Notes on Geometric Algebra

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June 21, 2018

Contents

Chapter 1

Introduction

These notes show an introductory description and serve as motivation for the subject of Geometric Algebra (GA). The Internet has plenty of good material in the subject. Wikipedia [1] is always a good source and it is my most cited website in these notes. However, I found the notes [2] by Eric Chilsom with an excellent treatment for the topic. Perhaps the most cited reference on the topic is the book *New Foundations for Classical Mechanics* by David Hestenes [4]. Hestenes, proposes GA as the preferred mathematical framework for physics. According to Wikipedia [3], David Hestenes is "the chief architect of geometric algebra as a unified language for mathematics and physics". Hestenes book provides, in its chapter 1, a great historical overview of the topic. Next I include a summary of Hestenes work, for those who do not want or cannot read his 30 pages of history.

Starting from the Greeks (where Euclid was a center figure) and their concept of geometry as measurements, Hestenes describes how the real line was filled starting with the natural numbers, 0 [4] attributed to India, fractions, which according to Hestenes "Diophantes (240 AD), the last of the great Greek mathematicians, was probably the fist to regard fractions as numbers", until finding numbers such as $\sqrt{2}$ which did not fit any of the previous types. He describes how numbers like $\sqrt{2}$, have issues while trying to measure a diagonal of a square using its side as a unit. The problem was numerically a mystery but geometrically was understood. In addition Hestenes illustrates how Greeks knew about multiplication using pure geometry. Jumping a few centuries, Descartes emphasized the mapping between a segment and a real number (the magnitude). Hestenes shows nice figures (from Euclid's Elements book) which explains how to compute geometrically the square root of any number, and indicated that Descartes had trouble with oriented segments (know today as vectors, including negative segments). Descartes was perhaps the center figure on mixing geometry with algebra and the inventor of analytic geometry, where

[1] http://en.wikipedia.org/wiki/Geometric_algebra
[2] http://arxiv.org/abs/1205.5935
[3] http://en.wikipedia.org/wiki/David_Hestenes
[4] http://en.wikipedia.org/wiki/0_(number)

algebraic equations could be mapped to geometry figures (the plot of a function). Numbers like $\sqrt{2}$ were added to the real line as *algebraic numbers* since they are solutions of algebraic equations such as $x^2 = 2$. However algebraic numbers did not fill the real line and Descartes did not know that. The finding on new numbers not yet classified did not happen until later. According to Wikipedia [5] "Leibniz introduced the name *transcendental"* in 1682, Euler was probably the first person to define transcendental number in the modern sense". In 1882, the German mathematician Ferdinand von Lindemann [6] proved that π is transcendental. A new type of number which is not algebraic, or is not a root of a non–zero polynomial equation with rational coefficients. Another transcendental number is the Euler number e (Euler and Lagrange conjectured that e was transcendental). Today it is accepted that the real line can be put in a one–to–one correspondence the real the union of *rational* (fractions) and *irrational* (the complement of the rational numbers) numbers, the *real* numbers which complete the continuum real line.

According to Hestenes, Hermann Grassman, in his book of 1844, developed the idea a *"directed number"* , "with precision and completeness that far surpassed the work of anyone else at the time". While for Descartes, two segments were congruent if their length matched, regardless the direction, for Grassman two segments were *"equivalent"* only if one is different from the other by its position. That is, we can take one into the other with a simple translation. The Cartesian segments were scalars, the Grassman segments where *"vectors"* . In this way a "directed line segment" is a *vector* . With this Grassman provides a new dimension to algebra and defined the algebraic operations on vectors that we know today. That is, sum, difference, product by scalar, and inner product. However the vector machinery was insufficient to provide full expression for geometrical ideas. The matrix algebra, tensor algebra, and other algebras were developed among others. For Hestenes this was not a proof of the richness of mathematical ideas but at the contrary a source of confusion. In his words "confusion about the *aims and principles of geometric algebra"* . He claims that GA is the natural continuation of the ideas of vector algebra which "facilitates a simple expression of the full range of geometical ideas". What is a product of vectors? An inner product is good for projections but it looses many things. It collapses whatever dimension the vectors have to 0, that is, to a scalar. After inner product there is no hint of which vectors originated it. It is well defined in one direction but very ambiguous in the other. Grassman introduced the outer (wedge $\wedge$) product as an alternative where it represents the area of the vectors being multiplied. In this sense it is an extension of the multiplication used by Euclid where he considered a rectangle as the representation of the multiplication ab of this sides. As a vector an outer product can be seen as an object that carries some information about the space where its members live (the spanning plane), and their size (the area). Additionally there is an orientation (positive or negative) in the sense that $a \wedge b = -b \wedge a$ (that is,

[5] http://en.wikipedia.org/wiki/Transcendental_number
[6] http://en.wikipedia.org/wiki/Ferdinand_von_Lindemann

the wedge product is anti–commutative). While the inner product is symmetric, the outer product is antisymmetric and it is the sum of this objects which will define the geometric product $ab = a \cdot b + a \wedge b$. While the inner product is an easy test for orthogonality ($a \cdot b = 0$, if and only if $a \perp b$), the outer product is an easy test for parallelism ($a \wedge b = 0$ if and only if $a \parallel b$).

In spite of comprehensive historical introduction to GA in Hestenes' book, he did not give credits to Clifford. [7] Geometric algebra is a branch of a more general mathematical topic know as Clifford Algebra [8], but where the scalars are real and it makes emphasis on its geometric interpretation and physical applications. According to Wikipedia [9] "A geometric algebra (GA) is the Clifford algebra of a vector space over the field of real numbers endowed with a quadratic form." It was Clifford who first use the name "Geometric Algebra" (GA). In this context I believe that Hestenes' historical introduction lacks an important part occurring during the second part of the nineteen century.

In addition to Hestenes' historical introduction, there are many online references, including the Wikipedia [10] site with good history data on GA. Chris J. L. Doran's [11] Ph.D. thesis offers another good historical introduction to GA.

Aragón et. al. [12] claim that the importance of of Clifford algebra was first recoginized in quantum field theory. We include some of the references from Aragón et. al. work about the applications of Clifford Algebra. On projective geometry [5], on electrodynamics [1], on manifolds and differential geometry [6], Hestenes book [4] and also its article David Hestenes. [13] also shows applications of Clifford Algebra to crystallography. . Finally, applications of Clifford algebra in robotics, neural and quantum computing, computing vision, computer graphics, engineering, etc., can be found in [2].

There is a good number of software tools to compute GA operations. Here is a short list of tools for that purpose:

This website [14] shows a list of packages that could be used to perform GA computations. Observe that, among others, Maxima, [15] Matlab, [16] and Maple, [17] allow packages which extend their functionality to GA. The software Mathematica [18] also allows for its use with GA. The document [19] by Aragón et. al., illustrates the use

[7] http://en.wikipedia.org/wiki/William_Kingdon_Clifford
[8] http://en.wikipedia.org/wiki/Clifford_algebra
[9] http://en.wikipedia.org/wiki/Geometric_algebra
[10] http://en.wikipedia.org/wiki/Geometric_algebra#History
[11] http://cosmologist.info/thesis.ps.gz
[12] https://arxiv.org/pdf/0810.2412.pdf
[13] http://geocalc.clas.asu.edu/pdf-preAdobe8/crystalsymmetry.pdf
[14] https://ga-explorer.netlify.com/index.php/ga-software/
[15] http://maxima.sourceforge.net/
[16] https://www.mathworks.com/
[17] https://www.maplesoft.com/
[18] https://www.wolfram.com/mathematica/
[19] https://arxiv.org/pdf/0810.2412.pdf

of Mathematica to solve GA problems. Finally Python supports several packages of GA. pymbolic [20] is a package to do symbolic mathematics including GA. Another package which supports GA is Sympy [21] which deals with symbolic Python.

[20] https://documen.tician.de/pymbolic/index.html
[21] https://docs.sympy.org/0.6.7/modules/galgebra/GA/GAsympy.html

Chapter 2

Why should we learn GA?

2.1 Introduction

Let me include a few of my personal views on the subject. Some of these points might have regions of overlap. For example symmetry and geometry have things in common. Symmetry can be explained algebraically as well as geometrically. After all, we are discussing *Geometric Algebra.*

I show next a list of items (in capital boldface) which highlights advantages of GA with respect to the traditional linear algebra. When the item in the list is long I will separate it into a section.

- I always said that we should not add apples to bananas. Here is an exception. The Geometric algebra mixes scalars, vectors, and tensors in general into a single element. Here for example, 0 means 0 as scalar, as a vector, or as a tensor. How can that be? What is the point?

  The point is **UNIFICATION.** The linear algebra is just a particular case of the Clifford algebra, when all the elements belong to the same space. In linear algebra $\boldsymbol{a} + \boldsymbol{b}$ can not be added if $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{b} \in \mathbb{R}^m$, for $m \neq n$. This is no necessarily the case in geometric algebra. Although this seems to go against the principles of adding equal types ("apples with apples"), there is plenty of evidence of the use of GA on the solution of problems involving geometry, algebra, physics, etc.

- **SIMPLIFICATION** Table 2.1 shows a few formulas that simplify some operations that are of common use in linear algebra but that look simplified in GA. Although in writing the rotation (last row) seems more simplified in linear algebra than in GA, the computation in linear algebra involves a matrix vector multiplication where the matrix is a $n \times n$ matrix, while in GA is the geometric product of three vectors. The reader should not worry about understanding all the symbols in this table since we will develop them on this chapter.

Table 2.1: Operations in linear algebra that are simplified in GA. In the last row R is a rotation matrix of dimension $n \times n$, and R is a vector (rotor) of dimension n.

name	**Linear Algebra**	**Geom. Algebra**
Orthogonal Projection	$P_n(u) = \frac{u \cdot n}{\lvert u\rvert^2} u$	$P_u(n) = (u.n)u^{-1}$
Orthogonal Rejection along u	$P_n^{\perp}(u) = \frac{u \cdot n^{\perp}}{\lvert u\rvert^2} u$	$P_n^{\perp}(u) = (u \wedge n)u^{-1}$
Reflection along u (Householder transformation)	$H_n(u) = u - \frac{2(nn^T)u}{\lvert n\rvert^2}$	$H_n(n) = -(un)u^{-1}$
Reflection across u	$T_n(u) = \frac{u}{\lvert u\rvert^2} u \cdot (n - n^{\perp})$	$T_n(u) = (un)u^{-1}$
Rotation u	$T_\theta(u) = \mathrm{R}u$	$T_\theta(u) = RuR^{-1}$

The name *orthogonal rejection* is taken from Eric Chilsom notes, [1] where he challenges the reader about the chance to be familiar with it. However, David Hestenes, in his New Foundations For Classical Mechanics [4] quotes “The new term 'rejection' has been introduced here in the absence of a satisfactory standard name for this important concept”. In linear algebra this is a projection along the normal to u (noted here as $u^{\perp}$). However, we should be aware that there is not a unique normal to a line in a 3D space and in general to a hyper–plane embedded in a higher dimensional space. So, $u^{\perp}$ is chosen so that $u = P_n(u) + P_n^{\perp}(u)$, with $P_n(u)$ the “horizontal” component along n and $P^{\perp}(u)$ the “vertical” component along n. So here n is an arbitrary direction orthogonal to u, but once this direction is fixed, then $P_n^{\perp}$ is well defined.

We provide further explanation about the properties described in Table 2.1 below starting in page 16.

- **SYMMETRY** There is a great deal of symmetry in the GA, and this is part of the reason why it is appropriate for the analysis in crystal symmetries, as shown by David Hestenes. [2]

[1] http://arxiv.org/abs/1205.5935
[2] http://geocalc.clas.asu.edu/pdf-preAdobe8/crystalsymmetry.pdf

– Let me start with the formulas in Table 2.1 we have

$(u \cdot n)u^{-1}$	Orthogonal projection
$(u \wedge n)u^{-1}$	Orthogonal rejection
$-(un)u^{-1}$	Householder Reflection.
$(un)u^{-1}$	Reflection Across Plane.
RuR^{-1}	Rotation an angle θ

The last property (Rotation) needs to be explained further and this will be done after equation 2.2.3 where R is a *rotor* rotating u by $\theta/2$ counterclockwise.

Since the GA is associative, I can afford to remove the parenthesis to unveil the symmetry on these equations. We can have different interpretations with or without parenthesis:

* Without parenthesis: They are all quadratic equations in u, written in the same way as similarity transformation are written for linear algebra. However here a dot $(\cdot)$ means dot (*inner*) product, a wedge $(\wedge)$ means *wedge* product, and no symbol in between is the *geometric* product of elements of the Clifford algebra.
* With parenthesis: All, but the last, are operations taken in u^{-1}. Projection is a dot ".", orthogonal rejection is a wedge $\wedge$, Householder reflection is a negative geometric product, and reflection across a plane is the positive geometric product. This representation is the same regardless as if the vector n is unit or not. If the vector is a unit vector then $u^{-1} = u$ and we can remove the negative power everywhere.

The symmetry of the reflection formula is clear. The minus "-" sign comes from the reflection as a switch to the other side. The vectors u and u^{-1} are in charge of performing the reflection so the reflected vector ends in the correct location.

The orthogonal projection is to the dot product sign "$\cdot$", as the orthogonal rejection is to the wedge sign "$\wedge$".

– The definition of *geometric* product of two vectors u, v is motivated by the following identity, which comes after understanding that the rules for multiplication, distribution, and addition of vectors are satisfied:

$$uv = \frac{1}{2}(uv + vu) + \frac{1}{2}(uv - vu).$$

The first expression is symmetric while the second is antisymmetric. The expression of a product as the sum of symmetric and antisymmetric components looks like the decomposition of an operator into its symmetric and antisymmetric forms. The formula displays lots of symmetry. The first part of the sum is an inner product and the second is an outer product. So we have that the geometric product is the sum of an inner and an outer product. That is the inner product is defined as

$$u \cdot v := \frac{1}{2}(uv + vu)$$

while the outer (wedge) product is defined as

$$u \wedge v := \frac{1}{2}(uv - vu)$$

so

$$uv = u \cdot v + u \wedge v$$

These can be used as the starting point to introduce all product definitions in GA.

The geometric product of two vectors u and v is the sum of a scalar $u \cdot v = |u||v| \cos\theta$ (this is not yet proven) plus the sum of a vector $u \wedge v = |u||v| \sin\theta$. [3] This representation is due to Grassman [3].

Note the symmetry here. In the geometric product uv, of two vectors, the scalar component $u \cdot v = 0$ means perpendicularity, while the vector component $u \wedge v = 0$ means parallelism. One carries a $\sin\theta$ in its magnitude and the other a $\cos\theta$. If two vectors are orthogonal then the geometric product is pure wedge product $uv = u \wedge v$. If two vectors are parallel the geometric product is pure dot product, that is $uv = u \cdot v$. See that this has similarities with the study of normal and shear forces in a body. In wave propagation, the divergence (which is a dot product between an operator and a vector field) measures compressional forces while the curl (which is a wedge product of an operator with a vector field) measures shear forces. This makes sense from the geometric point of view since tangent is associated with shear and normal with compressional. The Helmholtz decomposition theorem establishes that under certain conditions a vector field is the sum of a vector potential and a scalar potential. The Helmholtz decomposition theorem also resembles the representation of the geometric product as the contribution of two operators, one measuring parallelism and the other measuring perpendicularity. In physics is common to decompose a vector as the sum of two components, one parallel to a base vector and the other perpendicular to it. This also resembles the definition of geometric product. A dot product projects a vector into another (up to a scale), while a wedge product extends the vector. Figure 2.1 illustrates the meaning of the geometric product as a signed projection plus a signed area bivector.

[3] This is quite not right but almost true. That is, the magnitud is the area of the parallelogram which indeed is $|u||v| \sin\theta$. We will be the exact meaning of $w \wedge v$ later on the document.

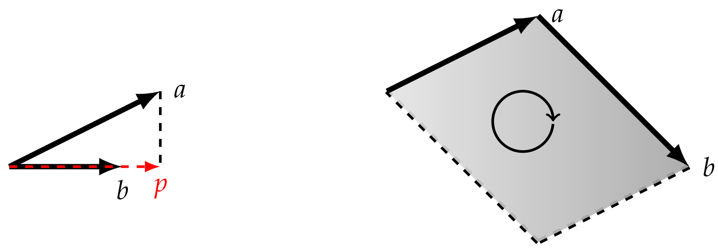

Figure 2.1: Given two vectors a and b, on the left the red vector p, is the projection $(a.b)b$ of the vector a into the unit vector b. On the right, the vector b is shifted by a away from the origin (so it really is $a+b$) the outer product $a \wedge b$ corresponds to the area of the parallelogram formed by the two vectors. The orientation is as shown in the picture (the clockwise orientation). The product ab is a sum of (i) the size of the projection p, with the sign of the cosine of the angle formed by the vectors a and b, and (ii) the area of the parallelogram formed by the vectors a and b, with the clockwise orientation.

The bivector $a \wedge b$ is not uniquely interpreted by the parallelogram in Figure 2.1. For example, any product $a \wedge (b + \lambda a)$ where λ is a scalar also produces the same value as $a \wedge b$. Figure 2.2 illustrates this for $\lambda = 0, 1, 2$.

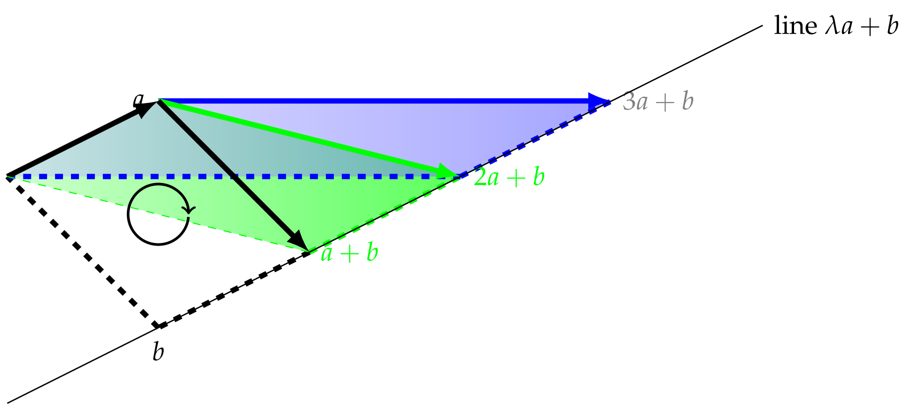

Figure 2.2: Given two vectors a and b, the bivectors $a \wedge (b + \lambda a)$ are represented by the three parallelograms as their area with the clock–wise orientation shown in the picture. The white parallelogram corresponds to $a \wedge b$, that is $\lambda = 0$, the green parallelogram corresponds to $a \wedge (a + b)$, that is $\lambda = 1$, and the blue parallelogram corresponds to $a \wedge (2a + b)$, that is $\lambda = 2$. All three are in the same plane, have the same area and orientation. See that for all three the base is $|a|$ and the height is the same, as the parallelogram sizes increases the angle becomes larger (and in the example larger than π), so the sine function gets reduced. Hence the area is constant. Even if $\lambda < 0$, the orientation, area, and plane of residence would be preserved. The figure would looked skewed in the other direction. In general for any point p in the line $b + \lambda a$, we have that $a \wedge p = a \wedge b$.

We see an ambiguity in the geometrical representation of the wedge product. However, if we think of it as an object on the plane spanned by the vectors a and b, with the size given by the area of the parallelogram and the sign given by the orientation, that representation would be unique. More generally, the shape of the object is not important. Three things are important here:

1. An *attitude* [4] : The subspace being represented (here the plane).
2. A *weight* the measure of the object: length,area, volume, etc.
3. An *orientation* positive or negative, forward or backward, counterclockwise or clockwise etc. In general plus or minus, since only two orientations are possible.

In this way, for 2D, all *bivectors* (vectors of the form $u \wedge v$) can be represented with circles with a given orientation, although the most common representation is that of a parallelogram determined by the vectors u and v, since at least there is a hint of where the components u and v might come from. On the other hand, from a circle such a hint is completely blurred. I would say that the representation of a bivector could be seen as a square, where the first vector is one side of the square. The second vector is good up to a point on the line which is separated from the first vector along the normal by the size of each side of the square. A line such the line $\lambda a + b$ in Figure 2.2. This object represents the bivector which satisfies the three properties above (attitude, weight, orientation), but even more, one of the vectors is in the object, and the other can never be determined unless it is provided. The figure (a square) has the largest symmetry and artistically should be the best candidate to represent a bivector. For 3D the candidate should be a cube and for higher dimensions hypercubes. In general, the three attributes are, I believe, the best way to uniquely characterize a geometric algebra object [5]. A vector has the attitude of the line it is sitting, the weight equal to its magnitude and the sign with respect to a provided orientation. A hypercube its sitting in a certain high dimensional space, it has some hyper–volume and some orientation (given by the order of the vectors that conform it).

The inner product reduces the *grade* (dimension of the object, rank of the tensor) by one from r to $r-1$, while the outer (wedge) product increases

[4] The word *attitude* is used by Eric Chilsom. Hestenes uses the word *direction,* instead. I believe direction is not appropriate after 3 dimensions. In 4 dimensions the bivector expands a plane, and 2 (no 1) dimensions are free of that plane, so there is another plane which is orthogonal to that plane. Is this second plane the *direction* of the first? what is it? To me direction implies a 1D object (a line). For example, the plane spanned by e_1, e_2, is orthogonal to the plane spanned by e_3, e_4. So, is this second plane the *direction* of the first? In this sense (I believe) Chilsom refers to *attitude* instead of direction.

[5] Actually we will see ahead that the objects that have these three atributes are known as blades. A general object in geometrical algebra could be a linear combination of blades and might not have the three attributes described above

the grade (dimension of the object, rank of the tensor) by one from r to $r+1$.
Chilsom [6] defines left and the right inner products of vector a with a r–vector (r–blade [7]) as

$$\begin{aligned} a \rfloor \boldsymbol{A_r} &:= \frac{1}{2}\left[a\boldsymbol{A_r} - (-1)^r \boldsymbol{A_r} a\right], \quad \text{left inner product} \\ \boldsymbol{A_r} \lfloor a &:= \frac{1}{2}\left[\boldsymbol{A_r} a - (-1)^r a \boldsymbol{A_r}\right], \quad \text{right inner product} \end{aligned} \tag{2.1}$$

I will discuss blade in detail later on.
Along the same lines Chilsom [8] defines the left and right outer products as

$$\begin{aligned} a \wedge \boldsymbol{A_r} &:= \frac{1}{2}\left[a\boldsymbol{A_r} + (-1)^r \boldsymbol{A_r} a\right], \quad \text{left outer product} \\ \boldsymbol{A_r} \wedge a &:= \frac{1}{2}\left[\boldsymbol{A_r} a + (-1)^r a \boldsymbol{A_r}\right], \quad \text{right outer product} \end{aligned}$$

Note that

$$\boldsymbol{A_r} \wedge a = (-1)^r a \wedge \boldsymbol{A_r}.$$

We will go back to these concepts later on the text with concise definitions.

2.2 Geometrical Insights, Connection With Complex Analysis

I introduce a few attributes which show the relationship between GA, analytic geometry, and complex analysis.

2.2.1 Building Blocks

Even scalars have three attributes: The space they leave (let us say $\mathbb{R}$), magnitude (absolute value) and direction (sign). Vectors have the same property, but how about tensors (such as for example matrices)? What is the direction of a matrix? What is the magnitude of a matrix?, what is the sign of a matrix?

Here is how geometric algebra builds its objects. Linear algebra (vector algebra) knows only scalars and vectors (and matrices). These two are the building blocks (cells) for geometric algebra. New objects are formed by the use of the outer (wedge)

[6] http://arxiv.org/abs/1205.5935
[7] Do not worry if you are not familiar with the concept of *blade* . I will get back to it in detail later.
[8] http://arxiv.org/abs/1205.5935

product. The first is a *bivector* . This is a vector of the form $w = u \wedge v$, as explained in figures 2.1 and 2.2. A *trivector* is of the form $u \wedge v \wedge w$, which can be seen as a parallelepiped defined by the three vectors as its sides. In general a product of n vectors $u_1 \wedge u_2 \cdots \wedge u_n$ for any natural number n is called a *blade* of grade r. We note this blade with the bold face symbol $\boldsymbol{A_r}$. A linear combination of blades with the same grade r is call an $r - vector$ and we use the notation like A_r for an r-vector (capital letter with a subindex indicating the grade) . A linear combination of blades and a scalar is called a *multivector* . We note multivectors with capital letters. A multivector is the most general object in a GA.

2.2.2 Square and Inverse of an object

The square of a vector is its square length. That is

$$uu = u \cdot u = |u|^2$$

(note that since u is parallel to u, then its wedge product is 0). So since

$$uu^{-1} = 1, \qquad \text{and} \qquad \frac{u^2}{u^2} = 1,$$

then

$$uu^{-1} = \frac{u^2}{u^2} = u\frac{u}{u^2} \Rightarrow u^{-1} = \frac{u}{u^2}$$

(if $u = 0$, then it does not have an inverse, so we assume $u \neq 0$)

So, in GA we can have that the inverse of a vector is another vector

$$u^{-1} = \frac{u}{u^2}.$$

This is a useful equation and shows one more potential of Geometric Algebra to extend functionality beyond regular linear algebra where there is not such a thing as the inverse of a vector.

We show a result which is used often. If two vectors a and b are orthogonal then $a \wedge b = ab$. Clearly $ab = a \cdot b + a \wedge b$, and since a is orthogonal to b then $a \cdot b = 0$, so $ab = a \wedge b$. In parcitular $e_i \wedge e_j = e_i e_j$ for $e_i \neq e_j$ base vectors. for $e_i \neq e_j$ base vectors. for $e_i \neq e_j$ base vectors. for $e_i \neq e_j$ base vectors.

