Finding the eigenvalues, and some eigenvectors of a few important tri-diagonal Toeplitz matrices

Herman Jaramillo
Universidad de Medellín

August 4, 2018

Abstract

The problem of finding the eigenvalues of a tri-diagonal Toeplitz matrix is of fundamental importance in several aspects of mathematics and engineering. An example is found in the solution of finite difference problems for the heat equation. The problem of solving for the eigenvalues of a tri-diagonal Toeplitz matrix is set up as a difference equation. Once the difference equation is in place there are at least three roads to its solution. One road is using Chebyshev polynomials and their roots, another is using the Z transform, and the other is using a more traditional method to solve difference equations with initial conditions. The last two roads are presented here while the first is just indicated in a footnote.

1 Introduction

We want to find the eigenvalues and eigenvectors of the following tri-diagonal Toeplitz matrix:

$$\begin{pmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & c & a & b \\ & & & & & c & a \end{pmatrix},$$

We start using a methodology based on the Z transform for some simpler matrices that lead us to the solution of the matrix above. In a second approach we solve a difference equation by setting up some solutions to a difference equations which satisfy some initial conditions.

2 An approach through the Z transform

2.1 A simple Toeplitz matrix

We start by finding the eigenvalues of the matrix

$$U = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & & \cdot & \cdot & \cdot \\ & & & & 1 & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix}.$$

The eigenvalues satisfy the characteristic equation

$$\det(U - \lambda I) = 0.$$

That is:

$$p_n(\lambda) = \begin{vmatrix} -\lambda & 1 & & & & \\ 1 & -\lambda & 1 & & & \\ & 1 & -\lambda & 1 & & \\ & & & \cdot & \cdot & \cdot \\ & & & & 1 & -\lambda & 1 \\ & & & & & 1 & -\lambda \end{vmatrix} = 0$$

While p_0 is not defined, we can define $p_0 = 1$, also for one dimension $p_1 = -\lambda$. For n dimensions $n \geq 2$ we expand the determinant along the first raw.

$$p_n(\lambda) = -\lambda\, p_{n-1} - p_{n-2}.$$

In this equation the first sample is taken at $n = 2$. To set up the problem using the typical definition of the unilateral Z transform, we want to have the first sample at $n = 0$. This is easily stablish by rewriting equation 1 as

$$p_{n+2}(\lambda) = -\lambda\, p_{n+1} - p_n.$$

$n = 0, 1 \cdots$. We want to solve the difference equation

$$p_{n+2}(\lambda) + \lambda\, p_{n+1} + p_n = 0$$

subject to the initial conditions:

$$p_0 = 1 \quad , \quad p_1 = -\lambda.$$

We take the unilateral Z transform [1] of equation 1 and obtain:

$$z^2 P(z) - z^2\, p_0 - z\, p + z\lambda P(z) - z\lambda p_0 + P(z) = 0.$$

That is

$$P(z)(z^2 + \lambda z + 1) = z(p_1 + \lambda p_0) + z^2 p_0 = z^2$$

and

$$P(z) = \frac{z^2}{z^2 + \lambda z + 1}.$$

We can find the inverse Z transform using partial fraction expansion. Let us first, find the roots of the quadratic equation in the denominator. We have

$$z_1 = -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4}}{2} \quad , \quad z_2 = -\frac{\lambda}{2} - \frac{\sqrt{\lambda^2 - 4}}{2}$$

We use the Gersgorin's Theorem to find, the bounds of the eigenvalues. Each eigenvalue is bound in absolute value by the sum of some raw of the matrix

[1]Instead of using the Z transform methods, or any more general difference equation methods, the reader can view the problem in a different way. By mapping $-\lambda = 2x$ we can rewrite the recursion 1 as

$$p_{n+2}(x) = 2xp_{n+1}(x) - p_n(x).$$

This is can be recognized as the Chebyshev polynomials of the second kind. A lot is known about them, including analytical closed from of their roots.

it came from, non–counting the diagonal element. In this particular case all eigenvalues are bounded by 2. That is

$$|\lambda| \le 2$$

Therefore we can define

$$\cos\theta = -\frac{\lambda}{2} \tag{1}$$

and,

$$\frac{\sqrt{\lambda^2-4}}{2} = \mathrm{i}\sqrt{1-\left(\frac{\lambda}{2}\right)^2} = \mathrm{i}\sqrt{1-\cos^2\theta} = \mathrm{i}\sin\theta$$

with $\mathrm{i} = \sqrt{-1}$.

