Advanced Electrodynamics

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Chapter 1

Lecture # 1

I did not typed this lecture. Only a few notes are here.

1.1 Curvilinear Coordinates

We need to go beyond the Cartesian coordinates and quite often we use cylindrical and spherical coordinates. Here I show a few insights into curvilinear coordinates in general.

1.1.1 The Basics

We assume some transformation

$$u = u(x, y, z) \quad , \quad v = v(x, y, z) \quad , \quad w = w(x, y, z).$$

where (x, y, z) is our old friend, the Cartesian coordinates. We assume that they can be inverted (this needs a one-to-one relationship in the working space). That is we want to be able to write

$$x = x(u, v, w) \quad , \quad y = y(u, v, w) \quad , \quad z = z(u, v, w).$$

The transformation above are very general and we want to impose a constraint. That is we want to assure that at any point in the coordinate system the coordinate lines, defined by $u = a$, $v = b$, and $w = c$, where (a, b, c) are constants, are orthogonal. For example, in cylindrical coordinates the radius is always orthogonal to the concentric circles, and the z component is always pointing perpendicular to the any horizontal plane in the (r, θ) space.

One important concept is how to expand a differential $d\boldsymbol{r}$ in the new coordinate system. This is defined by the equation

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{du}du + \frac{\partial \boldsymbol{r}}{dv}dv + \frac{\partial \boldsymbol{r}}{dw}dw. \tag{1.1}$$

Here the vectors $\partial\boldsymbol{r}/\partial u$, $\partial\boldsymbol{r}/\partial v$, and $\partial\boldsymbol{r}/\partial w$, are tangent to the coordinate lines. Let us call $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{v}}$, and $\hat{\boldsymbol{w}}$ the unit vectors tangent to the coordinate lines, and let us define the stretching factors

$$h_u = \left|\frac{\partial \boldsymbol{r}}{\partial u}\right| \quad . \quad h_v = \left|\frac{\partial \boldsymbol{r}}{\partial v}\right| \quad . \quad h_w = \left|\frac{\partial \boldsymbol{r}}{\partial w}\right| \quad . \tag{1.2}$$

we can write equation 1.1 as

$$d\boldsymbol{r} = h_u\hat{\boldsymbol{u}} + h_v\hat{\boldsymbol{v}} + h_w\hat{\boldsymbol{w}}$$

We use equation 1.2 to derive the stretching factors when changing from one coordinate to another. This are important to compute all kind of measures such as line differentials $d\ell$, area differentials dS and volume differentials dV. We will see how that is done.

As an example in the evaluation of the h factors, let us assume cylindrical coordinates in the two-dimensional space. That is, we use the transformation

$$x(r,\theta) = \rho\cos\theta \quad , \quad y(r,\theta) = \rho\sin\theta$$

Then we have that

$$\boldsymbol{r} = (x,y) \quad \Rightarrow \begin{cases} \frac{d\boldsymbol{r}}{d\rho} = (\cos\theta, \sin\theta) & h_\rho = \left|\frac{\partial \boldsymbol{r}}{\partial\theta}\right| = 1 \\ \\ \frac{d\boldsymbol{r}}{d\theta} = (-r\sin\theta, r\cos\theta) & h_\theta = \left|\frac{\partial \boldsymbol{r}}{\partial\theta}\right| = r \end{cases}$$

then

$$d\boldsymbol{r} = d\rho\boldsymbol{e}_\rho + \rho d\theta\boldsymbol{e}_\theta.$$

The line element is given by

$$d\ell = \sqrt{d\boldsymbol{r}\cdot\boldsymbol{r}} = \sqrt{d\rho^2 + \rho^2(d\theta)^2}$$

and the area element is given by

$$dS = h_\rho h_\theta d\rho d\theta = \rho d\rho d\theta.$$

more generally

$$d\ell = \sqrt{d\boldsymbol{r}\cdot\boldsymbol{r}} = \sqrt{(h_u du)^2 + (h_v dv)^2 + (h_w dw)^2}$$

and the volume element is:

$$dV = h_u h_v h_w du dv dw$$

The surface elements are given by

$$dS_u = h_v h_w dv dw \quad , \quad dS_v = h_v h_w du dw \quad , \quad dS_w = h_u h_v du dv.$$

1.1.2 The Gradient

Given a function $f(u, v, w)$ we write its differential its differential can be written as

$$df = \nabla f \cdot \boldsymbol{r}$$

or

$$\begin{aligned} df &= \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw \\ &= \frac{1}{h_u}\frac{\partial f}{\partial u}h_u du + \frac{1}{h_v}\frac{\partial f}{\partial v}h_v dv + \frac{1}{h_w}\frac{\partial f}{\partial w}h_w dw \\ &= \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{\boldsymbol{u}}\cdot d\boldsymbol{r} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{\boldsymbol{v}}\cdot d\boldsymbol{r} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{\boldsymbol{w}}\cdot d\boldsymbol{r} \\ &= \left(\frac{1}{h_u}\frac{\partial f}{\partial u}\hat{\boldsymbol{u}} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{\boldsymbol{v}} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{\boldsymbol{w}}\right)\cdot d\boldsymbol{r} \end{aligned}$$

from which we find that in the coordinates (u, v, w) we can write the gradient as

$$\nabla f = \quad = \quad \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{\boldsymbol{u}} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{\boldsymbol{v}} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{\boldsymbol{w}}.$$

1.1.3 The Divergence and the Laplacian

We use the integral definition of the divergence. This is, for a given point $\boldsymbol{r}$ in the space, the **divergence** is given by the following limit

$$\nabla \boldsymbol{F}(\boldsymbol{r}) \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS.$$

where V is a volume surrounding $\boldsymbol{r}$, and $\boldsymbol{n}$ is a normal pointing outward. For simplicity we consider a small cube around the point. The vector $\boldsymbol{F}$ has three components along the curvilinear coordinates $F = (F_u, F_v, F_w)$. In the small cube we consider the area elements around the point $\boldsymbol{r}$ at the center of the cube. Figure 1.1 illustrates the elements used in the derivation of the divergence formula. The field here goes to the right-up direction and the projection into the coordinate line with direction $\hat{\boldsymbol{v}}$ is shown. The area elelment $dS_u = h_v h_w dv dw$, and the flow that goes back at $u - du/2$ is $-F_u h_v h_w dv dw$, the flow that goes to front is $F_u h_v h_w dv dw$. Using Taylor series we have that

$$\int_{\text{surf}\perp u} \boldsymbol{F} \cdot \boldsymbol{n} dS = \frac{\partial F_u h_v h_w}{\partial u} du dv dw$$

likewise

$$\int_{\text{surf}\perp v} \boldsymbol{F} \cdot \boldsymbol{n} dS = \frac{\partial F_v h_u h_w}{\partial u} du dv dw$$

and

$$\int_{\text{surf}\perp w} \boldsymbol{F} \cdot \boldsymbol{n} dS = \frac{\partial F_w h_u h_v}{\partial u} du dv dw.$$

That is,

$$\oint_S \boldsymbol{F} \cdot \boldsymbol{n} dS = \left[\frac{\partial (F_u h_v h_w)}{\partial u} + \frac{\partial (F_v h_u h_w)}{\partial v} + \frac{\partial (F_w h_u h_v)}{\partial w} \right] du dv dw.$$

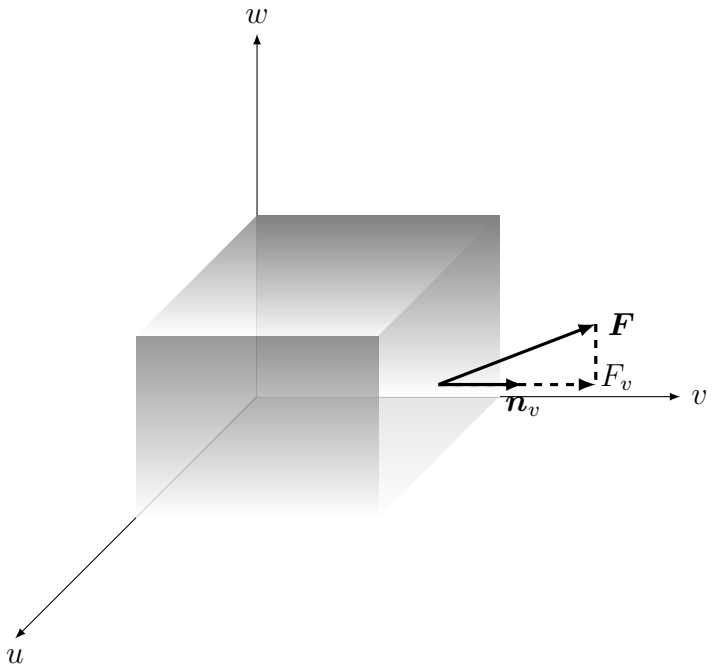

Figure 1.1: A cube to derive the divergence equation.

and usig $dV = dudvdwh_uh_vh_w$ we find

$$\nabla \cdot F = \frac{1}{h_u h_v h_w}\left[\frac{\partial(F_u h_v h_w)}{\partial u} + \frac{\partial(F_v h_u h_w)}{\partial v} + \frac{\partial(F_w h_u h_v)}{\partial v}\right].$$

and since $\nabla^2 f = \nabla \cdot \nabla f$, we see that calling $F = \nabla f$, we see that

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right]. \quad (1.3)$$

1.1.4 The Curl

The integral definition of the curl is given by

$$[\nabla \times \boldsymbol{F}]_n \equiv \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_\gamma \boldsymbol{F} \cdot d\boldsymbol{r},$$

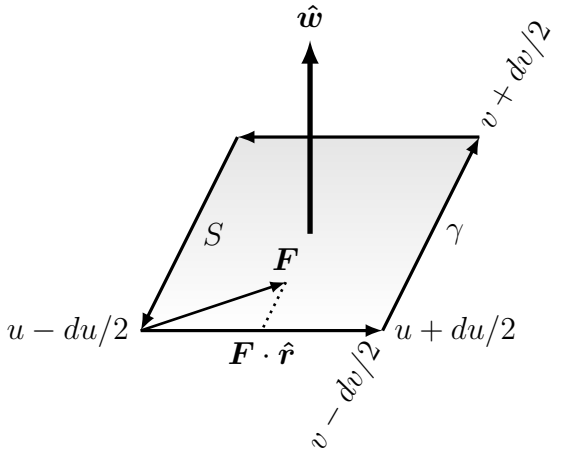

Figure 1.2: Figure to illustrate the derivation of the curl formula.

where the observation point is inside the surface S, and the curve γ encircles the surface. The vector $\boldsymbol{n}$ is notrmal to the surface. Let us start with the w component. We need to select an element perpendicular to $\hat{\boldsymbol{w}}$. This element is a loop. We can think of a square loop with the point at the center. Figure 1.2 shows an sketch of the loop.

Then in one side the field is circulating at the point $v - dv/2$ with the contribution $F_v h_u du$, and in the oposite side at $v + dv/2$ with contribution $-F_v h_u du$. These together produce

$$-\frac{\partial(h_u F_u)}{dv} dudv. \tag{1.4}$$

We need to account for the other two sides of the square loop. At $u + du/2$ we have the contribution $F_v hvdv$, and at $u - du/2$, $-F_v h_v dv$, for a total contribution of

$$\frac{\partial(F_v h_v)}{\partial u} dudv. \tag{1.5}$$

Then from 1.4 and 1.5

$$[\nabla \times \boldsymbol{F}]_{\hat{\boldsymbol{w}}} = \frac{1}{h_u h_v dudv} \left[\frac{\partial(F_v h_v)}{\partial u} - \frac{\partial(F_u h_u)}{\partial v} \right] dudv = \frac{1}{h_u h_v} \left[\frac{\partial(F_v h_v)}{\partial u} - \frac{\partial(F_v h_v)}{\partial v} \right].$$

The other two components can be derived in the same way. We can infer the formulas using the cyclip permutation $u \to v \to w \to u$. We find the formula

$$\nabla \times \boldsymbol{F} = \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u \hat{\boldsymbol{u}} & h_v \hat{\boldsymbol{v}} & h_w \hat{\boldsymbol{w}} \\ \partial/\partial u & \partial/\partial v & \partial/\partial w \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}.$$

As an example we apply the formulas above to derive the Laplacian in cylindrical and spherical coordinates.

1.1.5 The Laplacian in Cylindrical and Spherical Coordinates

Cylindrical Coordinates

The cylindrical coordinate transfromation is given by

$$x = \rho \cos\theta \quad , \quad y = \rho \sin\theta \quad , \quad z = z.$$

Now,

$$h_\rho = \sqrt{\frac{\partial \boldsymbol{r}}{\partial \rho} \cdot \frac{\partial \boldsymbol{r}}{\partial \rho}}$$

and

$$\frac{\partial \boldsymbol{r}}{\partial \rho} = (\cos\theta, \sin\theta, 0).$$

So, $h_\rho = \cos^2\theta + \sin^2\theta = 1$.

Now

$$\frac{\partial \boldsymbol{r}}{\partial \theta} = (-\rho \sin\theta, \rho \cos\theta, 0)$$

so $h_\theta = \rho(\sin^2\theta + \cos^2\theta) = \rho$. Finally $\partial \boldsymbol{r}/\partial z = (0, 0, 1)$ and $h_z = 1$.

We now use the Laplacian equation 1.3 in curvilinear coordinates

$$\begin{aligned}\nabla^2 u(\boldsymbol{r}) &= \frac{1}{h_\rho h_\theta h_{zeta}}\left[\frac{\partial}{\partial\rho}\left(\frac{h_\theta h_{zeta}}{h_\rho}\frac{\partial u}{\partial\rho}\right)+\frac{\partial}{\partial\theta}\left(\frac{h_\rho h_{zeta}}{h_\theta}\frac{\partial u}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(\frac{h_\rho h_\theta}{h_{zeta}}\frac{\partial u}{\partial z}\right)\right] \\ &= \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{\rho}\frac{\partial u}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(\rho\frac{\partial u}{\partial z}\right)\right]. \qquad (1.6)\end{aligned}$$

Spherical Coordinates

The spherical coordinates transformation is given by

$$x = r\sin\theta\cos\phi \quad , \quad y = r\sin\theta\sin\phi \quad , \quad z = r\cos\theta,$$

where θ is the azimuthal angle and ϕ the polar angle. We compute:

$$\frac{\partial \boldsymbol{r}}{\partial r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

and from here $h_r = 1$. Now,

$$\frac{\partial \boldsymbol{r}}{\partial \theta} = (r\cos\theta\cos\phi, r\cos\theta\sin\phi, -r\sin\theta),$$

and so $h_\theta = r$. Finally

$$\frac{\partial \boldsymbol{r}}{\partial \phi} = (-r\sin\theta\sin\phi, r\sin\theta\cos\phi, 0),$$

so $h_\phi = r\sin\theta$.

Now for the Laplacian in spherical coorindates we find

$$\begin{aligned}\nabla^2 u(\boldsymbol{r}) &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial\phi}\right)\right] \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\partial r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}. \qquad (1.7)\end{aligned}$$

1.2 Electro Statics

This section was not recorded here.

1.2.1 Coulomb's Law

For two charges

For any discrete number of charges

For a continuum of charges. Line, Surface, Volume

1.2.2 The Electric Field

Due to a charge

Due to a discrete number of charges

Due to a continuum of charges. Line, Surface, Volume

Example 1.2.1 *Find the electrical field due to an spherical shell with radius R shell and uniform charge σ.*

Solution: We assume that the sphere of radius R centered at 0. Let us assume an observation point o above the north pole of the sphere (by symmetry this should provide a good answer). We consider rings from the bottom up to the north polo $(0, 0, R)$. The ring at a high z, $-R \leq z \leq R$ has a radius $\rho = \sqrt{R^2 - z^2}$. We prefer to see the problem as a function of the polar angle from $-\pi/2$ to $\pi/2$. We have that $\rho = R\sin\theta$ with θ the polar angle. It is well known that for a ring with uniform charge density σ , radius r and an observation point in the axis of the ring at a distance d from the center (in the direction of the axis of the ring) produces the field

$$E(d, \rho) = \frac{\sigma \rho\, d}{2\epsilon_0(\rho^2 + d^2)^{3/2}}.$$

The distance between the observation point o and the ring at z height is $d = o - z$, and $z = R\sin\theta$, then we find

$$E(\theta) = \frac{\sigma(o - R\sin\theta)R\sin\theta}{2\epsilon_0(R^2\sin^2\theta + (o - R\sin\theta)^2)^{3/2}}.$$

We need to integrate along θ between 0 and π. Along the polar axis the element of integration is $d\ell = R\,d\theta$, so we will need to multiply by $R\,d\theta$.

$$E = \frac{\sigma}{2\epsilon_0}\int_{-\pi/2}^{\pi/2} \frac{(o - R\sin\theta)R^2\sin\theta}{(R^2\sin^2\theta + (o - R\sin\theta)^2)^{3/2}}d\theta. \tag{1.8}$$

Let us perform the following substitution

$$\begin{aligned} u = \cos\theta \quad , \quad du &= -\sin\theta d\theta \\ \theta = 0 &\implies u = 1 \\ \theta = \pi &\implies u = -1, \end{aligned}$$

then

$$E = \frac{\sigma R^2}{2\epsilon_0}\int_{-1}^{1} \frac{o - uR}{(R^2 + o^2 - 2oRu)^{3/2}}du$$

We split the integrand in two fractions (forget the coefficient for now).

$$\int_{-1}^{1} \frac{o}{(R^2 + o^2 - 2oRu)^{3/2}}du \quad \text{and} \quad -\int_{-1}^{1} \frac{uR}{(R^2 + o^2 - 2oRu)^{3/2}}du$$

For the first integral, let us make $x = R^2 + o^2 - 2oRu$, then $dx = -2oRdu$, and in terms of x,

$$-\frac{1}{2R}\int \frac{dx}{x^{3/2}} = \frac{1}{R\sqrt{x}},$$

Then the first integral is

$$\int_{-1}^{1} \frac{o}{(R^2 + o^2 - 2oRu)^{3/2}}du \;=\; \frac{1}{R\sqrt{R^2 + o^2 - 2oRu}}\Bigg|_0^1$$

Let us do the second integral usig integration by parts. We write

$$-\int_{-1}^{1}\frac{uR}{(R^2+o^2-2oRu)^{3/2}}du = -\frac{u}{o\sqrt{R^2+o^2-2oRu}} + \int\frac{1}{o\sqrt{R^2+o^2-2oRu}}du$$

Now,

$$\int\frac{1}{o\sqrt{R^2+o^2-2oRu}}du = -\frac{1}{o^2R}\sqrt{R^2+o^2-2oRu},$$

then

$$\begin{aligned}\int_{-1}^{1}\frac{uR}{(R^2+o^2-2oRu)^{3/2}}du &= \frac{u}{o\sqrt{R^2+o^2-2oRu}} + \frac{\sqrt{R^2+o^2-2oRu}}{o^2R}\\ &= \frac{R^2+o^2-oRu}{o^2R\sqrt{R^2+o^2-2oRu}}\end{aligned}$$

Putting the first and the second integrals back together we get

$$\frac{1}{R\sqrt{R^2+o^2-2oRu}} - \frac{R^2+o^2-oRu}{o^2R\sqrt{R^2+o^2-2oRu}} = \frac{-R^2+oRu}{o^2R\sqrt{R^2+o^2-2oRu}}$$

Hence we found that

$$\int\frac{o-uR}{(R^2+o^2-2oRu)^{3/2}}du = \frac{ou-R}{o^2\sqrt{o^2-2oRu+R^2}},$$

and so

$$\begin{aligned}\left.\frac{ou-R}{o^2\sqrt{o^2-2oRu+R^2}}\right|_{-1}^{1} &= \frac{o-R}{o^2\sqrt{o^2-2oR+R^2}} + \frac{o+R}{o^2\sqrt{o^2+2oR+R^2}}\\ &= \frac{o-R}{o^2|o-R|} + \frac{o+R}{o^2|o+R|}\end{aligned}$$

$$E = \frac{\sigma R^2}{2\epsilon_0}\left[\frac{o-R}{o^2|o-R|} + \frac{o+R}{o^2|o+R|}\right].$$

That is

$$E = \begin{cases} \frac{\sigma R^2}{o^2\epsilon_0} & o > R \\ \\ 0 & o < R \end{cases}$$

but

$$\frac{\sigma R^2}{o^2\epsilon_0} = \frac{4\pi\sigma R^2}{4\pi o^2\epsilon_0} = \frac{Q}{4\pi o^2\epsilon_0}$$

where $4\pi R^2\sigma$ is the total charge in the sphere. Then

$$E = \begin{cases} \frac{Q}{4\pi o^2\epsilon_0} & o > R \\ \\ 0 & o < R \end{cases}$$

What if $o = R$?

1.2.3 Gauss' Law

1.2.4 The Electric Potential

For one charge

The line integral representation

For any discrete number of charges

For a continuum of charges. Line, Surface, Volume

1.3 Problems

1.3.1 Problem 1

Let $\boldsymbol{r_{io}}$ be the separation vector from a fixed point $\boldsymbol{r}_i = (x_i, y_i, z_i)$ to the point $\boldsymbol{r}_0 = (x, y, z) = \boldsymbol{r}$, and let r_{io} be its length. Show the following identities:

$$\begin{aligned} \nabla r_{io}^2 &= 2\boldsymbol{r}_{io} = 2r_{io}\hat{\boldsymbol{r}}_{io} \\ \nabla(1/r_{io}) &= -1/{r_{io}}^2\hat{\boldsymbol{r}}_{\boldsymbol{io}} \qquad (1.9) \\ \nabla r_{io}^n &= n r_{io}^{n-1}\hat{\boldsymbol{r}}_{\boldsymbol{io}} \\ \nabla^2(1/r) &= -4\pi\delta(\boldsymbol{r}) \qquad (1.10) \end{aligned}$$

1.3.2 Problem 2

Given

$$\boldsymbol{r}_{io} = \boldsymbol{r} - \boldsymbol{r}_i \quad , \quad \hat{\boldsymbol{r}}_{io} = \frac{\boldsymbol{r} - \boldsymbol{r}_i}{\|\boldsymbol{r} - \boldsymbol{r}_i\|}$$

show that

$$\nabla \times \frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} = 0.$$

Chapter 2

Lecture # 2

2.1 Introduction

1. Discuss Evaluation.
2. Discuss Question from previous class. Can the electric field have a vector potential? If so what would happen?
3. Get e-mail addresses (and names) of the students to send homework on this class.

2.2 Review

2.2.1 More notes on the potential field

The electric potential due to a charge

Let us assume that we have a charge q in a point $\boldsymbol{r}_i$ in space. What would be the potential on an observation point $\boldsymbol{r}_o = \boldsymbol{r}$?.

We note the new notation of radii $\boldsymbol{r}_i, \boldsymbol{r}_o$, and then a new radius $\boldsymbol{r}_{io} = \boldsymbol{r}_o - \boldsymbol{r}_i$. Before, we were not careful because it did not make a difference in the computations but in this class we will observe that the distinction of the three vectors is important. Figure 2.1 shows an sketch of these vectors, which will be used all along the course in this way.

We know up to now that

$$V(\boldsymbol{r}) = -\int_{\ell} \boldsymbol{E} d\ell \tag{2.1}$$

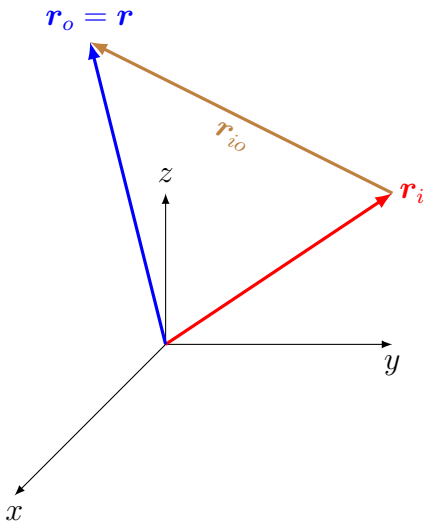

Figure 2.1: Radial vectors $\boldsymbol{r}_i$, $\boldsymbol{r}_o$, and their distance vector $\boldsymbol{r}_{io}$.

where ℓ is some path. Let us pick a path starting at some point $\boldsymbol{a}$ up to the point $\boldsymbol{r}$ in an straight line. From equation 2.1 we see that

$$V(\boldsymbol{r}) = -\int_{\boldsymbol{a}}^{\boldsymbol{r}} \frac{q}{4\pi\epsilon_0 r_{io}^2} d\ell = -\frac{q}{4\pi\epsilon_0}\int_{\boldsymbol{a}}^{\boldsymbol{r}} \frac{1}{r_{io}^2} d\boldsymbol{r}_i$$

with $r_{io} = \|\boldsymbol{r}_i - \boldsymbol{r}\|$. If we choose, without loss in generality the z direction to integrate the integral become simpler. That is,

$$V(z_o) = -\frac{q}{4\pi\epsilon_0}\int_a^{z_o} \frac{dz}{(z - z_i)^2} = \frac{q}{4\pi\epsilon_0}\left.\frac{1}{z - z_i}\right|_a^{z_o} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{z_o - z_i} - \frac{1}{z_i - a}\right).$$

It is customary to choose $a = \infty$ and to simplify the potential to the form

$$V(z_0) = \frac{q}{4\pi\epsilon_0(z_o - z_i)}$$

or in general

$$V(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0 r_{io}}$$

The potential due to several charges and a continuum of charges

Now, if we have a discrete number of charges q_i, we can use the superposition principle to find

$$V(\boldsymbol{r}) = \sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon_0 r_{io}} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_{io}},$$

or if we have a volume (surface, or line) of charges with charge density $\rho(\boldsymbol{r}_i)$ then

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\boldsymbol{r}_i)}{r_{io}} d\boldsymbol{r}_i. \tag{2.2}$$

Example 2.2.1 *Starting at equation 2.2 show that the electric potential satisfies the Poisson's equation.*

Review of multi-dimensional calculus Before showing that the potential satisfies the Poisson's equation we we solve several problems listed next.