2.2.3 Decomposition of a vector into is orthogonal projection and orthogonal rejection

We use the decomposition of a vector v as a component along a given vector n and a component along a normal to that given vector $v \wedge n$. In other words, an orthogonal projection and an orthogonal rejection.

$$\begin{aligned} v &= v(nn^{-1}) \\ &= (vn)n^{-1} \\ &= (v \cdot n)n^{-1} + (v \wedge n)n^{-1} \end{aligned} \tag{2.2}$$

We observe the orthogonal projection (as in Table 2.1) on the first term, and an orthogonal rejection (also in Table 2.1) on the second term. So a vector can be written as a decomposition along a direction space λu and its orthogonal complement.

One point, that we will highlight below, is that GA is *coordinate free.* This is not entirely true. At the end any implementation needs computations and those computations should be done in a reference frame. For example if we know a 2D vector from its magnitude and its direction we are using polar coordinates. If we use it by its Cartesian coordinates we are using Cartesian coordinates. So, *coordinate free* means that most (actually all) of the algebraic operations can be done without referring to any coordinate, however at the end, if we need numerical computations we should use some coordinates. It is hard to find (and I have not found one yet) a book or article in the internet where GA is explained with numerical examples. This make it harder to follow for those who like to touch the ground with mundane computations. I will try hard to make numerical examples for each concept used in this book. Let us start with the regular orthogonal projection.

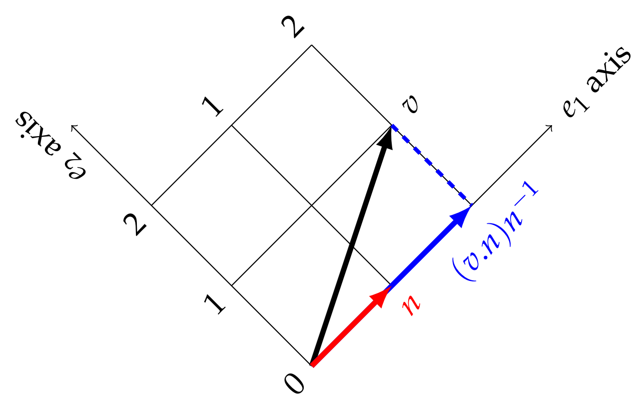

Figure 2.3: Given a vector v and a direction n (red), the projection $(v \cdot n)n^{-1} = 2e_1 = (2,0)$ (in blue) is found.

Example 2.2.1. Let us assume our coordinate axis tilted 45 degrees with respect to the horizontal text lines in this document. The coordinate axes are e_1 and e_2 as indicated in the figure. For the direction we pick $n = (1,0)$. This means $n = 1e_1 + 0e_2 = e_1$. Let us define $v = (2,1) = 2e_1 + 1e_2$. The vector $n^{-1} = (1,0) = e_1$, and so the orthogonal projection $v_{\|}$ [9] component is found by using the distributive property of

[9] we use the symbol $v_{\|}$ to indicate that the component is along (parallel) de vector v

the dot product.

$$v_{\|} = (v \cdot n)n^{-1} = [2e_1 \cdot 1e_1 + 1e_2 \cdot 1e_1]e_1 = 2e_1$$

where the term $1e_2 \cdot 1e_1 = 0$. From now on we will compute the dot product using the traditional $\sum a_i b_i$ formula where a_i are the components of the vector a and b_i the components of the vector b.

Let us now find the orthogonal rejection. That is, the component of v perpendicular to the vector n. This is given by the formula $(v \wedge n)n^{-1}$. In general, for a 2D vector $v = (v_1, v_2)$ and a direction $n = (n_1, n_2)$, and using the distributive, anticommutative, and the zero products of parallel vector properties, of the wedge product we find

$$\begin{aligned}
(v_1, v_2) \wedge (n_1, n_2) &= (v_1 e_1 + v_2 e_2) \wedge (n_1 e_1 + n_2 e_2) \\
&= v_1 n_1 \cancel{e_1 \wedge e_1} + v_1 n_2 e_1 \wedge e_2 + v_2 n_1 e_2 \wedge e_1 + \cancel{v_2 n_2 e_2 \wedge e_2} \\
&= v_1 n_2 e_1 \wedge e_2 - v_2 n_1 e_1 \wedge e_2 \\
&= (v_1 n_2 - v_2 n_1) e_1 \wedge e_2 \\
&= \det \begin{pmatrix} v_1 & v_2 \\ n_1 & n_2 \end{pmatrix} e_1 \wedge e_2. \qquad (2.3)
\end{aligned}$$

From now on, the magnitude of the wedge product in 2D is computed by using determinants.

In our numerical example $v = (2, 1)$ and $n = (1, 0)$, so $n^{-1} = (1, 0)$ and

$$(v \wedge n)n^{-1} = \det \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} (e_1 \wedge e_2)n = -(e_1 \wedge e_2)n.$$

We remark that since e_1 and e_2 are orthogonal then

$$e_1 e_2 = e_1 \cdot \mathbf{e_2} + e_1 \wedge e_2 = e_1 \wedge e_2.$$

Now since $n = (1, 0) = e_1$ we find that

$$(v \wedge n)n^{-1} = -(e_1 e_2)e_1 = -e_1(e_2 e_1) = e_1(e_1 e_2) = (e_1 e_1)e_2 = e_2.$$

That is $(v \wedge n)n^{-1} = (1, 0)$.

At the moment we know that $e_1 \wedge e_2$ is a blade and it represents the area 1 on the plane spanned by e_1 and e_2. However, from equation 2.2 we see that

$$(v \wedge n)n^{-1} = v - (v \cdot n)n^{-1} = v - v_{\perp},$$

and so we can interpret $(v \wedge n)n^{-1}$ as the orthogonal component (orthogonal rejection), that is

$$(v \wedge n)n^{-1} = v_{\perp} \quad , \quad v = v_{\|} + v_{\perp}.$$

Note the notation $v_{\perp}$ to indicate that the component is along the perpendicular direction to v. It is in this sense that Hestenes calls the expression $(v \wedge n)n^{-1}$ as the orthogonal rejection, and that is the interpretation that we provide to that expression from now on.

If $(1,0)(e_1 \wedge e_2) = (0,1)$, then we speculate that the blade $e_1 \wedge e_2$ is a counterclockwise 90 degree rotation operator, when used at the right of the vector $(0,1)$ and it would be a clockwise 90 degree rotation when used at the left of the vector $(1,0)$. That is, $e_1 \wedge e_2(1,0) = (-1,0)$. If we think that $\mathbb{R}^2$ is the set of complex coordinates, the blade $e_1 \wedge e_2$ on the right of a vector corresponds to the imaginary number $\mathrm{i} = \sqrt{-1}$.

Let us see this differently

$$(e_1e_2)^2 = e_1e_2e_1e_2 = -e_1e_1e_2e_2 = -1$$

So, if there is a square root of an element of a geometric algebra, we could make the association

$$(e_1e_2) = \pm\sqrt{-1} = \pm \mathrm{i}.$$

and think that $e_1 \wedge e_2$ represents the pure $\pm\mathrm{i} = \pm \mathrm{e}^{\mathrm{i}\pi/2}$. Which sign do we pick? To understand this let us assume that our base is $1, e_1 = (1,0), e_2 = (0,1), e_1e_2$. Here $e_1e_2 = \mathrm{i}$ and the system is right handed. If the base is, instead $1, e_1 = (0,1), e_2 = (1,0), e_1e_2$, then $e_1e_2 = -\mathrm{i}$ since the system is left handed. We can say that e_1e_2 together with 1 span the complex numbers and since any multiple of 1 is a scalar it commutes with $e_1e_2 = \mathrm{i}$ (or $e_1e_2 = -\mathrm{i}$). From now on we assume right handed systems where $e_1e_2 = \mathrm{i}$. In any case this wedge product applied to any complex number will rotate its phase by 90 degrees counterclockwise.

With this, the geometric algebra in 2D has the four *base vectors* (blades)

$$1 \quad , \quad e_1 \quad , \quad e_2 \quad , \quad e_1e_2.$$

The first base vector is pure stretching, the last vector (called also *pseudo–scalar* and can be written as $e_1 \wedge e_2 = e_1e_2 = e_{12} = \mathrm{i} = \mathrm{I}$. The I notation is chosen for dimensions bigger than 3. This last blade is pure rotation, the other two vectors define fix directions. We can see how the first and last base objects can generate the complex numbers spanned by the first and last base vectors. That is $\mathbb{C} = \mathrm{span}(1, e_{12})$. You can check that , in this context, the geometric product corresponds with the product of two complex numbers. We also interpret below that the product of two vectors u and v in $\mathbb{R}^2$ behave as the product of two complex numbers u and v in $\mathbb{C}$.

Figure 2.4 illustrates (the decomposition of a vector v into its parallel and orthogonal components.

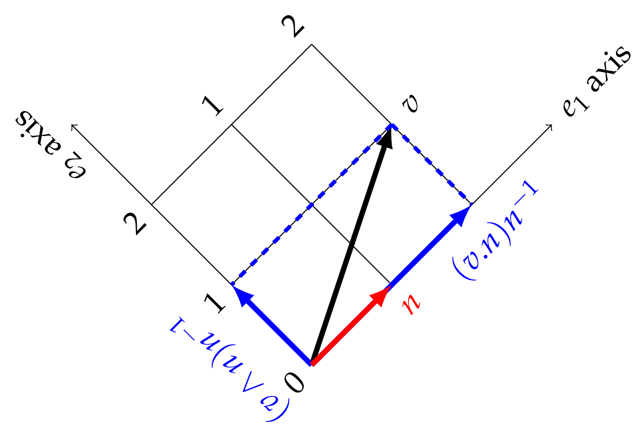

Figure 2.4: Given a vector v and a direction n (red), the vector can be decomposed in a $v = v_{\parallel} + v_{\perp}$, with the component along v being $v_{\parallel} = (v \cdot n)n^{-1} = 2e_1 = (2, 0)$ and the orthogonal rejection (normal to v) being $v_{\perp} = (v \wedge n)n^{-1} = \mathrm{e}_2 = (0, 1)$.

2.2.4 Reflections

Starting with the representation 2.2

$$v = (v \cdot n)n^{-1} + (v \wedge n)n^{-1}$$

we construct the following reflections.

A reflection along the vector n is built by reversing the sign of the orthogonal projection. That is

$$\begin{aligned} v' &= -(v \cdot n)n^{-1} + (v \wedge n)n^{-1} \\ &= -(n \cdot v)n^{-1} - (n \wedge v)n^{-1} \\ &= -(n \cdot v + n \wedge v)n^{-1} \\ &= -nvn^{-1}. \end{aligned} \tag{2.4}$$

This is the Householder reflection shown in the third row in Table 2.1

On the other hand, reversing the sign of the orthogonal rejection $(v \wedge n)n^{-1}$ we find

$$\begin{aligned} v' &= (v \cdot n)n^{-1} - (v \wedge n)n^{-1} \\ &= (n \cdot v)n^{-1} + (n \wedge v)n^{-1} \\ &= (n \cdot v + n \wedge v)n^{-1} \\ &= (nv)n^{-1} \end{aligned}$$

This is the reflection shown in the fourth row in Table 2.1

Figure 2.5 shows the results of the Householder transformation (reflection) and a reflection with respect to a plane with no coordinates. Note that the sign of the two reflections (blue vectors) is opposite as predicted from the computations above.

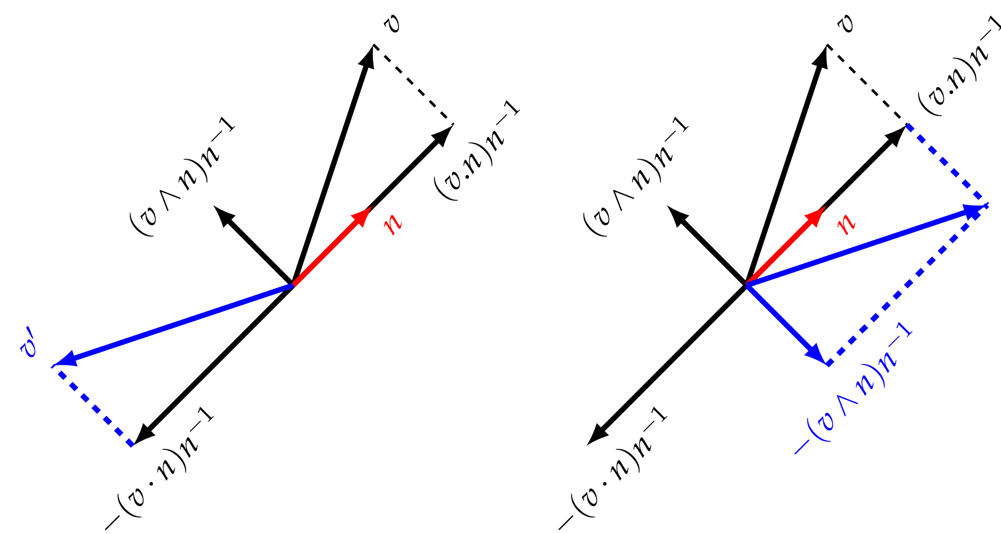

Figure 2.5: Given a vector v and a direction n (red), the vector v can be written as $(v \cdot n)n^{-1} + (v \wedge n)n^{-1}$, the first component is along the direction vector n, while the second is along the orthogonal (in the plane spanned by v and n) to that direction. A reflection along the direction n, (left frame) can be constructed by reversing the sign of the projection; that is, $v' = -(n \cdot v)n^{-1} - (n \wedge v)n^{-1} = -nvn^{-1}$. Similarly a reflection across the plane normal to $(v \wedge n)n^{-1}$, (right frame) can be constructed by reversing that component. That is, $v' = (v \cdot n)n^{-1} - (v \wedge n)n^{-1} = nvn^{-1}$.

The first approach in equation 2.4, is the *Householder* transformation shown in Table 2.1. Householder transformations have several advantages. For example, there is no ambiguity about which reflection we want to find. If we are in a 3D space, then reflection along the line as a Householder transformation is the same as reflection across the plane perpendicular to the vector n . However, in more than 3 dimensions there is no a unique plane normal to n, and still there is unique reflection along the line with direction n. Therefore Householder reflections are preferred. In addition a rotation can be built as a sequence of two cascaded Householder reflections.

2.2.5 Rotations

Let us assume we have a vector v and we want to rotate it by an angle θ in a given plane. Find two vectors n and m in that plane, which make an angle of $\theta/2$ between them and are in the same side of the rotation we want to perform. Let us assume that the first vector makes an angle ϕ smaller than $\theta/2$ with v. By doing two consecutive

Householder transformations (reflections) with respect to n and then with respect to m we position the vector v into a vector v'' which makes an angle θ with v and is on the same plane having v, m and n. We start with the vector v and perform a Householder transformation on it according to n, that is $-nvn^{-1}$, then apply a Householder transformation on this result based on m, that is $-m(-nvn^{-1})m^{-1} = (mn)v(mn)^{-1}$. If $R = mn$ then we write $v'' = RvR^{-1}$. R is called a *rotor.*

Figure 2.6 illustrates the situation. From the Figure we observe that the angle ϕ measures the aperture between the vector n and the vector m. It also measures the aperture between the first householder reflection (blue vector) and the negative y axis. The dotted black line which extends m to the third quadrant makes an angle $\theta/2 - \phi$ with the blue (householder reflection) vector. That same angle is made on the second quadrant between the second householder reflection (red vector) and the dotted line which extends the vector m. We then add the three angles between v and v''. That is, $\phi + \theta/2 + (\theta/2 - \phi) = \theta$ which verifies the rotation by θ from v to v''.

Figure 2.6 is actually a computation of the following example. Let us assume we start with a vector $v = (2, 3)$ in $\mathbb{R}^2$. We wish to rotate v by 90 degrees in the counter–clockwise direction. We know the answer is $(-3, 2)$ and this will serve to check the computations. Let us pick $n = (0, 1)$, a vector along the y axis, $m = (-1, 1)$ (note that this does not have to be a unit vector) making $\theta/2 = 45$ degrees with n. We will do two Householder reflections. One along n and then reflect the output of this once more, but this time along m. The reflection with respect to n is

$$v' = -nvn^{-1}.$$

First, $n^{-1} = (0, 1) = e_2$, so we can compute $-nvn$. That is,

$$nv = (0, 1) \cdot (2, 3) + (0, 1) \wedge (2, 3) = 3 - 2e_1 \wedge e_2.$$

so

$$\begin{aligned} -nvn^{-1} &= -(3 - 2e_1 \wedge e_2)n \\ &= -3n + 2e_1e_2n \\ &= -3e_2 + 2e_1e_2e_2 \\ &= 2e_1 - 3e_2 \\ &= (2, -3) \end{aligned}$$

Figure 2.6 shows this Householder transformation (ending in the blue vector v') with 2 units along e_1 direction and 3 units along the negative of the orthogonal rejection direction n.

We now proceed to do the second Householder transformation of v' along the vector m. This is $-mv'm^{-1}$. Since m is not unitary we find that

$$m^{-1} = \frac{m}{|m|^2} = \frac{m}{2}.$$

Now,

$$mv' = (-1,1) \cdot (2,-3) + (-1,1) \wedge (2,-3) = -5 + e_1 \wedge e_2,$$

and

$$\begin{aligned}
-mv'm^{-1} &= \frac{1}{2}(5m - e_1 \wedge e_2 m) \\
&= \frac{5}{2}(-1,1) - \frac{1}{2}e_1 \wedge e_2(-1,1) \\
&= \left(-\frac{5}{2}, \frac{5}{2}\right) - \frac{1}{2}e_1e_2(-e_1 + e_2) \\
&= \left(-\frac{5}{2}, \frac{5}{2}\right) + \frac{1}{2}e_1e_2e_1 - \frac{1}{2}e_1e_2e_2 \\
&= \left(-\frac{5}{2}, \frac{5}{2}\right) - \frac{1}{2}e_2 - \frac{1}{2}e_1 \\
&= \left(-\frac{5}{2}, \frac{5}{2}\right) + \left(-\frac{1}{2}, -\frac{1}{2}\right) \\
&= (-3,2).
\end{aligned}$$

as expected.

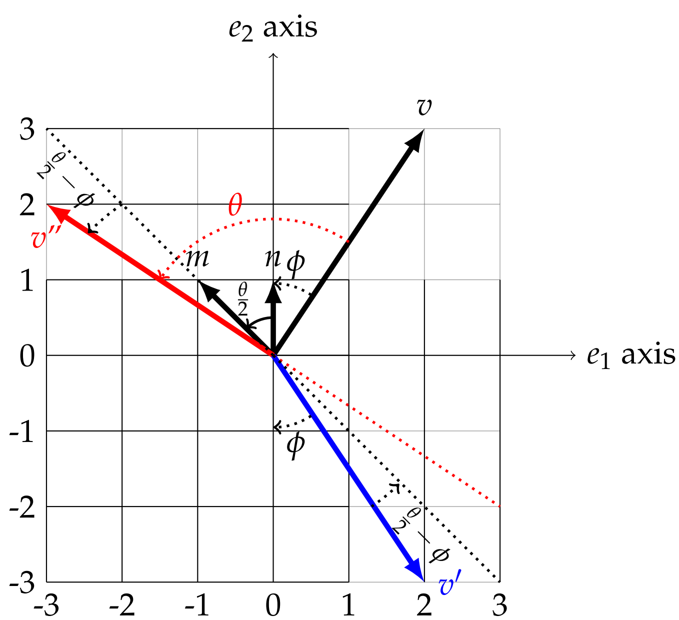

Figure 2.6: A rotation as a cascade of two Householder reflections. We start with the black vector v, the first Householder transformation along the direction n carries it to the blue vector v'. Then another Householder transformation to reflect v' along the vector m is done turning it into the red vector v''. This will result into the red vector v''. The vector v'' is a 90 degrees counterclockwise rotation of the vector v. The invention of this method is attributed to Hamilton.

The rotation $v'' = RvR$ is the last formula shown in Table 2.1 in the last row. Let us see in a different way how the rotor R can be interprated as a $\theta/2$ rotation using complex analysis.

We now show that the geometric product of two vectors in $\mathbb{R}^2$ bevahes as the product of two complex numbers.

The geometric product of two vectors is a complex number: Given two vectors u and v with an angle θ in between them,

$$uv = u \cdot v + u \wedge v \qquad vu = u \cdot v - u \wedge v$$

so multiplying the two products we find

$$(uv)(vu) = (u \cdot v)^2 - (u \wedge v)^2,$$

but $(uv)(vu) = u(vv)u = |u|^2|v|^2$, and $(u \cdot v)^2 = |u|^2|v|^2 \cos\theta$, so

$$(u \wedge v)^2 = -(1 - \cos^2\theta)|u|^2|v|^2 = -\sin^2\theta|u|^2|v|^2.$$

Then

$$u \wedge v = \mathrm{i}\sin\theta|u||v|.^{10}$$

and

$$uv = |u||v|(\cos\theta + \mathrm{i}\sin\theta) = |u||v|\mathrm{e}^{\mathrm{i}\theta}$$

We recognize here a product of two complex numbers. The product of a one complex number and the conjugate of another complex number. Think the numbers in polar form. That is $v = |v|\mathrm{iff}_1$, $u = |u|\mathrm{iff}_2$ then

$$vu^* = |v||u|\mathrm{e}^{\mathrm{i}(\alpha_2 - \alpha_1)} = |u||v|\mathrm{e}^{\mathrm{i}\theta}$$

wich $\theta = \alpha_2 - \alpha_1$ being the angle between the vector u and the vector v.

If the vectors u and v are unit vectors, then

$$u \cdot v = \cos\theta \qquad u \wedge v = \mathrm{i}\sin\theta$$

and

$$uv = \cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta}. \tag{2.5}$$

[10] where we use only the positive sign of the square root consistent with the right hand orientation.

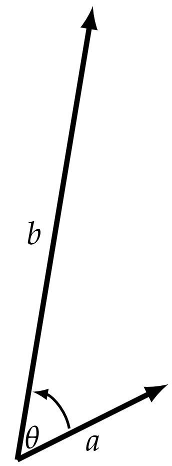

Figure 2.7: The transformation of the vector a into the vector b by extension and rotation.

The expression $e^{i\theta}$ is known as a *spinor* . It simplifies rotation by using the isomorphism betwen $\mathbb{R}^2$ and the complex plane $\mathbb{C}$. This also says that the product of two unit vectors only depends of the angle between them and not the absolute location of them. In other words, the product is rotation invariant under rotations. In particular a rotor $R = mn$, for unit vectors m and n, can be written as

$$R = mn = \cos\frac{\theta}{2} + \mathrm{i}\sin\frac{\theta}{2}.$$

Let us verify, following a different path, that the product of two vectors in $\mathbb{R}^2$ behaves the same as the product of two complex numbers.

$$\begin{aligned} u &= a_1e_1 + a_2e_2, \quad v = b_1e_1 + b_2e_2, \\ uv &= (a_1b_1 + a_2b_2) + (a_1b_2 - a_2b_1)e_1e_2 \end{aligned}$$

which is precisely the product of two complex numbers (a complex number b and the conjugate a^* of another complex number a) with the interpretation that $e_2e_2 = \mathrm{i}$, and $a = a_1 + \mathrm{i}a_2, b = b_1 + \mathrm{i}b_2$. We see that

$$a^*b = (a_1 - \mathrm{i}a_2)(b_1 + \mathrm{i}b_2) = (a_1b_1 + a_2b_2) + (a_1b_2 - a_2b_1)\mathrm{i}.$$

There are many ways to understand rotations. However here is a rotation idea which is useful. We can use equation 2.5 to turn a vector a into a vector b. Figure 2.7 shows the vectors a and b and the angle θ between them.

Clearly we can write

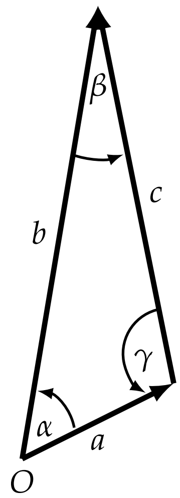

Figure 2.8: Triangle to show that the sum of interior angles is π radians. The angles are all in the same "positive" (right handed)orientation.

$$b = baa^{-1} = |b||a|e^{i\theta}a^{-1} = |b||a|e^{i\theta}\frac{a}{|a|^2} = \frac{|b|}{|a|}e^{i\theta}a. \tag{2.6}$$

In words we have the following two operations on a:

(i) Shrink or extend the size to fit length of a into length of b. That is achieved by dividing by the size of a (that is: $|a|$) and multiplying by the size of b (that is:$|b|$).

(ii) Rotate from a an angle θ until reaching the vector b.

We use the results in this section to show simple proofs of the law known properties in plane geometry such as: the sum of internal angles in a triangle being π radians, the law of sines, the law of cosines

2.2.6 Plane Geometry: Some simple proofs of important theorems

We start with the proof that in any plane triangle the sum of its internal angles is π radians (or 180 degrees). Then we show the law of sines and end the section with the proof of the law of cosines.