So we can rewrite

$$z_1 = e^{\mathrm{i}\theta} \quad , \quad z_2 = e^{-\mathrm{i}\theta}.$$

The partial fraction representation of 1 is:

$$\frac{z^2}{z^2+\lambda z+1} = z^2\left(\frac{A}{z-z_1} + \frac{B}{z-z_2}\right)$$

where

$$A = \frac{1}{z_1-z_2} = \frac{1}{2\,\mathrm{i}\sin\theta} \quad , \quad B = \frac{1}{z_2-z_1} = -\frac{1}{2\,\mathrm{i}\sin\theta}.$$

Then

$$P(z) = \frac{z^2}{z^2+\lambda z+1} = \frac{z^2}{2\mathrm{i}\sin\theta}\left(\frac{1}{z-z_1} - \frac{1}{z-z_2}\right)$$

or

$$P(z) = \frac{z}{2\mathrm{i}\sin\theta}\left(\frac{1}{1-z_1/z} - \frac{1}{1-z_2/z}\right).$$

We expand the geometric series (inverse Z transform) to obtain:

$$\begin{aligned} P(z) &= \frac{z}{2\mathrm{i}\sin\theta}[(1+z_1z^{-1}+z_1^2z^{-2}+\cdots+z_1^nz^{-n}\cdots) \\ &\quad -(1+z_2z^{-1}+z_2^2z^{-2}+\cdots+z_2^nz^{-n}+\cdots)] \\ &= \frac{1}{2\mathrm{i}\sin\theta}[(z_1-z_2)+(z_1^2-z_2^2)z^{-1}+\cdots+(z_1^n-z_2^n)z^{-n}+\cdots] \\ &= \frac{1}{\sin\theta}[\sin\theta+\sin 2\theta z^{-1}+\cdots+\sin(n+1)\theta z^{-n}+\cdots] \end{aligned}$$

and, from the inverse Z transform

$$p_n(\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$$

Here we find yet another representation of the Chebyshev polynomials of second kind, by mapping $x = \cos\theta = -\lambda/2$.

For the trivial (non–important) case of $n = 0$, $p_0(\theta) = p_0(\lambda) = 1$. The solution of $p_n(\theta) = 0$, $n > 0$, in terms of θ_k is given by

$$(n+1)\theta_k = k\pi$$

that is

$$\theta_k = \frac{k\pi}{n+1}$$

We now use equation 1 and find

$$\cos\theta_k = \cos\frac{k\pi}{n+1} = -\frac{\lambda_k}{2}$$

that is

$$\lambda_k = -2\cos\frac{k\pi}{n+1}$$

$k = 1, 2, \cdots, n$. If n is odd, then the root $\lambda_{(n+1)/2} = 0$, otherwise all roots are different from zero. Let us call $A = \{k/(n+1) \quad | \quad k = 1, 2, \cdots, n\}$. For n odd

$$A = \left\{\frac{1}{n+1}, \cdots \frac{n+1}{2(n+1)} = \frac{1}{2}, \frac{n+3}{2(n+1)}, \cdots \frac{n+2j-1}{2(n+1)}, \cdots, \frac{n}{n+1}\right\}$$

Now since $\cos\theta = -\cos(\pi - \theta)$ we want to map the second half of this set to the first half and simplify the set of roots.

