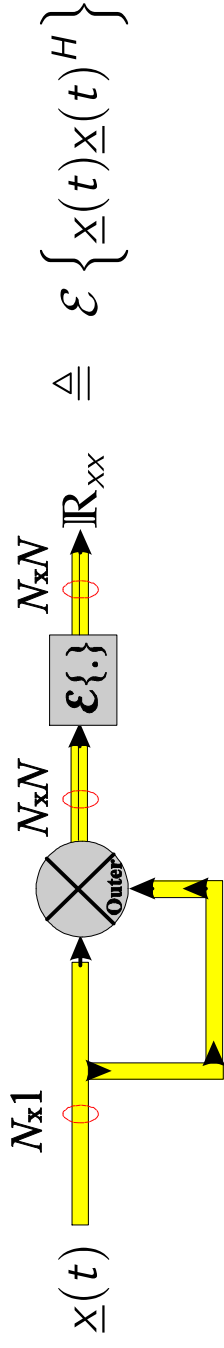
$$= \mathcal{E} \left\{ (\underline{S}\underline{m}(t) + \underline{n}(t)) \cdot (\underline{S}\underline{m}(t) + \underline{n}(t))^H \right\}$$

$$= \mathcal{E} \left\{ \underline{S}\underline{m}(t)\underline{m}(t)^H \underline{S}^H + \underline{n}(t)\underline{n}(t)^H + \underline{S}\underline{m}(t)\underline{n}(t)^H + \underline{n}(t)\underline{m}(t)^H \underline{S}^H \right\}$$

$$= \underbrace{\underline{S} \cdot \mathcal{E} \left\{ \underline{m}(t)\underline{m}(t)^H \right\} \cdot \underline{S}^H + \mathcal{E} \left\{ \underline{n}(t)\underline{n}(t)^H \right\}}_{\triangleq \mathbb{R}_{mm}} + \underbrace{\underline{S} \mathcal{E} \left\{ \underline{m}(t)\underline{n}(t)^H \right\} + \mathcal{E} \left\{ \underline{n}(t)\underline{m}(t)^H \right\} \underline{S}^H}_{\triangleq \mathbb{R}_{mn}} \quad (4)$$

$$= \underline{S} \cdot \mathbb{R}_{mm} \cdot \underline{S}^H + \mathbb{R}_{nn}$$

- i.e.



$$= \mathbf{S} \cdot \mathbb{R}_{mm} \cdot \mathbf{S}^H + \mathbb{R}_{nn}$$

where

$$\begin{aligned} \mathbb{R}_{mm} &\triangleq \mathcal{E} \left\{ \underline{\mathbf{m}}(t) \cdot \underline{\mathbf{m}}(t)^H \right\} = \text{2nd order statistics of } \underline{\mathbf{m}}(t) \text{ (unknown)} \\ &= \begin{bmatrix} \underbrace{\mathcal{E} \{ m_1(t) \cdot m_1(t)^* \}}_{\mathcal{E} \{ m_1(t)^2 \} = P_1}, & \mathcal{E} \{ m_1(t) \cdot m_2(t)^* \}, & \dots, & \mathcal{E} \{ m_1(t) \cdot m_M(t)^* \} \\ \mathcal{E} \{ m_2(t) \cdot m_1(t)^* \}, & \underbrace{\mathcal{E} \{ m_2(t) \cdot m_2(t)^* \}}_{\mathcal{E} \{ m_2(t)^2 \} = P_2}, & \dots, & \mathcal{E} \{ m_2(t) \cdot m_M(t)^* \} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E} \{ m_M(t) \cdot m_1(t)^* \}, & \mathcal{E} \{ m_M(t) \cdot m_2(t)^* \}, & \dots, & \underbrace{\mathcal{E} \{ m_M(t) \cdot m_M(t)^* \}}_{\mathcal{E} \{ m_M(t)^2 \} = P_M} \end{bmatrix} \end{aligned}$$

= an  $(M \times M)$  complex matrix (always Hermitian) - unknown

$$\begin{aligned}
 \mathbb{R}_{nn} &\triangleq \mathcal{E} \left\{ \underline{n}(t) \cdot \underline{n}(t)^H \right\} \text{ is 2nd order statistics of } \underline{n}(t) \\
 &= \begin{bmatrix} \underbrace{\mathcal{E} \{n_1(t) \cdot n_1(t)^*\}}_{\mathcal{E}\{n_1(t)^2\}=P_{n_1}}, & \underbrace{\mathcal{E} \{n_1(t) \cdot n_2(t)^*\}}_0, & \dots, & \underbrace{\mathcal{E} \{n_1(t) \cdot n_N(t)^*\}}_0 \\ \underbrace{\mathcal{E} \{n_2(t) \cdot n_1(t)^*\}}_0 & \underbrace{\mathcal{E} \{n_2(t) \cdot n_2(t)^*\}}_{\mathcal{E}\{n_2(t)^2\}=P_{n_2}}, & \dots, & \underbrace{\mathcal{E} \{n_2(t) \cdot n_N(t)^*\}}_0 \\ \dots, & \dots, & \dots, & \dots \\ \underbrace{\mathcal{E} \{n_N(t) \cdot n_1(t)^*\}}_0 & \underbrace{\mathcal{E} \{n_N(t) \cdot n_2(t)^*\}}_0 & \dots, & \underbrace{\mathcal{E} \{n_N(t) \cdot n_N(t)^*\}}_{\mathcal{E}\{n_N(t)^2\}=P_{n_N}} \end{bmatrix} \quad (5) \\
 &= \sigma_n^2 \mathbb{I}_N \\
 &= \text{an } (N \times N) \text{ complex matrix (always Hermitian) - unknown}
 \end{aligned}$$

- Note that, because we have assumed isotropic AWGN noise,

$$P_{n_1} = P_{n_2} = \dots = P_{n_N} = \sigma_n^2 \quad (6)$$



## Practical Covariance Matrix

- Consider that the signal  $\underline{x}(t) = \underline{S}\underline{m}(t) + \underline{n}(t)$  is observed over finite observation intervals equivalent to  $L$  snapshots.
- These  $L$  observations (snapshots) at times  $t_1, t_2, \dots, t_L$  (i.e. finite observation interval) are denoted as  $[\underline{x}(t_1), \underline{x}(t_2), \dots, \underline{x}(t_L)]$  and represented by the  $N \times L$  complex matrix  $\underline{X}$

i.e.