2.2.6.1 Sum of internal angles in a plane triangle

We use Figure 2.8 and equation 2.7 for to prove this. Starting at the side a we convert this into b using the transformation 2.7 2.7. That is we obtain

$$b = \frac{|b|}{|a|}e^{i\alpha}a$$

Likewise we convert $-b$ into $-c$ with the relation

$$-c = -\frac{|c|}{|b|}e^{i\beta}b$$

and finally c into $-a$ again

$$-a = \frac{|a|}{|c|}e^{i\gamma}c.$$

The sign a changed because we use O as the origin. The vector a that started has its initial point in O. When the cycle is completed, we projected c into a but following the angle γ which sends the upper point of the triangle into the origin O being this the last point of the side labeled as a. Take the last equation and replace c and b from the first two as follows

$$-a = \frac{|a|}{|c|}e^{i\gamma}\frac{|c|}{|b|}e^{i\beta}\frac{|b|}{|a|}e^{i\alpha}a = e^{i(\alpha+\beta+\gamma)}a.$$

From here $e^{i(\alpha+\beta+\gamma)} = -1$, and then $\alpha + \beta + \gamma = \pi$.

2.2.6.2 The law of sines

Assume $c = a + b$. Let us multiply this equation by a, b and c on the left, and since $a \wedge a = 0$ (and so for b and c) then

$$\begin{aligned} a \wedge c &= a \wedge b \\ b \wedge c &= b \wedge a \\ 0 &= c \wedge a + c \wedge b \end{aligned}$$

Only two of these equations are independent. For example the third equation can be obtained as the sum of the first two. We rewrite the first two equations as

$$a \wedge c = a \wedge b = c \wedge b$$

and taking magnitudes

$$|a \wedge c| = |a \wedge b| = |c \wedge b|$$

If abc is non–zero we can divide by $|a||b||c|$ and obtain

$$\frac{\sin A}{|a|} = \frac{\sin B}{|b|} = \frac{\sin C}{|c|},$$

The wedge (outer) product, which in vector algebra is only defined for the three–dimensional space is defined in geometric algebra for any dimension, where A, B and C are the angles between b, c; a, c; and a, b respectively.

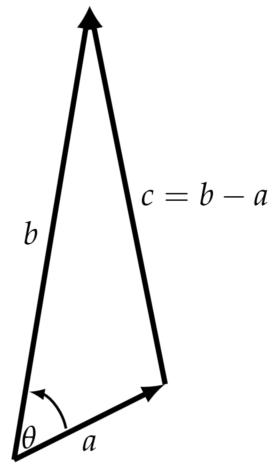

Figure 2.9: Three vectors a, b, c forming a triangle

2.2.6.3 The law of cosines

Take three vectors a, b, and c which make up a triangle. This happens if $c = b - a$ as shown in Figure 2.9.

$$\begin{aligned} c^2 &= (b-a)^2 \\ &= (b-a)(b-a) \\ &= bb - ba - ab + aa \\ &= a^2 + b^2 - 2a \cdot b \\ &= a^2 + b^2 - 2|a||b|\cos\theta \end{aligned}$$

where θ is the angle between a and b. This is the law of cosines for the square of the length of a side c, as a function of the lengths of the other two sides.

2.2.7 Geometric Product is Invariant Under Rotations:

We show that a rotation will leave the product of two base vectors $e_1'e_2'$ unchanged.

$$\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\begin{aligned} e_1'e_2' &= (\cos\theta e_1 + \sin\theta e_2)(-\sin\theta e_1 + \cos\theta e_2) \\ &= -\cos\theta\sin\theta + \sin\theta\cos\theta - \sin^2\theta e_2 e_1 + \cos^2\theta e_1 e_2 \\ &= e_1 e_2 \end{aligned}$$

So the product $e_1 e_2$ is preserved under rotation. In general from equation 2.5 we see that the geometric product of two vectors depends only on the size of the vectors and the angle between them.

2.3 Other geometrical attributes: Lines, Planes, Circles, and Spheres

2.3.1 Lines

In analytic geometry, given a direction vector $u \in \mathbb{R}^n$ and a point $x_0 \in \mathbb{R}^n$, we define the line through the point x_0 parametrically with respect to the parameter t as $x(t) = x_0 + tu$, where t is a scalar. In GA this is equivalent to say that $(x - x_0) \wedge u = 0$. If $x_0 = 0$ (the origin) then the equation is simply $x \wedge u = 0$.

2.3.2 Planes

The equation for the plane is similar to that of a line. Given a 2-blade U, we find that the plane passing through the point a is defined from the equation

$$(x - a) \wedge U = 0.$$

This comes from the fact that $x - a$ is aligned with any vector in the plane and the 2-blade U represents the span of two vectors in the plane (say $x_1 - a$, and $x_2 - a$) with x_1, x_2 in the plane. Hence $x - a$ is parallel to U and the wedge product is zero.

2.3.3 Spheres and Circles

As in linear algebra a sphere with center c and radius r is the set of points x such that $|x - c| = r$, or equivalent $(x - c)^2 = r^2$. Consider the plane $e_{12} = \mathrm{i}$. The intersection of this plane with the sphere is a great circle of radius r. Then from the definition of the plane and of the sphere we have

$$\begin{aligned} (x - c)^2 &= r^2 \\ (x - c) \wedge \mathrm{i} &= 0. \end{aligned}$$

We kind find circles in other planes such as $e_1 e_3$ and $e_2 e_3$ on $\mathbb{R}^3$. The concept can be generalized to $\mathbb{R}^n$ where instead we use the words hyper-sphere, hyper-circle, hyper-plane.

GA can be extended to Geometric Calculus (GC) by extending the definitions of calculus into multivectors. We will not address this extension here.

The inner product comes from the polarization identity.....

2.4 Solution of linear equations: Cramer's Rule

We set the problem initially as a simple 2 by 2 equation with two unknowns. Let us write

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

where we want to solve for x and y. We can write this as the system

$$\begin{pmatrix} a \\ d \end{pmatrix} x + \begin{pmatrix} b \\ e \end{pmatrix} y = \begin{pmatrix} c \\ f \end{pmatrix}$$

That is, we can write this equation as

$$ux + vy = w$$

where u is the column vector with entries a and d, v is the column vector with entries b and c, and w is the column vector with entries c and f. We can take the wedge product with u

$$u \wedge ux + u \wedge vy = u \wedge w.$$

Recall that x and y are scalars, and since $u \wedge u = 0$ we have

$$y = \frac{u \wedge w}{u \wedge v} \tag{2.7}$$

Likewise we can take wedge product with v and find

$$v \wedge ux + v \wedge vy = v \wedge w.$$

and since $v \wedge v = 0$,

$$x = \frac{v \wedge w}{v \wedge u} \tag{2.8}$$

As shown in equation 2.3 we can write

$$u \wedge w = \det \begin{pmatrix} a & c \\ d & f \end{pmatrix} e_1 e_2 \quad , \quad u \wedge v = \det \begin{pmatrix} a & b \\ d & e \end{pmatrix} e_1 e_2 \quad , \quad v \wedge w = \det \begin{pmatrix} b & c \\ e & f \end{pmatrix} e_1 e_2.$$

Then we have (since $v \wedge u = -u \wedge v$) that

$$x = -\det \begin{pmatrix} b & c \\ e & f \end{pmatrix} e_1e_2 / \det \begin{pmatrix} a & b \\ d & e \end{pmatrix} e_1e_2$$

Now, since $1/(e_1e_2) = (e_1e_2)/(e_1e_2)^2 = (e_1e_2)/(e_1e_2e_1e_2) = -e_1e_2$ then we have that

$$x = \frac{\det \begin{pmatrix} b & c \\ e & f \end{pmatrix}}{\det \begin{pmatrix} a & b \\ d & e \end{pmatrix}} e_1e_2e_1e_2 = -\frac{\det \begin{pmatrix} b & c \\ e & f \end{pmatrix}}{\det \begin{pmatrix} a & b \\ d & e \end{pmatrix}} = \frac{\det \begin{pmatrix} c & b \\ f & e \end{pmatrix}}{\det \begin{pmatrix} a & b \\ d & e \end{pmatrix}}$$

which is precisely Cramer's rule for the x solution. It is left to the reader to derive the y solution

$$y = \frac{\det \begin{pmatrix} a & c \\ d & f \end{pmatrix}}{\det \begin{pmatrix} a & b \\ d & e \end{pmatrix}}. \tag{2.9}$$

We will derive Cramer's rule for higher dimensional spaces later in the text. We see how equations 2.7 and 2.8 are much simpler than the Cramer's rule in terms of determinants. They both should evaluate to the same number, however the expressions using GA are simpler than those using linear algebra.

2.5 Geometric Algebra Is Coordinate Free

By *coordinate-free* we mean that the definitions and properties are represented by vectors and multi-vectors without referring to their coordinates. Observe that all derivations [11] were done without referring to coordinates.

Of course coordinates can be used, but if you observe the derivations so far, they have not need coordinates. Rotations, reflections, projections etc., can be done without using coordinates.

As discussed above, numerical implementations require the use of coordinates. The best we can do is to perform all the algebra without coordinates and as a last step write the result in a coordinate system for numerical implementation. In the next chapter we discuss the axiomatic treatment of GA.

[11] except those which were implemented numerically with the purpose of illustrate ideas such as for example Householder reflection, projections, etc.

Problems 2.5.1.

(1) Show that if a is orthogonal to b, then $a(ab) + (ab)a = 0$ and $a(ab) - (ab)a = 2a^2b$.

(2) Derive equation 2.9.

Chapter 3

The Axioms of GA

I list the axioms as in Chilsom [1] and add my own points of view on each topic. As the Axioms list advances, new objects are introduced. For example starting with the lowest dimensions, scalars, and then vectors, up to multivectors. Instead of a list of theorems as in Chilsom's work, I present a tutorial in the following chapter 4 of how the objects are built starting from vectors up to any general multivector, by using the products defined (left and right inner products, outer –wedge– products, and geometric products).

Chilsom's axiomatic approach to tackle the theory of GA is far from unique. Chris J. L. Doran's [2] Ph.D. thesis offers an informal approach where the axioms are all in the following paragraph " The above considerations lead us propose the following principle: *The axioms of an algebraic system should deal directly with the objects of interest.* That is to say, the axioms should offer some intuitive feel of the properties of the system they are defining. The central properties of a geometric algebra are the grading, which separates objects into different types, and the associative product between the elements of the algebra. With these in mind, we adopt the following definition. A Geometric Algebra GA is a graded linear space, the elements of which are called multivectors. The grade-0 elements are called scalars and are identified with the field of real numbers (we will have no cause to consider a geometric algebra over the complex field). The grade-1 elements are called vectors, and can be thought of as directed line segments. The elements of G are defined to have an addition, and each graded subspace is closed under this. A product is also defined which is associative and distributive, though non-commutative (except for multiplication by a scalar). The final axiom (which distinguishes a geometric algebra from other associative algebras) is that the square of any vector is a scalar. "

In the context of $\mathbb{R}^n$, Alan MacDonald in his notes [3] Defines GA is a set $\mathcal{G}$ with two operations, addition and multiplication (called *geometric product),* such that G is

[1] http://arxiv.org/abs/1205.5935

[2] http://cosmologist.info/thesis.ps.gz

[3] http://faculty.luther.edu/ macdonal/GAConstruct/GAConstruct.html

a ring [4]ring, and, to the ring axioms, he added the following couple of equations

$$ee = 1,$$
$$ef = -fe,$$

for e and f orthonormal vectors in $\boldsymbol{R}^n$

3.1 Axioms

Definition 3.1.1. A GA is a set $\mathcal{G}$ with two operations, addition and multiplication (called *geometric product*), that obey the following axioms.

- **Axiom 1.** *$\mathcal{G}$ is a ring [5]ring with unit. The additive identity is called 0 and the multiplicative identity is called 1. What this is saying is that The GA has the following set of properties:*

  (a) *Addition and multiplication in $\mathcal{G}$ are both associative. $A + (B + C) = (A + B) + C = A + B + C$ and $A(BC) = (AB)C = ABC$.*

  (b) *Both operations have identities. Addition 0, multiplication 1.*

  (c) *Every element has an aditive inverse (its negative) $A - A = 0$.*

  (d) *Addition commutes: $A + B = B + A$.*

  (e) *Multiplication is left and right distributive over addition. $A(B + C) = AB + AC, (B + C)A = BA + CA$.*

  A generic element is represented by a capital italic letter as above, that is, A, B,etc. These elements are called *multivectors* to differentiate from the vectors in linear algebra.

- **Axiom 2.** *$\mathcal{G}$ contains a field $\mathcal{G}_0$ of characteristic [6] zero which includes 0 and 1.*

  What this means is that the field [7] of scalars has the property that if 1 is the unit of addition, and 0 the modulus of addition, we can add $1 + 1 \cdots + 1$ as many times without ever reaching to 0 as a result. The scalars are denoted by small Greek symbols such as for example $\alpha, \beta, \cdots, \lambda, \mu$ etc.

- **Axiom 3.** *$\mathcal{G}$ contains a subset $\mathcal{G}_1$ closed under addition, and if $\lambda \in \mathcal{G}_0$, $v \in \mathcal{G}_1$ then $\lambda v = v\lambda \in \mathcal{G}_1$.*

  This Axiom introduces the set of vectors over the field $\mathcal{G}_0$. The vectors inherit from the GA a vector space [8] algebraic structure. Vectors are noted with small

[4]http://en.wikipedia.org/wiki/Ring_(mathematics)
[5]http://en.wikipedia.org/wiki/Ring_(mathematics)
[6]http://en.wikipedia.org/wiki/Characteristic_(algebra)
[7]http://en.wikipedia.org/wiki/Field_(mathematics)
[8]http://en.wikipedia.org/wiki/Vector_space

italic letters as $a, b, \cdots v, w$, etc. In ordinary linear algebra vectors can not be multiplied (except dot and cross products), in GA they can because they are elements of a ring. This will help us create new operations not yet defined in GA under this axiomatic treatment. We use the next axiom for that.

- **Axiom 4.** *The square of every vector is a scalar.*

  If the square of every vector is a scalar then $(u+v)^2$, u^2 and v^2 are scalars, so

  $$\frac{1}{2}(uv+vu) = \frac{1}{4}\left[(u+v)^2 - (u-v)^2\right]$$

  is a scalar. (check the algebra by expanding the right hand side)

  This is a very important result. We discussed that the left side of this expression is the inner product of vectors, and it is.

  In the context of Banach [9] and Hilbert [10] spaces there is a trick to get an inner product from the norm. This is based on an identity called the Polarization Identity [11]. For real scalars this is

  $$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

  (where the angle brackets are used to indicate inner product).

  So we can say that $u \cdot v = \frac{1}{2}(uv+vu) = \frac{1}{4}\left[(u+v)^2 - (u-v)^2\right]$ and the result is that the inner product defined in this context, is naturally induced by the norm.

  There are several notations for inner product. For example

  $$\langle x, y\rangle = x \cdot y = x \lfloor y = x \rfloor y.$$

  The symbols most used here are $\lfloor$ and $\rfloor$. While for vector to vector inner product they are equivalent, later when we study higher dimensional objects we will distinguish between the left and right inner product (floor) symbols above. The dot "." symbol is used commonly in computational linear algebra for a very specific definition. The angle brackets symbol is common in functional analysis, but here we use the angle brackets symbol to denote a projection from the general $\mathcal{G}$ space into a subspace of grade r (see equation 3.3). This explains the need for the new symbols here the left and right floor.

  Two vectors u and v are said *orthogonal* if $u \cdot v = 0$, u is a *unit* vector if $u^2 = \pm 1$, and u is *null* if $u^2 = 0$. If two vectors anti–commute then $uv = -uv$ so

[9] http://en.wikipedia.org/wiki/Banach_space
[10] http://en.wikipedia.org/wiki/Hilbert_space
[11] http://en.wikipedia.org/wiki/Polarization_identity

$$u \cdot v = \frac{1}{2}(uv + vu) = \frac{1}{2}(uv - uv) = 0.$$

That is $u \cdot v = 0$. Also if $u \cdot v = 0$ the vectors anti–commute.

If v is no null, then $v^{-1} = v/v^2$, since $v(v/v^2) = v^2/v^2 = 1$. We could use this expressiona as the definition of inner product for vector and build on top of it to extend it to inner product on other objects of GA. We have then the following definition:

Definition 3.1.2 (inner product)**.** The *inner product* between two vectors , u and v is noted as $u \cdot v$ and defined by the equation

$$u \cdot v := \frac{1}{2}(uv + vu) \tag{3.1}$$

Likewise we define the outer or wedge product.

Definition 3.1.3 (outer (wedge) product)**.** The *outer (wedge) product* between two vectors , u and v is noted as $u \wedge v$ and defined by the equation

$$u \wedge v := \frac{1}{2}(uv - vu) \tag{3.2}$$

Clearly, from the previous two equations

$$uv = u \cdot v + u \wedge v.$$

- **Axiom 5.** *The inner product is non–degenerate.*

  This means that the only vector orthogonal to the whole space is 0. We introduce some new definitions. For $r > 1$, an *r-blade* or *simple r-vector* is a product of r orthogonal (thus anti-commuting) vectors. A finite sum of r-blades is called an *r-vector* or *homogeneous multi–vector of grade r* , 2-vectors are called *bivectors* and 3-vectors *trivectors* . The set of *r-vectors* is called $\mathcal{G}_r$. It will be shown that the geometrical product of orthogonal vectors is equivalent to the outer product of arbitrary vectors. An *r-versor* is a product of r vectors. A *rotor* is the product of two invertible vectors, so a rotor is an invertible biversor. The set $\mathcal{G}_r$ is a vector space. Note here that this is not the usual subspace of linear algebra such a line, a plane, or a hypercube. This is the set of lines, planes, or hypercubes. So it is a much larger space and while every subspace generated by an r-blade is r-dimensional, the set $\mathcal{G}_r$ contains all of these r -dimensional objects.

- **Axiom 6.** *If $\mathcal{G}_0 = \mathcal{G}_1$, then $\mathcal{G} = \mathcal{G}_0$. Otherwise, $\mathcal{G}$ is the direct sum of all the $\mathcal{G}_r$.*

This axiom provides a clear separation of all grades. The grade $r = 0$ is the set of scalars. If the vectors equal the scalars is because the grade of the set $\mathcal{G}$ is 0 so it is $\mathcal{G}_0$, the set has 0 dimension. The whole space $\mathcal{G}$ is written as a direct sum of all grades from $r = 0$ (the scalars) up to $r = n$ the dimension of the space. That is, we can write any multivector A as $A = \sum_{r=0}^{n} \boldsymbol{A_r}$ with $\boldsymbol{A_r} \in \mathcal{G}_r$,

In traditional linear algebra, for the space $\mathbb{R}^3$, we have the set $\mathcal{B} = \{e_1, e_2, e_3\}$ as the cannonical basis that expands the whole 3D real space. In GA, the base is composed of the following objects:

$$\underbrace{1}_{\text{scalars}} \quad , \quad \underbrace{e_1 \ , \ e_2 \ , \ e_3}_{\text{vectors}} \quad , \quad \underbrace{e_1 \wedge e_2 \ , \ e_1 \wedge e_3 \ , \ e_2 \wedge e_3,}_{\text{bivectors}} \quad , \quad \underbrace{e_1 \wedge e_2 \wedge e_3 = \mathrm{I}}_{\text{trivector,pseudoscalar}}$$

Here resides the structure of GA and the big picture difference between this and a traditional vector space. For the field of scalars all products converge to the same object (in scalars, geometric product, dot product, wedge product, and product with a scalar is the same). When we get to the dimension 1, we have scalars and vectors and they can be added in GA, but they can not be added in traditional vector spaces. As the dimension increases the differences increase. Vector spaces stop in the second braced expression (the canonical base), GA stops until getting to the last (pseudoscalar) object. The grade is some kind of “wavelength” parameter in a harmonic decomposition. As the grade increases, the size (manifold dimension) of the object increases. A multivector could be a combination of many grades or just a simple grade.

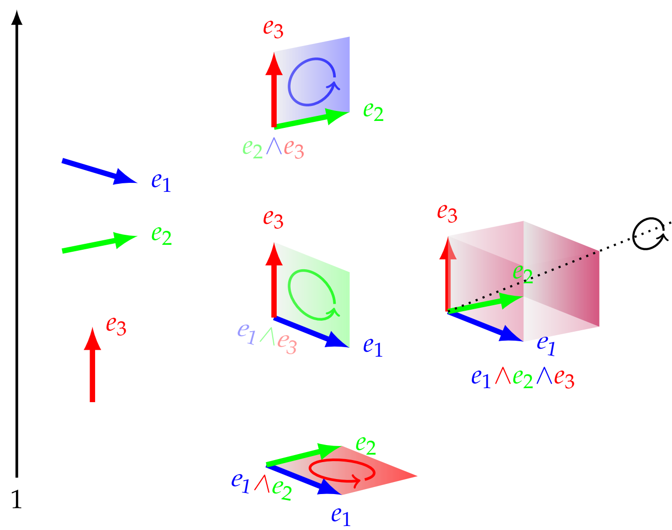

Figure 3.1: For the three dimensional space $\mathbb{R}^3$ we find 4 different grades. The 0 grade are the scalars represented here by the black arrow on the left and number 1 is the base for that space. Then comes the vectors which are e_1, e_2 , and e_3; linear combinations of these vectors provide any vector in the $\mathbb{R}^3$. Then come the bivectors (2–blades); these comes in pairs of base vectors e_i and e_j, for $1 \leq i < j \leq 3$. Finally, we see the only trivector with its spin represented around a main diagonal of the cube formed by the 3 base vectors.

In general, if we have an n-dimensional vector space, we can group the basis with ascending grades as follows

$$\underbrace{1}_{\text{scalars}}\ ,\ \underbrace{e_1\ ,\ e_2\ \cdots\ e_n}_{\text{vectors}}\ ,\ \underbrace{e_1 \wedge e_2\ ,\ e_1 \wedge e_3,\ \cdots\ , e_i \wedge e_j,}_{\text{bivectors}}\ \cdots, \underbrace{e_1 \wedge \cdots \wedge e_n}_{\text{n order blade.}}$$

Here $1 \leq i < j \leq n$. We can then use combinatory theory to count the number of base elements of grade r. The binomial cofficient counts in how many ways we can pick r objects from n objects. So the $\mathcal{G}_r$ space can be expanded with $\binom{n}{r}$ elements. For example, since the scalars have grade $r = 0$ and $\binom{n}{0} = 1$, similarly the pseudoscalar has a basis of $\binom{n}{n} = 1$ element. Since

$$(1+1)^n = \sum_{r=0}^{n} \binom{n}{r},$$

The total space $\mathcal{G}$ is spanned by 2^n objects of grades $0, 1, \cdots, n$. This illustrates the richness of the GA as compared with the linear algebra. While the dimension of the spaces in linear algebra grows linearly, it grows exponentially in GA.

The projection from the whole space $\mathcal{G}$ into $\mathcal{G}_r$ is defined by the operator

$$\langle\rangle_r : \mathcal{G} \mapsto \mathcal{G}_r, \tag{3.3}$$

We can write a multivector A in $\mathcal{G}_n$ as a linear combination of all the vector blades in the space as follows.

$$
\begin{aligned}
A &= \langle A\rangle_0 + \langle A\rangle_1 + \cdots + \langle A\rangle_n \\
&= \alpha_0 + \alpha_{11}e_1 + \cdots + \alpha_{1i_1}e_n + \alpha_{21}e_1e_2 + \cdots + \alpha_{2i_2}e_{n-1}e_n + \cdots + \alpha_{ni_n}\mathrm{I} \quad (3.4)
\end{aligned}
$$

with $\mathrm{I} = e_1e_2\cdots e_n$.

From now on and to simplify we will use Einstein repeated index notation [12] to simpliy writing. . That is, on each formula, indices are displayed in pairs and singles. No index can be written more than twice. Each repeated index implies a summation over it. We rewrite equation 3.4 as

$$A = \alpha_0 + \alpha_{1j}e_j + \alpha_{2jk}e_je_k + \cdots + \alpha_{rj_1j_2\cdots j_r}e_{j_1}e_{j_2}\cdots e_{j_r} + \cdots + \alpha_n\mathrm{I}$$

where $1 \le j_i < j_k \le n$, for $i < k$. That is, all indices are in ascending order. The $\langle A\rangle_r$ projection is then the r-fold sum

$$\langle A\rangle_r = \alpha_{rj_1j_2\cdots j_r}e_{j_1}e_{j_2}\cdots e_{j_r}$$

There are $\binom{n}{r}$ terms on this sum.

Let us illustrate the projection concept with a few simple examples.

Example 3.1.1 (Projections)**.** Sin $ab = a\cdot b + a\wedge b$ then

$$\langle ab\rangle = \langle ab\rangle_0 = a\cdot b \quad , \quad \langle ab\rangle_2 = a\wedge b$$

We see that the inner product is the projection of ab into its $1-1=0$ grade. The outer product is the projection of ab into its $1+1=2$ grade. Let us take the case of the projection of a tri-vector into the grade 1.

$$
\begin{aligned}
\langle abc\rangle_1 &= \langle a(b\cdot c + b\wedge c)\rangle_1 \\
&= \langle a(b\cdot c) + a(b\wedge c)\rangle_1 \\
&= a(b\cdot c) + \langle a\cdot(b\wedge c) + a\wedge(b\wedge c)\rangle_1 \\
&= a(b\cdot c) + \langle a\cdot(b\wedge c)\rangle_1 \\
&= a(b\cdot c) + a\cdot(b\wedge c).
\end{aligned}
$$

We leave to the reader to prove that

[12] https://en.wikipedia.org/wiki/Einstein_notation

$$a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b. \tag{3.5}$$

With this we have that

$$\langle abc \rangle_1 = a(b \cdot c) + (a \cdot b)c - (a \cdot c)b.$$

The following statements are left as a proof to the reader. Most proofs are a direct consequence of the linearity of the vector subspaces and of the independent representations for different grade r.