Start with the set $(n+2j-1)/2(n+1), j = 1, 2, \cdots (n+1)/2$, and fold it with respect to 1 as $1 - (n+2j-1)/2(n+1)$ which is

$$1 - \frac{n+2j-1}{2(n+1)} = \frac{n+3-2j}{2(n+1)}$$

$j = 1, 2, \cdots, (n+1)/2$. This half set of coefficients is

$$B = \left\{\frac{1}{n+1}, \cdots, \frac{n+1}{2(n+1)} = \frac{1}{2}\right\}$$

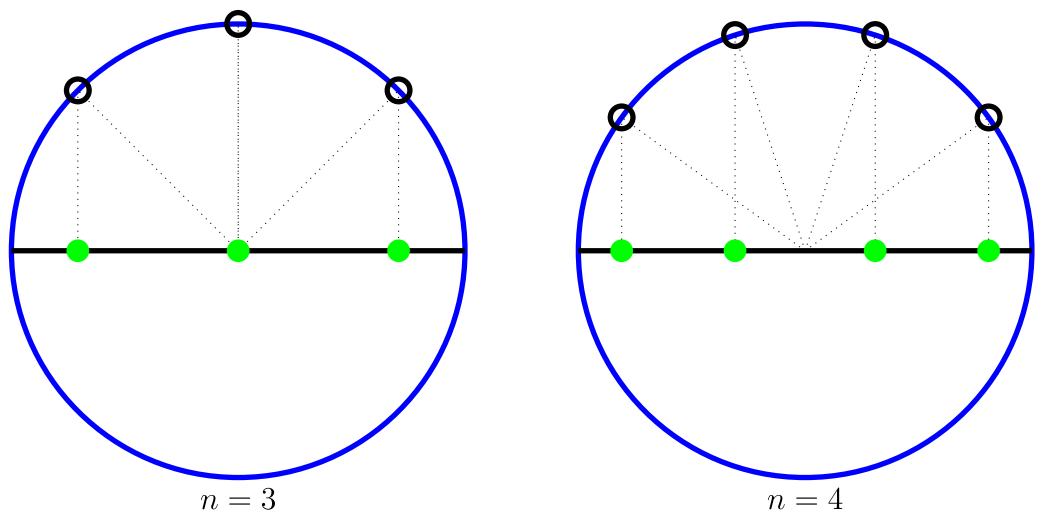

Figure 1: Illustration of the roots for the characteristic equation (eigenvalues of U) for $n = 3$ and $n = 4$. The black points on the circumferences represent the coordinates $(2\cos k\pi/(n+1), 2\sin k\pi/(n+1))$ and the projections on the x axis (green points) are the eigenvalues $2\cos k\pi/(n+1)$.

with $j = (n+1)/2, (n-1)/2, \cdots, 1$. We recognize that these angle coefficients are exactly those in the first half of the set A. Therefore the roots are complete characterized by the first half of angle coefficients in the set A, but for coefficients in the second half, the sign of the root should be flipped (again, because $\cos\theta = -\cos(\pi-\theta)$.) If n is even, the same argument applies but the zero root is not present.

This argument seems to be elaborated and could be explained in a different way. The Chebyshev polynomials with n even are even functions, and so the roots are symmetrically distributed with respect to the origin. For those polynomials 0 is not a root. The Chebyshev polynomials with n odd, are odd functions and so the roots are also symmetrically distributed with respect to the origin. For those odd polynomials 0 is of course a root (since $p_n(x) = -p_n(-x)$, that is $2\,p_n(0) = 0$.

Geometrically, the roots of the characteristic function are projections from a circle with radius 2, into the x–axis, where the points along the circles have simple angle description given by the $k\pi/(n+1)$ expression.

Figure 1 illustrate the distribution of the roots.