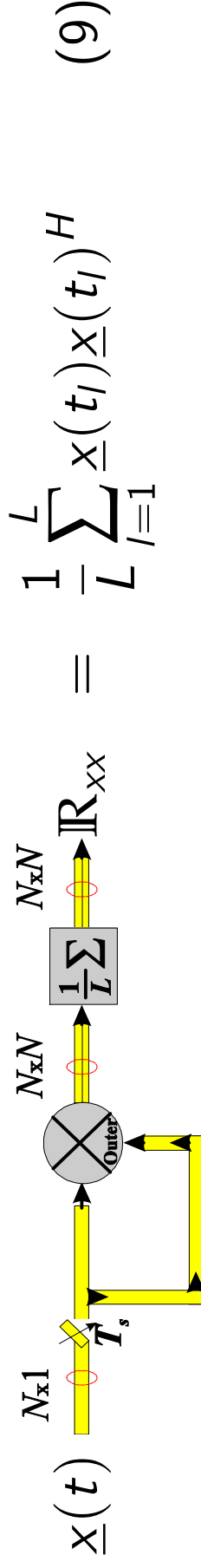
$$\underline{X} \triangleq [\underline{x}(t_1), \underline{x}(t_2), \dots, \underline{x}(t_L)] \quad (7a)$$

$$\begin{aligned} &= [\underline{S}\underline{m}(t_1) + \underline{n}(t_1), \underline{S}\underline{m}(t_2) + \underline{n}(t_2), \dots, \underline{S}\underline{m}(t_L) + \underline{n}(t_L)] \\ &= \underline{S}\underline{M} + \underline{N} \end{aligned} \quad (7b)$$

$$\begin{cases} \underline{S} = [\underline{S}_1, \underline{S}_2, \dots, \underline{S}_M] & (N \times M) \\ \text{with } \underline{M} = [\underline{m}(t_1), \underline{m}(t_2), \dots, \underline{m}(t_L)] & (M \times L) \\ \underline{N} = [\underline{n}(t_1), \underline{n}(t_2), \dots, \underline{n}(t_L)] & (N \times L) \end{cases} \quad (8)$$

where the matrices  $\underline{S}$ ,  $\underline{M}$  and  $\underline{N}$  (as well as the dimension  $M$ ) are unknown

- In this case the 2nd order statistics of  $\underline{x}(t)$  are estimated by the practical covariance matrix  $\mathbb{R}_{xx}$
- Practical Model:



$$\begin{aligned} \text{i.e. } \mathbb{R}_{xx} &= \frac{1}{L} \mathbf{X} \mathbf{X}^H \\ &= \underbrace{\mathbf{S} \frac{1}{L} \mathbf{M} \mathbf{M}^H \mathbf{S}^H}_{=\mathbb{R}_{mm}} + \underbrace{\frac{1}{L} \mathbf{N} \mathbf{N}^H}_{=\mathbb{R}_{nn}} \\ &= \mathbf{S} \cdot \mathbb{R}_{mm} \cdot \mathbf{S}^H + \mathbb{R}_{nn} \quad (10) \end{aligned}$$

### N.B.:

- In an array system the matrix  $\mathbb{R}_{xx}$  (theoretical or practical) contains all the **geometrical and other information** about the various sources relative to the array.
- Remember that, sometimes, we will use  $\hat{\mathbb{R}}_{xx}$  to denote the practical/estimated covariance matrix of the vector signal  $\underline{x}(t)$ .

## Generating $L$ Snapshots having a given Covariance Matrix

- To generate  $L$  snapshots of  $\underline{x}(t)$  having a predefined covariance matrix  $\mathbb{R}_{xx}$  the vectors  $\underline{x}(t_l)$  for  $l = 1, 2, \dots, L$  should be generated using the following expression

$$\underline{x}(t_l) = \mathbb{E} \mathbb{D}^{\frac{1}{2}} \underline{z}(t_l) \quad (11)$$

where

- ▶  $\mathbb{E}$  and  $\mathbb{D}$  are the eigenvector-matrix and eigenvalue-matrix of  $\mathbb{R}_{xx}$ , and
- ▶  $\underline{z}(t_l) \in \mathcal{C}^M$  is a Gaussian random complex vector of  $M$  elements of zero mean and variance 1, i.e.

$$\mathcal{E}\{\underline{z}(t_l) \cdot \underline{z}(t_l)^H\} = \mathbb{I}_M \quad (12)$$

- That is,

$$\begin{aligned} \mathbb{X} &= [\underline{x}(t_1), \underline{x}(t_2), \dots, \underline{x}(t_L)] \\ &= [\mathbb{E} \mathbb{D}^{\frac{1}{2}} \underline{z}(t_1), \mathbb{E} \mathbb{D}^{\frac{1}{2}} \underline{z}(t_2), \dots, \mathbb{E} \mathbb{D}^{\frac{1}{2}} \underline{z}(t_L)] \end{aligned} \quad (13)$$