Theorem 3.1.1. *Show the following list of properties:*

(a) if $r \neq s$, the space $\mathcal{G}_r \cap \mathcal{G}_s = 0$. The decomposition of a multivector A in terms of grades is unique. That is

$$A = \langle A \rangle_0 \oplus \langle A \rangle_1 \oplus \cdots \oplus \langle A \rangle_n.$$

Note that this actually Axiom 6. We can then relax Axiom 6 by saying that any multivector in $\mathcal{G}_n$ can be written as a sum of $r-$vectors and then show uniqueness of the representation.

(b) A is an r-vector iff $A = \langle A \rangle_r$.

(c) $\langle A + B \rangle_r = \langle A \rangle_r + \langle B \rangle_r$.

(d) $\langle \lambda A \rangle_r + \langle A \lambda \rangle_r = \lambda \langle A \rangle_r$.

(e) $\langle \langle A \rangle_r \rangle_s = \langle A \rangle_r \delta_{rs}$.

(f) We extend the definition of $\langle \rangle_r$ for $r < 0$, $\langle A \rangle_r = 0$ if $r < 0$ for all $A \in \mathcal{G}$. $\mathcal{G}$ is a direct sum of $\langle \mathcal{G} \rangle_+$ and $\langle \mathcal{G} \rangle_-$ where the subindex "+" means all the r-vectors for r even. Similarly "-" means all the r-vectors for r odd.

The default symbol $\langle \rangle$ means $\langle \rangle_0$, that is the set of scalars. A blade of order r is noted in boldface font as $\boldsymbol{A}_r$. If $r = 0, 1$ no boldface is used. Finally, $\langle A \rangle_+$ (or $\langle A \rangle_-$) is the even–grade (or odd–grade) part of A.

We showed that $\mathrm{I}^2 = -1$ and would like to have a way to find the actual norm of an object (which is a positive number). We introduce the following definition of the reverse of a product of GA objects, which help us in the definition of the norm of an object.

Definition 3.1.4 (reverse)**.** Given a product of GA objects $b = a_1 a_2 \cdots a_n$ we define the *reverse* of b as $b^\dagger$ given by

$$b^\dagger = (a_1 a_2 \cdots a_n)^\dagger = a_n a_{n-1} \cdots a_1.$$

We also assume that

$$\left(\sum_{i=1}^{n} a_i\right)^\dagger = \sum a_i^\dagger.$$

That is, we reverse the order of the product. We now observe that while $\mathrm{I}\mathrm{I} = -1$

$$\mathrm{I}\,\mathrm{I}^\dagger = (e_1 e_2)(e_2 e_1) = e_1 e_1 = 1.$$

We observe that $\mathrm{I}^\dagger = -\mathrm{I}$ and so the dagger symbol behaves like the conjugate on complex numbers.

From the definition we find that

Theorem 3.1.2. *If $\alpha \in \mathbb{R}$, A is a multivector and $a, a_1, a_2, \cdots a_n$ are vectors then*

(i) $\alpha^\dagger = \alpha$.

(ii) $a^\dagger = a$.

(iii) $(a_1 \wedge a_2)^\dagger = -a_1 \wedge a_2$.

(iv) $(a_1 \wedge a_2 \wedge a_3)^\dagger = -a_1 \wedge a_2 \wedge a_3$.

(v) $(A^\dagger)^\dagger = A$.

(vi) If $e_1, e_2, \cdots, e_n$ are anticommuting vectors then $(e_1 e_2 \cdots e_n)^\dagger = -1^{n(n-1)/2} e_1 e_2 \cdots e_n$.

We now address the following question: What would be the norm of a multivector A? We are tempted to say that it is $\sqrt{AA^\dagger}$. However we have the following problem.

(i) It could be that $AA^\dagger$ is not even a scalar.

(ii) It could happen that $AA^\dagger$ is negative.

We then need review each of the items above. To guarantee a scalar number we will project $A^\dagger A$ into the scalars by taking the real part $\langle A^\dagger A\rangle_0$. We now show that $\langle A^\dagger A\rangle_0 \geq 0$.

Theorem 3.1.3. *For any multivector A we have that*

$$\langle A^\dagger A\rangle_0 \geq 0.$$

Proof: For a scalar α or a vector a the result is immediate. Let assume that A is a multivector. The product $A^\dagger A$ is a sum of algebraic products of basis vectors vectors $e_1, e_2, \cdots, e_n$ times a scalar α. Now, if this product is non-zero

$$(e_{i_1} e_{i_2} \cdots e_{i_n})^\dagger (e_{i_1} e_{i_2} \cdots e_{i_n}) = e_1^2 e_2^2 \cdots e_n^2 = 1$$

So that $\langle A^\dagger A \rangle_0$ is the sum of 0 or positive terms and $\langle A^\dagger A \rangle_0 \geq 0$. We can now safely define the norm of a multivector.

Definition 3.1.5 (norm)**.** The norm of a multivector A is defined as

$$|A| = \sqrt{\langle A^\dagger A \rangle_0}$$

Equipped with this material, linearity, and the decomposition of any vector in term of orthonormal basis we can build a GA in $\mathbb{R}^n$ (that is $\mathcal{G}_n$). In the next chapter we show how this is done, starting at $\mathbb{R}$ and going up in dimensions.

Problems 3.1.1.

(1) Proof theorem 3.1.1

(2) Proof theorem 3.1.2

(3) Verify equation 3.5. (*hint: expand $a \cdot (b \wedge c)$ using equation 4.8) then add* $0 = \frac{1}{4}[bac - cab - bac + cab]$

Chapter 4

A Bottom up (inductive) approach

The idea is to start with small elements and combine them in a form that they will build the whole GA structure. Instead of a formal rigorous theory with a list of axioms and corollaries, I present a tutorial approach that shows the development, when we start going into higher dimensions and combining objects of different grades under all products defined here.

The building blocks of GA are vectors (grade 1) objects and scalars (grade 0) objects. Each vector generates a one dimensional space. We start with lower grades. Once we understand how this works, we go to three vectors and combine them in different forms, and after that a simple inductive argument will show the general behavior of multivectors under any of the products defined here. Note that the operation sum is not important for this chapter. Linearity helps and what is important is the grade of the object being treated.

4.1 $\mathcal{G}_0$:Scalars

Given a non zero number α, any non-real real number β can be written as $\beta = \alpha(\beta/\alpha)$ where β/α is a scale to produce the required output. In this particular case $\mathcal{G}_0 = \mathbb{R}$ and the GA is a field and a vector space at the same time. Here $ab = a \cdot b$. We will not say much about scalars at this point since they will come up again in the following sections.

4.2 $\mathcal{G}_2$: Scalars, Vectors, $\mathbb{R}^2$, and the complex numbers $\mathbb{C}$

The vectors are the same vectors of linear algebra in $\mathbb{R}^n$. In $\mathbb{R}^2$ we have the basis blades $1 = e_0, e_1, e_2, e_1e_2 = e_{12} = \mathrm{I} = \mathrm{i}$. We have a two fold representation of the complex numbers $\mathbb{C}$.

(i) The first (scalar) and last blade (the pseudoscalar i) generate the complex numbers.

(ii) The two middle blades e_1, e_2 behave as complex numbers in their sum and multiplication properties.

Table 4.1 shows of multiplication between blade base vectors: There are 16 entries

	1	e_1	e_2	I
1	1	e_1	e_2	I
e_1	e_1	1	I	e_2
e_2	e_2	-I	1	$-e_1$
I	I	-e_2	e_1	-1

Table 4.1: Products of blades in $\mathcal{G}_2$.

in total. Each base vector appears 4 times. The diagonal has the scalars (3 positive and 1 negative), the anti-diagonal has the pseudoscalars (3 positive and 1 negative).

Three with a positive sign and 1 with a positive sign. A general multivector in $\mathcal{G}_2$ is of the form

$$A = \alpha + \beta e_1 + \gamma e_2 + \delta \mathrm{I}$$

We want to write multivectors in terms of their base blades. That is, we want to use the coefficients (components) $\alpha, \beta, \gamma, \delta$ on each multivector to understand the general product of two multivectors. Hence, we need to use a better notation. Let us assume that the bold face vectors $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ correspond with the blades $1, e_1, e_2, e_{12}$ respectively.

Two multivectors A and B can be written

$$A = a_i \boldsymbol{e}_i \quad , \quad B = b_i \boldsymbol{e}_i$$

where we assume a sum over i. That is, $A = \sum_{i=0}^{3} a_i \boldsymbol{e}_i$, but from now on we will skip the $\sum$ symbol. Here a_i and b_i are the coefficients of each blade on A and on B. That is, they are the components along each blade.

The geometric product of A and B is then written by the double sum

$$AB \quad = \quad a_i b_j \boldsymbol{e}_i \boldsymbol{e}_j.$$

This is a sum of 16 terms. Each $\boldsymbol{e}_i \boldsymbol{e}_j$ is an entry in Table 4.1. We can sort all terms in 4 groups:

$$AB = \langle AB \rangle_0 + \langle AB \rangle_1 + \langle AB \rangle_2,$$

where the notation $\langle AB \rangle_r$, $r = 0, 1, 2$ indicates the $r - vectorpart$ of the multivector.

(i) **The scalar e_0.** From Table 4.1 above we see that scalars are obtained from products of repeated blades. We have that $1 = e_0 e_0 = e_1 e_1 = e_2 e_2 = e_3 e_3 = -II$ and so the scalar is given by $a_0 b_0 + a_1 b_1 + a_2 b_2 - a_3 b_3$.

(ii) **The coefficient of $e_1 = e_1$.** From Table 4.1 we see three positive and one negative contributions. Two of the three positive contributions result when one of the base vectors is the scalar $e_0 = 1$, the other from the product $Ie_2 = e_1 e_2 e_2 = e_1$. The negative contribution come from $e_2 I = e_2 e_1 e_2 = -e_1 e_2 e_2 = -e_1$. The coefficients turn out to be $a_0 b_1 + b_0 a_1 + a_3 b_2 - a_2 b_3$.

(iii) **The coefficient of $e_2 = e_2$.** From Table 4.1 we see three positive and one negative contributions. Two of the three positive contributions result when one of the base vectors is the scalar $e_0 = 1$, the other from the product $e_1 I = e_1 e_1 e_2 = e_2$. The negative contribution come from $Ie_1 = e_1 e_2 e_1 = -e_2 e_1 e_1 = -e_2$. The coefficients turn out to be $a_0 b_2 + b_0 a_2 + a_1 b_3 - a_3 b_1$.

(iv) **The coefficient of $e_3 =$ I.** From Table 4.1 we see three positive and one negative contributions. Two of the three positive contributions result when one of the base vectors is the scalar $e_0 = 1$, the other from the product $e_1 e_2 = \mathrm{I}$. The negative contribution come from $e_2 e_1 = -\mathrm{I}$. The coefficients turn out to be $a_0 b_3 + b_0 a_3 + a_1 b_2 - a_2 b_1$.

Clearly the multiplication becomes complicated if we want to perform it with n multi-vectors $A_1, A_2, \cdots, A_n$. We then consider only multiplications of multi-vectors with the same grade r.

(i) If the grade is $r = 0$ these are scalars and the problem is already known.

(ii) If the grade is $r = 1$ these are vectors. Let us then assume that we have n vectors $a_1, a_2, \cdots, a_n$ and want to find their geometric product. Each vector is a linear combination of the base vectors. That is,

$$a_i = \alpha_{ij} e_j$$

$i = 1, 2$ $j = 1, 2$, $\alpha_{ij} \in \mathbb{R}$. The product is then given by

$$a_1 a_2 \cdots a_n = (\alpha_{1j_1} e_{j_1})(\alpha_{2j_2} e_{j_2}) \cdots (\alpha_{nj_n} e_{j_n}) = \alpha_{1j_1} \alpha_{2j_2} \cdots \alpha_{nj_n} e_{j_1} e_{j_2} \cdots e_{j_n}$$

where $j_i = 1, 2$. It could happen that some of the e_{j_r} are repeated. Actually if $n \geq 3$ there need to be repeated e_{j_r} since we only have in $\mathbb{R}^2$ the base vectors e_1, e_2. Every pair e_1, e_1 produces a 1 and every pair e_2, e_2 produces a 1. Given a particular product $e_{j_1} e_{j_2} \cdots e_{j_n}$ we have the following options according to the decomposition of $n = s + t$ into s base vectors e_1 and t base vectors e_2.

(a) If s is even and t is even. After some k required permutations to sort all e_1 together and all e_2 together we find

$$e_{j_1}e_{j_2}\cdots e_{j_n} = (-1)^k\alpha_{1j_1}\alpha_{2j_2}\cdots\alpha_{nj_n}$$

(b) If s is odd and t is even. After some k required permutations to sort all e_1 together and all e_2 together we find

$$e_{j_1}e_{j_2}\cdots e_{j_n} = (-1)^k\alpha_{1j_1}\alpha_{2j_2}\cdots\alpha_{nj_n}e_1$$

(c) If s is even and t is odd. After some k required permutations to sort all e_1 together and all e_2 together we find

$$e_{j_1}e_{j_2}\cdots e_{j_n} = (-1)^k\alpha_{1j_1}\alpha_{2j_2}\cdots\alpha_{nj_n}e_2$$

(d) If s is odd and t is odd. After some k required permutations to sort all e_1 together and all e_2 together we find

$$e_{j_1}e_{j_2}\cdots e_{j_n} = (-1)^k\alpha_{1j_1}\alpha_{2j_2}\cdots\alpha_{nj_n}e_1e_2$$

(iii) If the grade is $r = 2$. We only have the pseudoscalar $I = e_{12} = e_1e_2$. In the product there are exactly n vectors e_1 and n vectors e_2. We have the following cases:

(a) If n is even

$$e_{j_1}e_{j_2}\cdots e_{j_n} = (-1)^k$$

where k is the required number of permutations to put each e_1 next to another e_1 or each e_2 next to another e_2 so that their product is one.

(b) If n is odd

$$e_{j_1}e_{j_2}\cdots e_{j_n} = (-1)^k e_1e_2.$$

where k is the required number of permutations to put each e_1 next to another e_1 or each e_2 next to another e_2 so that their product is one.

In particular let us find the geometric, inner, and wedge product for two two vectors $a = \alpha_1e_1 + \alpha_2e_2$, $b = \beta_1e_1 + \beta_2e_2$ in terms of their components (α_1, α_2) and (β_1, β_2).

$$\begin{aligned}
ab &= (\alpha_1 e_1)(\beta_1 e_1) + (\alpha_1 e_1)(\beta_2 e_2) + (\alpha_2 e_2)(\beta_1 e_1) + (\alpha_2 e_2)(\beta_2 e_2) \\
&= \alpha_1\beta_1 + \alpha_2\beta_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)e_1e_2 = \langle ab\rangle_0 + \langle ab\rangle_2.
\end{aligned}$$

From the inner product definition 3.1 we find

$$\begin{aligned}
a\cdot b &= \frac{1}{2}(ab+ba) \\
&= \frac{1}{2}(\alpha_1\beta_1 e_1e_1 + \alpha_1\beta_2 e_1e_2 + \alpha_2\beta_1 e_2e_1 + \alpha_2\beta_2 e_2e_2 \\
&\quad +\alpha_1\beta_1 e_1e_1 + \alpha_1\beta_2 e_2e_1 + \alpha_2\beta_1 e_1e_2 + \alpha_2\beta_2 e_2e_2 \\
&= \frac{1}{2}(\alpha_1\beta_1 + \cancel{\alpha_1\beta_2 e_1e_2} \cancel{-\alpha_2\beta_1 e_1e_2} + \alpha_2\beta_2 \\
&\quad +\alpha_1\beta_1 \cancel{-\alpha_1\beta_2 e_1e_2} + \cancel{\alpha_2\beta_1 e_1e_2} + \alpha_2\beta_2 \\
&= \alpha_1\beta_1 + \alpha_2\beta_2
\end{aligned}$$

as expected. Now, from the wedge product definition 3.2 we find

$$\begin{aligned}
a\wedge b &= \frac{1}{2}(ab-ba) \\
&= \frac{1}{2}(\alpha_1\beta_1 e_1e_1 + \alpha_1\beta_2 e_1e_2 + \alpha_2\beta_1 e_2e_1 + \alpha_2\beta_2 e_2e_2 \\
&\quad -\alpha_1\beta_1 e_1e_1 - \alpha_1\beta_2 e_2e_1 - \alpha_2\beta_1 e_1e_2\alpha_2\beta_2 e_2e_2 \\
&= \frac{1}{2}(\cancel{\alpha_1\beta_1} + \alpha_1\beta_2 e_1e_2 - \alpha_2\beta_1 e_1e_2 + \cancel{\alpha_2\beta_2} \\
&\quad \cancel{-\alpha_1\beta_1} + \alpha_1\beta_2 e_1e_2 - \alpha_2\beta_1 e_1e_2 + \cancel{-\alpha_2\beta_2} \\
&= (\alpha_1\beta_2 - \alpha_2\beta_1)e_1e_2 \\
&= \det(A)\ e_1e_2
\end{aligned}$$

where

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$$

We verify that

$$ab = a\cdot b + a\wedge b$$

as expected.

Blades: The vectors are the building blocks of blades. We define a *blade* or order r (or an *r-blade)* as the product of r orthogonal (and so anticommuting) vectors. A *versor* of grade r is the product of r vectors (so a blade is a particular versor where the factors are all mutually orthogonal).

Given that in a blade the component vectors need to be orthogonal we could not have more than two vectors in a blade. In this case the blade is simply $a_1 \wedge a_2$. We can operate a vector with a blade and find that the operation is anticommutative. Let us see

$$a(a \wedge b) = \frac{a}{2}(ab - ba) = \frac{1}{2}(a^2b - aba) = \frac{1}{2}(ba^2 - aba) = -\frac{1}{2}(ab - ba)a = -(a \wedge b)a.$$

We would like to extend the definition of inner and wedge products to other objects in $\mathcal{G}_2$. If α and β are scalars then we want to know what would be the inner product an outer product on scalars. Here we define the inner and outer product of a scalar α with a vector a as

$$\alpha \cdot a \;:=\; v \cdot \alpha := 0 \quad \text{inner} \tag{4.1}$$
$$\alpha \wedge a \;:=\; a \wedge \alpha := \alpha a \quad \text{outer} \tag{4.2}$$
$$\alpha a = a\alpha \;=\; \alpha \cdot a + \alpha \wedge a \quad \text{geometric.} \tag{4.3}$$

We justify definition later in the text.

4.3 $\mathcal{G}_3$: $\mathbb{R}^3$ and much more

4.3.1 Basis, table of blades, and products of multivectors

The basis for $\mathcal{G}_3$ is given by the set of blades

$$\{1\,,\, e_1\,,\, e_2\,,\, e_3\,,\, e_1e_2\,,\, e_1e_3\,,\, e_2e_3\,,\, \mathrm{I}\}$$

where $e_{ij} = e_ie_j$, $1 \leq i < j \leq 3$ and $\mathrm{I} = e_1e_2e_3$.

Table 4.2 shows a table of the product of blades in $\mathcal{G}_3$. There are 8 instances of each blade; 4 instances are positive and 4 instances are negative. The scalars are on the diagonal (4 negative and 4 positive), the pseudoscalars are in the anti-diagonal (4 negative and 4 positive).

A general multivector is of the form

$$A = \alpha_0 1 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_{12} + \alpha_5 e_{13} + \alpha_6 e_{23} + \alpha_7 \mathrm{I}.$$

	1	e_1	e_2	e_3	e_{12}	e_{13}	e_{23}	I
1	1	e_1	e_2	e_3	e_{12}	e_{13}	e_{23}	I
e_1	e_1	1	e_{12}	e_{13}	e_2	e_3	I	e_{23}
e_2	e_2	$-e_{21}$	1	e_{23}	$-e_1$	-I	e_3	$-e_{13}$
e_3	e_3	$-e_{13}$	$-e_{23}$	1	I	-e_1	-e_2	e_{12}
e_{12}	e_{12}	-e_2	e_1	I	-1	$-e_{23}$	e_{13}	-e_3
e_{13}	e_{13}	$-e_3$	-I	e_1	e_{23}	-1	-e_{12}	e_2
e_{23}	e_{23}	I	-e_3	e_2	-e_{13}	e_{12}	-1	-e_1
I	I	e_{23}	-e_{13}	e_{12}	-e_3	e_2	-e_1	-1

Table 4.2: Table of products of blades in $\mathcal{G}_3$.

The product of two arbitrary multivectors yields 64 terms which should be regrouped in a representation such as that of the vector A. For example, the product AB with

$$B = \beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 + \beta_4 e_{12} + \beta_5 e_{13} + \beta_6 e_{23} + \beta_7 \mathrm{I}.$$

yields the scalar $\alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 - \alpha_4\beta_4 - \alpha_5\beta_5 - \alpha_6\beta_6 - \alpha_7\beta_7$. The only scalars in the table are the diagonal elements (1 for the first 4 and -1 for the last 4). In the same way that we developed the explicit product of two arbitrary multivectors in $\mathcal{G}_2$ we can do it in $\mathcal{G}_3$. When many operations become tedious a symbolic computational package is handy. In the introduction we provide a list of symbolic packages that could be used to yield symbolic computations.

Let us now study the different products in $\mathcal{G}_3$.

4.3.2 Products of vectors in $\mathcal{G}_3$

Let $a = (a_1, a_2, a_3) = a_1 e_1 + a_2 e_2 + a_3 e_3$, and $b = (b_1, b_2, b_3) = b_1 e_1 + b_2 e_2 + b_3 e_3$. Then

$$ab = a_1 b_1 + a_2 b_2 + a_3 b_3 + (a_1 b_2 - a_2 b_1) e_{12} + (a_1 b_3 - a_3 b_1) e_{13} + (a_2 b_3 - a_3 b_2) e_{23}$$

By adding ab and ba we find

$$\begin{aligned} ab + ba &= a_1 b_1 + a_2 b_2 + a_3 b_3 + (a_1 b_2 - a_2 b_1) e_{12} + (a_1 b_3 - a_3 b_1) e_{13} + (a_2 b_3 - a_3 b_2) e_{23} \\ &+ a_1 b_1 + a_2 b_2 + a_3 b_3 - (a_1 b_2 - a_2 b_1) e_{12} - (a_1 b_3 - a_3 b_1) e_{13} - (a_2 b_3 - a_3 b_2) e_{23} \\ &= 2(a_1 b_1 + a_2 b_2 + a_3 b_3) \end{aligned}$$

and so

$$
\begin{aligned}
a \cdot b &= \frac{1}{2}(ab+ba) \\
&= a_1b_1 + a_2b_2 + a_3b_3
\end{aligned}
$$

as the regular dot product of two vectors. Note that if we assume that $e_i \cdot e_j = \delta_{ij}$ (the Kronecker delta) and $a = a_ie_i$, $b = b_je_j$ then, assuming the distributive law

$$a \cdot b = (a_ie_i) \cdot (b_je_j) = a_ib_je_i \cdot b_j = a_ib_j\delta_{ij} = a_ib_i.$$

Likewise we compute $ab - ba$

$$
\begin{aligned}
ab - ba &= a_1b_1 + a_2b_2 + a_3b_3 + (a_1b_2 - a_2b_1)e_{12} + (a_1b_3 - a_3b_1)e_{13} + (a_2b_3 - a_3b_2)e_{23} \\
&\quad - a_1b_1 - a_2b_2 - a_3b_3 + (a_1b_2 - a_2b_1)e_{12} + (a_1b_3 - a_3b_1)e_{13} + (a_2b_3 - a_3b_2)e_{23} \\
&= 2(a_1b_2 - a_2b_1)e_{12} + 2(a_1b_3 - a_3b_1)e_{13} + 2(a_2b_3 - a_3b_2)e_{23}
\end{aligned}
$$

Then

$$
\begin{aligned}
a \wedge b &= \frac{1}{2}(ab - ba) && (4.4) \\
&= (a_1b_2 - a_2b_1)e_{12} + (a_1b_3 - a_3b_1)e_{13} + (a_2b_3 - a_3b_2)e_{23}. && (4.5)
\end{aligned}
$$

Note that we can write this last equation as the determinant

$$a \wedge b = \begin{vmatrix} e_{23} & e_{31} & e_{12} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

We now relate the outer product $\wedge$ with the cross produt $\times$. By multiplying $a \wedge b$ by $-\mathrm{I}$ we find

$$
\begin{aligned}
-\mathrm{I}(a \wedge b) &= (a_2b_1 - a_1b_2)(e_1e_2e_3)(e_1e_2) + \\
&\quad (a_3b_1 - a_1b_3)(e_1e_2e_3)(e_1e_3) + (a_3b_2 - a_2b_3)(e_1e_2e_3)(e_2e_3).
\end{aligned}
$$

Now, from

$$
\begin{aligned}
(e_1e_2e_3)(e_1e_2) &= e_1e_3e_2e_2e_1 = -e_3 \\
(e_1e_2e_3)(e_1e_3) &= -e_1e_2e_3e_3e_1 = e_2 \\
(e_1e_2e_3)(e_2e_3) &= -e_1e_2e_3e_3e_2 = -e_1
\end{aligned}
$$

we find that

$$-\mathrm{I}a \wedge b = (a_2 b_3 - a_3 b_2)e_1 + (a_3 b_1 - a_1 b_3)e_2 + (a_1 b_2 - a_2 b_1)e_3$$

That is,

$$a \times b = -\mathrm{I}\, a \wedge b. \tag{4.6}$$

For a scalar α and vector b, from the definitions (now with $b \in \mathbb{R}^3$

$$\alpha \cdot b := 0 \quad , \quad \alpha \wedge b := \alpha b$$

where both operations are commutative. From equation 4.6 we have that

$$a \wedge b = Ia \times b$$

and then

$$ab = a \cdot b + \mathrm{I}a \times b = |a||b| \cos\theta + \mathrm{I}|a||b| \sin\theta = |a||b| \mathrm{e}^{\mathrm{I}\theta}.$$

4.3.3 Extension of the inner and outer product definitions

Let us define the inner and outer product of a vector a and a bivector B. We can write

$$aB = \frac{1}{2}(aB - Ba) + \frac{1}{2}(aB + Ba).$$

With this we define

$$a \cdot B := \frac{1}{2}(aB - Ba) \tag{4.7}$$
$$a \wedge B := \frac{1}{2}(aB + Ba), \tag{4.8}$$

and so

$$aB = a \cdot B + a \wedge B.$$

We note that in the definition $a \cdot B$ we picked the minus "-" sign instead of the plus "+" sign chosen when defining the inner product on vectors. The reason for this is that if B is a bi-vector then we should have $a \wedge B = B \wedge a$ because a needs to be rotated (permuted) twice to go through B and so its sign should not change. That is, we have

$$a \cdot B = -B \cdot a$$

Since $a \wedge B$ is the product of a vector by a bi-vector we can think of this product as $a \wedge (b \wedge c)$ where b and c are vectors and $B = b \wedge c$. We show that indeed the triple wedge product is associative. That is, we have

$$\begin{aligned}
(a \wedge b) \wedge c &= \frac{1}{2}\left[\frac{1}{2}(ab - ba)c + c\frac{1}{2}(ab - ba)\right] \\
&= \frac{1}{4}[abc - bac + cab - cba] \\
a \wedge (b \wedge c) &= \frac{1}{2}\left[a\frac{1}{2}(bc - cb)\frac{1}{2}(bc - cb)a\right] \\
&= \frac{1}{4}[abc - bcb + bca - cba].
\end{aligned}$$

We now take the difference of the two previous expressions and find

$$\begin{aligned}
(a \wedge b) \wedge c - a \wedge (b \wedge c) &= -\frac{1}{4}b(ac + ca) + \frac{1}{4}(ca + ac)b \\
&= -\frac{1}{2}b(a \cdot c) + \frac{1}{2}(a \cdot c)b \\
&= 0.
\end{aligned}$$

So, the triple wedge product is associative.