- **Find $\nabla\, r_{io}^n$.**
  We solve this problem in two different ways:

  (i) **Using spherical coordinates and the chain rule.** We solve first the easier problem ∇r^n. In spherical coordinates we have that

  $$\nabla r^n = \frac{1}{h_r}\frac{\partial r^n}{\partial r}\hat{\boldsymbol{r}} + \cancel{\frac{1}{h_\theta}\frac{\partial r^n}{\partial \theta}\hat{\theta}} + \cancel{\frac{1}{h_\phi}\frac{\partial r^n}{\partial \phi}}^{\,0}\hat{\boldsymbol{\phi}} = n r^{n-1}\hat{\boldsymbol{r}}.$$

  However we need to solve for the gradient of $r_{io} = \|\boldsymbol{r}_o - \boldsymbol{r}_i\|$. We can use the previous result and the chain rule. The chain rule for the composition of a scalar with a vector function is shown here.
  Let $f(\boldsymbol{x})$ a scalar field, and $\boldsymbol{g}(\boldsymbol{x})$ a vector field. Then

  $$\nabla f(g(\boldsymbol{x})) = A^T \nabla_{\boldsymbol{g}} f \tag{2.3}$$

  where A^T, the transpose of A is the matrix of partial derivatives $\partial g_j/\partial x_i$. This is easy to show. Choose an i-th component of the gradient. Then

$$\frac{\partial f}{\partial x_i} = \sum_j \frac{\partial f}{\partial g_j}\frac{\partial g_j}{\partial x_i}$$

and this, in matrix form is equation 2.3 above.
In our particular case $f(\boldsymbol{x}) = \|\boldsymbol{x}\| = r(\boldsymbol{x})$, and $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{r}_{io} = \boldsymbol{x} - \boldsymbol{x}_i$ (a simple shift). Then

$$\frac{\partial g_j}{\partial x_i} = \delta_{ij} \quad , \quad \text{so } A = I,$$

and then

$$\nabla r_{io}^n = \nabla_{r_{io}} r_{io}^n = n r_{io}^{n-1} \hat{\boldsymbol{r}}_{\boldsymbol{io}}.$$

(ii) **Using Cartesian coordinates.**

$$r_{io}^n = \left(\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}\right)^n,$$

then

$$\frac{\partial r_{io}^n}{\partial x} = n r_{io}^{n-1} \frac{2(x-x_0)}{2\, r_{io}^n} = n r_{io}^{n-2}(x-x_0).$$

we do the same for $y - y_0$ and $z - z_0$ and find

$$\nabla r_{io}^n = n r_{io}^{n-2} \boldsymbol{r}_{io} = n r_{io}^{n-1} \hat{\boldsymbol{r}}_{io}.$$

Here some important special cases.

(i) If $n = 0$, then $r_{io}^n = 1$ and the derivative is 0 as checked with the right side with $n = 0$.

(ii) If $n = 1$ we find that the gradient of the distance r_{io} is given by the unit radio $\boldsymbol{r}_{io}$ in the direction from $\boldsymbol{r}_i$ to $\boldsymbol{r}_{\boldsymbol{o}}$.

(iii) If $n = -1$ we find that $\nabla r_{io} = -(1/r_{io}^2)\boldsymbol{r}_{io}$. Note that in this case we should assume $r_{io} \neq 0$ to avoid division by 0. This is important for electric potentials. Actually we use this result to solve the Poisson's equation from the electric potential equation above.

- **Find $\nabla \cdot \frac{1}{r_{io}^2}$. for $r_{io} \neq 0$.**
We use, again, spherical coordinates. Let us start with the simpler problem

$$\begin{aligned}
\nabla \cdot (1/r^2) &= \frac{1}{h_r h_\theta h_\phi}\left[\frac{\partial h_2 h_3 (1/r^2)}{\partial r} + \cancel{\frac{\partial h_1 h_3(0)}{\partial \theta}} + \cancel{\frac{\partial h_2 h_3(0)}{\partial \phi}}^{\,0}\right] \\
&= \frac{1}{r^2 \sin\theta}\frac{\partial \cancel{r^2}\sin\theta(1/\cancel{r^2})}{\partial r} \\
&= 0.
\end{aligned}$$

Now for the chain rule of the divergence function. Assume we have a vector field $\boldsymbol{f}(\boldsymbol{x})$, and a vector field $g(\boldsymbol{x})$. Then, from the chain rule

$$\frac{\partial f_i}{\partial x_i} = \sum_k \frac{\partial f_i}{\partial g_k}\frac{\partial g_k}{\partial x_i}$$

and

$$\nabla \cdot \boldsymbol{f} = \sum_i \sum_k \frac{\partial f_i}{\partial g_k}\frac{\partial g_k}{\partial x_i}$$

In our problem $f(\boldsymbol{x}) = \|\boldsymbol{x}\| = r$, and $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{r} - \boldsymbol{r}_i$, so $\partial g_k/\partial x_j = \delta_{kj}$, and

$$\nabla \cdot f = \sum_i \sum_k \delta_{ki}\frac{\partial f_i}{\partial g_k} = \sum_i \frac{\partial f_i}{\partial g_i} = \nabla_g \cdot f.$$

That is

$$\nabla_r \cdot \frac{1}{r_{io}^2} = \nabla_{r_{io}} \cdot \frac{1}{r_{io}^2} = 0$$

- **Find:**

$$I = \int_V \nabla \cdot \left(\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right) d\boldsymbol{r}_i$$

where V is a sphere around the input point $\boldsymbol{r}_i$ touching to the output point $\boldsymbol{r}$. Let us do first a change of variable we call $\boldsymbol{\rho} = \boldsymbol{r}_i - \boldsymbol{r}$, the integration variable is $\boldsymbol{r}_i$. The Jacobian of the transformation (in 3D) is given by the determinant of the matrix with coordinates $\partial\boldsymbol{\rho}/\partial\boldsymbol{r}_i$ which is the identity. The center of the sphere is now at the point $\boldsymbol{x}_i - \boldsymbol{x}_i = 0$. So now we have the following (easier) integral

$$I = \int_{V_o} \nabla \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2} \right) d\boldsymbol{r},$$

with V_0 a sphere centered at the origin.

We use Gauss' theorem and convert the integral in a surface integral. That is,

$$\int_{V_o} \nabla \cdot \frac{\hat{\boldsymbol{r}}}{r^2} d\boldsymbol{r}_i = \int_{S_o} \frac{1}{\cancel{r^2}} \cancel{r^2} \sin\theta d\theta d\phi = 4\pi.$$

In summary we have

$$\begin{aligned} \nabla \cdot \left(\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right) &= 0 \quad , \quad r_{io} \neq 0 \\ \int_V \nabla \cdot \left(\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right) d\boldsymbol{r}_i &= 4\pi \end{aligned}$$

or what is the same

$$\begin{aligned} \frac{1}{4\pi} \nabla \cdot \left(\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right) d\boldsymbol{r}_i &= 0 \quad , \quad \boldsymbol{r} \neq \boldsymbol{r}_i \\ \frac{1}{4\pi} \int_V \nabla \cdot \left(\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right) d\boldsymbol{r}_i &= 1 \end{aligned}$$

This is the definition of a Dirac delta. That is, we found

$$\delta(\boldsymbol{r}-\boldsymbol{r}_i)=\frac{1}{4\pi}\left(\nabla\cdot\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2}\right). \tag{2.4}$$

If we want to be formal, we should put this in the context of theory of distributions [1]. However the study of the theory of distribution is out of the scope in this curse and we will accept this as a truthful statement. We note that since

$$\frac{\hat{\boldsymbol{r}}_{io}}{r^2}=-\nabla\left(\frac{1}{r_{io}}\right),$$

then we have that

$$\delta(\boldsymbol{r}-\boldsymbol{r}_i)=-\frac{1}{4\pi}\nabla^2\left(\frac{1}{r_{io}}\right). \tag{2.5}$$

We are now ready to solve the question that the electrical potential in equation 2.2 satisfies Poisson's equation. If we take the Laplacian on equation 2.2 we find

$$\nabla^2V(\boldsymbol{r})=\nabla\cdot\nabla V(\boldsymbol{r})=\frac{1}{4\pi\epsilon_0}\int\boldsymbol{\rho}(\boldsymbol{r}_i)\,\nabla\cdot\nabla\left(\frac{1}{r_{io}}\right)d\boldsymbol{r}_i.$$

Note that we moved the Laplacian inside the integral sign because it does not depend on the integration variable. Also, we moved the density $\boldsymbol{\rho}(\boldsymbol{r}_i)$ outside of the Laplacian operator because the operator is taken with respect to the output radius $\boldsymbol{r}_o=\boldsymbol{r}$ and $\rho(\boldsymbol{r}_i)$ depends only on the input radius $\boldsymbol{r}_i$. Now, since $\nabla 1/r_{io}=-\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2}$ we have that

$$\nabla^2V(\boldsymbol{r})=-\frac{1}{4\pi\epsilon_0}\int\boldsymbol{\rho}(\boldsymbol{r}_i)\,\nabla\cdot\left(\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2}\right)d\boldsymbol{r}_i.$$

and from equation 2.4,

$$\nabla^2V(\boldsymbol{r})=-\frac{1}{4\pi\epsilon_0}\int\boldsymbol{\rho}(\boldsymbol{r}_i)(4\pi)\delta(\boldsymbol{r}-\boldsymbol{r}_i)d\boldsymbol{r}_i=-\frac{\boldsymbol{\rho}(\boldsymbol{r})}{\epsilon_0}.$$

which is Poisson's equation.

[1] https://en.wikipedia.org/wiki/Distribution_(mathematics)

A differential proof of Gauss' law

We take advantage of the machinery developed here to derive the Gauss's law using the Dirac delta distribution. We assume a volume or charges distributed with a density $\rho(\boldsymbol{r})_i$. Let us find $\nabla \cdot \boldsymbol{E}$. We start with the representation

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}_i)\hat{\boldsymbol{r}}_i}{r_{io}^2} d\boldsymbol{r}_i.$$

Then

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\boldsymbol{r}_i)\nabla \cdot \left(\frac{\hat{\boldsymbol{r}}_i}{r_{io}^2}\right) d\boldsymbol{r}_i$$

That is

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\boldsymbol{r}_i)(4\pi)\delta(\boldsymbol{r}-\boldsymbol{r}_i) d\boldsymbol{r}_i = \frac{\rho(\boldsymbol{r})}{\epsilon_0}$$

which is Gauss' law.

2.3 More on Poisson's equation

The computation of the electrical field is difficult when there are no symmetries, and we can not use Gauss' law. The integrals for non-symmetric problems could be tedious and the computation of the scalar potential based on Coulomb's law is not easy either. We now go back to the analysis on Poisson's equation

$$\nabla^2 V(\boldsymbol{r}) = -\frac{\rho(\boldsymbol{r})}{\epsilon}.$$

The Poisson's equation is a special case of a group of (PDE) partial differential equations known as elliptical equations. There is a vast theory on the subject but we will consider only a few isolated features.

We start by assuming that the charge density is 0, so we are free of charges. The Poisson's equation becomes the Laplace's equation. We want to solve this equation and this is the task of the following lines.

$$\nabla^2 V(\boldsymbol{r}) = 0. \tag{2.6}$$

2.3.1 The Green's Function

The Green's function is the solution of the equation

$$\nabla^2 G(\boldsymbol{r}, \boldsymbol{r}_i) = -\delta(\boldsymbol{r} - \boldsymbol{r}_i). \tag{2.7}$$

Finding the Green's function is like finding the inverse of the operator. It is as if in a matrix problem $Ax = b$, we find A^{-1}, then we can solve the equation as $x = A^{-1}b$ for any source term b.

Let us ses how we can use the Green's function to find solutions of the differential equation. If we know $G(\boldsymbol{r}, \boldsymbol{r}_i)$ which satisfies equation 2.7 we can use the Green's second identity

$$\int_D (u\nabla^2 v - v\nabla^2 u)dV = \int_{\partial D} \left[u\frac{dv}{dn} - v\frac{du}{dn}\right] dS. \tag{2.8}$$

Let us assume that $v = G(\boldsymbol{r}, \boldsymbol{r}_i)$ is a Green's function which satisfies equation 2.7, and $u(\boldsymbol{r}_i)$ satisfies Laplace's equation. That is $\nabla^2 u(\boldsymbol{r}_i) = 0$. Then by plugging these two functons in the Green's second second [2] identity above, we find that

$$\int_D [-u(\boldsymbol{r}_i)\delta(\boldsymbol{r} - \boldsymbol{r}_i) - \cancelto{0}{G(\boldsymbol{r}, \boldsymbol{r}_i)\nabla^2 u(\boldsymbol{r}_i)}]dV = \int_{\partial D} \left[u(\boldsymbol{r}_i)\frac{d}{dn}G(\boldsymbol{r}, \boldsymbol{r}_i) - G(\boldsymbol{r}, \boldsymbol{r}_i)\frac{du}{dn}\right] dS$$

Using the sifting property of the Dirac delta distribution we find that

$$u(\boldsymbol{r}) = -\int_{\partial D} \left[u(\boldsymbol{r}_i)\frac{d}{dn}G(\boldsymbol{r}, \boldsymbol{r}_i) - G(\boldsymbol{r}, \boldsymbol{r}_i)\frac{du(\boldsymbol{r}_i)}{dn}\right] dS. \tag{2.9}$$

This is a solution of the Laplace's equation assuming that we know in addition to $G(\boldsymbol{u}_r, \boldsymbol{u}_i)$, $u(\boldsymbol{r}_i)$, and their normal derivative at the boundary.

There are three types of boundary conditions

[2] https://en.wikipedia.org/wiki/Green%27s_identities

1. **Dirichlet Boundary Conditions:**
   These indicate that we should know the function in a closed boundary. So, if the region where we want to solve the problem is Ω and its boundary $\partial\Omega$, then we should know the solution $u(x, y)$ to be of some form

$$u(x, y) = h(x, y) \quad , \quad (x, y) \in \partial\Omega.$$

   If the solution of the Laplace's equation is a surface, the Dirichlet boundary conditions fixed the surface to a closed boundary and this guarantees uniqueness of the problem.

2. **Neumann Boundary Conditions:**
   In this case the derivative of the function is prescribed in the boundary.

3. **Mixed boundary conditions.**
   In this case the function is specified in one part of the boundary and its derivative in the complement.

Let us now find Green's functions, from equation 2.7, in general for any dimensions. We observe first that a translation of the coordinates will not change the equation. We showed that both the gradient and the divergence will not change until a translation. That is, we can consider $\boldsymbol{r}_i = 0$ and the equation should still apply. Then

$$\nabla^2 G(\boldsymbol{r}, 0) = -\delta(\boldsymbol{r}). \tag{2.10}$$

Now, we also see that the Green's function is isotropic. That is, the equation should be rotationally invariant, hence we consider only the radial part of the Laplacian. For n dimensions we can think of a function which depends only on r. That is, let us assume we have $f(r)$, and want to find the Laplacian. We have $r = \sqrt{x_1^2 + \cdots x_n^2}$, and

$$\frac{\partial f(r)}{\partial x_i} = \frac{\partial f}{\partial r}\frac{x_i}{r}$$

now for the second derivative

$$\frac{\partial^2 f(r)}{\partial x_i^2} = \frac{\partial^2 f}{\partial r^2}\frac{x_i^2}{r^2} + \frac{\partial f}{\partial r}\frac{r - x_i^2/r}{r^2} = \frac{x_i^2}{r^2}\frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{\partial r}\frac{r^2 - x_i^2}{r^3}$$

We add over all components i, to find

$$\begin{aligned}\sum_{i=1}^{n}\frac{\partial^2 f(r)}{\partial x_i^2} &= \frac{\partial^2 f}{\partial r^2}+\frac{n}{r}\frac{\partial f}{\partial r}-\frac{1}{r}\frac{\partial f}{\partial r}\\ &= \frac{\partial^2 f}{\partial r^2}+\frac{n-1}{r}\frac{\partial f}{\partial r}.\end{aligned}$$

This can be written in the compact form as

$$\nabla^2 f(r)=\frac{1}{r^{n-1}}\frac{d}{dr}r^{n-1}\frac{df}{dr}. \tag{2.11}$$

where we used total derivative since f is only function of r.

For n dimensions in an isotropic function we found that Laplacian is given, from equation 2.11 by

$$\nabla^2 G(\boldsymbol{r},0)=\frac{1}{r^{n-1}}\frac{d}{dr}\left[r^{n-1}\frac{dG}{dr}\right]=-\delta(\boldsymbol{r}).$$

Away from the origin the equation is

$$\nabla^2 G(\boldsymbol{r},0)=\frac{1}{r^{n-1}}\frac{d}{dr}\left[r^{n-1}\frac{dG(\boldsymbol{r},0)}{dr}\right]=0$$

from which we get

$$r^{n-1}\frac{dG(\boldsymbol{r},0)}{dr}=c_n$$

or

$$\frac{dG(\boldsymbol{r},0)}{dr}=\frac{c_n}{r^{n-1}},$$

and

$$G(\boldsymbol{r},0)=c_n\ln r,\quad ,\quad n=2,\qquad G(\boldsymbol{r},0)=-\frac{c_n}{(n-2)r^{n-2}},\quad n>2.$$

We still need to find c_n. To to that we integrate equation 2.10 choosing a small sphere of radius $r = \epsilon$ around 0 and using Gauss' theorem. That is

$$-1 = \int_{r\leq\epsilon} \nabla\cdot\nabla G(\boldsymbol{r},0)dr = \int_{r=\epsilon} \frac{dG(\boldsymbol{r},0)}{dr}dS = \int_{r=\epsilon} \frac{c_n}{r^{n-1}}dS = \int_{r=\epsilon} c_n d\Omega_n.$$

Here $d\Omega_n$ denotes the differential solid angle in n dimensions, and $dS = r^{n-1}d\Omega_n$. For examplle, in two an three dimensions we have

$$d\Omega_2 = d\theta \quad , \quad d\Omega_3 = \sin\theta d\theta d\phi.$$

It can be shown (this is left as an exercise) that in n dimensions, the solid angle Ω_n is given by $2\pi^{n/2}/\Gamma(n/2)$, where Γ is the Gamma function [3].

Then,

$$c_n = -\frac{\Gamma(n/2)}{2\pi^{n/2}},$$

and shifting back to the new origin $\boldsymbol{r}_i$,

$$G(\boldsymbol{r},\boldsymbol{r}_i) = \begin{cases} -\frac{1}{2\pi}\log r_{oi} & n=2, \\ \frac{1}{4\pi r_{oi}} & n=3, \\ \frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}} r_{oi}^{n-2} & n\geq 3, \end{cases} \tag{2.12}$$

We now show an important property known as the mean value theorem of the Laplacian.

The mean value property of harmonic functions

We will prove below that the harmonic function present the striking property that the value at any point is the average of the values of the function in a circumference around it or the average of the values of a circle around it, and this happens in any number of dimensions.

[3] https://en.wikipedia.org/wiki/Gamma_function

Let us assume that we have a function that satisfies the Laplace's equation at a point $\boldsymbol{r}_i$, and inside a volume D. We also assume that we want to integrate this function in a ball of radius a around the point $\boldsymbol{r}_i$, contained in D. We define a new Green's function

$$G(\boldsymbol{r}, \boldsymbol{r}_i) = \begin{cases} -\frac{1}{2\pi} \log r_{oi} + \frac{1}{2\pi} \log a & n = 2, \\ \frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}} r_{oi}^{n-2} - \frac{\Gamma(n/2)}{2(n-2)\pi^2 n/2} a^{n-2} & n \geq 3, \end{cases}$$

Note that this is again a Green's function. Green's functions are not necessary unique. We can add any constant to a Green's function of the Laplacian and still satisfies the requirements of the Green's function. This new Green's function vanishes at boundary of the sphere. Then from equation 2.9 and from $c_n = -1/\Omega_n$,

$$u(\boldsymbol{r}) = -\int_{r=a} u(\boldsymbol{r}_i) \frac{d}{dn} G(\boldsymbol{r}, \boldsymbol{r}_i) = -\int_{r=a} u(\boldsymbol{r}_i) \frac{c_n}{r^{n-1}} dS = \int_{r=a} u(\boldsymbol{r}_i) \frac{1}{\Omega_n r^{n-1}} dS$$

That is

$$u(\boldsymbol{r}) = \frac{1}{\Omega_n a^{n-1}} \int_{r=a} u(\boldsymbol{r}_i) dS = \frac{1}{\Omega_n} \int_{r=a} u(\boldsymbol{r}_i)\, d\Omega_n$$

We see that $u(\boldsymbol{r})$ is the mean of its value on a sphere centered at $\boldsymbol{r}_i$. We also conclude that for any point in D (the domain of validity of the Laplace's equation) there can not be an isolated maximum or minimum point, since then the average on any sphere around this point would not be larger or smaller than all its neighbors.

We study the Laplace's equation in one, two, and three dimensions.

2.3.2 Laplace's Equation in One Dimension

$$\frac{d^2 V}{dx^2} = 0,$$

with a general (linear) solution $V(x) = mx + b$. To be able to determine m and b we need to use two boundary conditions (values of V or its derivative at one point or V at two points). We will study the boundary conditions for the electrical potential later. For now we will study the structure of the equation. We have two observations:

(i) Being linear, any point inside is the average of two neighbor points located at an equal distance. That is

$$V(x) = \frac{1}{2}[V(x-d) + V(v+d)]$$

(ii) There are no local minima or maxima. This is obvious here but it will be useful in several dimensions.

2.3.3 Laplace's Equation in Two Dimension

Here the Laplace's equation is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

The solutions of the Laplace's equation with second order continuous derivatives, are known as **harmonics** .

Laplace's equation and analytic functions

We recall from complex variable that an analytic function satisfies the Riemann-Cauchy conditions. That is, let $u(x, y)$ the real part and $v(x, y)$ the imaginary part of the function, that is let

$$f(x, y) = u(x, y) + \mathrm{i}v(x, y),$$

then the Riemann-Cauchy conditions are

$$u_x = v_y \quad , \quad v_x = -u_y.$$

From here

$$(u_x)_x = (v_y)_x = (v_x)y = (-u_y)_y = -u_{yy}$$

so

$$u_{xx} + u_{uu} = 0.$$

similarly we could show that $v_{xx} + v_{yy} = 0$. Then both the real and imaginary parts of an analytic complex function satisfy the Laplace's equation.

The converse is also true but with more restrictions. That is, if $h(x, y)$ is harmonic we could find an analytic function f such that $h(x, y)$ is the real part of it. That is, such that $\phi(x, y) = \mathrm{Re} f(x, y)$.

Laplace's equation boundary conditions

The previous section showed that any analytical function satisfies the Laplace's equation and there are an infinity non-countable set of analytic functions. Note that all constant functions are harmonic, all linear polynomials of the form $ax + by$ are harmonic, all quadratic polynomials of the form $x^2 - y^2$ and xy are harmonic, linear combination of these such as for example.

$$u(x, y) = a(x^2 - y^2) + bxy \quad (a, b, \text{ constants})$$

are harmonic. Functions with radial symmetry such as $\phi(r) = \ln r$ with $r = \sqrt{x^2 + y^2}$, and its multiples are harmonic. Functions growing or decaying exponentially such as $\mathrm{e}^{kx} \sin ky$, are harmonic. To be able to guarantee uniqueness of solutions we need to fix some boundary conditions.