2.2 A bit more complicated Toeplitz matrix

We extend the matrix in the previous section to the following Toeplitz matrix:

$$U = \begin{pmatrix} a & 1 & & & & \\ 1 & a & 1 & & & \\ & 1 & a & 1 & & \\ & & & \cdot & \cdot & \cdot \\ & & & & 1 & a & 1 \\ & & & & & 1 & a \end{pmatrix}.$$

The characteristic equation is given by the determinant

$$p_n(\lambda) = \det(U - \lambda I) = \begin{vmatrix} a-\lambda & 1 & & & & \\ 1 & a-\lambda & 1 & & & \\ & 1 & a-\lambda & 1 & & \\ & & & \cdot & \cdot & \cdot \\ & & & & 1 & a-\lambda & 1 \\ & & & & & 1 & a-\lambda \end{vmatrix} = 0$$

Reusing all the work in the previous section we find that the roots are such that

$$a - \lambda_k = 2\cos\frac{k\pi}{n+1}.$$

So,

$$\lambda_k = a - 2\cos\frac{k\pi}{n+1}.$$

Geometrically, all the roots are projections of equally distributed points on the upper part of the circle with radius 2, but this time shifted horizontally a distance a. Note that we could have used a plus "+" sign instead of a minus "-" in the right hand side of equation 2 due to the symmetry o the solution set with respect to the axis $x = a$ as shown in the previous section.

2.3 A $b\ a\ b$ Toeplitz matrix

Let us now consider the matrix:

$$U = \begin{pmatrix} a & b & & & & \\ b & a & b & & & \\ & b & a & b & & \\ & & & \cdot & \cdot & \cdot \\ & & & & b & a & b \\ & & & & & b & a \end{pmatrix}.$$

Assuming that $b \neq 0$ we can divide by b the previous matrix and obtain:

$$U_s = \begin{pmatrix} a/b & 1 & & & & \\ 1 & a/b & 1 & & & \\ & 1 & a/b & 1 & & \\ & & & \cdot & \cdot & \cdot \\ & & & & 1 & a/b & 1 \\ & & & & & 1 & a/b \end{pmatrix}.$$

and resting on the result of the previous section we find that the eigenvalues of the scaled matrix U_s are:

$$\gamma_k = \frac{a}{b} - 2\cos\frac{k\pi}{n+1}.$$

and so the eigenvalues of the original matrix U (scaling back) are:

$$\lambda_k = a - 2b\cos\frac{k\pi}{n+1}. \tag{2}$$

2.4 Eigenvectors of the matrix Toeplitz matrix $b\ a\ b$

let us note the eigenvector k–th eigenvector of the Toeplitz $b\ a\ b$ matrix U as $\boldsymbol{v_k}$. We have that

$$(U - \lambda_k I)\boldsymbol{v_k} = 0$$

The i–th row of this system is given by:

$$b\, v_{ki-1} + (a - \lambda_k)v_{ki} + b\, v_{ki+1} = 0. \tag{3}$$

$i = 1, \cdots, n$. We rewrite equation 3 by renaming indices as

$$b\,v_{ki} + (a - \lambda_k)\,v_{ki+1} + b\,v_{ki+2} = 0. \tag{4}$$

$i = 0, \cdots, n-1$. We padded the $\boldsymbol{v_k}$ array with two extra components $v_{k0} = 0$ and $v_{kn+1} = 0$ without changing any of the equations.

Again we have here a difference equation that we could solve by applying the unilateral Z transform. After taking the unilateral Z transform on equation 4 we find:

$$b\,V_k(z) + (a - \lambda_k)(zV_k(z) - zv_{k0}) + b\,(z^2 V_k(z) - z^2 v_{k0} - zv_{k1}) = 0$$

That is,

$$V_k(z)[b + z\,(a - \lambda_k) + z^2 b] = (a - \lambda_k) z v_{k0} + b z^2 v_{k0} + z v_{k1}. \tag{5}$$

We need an extra initial condition for v_{ki}. If we set up $v_{k1} = 0$ then the recursion will result in $\boldsymbol{v_k} = 0$. So, in order to avoid zero eigenvectors we need to set up $v_{k1} = m_k \neq 0$. m_k could be any constant. An eigenvector does not change its condition of eigenvector after being scaled by any constant. The eigenvector equation 5 becomes

$$V_k(z)\left[b + (a - \lambda_k)\,z + b\,z^2\right] = z\,m_k$$

The solution eigenvector is given by the inverse Z transform of

$$V_k(z) = \frac{z\,m_k}{b + (a - \lambda_k)\,z + b\,z^2}.$$

As we did before, let us find the roots of the quadratic equation in the denominator.