We use the notation, with $B = A_2$,

$$\begin{aligned}
a \cdot A_2 &= \frac{1}{2}[aA_2 - (-1)^2 aA_2] \\
A_2 \cdot a &= \frac{1}{2}[A_2 a - (-1)^2 A_2 a] = -a \cdot A_2.
\end{aligned}$$

It seems silly to write the product in this way since $(-1)^2 = 1$ but when we consider $\boldsymbol{A_r}$ with $r \geq 3$ this notation will make sense. Since there is no a unique inner product we are in need to introduce new symbols and define two different inner products. The *left inner product* or *left contractoion* defined by

$$a \rfloor A_2 := \frac{1}{2}[aA_2 - (-1)^2 aA_2] \tag{4.9}$$

and the *right inner product* or *right contraction* defined by

$$A_2 \lfloor a := \frac{1}{2}[A_2 a - (-1)^2 A_2 a].$$

Also, if A_2 is any 2-vector we see that

$$\begin{aligned} a \wedge A_2 &= \frac{1}{2}[aA_2 + (-1)^2 aA_2] \\ A_2 \wedge a &= \frac{1}{2}[A_2 a + (-1)^2 A_2 a] = a \wedge A_2. \end{aligned} \tag{4.10}$$

We do not use two symbols for the wedge product. After all, we know that the wedge product is not commutative and we have, since the beginning understood that $u \wedge v$ and $v \wedge u$ are not the same. We simply say that the first is a wedge product of a vector with a 2-vector, and the second is the wedge product of a 2-vector with a vector. In particular let us study the triple wedge product $a_1 \wedge a_2 \wedge a_3$ of three vectors.

It is interesting that the cross product defined in equation 4.6 is not associative and the wedge product is associative. To show that the cross product "$\times$" is non associative we use a simple example. Assume $i = (1,0,0)$, $j = (0,1,0)$ and $k = (0,0,1)$ then

$$\begin{aligned} (i \times i) \times j &= 0 \\ i \times (i \times j) &= i \times k = -j. \end{aligned}$$

from which $(i \times i) \times j \neq i \times (i \times j)$.

The algebra of $\mathcal{G}_3$ is attached to the concepts of quaternions as shown next:

4.3.4 From quaternions to vectors to the geometric product

The history of geometric algebra is linked to the introduction of quaternions by William Hamilton in 1843.

While looking for an extension of a complex number $a + \mathbf{i}\, b$ to the three dimensional space, Hamilton arrived to $a + \mathbf{i}\, b + \mathbf{j}\, c + \mathbf{k} d$, where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Then he introduced the concept of vector for a pure quaternion (that is where the scalar was not existent, or $a = 0$.)

The product of two quaternions followed the anticonmutative rule for the “base” vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, that is $\mathbf{ij} = -\mathbf{ji}$ and so forth.

The quaternions $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ can be identified with the base blade vectors $1, e_{12}, e_{23}, e_{13}$. That is, we make the mapping

$$\begin{aligned} 1 &\leftrightarrow 1 \\ e_{23} = e_2 e_3 &\leftrightarrow \mathbf{i} \\ e_{13} = e_1 e_3 &\leftrightarrow \mathbf{j} \\ e_{12} = e_1 e_2 &\leftrightarrow \mathbf{k}. \end{aligned}$$

The four base vectors $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ make a subalgebra which is the quaternions algebra.

$$\begin{aligned} e_{23}e_{13} &= e_2e_3e_1e_3 = e_1e_2 \\ e_{13}e_{12} &= e_1e_3e_1e_2 = e_2e_3 \\ e_{12}e_{23} &= e_1e_2e_2e_3 = e_1e_3 \end{aligned}$$

we see that $\mathbf{ij} = \mathbf{k}$, $\mathbf{jk} = \mathbf{i}$ and $\mathbf{ki} = \mathbf{j}$.

We can compute, the product of two vectors:

$$\boldsymbol{ab} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = -a_ib_i + c$$

(sum over repeated index i). With

$$\boldsymbol{c} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

We recognize in the first term of the product $\boldsymbol{ab}$ as the negative of the inner product $\boldsymbol{a} \cdot \boldsymbol{b}$ and the second term as the cross–product $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}$. Except for the sign of the inner product we see here the structure of the geometric product $\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b}$.

Table 4.3 shows of multiplication the quaternions

	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	-1	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	-1

Table 4.3: Products of quaternions.

The algebra of quaternions is related to the Pauli matrices [1] from the study of quantum mechanics, and to three-dimensional rotations. We will not go any deeper on this subject but recomend the article by Krishnaswami and Sachdev [7] which shows nicely some of the relations indicated here and more.

[1] https://en.wikipedia.org/wiki/Pauli_matrices

4.4 $\mathcal{G}_n$: The general GA in $\mathbb{R}^n$

As indicated already, the blade vectors basis induced by the space $\mathbb{R}^n$ are

$$\underbrace{1}_{\text{scalars}} \quad , \quad \underbrace{e_1 \, , \, e_2 \, , \, \cdots \, , \, e_n}_{\text{vectors}} \quad , \quad \underbrace{e_1 \wedge e_2 \, , \, e_1 \wedge e_3, \, \cdots \, , e_i \wedge e_j,}_{\text{bivectors}} \quad \cdots \, , \underbrace{e_1 \wedge \cdots \wedge e_n}_{\text{n order blade.}}$$

The base blade 1 generate scalars. However this is not the only way to create scalars. Remember, for example, that $e_1e_2e_1e_2 = -1$. We can group the basis of $\mathcal{G}_n$ in subbasis as follows. Basis blades of grade 0:the scalars, basis blades of grade 1: vectors, basis blades of grade 2: bivectors, and so on basis blades of an arbitrary order r, $(0 \leq r \leq n)$ creating $r-$vectors. The number of r-vectors in each basis group of grade r is given by the binomial coefficient $\binom{n}{r}$. Note that since the binomial coefficient is symmetric we have $\binom{n}{0} = \binom{n}{n} = 1$ so that the scalar and the pseudoscalar are unique up to a scale factor. Likewise since $\binom{n}{1} = \binom{n}{n-1} = n$ we have n vectors in the first case and n, (n-1)-pseudovectors in the second. The number of base blades increases toward the center as the binomial distribution does.

As indicated in equation 3.4 a multivector can be written as a sum of its projections on the blade basis. That is,

$$\begin{aligned} A &= \alpha_0 + \alpha_{11}e_1 + \cdots + \alpha_{1i_1}e_n + \alpha_{21}e_1e_2 + \cdots + \alpha_{2i_2}e_{n-1}e_n + \cdots + \alpha_{ni_n}\mathrm{I} \\ &= \langle A \rangle_0 + \langle A \rangle_1 + \cdots + \langle A \rangle_n \end{aligned}$$

The geometric product of two multivectors A and B can be written as

$$C = AB = \langle C \rangle_0 + \langle C \rangle_1 + \cdots + \langle C \rangle_n.$$

Finding the projection $\langle C \rangle_i$ could be difficult. We will only study a few simple cases.

4.4.1 Geometrical product of r vectors

To write the geometrical product of r vectors a_i in the space of n dimensions we write

$$a_i = \alpha_{ij}e_j.$$

Then

$$\begin{aligned} A = a_1a_2\cdots a_r &= (\alpha_{1j_1}e_{j_1})(\alpha_{2j_2}e_{j_2})\cdots(\alpha_{rj_r}e_{j_r}) \\ &= \alpha_{1j_1}\alpha_{2j_2}\cdots\alpha_{rj_r}e_{j_1}e_{j_2}\cdots e_{j_r}. \end{aligned}$$

Observing the last product we can have the following cases.

(i) All e_{j_k} are different. Then there are exactly r base vectors e_{j_k} ($k = 1, 2, \cdots, r$) on each factor of the $r!$ permutations. We can sort all in ascending order with indices $J_1 < J_2 < \cdots J_r$, and write

$$e_{j_1} e_{j_2} \cdots e_{j_r} \quad = \quad \epsilon_{j_1 j_2 \cdots j_r} e_{J_1} e_{J_2} \cdots e_{J_r}.$$

where

$$\epsilon_{j_1 j_2 \cdots j_r} = \begin{cases} 1 & \text{if } j_1 j_2 \cdots j_r \text{ is an even permutation of } J_1 J_2 \cdots J_r \\ -1 & \text{if } j_1 j_2 \cdots j_r \text{ is an odd permutation of } J_1 J_2 \cdots J_r. \end{cases}$$

Note that r should be smaller or equal than n (the dimension of the space) otherwise, if $r > n$ then at least two e_{j_k} should be repeated in the expression.

If $r = n$ then we have that $J_1 J_2 \cdots J_n = 12 \cdots n$. We then can write

$$\langle A \rangle_r = \epsilon_{j_1 j_2 \cdots j_r} \alpha_{1 j_1} \alpha_{2 j_2} \cdots \alpha_{r j_r} e_{J_1} e_{J_2} \cdots e_{J_r} = \det \boldsymbol{A_r} \; e_{J_1} e_{J_2} \cdots e_{J_r}$$

where $\boldsymbol{A_r}$ is the matrix with entries $\alpha_{i j_i}$.

(ii) Two e_{j_k} are equal and all other different. The equal e_{j_k} will collapse into $(-1)^m$ where m is the required number of permutations to get them together. Then we write

$$e_{j_1} e_{j_2} \cdots e_{j_r} \quad = \quad (-1)^m \epsilon_{j_1 j_2 \cdots j_{r-2}} e_{J_1} e_{J_2} \cdots e_{J_{r-2}}.$$

where, if necessary, we renamed the j_i indices. The projection $\langle A \rangle_{r-2}$ is not that easy to obtain. There are $\binom{n}{r-2}$ possible ways to choose $r-2$ vectors e_{j_i}. Once we choose a particular ordered set with indices $J_1 < J_2 < \cdots < J_{r-2}$ and then consider all its added permutations of the form

$$(-1)^m \epsilon_{j_1 j_2 \cdots j_{r-2}} \alpha_{1 j_1} \alpha_{2 j_2} \cdots \alpha_{r-2 j_{r-2}} e_{J_1} e_{J_2} \cdots e_{J_{r-2}}.$$

There are $\binom{n}{r-2}$ sums like this which build $\langle A \rangle_{r-2}$.

(iii) In general, let us assume $2k$ pairs of equal e_{j_i} vectors ($2k < r$). Then we get the building blocks of $\langle A \rangle_{r-2k}$. We find

$$e_{j_1} e_{j_2} \cdots e_{j_r} \quad = \quad (-1)^{m_1 + m_2 + \cdots m_k} \epsilon_{j_1 j_2 \cdots j_{r-2k}} e_{J_1} e_{J_2} \cdots e_{J_{r-2k}}.$$

where $m_1 + m_2 \cdots m_k$ indicate the number of permutations required to put all pairs next to each other.

We need $\binom{n}{r-2k}$ combinations of sums such as

$$(-1)^{m_1+m_2+\cdots+m_k} \epsilon_{j_1 j_2 \cdots j_{r-2k}} \alpha_{1j_1} \alpha_{2j_2} \cdots \alpha_{r-2j_{r-2k}} e_{J_1} e_{J_2} \cdots e_{J_{r-2}}.$$

to build $\langle A \rangle_{r-2k}$.

If r is even we could continue to collapse equal base vectors until we get to a scalar. That is, $r - 2k = 0$. The only projections are of grades $0, 2, 4, \cdots, r$. If r is odd we can collapse all but one base vectors up to a vector of grade 1. In this case the lower grade is $r - 2k = 1$, and we will have projections of grades $1, 3, 5, \cdots, r$.

4.4.2 Outer product of r vectors

We study the product $a_1 \wedge a_2 \wedge \cdots \wedge a_n$. Initially, as done in the previous section, we write each a_i as a linear combination of base vectors e_j. That is,

$$a_i = \alpha_{ij} e_j$$

then

$$\begin{aligned} \boldsymbol{A_r} = a_1 \wedge a_2 \wedge \cdots \wedge a_r &= (\alpha_{1j_1} e_{j_1}) \wedge (\alpha_{2j_2} e_{j_2}) \wedge \cdots (\alpha_{rj_r} e_{j_r}) \\ &= \alpha_{1j_1} \alpha_{2j_2} \cdots \alpha_{rj_r} e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_r}. \end{aligned}$$

If $r = n$, the dimension of the space $\mathbb{R}^n$, then

$$\boldsymbol{A_r} = \epsilon_{j_1 j_2 \cdots j_n} \alpha_{1j_1} \alpha_{2j_2} \cdots \alpha_{nj_n} e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_n} = \det A e_1 e_2 \cdots e_n.$$

What if $r < n$? We can think of the Gram-Schmidt process [2] to write $a_1, a_2, \cdot a_r$ as a linear combination of the vectors $e_1, e_2, \cdots e_r$. That is, we can write

$$a_i = \alpha_{ij} e_j$$

where $j = 1, 2, \cdots, r$ and $R = (\alpha_{ij})$ is the matrix of transformation between the vectors e_i and the vectors a_i. Then we can write

[2] https://en.wikipedia.org/wiki/Gram%E2%80%93Schmidt_process

$$
\begin{aligned}
\boldsymbol{A_r} = a_1 \wedge a_2 \wedge \cdots \wedge a_r &= (\alpha_{1j_1} e_{j_1}) \wedge (\alpha_{2j_2} e_{j_2}) \wedge \cdots (\alpha_{rj_r} e_{j_r}) \\
&= \alpha_{1j_1}\alpha_{2j_2}\cdots\alpha_{rj_r} e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_r} \\
&= \alpha_{1j_1}\alpha_{2j_2}\cdots\alpha_{rj_r} \epsilon_{j_1 j_2 \cdots j_r} e_1 \wedge e_2 \wedge e_r \\
&= \epsilon_{j_1 j_2 \cdots j_r} \alpha_{1j_1}\alpha_{2j_2}\cdots\alpha_{rj_r} e_1 \wedge e_2 \wedge \cdots \wedge e_r \\
&= \det R \; e_1 e_2 \cdots e_r. \qquad (4.11)
\end{aligned}
$$

Note several things:

(i) The vectors a_i are linearly independent. Otherwise the outer product is zero. This is easy shown by writing a_i as a lineal combination of the e_i vectors (use $a_i = \alpha_{ij} e_j$ for some indices j). Then in each wedge product they will appear repeated indices which leave the wedge product (being fully anti-symmetric) 0. The details of this are left to the reader in the exercises at the end of this chapter.

(ii) If the vectors are orthogonal then the determinant of R is either 1 or -1 (depending on the order of the vectors a_i. Left or right handed order).

(iii) If the vectors are not orthogonal then there would be a scaling factor (Jacobian-like) to correct from the transformation of an orthogonal to a non-orthogonal basis. Geometrically from a cube to a parallelepiped.

We want to write $a_1 \wedge a_2 \wedge \cdots \wedge a_r$ in terms of the vectors $a_1, a_2, \cdots, a_r$. Let us start with the simple case of two base vectors e_1, e_2. We know that

$$e_1 \wedge e_2 = \frac{1}{2}(e_1 e_2 - e_2 e_1) = \frac{1}{2}\epsilon_{ij} e_i e_j,$$

with $i, j = 1, 2$. In the same way we can see that

$$a_1 \wedge a_2 = \frac{1}{2} e_{ij} a_i \wedge a_j.$$

If a_i and a_j are orthogonal then

$$a_i \wedge a_j = a_i a_j - a_i \cdot a_j = a_i a_j.$$

So we can write

$$a_1 \wedge a_2 = \frac{1}{2} e_{ij} a_i a_j.$$

We want to generalize this to n vectors a_i. Let us assume initially that they are linearly idependent. Then, assuming that the wedge product is fully anti-symmetric

$$a_{i_1} \wedge a_{i_2} \wedge a_{i_n} = \epsilon_{i_1 i_2 \cdots i_n} a_1 \wedge a_2 \cdots \wedge a_n, \tag{4.12}$$

and multiplying both sides by $\epsilon_{i_1 i_2 \cdots i_n}$ we find

$$e_{i_1 i_2 \cdots i_n} a_{i_1} \wedge a_{i_2} \wedge a_{i_n} = \epsilon_{i_1 i_2 \cdots i_n} \epsilon_{i_1 i_2 \cdots i_n} a_1 \wedge a_2 \cdots \wedge a_n,$$

Now

$$\epsilon_{i_1 i_2 \cdots i_n} \epsilon_{i_1 i_2 \cdots i_n} = n!$$

since there are $n!$ permutations of $i_1, i_2, \cdots, i_n$ and all the signs of this n permutations are squared to 1. We then, assuming a fully anti-symmetric product have

$$a_1 \wedge a_2 \wedge \cdots \wedge a_n = \frac{1}{n!} \epsilon_{i_1 i_2 \cdots i_n} a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_n}.$$

In the same way we did for just two vectors we claim that if all vectors here are mutually orthogonal then the wedge product is equal to the geometrical product. That is, we have

$$a_1 \wedge a_2 \wedge \cdots \wedge a_n := \frac{1}{n!} \epsilon_{i_1 i_2 \cdots i_n} a_{i_1} a_{i_2} \cdots a_{i_n}.$$

This expression is taken as the definition of the wedge product of n vectors. If they are linearly dependent both sides produce the vector 0.

In particular is $a_i = e_i$ we can order the last strings with factors a_{i_k} by introducing an additional Levi-Civita symbol $\epsilon_{i_1 i_2 \cdots i_n}$. That is,

$$e_1 \wedge e_2 \wedge \cdots \wedge e_n := \frac{1}{n!} \epsilon_{i_1 i_2 \cdots i_n} \epsilon_{i_1 i_2 \cdots i_n} e_1 e_2 \cdots e_n.$$

And since $\epsilon_{i_1 i_2 \cdots i_n} \epsilon_{i_1 i_2 \cdots i_n} = n!$ we find

$$e_1 \wedge e_2 \wedge \cdots \wedge e_n = e_1 e_2 \cdots e_n.$$

Then we can say that the wedge product of a set of orthogonal vectors is a blade as well. What if the vectors are not orthogonal? Let us revisit equations 4.11. This tells us that the wedge product of n, a_i linearly independent vectors is an r-blade, or better a scalar times the r-blade $e_1 e_2 \cdots e_r$.

We then can say that $a_1 \wedge a_2 \wedge \cdots \wedge a_r$ spans the r-dimensional space generated by its vectors. Each point in this r-dimensional space can be seen as linear combination of the sides of an r-parallelepiped formed by the vectors a_i.

We prove an important theorem that helps us understand when a vector lies in the span represented by a blade $\boldsymbol{A_r}$.

Theorem 4.4.1. *A vector a is in the r-dimensional space spanned by $a_1, a_2, \cdots, a_r$ with $\boldsymbol{A_r} = a_1 \wedge a_2 \wedge \cdots a_r$ if and only if*

$$a \wedge \boldsymbol{A_r} = 0. \tag{4.13}$$

proof

(i) "$\Rightarrow$"
This is so because when we span a in terms of the vectors $e_1, e_2, \cdots, e_r$ and since $\boldsymbol{A_r} = \det R e_1 \wedge e_2 \wedge \cdots e_r$, then in each term of the wedge product there would be a repeated vector e_j which cancels the product to 0.

(ii) "$\Leftarrow$"
On the other hand, if $a \wedge \boldsymbol{A_r} = 0$ we show that a is linearly dependent with the set the vectors a_i which make up $\boldsymbol{A_r}$. Asume that a is linearly independent with that set. Then the set $a_1, \ldots, a_r, a = a_{r+1}$ is linearly independent. There is a matrix R which transforms the vectors $a_1, a_2, \cdots a_{r+1}$ to some orthogonal new vectors $e_1, e_2, \cdots e_{r+1}$. This transformation is written in matrix form as

$$A_{r+1} = Q_{r+1} R$$

where Q_r is an orthonormal matrix and R is a non-singular triangular matrix. Since we can write

$$\boldsymbol{A_{r+1}} = \det R e_1 e_2 \cdots e_{r+1},$$

and e_i are all different orthogonal vectors and R is non-singular, then $\det R \neq 0$ so $A_{r+1} = a \wedge A_r \neq 0$ which contradicts the hypothesis. Then a is linearly dependent with the set of vectors $a_1, a_2, \cdots a_r$. In other words a is in the span of $a_1, a_2, \cdots, a_r$.

4.4.3 Outer and inner product between vectors and blades

We motivate the definition of the outer product of a vector with an r-blade as follows. Let us think of the vector e_j in the base $\{e_1, e_2, \cdots, e_n\}$. We consider an r-blade $\boldsymbol{A_r} = \beta e_1 e_2 \cdots e_r$ where $\beta = \det R$ and R is the matrix of transformation from the base vectors e_i to the base vectors a_i. That is, $a_i = \beta_{ij} e_i$ and the columns of A are the vectors of the blade $\boldsymbol{A_r}$. Then

$$\begin{aligned} e_j \wedge \boldsymbol{A_r} &= e_j \wedge \beta e_1 \wedge e_2 \wedge \cdots \wedge e_r \\ &= \beta e_j e_1 e_2 \cdots e_r \\ &= e_j \beta e_1 e_2 \cdots e_r \\ &= e_j \boldsymbol{A_r}. \end{aligned}$$

Note that if e_j is in the set $\{e_1, e_2, \cdots, e_r\}$ then both sides are equal to 0.

On the other hand

$$\begin{aligned} e_j \wedge \boldsymbol{A_r} &= \beta e_j \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_r \\ &= (-1)^r \beta \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_r e_j \\ &= (-1)^r \boldsymbol{A_r} e_j \end{aligned}$$

then adding the previous two equations we find

$$2\, e_n \wedge \boldsymbol{A_r} = e_n \boldsymbol{A_r} + (-1)^r \boldsymbol{A_r} e_n.$$

or

$$e_n \wedge \boldsymbol{A_r} = \frac{1}{2}[e_n \boldsymbol{A_r} + (-1)^r \boldsymbol{A_r} e_n].$$

This is valid for any base vector e_j. That is,

$$e_j \wedge \boldsymbol{A_r} = \frac{1}{2}[e_j \boldsymbol{A_r} + (-1)^r \boldsymbol{A_r} e_j].$$

If, instead of a vector e_n, we have a general vector $a = \alpha_{ij} e_j$ then by multiplying (and adding) the previous equation by α_{ij} we find (in each step α_{ij} is a scalar an can be moved to any position within the strings of products)

$$\alpha_{ij} e_j \wedge \boldsymbol{A_r} = \frac{1}{2}[a_{ij} e_j \boldsymbol{A_r} + (-1)^r \boldsymbol{A_r} \alpha_{ij} e_j]$$

That is,

$$a \wedge \boldsymbol{A_r} := \frac{1}{2}[a\boldsymbol{A_r} + (-1)^r \boldsymbol{A_r} a]. \tag{4.14}$$

Along the same lines we can consider an r-vector A_r as a supeposition of blades and adding expressiones such as the previous for a set of r-vectors we find

$$a \wedge A_r := \frac{1}{2}[aA_r + (-1)^r A_r a] \tag{4.15}$$

where now A_r is an r-vector.