We found the Green's function

$$G(\boldsymbol{r}, \boldsymbol{r}_i) = -\frac{1}{2\pi} \log r,$$

for the two-dimensional case. Using the Green's function we showed that the value at any point of the domain of validity of the solution, the integral around it and along a circle, provides the mean value along the values taken on the circle by the function. This implies that there are no maximum or minimum inside the chosen circle.

2.3.4 Laplace's Equation in Three Dimension

In Cartesian coordinates this is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

We found the Green's function

$$G(\boldsymbol{r}, \boldsymbol{r}_i) = \frac{1}{4\pi r_{io}}$$

for this Laplacian. Again, we used the Green's function to show that the value of the solution at any point in the domain is the average of the values around a sphere where the point is the center. This implies, once again, that there are no local minima or maxima on the solution of the Laplace's equation.

2.4 Problems:

2.4.1 Problem 1

Let us assume that we have a surface of cone which is standing on its apex on a plane perpendicular to the axis. The height and radius of the cone is given by h and $r = h$. The charge density in the surface of the cone is given by σ. We also assume that we want to evaluate the electrical potential of the cone in the center of the base of the cone (the circle) directly above the apex. We use the formula for the electric potential.

$$V(\boldsymbol{p}) = \frac{1}{4\pi\epsilon_0} \int_{dS} \frac{\sigma dS}{r}. \tag{2.13}$$

where S is the surface with homogeneous charge density σ.

There are at least two ways to interpret this problem.

(i) Figure 2.2 shows the cone with its observation point on $\boldsymbol{p}$ and a ring through $\boldsymbol{c}$ which shows the differential element of the surface use for the integration.

On this first approach we focus on the triangle that joints the points $\boldsymbol{p}$, $\boldsymbol{\ell}$, and $\boldsymbol{c}$. We choose the domain of integration along the direction on ℓ with the differential element $d\ell$ between 0 in $\boldsymbol{p}$ and h in the point $\boldsymbol{a}$ (in transparent blue). We identify the magnitudes of the triangle sides as:

ℓ : distance between $\boldsymbol{p}$ and $\boldsymbol{\ell}$

ρ : radius of ring

r : distance between point of observation $\boldsymbol{p}$ and charge on $\boldsymbol{c}$.

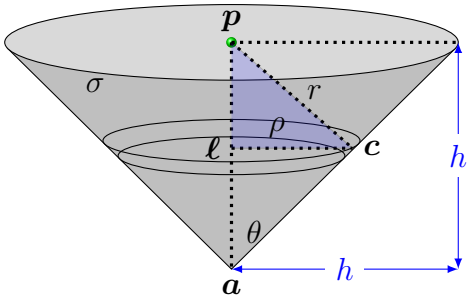

Figure 2.2: Cone for problem 1 (i)

Since our variable of integration is ℓ, we should write the lengths of the triangle (shaded) sides in terms of ℓ. The distance between $\boldsymbol{r}$ and $\boldsymbol{\ell}$ is ℓ, since our 0 point is $\boldsymbol{p}$. For the radius ρ of the ring that goes through $\boldsymbol{c}$ we see that

$$\tan\theta = \frac{\|\boldsymbol{c}-\boldsymbol{\ell}\|}{\|\boldsymbol{\ell}-\boldsymbol{a}\|} = \frac{\|\boldsymbol{c}-\boldsymbol{\ell}\|}{h-\ell} = \frac{\rho}{h-\ell}$$

where $\|\boldsymbol{x}\|$ measures the length of the segment $\boldsymbol{x}$. Now, since the height of the cone h is equal to its radius we have that $\tan\theta = 1$, and so

$$\rho = h - \ell.$$

The blue triangle is a right triangle, so using Pythagoras theorem we find

$$r = \sqrt{\rho^2 + \ell^2} = \sqrt{\ell^2 + (h-\ell)^2}.$$

We have all the elements to evaluate integral 2.13. We use the symmetry of the problem. That is, each point on ℓ along the center of the rings of integration provides the same value independent of the rotation angle. The cone is rotationally invariant. The contribution on the integration is given by the length of the circle multiplied by the charge density. That is, $2\pi\rho = 2\pi(h-\ell)$. Note that since in ring r is constant we can factor it out. Hence we reduce the transversal (horizontal) dimension and we only integrate along the vertical ℓ. We are changing a surface integral by a line integral over $d\ell$ between 0 and h.

$$V(\boldsymbol{p}) = \frac{1}{4\pi\epsilon_0}\int_0^h \frac{2\pi(h-\ell)\sigma d\ell}{\sqrt{\ell^2+(h-\ell)^2}}.$$

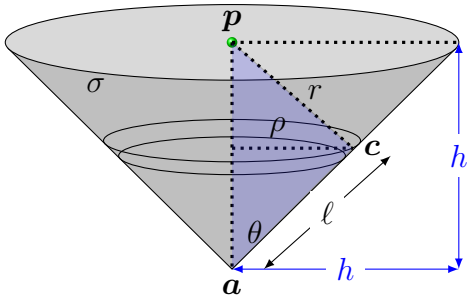

Figure 2.3: Cono para el problema 1 (ii)

(ii) Figure 2.3 shows a different way to interpret this problem. This time we look at the blue triangle formed by the vertices $\boldsymbol{p}$, $\boldsymbol{c}$, and $\boldsymbol{a}$. The integration is assumed this time along the cone generating line. That is we integrage along the segment that goes through the points $\boldsymbol{a}$, and $\boldsymbol{c}$. It is clear that the path of integration (which is straight) has length $\sqrt{2}h$ (the diagonal of a square with side h.) In this case, the the radius of the ring can be computed with respect to our path of integration. That is, $\ell = \|\boldsymbol{c} - \boldsymbol{a}\|$, and

$$\sin\theta = \frac{\rho}{\ell},$$

where, of course, $\sin\theta = \sqrt{2}/2$,

$$\rho = \frac{\sqrt{2}}{2}\ell.$$

To find r we use the cosine law on the blue triangle.

$$r = \sqrt{h^2 + \ell^2 - 2h\ell\cos\theta} = \sqrt{h^2 + \ell^2 - 2h\ell\sqrt{2}/2} = \sqrt{h^2 + \ell^2 - h\ell\sqrt{2}}.$$

We now have all elements to compute the electric potential using the integral 2.13. We find

$$V(\boldsymbol{p}) = \frac{1}{4\pi\epsilon_0}\int_0^{\sqrt{2}h} \frac{2\pi\sigma(\sqrt{2}/2)\,\ell\,d\ell}{\sqrt{h^2 + \ell^2 - \sqrt{2}\,h\ell}}.$$

The question to this problem is: Are the two formulations of the solution right? If so evaluate the integrals and show that they are correct. If any of the two is incorrect please explain why.

2.4.2 Problem 2

Let us consider the equation of the electric potential for a given volume

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\omega(\boldsymbol{r}')}{r'}d\boldsymbol{r}'. \tag{2.14}$$

where r' is the distance from an observation point $\boldsymbol{r}$ to each poit of the domain of integration. Here $\omega(\boldsymbol{r}')$ is the charge density [4].

We want to find the electric potential along a solid sphere with uniform charge density σ, from an interior point in the sphere. We can assume coordinates $(0, 0, z)$. It can be shown that if we have a disc with a uniform charge density σ, and radius ρ, and the observation point is located a distance w along the axis of the disc, the electric potential is given by the equation

$$V_d(w) = \frac{\sigma(\sqrt{\rho^2 + w^2} - w)}{2\epsilon_0}. \tag{2.15}$$

By using the superposition principle we can think of a sphere as continuum set of discs. If we know the potential for a disc we can say some integration work from three dimensions to one dimension, by converting a volume integral to a line integral. Let us see how we can solve this problem. We define a polar angle θ shown here between the radius at the sphere edge and the segment from the origin with the center of the disc. At the north pole of the sphere $\theta = 0$. When $\theta = \pi/2$, the disc is seen as sliced from the north pole to the equator with maximum radius $\rho = R$, where it starts getting smaller until it becomes 0 at the south pole. Figure 2.4 shows a vertical section (projection onto the plane $x = 0$) of the sphere. At this instance of the integration the disc has a red color. The distance w to the disc is in blue. The disc radius is ρ and it is given by

$$\rho = R\sin\theta.$$

The distance w is given by

[4]The symbol ρ is used here to identify the radius of the disc

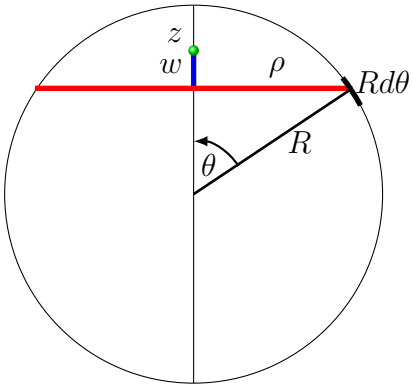

Figure 2.4: Sección vertical para el problema 2.

$$w = z - R\cos\theta.$$

The differential element of integration is given by $ds = Rd\theta$. We now have all elements to compute the potential. We replace the obtained values above in equation 2.15 to find

$$V_d(w) = \frac{\sigma(\sqrt{(R^2\sin^2\theta + (z - R\cos\theta)^2} - R\cos\theta)}{2\epsilon_0}.$$

The electric potential would then be given by the integral in θ, between 0 and π

$$V_S(z) = \int_0^\pi \frac{\sigma(\sqrt{(R^2\sin^2\theta + (z - R\cos\theta)^2} - R\cos\theta)}{2\epsilon_0} Rd\theta.$$

where we assume that the uniform electric charge density of the sphere is σ (the same as that of the disc. Some constant value in corresponding surface units) We simplify the integral.

$$V_S(z) = \frac{\sigma}{2\epsilon_0}\int_0^\pi \sigma(\sqrt{(R^2 + z^2 - 2R\cos\theta} - R\cos\theta)R\,d\theta.$$

Even though this integral looks relatively simple, it is not. The purpose of this exercise is:

1. Investigate how to solve this integral. It is not necessary to provide a soltuion, but you should explain what ever you could from it. If you can solve it please do.

2. This problem, even though it looks a contradiction, is easier to solve evaluating directly the volume 2.14 formula. Solve it this way.

3. What conclusions we would obtain from this?

2.4.3 Problema 3

Find the surface area and volume of a sphere in the space of n dimensions.

2.4.4 Problema 4

Show that for a sphere with uniform charge density ρ the potential is given by the formula

$$V(r) = \frac{q}{2R(4\pi\epsilon_0)}\left(3 - \frac{r^2}{R^2}\right) \quad , \quad r \leq R. \tag{2.16}$$

Chapter 3

Lecture # 3

In the previous lecture we found a representation of the solution for the Laplace's equation in terms of Green's functions using Green's second identity. This is shown in equation 2.9 which is rewritten here.

$$u(\boldsymbol{r}) = -\int_{\partial D}\left[u(\boldsymbol{r}_i)\frac{d}{dn}G(\boldsymbol{r},\boldsymbol{r}_i) - G(\boldsymbol{r},\boldsymbol{r}_i)\frac{du(\boldsymbol{r}_i)}{dn}\right]dS. \tag{3.1}$$

We found Green's functions $G(\boldsymbol{r},\boldsymbol{r}_i)$ which satsfy this equation and realized that the are uncountable infinite many solutions that fit the Laplace's equation. We also show that we could adjust the Green's function $G(\boldsymbol{r}_i,\boldsymbol{r})$ to fit a boundary condition which would leads us to a solution of $u(\boldsymbol{r})$ only in terms of its values at the boundary ∂D. There are three main type of boundary conditions. We could fit $G(\boldsymbol{r},\boldsymbol{r})_s$ to the boundary (Dirichlet Boundary Condition) , or fit $dG(\boldsymbol{r},\boldsymbol{r}_i)/dn$ to the boundary (Neumann Boundary Condition) , or fit a combination of these two to the boundary (Mixed boundary condition). In addition, if both Dirichlet and Neumann boundary conditions are set we say that we have Cauchy boundary conditions. In particular we showed how by finding a Green's function which is zero in the boundary we solved the problem and proved the mean value theorem for the solutions of the Laplace's equation. Setting $G(\boldsymbol{r},\boldsymbol{r}_s) = 0$ to the boundary produces.

$$u(\boldsymbol{r}) = -\int_{\partial D} u(\boldsymbol{r}_i)\frac{d}{dn}G(\boldsymbol{r},\boldsymbol{r}_i)dS$$

Instead of making the Green's function to vanish in the boundary we can make its derivative to vanish at the boundary. That is $dG/dn = 0$. This would produce a solution

of the form.

$$u(\boldsymbol{r}) = \int_{\partial D} G(\boldsymbol{r}, \boldsymbol{r}_i) \frac{du(\boldsymbol{r}_i)}{dn} dS.$$

3.1 Uniqueness of solutions for the Laplace's equation

3.1.1 Cauchy Boundary Conditions

Now let us assume that the function $u(\boldsymbol{r}_i)$ and its normal derivative vanish at the boundary (zero Cauchy boundary conditions). That is, the Dirichlet and Neumann boundary conditions are zero. Then from equation 3.1 we observe that $u(\boldsymbol{r}) = 0$ since both terms inside the integral vanish. This is important because it proves that the solution of the Poisson's equation for prescribed boundary conditions its solution unique. If we assume that a solution $u(\boldsymbol{r}_i)$ of the Laplace's equation which satisfies the Dirichlet and Neumann boundary conditions is unique. The proof is simple. Let us assume that there are two solutions u_1 and u_2 which are solutions of the Laplace's equation which satisfy the boundary conditions. Then we construct a third solution which is $u = u_1 - u_2$. The third solutions would have zero boundary conditions (because they are subtracted) and so the solution u is identically 0. That is $u_1 = u_2$. Hence the solution of the Laplace's equation is unique under Cauchy boundary conditions.

3.1.2 Dirichlet Boundary Conditions

Let us relax more the problem and assume that only Dirichlet boundary conditions are imposed. Again, as in the previous case let us assume that two different solutions $u_1(\boldsymbol{r})$ and $u_2(\boldsymbol{r})$ exist and that we call $u = u_1 - u_2$. Since both u_1 and u_2 satisfy the Laplace's equation u also satisfies the Laplace's equation. In addition, since both u_1 and u_2 satisfy the same (Dirchlet) boundary condition then $u = 0$ in the boundary. Consider the expression

$$J = \int_V (\nabla u \cdot \nabla u)\, dV = \int_V \|\nabla u\|^2 dV.$$

It happens that $J = 0$ only if $u =$ constant, since otherwise the gradient ∇u would not be zero and its norm is positive, the integral of positive number is a positive number. So let us assume that u is not a constant and then $J > 0$. Let us now consider the identity

$$\nabla \cdot (u\nabla u) = (\nabla u) \cdot (\nabla u) + u\nabla^2 u$$

Now, since $\nabla^2 u = 0$ on the volume we have that

$$\|\nabla u\|^2 = \nabla \cdot (u\nabla u).$$

and applying the divergence theorem on J we see that

$$\int_V \|\nabla u\|^2 = \int_S (u\nabla u)\, d\boldsymbol{S} = \int_S u\frac{\partial u}{\partial n} dS \tag{3.2}$$

but since $u = 0$ in the boundary (the Dirichlet boundary condition) then $J = 0$. So inside the volume u needs to be a constant. However the limit of a constant in the boundary should be a constant, and since in the boundary the value is 0, the value inside is 0. This means that $u = u_1 - u_2 = 0$, and $u_1 = u_2$ which proves uniqueness.

3.1.3 Neumann Boundary Conditions

We show in this particular case uniqueness up to a constant. Let us assume assume, as before, that there are two solutions $u_1(\boldsymbol{r})$ and $u_2(\boldsymbol{r})$ that satisfies the Neumann boundary conditions equally. That is,

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \quad , \quad \text{for all points } \boldsymbol{r} \in S.$$

We follow the same procedure above. That is, we define $u = u_1 - u_2$ which satisfies $\nabla u = 0$, and $\partial u/\partial n = 0$. We define J as above, and equation 3.2 is still valid. We find that $J = 0$ and that $\partial u/\partial n = 0$ for all $\boldsymbol{r} \in S$. Then $u(\boldsymbol{r})$ is constant inside , so that $u_1 - u_2 =$ constant. The condition $\partial u/\partial n = 0$ does not guarantee that the constant is 0. Since we are interested in this course in tools to find the electric field $\boldsymbol{E}$ and this is the gradient of the potential (solution of the Laplace's equation), then we do not care of this constant, and the solution of the Laplace's equation still provides a unique electric field.

Note that since the electrical field is the gradient of the potential, the Neumann boundary conditions (which act on the gradient) are natural to this problem. We leave as an exercise to prove that in the case of mixed boundary conditions the Laplace's equation has a unique solution.

3.2 Solution of the Poisson's Equation using the Green's function method

In general if we have an operator of the form

$$Lu = f \tag{3.3}$$

and we know the corresponding Green's function, that is we know $G(\boldsymbol{r}, \boldsymbol{r}_i)$ such that

$$LG(\boldsymbol{r}, \boldsymbol{r}_i) = -\delta(\boldsymbol{r} - \boldsymbol{r}_i) \tag{3.4}$$

then we can combine the two equations above to get a solution of u in terms of the Green's function G. We do this by multiplying equation 3.3 by $G(\boldsymbol{r}, \boldsymbol{r}_i)$ and equation 3.4 by $u(\boldsymbol{r}_i)$ and subtract and then integrate over a volume having the input points $\boldsymbol{r}_i$. This is,

$$\int_V G(\boldsymbol{r}, \boldsymbol{r}_i) L\, u(\boldsymbol{r}_i) - u(\boldsymbol{r_i}) L\, G(\boldsymbol{r}, \boldsymbol{r}_i) d\boldsymbol{r}_i = \int_V [G(\boldsymbol{r}, \boldsymbol{r}_i) f(\boldsymbol{r}_i) + u(\boldsymbol{r}_i)\delta(\boldsymbol{r} - \boldsymbol{r}_i)] d\boldsymbol{r}_i.$$

and since

$$\int_V u(\boldsymbol{r}_i)\delta(\boldsymbol{r} - \boldsymbol{r}_i) d\boldsymbol{r}_i. = u(\boldsymbol{r}),$$

we find that

$$u(\boldsymbol{r}) = \int_V [G(\boldsymbol{r}, \boldsymbol{r}_i) L u(\boldsymbol{r}_i) - u(\boldsymbol{r_i}) L G(\boldsymbol{r}, \boldsymbol{r}_i)] d\boldsymbol{r}_i - \int_V [G(\boldsymbol{r}, \boldsymbol{r}_i) f(\boldsymbol{r}_i) d\boldsymbol{r}_i$$

In particular, for the Poisson's equation with charge density $f(\boldsymbol{r}_i) = \rho(\boldsymbol{r}_i)$ and $L = \nabla^2$ we find

$$u(\boldsymbol{r}) = \int_V [G(\boldsymbol{r}, \boldsymbol{r}_i) \nabla^2 u(\boldsymbol{r}_i) - u(\boldsymbol{r_i}) \nabla^2 G(\boldsymbol{r}, \boldsymbol{r}_i)] d\boldsymbol{r}_i - \int_V [G(\boldsymbol{r}, \boldsymbol{r}_i) \rho(\boldsymbol{r}_i) d\boldsymbol{r}_i$$

We can apply again Green's second identity 2.8 second [1] to reduce the first integral from a volume to a surface integral. That is

$$u(\boldsymbol{r}) = \int_S \left[G(\boldsymbol{r}, \boldsymbol{r}_i) \frac{\partial u(\boldsymbol{r}_i)}{\partial n} - u(\boldsymbol{r_i}) \frac{\partial G(\boldsymbol{r}, \boldsymbol{r}_i)}{\partial n} \right] d\boldsymbol{r}_i - \int_V G(\boldsymbol{r}, \boldsymbol{r}_i) \rho(\boldsymbol{r}_i) d\boldsymbol{r}_i. \tag{3.5}$$

Then we found a solution of the Poisson's equation in terms of a surface integral over some known boundary conditions for the Green's function and the function $u(\boldsymbol{r}_i)$ and a volume integral over the source, multiplied by the Green's function. Note that we can have a reduction in two cases here:

(i) **Zero Dirichlet Boundary Conditions:**
In this case $u(\boldsymbol{r}_i) = 0$ in the surface S and equation 3.5 reduces to

$$u(\boldsymbol{r}) = \int_S G(\boldsymbol{r}, \boldsymbol{r}_i) \frac{\partial u(\boldsymbol{r}_i)}{\partial n} dS - \int_V G(\boldsymbol{r}, \boldsymbol{r}_i) \rho(\boldsymbol{r}_i) d\boldsymbol{r}_i$$

(ii) **Zero Neumann Boundary Conditions:**
In this case $\partial u / \partial n = 0$ and equation 3.5 reduces to

$$u(\boldsymbol{r}) = \int_S u(\boldsymbol{r_i}) \frac{\partial G(\boldsymbol{r}, \boldsymbol{r}_i)}{\partial n} d\boldsymbol{r}_i - \int_V G(\boldsymbol{r}, \boldsymbol{r}_i) \rho(\boldsymbol{r}_i) d\boldsymbol{r}_i. \tag{3.6}$$

3.2.1 Uniqueness of solutions of the Poisson's Equation

We can reproduce the same proofs for the uniqueness of the solutions of the Laplace's equation here (note that source term $\rho(\boldsymbol{r})$) gets subtracted in $u = u_1 - u_2$). However we have to be careful with the Cauchy boundary conditions. We show how the problem for Cauchy boundary conditions could ill-posed (that is, it might not have a solution).

Let us assume for the moment that we neglect the data corresponding to the derivative $\partial u / \partial n$ on S (Neumann zero boundary conditions) and assume some Dirichlet boundary conditions in solution 3.5. This equation looks now as equation 3.6, where we apply now a Dirichlet boundary condition (recall that in Cauchy boundary condition both, Dirichlet and Neumann are used). Then from equation 3.6 we can compute $\partial u / \partial n$ inside the volume and take the limit along the boundary S. How can we guarantee that $\partial u / \partial n = 0$ as assumed initially? Note that we have the freedom of choosing any charge density $\rho(\boldsymbol{r}_i)$ as a source.

[1] https://en.wikipedia.org/wiki/Green%27s_identities

3.3 Orthogonal Functions

Here we drop the convention of **bold** on vectors since the concept of vector will be very general and sometimes the scalars in this context are vectors as well.

3.3.1 Motivation

Let us consider the two dimensional space $\mathbb{R}^2$ with a basis of two vectors $\{v, w\}$. Hence any two–dimensional $u = (x, y)$ vector can be written as

$$u = c_1 v + c_2 w.$$

For example, let $v = (1, 1/2)^T$, and $w = (-1/2, 2)^T$. We want to find a representation of the vector $(2, 3)^T$ in these basis. For that we need to solve the matrix equation

$$\begin{pmatrix} 1 & -1/4 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

The solution of this system is given by

$$c_1 = 22/9 \quad , \quad c_2 = 16/9$$

Figure 3.1 illustrates the two basis vectors, the input vector, and its parallel projections (components along the basis), with the coefficients c_1, c_2.

What would happen if the two basis vectors are orthogonal (and normalized?). For example, consider the vectors $(\sqrt{2}/2, \sqrt{2}/2)$, and $(-\sqrt{2}/2, \sqrt{2}/2)$? Certainly we could set up a 2x2 matrix system such as the one above, but this problem can be solved much more easily in a different form. Let the two orthonormal vectors be v, w, We want to write

$$u = c_1 v + c_2 w.$$

Take the dot product with v and find

$$u \cdot v = c_1 v \cdot v + c_2 w \cdot v.$$

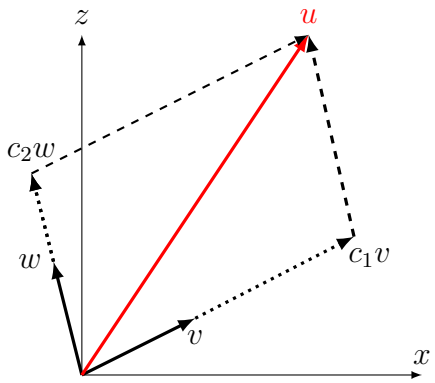

Figure 3.1: The red vector u shown as a linear combination of the basis vectors v and w. The stretching factors (later to be known as Fourier coefficients, c_1, c_2 are needed for the representation.

But $v \cdot v = 1$ since the two vectors are unitary, and $w \cdot v = 0$ since the two vectors are orthogonal. We found then

$$c_1 = u \cdot v.$$

Likewise $c_2 = u \cdot w$. We did not have to solve any equation to get to this result. Just take inner product (projections) of the output vector u, with the base vectors v and w.