$$z_\pm = \frac{-(a - \lambda_k)}{2b} \pm \frac{\sqrt{(a - \lambda_k)^2 - 4b^2}}{2b}$$

Now, from equation 2,

$$a - \lambda_k = 2b \cos\frac{k\pi}{n+1}.$$

so

$$z_\pm = \cos\frac{k\pi}{n+1} \pm \mathrm{i}\sqrt{1 - \cos^2\frac{k\pi}{n+1}}$$

so after defining $\theta_k = k\pi/(n+1)$ we have

$$z_\pm = e^{\pm \mathrm{i}\theta_k}$$

We now proceed to expand equation 6 into partial fractions. That is

$$V_k(z) = \frac{z\, m_k}{b + (a-\lambda_k)\, z + b\, z^2} = z\, m_k \left(\frac{A}{z - z_+} + \frac{B}{z - z_-} \right)$$

We find

$$A = \frac{1}{z_+ - z_-} = \frac{1}{2\mathrm{i}\sin\theta_k}$$
$$B = -\frac{1}{z_+ - z_-} = -\frac{1}{2\mathrm{i}\sin\theta_k}$$

and

$$\begin{aligned}
V_k(z) &= \frac{z\, m_k}{2\mathrm{i}\sin\theta_k}\left(\frac{1}{z - z_+} - \frac{1}{z - z_-}\right) \\
&= \frac{m_k}{2\mathrm{i}\sin\theta_k}\left(\frac{1}{1 - z_+/z} - \frac{1}{1 - z_-/z}\right) \\
&= \frac{m_k}{2\mathrm{i}\sin\theta_k}[(1 + z_+ z^{-1} + z_+^2 z^{-2} + \cdots + z_+^n z^{-n} + \cdots \\
&\quad -(1 + z_- z^{-1} + z_-^2 z^{-2} + \cdots + z_-^n z^{-n} + \cdots] \\
&= \frac{m_k}{\sin\theta_k}\left(\sin\theta_k z^{-1} + \sin 2\theta_k z^{-2} + \cdots + \sin n\theta_k z^{-n} + \cdots\right)
\end{aligned}$$

If we pick $m_k = \sin\theta_k$ (remember that up to a constant, the eigenvector is still an eigenvector). We find the k eigenvector

$$v_{ki} = \sin i\,\theta_k = \sin\frac{ki\pi}{n+1} \quad i = 1, 2, \cdots n.$$

2.5 A Toeplitz matrix $c\ a\ b$

We now deal with the most general Toeplitz matrix on this document.

$$U = \begin{pmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & & \cdot & \cdot & \cdot \\ & & & c & a & b \\ & & & & c & a \end{pmatrix}.$$

Let us define a new matrix:

$$U_1 = \begin{pmatrix} \frac{a}{\sqrt{bc}} & \frac{b}{\sqrt{bc}} & & & & & \\ \frac{c}{\sqrt{bc}} & \frac{a}{\sqrt{bc}} & \frac{b}{\sqrt{bc}} & & & & \\ & \frac{c}{\sqrt{bc}} & \frac{a}{\sqrt{bc}} & \frac{b}{\sqrt{bc}} & & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \frac{c}{\sqrt{bc}} & \frac{a}{\sqrt{bc}} & \frac{b}{\sqrt{bc}} \\ & & & & & \frac{c}{\sqrt{bc}} & \frac{a}{\sqrt{bc}} \end{pmatrix}.$$

The eigenvalues γ_k of U_1, are related to those of U (λ_k) by the equation

$$\gamma_k = \frac{\lambda_k}{\sqrt{bc}}.$$

Now since the product (entry by entry) of the off–diagonal entries is equal to 1, the recursion characteristic polynomial would be a Chebyshev polynomials of the second kind. That is, here

$$p_{n+2} = \left(\frac{a}{\sqrt{bc}} - \gamma_k \right) p_{n+1} - p_n$$

The roots of these polynomials satisfy:

$$\frac{a}{\sqrt{bc}} - \gamma_k = -2 \cos \frac{k\pi}{n+1}$$

so

$$\gamma_k = \frac{a}{\sqrt{bc}} - 2 \cos \frac{k\pi}{n+1}$$

and so

$$\lambda_k = a - 2\sqrt{bc} \cos \frac{k\pi}{n+1}$$

Due to the symmetry relationships shown in section 2.1 we can rewrite

$$\lambda_k = a + 2\sqrt{bc} \cos \frac{k\pi}{n+1}$$

and this produce the same set of eigenvalues. This is a most common form for the expression of the eigenvalues of the Toeplitz matrix c a b.