What we show here using the superposition principle, linear algebra and the definition of wedge product for a set of vectors is used in other places as a definition. Following the same method we can show (this is left as an exercise to the student) that

$$A_r \wedge a = \frac{1}{2}[A_r a + (-1)^r a A_r] \tag{4.16}$$

and so

$$a \wedge A_r = (-1)^r A_r \wedge a.$$

Then, if we define the geometric product of a vector and a blade, as the sum of two products (the inner and outer products). That is

$$aA_r = a \cdot A_r + a \wedge A_r,$$

then

$$\begin{aligned} a \cdot A_r &= aA_r - a \wedge A_r \\ &= aA_r - \frac{1}{2}(aA_r + (-1)^r A_r a) \\ &= \frac{1}{2}[aA_r - (-1)^r A_r a]. \end{aligned}$$

The equation

$$a \cdot A_r = \frac{1}{2}[aA_r - (-1)^r A_r a],$$

is taken often as a definition. In fact equation 4.9 was defined as an inner product of a with A_2. It is common to find the notation

$$a\rfloor A_r := \frac{1}{2}[aA_r - (-1)^r A_r a], \tag{4.17}$$

where the foot of the symbol $\rfloor$ is pointing towards the vector. This operation is called also a *left inner product* . Likewise we define the *right inner product* as

$$A_r\lfloor a := \frac{1}{2}[A_r a - (-1)^r a A_r]. \tag{4.18}$$

Clearly

$$a\rfloor A_r = (-1)^{r+1} A_r \lfloor a.$$

If we add equations 4.16 and 4.18 we find

$$A_r\, a = A_r\lfloor a + A_r \wedge a \tag{4.19}$$

Note that we defined two inner products and just one outer product. Why is this? The outer product of a vector with a blade just extend the "box" one dimension. Whether it is on the left or on the right, there is a possible sign change but the geometrical meaning is the same. However the geometrical meaning of inner product changes for inner product. We discuss this in Theorem 4.4.4.

We show now that the wedge product between vectors and k blades as defined above are in effect grade extension operators. That is, if $\boldsymbol{A_r} = a_1 a_2 \cdots a_r$ is a blade of grade r, $a \wedge \boldsymbol{A_r}$ and $A_l \wedge a$ are of grade $r+1$ or 0. a lies in the span of the vectors $a_1, a_2, \cdots a_r$ then we showed (see equation 4.13) that $a \wedge \boldsymbol{A_r} = 0$. Let us then assume that a is not in the span of the vectors $a_1, a_2, \cdots, a_r$. Then $a = \alpha_j e_j$ with $j = 1, 2, \cdots n$. Let us write $\boldsymbol{A_r} = \beta e_1 e_2 \cdots e_r$, and

$$a\boldsymbol{A_r} = \alpha_j e_j (\beta e_1 e_2 \cdots e_r)$$

For each term of this sum, let us consider two cases:

(i) $j = J$ is such that $1 \leq J \leq r$ then one of the vectors e_i, ($1 \leq i \leq r$) is repeated and we find the corresponding term to be

$$\alpha_J e_J (\beta e_1 e_2 \cdots e_J \cdots e_r) = \beta \alpha_J (-1)^J e_1 e_2 \cdots e_J e_J \cdots e_r = \beta \alpha_J (-1)^{J-1} e_1 e_2 \check{e}_J \cdots e_r$$

where the symbol $\check{e}_J$ means that this vector is taken out of the string. If, instead of $a\boldsymbol{A_r}$ we would have taken $\boldsymbol{A_r}a$ then we would have found the corresponding J term to be

$$(\beta e_1 e_2 \cdots e_r)(\alpha_j e_j) = (\beta \alpha_J e_1 e_2 \cdots e_r)e_j = \beta\alpha_J(-1)^{r-J} e_1 e_2 \cdots \check{e}_J \cdots e_r$$

and in the sum $a\boldsymbol{A_r} + (-1)^r \boldsymbol{A_r}a$ we would have found (for this J term)

$$\beta\alpha_J(-1)^{J-1} e_1 e_2 \check{e}_J \cdots e_r + (-1)^r \beta\alpha_J(-1)^{r-J} e_1 e_2 \cdots \check{e}_J \cdots e_r = 0, \qquad (4.20)$$

and this happens for each $J = 1, 2, \cdots r$. So, the only non zero contributions are obtained from the second case

(ii) $J > r$. In this case each term of $a\boldsymbol{A_r}$ is like

$$\alpha_J e_J(\beta e_1 e_2 \cdots e_J \cdots e_r) = \beta\alpha_J(-1)^r e_1 e_2 \cdots\cdots e_r e_J \qquad (4.21)$$

and each term of $\boldsymbol{A_r}a$ is like

$$(\beta e_1 e_2 \cdots e_r)(\alpha_j e_j) = (\beta\alpha_J e_1 e_2 \cdots e_r)e_J \qquad (4.22)$$

so, by adding over j (change J by j)

$$\frac{1}{2}(a\boldsymbol{A_r} + (-1)^r \boldsymbol{A_r}a) = (-1)^r \, \beta\alpha_j e_1 e_2 \cdots e_r e_j = \beta\alpha_j e_j e_1 e_2 \cdots e_r$$

and the wedge product $a \wedge A$ increases the grade of A by 1. We would like to write this expression in terms of a_i, and a, and not of e_i. First, we note that $a = \alpha_j e_j$ and $\beta = \det R = \epsilon_{j_1 j_2 \cdots j_r} \alpha_{1j_1} \alpha_{2j_2} \cdots \alpha_{rj_r}$, so

$$\begin{aligned}\frac{1}{2}(a\boldsymbol{A_r} + (-1)^r \boldsymbol{A_r}a) &= a\epsilon_{j_1 j_2 \cdots j_r} \alpha_{1j_1} \alpha_{2j_2} \cdots \alpha_{rj_r} e_1 e_2 \cdots e_r \\ &= a\, \alpha_{1j_1} \alpha_{2j_2} \cdots \alpha_{rj_r} e_{j_1} e_{j_2} \cdots e_{j_r} \\ &= a\alpha_{1j_1} e_1\, \alpha_{2j_2} e_2 \,\cdots\, \alpha_{j_r} e_r.\end{aligned}$$

Here we used the relation 4.12 with e_i instead of a_i. Now, since $\alpha_{ij} e_j = a_j$ we can write

$$\frac{1}{2}(a\boldsymbol{A_r} + (-1)^r \boldsymbol{A_r}a) \;=\; a a_1 a_2 \cdots a_r. \qquad (4.23)$$

That is

$$a \wedge \boldsymbol{A_r} = \langle a\boldsymbol{A_r}\rangle_{r+1}. \tag{4.24}$$

We leave to the reader to show that

$$\boldsymbol{A_r} \wedge a = \langle \boldsymbol{A_r} a\rangle_{r+1}. \tag{4.25}$$

Expressions 4.24 and 4.25 are sometimes used as definitions for outer product between a vector a and a r-vector $\boldsymbol{A_r}$. By using superposition we can generalize the previous two expressions to r-vectors. That is, we have

$$a \wedge A_r = \langle aA_r\rangle_{r+1}.$$

We leave to the reader to show that

$$A_r \wedge a = \langle A_r a\rangle_{r+1}.$$

We now show that the left inner of a vector and an r-vector $\boldsymbol{A_r}$ product reduces the grade of the $\boldsymbol{A_r}$ vector by one. Let us revisit equation 4.20. Since we want to estimate $\boldsymbol{A_r}a - (-1)^r a\boldsymbol{A_r}$, then, instead of 0, we find that the J-th ($1 \leq J \leq r$) term of the sum produced by $a\boldsymbol{A_r} - (-1)^r \boldsymbol{A_r} a$ we produce

$$\begin{aligned} \beta\alpha_J(-1)^{J-1} e_1 e_2 \check{e}_J \cdots e_r - (-1)^r \beta\alpha_J (-1)^{r-J} e_1 e_2 \cdots \check{e}_J \cdots e_r &= \\ 2\beta\alpha_J(-1)^{J-1} e_1 e_2 \check{e}_J \cdots e_r. \end{aligned}$$

We want to consider the case of $J > r$. In this case we use equation 4.21 for the J-th term of the sum of $a\boldsymbol{A_r}$ in terms of the basis vectors e_j, and equation 4.22 for the J-th term of the sum of $\boldsymbol{A_r}a$ in terms of the basis vectors e_j. Then by using substraction instead of addition we find

$$\frac{1}{2}(a\boldsymbol{A_r} - (-1)^r \boldsymbol{A_r} a) = 0$$

In conclusion we write $a \rfloor \boldsymbol{A_r}$ as the sum

$$a \rfloor \boldsymbol{A_r} = \beta\alpha_j(-1)^{j-1} e_1 e_2 \cdots \check{e}_j \cdots e_r$$

where now we use j instead J to indicate a sum over all $1 \leq j \leq r$.

We want to write this expression in terms of a_i and not base vectors e_i, as we did to get to equation 4.23. That is, we use the fact that $a = \alpha_j e_j$ and $\beta = \det R =$

$\epsilon_{j_1 j_2 \cdots j_r} \alpha_{1 j_1} \alpha_{2 j_2} \cdots \alpha_{r j_r}$. We note that since the factor $\check{e}_j$ is not included in the product j will not count as a repeated index over the sum but still we have a sum over j. Actually we will sum over j but the sum will be left for later. Right now we will fix the index J and we will ignore repeated indices in J as a sum. Let us expand the determinant of R in its cofactors over the column J. That is

$$\beta = \beta_{kJ} A_{kJ}.$$

(sum only over k) where A_{kj} is the kJ cofactor of the matrix R. See that $\beta_{kJ}\alpha_k$ is the inner product between the column k of R (that is of the vector a_k with a. That is we write $\alpha_k \beta_{kJ} = a \rfloor a_J = a \cdot a_J$ and

$$a \rfloor \boldsymbol{A_r} = (-1)^{J-1}\ a \rfloor a_J\ A_{kJ} e_1 e_2 \cdots \check{e}_J \cdots e_r$$

Since the cofactor A_{kJ} is the determinant of the minor matrix produced by eliminating the row k and the column J of the matrix, this can be written as

$$A_{kJ} = \epsilon_{i_1 i_2 \cdots \check{J} \cdots i_r} \beta_{1 i_1} \beta_{2 i_2} \cdots \check{\beta_{J i_J}} \cdots \beta_{r i_r}$$

where by $\check{J}$ and $\check{J i_J}$ we mean to take out those indices and factors. That is, this is an $r-1 \times r-1$ rolling along the J column as a cofactor. Then using identity 4.12 we can write

$$\begin{aligned}
J^{\text{th}} \text{ term of } a \rfloor \boldsymbol{A_r} &= (-1)^{J-1}\ a \rfloor a_J\ \epsilon_{i_1 i_2 \cdots \check{J} \cdots i_r} \beta_{1 i_1} \beta_{2 i_2} \cdots \check{\beta_{J i_J}} \cdots \beta_{ri} e_1 e_2 \cdots \check{e}_J \cdots e_r \\
&= (-1)^{k-1}\ a \rfloor a_J\ \beta_{1 i_1} \beta_{2 i_2} \cdots \check{\beta_{J i_J}} \cdots \beta_{ri}\ e_{i_1} e_{i_2} \cdots \check{e}_J \cdots e_{i_r} \\
&= (-1)^{k-1} (a \rfloor a_J)\, a_1 a_2 \cdots \check{a}_J \cdots a_r.
\end{aligned}$$

At this point we can add all J-terms and find

$$a \rfloor \boldsymbol{A_r} \quad = \quad (-1)^{j-1} (a \rfloor a_j)\, a_1 a_2 \cdots \check{a}_j \cdots a_r. \tag{4.26}$$

The interpretation of this is that $a \rfloor \boldsymbol{A_r}$ is taking inner product with every vector of $\boldsymbol{A_r}$ then scaling the other vectors by this inner product (a projection into the remaining $r-1$ dimensional space) iterating signs, starting with a "+" sign. It is the projection of a into the $\boldsymbol{A_r}$ blade so that

$$a \rfloor \boldsymbol{A_r} = \langle a \boldsymbol{A_r} \rangle_{r-1}.$$

After adding equations 4.17 and 4.14 we find

$$a\boldsymbol{A_r} = a\rfloor\boldsymbol{A_r} + a \wedge \boldsymbol{A_r}. \tag{4.27}$$

In other words we decompose the product $a\boldsymbol{A_r}$ into two parts

$$\begin{aligned} a\rfloor\boldsymbol{A_r} &= \langle a\boldsymbol{A_r}\rangle_{r-1} \\ a \wedge \boldsymbol{A_r} &= \langle a\boldsymbol{A_r}\rangle_{r+1} \end{aligned} \tag{4.28}$$

We leave to the reader to show the relation

$$\boldsymbol{A_r}\lfloor a = \langle \boldsymbol{A_r} a\rangle_{r-1}. \tag{4.29}$$

Equations 4.28 and 4.29 are sometimes used as definitions of the left and right contractions (inner products). As we did befofre we can use superposition and generalize the previous equations to r-vectors. That is,

$$\begin{aligned} a\rfloor A_r &= \langle aA_r\rangle_{r-1} \\ a \wedge A_r &= \langle aA_r\rangle_{r+1} \end{aligned}$$

and

$$A_r\lfloor a = \langle A_r a\rangle_{r-1}.$$

We use equation 4.26 to show what we expect in dealing with vectors and r-blades, mainly, when we can say that a vector is orthogonal to a blade. Being orthogonal to a blade means to be orthogonal to each vector in the blade. We show the following important result The following part up to the end of this chapter follows closely the work in Chilsom [3] notes.

Theorem 4.4.2.

(i) The vector a is orthogonal to $\boldsymbol{A_r}$ if and only if $a\rfloor\boldsymbol{A_r} = 0$.

(ii) The set $a\rfloor\boldsymbol{A_r}$ is the orthogonal complement of a in $\boldsymbol{A_r}$, for $r \geq 2$.

Proof:

(i)
- $\Rightarrow$ If a is orthogonal to $\boldsymbol{A_r}$ then it is orthogonal to each a_i that forms the blade $\boldsymbol{A_r}$. Then $a\rfloor a_i = 0$ for each $i = 1, 2, \cdots, r$. Then using equation 4.26 we see that $a\rfloor\boldsymbol{A_r} = 0$.

[3] http://arxiv.org/abs/1205.5935

- $\Leftarrow$ On the other hand, if $a\rfloor \boldsymbol{A_r} = 0$ we have the following possibilities:
  - All but one a_k, vector of $\boldsymbol{A_r}$ are not orthogonal to a. Then using again equation 4.26 we find $a\rfloor \boldsymbol{A_r} = (-1)^{k-1}(a\rfloor a_k)a_1 a_2 \cdots a_{k-1}a_{k+1}\cdots a_r$. Since $a\rfloor \boldsymbol{A}_r = 0$ and the a_i vectors are linearly independent then $a_1 a_2 \cdots a_{k-1}a_{k+1}\cdots a_r \neq 0$ and so $a\rfloor a_k = 0$ so a_k is orthogonal to $\boldsymbol{A}_r$.
  - There are a number of a_is such that none of them are orthogonal to $\boldsymbol{A_r}$. Without loss in generality and changing the index if necessary we can say that the first $a_1, a_2, \cdots, a_s$ are not orthogonal to $\boldsymbol{A_r}$. We can build the following sequence of vectors b_j, $j = 2, 3, \cdots, s$

$$b_j = a_j - \left(\frac{a\rfloor a_j}{a\rfloor a_1}\right) a_1 \tag{4.30}$$

Each b_j is nonzero because all a_i are linearly independent and $j > 1$. We see that a is orthogonal to each b_j, $j = 2, 3, \cdots s$, since

$$a\rfloor b_j = a\rfloor a_j - \left(\frac{a\rfloor a_j}{\cancel{a\rfloor a_1}}\right)\cancel{a\rfloor a_1} = a\rfloor a_j - a\rfloor a_j = 0.$$

Now the product $a_1 \wedge a_2 \wedge \cdots \wedge a_r$ does not change if we replace a_j ($j = 2, 3, \cdots s$) by b_j. That is

$$a_1 \wedge a_2 \wedge \cdots \wedge a_r = a_1 \wedge b_2 \wedge \cdots \wedge b_s \wedge a_{s+1} \cdots \wedge a_r.$$

However now we are in the first case (i) when we only had the first (a_1) vector no orthogonal to the vector a. In that case we showed that $a\rfloor a_1 = 0$. So the space $\boldsymbol{A_r}$ is orthogonal to a.

(ii) We define the **orthogonal complement** space to a subspace represented by the blade $\boldsymbol{A}$ as the set of all vectors a perpendicular to $\boldsymbol{A}$. That is exactly the meaning of the set of vectors a such that $a\rfloor \boldsymbol{A_r} = 0$. We would be required to find then the set of a such that $a\rfloor \boldsymbol{A_r} = 0$. Now we show that $a\rfloor \boldsymbol{A_r}$ is the orthogonal complement of a in $\boldsymbol{A_r}$. That is, we show that that $a\rfloor(a\rfloor \boldsymbol{A_r}) = 0$. We know, from equation 4.26 that

$$a\rfloor \boldsymbol{A_r} \;\; = \;\; (-1)^{j-1}(a\rfloor a_j)\, a_1 a_2 \cdots \breve{a}_j \cdots a_r.$$

Let us first consider a few particular cases. Starting with $a\rfloor \boldsymbol{A_2} = a\rfloor a_1 \wedge a_2$. We have, that since

$$a\rfloor a_1 \wedge a_2 = (a\rfloor a_1)a_2 - (a\rfloor a_2)a_1$$

Then

$$a\rfloor(a\rfloor a_1 \wedge a_2) = (a\rfloor a_1)(a\rfloor a_2) - (a\rfloor a_2)(a\rfloor a_1) = \det\begin{pmatrix} a\rfloor a_1 & a\rfloor a_2 \\ a\rfloor a_1 & a\rfloor a_2 \end{pmatrix} = 0.$$

Now we consider $a\rfloor \boldsymbol{A_3} = a\rfloor a_1 \wedge a_2 \wedge a_3$. This is

$$a\rfloor a_1 \wedge a_2 \wedge a_3 = (a\rfloor a_1)a_2a_3 - (a\rfloor a_2)a_1a_3 + (a\rfloor a_3)a_1a_2$$

Then

$$\begin{aligned} a\rfloor(a\rfloor a_1 \wedge a_2 \wedge a_3) &= (a\rfloor a_1)(a\rfloor a_2)a_3 - (a\rfloor a_1)(a\rfloor a_3)a_2 - (a\rfloor a_2)(a\rfloor a_1)a_3 \\ &+ (a\rfloor a_2)(a\rfloor a_3)a_1 + (a\rfloor a_3)(a\rfloor a_1)a_2 - (a\rfloor a_3)(a\rfloor a_2)a_1 \\ &= a_1[(a\rfloor a_2)(a\rfloor a_3) - (a\rfloor a_3)(a\rfloor a_2)] + a_2[(a\rfloor a_3)(a\rfloor a_1) - \\ &\quad (a\rfloor a_1)(a\rfloor a_3)] + a_3[(a\rfloor a_1)(a\rfloor a_2) - (a\rfloor a_2)(a\rfloor a_1)] \\ &= \det\begin{pmatrix} a\rfloor a_1 & a\rfloor a_2 & a\rfloor a_3 \\ a\rfloor a_1 & a\rfloor a_2 & a\rfloor a_3 \\ a_1 & a_2 & a_3 \end{pmatrix} = 0. \end{aligned}$$

We see a pattern of a determinant with two rows repeated. To better defined the pattern let us go one more step. That is let us consider $a\rfloor \boldsymbol{A_r} = a\rfloor a_1 \wedge a_2 \wedge a_3 \wedge a_4$.

$$a\rfloor a_1 \wedge a_2 \wedge a_3 \wedge a_4 = (a\rfloor a_1)a_2a_3a_4 - (a\rfloor a_2)a_1a_3a_4 + (a\rfloor a_3)a_1a_2a_4 - (a\rfloor a_4)a_1a_2a_3.$$

If we take one more left inner product with a we find that

$$a\rfloor(a\rfloor \boldsymbol{A_4}) = E - F + G - H$$

with

$$\begin{aligned} E &= (a\rfloor a_1)a\rfloor a_2a_3a_4 \\ &= (a\rfloor a_1)(a\rfloor a_2)a_3a_4 - (a\rfloor a_1)(a\rfloor a_3)a_2a_4 + (a\rfloor a_1)(a\rfloor a_4)a_2a_3 \\ F &= (a\rfloor a_2)(a\rfloor a_1)a_3a_4 - (a\rfloor a_2)(a\rfloor a_3)a_1a_4 + (a\rfloor a_2)(a\rfloor a_4)a_1a_3 \\ G &= (a\rfloor a_3)(a\rfloor a_1)a_2a_4 - (a\rfloor a_3)(a\rfloor a_2)a_1a_4 + (a\rfloor a_3)(a\rfloor a_4)a_1a_2 \\ H &= (a\rfloor a_4)(a\rfloor a_1)a_2a_3 - (a\rfloor a_4)(a\rfloor a_2)a_1a_3 + (a\rfloor a_4)(a\rfloor a_3)a_1a_2 \end{aligned}$$

We are missled to think that $a\rfloor(a\rfloor \boldsymbol{A_2})$ is a 4×4 determinant but that is no true. The 4×4 determinant has $4! = 24$ terms and we only have 12 which is half. The reason is that products of vectors $a_i a_j$ preserve the ascending order. However we observe that for each product $(a\rfloor a_j)(a\rfloor a_k)a_l a_m$ with $j<k$, $l<m$ there is a product $(a\rfloor a_k)(a\rfloor a_j)a_l a_m$ with opposite sign. Hence all 12 terms add to zero. We can use short hand notation and write

$$a\rfloor(a\rfloor \boldsymbol{A_4}) = \epsilon_{ijkl}(a\rfloor a_i)(a\rfloor a_j)b_{ijkl}$$

where

$$b_{ijkl} = \begin{cases} a_k a_l \breve{a}_j \breve{a}_k & k<l \\ 0 & \text{otherwise} \end{cases}$$

For each $k<l$ fixed the coefficient of b_{ijkl} is 0 since ϵ_{ijkl} is anti-symmetric in ij and the product $(a\rfloor a_i)(a\rfloor a_j)$ is symmetric in ij. We now move the general case. Applying 4.26 twice we find that

$$\begin{aligned} a\rfloor(a\rfloor \boldsymbol{A_r}) &= (-1)^{k_1-1}(-1)^{j-1}(a\rfloor a_{k_1})(a\rfloor a_j)a_1a_2\cdots \breve{a_{k_1}}\cdots \breve{a_j}\cdots a_r \\ &+ (-1)^{k_2-2}(-1)^{j-1}(a\rfloor a_{k_2})(a\rfloor a_j)a_1a_2\cdots \breve{a_j}\cdots \breve{a_{k_2}}\cdots a_r \end{aligned}$$

where $k_1 < j$ and $k_2 > j$. Note that the exponent in $(-1)^{k_2-2}$. The reason for this is that there is one less a_i on the left of k_2 and so there is one les (-1) in the blades to the left of the index location k_2. Both terms on this expression have an equal amount of sums. Each couple of indices (l,m) ($l\neq m$) has a corresponding (m,l) pair in the sum. For example let us choose the indices $(3,5)$ with $r=7$, then we have that the contributions of $(3,5)$ and $(5,3)$. In the first case $j=5, k_1=3$, in the second $j=3, k_2=5$. The two terms in the sum are given by

$$(-1)^{3-1}(-1)^{5-1}(a\rfloor a_3)(a\rfloor a_5)a_1a_2a_4a_6a_7 + (-1)^{3-2}(-1)^{5-1}(a\rfloor a_5)(a\rfloor a_3)a_1a_2a_4a_6a_7$$

and this is 0. In general to each term in the first sum (with $k_1<j$) there is a symmetric term in the right (with $j<k_2$) wich exactly cancel out.

We found then that

$$a\rfloor(a\rfloor \boldsymbol{A_r}) = 0,$$

and the Theorem is proven.

Together with equation 4.13, Theorem 4.4.2 tells us that to check for parallelism (vector in the span) we verify equation 4.13, that is $a \wedge \boldsymbol{A_r} = 0$ and to check for orthogonality we check Theorem 4.4.2 and more particularly $a \rfloor \boldsymbol{A_r} = 0$.

Problems 4.4.1.

(1) Show that if $a_1, a_2, \cdots a_r$ are linearly dependent vectors then $a_1 \wedge a_2 \wedge \cdots a_r = 0$.

(2) Derive equation 4.16.

(3) Derive equation 4.25.

(4) Derive equation 4.29.

4.4.4 Algebra of blades. Relation between blades, Inner and outer products between blades. Orthogonality and Parallelism in spaces.

When dealing with vectors we know that a way to show parallelism is by testing with the wedge product of two vectors. If the wedge product of two vectors is 0 then both vectors are parallel. We extend the idea of parallelism to higher dimensions. That is, we indicate that a vector a is in a space represented by a blade $\boldsymbol{A_r}$ if the wedge product of the vector with the blade $\boldsymbol{A_r}$ is zero. We extend the idea of "parallel" spaces by testing wedge product to 0 of blades. Similarly we show orthogonality in vectors by testing the inner product. If the inner product is 0 then two vectors are orthogonal. We will show below an extension of this idea for the case of vectors with blades and to the most general case of blades with blades.

According with equation 4.13, given a blade $\boldsymbol{A_r}$ we characterize the space of this blade as the set of all vectors a such that $a \wedge \boldsymbol{A_r} = 0$.

If $\boldsymbol{A_r} = \lambda \boldsymbol{B_r}$ then clearly $\boldsymbol{A_r}$ and $\boldsymbol{B_r}$ represent the same space since $a \wedge \boldsymbol{B_r} = 0$ is equivalent to say $a \wedge \boldsymbol{A_r} = a\lambda \wedge \boldsymbol{B_r} = 0$. Let us now show that if $\boldsymbol{A_r} = a_1 \wedge a_2 \wedge \cdots \wedge a_r$ and $B = b_1 \wedge b_2 \wedge \cdots b_r$ represent the same space (that is the span generated by $a_1, a_2, \cdots a_r$ is the same as the span generated by $b_1, b_2, \cdots b_r$) then necessarily $A = \lambda B$ for some non-zero scalar λ. Let us see.