In general in any finite dimensional space of n dimensions if we have an orthogonal basis $\{b_1, b_2, \cdots, b_n\}$, we ca write any vector u in the space as

$$u = c_1 b_1 + c_2 b_2 + \cdots c_n b_n,$$

where the j-th coefficient is given by

$$c_j = u \cdot b_j$$

or

$$u = (u \cdot b_1) b_1 + (u \cdot b_2) b_2 + \cdots (u \cdot b_n) b_n,$$

or

$$u = \sum_{j=1}^{n} (u \cdot b_j) b_j \tag{3.7}$$

So, we find a decomposition of the vector u in the orthogonal basis by figuring out that the coefficients are the orthogonal projections (here just a dot product) of the output vector u with each of the base vectors. The coefficients $c_j = u \cdot b_j$ are known as the **Fourier** coefficients, for reasons that will be clear soon.

Equation 3.7 is very important. It is a representation of the (output) vector u in terms of some basis $\{b_j\}$. This is the hearth of Fourier series and more generally orthogonal representations of functions. We want to generalize this idea to functions. For that we need to extend the dot product from finite dimensional spaces to infinite dimensional vector spaces or spaces of functions. We start by the definition of an important inner product space (Hilbert space).

3.3.2 Hilbert Spaces

The Hilbert spaces belong to the theory of functional analysis. Here we only make a small survey of some ideas that are needed to develop solutions to differential equations such as the Poisson's, Laplace's, and wave equations needed to cover the theory of electrodynamics.

A Hilbert space [2] $\mathcal{H}$, real or complex is a complete metric space with respect to the distance which is induced by its inner product. Think of a vector space where an inner product operation (defined below) induces a metric (distance). Let x, y, z elements of the Hilbert space and a, b scalars (complex or real. For generality we assume them to be complex since any real number is complex with 0 imaginary component). Here are the axiomatic properties that define an **inner product.**

(i) The inner product between two elements of space x, y (noted with brackets $\langle x, y \rangle$) turns into the conjugate when we reverse the order. That is

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

(ii) The inner product is linear in its first element. That is,

[2] https://en.wikipedia.org/wiki/Hilbert_space

$$\langle a\,x + b\,y, z\rangle = a\langle x, y\rangle + b\langle x, y\rangle.$$

(iii) The inner product of an element x with itself is 0 only if $x = 0$, otherwise is positive. That is,

$$\begin{aligned} \langle x, x\rangle = 0 \quad &\Longleftrightarrow \quad x = 0, \\ \langle x, x\rangle > 0 \quad &\Longleftrightarrow \quad x \neq 0, \end{aligned}$$

We see that

$$\langle x, ay + bz\rangle = \overline{\langle ay + bz, x\rangle} = \overline{\langle ay, x\rangle + \langle bz, x\rangle} = \overline{\langle ay, x\rangle} + \overline{\langle bz, x\rangle}$$

That is, after reducing the last expression we find

$$\langle x, ay + bz\rangle = \overline{a}\langle x, y\rangle + \overline{b}\langle x, z\rangle.$$

The norm (size) of an element of the space is given by $\|x\| = \sqrt{\langle x, x\rangle}$, and the distance (metric) between two elements x and y is given by

$$d = \|x - y\| = \sqrt{\langle x - y, x - y\rangle}.$$

The distance is a metric and as such it satisfies the triangular inequality $d(x, z) \leq d(x, y) + d(y, z)$.

Example 3.3.1 1. The space $\mathbb{R}^n$ with

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \quad , \quad x = (x_1, \cdots, x_n) \quad , \quad y = (y_1, \cdots, y_n).$$

2. The space $\mathbb{C}^n$ with

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i} \quad , \quad x = (x_1, \cdots, x_n) \quad , \quad y = (y_1, \cdots, y_n).$$

3. The ℓ_p spaces. We can have infinity sequences $a = (a_i)$ such that $(\sum_{i=1}^{\infty} |a_i|^p)^{1/p} < \infty$, $0 < p < \infty$.

   Choose two sequences $a = (a_i)$, $b = (b_i)$, with a_i, b_i, complex numbers. Define the inner product as $\langle a, b\rangle = \sum_{i=0}^{\infty} a_i \overline{b_i}$.

   The $p-$norm of an element $a = (a_i)$, is given by

$$\|a\|_p = \left(\sum_{n=0}^{\infty} |a_i|^p \right)^{1/p}.$$

   Up to this point all the examples belong to discrete spaces. We now list examples in the continuous spaces.

4. The set of real valued functions in an interval $[a, b]$,

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx.$$

5. the $L^p[a, b]$ spaces (Lebesgue integrable functions) in an interval $[a, b]$. Choose the set of functions f such that

$$\int_a^b |f(x)|dx < \infty$$

   Then define the inner product of two functions f and g as

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

   and the $p-$norm as

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}.$$

6. Weighted $L^p_w([a,b])$ space. Provided a weight function which is integrable then define the inner product of two functions f and g as

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx,$$

and the norm as

$$\|f\|_p = \left(\int_a^b w(x)|f(x)|^p dx\right)^{1/p}.$$

Orthogonality

We say that two elements f and g of a Hilbert space $\mathcal{H}$ are **orthogonal** if $\langle f, g\rangle = 0$. We are familiar with orthogonality in $\mathbb{R}^n$. We show some examples of orthogonality in the continuum.

- Let us define the inner products for integrable functions in the interval $[-\pi, \pi]$. Then the set of functions defined $f_n(x) = \sin nx$, has orthogonality for any two functions such that $n \neq m$. We show this. Pick n, m, $n \neq m$, then

$$\langle f_n, f_m\rangle = \int_{-\pi}^{\pi} \sin mx \sin nx dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x]dx = 0. \quad (3.8)$$

since $\cos x$ is periodic in the interval $[-\pi, \pi]$.

- Let us consider the complex exponential function $f_n(t) = \mathrm{e}^{\mathrm{i}n(2\pi t)}$ in the interval $[0, 1]$. We compute $\langle f_n, f_m\rangle$ for $m \neq n$. That is,

$$\langle f_n, f_m\rangle = \int_0^1 \mathrm{e}^{\mathrm{i}(2\pi nt)}\mathrm{e}^{-\mathrm{i}(2\pi mt)}dt = \int_0^1 \mathrm{e}^{\mathrm{i}(2\pi(n-m)t)}dt = \frac{1}{2\pi(n-m)}\mathrm{e}^{2\mathrm{i}(n-m)\pi t}\Big|_0^1.$$

Now, $\mathrm{e}^{\mathrm{i}(2\pi kt)}$ for any $k \in \mathbb{Z}$, evaluates to 1 in $t = 1$, and in $t = 0$, so $\langle f_n, f_m\rangle = 0$.

- If we define the inner product as $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$, then the polynomials 1 and x are orthogonal, since $\int_{-1}^{1} x\, dx = 0$.

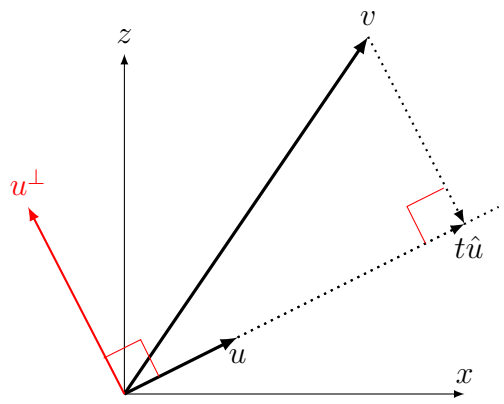

Figure 3.2: The Gram-Schmidt process....

As shown in the motivation section, orthogonal functions are important for representation of functions in different coordinates. Here the base functions are coordinates on a new space. We show next how to convert a set of non orthogonal functions to a set of orthogonal functions.

The Gram-Schmidt Process

We recall how, from two independent vectors which are not orthogonal we can generate a set of two orthogonal vectors which span the same space. Then this process can be extended in the same way to any number of vectors. We use the Figure 3.2 to explain the process.

- Let us assume that we start with two vectors u, v as shown in the figure. The vectors are not orthogonal. We start with the vector u as the first vector on the new collection of orthogonal vectors representing the same spanned space.

- We normalize u, that is we find

$$\hat{u} = \frac{u}{\|u\|}.$$

- We project v into the vector line spanned by u. That is we take the dot product of v with $\hat{u}$.

$$t = v \cdot \hat{u} = \frac{v \cdot u}{\|u\|}.$$

The vector $t\hat{u}$ provides the foot of the perpendicular from v to the line having u (because we made the projection. See the figure). If now we subtract $t\hat{u}$ from v we should get $u^{\perp}$, which is perpendicular to u. That is,

$$u^{\perp} = v - t\hat{u} = v - \frac{v \cdot u}{\|u\|^2}u.$$

Let us verify that $u^{\perp}$ is indeed perpendicular to u. We take the inner (dot) product between the two vectors.

$$u \cdot u^{\perp} = u \cdot v - u \cdot \frac{v \cdot u}{\|u\|^2}u. = u \cdot v - u \cdot v\frac{\cancel{u \cdot u}}{\cancel{\|u\|^2}} = 0.$$

If we would have three vectors, the third vector w, should be projected to the plane spanned by the previous two vectors and that projection be removed from it. Removing projections is to straight up the vector that it is now perpendicular to the subspace spanned by the previous vectors.

The algorithm is as follows: Given a set of vectors $\{u_1, u_2, \cdots u_n\}$ find new vectors $\{v_1, v_2, \cdots v_n\}$ such that they are orthogonal and span the same space. Define

$$\text{proj}_u(v) = (v \cdot \hat{u})\hat{u} = \left(v \cdot \frac{u}{\|u\|}\right)\frac{u}{\|u\|},$$

- $v_1 = u_1$ $\qquad\qquad$ $\hat{v}_1 = \frac{v_1}{\|v_1\|}$
- $v_2 = u_2 - \text{proj}_{v_1}(u_2)$ $\qquad\qquad$ $\hat{v}_2 = \frac{v_2}{\|v_2\|}$
- $v_3 = u_3 - \text{proj}_{v_2}(u_3) - \text{proj}_{v_1}(u_3)$ $\qquad\qquad$ $\hat{v}_3 = \frac{v_3}{\|v_3\|}$,

  and for the vector v_k,
- $v_k = u_k - \sum_{i=1}^{k-1} \text{proj}_{v_k}(u_k)$ $\qquad\qquad$ $\hat{v}_k = \frac{v_k}{\|v_k\|}$,

The same idea applies for the Gram-Schmidt process on functions on a Hilbert space. I leave as an exercise to find the orthogonal functions using the Gram-Schmidt method starting at $\{1, x, x^2, x^3\}$, and using the definition of inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

Spectral Expansion

Let us assume that we have a basis $\{\boldsymbol{\phi}_j\}$ that could be infinite countable (that is j could go from 0 to ∞), which spans a given vector in the space. Let us assume also that the functions are orthonormal. Then

$$u = \sum_j c_j \boldsymbol{\phi}_j.$$

After multiplying (inner product) by $\boldsymbol{\phi}_i$ we find

$$\langle u, \boldsymbol{\phi}_i\rangle = \sum_j c_j \langle \boldsymbol{\phi}_j, \boldsymbol{\phi}_i\rangle = \sum c_j \delta_{ij} = c_i,$$

since the functions are orthonormal. We then write

$$u = \sum_i \langle u, \boldsymbol{\phi}_i\rangle \boldsymbol{\phi}_i. \tag{3.9}$$

This equation is very powerful and it is used often in the theory of approximation by series. We assume are assuming that the index i goes through all the collection of base functions so that the equation is exact. See that if the inner product is defined as an integral form we can write equation 3.9 as

$$u(x) = \left(\sum_i \int_a^b u(y)\overline{\phi}_i(y)dy\right)\phi_i(x) = \int_a^b u(y)\left(\sum_i \overline{\phi}_i(y)\phi_i(x)\right)dy$$

and so, from the definition of the Dirac delta we observe that it should happen that

$$\delta(y-x) = \sum_i \phi_i(x)\overline{\phi}_i(y). \tag{3.10}$$

This expression is right only if the expansion 3.9 is exact, for all possible functions $u \in \mathcal{H}$. In that case we say that the set of orthogonal function is **complete** .

We show the classical example of expansion using trigonometrical polynomials.

Let us assume that $\boldsymbol{\phi}_i$ is defined as

$$\boldsymbol{\phi}_j = \alpha \mathrm{e}^{\mathrm{i}\omega j t},$$

in the interval $[0, T]$. The circular frequency is given by $\omega = 2\pi/T$. The coefficient α is such that the function is normalized. Let us find α. We want

$$\langle \boldsymbol{\phi}_j, \boldsymbol{\phi}_j \rangle = 1$$

That is

$$\alpha^2 \int_0^T \mathrm{e}^{\mathrm{i}\omega_j t} \mathrm{e}^{-\mathrm{i}\omega_j t} = \alpha^2 \int_0^T \mathrm{e}^0 = \alpha^2 T = 1.$$

and from here $\alpha = 1/\sqrt{T}$. Note that α is independent of j. Then our normalized basis is given by

$$\boldsymbol{\phi}_j = \frac{1}{\sqrt{T}} \mathrm{e}^{\mathrm{i}j\omega t}.$$

We already showed that $\langle \boldsymbol{\phi}_j, \boldsymbol{\phi}_k \rangle = \delta_{jk}$. That is the basis is orthonormal. We then can use equation 3.9 to write any function $u(t)$ as

$$u = \sum_i \langle u, \boldsymbol{\phi}_i \rangle \boldsymbol{\phi}_i = \sum_{j=-\infty}^{\infty} c_j \mathrm{e}^{\mathrm{i}j\omega t} dt \tag{3.11}$$

with the Fourier coefficient

$$c_j = \frac{1}{T} \int_0^T u(t) \mathrm{e}^{\mathrm{i}j\omega t}$$

Equation 3.11 is the Fourier series expansion of the function u, in terms of complex exponential functions.

Linear Operators

A linear operator in a Hilbert space $\mathcal{H}$, is a transformation $T : \mathcal{H} \to \mathcal{H}$ such that $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$. In the space $\mathbb{R}^n$, the linear transformation T is associated with an $n \times n$ matrix. A simple example in any Hilbert space is a multiplication by a scalar. That is if $T(x) = \alpha x$, then clearly T is linear. If $\mathcal{H}$ is the space of $C^\infty[a, b]$, functions, that is the space of functions which have an infinite number of derivatives in $\mathbb{R}^n$, then the any partial derivative $T(u) = \partial u/\partial x_i$, is a linear operator. In general derivatives and integrals are linear operators but we need to be careful that they are well defined everywhere in the space. For example, while the Heaviside function $H(x) = 0$, for $x < 0$, and $H(x) = 1$, for $x \geq 0$, is well defined everywhere, its derivative (a Dirac delta) is not a function in the same space. Some sort of closure is important for linear operators. We can extend the definition of linear operator T from a one space $\mathcal{H}_1$ to another space $\mathcal{H}_2$, as long as the operator has the property $T(\alpha u + \beta v) = \alpha T(u + \beta v)$, for all α, β scalars (in $\mathbb{R}$ or $\mathbb{C}$) and $u, v \in \mathcal{H}_1$.

Eigenvalues-Eigenfunctions

For a linear operator T in a Hilbert space, we say that λ is an eigenvalue and u is an eigenfunction if

$$Tu = \lambda u.$$

We are interested in solutions of this equation other than the trivial $u = 0$ solution. The eigenvalue/eigenvector theory is very important since it will provide a way to generate orthogonal basis used for representation of functions in other forms.

Adjoint of a Linear Operator

The adjoint of a linear operator is the generalization of the conjugate transpose (Hermitian) of a matrix to general operators in Hilbert spaces. Assume that we have an operator $A : \mathcal{H} \to \mathcal{H}$, and that there exists another operator $A^\star : \mathcal{H} \to \mathcal{H}$ such that

$$\langle Au, v\rangle = \langle u, A^\star v\rangle \quad , \quad \text{for all } u, v \in \mathcal{H} \tag{3.12}$$

then we say that $A^\star$ is the **adjoint** of A. We can extend this definition to linear operators from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$. Then in this case the adjoint A would go from $\mathcal{H}_2$ into $\mathcal{H}_1$. For example a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be seen as an $m \times n$

matrix of real numbers. The adjoint would be an operator from $\mathbb{R}^m$ to $\mathbb{R}^n$ represented by the trasnposed $n \times m$ matrix or real numbers.

While finding the adjoint of matrices is easy (transpose and conjugate) that is not the case of differential operators. We provide some simple examples about how to find the adjoint of a differential operator and then show a table of important adjoint operators.

The adjoint of a differential operator is usually done using integration by parts in scalar functions or different versions of Stokes and Gauss (divergence) theorems for higher dimensional spaces. Let us think of the derivative operator on polynomials of degree n to polynomials of degree $n-1$ in the one-dimensional real space. That is, we define $T : \mathcal{H}_n \to \mathcal{H}_{n-1}$ where $\mathcal{H}_n$ is the Hilbert space of polynomials of order n, and $\mathcal{H}_{n-1}$ the corresponding Hilbert space of polynomials of order $n-1$, and T such that. $T(p(x)) = p'(x)$. Then, considering the definition of inner product as the integral of the product on the interval $[-1, 1]$, and doing integration by parts

$$\begin{aligned}
\langle\, T(p), q\,\rangle &= \int_{-1}^{1} p'(x)q(x)dx \\
&= p(x)q(x)\Big|_{-1}^{1} - \int_{-1}^{1} p(x)q'(x)dx \\
&= p(x)q(x)\Big|_{-1}^{1} - \langle\, p, T(q)\,\rangle \\
&= p(1)q(1) - p(-1)q(-1) - \langle\, p, T(q)\,\rangle
\end{aligned}$$

Let us, for the moment assume that the boundary values evaluate to zero. That is,

$$p(1)q(1) - p(-1)q(-1) = 0.$$

Then we would have

$$\langle\, T(p), q\,\rangle = \langle\, p, -T(q)\,\rangle$$

and from the definition of adjoint we would have that the adjoint of $T = dp/dx$, is $-T = -dp/dx$. To include the boundaries is a bit harder. We need to define an extra operator B such that

$$\langle p, Bq\rangle = p(1)q(1) - p(-1)q(-1)$$

and in that case the adjoint of T would be $-T + B$, where now the boundaries are incorporated. Such an operator B would be

$$B(x) = \delta(x-1) - \delta(x+1)$$

Since

$$\begin{aligned}\langle p(x), [\delta(x-1) - \delta(x+1)]q(x)\rangle &= \langle p(x), \delta(x-1)q(x)\rangle - \langle p(x), \delta(x+1)q\rangle \\ &= p(1)q(1) - p(-1)q(-1).\end{aligned}$$

Note that we are extending the space of polynomials to distributions. In general, adjoint operators are computed with the boundary terms vanishing.

An operator that is equal to its adjoint is called a **self-adjoint operator.** We showed that the adjoint of a first order derivative is its negative. If we take a second order derivative then we return to the original operator. Then the adjoint of a second derivative is self-adjoint. Another symple self-adjoint operator is a multiplication by a real scalar function. That is, let us define the operator $T(u(x)) = a(x)u(x)$, then

$$\langle Tu, v\rangle = \langle au, v\rangle = \langle u, av\rangle = \langle u, Tv\rangle,$$

so $T = T^\star$. That is, T is self-adjoint. One last example for self-adjoint operator is the operator T defined as

$$Tu = \frac{d}{dx}p(x)\frac{du(x)}{dx}, \tag{3.13}$$

with $p(x)$ a real function, and the boundaries evaluate to 0. Let us see

$$\langle Tu, v\rangle = \int_a^b \frac{d}{dx}p(x)\frac{du(x)}{dx}\, v(z)dx$$

we do integration by parts, and have that

$$\langle Tu, v\rangle = \cancel{p(x)\frac{du(x)}{dx}v(z)\Big|_a^b}^{\to 0} - \int p(x)\frac{du(x)}{dx}\frac{dv(x)}{dx}dx$$

We do one more integration by parts on du/dx, to find

$$\langle Tu, v\rangle = -\cancel{u(x)\frac{d}{dx}p(x)\frac{dv(x)}{dx}\Big|_a^b}^{\to 0} + \int_a^b \frac{d}{dx}p(x)\frac{dv(x)}{dx}.$$

but this is precisely $\langle u, Tv\rangle$ so we found

$$\langle Tu, v\rangle = \langle u, Tv\rangle$$

and $T = T^*$, so T is self-adjoint operator. We add that the sum of self-adjoint operators is as well a self-adjoint operator. In fact here are a few properties of self-adjoint operators which are easy to show

(i) $I^* = I$, where I is the identity operator.

(ii) $(A^*)^* = A$.

(iii) $(A + B)^* = A^* + B^*$.

(iv) $(\alpha A)^* = \overline{\alpha}A^*$.

(v) $(AB)^* = B^*A^*$.

(vi) If A^{-1} exists then $(A^*)^{-1} = (A^{-1})^*$.

We only show the last property. Let us assume that A^{-1} exists, then

$$\langle x, y\rangle = \langle Ix, y\rangle = \langle A^{-1}Ax, y\rangle = \langle Ax, (A^{-1})^*y\rangle = \langle x, A^*(A^{-1})^*y\rangle$$

Then $A^*(A^{-1})^*$ is the adjoint the identity. But the adjoint of the identity is the identity, so $A^*(A^{-1})^* = I$, or $(A^*)^{-1} = (A^{-1})^*$. Table 3.1 shows a few operators and their adjoints. All of them can be derived from the definition of adjoint 3.12.

We show an important theorem for function representations.

- **Theorem:** If an operator is self-adjoint then its eigenvalues are real and the eigenfunctions corresponding to different eigenvalues are orthogonal.

Table 3.1: Some common operators and its adjoint pairs

Gradient/Divergence	$a(\boldsymbol{x})\nabla u(\boldsymbol{x})$	$-\nabla \cdot [a(\boldsymbol{x})\boldsymbol{G}(\boldsymbol{x})]$
Curl	$a(\boldsymbol{x})\,\nabla \times \boldsymbol{F}(\boldsymbol{x})$	$\nabla \times [a(\boldsymbol{x})\boldsymbol{F}(\boldsymbol{x})]$
Laplacian	$a(\boldsymbol{x})\nabla^2 u(\boldsymbol{x})$	$\nabla^2[a(\boldsymbol{x})g(\boldsymbol{x})]$
Mixed	$\nabla \cdot a(\boldsymbol{x})\nabla u(\boldsymbol{x})$	$\nabla \cdot a(\boldsymbol{x})\nabla g(\boldsymbol{x})$

- **Proof:**
  Let us assume that A is self-adjoint and u, v are eigenfunctions with eigenvalues λ, σ. Then

$$\langle Au, v\rangle = \langle \lambda u, v\rangle,$$

  and since A is self-adjoint we have that

$$\langle Au, v\rangle = \langle u, Av\rangle = \langle u, \sigma v\rangle.$$

  Then

$$(\lambda - \overline{\sigma})\langle u, v\rangle = 0$$

  Then since $\lambda \neq \sigma$ we find that $\langle u, v\rangle = 0$, so u is orthogonal to v.

  Now, from

$$\begin{aligned}\langle Au, u\rangle &= \lambda\langle u, u\rangle = \lambda\|u\|^2\\ \langle u, Au\rangle &= \overline{\lambda}\langle u, u\rangle = \overline{\lambda}\|u\|^2,\end{aligned}$$

  $\lambda = \overline{\lambda}$, and $\lambda \in \mathbb{R}$. This shows the theorem.

The Sturm-Liouville Problem

Let now us consider the Sturm-Liouville [3] differential equation

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u(x) = -\lambda w(x)u(x). \tag{3.14}$$

with $u(x)$, a function with second continuous derivatives in $(0,1)$ with some boundary conditions at $x = 0, 1$. The function $w(x) > 0$, is known as weight or density function and assumed continuous in $[a, b]$, p is continuously differentiable, and q is continuous in $[a, b]$.

The importance of the operator L cannot be underestimated, it encompasses a wealth of number of problems appearing from the method of separation of variables in partial differential equations. They have the nice property of self-adjointness which provides real (physical) distinc eigenvalues with orthogonal eigenfunctions. W.N. Everitt [4] shows a catalog of Sturm-Liouville differential equations.

The Sturm-Liouville operator 3.14 is written as a "normalized" operator by dividing it by $w(x)$ (we assume $w(x) \neq 0$), and making it into an eigenvalue/eigenvector operator which is self adjoing and so its eigenfunctions are orthogonal and its eigenvalues are real. That is, we can write equation 3.14 as

$$Lu = -\frac{1}{w(x)}\left[\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u\right]. \tag{3.15}$$

Observe that the $Lu = \lambda u$ is equivalent to equation 3.14, but now it is an eigenfunction/eigenvalue problem. We showed in section 3.3.2 that the product with a real scalar operator is a self adjoint operator, that the operator $[(d/dx)p(d/dx)]$ (see equation 3.13) is a self-adjoint operator and since the sum of self-adjoint operators is itself a self-adjoint operator then the Sturm-Liouville operator L defined above is self-adjoint.