3 An approach following a more standard method to solve difference equations

The eigenvalues of U are roots of the characteristic polynomial That is

$$p(\lambda) = \det(U - \lambda I) = 0.$$

That is,

$$p_n(\lambda) = \det \begin{pmatrix} a-\lambda & c & & & & & \\ b & a-\lambda & c & & & & \\ & b & a-\lambda & c & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & b & a-\lambda & c \\ & & & & & b & a-\lambda \end{pmatrix} = 0$$

where n is the number of rows (or columns) of A. At the moment we assume $n \geq 2$. We expand the determinant through the first row to find

$$p(\lambda) = (a - \lambda)p_{n-1} - c \det B$$

where B is the matrix

$$\begin{pmatrix} b & c & & & & & \\ & a-\lambda & c & & & & \\ & b & a-\lambda & c & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & b & a-\lambda & c \\ & & & & & b & a-\lambda \end{pmatrix}$$

Then we expand the determinant of B through the first column to find

$$p_n(\lambda) = (a - \lambda)p_{n-1} - bcp_{n-2},$$

and

$$p_n(\lambda) - (a - \lambda)p_{n-1} - bcp_{n-2} = 0. \tag{6}$$

This is a linear difference equation. [2] We want to find an analytic solution

[2] https://en.wikipedia.org/wiki/Linear_difference_equation

for the function $p_n(\lambda)$ which is valid for all n. This equation needs two initial conditions due to the fact that p_n depends on two previous instances p_{n-1} and p_{n-2}. That is, for $n = 1$, $p_1(\lambda) = a - \lambda$ and for $n = 2$

$$p_2(\lambda) = \det \begin{pmatrix} a - \lambda & c \\ b & a - \lambda \end{pmatrix} = (a - \lambda)^2 - bc, \tag{7}$$

This could be the initial conditions. All other values of $p_n(\lambda)$ could be found starting with these two initial conditions on the recursive equation 6. However, and to simplify operations we can state simpler initial conditions. Although there are no matrices with $n = 0$ rows we could define $p_0(\lambda) = 1$, and keep the initial condition $p_1(\lambda) = a - \lambda$. Using the recursion 6 we find that

$$p_2(\lambda) = (a - \lambda)^2 - bc$$

which is exactly equation 7. The solution of $p_n(\lambda)$ for higher n values will not change. We then have the initial conditions

$$p_0(\lambda) = 1 \quad , \quad p_1(\lambda) = a - \lambda. \tag{8}$$

We review the very basic rules to solve linear difference equation problems. While the linear Ordinary Differential Equations (ODE) with constant coefficients can be solved by taking the Laplace transform on the equation to solve, the solution of linear difference equations with constant coefficients could be done taking the Z transform as we did in the previous section. This method provides solutions to the general equation that can be adjusted using the initial conditions. That is, for example, in the case of linear ODE with constant coefficients for the function $y = y(t)$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots a_0 y = 0. \tag{9}$$

we find (using for example the Laplace transform) solutions of the form $y(t) = \mathrm{e}^{st}$. For the case of linear difference equations with constant coefficients we have that

$$a_n y_n + a_{n-1} y_{n-1} + \cdots a_0 y_0 = 0. \tag{10}$$

can be solved by assuming solutions of the form $y_n = s^n$ (this can be justified using the Z transform). The substitution $y_n = s^n$ in the difference equation 10 produces