If $b_i = \alpha_{ij} a_j$, then

$$\begin{aligned}
\boldsymbol{B_r} &= b_1 \wedge b_2 \cdots \wedge b_r \\
&= \alpha_{1j_1} a_{j_1} \wedge \alpha_{2j_2} a_{j_2} \cdots \wedge \alpha_{rj_r} a_{j_r} \\
&= \alpha_{1j_1} \alpha_{2j_2} \cdots \alpha_{rj_r}\, a_{j_1} \wedge a_{j_2} \wedge \cdots \wedge a_{j_r}
\end{aligned}$$

If any two a_{j_i} and a_{j_k} are repeated the product is zero, so at the end we only end up with products of all distinct a_i. We can sort them in ascending order. Then we can write

$$\begin{aligned}\boldsymbol{B_r} &= \epsilon_{j_1 j_2 \cdots j_r} \alpha_{1 j_1} \alpha_{2 j_2} \cdots \alpha_{r j_r} a_1 \wedge a_2 \wedge \cdots \wedge a_r \\ &= \lambda a_1 \wedge a_2 \wedge \cdots \wedge a_r \\ &= \lambda \boldsymbol{A_r}\end{aligned}$$

with $\lambda = \epsilon_{j_1 j_2 \cdots j_r} \alpha_{1 j_1} \alpha_{2 j_2} \cdots \alpha_{r j_r} = \det A$, where A is the matrix of the transformation from the base of vectors a_i to the base of vectors b_i. We then say that two blades characterize the same subspace if and only if one is a scalar multiple of the other .

We now show that if $\boldsymbol{A_r} = a_1 \wedge a_2 \wedge \cdots \wedge a_r$ represents a proper subspace of $\boldsymbol{A_s}$ ($r < s$), then we can extend $\boldsymbol{A_r}$ in a way that $\boldsymbol{A_s} = \boldsymbol{A_r} \wedge \boldsymbol{A_{s-r}}$. If $a_1, a_2, \cdots a_r$ are linearly independent vectors representing the space associated with $\boldsymbol{A_r}$, then we can consider new vectors $a_{r+1}, a_{r+2}, \cdots, a_s$, so that the blade $\boldsymbol{A_r} \wedge \boldsymbol{A_{s-r}}$ spans the same space represented by $\boldsymbol{A_s}$. Then $\boldsymbol{A_s}$ and $\boldsymbol{A_r} \wedge \boldsymbol{A_{s-r}}$ represent the same space and they can only differ a scalar product. That is, $\boldsymbol{A_s} = \lambda \boldsymbol{A_r} \wedge \boldsymbol{A_{s-r}}$. Now we can absorb the scalar λ into one of the vectors of $\boldsymbol{A_{s-r}}$ and rename if necessary (if $\lambda \neq 1$) the blade $\boldsymbol{A_{s-r}}$. That is, we can say that exists $\boldsymbol{A_{s-r}}$ such that

$$\boldsymbol{A_s} = \boldsymbol{A_r} \wedge \boldsymbol{A_{s-r}}.$$

Let us characterize and define the different products between multivectors.

4.4.4.1 Geometrical Product between multivectors

We show that if $A = \sum_r A_r, B = \sum_s B_s$ then $AB = \sum_{r,s} A_r B_s$. We now consider the geometrical product between two blades. We claim that

$$A_r B_s = \sum_{j=0}^{\min\{r,s\}} \langle A_r B_s \rangle_{|r-s|+2j}. \tag{4.31}$$

If $r = s = 0$ we have the product of two scalars which is $A_0 B_0 = \langle A_0 B_0 \rangle_0$. Let us then assume that $0 < r \leq s$. We assume that A_r is a blade $\boldsymbol{A}_r$. If $r = 1$ then we have the product of a vector $a = \boldsymbol{A_1}$ with B_s. That is,

$$\boldsymbol{A_1} B_s = a B_s = a \cdot B_s + a \wedge B_s.$$

Since here $r = 1$ then $\boldsymbol{A_r} B_s = \langle A_1 B_s \rangle_{s-1} + \langle A_1 B_s \rangle_2$. This matches equation 4.35. If A_r is an r-vector, by superposition the equation is valid as well. We show that equation 4.35 is true for any $\boldsymbol{A}_r$ by induction over r. That is, let us assume the equation to be true for some r. Then, writing $\boldsymbol{A_{r+1}} = a \boldsymbol{A_r}$ where a is a vector, and since $s \leq r$,

$$\begin{aligned} \boldsymbol{A_{r+1}} B_s &= a \sum_{j=0}^{\min\{r,s\}} \langle \boldsymbol{A_r} B_s \rangle_{|r-s|+2j} \\ &= \sum_{j=0}^{r} a \langle \boldsymbol{A_r} B_s \rangle_{s-r+2j}. \end{aligned}$$

We can call $k = s - r + 2j$, and $C_k = \langle A_r B_s \rangle_k$. Then C_k is a k-vector so we can write

$$aC_k = a \rfloor C_k + a \wedge C_k = \langle aC_k \rangle_{k-1} + \langle aC_k \rangle_{k+1}.$$

That is, we find

$$\begin{aligned} \boldsymbol{A_{r+1}} B_s &= \sum_{j=0}^{r} \langle \boldsymbol{A}_{r+1} B_s \rangle_{s-r+2j-1} + \sum_{j=0}^{r} \langle \boldsymbol{A}_{r+1} B_s \rangle_{s-r+2j+1} \\ &= \langle \boldsymbol{A}_{r+1} B_s \rangle_{s-r-1} + 2 \langle \boldsymbol{A}_{r+1} B_s \rangle_{s-r-1} \cdots \\ &\quad +2 \langle \boldsymbol{A}_{r+1} B_s \rangle_{s-r-1+2(r-1)} + \langle \boldsymbol{A}_{r+1} B_s \rangle_{s-r-1+2r} \end{aligned}$$

We can write the set of indices as $|r+1-s|+0, |r+1-s|+2, \cdots |r+1-s|+2(r-1), |r+1-s|+2r,$ [4] and so the sum can be written as

$$\boldsymbol{A_{r+1}} B_s = \sum_{j=0}^{\min\{s,r+1\}} \alpha_j \langle \boldsymbol{A_{r+1}} B_s \rangle_{|r+1-s|+2j}$$

where $\alpha_j = 1$ for $i = 0, \min\{s, r+1\}$ and 2 for $i = 1, 2, \cdots r$ The factor α_j can be absorbed by the $\langle . \rangle$ operator. We showed that $\boldsymbol{A_r} B_s$ can be written as a sum of projections with grades jumping by 2. By superposition we have the result valid for r-vectors. That is,

$$A_r B_s = \sum_{j=0}^{\min\{s,r\}} \alpha_j \langle A_r B_s \rangle_{|r-s|+2j}.$$

While $A_r B_s$ is not necessarily a k-vector it is a sum of k-vectors. If $r+s$ is even then the terms of $A_{r+1} B_s$ are all even and if $r+s$ is odd then the terms of $A_{r+1} B_s$ are all odd.

Now

[4] recall that if $s-r-1=0$ then $\langle \boldsymbol{A}_{r+1} B_s \rangle_{s-r-1} = 0$.

$$
\begin{aligned}
a\rfloor(A_rB_s) &= a\rfloor\sum_{j=0}^{\min\{s,r\}}\alpha_j\langle A_rB_s\rangle_{|r-s|+2j}\\
&= \sum_{j=0}^{\min\{s,r\}}\frac{1}{2}\alpha_j[a\langle A_rB_s\rangle_{|r-s|+2j}-(-1)^{|r-s|+2j}\langle A_rB_s\rangle a_{|r-s|+2j}]\\
&= \frac{1}{2}[a(A_rB_r)-(-1)^{|r-s|}(A_rB_s)a].
\end{aligned}
$$

Likewise it is easy to show that

$$a\wedge(A_rB_s)=\frac{1}{2}(a(A_rB_s)+(-1)^{|r-s|}A_rB_sa). \tag{4.32}$$

4.4.4.2 Left, right inner products and outer products between multivectors

The general definition for left, right inner products and outer products between multivectors is given as follows:

Definition 4.4.1. If $A=\sum_r A_r$ and $B=\sum_s B_s$ then

$$
\begin{aligned}
A\rfloor B &:= \sum_{r,s}\langle A_rB_r\rangle_{s-r}\\
A\lfloor B &:= \sum_{r,s}\langle A_rB_r\rangle_{r-s}\\
A\wedge B &:= \sum_{r,s}\langle A_rB_r\rangle_{r+s}.
\end{aligned}
$$

All definitions of inner and outer products are special cases of the definitions above. We have the following facts related to these definitions.

(i) A scalar. $A_r\rfloor B_r=A_r\lfloor B_r=\langle A_rB_r\rangle$.

(ii) If $r>s$ then $A_r\rfloor B_s=B_s\lfloor B_r=0$ since multivectors with negative grade are 0.

(iii) The lowest grade of A_rB_r is $A_r\rfloor B_s$ if $r\leq s$ and $A_r\lfloor B_s$ if $r\geq s$ and the highest grade term is is $A_r\wedge B_s$.

Here are some geometrical facts about outer products. We first see that if the blades $\boldsymbol{A_r}$ and $\boldsymbol{A_s}$, $r,s\geq 1$ share at least a non-zero vector their outer product is 0. This is so, since this would mean that the vectors that compose $\boldsymbol{A_r}$ and $\boldsymbol{A_s}$ are no linearly independent. We can verify this by using equation 4.13. If $\boldsymbol{A_r}\wedge\boldsymbol{A_s}\neq 0$ then, the only common element to the spaces representing $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ is 0 and so the whole space $\boldsymbol{A_r}\wedge\boldsymbol{A_s}$ can be written as a direct sum of the corresponding subspaces.

We nos show a theorem of the various associative laws for inner and outer products.

Theorem 4.4.3 (Associative Laws between r-vectors). *Let A_r, B_s and C_t, r,s,t vectors respectively. Then we have the following associative laws:*

(i) $A_r \wedge (B_t \wedge C_s) = (A_r \wedge B_t) \wedge C_s$

(ii) $A_r \rfloor (B_t \rfloor C_s) = (A_r \wedge B_t) \rfloor C_s$

(iii) $A_r \rfloor (B_t \lfloor C_s) = (A_r \rfloor B_t) \lfloor C_s.$

(iv) $A_r \lfloor (B_t \wedge C_s) = (A_r \lfloor B_t) \lfloor C_s$

Particularly we observe that nor $\rfloor$ neither $\lfloor$ are associative, however, as seen in ii, $\rfloor$ is associative when putting together with $\lfloor$. Also, he have that, for example

$$A_r \wedge (B_s \wedge C_t) = \langle A_r B_s C_t \rangle_{r+s+t},$$

and in particular

$$a_1 \wedge a_2 \wedge \cdots \wedge a_r = \langle a_1 a_2 \cdots a_r \rangle_r.$$

Proof:

(i) We have the following sequence of operations:

$$\begin{aligned} A_r \wedge (B_s \wedge C_t) &= A_r \wedge \langle B_s C_t \rangle s + t \\ &= \langle A_r \langle B_s C_t \rangle s + t \rangle_{r+s+t} \\ &= \langle A_r B_s C_t \rangle_{r+s+t} \\ &= \langle \langle A_r B_s \rangle_{r+s} C_t \rangle_{r+s+t} \\ &= (A_r \wedge B_s) \wedge C_t. \end{aligned}$$

(ii) We have

$$\begin{aligned} A_r \rfloor (B_s \rfloor C_t) &= A_r \rfloor \langle B_s C_t \rangle_{t-s} \\ &= \langle A_r \langle B_s C_t \rangle_{t-s} \rangle_{t-(r+s)}. \end{aligned}$$

If $t < s$ then both sides vanish. If $t > s$ but $t < r + s$ then again both sides vanish. Let us then assume that $t \geq r + s$. Then

$$\begin{aligned} A_r \rfloor (B_s \rfloor C_t) &= \langle A_r B_s C_t \rangle_{t-(r+s} \\ &= \langle \langle A_r B_s \rangle_{r+s} C_t \rangle_{t-(r+s)} \\ &= \langle A_r B_s \rangle_{r+s} \rfloor C_t \\ &= (A_r \wedge B_s) \rfloor C_t. \end{aligned}$$

(iii) We first write

$$A_r\rfloor(B_s\lfloor C_t) \quad = \quad A_r\rfloor(\langle B_sC_t\rangle_{s-t}$$

We consider several sceneries. If $s < t$ then $s - t < 0$, and $s - t - r < 0$ and both sides are 0. If $s > t$ but $r > s - t$ then also both sides are 0. We are left only with the case $r \leq s - t$. We have

$$\begin{aligned} A_r\rfloor(B_s\lfloor C_t) &= \langle A_r\langle B_sC_t\rangle_{s-t}\rangle_{s-(t+r)} \\ &= \langle A_rB_sC_t\rangle_{(s-r)-t} \\ &= \langle\langle A_rB_s\rangle_{s-r}C_t\rangle_{(s-r)-t} \\ &= \langle A_rB_s\rangle_{s-r}\lfloor C_t \\ &= (A_r\rfloor B_s)\lfloor C_t. \end{aligned}$$

(iv) This is left as an exercise to the reader.

We now provide the definition of orthogonality between spaces.

Definition 4.4.2 (orthogonaliy between spaces)**.** Given two spaces which represent a blade $\boldsymbol{A_r}$ and a blade $\boldsymbol{B_s}$. We say that they are orthogonal if each vector on the first space is orthogonal to the second space. That is, each vector on the first space is orthogonal to each in the second space.

In what follows we prove a list of theorems where we only use the left inner product "$\rfloor$". All theorems could be re-written to consider instead the right inner product "$\lfloor$”, using a symmetric reasoning. We will skip that list of theorems and their proof.

Next, let us show a theorem which is a generalization of Theorem 4.4.1:

Theorem 4.4.4 (About left inner product on blades)**.** *Let us assume that $\boldsymbol{A_r}$ and $\boldsymbol{B}_s$ are nonzero blades with $r, s \geq 1$. Then:*

(i) $\boldsymbol{A_r}\rfloor\boldsymbol{B_s} = 0$ if and only if $\boldsymbol{A_r}$ contains a nonzero vector orthogonal to $\boldsymbol{B_s}$.

(ii) If $r < s$ and $\boldsymbol{A_r}\rfloor\boldsymbol{B_s} \neq 0$, this left inner product is a blade representing the orthogonal complement of $\boldsymbol{A_r}$ in $\boldsymbol{B_s}$. This means that the blade $\boldsymbol{A_r}\rfloor\boldsymbol{B_s}$ of grade $s - r$ is a subspace of the blade $\boldsymbol{B}_s$ and each vector of this blade is orthogonal to $\boldsymbol{A_r}$.

Note here that geometrically $\boldsymbol{A_r}\rfloor\boldsymbol{B_s}$ would have a different meaning of $\boldsymbol{A_r}\lfloor\boldsymbol{B_s}$ and this justifies two different definitions for inner products.

Proof;

(i) We prove this by induction on r. If $r = 1$ then we have a vector $a = a_1$ and consider $a \rfloor \boldsymbol{B_s}$. If $a \rfloor \boldsymbol{B_s} = 0$ then this is just a particular case of the Theorem 4.4.2.

Let us now assume the the theorem is valid for a blade $\boldsymbol{A_r}$ and we show that it is also valid for a blade of order $\boldsymbol{A_{r+1}}$. That is, we assume

$$\boldsymbol{A_r} \rfloor \boldsymbol{B_s} = 0,$$

and so

$$\boldsymbol{A_{r+1}} \rfloor \boldsymbol{B_s} = \boldsymbol{A_r} \wedge a_{r+1} \rfloor \boldsymbol{B_s}.$$

We now use equation *(ii)* in Theorem 4.4.3. That is, we write

$$\boldsymbol{A_{r+1}} \rfloor \boldsymbol{B_s} = \boldsymbol{A_r} \rfloor (a_{r+1} \rfloor \boldsymbol{B_s}).$$

Let us show the "$\Leftarrow$" implication. Assume that $\boldsymbol{A_{r+1}}$ has a vector which is orthogonal to $\boldsymbol{B_s}$. We can assume that this vector is a_{r+1} (or commute if not) and so $a_{r+1} \wedge \boldsymbol{B_s} = 0$, so $\boldsymbol{A_{r+1}} \rfloor \boldsymbol{B_s} = 0$.

Now, For the "$\Rightarrow$" implication let us assume that $\boldsymbol{A_{r+1}} \rfloor \boldsymbol{B_s} = 0$; then it follows that $\boldsymbol{A_r} \rfloor (a_{r+1} \rfloor \boldsymbol{B}_s) = 0$. There are three options:

(a) If $a_{r+1} \rfloor \boldsymbol{B_s} = 0$ then a_{r+1} is orthogonal to $\boldsymbol{B}_s$.

(b) Otherwise we apply the induction hypothesis in $\boldsymbol{A}_r$ and so $\boldsymbol{A_r}$ has a vector a which is orthogonal to $a_{r+1} \rfloor \boldsymbol{B_s}$ [5]. If a is orthogonal to $\boldsymbol{B_s}$ we are done, so let us assume that a is not orthogonal to $\boldsymbol{B_s}$ this yields the third case.

(c) Recall the mapping 4.30. For $\boldsymbol{B_s}$ we write it as

$$c_j = b_j - \left(\frac{a_{r+1} \rfloor b_j}{a_{r+1} \rfloor b_1} \right) b_1$$

and then we write $\boldsymbol{B_s} = b_1 c_2 \cdots c_s$ where a_{r+1} is orthogonal to each c_i, $i = 2, \cdots r$. Then we can write

$$a_{r+1} \rfloor \boldsymbol{B}_s = (a_{r+1} \rfloor b_1) c_2 \wedge \cdots \wedge c_s. \tag{4.33}$$

[5] note that $a_{r+1} \rfloor \boldsymbol{B_s}$ is not necessarily a blade but a sum of blades. We can use the superposition principle to pass from an $\boldsymbol{B_s}$ blade to an s-vector B_s.

We construct

$$c = a_{r+1} - \left(\frac{a_{r+1}\rfloor b_1}{a\rfloor b_1}\right) a. \tag{4.34}$$

$c \neq 0$ since $a \in \boldsymbol{A_r}$ and a_{r+1} is linearly independent with each vector of $\boldsymbol{A_r}$. Now, since a is orthogonal to $a_{r+1}\rfloor \boldsymbol{B_s}$ and looking into equation 4.33 we see that a is orthogonal to $c_2, c_3, \cdots, c_s$. Now by construction in equation 4.34

$$c\rfloor b_1 = a_{r+1}\rfloor b_1 - a_{r+1}\rfloor b_1 = 0.$$

Then $c \in \boldsymbol{A_{r+1}}$ is orthogonal to $\boldsymbol{B_s}$. So we found a vector in $\boldsymbol{A_{r+1}}$ which is orthogonal to $\boldsymbol{B}_s$. This finishes the proof.

(ii) We use induction again on r. If $r = 1$ we have from 4.4.2 that given $a\rfloor \boldsymbol{B_s} \neq 0$, with $a = A_1$, is the orthogonal complement of a in $\boldsymbol{B_s}$. Now, let us assume that if $\boldsymbol{A_r}\rfloor \boldsymbol{B_s} \neq 0$ then $\boldsymbol{A_r}\rfloor \boldsymbol{B_s}$ is the orthogonal complement of $\boldsymbol{A_r}$ in $\boldsymbol{B_s}$. Let us then assume that $\boldsymbol{A_{r+1}}\rfloor \boldsymbol{B_s} \neq 0$. Now, as we did above, using an "associative law" $\boldsymbol{A_{r+1}}\rfloor \boldsymbol{B_s} = \boldsymbol{A_r}\rfloor (a_{r+1}\rfloor \boldsymbol{B_s}) \neq 0$. By using the induction hypothesis $\boldsymbol{A_{r+1}}\rfloor \boldsymbol{B_s}$ is the orthogonal complement of $\boldsymbol{A_r}$ in $a_{r+1}\rfloor \boldsymbol{B_s}$, and $a_{r+1}\rfloor \boldsymbol{B_s}$ is the orthogonal complement of a_{r+1} in $\boldsymbol{B_s}$. This means that a vector a is in $\boldsymbol{A_{r+1}}\rfloor \boldsymbol{B_s}$ if and only it lies in $\boldsymbol{B_s}$ and it is orthogonal to both a_{r+1} and $\boldsymbol{A_r}$. That is, it is orthogonal to $\boldsymbol{A_{r+1}}$ in $\boldsymbol{B_s}$.

Example 4.4.1. Let us assume $\boldsymbol{A_2} = e_1e_2$ and $\boldsymbol{B_3} = e_3e_4e_5$, then

$$\begin{aligned}
\boldsymbol{A_2}\rfloor \boldsymbol{B_3} &= (e_1e_2)\rfloor (e_3e_4e_5) \\
&= e_1\rfloor [e_2\rfloor e_3e_4e_5)] \\
&= e_1\rfloor [\cancelto{0}{(e_2\rfloor e_3)}e_4e_5 + \cancelto{0}{(e_2\rfloor e_4)}e_3e_5 + \cancelto{0}{(e_2\rfloor e_5)}e_3e_4] \\
&= 0
\end{aligned}$$

This makes sense. The blade $e_3e_4e_5$ is orthogonal to the blade e_1e_2. Let us now consider a case where the left contraction is non-zero. Use $\boldsymbol{B_3} = e_1e_2e_3$, then

$$\begin{aligned}
\boldsymbol{A_2}\rfloor \boldsymbol{B_3} &= (e_1e_2)\rfloor (e_1e_2e_3) \\
&= e_1\rfloor [e_2\rfloor e_1e_2e_3)] \\
&= e_1\rfloor [\cancelto{0}{(e_2\rfloor e_1)}e_2e_3 + (e_2\rfloor e_2)e_1e_3 + \cancelto{0}{(e_2\rfloor e_3)}e_1e_2] \\
&= e_1\rfloor [e_1e_3] \\
&= e_3
\end{aligned}$$

which is indeed the orthogonal complement of $\boldsymbol{A_2} = e_1e_2$ in $\boldsymbol{B_3} = e_1e_2e_3$.
We now state a list of important properties about blades in the following theorem.

Theorem 4.4.5. *Here we deal with blades $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ with $r, s \geq 1$.*

(i) $\boldsymbol{A_r}^2$ *is a scalar.*

(ii) $\boldsymbol{A_r}$ *is invertible if and only if* $\boldsymbol{A_r} \neq 0$ *and* $\boldsymbol{A_r}^{-1} = \boldsymbol{A_r}/\boldsymbol{A_r}^2$. *Then* $\boldsymbol{A_r}$ *and* $\boldsymbol{A_r}^{-1}$ *represent the same subspace*

(iii) $\boldsymbol{A_r}$ *is invertible if and only if the inner product is nonedegenerate on* $\boldsymbol{A_r}$.

(iv) If $1 \leq r \leq s$, *then*

- *(a) If* $\boldsymbol{A_r}$ *represents a subspace of* $\boldsymbol{B_s}$, *then* $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r} \rfloor \boldsymbol{B_s}$.
- *(b) The converse if true if:*
  - *(1)* $r = 1$ *or* $s = 1$, *or*
  - *(2)* $\boldsymbol{A_r}$ *or* $\boldsymbol{B_s}$ *is invertible*

(v)
- *(a) If* $\boldsymbol{A_r}$ *and* $\boldsymbol{B_s}$ *are orthogonal, then* $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r} \wedge \boldsymbol{B_s}$.
- *(b) The converse is true if either:*
  - *(1)* $r = 1$ *or* $s = 1$, *or*
  - *(2)* $\boldsymbol{A_r}$ *or* $\boldsymbol{B_s}$ *is invertible.*

Proof:

(i) If A_r is an r-blade then $A_r = a_1a_2\cdots a_r$ where the vectors a_i are orthogonal. Then

$$\boldsymbol{A_r}^2 = (a_1a_2\cdots a_r)(a_1a_2\cdots a_r)$$

To move a_1 from the second product to the place, after a_1, in the first product we need $r-1$ permutations, then to move a_2 to the place, after a_2 in first factor we need $r-2$ permutations. We keep on moving the a_is from the second product so that they are sitting after each a_i on the first product until the a_{r-1} on the second product is moved 1 place to put it before a_r in the first product. That is a total of $1+2+\cdots+r-1 = r(r-1)/2$ permutations. Then we find

$$\boldsymbol{A_r}^2 = (-1)^{\frac{r(r-1)}{2}}\ a_1^2a_2^2\cdots a_r^2$$

which is a scalar.

(ii) Let us assume $\boldsymbol{A_r} \neq 0$ invertible, this is equivalent to say that there is $\boldsymbol{A_r}^{-1}$ such that $\boldsymbol{A_r}\boldsymbol{A_r}^{-1} = 1$. To find $\boldsymbol{A_r}^{-1}$ we observe that

$$\boldsymbol{A_r}\frac{\boldsymbol{A_r}}{\boldsymbol{A_r}^2} = \frac{\boldsymbol{A_r}^2}{\boldsymbol{A_r}^2} = 1.$$

That is, $\boldsymbol{A_r}/\boldsymbol{A_r}^2$ is $\boldsymbol{A_r}^{-1}$.