Table 3.2 illustrates a set of eigenfunctions and their eigenvalues associated to the Sturm-Liouville eigenvalue/eigenfunction problem 3.15. Note the collection of orthogonal eigenfunctions: Fourier, Bessel, Hermite, Legendre, Laguerre, and Tchebysheff.

[3] https://en.wikipedia.org/wiki/Sturm%E2%80%93Liouville_theory
[4] http://www.math.niu.edu/SL2/papers/birk0.pdf

Name associated with function	$p(x)$	$q(x)$	$w(x)$	interval	Boundary conditions	eigenvalues n=1,2...,	Normalized eigenvectors
Fourier	1	0	1	$-[\pi,\pi]$	$y(\pi)=y(-\pi)$ $y'(\pi)=y'(-\pi)$	$0, n^2$	$1, \sin nx, \cos nx$
Bessel	x	$-k^2x^{-1}$	x	$[0,1]$	$y(0)=$ finite $y(1)=0$	$\mu_n, J_k(\mu_n)=0$	$J_k(\mu_n(x)$
Legendre polynomials	$1-x^2$	0	1	[-1,1]	$y(-1)$ and $y(1)$ finite	$n(n+1)$	$P_n(x)$
Tchebycheff polynomials	$(1-x^2)^{1/2}$	0	$(1-x^2)^{-1/2}$	[-1,1]	$y(-1)$ and $y(1)$ finite	n^2	$T_n(x)$
Hermite polynomials	$\exp(-x^2)$	0	$\exp(-x^2)$	$(-\infty,\infty)$	As $\|x\|\to\infty$, $y=\mathcal{O}(x^k)$ for some $k>0$	$0, 2n$	$H_0(x), H_n(x)$
Laguerre	$x\exp(-x)$	0	$x\exp(-x)$	$[0,\infty)$	$y(0)$ finite; as $x\to\infty$, $y=\mathcal{O}(x^k)$ for some $k>0$	n	$L_n(x)$

Table 3.2: A few examples of the Sturm–Liouville operator.

3.4 Problems

3.4.1 Problem 1

Show that the Poisson's equation has a unique solution if the data are fully prescribed with mixed boundary conditions.

3.4.2 Problem 2

If Cauchy boundary conditions are imposed in the Poisson's equation can this leads to a problem with no solution? That is, is this an ill-posed problem?

3.4.3 Problem 3

Given the set of elements of a Hilbert space in the space of polynomials $\{1, x, x^2, x^3, x^4\}$ with the inner product defined as

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

Use the Gram-Schmidt method to find a corresponding orthogonal representation of this set.

3.4.4 Problem 4

Equation 3.8 shows that the set of functions $\{\sin nx\}$, $n \in \mathbb{Z}$, is orthogonal, under the inner product defined in the same equation. Find the norm $s = \|\sin nx\|$. Then according to Dirac delta representation 3.10. We can say that

$$\delta(x-y) = \frac{1}{s^2}\sum_{j\mathbb{Z}}(\sin jx)(\sin jy).$$

Is this true? why?

Chapter 4

Lecture #4

4.1 Separation of Variables

We showed how to solve the Laplace's and Poison's equation using Green functions. We now show a different method called **separation of variables** . According to Morse and Feshbach [3] there are eleven coordinate system which yields separation of variables in the Laplacian operator. We focus in the most used coordinate systems which are the rectangular, the cylindrical and the spherical. Each coordinate system is used according to the symmetry of the problem. Let us start.

4.1.1 The Rectangular Coordinate System

The Laplace's equation in rectangular coordinates is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{4.1}$$

We already showed that if we specify the potential u in the surface (Dirichlet boundary conditions) the solution is unique. Then we will assume that together with the Laplace's equation we have some boundary conditions in a rectangular box. It needs to be a rectangular box because the particular problem we want to solve. As indicated above Morse and Feshbach showed eleven coordinate systems and the rectangular system only works for problem defined in rectangular boxes. For example, problems with spherical symmetry should work, or course, in spherical coordinates. We then indicate that there are some boundary conditions which the function should satisfy.

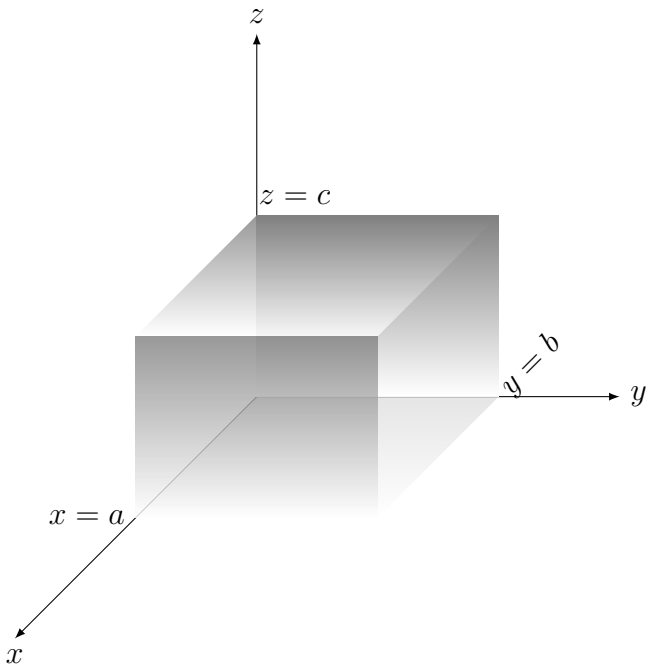

Figure 4.1: A box with potential defined in its 6 faces.

We consider a potential u which is known in the six faces of a rectangular box. Figure 4.1 illustrates the box with one vertex at 0. We want to find u everywhere inside the box. Here are the boundary conditions:

$$\begin{aligned} u(x, y.0) &= V_{\text{bottom}}(x, y) \\ u(x, y.c) &= V_{\text{top}}(x, y) \\ u(x, 0.z) &= V_{\text{left}}(x, z) \\ u(x, b.z) &= V_{\text{right}}(x, z) \\ u(0, y.z) &= V_{\text{back}}(y, z) \\ u(a, y.z) &= V_{\text{front}}(y, z) \end{aligned} \tag{4.2}$$

The method of separation of variables converts a partial differential equation into a system of three ordinary differential equations by assuming that the solution has the specific form

$$u(x, y, z) = X(x)Y(y)Z(z).$$

Note that it is assumed that the solution is factored out in three different factors each depending only on one of the coordinates. If we substitute this equation into equation 4.1 and divide the result by $X(x)Y(y)Z(z)$ we find

$$\frac{1}{X(x)}\frac{d^2X}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y}{dy^2}+\frac{1}{Z(z)}\frac{d^2Z}{dz^2}=0.$$

Note here that the derivatives are not partial derivatives any more. If the sum of of three functions of different variables is 0 is because each function is a constant. Otherwise we could find 0 for some value of one of the three functions and by changing its values (since it is variable) we would get a non-zero value. We have then the three problems:

$$\begin{aligned}
\frac{1}{X(x)}\frac{d^2X}{dx^2} &= \lambda \quad , \quad X(0)=V_{\text{back}}(0,0)=x_0 \quad , \quad X(a)=V_{\text{front}}(0,0)=x_a \\
\frac{1}{Y(y)}\frac{d^2Y}{dy^2} &= \mu \quad , \quad Y(0)=V_{\text{left}}(0,0)=y_0 \quad , \quad Y(b)=V_{\text{right}}(0,0)=y_a \\
\frac{1}{Z(z)}\frac{d^2Z}{dz^2} &= \nu \quad , \quad Z(0)=V_{\text{bottom}}(0,0)=z_0 \quad , \quad Z(c)=V_{\text{top}}(0,0)=z_a
\end{aligned}$$

where λ, μ, and ν are constants (do not depend on x, y or z). Observe also that since $\lambda+\mu+\nu=0$, $\nu=-(\lambda+\mu)$. We reduce then the problem of one partial differential equation into three ordinary differential equations. We can write these three equations differently as

$$\frac{d^2X}{dx^2}=\lambda X(x) \quad , \quad \frac{d^2Y}{dy^2}=\mu Y(y) \quad , \quad \frac{d^2Z}{dz^2}=\nu Z(z),$$

and see that the three equations can be seen as three eigenvalue/eigenvector problems of the same operator with different variable. That is, we write

$$L_xX=\lambda X \quad , \quad L_yY=\mu Y \quad , \quad L_zZ=\nu Z$$

Note that each one of these operators corresponds to a Sturm-Liouville problem 3.15 with $p(x)=1, q(x)=0, w(x)=1$. Since we are dealing with real functions we assume that the eigenvalues λ, μ, and ν are real. The boundary conditions imposed will indicate the

type of eigenfunctions and define if the operator is self-adjoint or not. Zero boundary conditions on second derivative operators always indicate self-adjointess. Also if the boundary condition is non-zero but the total contribution (total flow) in the boundary is 0 the second derivative operators are self-adjoint. It is important to be able to identify self-adjoint operators because they guarantee orthogonality of their eigenfunctions.

The eigenfunctions of the second derivative operator are complex exponentials. That is, if

$$\frac{d^2X}{dx^2} = \lambda X(x), \tag{4.3}$$

then $X(x) = \mathrm{e}^{\pm\sqrt{\lambda}x}$. Now, for the boundary conditions we have that $X(0) = x_0$ and $X(a) = x_a$. We can write the general solution of equation 4.3 as the linear combination of the two solutions. That is,

$$X(x) = A\mathrm{e}^{\sqrt{\lambda}x} + B\mathrm{e}^{-\sqrt{\lambda}x} \tag{4.4}$$

and find A and B by using the two boundary conditions. That is

$$\begin{aligned} A + B &= x_0 \\ A\mathrm{e}^{\sqrt{\lambda}a} + B\mathrm{e}^{-\sqrt{\lambda}a} &= x_a. \end{aligned} \tag{4.5}$$

Likewise we find

$$Y(y) = C\mathrm{e}^{\sqrt{\mu}y} + D\mathrm{e}^{\sqrt{\mu}y} \tag{4.6}$$

with C and D satisfying

$$\begin{aligned} C + D &= y_o \\ C\mathrm{e}^{\sqrt{\mu}b} + D\mathrm{e}^{-\sqrt{\mu}b} &= y_b \end{aligned}$$

and

$$Z(z) = E\mathrm{e}^{\sqrt{-\lambda-\mu}z} + F\mathrm{e}^{-\sqrt{-\lambda-\mu}z} \tag{4.7}$$

with E and F satisfying

$$\begin{aligned} E + F &= z_0 \\ E\mathrm{e}^{\mathrm{i}\sqrt{-\lambda-\mu}c} + D\mathrm{e}^{-\mathrm{i}\sqrt{-\lambda-\mu}c} &= z_c \end{aligned} \tag{4.8}$$

Recall that $\nu = -\lambda - \mu$. At least one of the representations 4.4, 4.6, and 4.7 should have complex exponentials and one should have real exponentials. Observe that under the square roots are positive signs for the first two and negative signs for the third. For the problem to be practical the boundary solutions should be such that provide self-adjoint operators which will lead to orthogonal functions. The second order differential operators are self-adjoint but we need the data to vanish in the boundaries. That is why most of the problems solved this way have a few boundaries where the data vanishes, otherwise the problems would be rather difficult and the method would not be practical. We show in the examples an analysis of the values that the eigenvalues can take.

We note that if λ, μ and $\nu = -\lambda - \mu$, are not zero the three two-by-two systems above have solution and this will provide solutions for each of the three ordinary differential equations. However this does not guarantee uniqueness in the general partial differential equation where higher dimensional spaces are involved. Recall that the constants λ, μ, and ν are arbitrary. Actually there are only two constants that we need to solve since $\lambda + \mu + \nu = 0$. We can choose λ and μ as the constants we need to solve for.

The next step is to use the superposition principle satisfied by linear differential equations. That is, we can add all solutions and the sum would still be a solution. We do not know yet if the values of λ, and μ are defined in the continuum or in the discrete. Actually since the cube is finite it can be shown that λ and μ can assume only particular values (in the discrete) related to the sides of the rectangular box. As the side of the box goes to infinity the values of λ and μ can take over smaller and smaller intervals up to the continuum when the side of the rectangular box reaches infinity. For generality we write the solution in integral form as

$$\begin{aligned} u(x,y,z) \;=\; & \int_S f(\lambda,\mu)\left(A\mathrm{e}^{\mathrm{i}\sqrt{-\lambda}x} + B\mathrm{e}^{-\mathrm{i}\sqrt{-\lambda}x}\right)\left(C\mathrm{e}^{\mathrm{i}\sqrt{-\mu}y} + D\mathrm{e}^{-\mathrm{i}\sqrt{-\mu}y}\right) \\ & \left(E\mathrm{e}^{\mathrm{i}\sqrt{\lambda+\mu}z} + F\mathrm{e}^{-\mathrm{i}\sqrt{\lambda+\mu}z}\right) d\lambda d\mu. \end{aligned}$$

where we already know the coefficients A, B, C, D, E, F, and $f(\lambda, \mu)$ is a weight function for each λ and μ chosen. The surface of integration S is the set of all possible λ and μ

values. The problem is not solved until we find $f(\lambda, \mu)$ and here is where we go back to the boundary conditions for the general 3D problem.

In the particular case of the cube that we are considering, we have 6 surfaces and each surface has a boundary condition as shown in the system of equations 4.2. This provides six equations that we need to solve for.

We have, for example that one of those equations would be

$$V_{\text{bottom}}(x, y) = \int_S f(\lambda, \mu) \left(A\mathrm{e}^{\sqrt{\lambda}x} + B\mathrm{e}^{-\sqrt{\lambda}x} \right) \left(C\mathrm{e}^{\sqrt{\mu}y} + D\mathrm{e}^{\sqrt{\mu}y} \right) (E + F)\, d\lambda d\mu.$$

and with this and the other 5 equations solve the integral equation for $f(\lambda, \mu)$. This is in general difficult but here is why this problem is solvable:

1. For finite dimensional volumes (such as the rectangular box here) the integral is not an integral but a two fold sum over the discrete values λ_i and μ_i.

2. For the problems that we solve some functions on the right are orthogonal, so the coefficients of these sums are found by making inner product of the data on the left with the orthogonal eigenfunctions. This will provide Fourier coefficients $f(\lambda, \mu)$. In general the values of λ and μ are known as the spectrum (frequencies for temporal variables or wave-numbers for spacial variables such as the ones we are dealing with now).

3. In the continuum the integral for each boundary is a two dimensional Fourier transform which is easy to invert (using the inverse Fourier transform). In the discrete the integral turns into an (infinite) Fourier series which are also easy to invert.

We now illustrate the methodology shown above with an example. Assume a box with dimensions a, b, and c, and with potential 0 in all faces except in the top face where $u(x, y, c) = V(x, y)$. See Figure 4.2. Find the potential inside the box.

The boundary conditions for this problem are:

$$\begin{aligned} u(x, y.0) &= 0 \\ u(x, y.c) &= V(x, y) \\ u(x, 0.z) &= 0 \\ u(x, b.z) &= 0 \\ u(0, y.z) &= 0 \\ u(a, y.z) &= 0. \end{aligned}$$

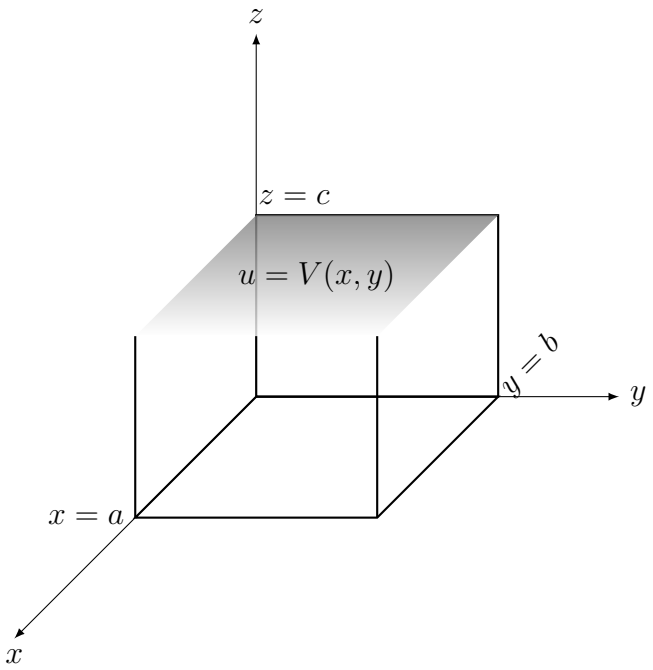

Figure 4.2: A box with potential zero in all faces except for the top faces where the potential is some function $V(x, y)$.

We start with equation 4.4 and the system 4.5 created by the one-dimensional boundary conditions. In this problem both $x_o = x_a = 0$, then $A = -B$, and we have that

$$A(\mathrm{e}^{\sqrt{\lambda}a} - \mathrm{e}^{-\sqrt{\lambda}a}) = 0$$

One way for this to happen is that $A = 0$. We want to avoid this which does not lead us to anything interesting. Then

$$\mathrm{e}^{\sqrt{\lambda}a} - \mathrm{e}^{-\sqrt{\lambda}a} = 0.$$

If $\lambda > 0$ then these two are real exponentials and they only are equal at $\lambda = 0$. Remember we are not interested in zero eigenvalues of the operator. Let us then assume that $\lambda < 0$, then we have the equation

$$2\mathrm{i} \sin \sqrt{|\lambda|}a^2 = 0. \tag{4.9}$$

This equation has an infinite number of solutions. If $\sqrt{|\lambda|}a = n\pi$, we have that

$$-\lambda_n = \frac{n^2\pi^2}{a^2}$$

is a solution of equation 4.9. We have an infinite set of eigenvalues and the corresponding eigenfunction is

$$X_n(x) = \sin\sqrt{|\lambda|_n}x.$$

We are interested only in positive values of $-\lambda$. As we claimed above the values of λ are discrete and can be made smaller by making a large. When a goes to ∞, then λ changes to be a discrete parameter to a continuum parameter.

We now look at the solutions in the y direction. From equations 4.6 and 4.7 and $y_o = y_b = 0$ we find that $C = -D$ and similarly as in the previous case we find

$$\mathrm{e}^{\sqrt{\mu}b} - \mathrm{e}^{-\sqrt{\mu}b} = 0.$$

We find, again here that the eigenvalues are given by the formula $-\mu_m = m^2\pi^2/b^2$, and the eigenfunctions by

$$Y_n(y) = \sin\sqrt{|\mu|_n}x.$$

With the z coordinate happens something different. Here instead of $-\lambda$ or $-\mu$, we have the combined eigenvalue $\lambda + \mu$, and most important, the condition at $z = c$ is non-zero. That is $z_c \neq 0$. Then from equation 4.7 we find; from $z_o = 0$, $E = -F$ and so the solution 4.8 is given by

$$Z_{mn}(z) = E(\mathrm{e}^{\sqrt{-\lambda-\mu}c} - \mathrm{e}^{-\sqrt{-\lambda-\mu}c}) = E\sinh\sqrt{|\lambda_n + \mu_m|}c.$$

since now $\lambda_n + \mu_m < 0$. In summary we found the following collections of eigenvalues with their respective eigenfunctions:

$$\begin{aligned}
\lambda_n &= -\frac{n^2\pi^2}{a^2} \quad , \quad X_n(x) = \sin\sqrt{|\lambda_n|}x \\
\mu_m &= -\frac{m^2\pi^2}{b^2} \quad , \quad Y_n(y) = \sin\sqrt{|\mu_m|}y \\
&\qquad Z_{nm}(z) = \sinh\sqrt{|\lambda_n + \mu_m|}z
\end{aligned}$$

and this generate a two index (on m and n) functions

$$u_{nm}(x,y,z) = (\sin\sqrt{|\lambda_n|}x)(\sin\sqrt{|\mu_m|}y)(\sinh\sqrt{|\lambda_n+\mu_m|}z).$$

We now claim that the superposition of all solutions of this type satisfies the Laplace's equation inside and in the boundary. However the superposition has unknown coefficients A_{mn} which we should find. That is, we claim that

$$u(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm}(\sin\sqrt{|\lambda_n|}x)(\sin\sqrt{|\mu_m|}y)(\sinh\sqrt{|\lambda_n+\mu_m|}z). \tag{4.10}$$

This is the point where we use the multi-dimensional boundary conditions to find A_{mn}. That is, we know that at $z=c$, $\quad u(x,y,c) = V(x,y)$. Then this is

$$V(x,y) = \sum_{n,m=1}^{\infty} A_{nm}\left[\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sinh\left(\sqrt{\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}}\;c\right)\right].$$

To solve for this equation we recognize that the sine functions are orthogonal. We will find their norm and use equation 3.9 to extract the A_{nm} coefficients. The interval of integration along the x direction is $[0,a]$. Then we use the definition

$$\langle f,g\rangle = \int_0^a f(x)g(x)dx.$$

Pick any n, and then

$$\begin{aligned}
\left\langle \sin\left(\frac{n\pi}{a}x\right), \sin\left(\frac{n\pi}{a}x\right)\right\rangle &= \int_0^a \sin^2\left(\frac{n\pi}{a}x\right)dx \\
&= \int_0^a \frac{1-\cos(2n\pi/a)x}{2}dx \\
&= \frac{a}{2} - \frac{\sin(2n\pi/a)x}{2}\bigg|_0^a \\
&= \frac{a}{2}
\end{aligned}$$

Then the normalization factor is $\sqrt{a/2}$. Likewise the normalization of the sine functions along the y direction is given by the factor $b/2$. We divide all the orthogonal functions by $\sqrt{ab}/2$ and then they will be orthonormal.

That is, we write now

$$V(x,y) = \frac{2}{\sqrt{ab}} \sum_{n,m=1}^{\infty} A_{nm} \left[\sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sinh\left(\sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}\, c \right) \right].$$

where we recycle the symbol A_{mn}. We now multiply (inner product) . We now take the 2D inner product of $V(x,y)$ with the orthonormal 2D function

$$\Phi_{jk}(x,y) = \frac{2}{\sqrt{ab}} \sin\left(\frac{j\pi}{a}x\right) \sin\left(\frac{k\pi}{b}y\right)$$

$$\begin{aligned} \langle V(x,y), \Phi_{jk}(x,y) \rangle &= \sum_{n,m=1}^{\infty} A_{nm}\delta_{nj}\delta_{mk} \sinh\left(\sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}\, c \right) \\ &= A_{jk} \sinh\left(\sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}\, c \right) \end{aligned}$$

From here

$$\begin{aligned} A_{jk} &= \frac{1}{\sinh\left(\sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}\, c \right)} \langle V(x,y), \Phi_{jk}(x,y) \rangle \\ &= \frac{1}{\sinh\left(\sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}\, c \right)} \int_0^a dx \int_0^b dy \frac{2}{\sqrt{ab}} V(x,y) \sin\left(\frac{j\pi}{a}x\right) \sin\left(\frac{k\pi}{b}y\right). \end{aligned}$$

The solution then is given by 4.10 where A_{nm} is defined as

$$A_{nm} = \frac{2}{\sqrt{ab}\sinh\left(\sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}\, c \right)} \int_0^a dx \int_0^b dy V(x,y) \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right).$$

4.1.2 The Cylindrical Coordinate System

We know the Laplacian in cylindrical coordinates from equation 1.6

$$\nabla^2 u = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\rho}\frac{\partial u}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial u}{\partial z}\right)\right] \tag{4.11}$$

We show the general method for arbitrary boundary conditions on the cylinder and then a couple of examples with simple boundary conditions.