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0 = 0. \tag{11}$$

This equation is known as the characteristic equation associated with the linear difference equations 10. The same equation shows up in the solution of the ODE 9, with substitution of exponential solutions on the homogeneous difference equation 10. As done in ODE methods the general solution is a linear combination of all the solutions t^n obtained from the roots t of the characteristic equation 11. If there are no repeated roots the solutions are of the form

$$y_n = c_1 t_1^n + c_2 t_2^n + \cdots c_n t_n^n.$$

On the other hand, if there are repeated roots we use combinations of the form $n^j t_k$ where we assume that t_k is repeated (say m times) with $k = 0, 1, 2, \cdots, m-1$. For example, if t_4 is repeated 3 times the solutions associated with this root would be $t_4, nt_4, n^2 t_4, n^3 t_4$. The The undetermined coefficients c_i, $i = 1, 2, \cdots, n$ can be found using the initial conditions.

We apply this bit of theory to obtain the solution of our problem. Returning to equation 6 and replacing $p_i(\lambda)$ by t^i we find the characteristic equation

$$t^n - (a-\lambda)t^{n-1} - bct^{n-2} = t^{n-2}[t^2 - (a-\lambda)t - bc] = 0.$$

That is, we need to solve the quadratic equation

$$t^2 - (a-\lambda)t - bc = 0. \tag{12}$$

The two solutions are

$$t_\pm = \frac{(a-\lambda) \pm \sqrt{(a-\lambda)^2 - 4bc}}{2}. \tag{13}$$

The roots could be equal or different. We consider two cases;

(i) **No repeated roots:** The general solution $p_n(\lambda)$ can be written as

$$p_n(\lambda) = c_1 t_+^n + c_2 t_-^n. \tag{14}$$

From the initial conditions 8 we have

$$\begin{aligned} 1 &= c_1 + c_2 \\ a - \lambda &= c_1 t_+ + c_2 t_-. \end{aligned} \tag{15}$$

This system has solution if its determinant does not vanish. That is, if

$$\Delta = \det\begin{pmatrix} 1 & 1 \\ t_+ & t_- \end{pmatrix} = t_- - t_+ = -\sqrt{(a-\lambda)^2 - 4bc} \neq 0.$$

Please observe that this is consistent with the fact that $t_+ \neq t_-$. We have then that $(a-\lambda)^2 - 4bc \neq 0$. To solve system 15 we observe that $c_2 = 1 - c_1$ and

$$a - \lambda = c_1 t_+ + (1-c_1)t_- = c_1(t_+ - t_-) + t_-,$$

That is,

$$c_1 = \frac{(a-\lambda) - t_-}{t_+ - t_-} \quad , \quad c_2 = 1 - c_1 = \frac{t_+ - (a-\lambda)}{t_+ - t_-}, \tag{16}$$

so that the solution 14 is given by

$$p_n(\lambda) \quad = \quad t_+^n \frac{(a-\lambda) - t_-}{t_+ - t_-} + t_-^n \frac{t_+ - (a-\lambda)}{t_+ - t_-}. \tag{17}$$

Now, since the sum of the roots of equation 13 is given by $t_+ + t_- = (a-\lambda)$ we write 16

$$p_n(\lambda) = t_+^n \frac{t_+ + \cancel{t_-} - \cancel{t_-}}{t_+ - t_-} + t_-^n \frac{\cancel{t_+} - \cancel{t_+} - t_-}{t_+ - t_-} \quad = \quad \frac{t_+^{n+1} - t_-^{n+1}}{t_+ - t_-}.$$

This is the analytic solution of 6 with initial conditions 8 . Since we need to know λ such that $p_n(\lambda = 0)$ we have that the eigenvalues λ satisfy

$$t_+^{n+1} = t_-^{n+1}. \tag{18}$$

Note that equation $t_+^{n+1} - t_-^{n+1} = 0$ is of degree $n+1$ in λ since $t_+^{n+1} - t_-^{n+1} = (t_+ - t_-)p_n(\lambda)$. We introduced a new root to the original problem. This root satisfy the equation $t_+ - t_- = 0$. We are assuming $t_+ - t_- \neq 0$, λ can not be a root of $\Delta = 0$. Please observe that $\Delta = 0$ implies $(a-\lambda)^2 - 4bc = 0$. That is, the roots