(iii) To show that the inner product is nonedegenerate in $\boldsymbol{A_r}$ we show that for any vector such that $a\rfloor\boldsymbol{A_r} = 0$ then $a = 0$.

We know that $a\boldsymbol{A_r} = a\rfloor\boldsymbol{A_r} + a \wedge \boldsymbol{A_r} = a \wedge \boldsymbol{A_r}$. We multiply both sides by $\boldsymbol{A_r}^{-1}$ to both sides on the right to find

$$a = (a \wedge \boldsymbol{A_r})\boldsymbol{A_r}^{-1} = (a \wedge \boldsymbol{A_r}\rfloor\boldsymbol{A_r}^{-1}) + (a \wedge \boldsymbol{A_r}) \wedge \boldsymbol{A_r}^{-1} = 0,$$

since both terms in the last sum are zero. The first because $a \wedge \boldsymbol{A_r}$ is an r+1-blade contracted against an r-grade. The second term is a wedge product with at least a repeated vector (up to a scalar).

(iv) Let us assume that $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ are nonzero blades with $1 \leq r \leq s$. Then

(a) We assume that $\boldsymbol{A_r}$ represent a subspace of $\boldsymbol{B_s}$ and show that $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r}\rfloor\boldsymbol{B_s}$. As usual we use induction over r. If $r = 1$ then we have that $\boldsymbol{A_1} = a$ is a subspace of $\boldsymbol{B_s}$ and so $a \wedge \boldsymbol{B_s} = 0$, so from the equation $a\boldsymbol{B_s} = a\rfloor\boldsymbol{B_s} + a \wedge \boldsymbol{B_s}$ we find that $a\boldsymbol{B_s} = a\rfloor\boldsymbol{B_s}$, which shows the validity of the result for $r = 1$. Let us now assume that the result is valid for some r. That is,

$$\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r}\rfloor\boldsymbol{B_s} \tag{4.35}$$

Then

$$\boldsymbol{A_{r+1}}\boldsymbol{B_s} = \boldsymbol{A_r}(a_{r+1}\boldsymbol{B_s}) = \boldsymbol{A_r}(a_{r+1}\rfloor\boldsymbol{B_s}) = \boldsymbol{A_r}\rfloor(a_{r+1}\rfloor\boldsymbol{B_s}) = \boldsymbol{A_{r+1}}\rfloor\boldsymbol{B_s}.$$

Note that we used the induction hypothesis assuming that $a_{r+1}\rfloor\boldsymbol{B_s}$ is a blade (of grade $s+1$). This assumption is justified as in equation 4.33.

(b) (1) Let us assume $r = 1$, , $\boldsymbol{A_r} = a_1$, and $a_1\boldsymbol{B_s} = a_1\rfloor\boldsymbol{B_s}$ then from $a_1\boldsymbol{B_s} = a_1\rfloor\boldsymbol{B_s} + a_1 \wedge \boldsymbol{B_s}$ then $a_1 \wedge \boldsymbol{B_s} = 0$ and so $a_1 \in \boldsymbol{B_s}$. Likewise, if $s = 1$ then since $1 \leq r \leq s$ then $r = s = 1$ and we have that given $a_1 b_1 = a_1\rfloor b_1$ then $a_1\rfloor b_1 = 0$ a_1 is parallel to b_1 and the space representing a_1 is the same (and so a subspace) of the space representing b_1. That is, $\boldsymbol{A_r}$ is a subspace of $\boldsymbol{B_s}$ as desired.

(2) Let us now assume that $\boldsymbol{A_r}$ is invertible. Then, from $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r}\rfloor\boldsymbol{B_s}$ we find $\boldsymbol{B_s} = \boldsymbol{A_r}^{-1}\boldsymbol{A_r}\rfloor\boldsymbol{B_s}$. If $r = s$ then $\boldsymbol{B_s}$ is a multiple of A_r^{-1} which is multiple of $\boldsymbol{A_r}$. That is, both $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ represent the same subspace and we can say that $\boldsymbol{A_r}$ is a subspace of $\boldsymbol{B_s}$. Let us now assume that $r < s$. Given that $\boldsymbol{B_s}$ has grade s, then $\boldsymbol{A_r}^{-1}\boldsymbol{A_r}\rfloor\boldsymbol{B_s}$ has grade s as well. No, the grade s is the highest grade of this product happens at $\boldsymbol{A_r}^{-1}\wedge \boldsymbol{A_r}\rfloor\boldsymbol{B_s}$. Then $\boldsymbol{B_s}$ is a direct sum of $\boldsymbol{A_r}^{-1}$ and $\boldsymbol{A_r}\rfloor\boldsymbol{B_s}$. Then $\boldsymbol{A_r}$ represent a subspace of $\boldsymbol{B_s}$, since $\boldsymbol{A_s}^{-1}$ represents a subspace of $\boldsymbol{A_s}$. On the other hand if, instead of assuming $\boldsymbol{A_r}$ invertible let us assume $\boldsymbol{B_s}$ invertible. Then multiplying both sides of equation 4.35, on the right, by $\boldsymbol{B_s}^{-1}$ we find

$$\boldsymbol{A_r} = (\boldsymbol{A_r}\rfloor\boldsymbol{B_s})\boldsymbol{B_s}^{-1}.$$

Now, both sides of this equation have grade r which is the lowest grade . This grade is achieved in the right hand side with the expression $(\boldsymbol{A_r}\rfloor\boldsymbol{B_s})\rfloor\boldsymbol{B_s}^{-1}$ where we see that $\boldsymbol{A_r}$ is a subspace of $\boldsymbol{B_s}^{-1}$ and hence a subspace of $\boldsymbol{B_s}$.

(v) (a) Let us assume that $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ are orthogonal. We simply write the product as

$$\begin{aligned}\boldsymbol{A_r}\boldsymbol{B_s} &= a_1a_2\cdots a_rb_1b_2\cdots b_r\\ &= a_1\wedge a_2\wedge\cdots\wedge a_r\wedge b_1\wedge b_2\wedge\cdots\wedge b_r = \boldsymbol{A_r}\wedge\boldsymbol{B_s},\end{aligned}$$

since by the definition of blade all vectors are orthogonal

(b) We have two choices

(1) Let us assume that $r = 1$ or $s = 1$. If $r = 1$ then $\boldsymbol{A_r} = a_1$ and from $a_1\boldsymbol{B_s} = a\wedge\boldsymbol{B_s}$ and $a_1\boldsymbol{B_s} = a_1\rfloor\boldsymbol{B_s} + a_1\wedge\boldsymbol{B_s}$ we have that $a_1\rfloor\boldsymbol{B_s} = 0$. That is, a_1 is orthogonal to $\boldsymbol{B_s}$, or what is the same, $\boldsymbol{A_r}$ is orthogonal to $\boldsymbol{B_s}$. If $s = 1$, then $\boldsymbol{B_s} = b_1$ and from $\boldsymbol{A_r}b_1 = \boldsymbol{A_r}\wedge b_1$ and using equation 4.19 with b_1 instead of a,

$$\boldsymbol{A_r}b_1 = \boldsymbol{A_r}\lfloor b_1 + \boldsymbol{A_r}\wedge b_1$$

we find $\boldsymbol{A_r}\lfloor b_1 = 0$ and so $\boldsymbol{A_r}$ is orthogonal to b_1. That is, the space represented by $\boldsymbol{A_r}$ is orthogonal to that represented by $\boldsymbol{B_s}$.

(2) Let us assume that $\boldsymbol{A_r}$ is invertible. Then, from $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r}\wedge\boldsymbol{B_s}$ and multiplying both sides on the left by $\boldsymbol{A_r}^{-1}$ we find

$$\boldsymbol{B_s} = \boldsymbol{A_r}^{-1}(\boldsymbol{A_r}\wedge\boldsymbol{B_s})$$

Now $\boldsymbol{A}_r^{-1}$ represents the same subspace that $\boldsymbol{A}_r$, and $\boldsymbol{A}_r$ represents a subspace of that represented by $\boldsymbol{A}_r \wedge \boldsymbol{B}_s$. We then use proposition *(iv)-a* on this theorem to write

$$\boldsymbol{B}_s = \boldsymbol{A}_r^{-1}(\boldsymbol{A}_r \wedge \boldsymbol{B}_s) = \boldsymbol{A}_r^{-1} \rfloor \boldsymbol{A}_r \wedge \boldsymbol{B}_s,$$

then from Theorem 4.4.4 $\boldsymbol{B}_s$ is orthogonal to $\boldsymbol{A}_r^{-1}$, and so it is orthogonal to $\boldsymbol{A}_r$.
If, on the other hand, $\boldsymbol{B}_s$ is invertible we can postmultiply $\boldsymbol{A}_r\boldsymbol{B}_s = \boldsymbol{A}_r \wedge \boldsymbol{B}_s$ by $\boldsymbol{B}_s^{-1}$ and find

$$\boldsymbol{A}_r = (\boldsymbol{A}_r \wedge \boldsymbol{B}_s)\boldsymbol{B}_s^{-1}.$$

We cannot use the proposition *(iv)-(a)* on this theorem since we need now $\boldsymbol{B}_s$ to be a subspace of $\boldsymbol{A}_r$, or better (now in this context) $\boldsymbol{B}_s^{-1}$ is a subspace of $\boldsymbol{A}_r \wedge \boldsymbol{B}_s$. Instead of proposition *(iv) -(a)* we require the following proposition (left to the reader)

$$\textit{If } \boldsymbol{B}_s \textit{ is a subspace of } \boldsymbol{A}_r \textit{ then} \boldsymbol{A}_r\boldsymbol{B}_s = A_r \lfloor \boldsymbol{B}_s. \tag{4.36}$$

Now then with that proposition we have, in the actual context, that

$$\boldsymbol{A}_r = (\boldsymbol{A}_r \wedge \boldsymbol{B}_s) \lfloor \boldsymbol{B}_s^{-1},$$

and using the following proposition not shown (since as we said above, we are not proving propositions with "$\lfloor$")

$$\begin{aligned}&\textit{If } r > s \textit{ and } \boldsymbol{A}_r \lfloor \boldsymbol{B}_s \neq 0 \textit{ then } \boldsymbol{A}_r \lfloor \boldsymbol{B}_s \textit{ of grade } r - s\\ &\textit{is a subspace of the blade } \boldsymbol{A}_r\\ &\textit{which is the orthogonal complement of } \boldsymbol{B}_s\end{aligned} \tag{4.37}$$

then $\boldsymbol{A}_r$ is orthogonal to $\boldsymbol{B}_s^{-1}$; that is, $\boldsymbol{A}_r$ is orthogonal to $\boldsymbol{B}_s$. We leave the proof of the proposition 4.37 to the reader as an exercise.

Finally we show two important identities.

Theorem 4.4.6. *If* $1 \leq r \leq s$ *and the nonzero blades* $\boldsymbol{A}_r$ *and* $\boldsymbol{B}_s$ *satisfy* $\boldsymbol{A}_r\boldsymbol{B}_s = \boldsymbol{A}_r \rfloor \boldsymbol{B}_s$ *then*

(i)

$$\boldsymbol{A}_r^2\boldsymbol{B}_s = \boldsymbol{A}_r \wedge (\boldsymbol{A}_r \rfloor \boldsymbol{B}_s)$$

(ii)

$$\boldsymbol{B}_s = \boldsymbol{A}_r^{-1} \wedge (\boldsymbol{A}_r \rfloor \boldsymbol{B}_s) = \boldsymbol{A}_r \wedge (\boldsymbol{A}_r^{-1} \rfloor \boldsymbol{B}_s)$$

Proof:

(i)

$$A_r^2 B_s = A_r(A_r B_s) = A_r(A_r \rfloor B_s).$$

Now, since $A_r \rfloor B_s$ is orthogonal to A_r then using the item $(v) - a$ of the previous theorem we have that $A_r(A_r \rfloor B_s) = A_r \wedge (A_r \rfloor B_s)$. Then

$$A_r^2 B_s = A_r \wedge (A_r \rfloor B_s),$$

as desired.

(ii) If $A = 0$ then both sides of the previous equation are zero. Otherwise we can divide both sides by the scalar A_r^2 and hence

$$B_s = \frac{A_r}{A_r^2} \wedge (A_r \rfloor B_s) = A_r^{-1} \wedge (A_r \rfloor B_s) = A_r \wedge \left(\frac{A_r}{A_r^2} \rfloor B_s \right) = A_r \wedge (A_r^{-1} \rfloor B_s).$$

Problems 4.4.2.

(1) Derive equation 4.32.

(2) Proof equation (iv) of Theorem 4.4.3.

(3) Proof proposition 4.36.

(4) Proof proposition 4.37.

4.5 Projection of Subspaces

We now present a generalization of the chain of equations 2.2. For any vector a and blade A_r we can write

$$a = (aA_r)A_r^{-1} = (a \rfloor A_r + a \wedge A_r)A_r^{-1} = (a \rfloor A_r)A_r^{-1} + (a \wedge A_r)A_r^{-1}.$$

We interpret, as in the discussion behind the equation 2.2, the first term as the *projection* of a into the space A and the second as the *rejection* of a by A. When A is a vector n we obtain again equation 2.2. Let us analyze both components.

Given that $A_r^{-1} = A_r / A_r^2$ we can write

$$(a\rfloor \boldsymbol{A_r})\boldsymbol{A_r}^{-1} = \left(a\rfloor \frac{\boldsymbol{A_r}}{\sqrt{\boldsymbol{A_r^2}}}\right)\frac{\boldsymbol{A_r}}{\sqrt{\boldsymbol{A_r^2}}} = (a\rfloor \boldsymbol{U}_r)\boldsymbol{U_r},$$

with $\boldsymbol{U_r} = \boldsymbol{A_r}/\sqrt{\boldsymbol{A_r^2}}$. We see that $\boldsymbol{U_r}$ makes the role of a "unitary" blade along $\boldsymbol{A_r}$. Both $\boldsymbol{U_r}$ and $\boldsymbol{A_r}$ span the same subspace. Now, since $a\rfloor\boldsymbol{U_r}$ represents a subspace of $\boldsymbol{U_r}$ we can write this first term as

$$(a\rfloor \boldsymbol{A_r})\boldsymbol{A_r}^{-1} = (a\rfloor \boldsymbol{U_r})\rfloor \boldsymbol{U_r}.$$

If $a\rfloor\boldsymbol{U_r} = 0$ then a is orthogonal to $\boldsymbol{U_r}$ and hence to $\boldsymbol{A_r}$ and the orthogal projection of a into $\boldsymbol{A_r}$ is 0. Otherwise $a\rfloor\boldsymbol{U_r}$ is the orthogonal complement of a in $\boldsymbol{U_r}$. This is a blade of grade $r-1$ which is orthogonal to a in $\boldsymbol{A_r}$. Then $(a\rfloor\boldsymbol{U_r})\rfloor\boldsymbol{U_r}$ is the orthogonal complement of the orthogonal complement of a in $\boldsymbol{U_r}$, which is the part of a which lies in $\boldsymbol{U_r}$. A grade of order $r-(r-1)=1$.

Likewise, we can write

$$(a\wedge \boldsymbol{A_r})\boldsymbol{A_r}^{-1} = \left(a\wedge \frac{\boldsymbol{A_r}}{\sqrt{\boldsymbol{A_r^2}}}\right)\frac{\boldsymbol{A_r}}{\sqrt{\boldsymbol{A_r^2}}} = (a\wedge \boldsymbol{U}_r)\boldsymbol{U_r},$$

Now, since $\boldsymbol{U_r}$ represents a subspace of $a\wedge\boldsymbol{U_r}$ we can write this second term as

$$(a\wedge \boldsymbol{A_r})\boldsymbol{A_r}^{-1} = (a\wedge \boldsymbol{U_r})\lfloor \boldsymbol{U_r}.$$

If $a\wedge\boldsymbol{U_r} = 0$ then a is in the space generated by $\boldsymbol{U_r}$ and there is no (zero) rejection of a in $\boldsymbol{U_r}$. Otherwise if $a\wedge\boldsymbol{U_r} \neq 0$ and we have that the second term is the orthogonal complement of $\boldsymbol{U_r}$ on $a\wedge\boldsymbol{U_r}$. A vector or grade $(r+1)-r=1$ orthogonal to $\boldsymbol{U_r}$; that is, orthogonal to $\boldsymbol{A_r}$. We illustrate this with a simple example.

Example 4.5.1. Let us assume $\mathbb{R}^3$ as the reference space. Choose $a=(1,3,-2)$ and $\boldsymbol{A_2} = e_1e_2$. $\boldsymbol{A_2}^{-1} = (e_1e_2)/(e_1e_2e_1e_2) = -e_1e_2$. Then

$$\begin{aligned}
a\rfloor \boldsymbol{A_2} &= (e_1+3e_2-2e_3)\rfloor e_1e_2 \\
&= (e_1\rfloor e_1)e_2 - (e_1\rfloor e_2)e_1 + 3(e_2\rfloor e_1)e_2 - 3(e_2\rfloor e_2)e_1 - 2(e_3\rfloor e_1)e_2 + 2(e_3\rfloor e_2)e_1 \\
&= e_2 - 3e_1 = (-3,1,0)
\end{aligned}$$

y

$$(a\rfloor \boldsymbol{A_2})\boldsymbol{A_2}^{-1} = (e_2-3e_1)(-e_1e_2) = -e_2e_1e_2 + 3e_1e_1e_2 = e_1+3e_2 = (1,3,0)$$

which in effect is the projection of $(1, 3, -2)$ along the horizontal plane spanned by e_1 and e_2.

Now

$$a \wedge \boldsymbol{A_2} = (e_1 + 3e_2 - 2e_3) \wedge (e_1e_2) = \cancelto{0}{e_1 \wedge (e_1e_2)} + \cancelto{0}{3e_2 \wedge (e_1e_2)} - 2e_3 \wedge (e_1e_2) = -2\mathrm{I}$$

So

$$(a \wedge \boldsymbol{A_2})\boldsymbol{A_2}^{-1} = (-2\mathrm{I})(-e_1e_2) = 2e_1e_2e_3e_1e_2 = -2e_3 = (0, 0, -2)$$

which confirms the decomposition of $a = (1, 3, -2)$ en $(1, 3, 0)$ in $\boldsymbol{A_2}$ and $(0, 0, -2)$ orthogonal to $\boldsymbol{A_2}$.

Along the same lines of projections we could analyze reflections in high dimensional spaces but we will omit this analysis.

Problems 4.5.1.

(1) Assuming $\boldsymbol{R}^5$ as the space of reference and $A_3 = e_1e_2e_4$ find the decomposition of $a = (3, 1, 5, 3, 0)$ in terms of a projection into A_3 and a rejection on A_3.

4.6 Crammer's Rule

We now derive Crammer's rule by extending the work did in 2.4.

Let us assume a linear system

$$a_{ij}x_j = b_i \tag{4.38}$$

We can write this system as

$$x_1A_1 + x_2A_2 + \cdots + x_nA_n = b$$

where A_i is the i-th column of the matrix $A = (a_{ij})$ and $x = (x_1, x_2, \cdots, x_n)$ is the vector solution which we want to find. We assume that the vectors A_i are linearly independent (otherwise we do not have a unique solution). The idea is that to knock down the variables $x_2, x_3, \cdots, x_n$ we should take wedge product with their corresponding vectors to the right. That is,

$$x_1 A_1 \wedge A_2 \cdots \wedge A_n = b \wedge A_2 \wedge A_3 \wedge \cdots \wedge A_n.$$

See that for each x_i with $i > 1$ there is always a repeated vector A_i in the chain of wedge products, hence all those terms are zero. We can write these chains of wedge products in terms of the cannonical vectors e_i and find

$$x_1 \det A\mathrm{I} = \det B_1$$

with $\mathrm{I} = e_1 e_2 \cdots e_n$. Here B_1 is the matrix that has b in the first column and $A_2, A_3, \cdots, A_n$ in the other columns. That is

$$x_1 = \frac{\det B_1}{\det A}.$$

In general, and using the same idea (multiply by all, but the A_i next to each x_i) we find

$$x_i = \frac{\det B_i}{\det A},$$

where B_i is the matrix that has b in the i column and $A_1, A_2, \cdots A_{i-1}, A_{i+1}, A_n$ in the other columns. This is precisely Crammer's rule to solve the linear system 4.38.

Chapter 5

What is next?

There is a lot of material that we could add to these notes. I make a wish list which could be filled up in the future.

(i) Extend the concept of projection and rejection to arbitrary r-vectors and in general to multivectors.

(ii) Include the concept of rotations for arbitrary r-vectors and in general to multivectors.

(iii) Include the concept of reflections for arbitrary r-vectors and in general to multivectors.

(iv) Introduce and extend concepts such as

   (a) grade Involution
   (b) reversion
   (c) Clifford conjugation
   (d) scalar product
   (e) versor
   (f) dual
   (g) commutator and
   (h) frames and basis

   among others.

After this could think of transformations between GA objects. Among those transformations the linear transformation are of great importance. This should take us into connections with linear algebra. Differential and integral calculus could be considered in the context of GA and more generally differential geometry and how GA can be extended to manifolds.

There are many applications of GA to physics and engineering. For example in crystallography...

Appendices

Appendix A

A counter–example

This example shows that non every multivector of the geometric algebra has an inverse. Problem taken from David Hestenes' book [4].

Let $\boldsymbol{A} = \alpha + \boldsymbol{a}$ where α is a scalar and $\boldsymbol{a}$ is a non–zero vector.

- Find $\boldsymbol{A}^{-1}$ as a function of α and $\boldsymbol{a}$. What conditions on α and $\boldsymbol{a}$ imply that $\boldsymbol{A}^{-1}$ does not exist?

solution Let us assume that there exists an object $\beta + \boldsymbol{b}$ which is the inverse of $\alpha + \boldsymbol{a}$. That is

$$(\alpha + \boldsymbol{a})(\beta + \boldsymbol{b}) = 1$$

Expanding this

$$\alpha\beta + \alpha\boldsymbol{b} + \beta\boldsymbol{a} + \boldsymbol{a}\boldsymbol{b} = 1$$

Since 1 is scalar, then we need $\boldsymbol{b} = \gamma\boldsymbol{a}$ for some γ, so then

$$\alpha\beta + \alpha\gamma\boldsymbol{a} + \beta\boldsymbol{a} + \gamma\boldsymbol{a}^2 = 1$$

At this moment we need

$$\alpha\gamma = -\beta \qquad \alpha\beta + \gamma\boldsymbol{a}^2 = 1.$$

From the first equation into the second

$$\alpha\beta - \frac{\beta}{\alpha}\boldsymbol{a}^2 = 1$$

No matter which α or which β, if $\boldsymbol{a}^2 = \alpha^2$, we get

$$\alpha\beta - \alpha\beta = 1 \quad \Rightarrow \quad 0 = 1,$$

which is a contradiction, so it can never happen that $a^2 = \alpha^2$, if we want $\boldsymbol{A}$ to be invertible.

Here is a short cut to this solution, that assumes that we use division [1] of geometric algebra objects

$$(\alpha + \boldsymbol{a})^{-1} = \frac{1}{\alpha + \boldsymbol{a}} = \frac{1}{\alpha + \boldsymbol{a}} \frac{\alpha - \boldsymbol{a}}{\alpha - \boldsymbol{a}} = \frac{\alpha - \boldsymbol{a}}{\alpha^2 - \boldsymbol{a}^2}.$$

This is undefined if $\alpha^2 = \boldsymbol{a}^2$.

- Show that if $\boldsymbol{A}^{-1}$ does not exist, then $\boldsymbol{A}$ can be normalized so that

$$\boldsymbol{A}^2 = \boldsymbol{A}.$$

A quantity with this property is said to be *idempotent* .

solution If $\alpha^2 = \boldsymbol{a}^2$ let $\hat{\boldsymbol{A}} = \boldsymbol{A}/2\alpha$ then

$$\hat{\boldsymbol{A}}\hat{\boldsymbol{A}} = \left(\frac{\alpha + \boldsymbol{a}}{2\alpha}\right)\left(\frac{\alpha + \boldsymbol{a}}{2\alpha}\right) = \frac{\alpha^2 + 2\alpha\boldsymbol{a} + \boldsymbol{a}^2}{4\alpha^2} = \frac{2\alpha^2 + 2\alpha\boldsymbol{a}}{4\alpha^2} = \frac{\alpha + \boldsymbol{a}}{2\alpha} = \hat{\boldsymbol{A}}$$

So $\hat{\boldsymbol{A}}$ is idempotent.

- Show that if $\boldsymbol{A} \neq 1$ is idempotent, its product with any other multivector is not invertible. It can be proved that every multivector which does not have an inverse has an idempotent for a factor.

solution Assume $\boldsymbol{B}$ is a multivector and $\boldsymbol{A}$ is idempotent. Then if $\boldsymbol{AB}$ is invertible then

$$\frac{1}{\boldsymbol{AB}} = \frac{\boldsymbol{A}}{\boldsymbol{A}^2\boldsymbol{B}} = \frac{\boldsymbol{A}}{\boldsymbol{AB}} = \frac{1}{\boldsymbol{B}}$$

That is $\boldsymbol{AB} = \boldsymbol{B}$, but $\boldsymbol{A} \neq 1$, so this is a contradiction, unless $\boldsymbol{B} = 0$, for which $\boldsymbol{AB} = 0$, which can not have an inverse.

- Find an idempotent which does have an inverse. Let us assume A is idepotent and has an inverse. That is $A^2 = A$ and $A\,A^{-1} = 1$. Multiply the first equality by A^{-1}, so $A = 1$. Indeed 1 is invertible and nilpotenent and it is the only multivector which is idemponent and has an inverse.

[1] Division is justified if there is an inverse.

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