The General Case

We want to impose some boundary conditions on the cylinder as three conditions for its side, top and bottom. Figure 4.3 illustrates the cylinder, its dimensions and its location. The boundary conditions are specified as follow:

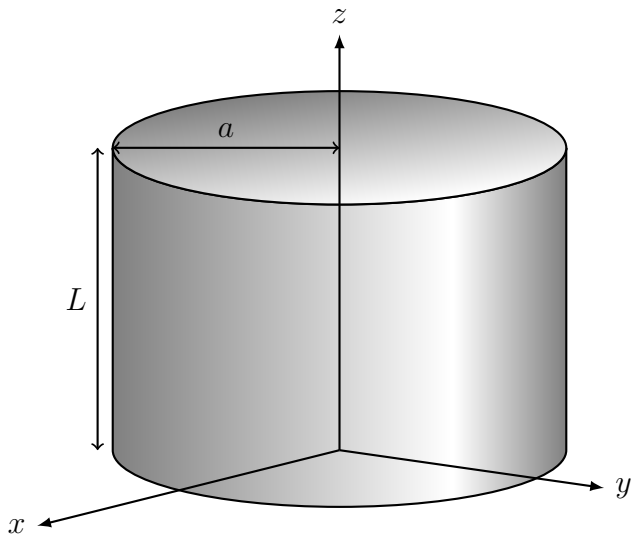

Figure 4.3: A cylinder with the potential defined on its surface.

$$\begin{aligned} u(\rho,\theta,0) &= V_0(\rho,\theta) \\ u(\rho,\theta,L) &= V_L(\rho,\theta) \\ u(a,\theta,z) &= V_a(\theta,z) \\ \lim_{\rho\to 0} u(\rho,\theta,L) &< \infty. \end{aligned}$$

We write equation 4.11 as

$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial^2 z} \tag{4.12}$$

We consider $u(r,\theta,z)$ as

$$u(r,\theta,z) = R(\rho)T(\theta)Z(z),$$

and plug this into equation 4.12 to find

$$\frac{TZ}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{RZ}{\rho^2}\frac{d^2T}{d\theta^2} + RT\frac{d^2Z}{dz^2} = 0.$$

We divide both sides of this equation by $R(r)T(\theta)Z(z)$ (assuming we do not have a zero solution) and move the z term to the right, to find

$$\frac{1}{R\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2 T}\frac{d^2T}{d\theta^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2}.$$

The left hand side depends on ρ,θ, while the right hand side depends only on z. The only way that this can happen is that both sides are constant. Let us call that constant $-\lambda$, and write

$$\begin{aligned} \frac{1}{R\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2 T}\frac{d^2T}{d\theta^2} &= -\lambda \\ -\frac{1}{Z}\frac{d^2Z}{dz^2} &= -\lambda. \end{aligned} \tag{4.13}$$

Let us start with the last equation.

$$\frac{d^2Z}{dz^2} = \lambda Z. \tag{4.14}$$

This is an eigenvalue/eigenfunction equation for a second derivative operator. Provided some boundary conditions the operator is self-adjoint and its eigenfunctions orthogonal. This problem corresponds to one of the Sturm-Liouville operators with $p = 1$, $q = 0$, and $w = 1$. Equation 4.14 can be written as

$$\frac{d^2Z}{dz^2} - \lambda Z = 0. \tag{4.15}$$

The solutions to this equation are of the form $\mathrm{e}^{\pm\sqrt{\lambda}z}$, and a general solution can be written as

$$Z_\lambda(z) = A\mathrm{e}^{\sqrt{\lambda}z} + B\mathrm{e}^{-\sqrt{\lambda}z} \tag{4.16}$$

Since a second order equation needs two boundary conditions we can use them to solve for A and B. At the moment we have not said if $\lambda > 0$ or $\lambda < 0$. If $\lambda > 0$ the solutions are hyperbolic functions which do not oscillate, if $\lambda < 0$ then solutions are trigonometrical (sines and cosines) function which oscillate. The type of boundary conditions will determine the sign of λ.

Now, for the other part of the equation 4.13 we multiply the equation by ρ^2 and move the term on T to the right to find

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \lambda\rho^2 = -\frac{1}{T}\frac{d^2T}{d\theta^2} \tag{4.17}$$

Since the expression on the left is function only of ρ (forget about constants) and the expression on the right is only a function of θ, we find that both expressions should be equal to a constant. We call the constant μ, and find that

$$-\frac{1}{T}\frac{d^2T}{d\theta^2} = \mu$$

or

$$-\frac{d^2T}{d\theta^2} = \mu T. \tag{4.18}$$

We see that T is an eigenfunction of a self-adjoint operator. The T eigenfunctions that satisfy this should be orthogonal (depending on the boundary conditions used). As in the previous case we find that

$$T_\mu = C\mathrm{e}^{\sqrt{\mu}\mathrm{i}\theta} + D\mathrm{e}^{-\sqrt{\mu}\mathrm{i}\theta}$$

We can say, ahead of time, that $\mu > 0$, because the function should oscillate in the angle θ. Since it is a cylinder it should be periodic on θ with period at most 2π. Now, from equation 4.17 and using μ in the right hand side we see that

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \lambda\rho^2 = \mu$$

and multiplying by R,

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \lambda R\rho^2 = \mu R \tag{4.19}$$

Let us expand the differentiation to find that

$$\rho^2\,\frac{d^2R}{d\rho^2} + \rho\,\frac{dR}{d\rho} + (\lambda\rho^2 - \mu)R = 0 \tag{4.20}$$

We have not yet specified the sign of λ. We first assume the case of $\lambda > 0$, and then in example 4.1.2 we show what happens when $\lambda < 0$.

To make the association of this function with a Bessel function it is convenient to define, first the mapping

$$\begin{aligned} \lambda &\mapsto \ell^2 \\ \mu &\mapsto \alpha^2 \end{aligned} \tag{4.21}$$

then equation 4.20 turns into

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho \frac{dR}{d\rho} + (\ell^2\rho^2 - \alpha^2)R = 0 \tag{4.22}$$

This equation is known as the **parametric Bessel** equation and we show how this equation is part of the general Sturm-Liouville operator. Let us divide by ρ to find

$$\rho \frac{d^2R}{d\rho^2} + \frac{dR}{d\rho} + \left(\ell^2\rho - \frac{\alpha^2}{\rho}\right) R = 0. \tag{4.23}$$

Now, since $(xy')' = xy'' + y'$ we can collapse the first two terms to find

$$\frac{d}{d\rho}\left[\rho \frac{dR}{d\rho}\right] - \frac{\alpha^2}{\rho}R = -\ell^2\rho$$

This is the Sturm-Liouville differential equation 3.14 with

$$p(\rho) = \rho \quad , \quad q(\rho) = -\frac{\alpha^2}{\rho} \quad , \quad w(\rho) = \rho \quad , \quad \lambda = \ell^2.$$

If we provide the right boundary conditions this operator is self-adjoint under the inner product defined by the equation

$$\langle f(\rho), g(\rho)\rangle = \int_0^a f(\rho)g(\rho)\rho d\rho,$$

where in the cylinder that we consider, we know that $\rho \in [0, a]$.

If in equation 4.23 we say that $x = \rho\ell$ we find

$$x^2 \frac{d^2R}{dx^2} + x \frac{dR}{dx} + (x^2 - \alpha^2)R = 0 \tag{4.24}$$

which is Bessel [1] differential equation. The solution of the Bessel differential equation can be found using the Frobenius method [2]. We should find two solutions $J_\alpha(x)$ and $Y_\alpha(x)$. They are known as Bessel function of first kind and second kind respectively. With this, a solution for R is given as

[1] https://en.wikipedia.org/wiki/Bessel_function
[2] https://en.wikipedia.org/wiki/Frobenius_method

$$R_{\lambda,\mu} = E\ J_{\sqrt{\mu}}(\sqrt{\lambda}\rho) + F\ Y_{\sqrt{\mu}}(\sqrt{\lambda}\rho).$$

We are ready to do superposition and say that the general solution is a superposition of the products of the solutions found here. That is,

$$\begin{aligned} u(\rho,\theta,z) &= \int_S d\lambda\, d\mu f(\lambda,\mu) \left(A\mathrm{e}^{\sqrt{\lambda}\mathrm{i}z} + B\mathrm{e}^{-\sqrt{\lambda}\mathrm{i}z}\right)\left(C\mathrm{e}^{\sqrt{\mu}\theta} + D\mathrm{e}^{-\sqrt{\mu}\theta}\right) \\ &\quad \left(E\ J_{\sqrt{\mu}}(\sqrt{\lambda}\rho) + F\ Y_{\sqrt{\mu}}(\sqrt{\lambda}\rho)\right). \end{aligned}$$

Observe that we do not yet know the sign of λ. In equation 4.21 we implicitly assumed that $\lambda > 0$. However λ could be negative and in this case we would obtain a different differential equation. Actually the second example 4.1.2 below illustrates this case.

As in the rectangular coordinates we can have the λ and μ varying over the continuum or the discrete. If the volume (domain) of the function where the boundary conditions are applied is finite then the set of λ and μ are a discrete (countable). If the dimensions of the cylindrical surface go to ∞, then the λ or μ, or both vary over the continuum. In the case of the continuum transforms need to be used to invert (for $f(\lambda,\mu)$) the integral equation. The complex exponentials are inverted with Fourier transform and the Bessel functions with Hankel transforms.

First Example

We assume the simple case of a cylinder where there is only non-zero potential specified in the upper face $z = L$. Figure 4.4 sketches this.

The boundary conditions for this case are:

$$\begin{aligned} V(\rho,\theta,0) &= 0 \\ V(\rho,\theta,L) &= V(\rho,\theta) \\ V(a,\theta,z) &= 0 \\ \lim_{\rho\to 0} u(\rho,\theta,L) &< \infty. \end{aligned}$$

Let us start with equation 4.14

$$\frac{d^2Z}{dz^2} = \lambda Z.$$

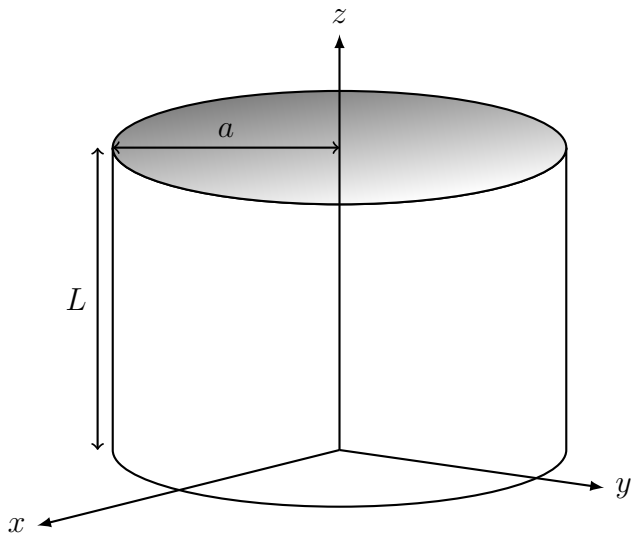

Figure 4.4: A cylinder with potential non-zero only in the upper cap $z = L$.

along the z coordinates, and its general solution.

$$Z_\lambda(z) = A\mathrm{e}^{\sqrt{\lambda}z} + B\mathrm{e}^{-\sqrt{\lambda}z} \tag{4.25}$$

From $Z_\lambda(0) = 0$ we find that $A = -B$, and so

$$Z_\lambda(z) = A\sinh\sqrt{\lambda}z, \tag{4.26}$$

where we are assuming that $\lambda > 0$. We do not expect oscillations along the z direction.

Now, along the θ axis we choose equation 4.18 which should oscillate since the function is periodic here with period 2π.

$$\frac{d^2T}{d\theta^2} + \mu T = 0$$

The general solution is of the form

$$T_\mu(\theta) = C\sin\sqrt{\mu}\theta + D\cos\sqrt{\mu}\theta.$$

Since the function is periodic with period 2π, we have that

$$\sqrt{u} = n, \quad , \quad n \geq 0.$$

and we can write

$$T_n(\theta) = C \sin n\theta + D \cos n\theta.$$

Finally for the equation along ρ. We use equation 4.19

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho \frac{dR}{d\rho} + (\lambda\rho^2 - \mu)R = 0$$

and the solution

$$R_{\lambda,\mu} = E\ J_{\sqrt{\mu}}(\sqrt{\lambda}\rho) + F\ Y_{\sqrt{\mu}}(\sqrt{\lambda}\rho).$$

Since we assume that at $\rho = 0$ we have a finite value of the potential then $F = 0$ (the function $Y_{\sqrt{\nu}}$ diverges in 0), and so

$$R_{\lambda,\mu} = E\ J_{\sqrt{\mu}}(\sqrt{\lambda}\rho) = E\ J_n(\sqrt{\lambda}\rho).$$

From the boundary condition that the potential is 0 at $\rho = a$, we have that

$$R_{\lambda,\mu} = E\ J_n(\sqrt{\lambda}a) = 0$$

and so

$$\sqrt{\lambda}a = x_{mn}$$

where x_{mn} is the m root of $J_n(x)$, $m \geq 1$. That is, we can say

$$\lambda = \frac{x_{mn}^2}{a^2},$$

and revisit equation 4.26 to write

$$Z_{mn}(z) = A \sinh\left(\frac{x_{mn}z}{a}\right) \quad , \quad R_{mn} = E\ J_n\left(\frac{x_{mn}\rho}{a}\right)$$

We now use the superposition principle to write

$$u(\rho,\theta,z) = \sum_{n=0,m=1}^{\infty} \sinh\left(\frac{x_{mn}z}{a}\right) J_n\left(\frac{x_{mn}\rho}{a}\right)(A_{mn}\sin n\theta + B_{mn}\cos n\theta).$$

Let us normalize the orthogonal functions $\sin n\theta$ and $\cos n\theta$.

$$\langle \sin n\theta, \sin n\theta \rangle = \int_0^{2\pi} \sin^2 n\theta d\theta = \frac{1}{2}\int_0^{2\pi}(1-\cos 2\theta)d\theta = \pi.$$

Then $\|\sin n\theta\| = \sqrt{\pi}$, and the same norm has $\cos n\theta$. We then write instead of the equation above.

$$u(\rho,\theta,z) = \sum_{n=0,m=1}^{\infty} \sinh\left(\frac{x_{mn}z}{a}\right) J_n\left(\frac{x_{mn}\rho}{a}\right)\left(A_{mn}\frac{\sin n\theta}{\sqrt{\pi}} + B_{mn}\frac{\cos n\theta}{\sqrt{\pi}}\right). \tag{4.27}$$

Let us now apply the boundary condition on top. That is at $z = L$, we have that

$$u(\rho,\theta,L) = V(\theta,\rho) = \sum_{n=0,m=1}^{\infty} \sinh\left(\frac{x_{mn}L}{a}\right) J_n\left(\frac{x_{mn}\rho}{a}\right)\left(A_{mn}\frac{\sin n\theta}{\sqrt{\pi}} + B_{mn}\frac{\cos n\theta}{\sqrt{\pi}}\right). \tag{4.28}$$

To find A_{mn} and B_{mn} we take inner product with $\sin i\theta$ and $\cos i\theta$ respectively. That is,

$$\begin{aligned}\left\langle V(\theta,\rho), \frac{\sin i\theta}{\sqrt{\pi}} \right\rangle &= \sum_{n=0,m=1}^{\infty} \sinh\left(\frac{x_{mn}L}{a}\right) J_i\left(\frac{x_{mn}\rho}{a}\right) A_{mn}\delta_{ni} \\ &= \sum_{m=1}^{\infty} \sinh\left(\frac{x_{mi}L}{a}\right) J_i\left(\frac{x_{mi}\rho}{a}\right) A_{mi}\end{aligned}$$

or

$$\int_0^{2\pi} V(\theta,\rho)\frac{\sin i\theta}{\sqrt{\pi}}d\theta = \sum_{m=1}^{\infty}\sinh\left(\frac{x_{mn}L}{a}\right)J_i\left(\frac{x_{mi}\rho}{a}\right)A_{mi} \tag{4.29}$$

Here we use the orthogonality property of the Bessel functions. It can be shown that

$$\left\langle J_i\left(\frac{x_{mj}\rho}{a}\right), J_i\left(\frac{x_{mk}\rho}{a}\right)\right\rangle = \int_0^a \rho J_i\left(\frac{x_{mj}\rho}{a}\right)J_i\left(\frac{x_{mk}\rho}{a}\right)d\rho = \frac{a^2}{2}\left[J_{i+1}\left(\frac{x_{mk}\rho}{a}\right)\right]^2\delta_{jk},$$

We then take the inner product of 4.29 with the orthogonal Bessel function $J_i(x_{ki}\rho/a)$. That is,

$$\begin{aligned}\int_0^a \rho\int_0^{2\pi} V(\theta,\rho)\frac{\sin i\theta}{\sqrt{\pi}}J_i\left(\frac{x_{ki}\rho}{a}\right)d\theta d\rho &= \sum_{m=1}^{\infty}\frac{a^2}{2}\left[J_{i+1}\left(\frac{x_{ki}\rho}{a}\right)\right]^2\sinh\left(\frac{x_{mn}L}{a}\right)\delta_{km}A_{mi}\\ &= \frac{a^2}{2}\left[J_{i+1}\left(\frac{x_{ki}\rho}{a}\right)\right]^2\sinh\left(\frac{x_{kn}L}{a}\right)A_{ki}.\end{aligned}$$

Then

$$A_{ki} = \frac{2}{\sqrt{\pi}a^2\left[J_{i+1}\left(\frac{x_{ki}\rho}{a}\right)\right]^2\sinh\left(\frac{x_{kn}L}{a}\right)}\int_0^a \rho\int_0^{2\pi} V(\theta,\rho)\sin i\theta J_i\left(\frac{x_{ki}\rho}{a}\right)d\theta d\rho.$$

Now for the B coefficients we take the inner product in equation 4.28 with $\cos(i\theta)/\sqrt{\pi}$ to find

$$\begin{aligned}\int_0^{2\pi} V(\theta,\rho)\frac{\cos i\theta}{\sqrt{\pi}}d\theta &= \sum_{n=0,m=1}^{\infty}\sinh\left(\frac{x_{mn}L}{a}\right)J_i\left(\frac{x_{mn}\rho}{a}\right)B_{mn}\delta_{in}\\ &= \sum_{m=1}^{\infty}\sinh\left(\frac{x_{mi}L}{a}\right)J_i\left(\frac{x_{mi}\rho}{a}\right)B_{mi}.\end{aligned}$$

Now we take the inner product with $J_i(x_{ki}\rho)/a$ to find

$$\begin{aligned}\int_0^a \rho\int_0^{2\pi} V(\theta,\rho)\frac{\cos i\theta}{\sqrt{\pi}}J_i\left(\frac{x_{ki}\rho}{a}\right)d\theta d\rho &= \sum_{m=1}^{\infty}\frac{a^2}{2}\left[J_{i+1}\left(\frac{x_{ki}\rho}{a}\right)\right]^2\sinh\left(\frac{x_{mn}L}{a}\right)\delta_{km}B_{mi}\\ &= \frac{a^2}{2}\left[J_{i+1}\left(\frac{x_{ki}\rho}{a}\right)\right]^2\sinh\left(\frac{x_{ki}L}{a}\right)B_{ki}\end{aligned}$$

and so

$$B_{ki} = \frac{2}{\sqrt{\pi}a^2\left[J_{i+1}\left(\frac{x_{ki}\rho}{a}\right)\right]^2 \sinh\left(\frac{x_{ki}L}{a}\right)} \int_0^a \rho \int_0^{2\pi} V(\theta,\rho)\cos i\theta J_i\left(\frac{x_{ki}\rho}{a}\right) d\theta d\rho.$$

We re-index A_{ki} and B_{ki} in terms of A_{nm} and B_{mn} and equation 4.27 is solved for the potential $u(\rho,\theta,z)$.

Second Example

We now show another example where the modified Bessel functions are involved. The hollow cylinder in Figure 4.5 with radius a and height L, has potential $V(\theta,z)$ at its surface $\rho = a$, zero potential on top and bottom. The cylinder has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$.

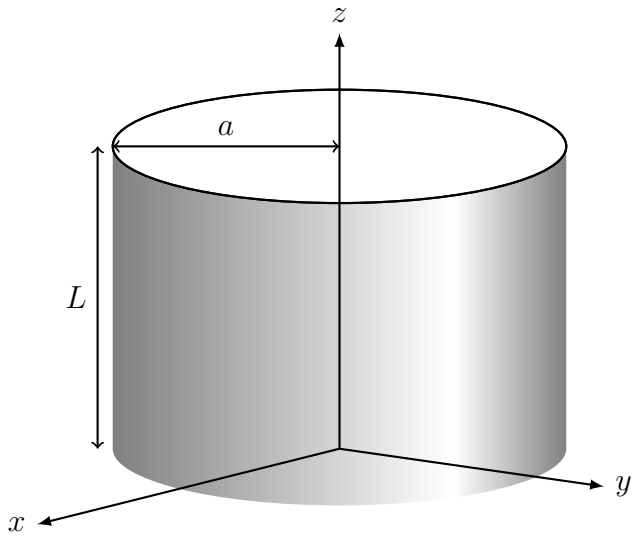

Figure 4.5: A cylinder with charges only along its side $\rho = a$.

The boundary conditions in formal language are:

$$\begin{aligned} V(\rho,\theta,0) &= 0 \\ V(\rho,\theta,L) &= 0 \\ V(a,\theta,z) &= V(\theta,z) \\ \lim_{\rho\to 0} u(\rho,\theta,L) &< \infty. \end{aligned}$$

We start with the ordinary differential equation 4.15 along Z,

$$\frac{d^2Z}{dz^2} - \lambda Z = 0 \qquad Z(0) = 0 \quad , \quad Z(L) = 0.$$

The general solution of this equation is given by equation 4.25. This is

$$Z_\lambda(z) = A\mathrm{e}^{\sqrt{\lambda}z} + B\mathrm{e}^{-\sqrt{\lambda}z}.$$

We use the boundary conditions to find A and B. That is, from $Z_\lambda(0) = 0$, we see that $A = -B$, and then

$$Z_\lambda(z) = A(\mathrm{e}^{\sqrt{\lambda}z} - \mathrm{e}^{-\sqrt{\lambda}z}).$$

Since the operator L is a double derivative and we have 0 boundary conditions the functions are orthogonal under the definition of inner product and the eigenvalue λ is real (actually we always assume that the eigenvalues are real for these problems).

$$\langle f, g\rangle = \int_0^L f(x)g(x)dx.$$

We have not indicated if λ is positive or negative. If $\lambda > 0$, then the two exponentials above are real (no complex). We use the second condition $Z_\lambda(L) = 0$. That is

$$Z_\lambda(z) = A(\mathrm{e}^{-\sqrt{\lambda}L} - \mathrm{e}^{\sqrt{\lambda}L}) = 0.$$

Making $A = 0$ is not an option because it produces a zero solution that we want avoid. The only way that these two exponential functions cancel is by having $\lambda = 0$. However we do not want 0 eigenvalues. Then necessarily $\lambda < 0$. With this

$$Z_\lambda(z) = 2A\sin\sqrt{|\lambda|}z$$

and from $z_\lambda(L) = 0$ we find

$$Z_\lambda(L) = 2A\sin\sqrt{|\lambda|}L = 0.$$

That is,

$$\sqrt{|\lambda|}L = \pi n,$$

where $n = 1, 2, \cdots$. From here the eigenvalues have the form

$$\lambda = -\frac{\pi^2 n^2}{L^2}, \tag{4.30}$$

and

$$Z_n(z) = 2A \sin \frac{\pi n z}{L} \quad , \quad n = 1, 2, \cdots$$

where the case of $n = 0$ will not provide any contribution. At this point we can drop the value of $2A$ and instead find a coefficient that normalizes the function. That is, we find first

$$\left\langle \sin \frac{\pi n z}{L}, \sin \frac{\pi n z}{L} \right\rangle = \int_0^L \sin^2 \frac{\pi n z}{L} dz = \frac{1}{2} \int_0^L \left(1 - \cos \frac{2\pi n z}{L} \right) dz = \frac{L}{2}.$$

Then we write the normalized eigenfunctions

$$Z_n(z) = \sqrt{\frac{2}{L}} \sin \frac{\pi n z}{L}.$$

Let us now work in the θ coordinate. We write equation 4.18

$$\frac{d^2 T}{d\theta^2} + \mu T = 0.$$

Two linearly independent solutions are $\mathrm{e}^{\pm \mathrm{i}\sqrt{\mu}}$, and their combination provides

$$T_\mu(\theta) = C\mathrm{e}^{\mathrm{i}\sqrt{\mu}\theta} + D\mathrm{e}^{-\mathrm{i}\sqrt{\mu}\theta}$$

Since T_μ is periodic (in the cylinder $T_\mu(0) = T_\mu(2\pi)$ the exponentials are oscillating. That is $\mu > 0$ (otherwise, if $\mu < 0$ they would be hyperbolic functions which do not oscillate). Each exponential has the property that

$$e^{i\sqrt{\mu}\theta_0} = e^{i\sqrt{\mu}(\theta_0+2\pi)}$$

and so

$$e^{2\pi i\sqrt{\mu}} = 1$$

so

$$2\pi\sqrt{u} = 2m\pi,$$

and $m = \sqrt{u}$ so we have a set of eigenvalues $\mu = m^2$, and a set of solutions

$$T_n(\theta) = Ce^{im\theta} + De^{-im\theta} \tag{4.31}$$

Since we are assuming complex solutions we can as well assume complex coefficients. Equation 4.31 can be expanded as

$$\begin{aligned} T_n(\theta) &= C_r\cos m\theta - C_i\sin m\theta + D_r\cos m\theta - D_i\sin m\theta + \\ &\quad i[C_r\sin m\theta + C_i\cos m\theta - D_r\sin m\theta + D_i\cos m\theta] \end{aligned}$$

Since we are looking for real solutions we only consider the first part of the equation above. That is,

$$T_n(\theta) = (C_r + D_r)\cos m\theta - (C_i - D_i)\sin m\theta,$$

and recycling symbols we can just say that

$$T_m(\theta) = C\sin m\theta + D\cos m\theta \quad , \quad m = 0, 1, \cdots .$$

Note that we could have said from the beginning that two linearly independent solutions of the equation $T'' + \mu T = 0$ are $\sin\sqrt{\mu}\theta$, $\cos\sqrt{\mu}\theta$, which are real functions, but we wanted to do all the work in terms of exponentials to show another approach.