$$\lambda = a \pm 2\sqrt{bc} \tag{19}$$

cannot be included in the set of solutions to the problem. Now,

$$\frac{t_+}{t_-} = \frac{t_+^2}{t_+ t_-} = \frac{t_+^2}{bc} = \left(\frac{t_+}{\sqrt{bc}}\right)^2 \tag{20}$$

where we used the product of the two roots of the quadratic equation 12 $t_+ t_- = bc$. We find, using 18

$$\left(\frac{t_+}{\sqrt{bc}}\right)^{2n+2} = 1.$$

The solution of this equation corresponds with $2n + 2$ roots located uniformly along the unit circle. These roots are represented by

$$\frac{(t_+)_k}{\sqrt{bc}} = \mathrm{e}^{\frac{k\pi \mathrm{i}}{2n+2}} \quad , \quad k = 0, 1, \cdots, 2n - 1.$$

so that $(t_+)_k = \sqrt{bc}\,\mathrm{e}^{k\pi\mathrm{i}/(2n+2)}$. From $t_+ t_- = bc$ we find $(t_-)_k = \sqrt{bc}\,\mathrm{e}^{-k\pi\mathrm{i}/(2n+2)}$ and from $\mathrm{e}^{\theta} + \mathrm{e}^{-\theta} = 2\cos\theta$,

$$(t_+)_k + (t_-)_k = 2\sqrt{bc}\cos\frac{k\pi}{2n+2}.$$

Now, since $(t_+)_k + (t_-)_k = a - \lambda$ we have that

$$a - \lambda_k = 2\sqrt{bc}\cos\frac{k\pi}{2n+2}$$

and

$$\lambda_k = a - 2\sqrt{bc}\cos\frac{k\pi}{2n+2} \quad , \quad k = 0, 1, \cdots, 2n + 1.$$

However $p_n(\lambda) = 0$ only has n roots and we found $2n + 2$. We already eliminated two roots 19, corresponding to $k = 0$ and $k = n + 1$. We need to exclude n other roots. If we observe equation 20 we see that the function t_+ was squared when substituting t_- for $\sqrt{cd}/t_+$. This introduced n new roots, corresponding to $k = 1, 3, 5, \cdots, 2n+1$, which are solutions of the equation $(t_+/\sqrt{bc})^{2n+2}$ instead of the corresponding equation $(t_+/\sqrt{bc})^{n+1}$ where t_+ is not squared. With this, the solutions λ_k are given by the set

$$\lambda_k = a - 2\sqrt{bc}\cos\frac{k\pi}{2n+2} \quad , \quad k = 2, 4, \cdots, 2n,$$

or

$$\lambda_k = a - 2\sqrt{bc}\cos\frac{k\pi}{n+1} \quad , \quad k = 1, 2, \cdots, n.$$

(ii) **Repeated solutions:**

We have $t_+ = t_- = (a - \lambda)/2$. That is, the quadratic equation is a perfect square. The recursive equation 6 has a solution of the form

$$p_n(\lambda) = c_1 t_+^n + c_2 n t_+^n.$$

Again, to find c_1 y c_2 we need to apply the initial conditions to find a two-by-two linear system of equations.

$$\begin{aligned} p_0(\lambda) &= 1 = c_1 \\ p_1(\lambda) &= a - \lambda = (c_1 + c_2)t_+ \end{aligned}$$

and so

$$c_2 = \frac{a - \lambda}{t_+} - 1 = 1.$$

Then

$$p_n(\lambda) = t_+^n + n t_+^n = \left(\frac{a - \lambda}{2}\right)^n (1 + n).$$

From here we find that all eigenvalues are equal.

Recall that $\Delta = 0$ means $\lambda = a \pm 2\sqrt{bc}$, and given that $\lambda = a$ we have that $a = 0$ or $b = 0$. That is, the original matrix is lower triangular or upper triangular where all its eigenvalues are sitting along the diagonal.