Let us normalize these solutions. Define an inner product

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)dx.$$

Then compute

$$\langle \sin m\theta, \sin m\theta\rangle = \int_0^{2\pi} \sin^2 m\theta d\theta = \frac{1}{2}\int_0^{2\pi} (1-\cos 2\theta)d\theta = \int_0^{2\pi} d\theta = \pi$$

then $\|\sin m\theta\| = \sqrt{\pi}$. The same occurs with $\cos m\theta$. Then we write

$$T_n(\theta) = C\frac{\sin m\theta}{\sqrt{\pi}} + D\frac{\cos m\theta}{\sqrt{\pi}}.$$

Finally we look at the radial component. In equation 4.20 we replace λ by $-\pi^2 n^2/L^2$ (from equation 4.30), and $\mu = m^2$ to obtain

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho \frac{dR}{d\rho} - \left(\frac{\pi^2 n^2 \rho^2}{L^2} + m^2\right) R = 0 \tag{4.32}$$

This is the differential equation for the modified Bessel function. Let us reduce more by defining

$$x = \frac{\pi n \rho}{L}$$
$$dx = \frac{\pi n d\rho}{L}$$

From the chain rule

$$\rho\frac{dR}{d\rho} = \rho\frac{dR}{dx}\frac{dx}{d\rho} = \rho\frac{dR}{dx}\frac{\pi n}{L} = \frac{dR}{dx}\frac{\pi n \rho}{L} = x\frac{dR}{dx}$$

We can use symbols above and say that

$$\frac{\rho}{d\rho} = \frac{x}{dx}$$

and so from here

$$\frac{\rho^2}{d\rho^2} = \frac{x^2}{dx^2}.$$

This is not correct mathematical syntax but good mnemonic. We find then that equation 4.32 reduces to

$$x^2 \frac{d^2R}{dx^2} + x \frac{dR}{dx} - (x^2 + m^2)R = 0$$

which is the differential equation for the modified Bessel function [3]. As shown in the Wikipedia page the two solutions of this differential equation are the modified Bessel functions $I_m(x)$ and $K_m(x)$ and in our case, our general solution

$$R_{m,n} = EI_m\left(\frac{\pi n\rho}{L}\right) + FK_m\left(\frac{\pi n\rho}{L}\right).$$

From the condition as $\rho \to 0$ we see that the solution should be finite at the origin. Since the modified Bessel function K_m diverges at $x = 0$, we require $F = 0$, and so

$$R_{m,n} = EI_m\left(\frac{\pi n\rho}{L}\right)$$

A solution for n and m of the Laplacian equation is given by

$$u_{nm}(\rho, \theta, z) = \left[EI_m\left(\frac{\pi n\rho}{L}\right)\right]\left[C\frac{\sin m\theta}{\sqrt{\pi}} + D\frac{\cos m\theta}{\sqrt{\pi}}\right]\left[\frac{2A}{L}\sin\frac{\pi nz}{L}\right].$$

We are ready to do the superposition. The constants A and E can be absorbed into the coefficients C and D given them another dimension. That is, instead of C_n we now will write C_{nm}, and the same for D. We can write the solution as the superposition of the solutions above. That is, we can write:

$$u(\rho, \theta, z) = \sum_{m=0,n=1}^{\infty}\left[I_m\left(\frac{\pi n\rho}{L}\right)\right]\left[C_{mn}\frac{\sin m\theta}{\sqrt{\pi}} + D_{mn}\frac{\cos m\theta}{\sqrt{\pi}}\right]\left[\sqrt{\frac{2}{L}}\sin\frac{\pi nz}{L}\right]. \quad (4.33)$$

[3] https://en.wikipedia.org/wiki/Bessel_function

where C_{mn} and D_{mn} are coefficients that we need to find. For this we need to use the multi-dimensional boundary conditions. At $\rho = a$ we find

$$u(a,\theta,z) = \sum_{m=0,n=1}^{\infty} \left[I_m\left(\frac{\pi n a}{L}\right)\right]\left[C_{mn}\frac{\sin m\theta}{\sqrt{\pi}} + D_{mn}\frac{\cos m\theta}{\sqrt{\pi}}\right]\left[\sqrt{\frac{2}{L}}\sin\frac{\pi n z}{L}\right] = V(\theta,z).$$

Since the functions $\sqrt{2/L}\sin(\pi n z/L)$ are orthonormal we take the inner product with these functions and find

$$\begin{aligned}\left\langle V(\theta,z), \sqrt{\frac{2}{L}}\sin\frac{\pi i z}{L}\right\rangle &= \sum_{m=0,n=1}^{\infty}\left[I_m\left(\frac{\pi n a}{L}\right)\right]\left[C_{mn}\frac{\sin m\theta}{\sqrt{\pi}} + D_{mn}\frac{\cos m\theta}{\sqrt{\pi}}\right]\delta_{ni} \\ &= \sum_{m=0}^{\infty}\left[I_m\left(\frac{\pi i a}{L}\right)\right]\left[C_{mi}\frac{\sin m\theta}{\sqrt{\pi}} + D_{mi}\frac{\cos m\theta}{\sqrt{\pi}}\right].\end{aligned}$$

That is

$$\sqrt{\frac{2}{L}}\int_0^L V(\theta,z)\,\sin\frac{\pi i z}{L}dz = \sum_{m=0}^{\infty}\left[I_m\left(\frac{\pi i a}{L}\right)\right]\left[C_{mi}\frac{\sin m\theta}{\sqrt{\pi}} + D_{mi}\frac{\cos m\theta}{\sqrt{\pi}}\right]. \quad (4.34)$$

We now take inner product with $\sin(j\theta)/\sqrt{\pi}$ and from the orthogonality conditions find

$$\sqrt{\frac{2}{\pi L}}\int_0^{2\pi} d\theta \int_0^L V(\theta,z)\,\sin\left(\frac{\pi i z}{L}\right)\sin j\theta\, dz = \sum_{m=0}^{\infty} I_m\left(\frac{\pi i a}{L}\right) C_{mi}\delta_{mj} = I_j\left(\frac{\pi i a}{L}\right) C_{ij},$$

and from here

$$C_{ij} = \sqrt{\frac{2}{\pi\, L}}\frac{1}{I_j(\pi i a/L)}\int_0^{2\pi} d\theta \int_0^L V(\theta,z)\,\sin\left(\frac{\pi i z}{L}\right)\sin j\theta dz$$

Finally in equation 4.34 we take the orthogonal product with $\cos(j\theta)/\sqrt{\pi}$ to find

$$\sqrt{\frac{2}{\pi\, L}}\int_0^{2\pi} d\theta \int_0^L V(\theta,z)\,\sin\left(\frac{\pi i z}{L}\right)\cos j\theta dz = \sum_{m=0}^{\infty}\left[I_m\left(\frac{\pi i a}{L}\right)\right] D_{mi}\delta_{mj}$$
$$= I_j\left(\frac{\pi i a}{L}\right) D_{ji}$$

and from here

$$D_{ji} = \sqrt{\frac{2}{\pi\, L}}\frac{1}{I_j(\pi i a/L)}\int_0^{2\pi} d\theta \int_0^L V(\theta,z)\,\sin\left(\frac{\pi i z}{L}\right)\cos j\theta dz$$

The final solution is given by 4.33 where C_{mn} and D_{mn} above are inserted. Let us, for example assume that $u(a,\theta,z) = V(\theta,z) = V_0$, so

$$\begin{aligned}
C_{ij} &= \sqrt{\frac{2}{\pi\, L}}\frac{1}{I_j(\pi i a/L)}\int_0^{2\pi} d\theta \int_0^L V_0 \sin\left(\frac{\pi i z}{L}\right)\sin j\ \theta dz \\
&= \sqrt{\frac{2}{\pi L}}\frac{V_0}{I_j(\pi i a/L)}\int_0^{2\pi} \sin j\theta\ d\theta \int_0^L \sin\left(\frac{\pi i z}{L}\right) dz \\
&= \sqrt{\frac{2}{\pi L}}\frac{V_0}{I_j(\pi i a/L)}\frac{\cos j\theta}{j}\Bigg|_0^{2\pi}\frac{L}{\pi i}\cos\left(\frac{\pi i z}{L}\right)\Bigg|_0^L \\
&= \frac{\sqrt{2L}V_0}{ij\pi^{3/2}}(1-1)[(-1)^i - 1] \\
&= 0.
\end{aligned}$$

Note that if $j = 0$ from the first line of the chain of equations above, then $C_{ij} = 0$, so we are dividing by j with a good reason. so $C_{ij} = 0$. Let us find now D_{ji},

$$\begin{aligned}
D_{ji} &= \sqrt{\frac{2}{\pi\, L}}\frac{1}{I_j(\pi i a/L)}\int_0^{2\pi} d\theta \int_0^L V_0\ \sin\left(\frac{\pi i z}{L}\right)\cos j\theta\ dz \\
&= \sqrt{\frac{2}{\pi L}}\frac{V_0}{I_j(\pi i a/L)}\int_0^{2\pi} \cos j\theta d\theta \int_0^L \sin\left(\frac{\pi i z}{L}\right) dz
\end{aligned}$$

We see that the only non zero value is produced by $j = 0$. That is

$$D_{0i} = -\sqrt{\frac{2}{\pi L}}\frac{2\pi V_0}{I_0(\pi i a/L)}\frac{L}{\pi i}\cos\left(\frac{\pi i z}{L}\right)\Bigg|_0^L = -\frac{2\sqrt{2L}V_0}{i\sqrt{\pi}I_0(\pi i a/L)}[(-1)^i - 1]$$

So only for i odd $D_{0i} \neq 0$, and we have from the final solution 4.33

$$u(\rho,\theta,z) = \sum_{n=1}^{\infty} I_m(0)\left[D_{0n}\frac{1}{\sqrt{\pi}}\right]\sqrt{\frac{2}{L}}\sin\left(\frac{\pi n z}{L}\right) = \frac{8V_0}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{1}{m}\frac{I_0(\pi m\rho/L)\sin(\pi n z/L)}{I_0(\pi i\rho/L)}$$

As a summary it is good to indicate when the Bessel functions are used and when the modified Bessel functions are used instead, we see that if the function along the z axis behave like a real exponential (do not oscillate) then we get to a Bessel function. That is

$$\nabla(R(\rho)\mathrm{e}^{\sqrt{\lambda}z}\mathrm{e}^{\mathrm{i}\sqrt{\mu}} = \rho^2 R'' + \rho R' + (\lambda\rho^2 - \mu^2)R = 0,$$

where, since $\lambda > 0$ is a Bessel function. Now, if on the other hand the function along the z axis oscillate we will find a modified Bessel function.

That is

$$\nabla(R(\rho)\mathrm{e}^{\mathrm{i}\sqrt{\lambda}z}\mathrm{e}^{\mathrm{i}\sqrt{\mu}} = \rho^2 R'' + \rho R' - (\lambda\rho^2 + \mu^2)R = 0,$$

Note that in any case the function oscillates along the θ direction due to the periodicity of the cylinder by 2π.

4.2 Problems

4.2.1 Problem 1

Here is how Jackson [2] states the separation of variables in rectangular coordinates. Set the potential $\Phi(x, y, z) = X(x)Y(y)Z(z)$. Plug this into the Laplace's equation and find three system of ordinary differential equations:

$$\begin{aligned}\frac{1}{X}\frac{\partial^2 X}{\partial x} &= -\alpha^2 \\ \frac{1}{Y}\frac{\partial^2 Y}{\partial y} &= -\beta^2 \\ \frac{1}{Z}\frac{\partial^2 Z}{\partial z} &= \gamma^2\end{aligned}$$

With $\alpha^2 > 0$, $\beta^2 > 0$, and $\alpha^2 + \beta^2 = \gamma^2$. Then a solution depending on the parameters α, γ, β is given by

$$\Phi = \mathrm{e}^{\pm \mathrm{i}\alpha^2 x}\mathrm{e}^{\pm \mathrm{i}\beta^2 y}\mathrm{e}^{\pm \mathrm{i}\sqrt{\alpha^2+\beta^2}z}.$$

Then he goes into choosing a box with lengths with sides $x = 0, a$, $y = 0, b$, and $z = 0, c$ (see Figure 4.2). Set the boundary conditions as follows: $\Phi(x = 0, y, z) = \Phi(x = a, y, z) = \Phi(x, y = 0, z) = \Phi(x, y = b, z) = \Phi(x, y, z = 0) = 0$, and $\Phi(x, y, z = c) = V(x, y)$.

With that we can say that, since the potential is zero in the line for $y = z = 0$,then $\Phi(x = 0, y, z) = 0$, $0 \leq x \leq a$, and we have the functions for this particular case reduced to a linear combination of the two solutions for $X(x)$, that is

$$\Phi(x, 0, 0) = A\mathrm{e}^{\mathrm{i}\alpha^2 x} + B\mathrm{e}^{-\mathrm{i}\alpha^2 x}$$

and since at $x = 0$ we have that $\Phi(0, 0, 0) = 0$, then $B = -A$ and

$$\Phi(x, 0, 0) = 2\,\mathrm{i}\sin\alpha^2 x. \tag{4.35}$$

Since the boundary conditions indicate that $\Phi(x, 0, 0) = 0$, for all $x \in [0, a]$, then the only α that satisfies equation 4.35 is $\alpha = 0$ and we have that $X(x) = 0$, so $\Phi(x, y, z) = 0$ is the only solution to the Poisson's equation by using Jackson's method of separation of variables.

Question: What is wrong this this approach?

4.2.2 Problem 2

Find the eigenvalues and the eigenfunctions of the operator L defined as

$$Lu = \frac{d^2u}{dx^2}$$

subject to the boundary conditions $u(-1) = u(1) = 0$. Verify that the eigenfunctions are orthogonal since the operator L in the interval $[-1, 1]$ is self-adjoint

4.2.3 Problem 3

In class we studied how to solve the Laplace's equation in three dimentions with Direchlet boundary conditions. Here we will solve the problem in two dimensiones with mixed boundary conditions. Think of a coaxial cable or a cylindier with infinite height where there is no variation along the z axis. We use Dirichlet boundary conditions for the internal surface and Neumann for the external surface. Figure 4.6 illustrates the problem. The interior radius is 1 y the exterior is 2. We want to find $u(r, \theta)$, $1 < r < 2$, $0 \leq \theta \leq 2\pi$,

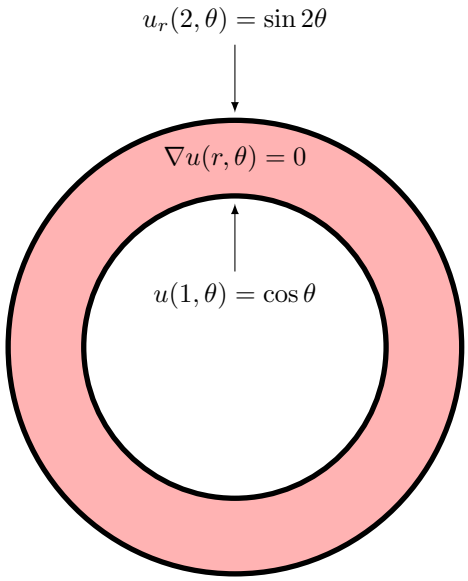

Figure 4.6: Illustration for problem 1. Two concentric discs with boundary mixed conditions in the edges.

such that

$$\begin{cases} \nabla^2 u(r,\theta) = 0, & 1 < r < 2 \quad , 0 \leq \theta \leq 2\pi \\ u(1,\theta) = \cos\theta & 0 \leq \theta \leq 2\pi \\ u_r(2,\theta) = \sin 2\theta & 0 \leq \theta \leq 2\pi \end{cases}$$

where $u_r = \partial u / \partial r$.

4.2.4 Problem 4

Use Gauss's law to find the electric field inside and outside of hollow cylinder with uniform charge in the surface. Check that the boundary conditions (tangential and normal) shown in class are valid here.

Chapter 5

Lecture #5: Continuation on Separation of Variables

5.1 The Spherical Coordinate System

We use equation 1.7

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}.$$

As usual we express the solution $u(r,\theta,\phi)$ as the product of three functions each on only one of the coordinates. That is,

$$u(r,\theta,\phi) = R(r)T(\theta)\Phi(\phi),$$

and insert this in the $\nabla^2 u = 0$ above, to find

$$\begin{aligned}\nabla^2 u(r,\theta,\phi) &= \frac{1}{r^2}\frac{d^2}{dr^2}\left(r^2\frac{d^2R(r)}{dr^2}\right)T(\theta)\Phi(\phi) + R(r)\,\frac{1}{r^2\sin\theta}\frac{d^2}{d\theta^2}\left(\sin\theta\frac{d^2T(\theta)}{d\theta^2}\right)\Phi(\phi)\\ &+ R(r)\Theta(\theta)\,\frac{1}{r^2\sin^2\theta}\frac{d^2u}{d\phi^2} = 0.\end{aligned}$$

We now divide the previous expression for the triple product $R(r)T(\theta)\Phi(\phi)$ to find

$$\begin{aligned}\nabla^2 u(r,\theta,\phi) &= \frac{1}{R(r)}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+\frac{1}{T(\theta)}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT(\theta)}{d\theta}\right)\\ &+ \frac{1}{\Phi(\phi)}\frac{1}{r^2\sin^2\theta}\frac{d^2\Phi(\phi)}{d\phi^2}=0.\end{aligned}$$

Let us multiply the expression above by $r^2\sin^2\theta$ and move the last term to the right,

$$\frac{\sin^2\theta}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+\frac{\sin\theta}{T(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dT(\theta)}{d\theta}\right)=-\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}. \tag{5.1}$$

The expression on the left is all function of r and θ, while that on the right is only function of ϕ. Then the expression on the right needs to be a constant. What we did is to isolate the expressions as in the previous coordinate systems. We have then that from the last expression equal to a constant μ,

$$-\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}=\mu \tag{5.2}$$

or

$$\frac{d^2\Phi(\phi)}{d\phi^2}+\mu\Phi(\phi)=0.$$

This equation has two general, linear independent solutions,

$$\Phi(\phi)=\mathrm{e}^{\pm\mathrm{i}\sqrt{\mu}\phi}.$$

Since Φ is periodic ($\Phi(\phi)=\Phi(\phi+2\pi)$) (due to the spherical symetry) we know that the function oscillates and so the solution for this differential equation is of the form

$$\Phi_\mu(\phi)=A\mathrm{e}^{\mathrm{i}\sqrt{\mu}\phi}+B\mathrm{e}^{-\mathrm{i}\sqrt{\mu}\phi}.$$

with $\mu>0$. As we did before we show that $\sqrt{\mu}=m$ for some integer number m. That is, if the solution oscillates with period 2π, then it has components of the form

$$\mathrm{e}^{\mathrm{i}\sqrt{\mu}\phi}=\mathrm{e}^{\mathrm{i}\sqrt{\mu}(\phi+2\pi)}=\mathrm{e}^{\mathrm{i}\sqrt{\mu}\phi}\mathrm{e}^{\mathrm{i}\sqrt{\mu}2\pi},$$

from which

$$e^{i\sqrt{\mu}(2\pi)} = 1,$$

and from here $\sqrt{\mu}(2\pi) = 2m\pi$, and $\sqrt{\mu} = m$, or $\mu = m^2$. We then have azimuthal eigenfunctions

$$\phi_m(\phi) = A\cos m\phi + B\sin m\phi.$$

We now divide equation 5.1 by $\sin^2\theta$, and replace the right hand side term with μ to find

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + \frac{1}{\sin\theta T(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dT(\theta)}{d\theta}\right) = \frac{\mu}{\sin^2\theta}$$

We now move the second term to the right and separate the expression on dependence on r on the left and θ on the right. That is

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) = -\frac{1}{\sin\theta\, T(\theta)}\frac{d}{d\theta}\left(\sin\theta\,\frac{dT(\theta)}{d\theta}\right) + \frac{\mu}{\sin^2\theta}$$

Each term of this equation should be a constant. Let us assume the constant to be λ. We then have two ordinary differential equations

$$\begin{aligned}\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) &= \lambda \\ \frac{1}{\sin\theta\, T(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dT(\theta)}{d\theta}\right) - \frac{\mu}{\sin^2\theta} &= -\lambda\end{aligned} \tag{5.3}$$

Let us multiply by $R(r)$ in the first equation to find

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) - \lambda R(r) = 0$$

or expanding the derivatives

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-\lambda R=0. \tag{5.4}$$

This an EulerB.1 equation. Appendix B shows how to solve this equation. The characteristic polynomial for this equation is

$$n^2+n-\lambda=0.$$

The solutions of this equation are

$$n=\frac{-1\pm\sqrt{1+4\lambda}}{2}$$

If $n_1\neq n_2$ and $n_1=\mu_1+\mathrm{i}\nu_1$, $n_2=\mu_2+\mathrm{i}\nu_2$, then the two roots produce two solutions $r^{\mu_1}\cos(\nu_1\ln(r))$ and $r^{\mu_2}\sin(\nu_2\ln(r))$, if the roots are equal (and real) we have that the solutions are r^{n_1} and $(\ln r)r^{n_2}$. We do not consider this case for physical reasons. In this case both solutions diverge as $r\to\infty$ and we want to study physical solutions in finited bound potentials.

The sum of the roots is $n_1+n_2=-1$, and the product $n_1n_2=-\lambda$. It is common in the literature to assume that the roots are real and different and so, if one root is n the other root is $-1-n$, and write the general solution as

$$R_\lambda(r)=Ar^n+B\frac{1}{r^{n+1}}$$

Also, since $-n(n+1)=-\lambda$ it is common to use $n(n+1)$ instead of λ, in equations 5.3. We now rewrite these equations with this new finding.

$$\begin{aligned}\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)&=n(n+1)\\ \frac{1}{\sin\theta T(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dT(\theta)}{d\theta}\right)-\frac{\mu}{\sin^2\theta}&=-n(n+1).\end{aligned}$$

We write the final equation to solve

$$\frac{1}{\sin\theta T(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dT(\theta)}{d\theta}\right)-\frac{\mu}{\sin^2\theta}=-n(n+1).$$

We multiply by $T(\theta)$ to find

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT(\theta)}{d\theta}\right)+\left[n(n+1)-\frac{\mu}{\sin^2\theta}\right]T(\theta)=0 \tag{5.5}$$

This equation is transformed into a more familiar equation with the substitution $x=\cos\theta$. In terms of this transformation we have

$$\frac{d}{d\theta}=\frac{d}{dx}\frac{dx}{d\theta}=-\sin\theta\frac{d}{dx}.$$

so in terms of x equation 5.5 turns into

$$\frac{d}{dx}\left[(1-x^2)\frac{dT}{dx}\right]+\left[n(n+1)-\frac{\mu}{1-x^2}\right]T=0. \tag{5.6}$$

This equation is the generalized Legendre equation, and its solutions are the **associated Legendre functions,** knowing in this context as **spherical harmonics** . We first assume $\mu=0$ and reduce the equation to

5.1.1 Azimuthal Independent Potentials

$$\frac{d}{dx}\left[(1-x^2)\frac{dT}{dx}\right]+n(n+1)T=0. \tag{5.7}$$

This equation is the Legendre differential [1] equation. It can be solved by the Frobenius method (power series solution). Appendix C shows that the finite solutions of this equation are in the form of polynomials. Here are the first 4 polynomials.

$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= \frac{1}{2}(3x^2-1)\\
P_3(x) &= \frac{1}{2}(5x^3-3x).
\end{aligned}$$

[1] http://mathworld.wolfram.com/LegendreDifferentialEquation.html

These polynomials are not normalized against its inner product. Since $x \in [-1,1]$ ($\cos\theta \in [-1,1]$) we have the norm defined as

$$\|T_i(x)\| = \sqrt{\langle T_i(x), T_i(x)\rangle} = \sqrt{\int_{-1}^{1} T_i^2(x)dx}.$$

Actually the Legendre polymoials as derived from its differential equation satisfy the orthogonality condition C.11

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{nm}.$$

In fact, the Legendre polymials are complete set in the space of piece-wise smooth functions in the interval $[-1,1]$. That is, any piecewise smooth function in the interval $[-1,1]$ can be expanded using Legendre polynomials. The coefficients are obtained from the equation C.14 pair of (Fourier-Legendre) equations:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_i T_i(x) \\ c_i &= \left\langle f(x), \frac{T_i(x)}{(2/(2i+1)} \right\rangle = \frac{2i+1}{2}\int_{-1}^{1} f(x)T_i(x)dx. \end{aligned}$$

We are ready to show the general way to solve a problem with azimuthal symmetry ($m = 0$).

Initially, the superposition principle stays that the potential is given by

$$u(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + B_n \frac{1}{r^{n+1}}\right) P_n(\cos\theta).$$

Note that since we are knocking out one dimension the sum is only one-fold. We now assume that we have data on the surface of a sphere $r = a$. That is, we know the potential $u(a,\theta,\phi)$ in a sphere of radius $r = a$, and that the potential is independent of ϕ [2]

[2] in equation 5.2 we assume $\mu = 0 = m$.

We say that the potential is $V(\theta) = u(a, \theta, \phi = 0)$, and want to know the potential inside the sphere $r < a$. We also assume that the potential is finite at the origin. That is $\lim_{r\to 0} u(r, \theta, \phi) = 0$. Then the coefficients B_n above are zero (since at zero the expression $1/r^{n+1}$ blows up for $n \geq 0$). Then we have the problem

$$V(\theta) = \sum_{n=0}^{\infty} A_n a^n T_n(\cos\theta). \tag{5.8}$$

We normalize the Legendre polynomials and redefine A_n. That is we write

$$V(\theta) = \sum_{n=0}^{\infty} A_n a^n \left[\frac{2n+1}{2} T_n(\cos\theta)\right].$$

Take the inner product with $[(2j+1)/2]T_j(\cos\theta)$, That is

$$\left\langle V(\theta)\ ,\ \frac{2j+1}{2} T_j(\cos\theta)\right\rangle = \sum_{n=0}^{\infty} A_n a^n \delta_{jn} = A_j a^j.$$

Now

$$\left\langle V(\theta)\ ,\ \frac{2j+1}{2} T_j(\cos\theta)\right\rangle = \frac{2j+1}{2}\int_0^{\pi} V(\theta) T_j(\cos\theta)\sin\theta\, d\theta$$

where the $\sin\theta$ appears since we made a change of variable $x = \cos\theta$, $dx = -\sin\theta d\theta$. The minus is fixed with the integration limits being reversed. We found then that

$$A_j = \frac{2j+1}{2a^j}\int_0^{\pi} V(\theta) T_j(\cos\theta)\sin\theta\, d\theta.$$

What if we want to find the potential outside of the sphere? that is for $a < r$? We need to add a constraint here is that $\lim_{r\to\infty} u(r, \theta, \phi)$ is finite. If this is the case then we only have functions in the radial direction of the type r^{-k}, for $k > 0$. This means that the coefficient A_n should be zero and the coefficient $B_n \neq 0$. If we already solve the problem inside the sphere, the problem outside would be the same solution after replacing $(r/a)^n$, by $(r/a)^{-(n+1)}$.

5.1.2 Azimuthal Dependent Potentials

Equation 5.6 states that

$$\frac{d}{dx}\left[(1-x^2)\frac{dT}{dx}\right]+\left[n(n+1)-\frac{\mu}{1-x^2}\right]T=0.$$

The case $\mu = 0$ corresponds to the regular Legendre equation. This is the generalized Legendre equation and its solutions now depend on μ as well as of λ. Recall $\mu = m^2$ and $\lambda = n(n+1)$. We call the solutions of this as $P_n^m(x)$. This functions are called the associated Legendre functions. From equation C.22 we write.

$$P_n^m(x) \quad = \quad (1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m}$$

Note that $m > n$ $P_n^m = 0$, since we are taking higher derivavies than the highest order of the polynomial, so we should have $n \leq m$. Some authors (see for example Jackson, [2] include the coefficient $(-1)^n$. This coefficient is controversial and it is known as the Condon Shortlye phase [3]. Since the generalized Legendre equation is a Sturm-Liouville operator we can expect orthogonality of the associated Legendre functions. We show in section C.3.2 the orthogonality of the associated Legendre functions. This is equation C.24

$$\int_{-1}^{1} P_i^i(x)P_j^m(x)dx = \delta_{ij}\frac{2}{2i+1}\frac{(i+m)!}{(i-m)!}.$$

We are ready to construct the solutions of the Laplace equation with data given on a sphere. The solution is the supperposition of products of three factors in r, θ, and ϕ. It is convenient to combine both angular factors into one function. Jackson calls this combination **spherical harmonics** but he recognize that the term is also used as a solution of the generalized Legendre equation, and that what he calls sphereical harmonics are known as **tesseral harmonics.** The functions $\Phi(\phi)$ are given by

$$\Phi(\phi) = \mathrm{e}^{\pm \mathrm{i}\sqrt{\mu}\phi}.$$

where we showed that $\mu = m^2$ and so we can call them

$$\Phi_m(\phi) = \mathrm{e}^{\mathrm{i}m\phi}.$$

[3] https://en.wikipedia.org/wiki/Spherical_harmonics#Condon.E2.80.93Shortley_phase

The orthogonality condition for the associated Legendre functions is shown in Appendix C, see equation C.24 We have

$$\int_{-1}^{1} P_n^m(x) P_l^m(x) = \delta_{nl} \frac{2}{2l+1} \frac{(n+m)!}{(n-m)!}.$$

These functions form a complete set of orthogonal functions in the index m. The norm of the complex exponential functions above, on the interval $[-\pi, \pi]$ is given by $\sqrt{2\pi}$, and so we can combine the azimuthal solutions with the polar solutions into orthonormal solutions

$$Y_{nm}(\theta, \phi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_n^m(\cos\theta) \mathrm{e}^{\mathrm{i}m\phi}.$$

Note that if $m = 0$ then

$$Y_{n0}(\theta, \phi) = \sqrt{\frac{(2n+1)}{4\pi}} P_n(\cos\theta)$$

Note that since m can take negative values we can consider $-m \leq 0 \leq m$, $m \geq 0$, and extend Y_{nm} to negative values of m using equation C.23. That is, we have

$$Y_{n,-m}(\theta, \phi) = (-1)^m \overline{Y}_{nm}(\theta, \phi)$$

where $\overline{Y}$ is the conjugate of Y due to the conjugate of the complex exponential $\mathrm{e}^{\mathrm{i}m\phi}$.

We have then that since the product of orthogonal functions forms a new set of orthogonal functions,

$$\int_0^{2\pi} \int_0^{\pi} d\theta \sin\theta \overline{Y}_{nm}(\theta, \phi) Y_{op}(\theta, \phi) = \delta_{no}\delta_{mp}. \tag{5.9}$$

Recall that $\sin\theta$ comes from the change of variables $x = \cos\theta$.

We have now all the elements to solve problems with data on a sphere. If a function $f(\theta, \phi)$ needs to be expanded in terms of spherical harmonics, we have that

$$f(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn} Y_{nm}(\theta, \phi).$$

and taking inner product with $Y_{op}(\theta, \phi)$ and the orthogonality relation 5.9 we find

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta f(\theta, \phi)\overline{Y}_{op}(\theta, \phi)d\theta = \sum_{n=0^{\infty}} \sum_{m=-n}^{n} A_{nm}\delta_{no}\delta_{mp}$$

That is,

$$A_{op} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta f(\theta, \phi)\overline{Y}_{op}(\theta, \phi)d\theta.$$

The general solution of the boundary-value problem in spherical coordinates is written as

$$u(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[A_{nm}r^n + B_{nm}\frac{1}{r^{n+1}} \right] Y_{nm}(\theta, \phi).$$

With this, the Laplace equation in spherical coordinates can be solved, knowing the data on a sphere and the behavior of the potential at 0 and at ∞.

Chapter 6

Lecture #6

6.1 Electrostatic Boundary Conditions

We establish normal and tangential electrostatic behavior on a surface boundary.

6.1.1 Normal Boundary Conditions

For the normal boundary conditions we use the Gauss law. Let us assume a small thin Gauss pix box drawn around the surface. We apply Gauss's law to find

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}.$$

Even if the surface curves we can adjust the Gaussian box to fit the surface. Then we let the thickness of the pill box to approximate zero and find that the variation of the electric field across the surface is σ_0/ϵ_0. That is

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}.$$

where we can think that the surface is looked from a reference point such that $E^{\perp}_{\text{above}}$ is the normal component directly above the surface and $E^{\perp}_{\text{below}}$ is the normal component directly below the surface. We found then that the normal component of the electrical wavefield is discontinuous across the surface and the amount of discontinuity is given σ/ϵ_0. We now study how the tangential component of the electric field varies when moving across a surface.

6.1.2 Tangential Boundary Conditions

Let us, for the sake of simplicity, assume that we have only one point charge q. Then we use the superposition principle to apply the findings to more general charge distributions.

We know that the electrical field is conservative and so around a closed loop

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = 0$$

Now, we can think of a small rectangular loop around a piece of surface with charge density γ. The loop is thin and by taking the thickness go to zero we see that since the line integral is 0 and

$$\boldsymbol{E}^{\parallel}_{\text{above}} = \boldsymbol{E}^{\parallel}_{\text{below}}.$$

Likewise for the potential we find that the potential normal derivatives of the potential are discontinuous across the surface while the tangential are continuous. That is

$$\nabla V_{\text{above}} \cdot \boldsymbol{n} - \nabla V_{\text{below}} \cdot \boldsymbol{n} = -\frac{\sigma}{\epsilon_0},$$

and

$$\nabla V_{\text{above}} \cdot \boldsymbol{t} - \nabla V_{\text{below}} \cdot \boldsymbol{t} = 0$$

where $\boldsymbol{n}$ is a vector normal to the surface and $\boldsymbol{t}$ tangent to the surface.

6.2 Work and Energy in Electrostatics

As it is usual we introduce work and energy for a particle, for a discrete system of particles, and then for a continuum of particle.

6.2.1 Work to move a Charge

We asssume that we have a test charge Q and want to move it from point $\boldsymbol{a}$ to point $\boldsymbol{b}$ in the presense of an electric field $\boldsymbol{E}$. By definition of work

$$W = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{F} \cdot d\boldsymbol{\ell} = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} Q\boldsymbol{E} \cdot d\boldsymbol{\ell} = Q\int_{\boldsymbol{a}}^{\boldsymbol{b}} \nabla V \cdot d\boldsymbol{\ell} = Q[V(\boldsymbol{b}) - V(\boldsymbol{a}].$$

We can then say that

$$V(\boldsymbol{b}) - V(\boldsymbol{a}) = \frac{W}{Q}.$$

That is, the potential difference for a test charge Q between two points $\boldsymbol{a}$ and $\boldsymbol{b}$ is equivalent to the work done by moving the charge from $\boldsymbol{a}$ to $\boldsymbol{b}$. In particular if $\boldsymbol{a}$ is the reference point at ∞ when the potential is zero, we have that

$$W = QV(\boldsymbol{b}).$$

We see that potential here is seen as potential energy and the interaction between work and energy. We now consider a discrete number of charges and find the work done to bring the system to a given configuration.

6.2.2 Work of a discrete system of charges

We move a system of charges n charges q_i to some positions $\boldsymbol{r}_i$ and want to find out the total amount of work to set the system to the given configuration. We assume that initially there are no charges and so $\boldsymbol{E} = 0$. The first charge can be moved to any location without any effort. That is, the work to move the first charge is 0. Let us now move the second charge to a position $\boldsymbol{r}_2$. Since now there is an electrical field due to the first charge, there will be some work determined by

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \frac{q_1}{r_{12}} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}} \tag{6.1}$$

where r_{12} is the relative position of particle q_2 with respect to particle location q_1. To bring q_3 to a position r_3 we now have that it needs to work against the field created by the two previous charges. That is

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}} \right).$$

In general to bring the q_n charge to the system will require the work

$$W_n = \frac{1}{4\pi\epsilon_0} q_n \left(\frac{q_1}{r_{1n}} + \frac{q_2}{r_{2n}} + \cdots \frac{q_n}{r_{(n-1)n}} \right),$$

and the total work for moving all n charges to the configuration is given by

$$W = \sum_{j=1}^{n} W_j = \frac{1}{8\pi\epsilon_0} \sum_{j=1}^{n} \sum_{i \neq j}^{n} \frac{q_i q_j}{r_{ij}}. \tag{6.2}$$

where we have an extra division by 2 to avoid counting the ij combination twice (ij and ji. The order does not change the amount of work contributed from the particles i and j.)

We can factor out, for example, the j-th term here and rewrite

$$W = \frac{1}{2} \sum_{j=1}^{n} q_j \sum_{i \neq j}^{n} \frac{q_i}{4\pi\epsilon_0 \, r_{ij}}.$$

The term in parenthesis is the potential at $\boldsymbol{r}_j$ due to all charges (except by q_j) in the system. That is, we write

$$W = \frac{1}{2} \sum_{j=1}^{n} q_j V_j. \tag{6.3}$$

So the total energy required to assemble or disassemble the system of n particles.

6.2.3 Work of a continuum of charges

Equation 6.3 can be used to infere the work to assemble a continuum of charges. By having a continuum of charges with charge density ρ on a given volume we can write the work to assemble the charges in their location as

$$W = \frac{1}{2} \int_{\text{Vol}} \rho V(\boldsymbol{r}) d\boldsymbol{r} \tag{6.4}$$

where Vol is the volume of integration. The volume could be a surface or a line elment as well.

We use Gauss' law to find another represention of the integral above. That is, given that

$$\rho = \epsilon_0 \nabla \cdot \boldsymbol{E}$$

$$W = \frac{\epsilon_0}{2} \int_{\text{Vol}} (\nabla \cdot \boldsymbol{E}) V d\boldsymbol{r}.$$

Integration by parts can be done with the identity

$$\nabla \cdot (V\boldsymbol{E}) = \nabla V \cdot \boldsymbol{E} + V \nabla \cdot \boldsymbol{E}.$$

That is

$$(\nabla \cdot \boldsymbol{E}) V = \nabla \cdot (V\boldsymbol{E}) - \nabla V \cdot \boldsymbol{E}.$$

Then

$$W = \frac{\epsilon_0}{2} \int_{\text{Vol}} [\nabla \cdot (V\boldsymbol{E}) - \nabla V \cdot \boldsymbol{E}] d\boldsymbol{r},$$

and applying the divergence theorem

$$W = \frac{\epsilon_0}{2} \int_S (V\boldsymbol{E}) \cdot d\boldsymbol{S} - \frac{\epsilon_0}{2} \int_{\text{Vol}} \nabla V \cdot \boldsymbol{E} d\boldsymbol{r}.$$

The surface S surrounds the volume of integration S. Since $\nabla V = -\boldsymbol{E}$ we have that $\nabla V \cdot \boldsymbol{E} = -\boldsymbol{E} \cdot \boldsymbol{E} = -E^2$, so

$$W = \frac{\epsilon_0}{2} \int_S (V\boldsymbol{E}) \cdot d\boldsymbol{S} + \frac{\epsilon_0}{2} \int_{\text{Vol}} E^2 d\boldsymbol{r},$$

We need that the surface S enclose all charges and so outside the surface S, $\rho = 0$. We can then let S to recede to infinity. Since V behaves as $1/r$, $\boldsymbol{E}$ as $1/r^2$ and $d\boldsymbol{S}$ as r^2 we find that the integral behaves as $1/r$ and as $r \to \infty$ the integral becomes zero. Then we can write

$$W = \frac{\epsilon_0}{2} \int_{\mathbb{R}^3} E^2 d\boldsymbol{r}. \tag{6.5}$$

where now the integration is along the whole space $\mathbb{R}^3$.

6.3 Discussion

- In the equations for work above we freely interchange the word "work" for "energy". That is, we can refer them to potential energy as well as work.
- Equation 6.4 provides a way to compute the energy from integration on the charge distribution while equation 6.5 computes the energy from the field lines $\boldsymbol{E}$.
- There seems to be a contradiction between equations 6.5 and equation 6.1. Integral 6.5 is positive definite while equation 6.1 can be negative if the sign of charges q_1 and q_2 is opposite. We need to clarify this point. For that matter let us compute the integral of E^2 based on only the field lines of the two charges and look for the difference results between the two approaches.

  Given two charges q_1 and q_2, the electrical field $\boldsymbol{E}$ due to these two charges is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q_1}{4\pi\epsilon_0\, r_1^2}\hat{\boldsymbol{r}}_1 + \frac{q_2}{4\pi\epsilon_0\, r_2^2}\hat{\boldsymbol{r}}_2$$

  where $r_1 = \|\boldsymbol{r} - \boldsymbol{r}_1\|$, $r_2 = \|\boldsymbol{r} - \boldsymbol{r}_2\|$, $\hat{\boldsymbol{r}}_1 = (\boldsymbol{r} - \boldsymbol{r}_1)/r_1$, and $\hat{\boldsymbol{r}}_2 = (\boldsymbol{r} - \boldsymbol{r}_2)/r_2$.

  Then

$$E^2 = \frac{q_1^2}{16\pi^2\epsilon_0^2 r_1^4} + \frac{q_2^2}{16\pi^2\epsilon_0^2 r_2^4} + \frac{2\, q_1 q_2}{16\pi^2\epsilon_0^2 r_1^2 r_2^2}\hat{\boldsymbol{r}}_1 \cdot \hat{\boldsymbol{r}}_2. \tag{6.6}$$

  We want to compute $\int E^2 d\boldsymbol{r}$ over all space. However we will ignore the self-charge terms (the first two terms) which are ignored in the discrete problem, and estimate

$$I = \int_{\text{allspace}} \frac{q_1 q_2}{16\pi^2\epsilon_0^2 r_1^2 r_2^2}\hat{\boldsymbol{r}}_1 \cdot \hat{\boldsymbol{r}}_2 d\boldsymbol{r} = \frac{q_1 q_2}{16\pi^2\epsilon_0^2}\int_{\text{allspace}} \frac{\hat{\boldsymbol{r}}_1 \cdot \hat{\boldsymbol{r}}_2 d\boldsymbol{r}}{r_1^2 r_2^2}.$$

  We now use the identity 1.9 to simplify this integral. That is, since $\nabla 1/r = (-1/r^2)\hat{\boldsymbol{r}}$, then

$$I = \frac{q_1 q_2}{16\pi^2\epsilon_0^2}\int_{\text{allspace}} \nabla\left(\frac{1}{r_1}\right) \cdot \nabla\left(\frac{1}{r_2}\right) d\boldsymbol{r}$$

We now use integration by parts as follows. From

$$\nabla \cdot \left[\frac{1}{r_1}\nabla\left(\frac{1}{r_2}\right)\right] = \nabla\left(\frac{1}{r_1}\right)\cdot\nabla\left(\frac{1}{r_2}\right) + \frac{1}{r_1}\nabla^2\left(\frac{1}{r_2}\right),$$

$$\int_{\mathbb{R}^3} \nabla\left(\frac{1}{r_1}\right)\cdot\nabla\left(\frac{1}{r_2}\right) d\boldsymbol{r} = \int_{\mathbb{R}^3} \nabla \cdot \left[\frac{1}{r_1}\nabla\left(\frac{1}{r_2}\right)\right] d\boldsymbol{r} - \int_{\mathbb{R}^3} \frac{1}{r_1}\nabla^2\left(\frac{1}{r_2}\right) d\boldsymbol{r},$$

and using the divergence theorem

$$\int_{\mathbb{R}^3} \nabla\left(\frac{1}{r_1}\right)\cdot\nabla\left(\frac{1}{r_2}\right) d\boldsymbol{r} = \int_{S} \left[\frac{1}{r_1}\nabla\left(\frac{1}{r_2}\right)\right] dS - \int_{\mathbb{R}^3} \frac{1}{r_1}\nabla^2\left(\frac{1}{r_2}\right) d\boldsymbol{r}.$$

The first integral after the equal sign is 0, since the first factor behaves as $1/r$, the second factor as $1/r^2$ and the surface as r^2 so in the limit as $r \to \infty$, the integral becomes 0. We then have that

$$\int_{\mathbb{R}^3} \nabla\left(\frac{1}{r_1}\right)\cdot\nabla\left(\frac{1}{r_2}\right) d\boldsymbol{r} = -\int_{\mathbb{R}^3} \frac{1}{r_1}\nabla^2\left(\frac{1}{r_2}\right) d\boldsymbol{r}.$$

and using equation 2.5 we see that

$$\int_{\mathbb{R}^3} \nabla\left(\frac{1}{r_1}\right)\cdot\nabla\left(\frac{1}{r_2}\right) d\boldsymbol{r} = \int_{\mathbb{R}^3} \frac{1}{r_1}[4\pi\delta(\boldsymbol{r}-\boldsymbol{r}_2)]d\boldsymbol{r}.$$

and so

$$\int_{\mathbb{R}^3} \nabla\left(\frac{1}{r_1}\right)\cdot\nabla\left(\frac{1}{r_2}\right) d\boldsymbol{r} = \frac{4\pi}{r_{12}}.$$

with $r_{12} = \|\boldsymbol{r}_1 - \boldsymbol{r}_2\|$.

Then

$$W' = \frac{\epsilon_0}{2}\int_{\text{allspace}} E^2 d\boldsymbol{r} = \frac{\epsilon_0}{2}(2I) = \frac{q_1 q_2}{16\pi^2\epsilon_0}\frac{4\pi}{r_{12}} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}},$$

and this is precisely equation 6.1 for the work done in moving two charges to a relative distance r_{12}.

Note that we used the symbol W' for the energy where we excluded the two self-charge therms in equation 6.6.

- Integral 6.4 is actually a two-fold integral since $V(\boldsymbol{r})$ is itself an integral. That is,

$$W = \frac{1}{2}\int \rho(\boldsymbol{r}) \left(\int \frac{\rho(\boldsymbol{r}_i)}{4\pi\epsilon_0 r_{io}} d\boldsymbol{r}_i \right) d\boldsymbol{r} = \frac{1}{8\pi\epsilon_0}\int d\boldsymbol{r} d\boldsymbol{r}_i \frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r}_i)}{r_{io}}. \tag{6.7}$$

  with $r_{io} = \|\boldsymbol{r} - \boldsymbol{r}_i\|$. See that this equation is the continuum form of equation 6.2. It is now clear from comparing equation 6.7 with equation 6.2 that in the discrete version we are not taking the self-charge temrs ($i = j$), while in the continuum we include them. A self charge term seems to contribute infinite energy and it does in the discrete since there $r_{ij} = 0$ and we would be dividing by zero. In the continuum the differential of charge gets infinitely small and the integral could converge even with an $r \to 0$, in the denominator. For example $\int (1/\sqrt{x})dx = 2\sqrt{x}$, which converges as $x \to 0$.

- We see that equation 6.5 violates the principle of superposition. That is, if we want to compute W as a function of E we can not simply say that

$$W(E_1 + E_2) = W(E_1) + W(E_2),$$

  since the relation is quadratic. More explicitly

$$\begin{aligned} W(E_1 + E_2) &= \frac{\epsilon_0}{2}\int_{\text{Vol}} (E_1 + E_2)^2 d\boldsymbol{r} \\ &= \frac{\epsilon_0}{2}\int_{\text{Vol}} E_1^2 d\boldsymbol{r} + \frac{\epsilon_0}{2}\int_{\text{Vol}} E_2^2 d\boldsymbol{r} + 2\frac{\epsilon_0}{2}\int_{\text{Vol}} E_1 E_2 d\boldsymbol{r} \\ &= W(E_1) + W(E_2) + \epsilon_0 \int_{\text{Vol}} E_1 E_2 d\boldsymbol{r} \end{aligned}$$

The crossed term at the end violates linearity and so the superposition principle. We can not divide our problem in small problems and think that adding them will get the whole. For example, finding the energy stored in a sphere can not be done using the formula for the field representation (in terms of E^2) and slice the sphere in discs or concentric shells, for example.

6.4 Examples

In this section we show one example which illustrates the different equations derived here and, in addition, shows new ways to think of solving electric potential energy problems. In the development of this example we find a way to compute an interesting integral.

The example is as follows. Let us assume that we want to find the energy required to assemble an sphere of charges with uniform charge density ρ. Figure 6.1 illustrates this. We do this in several ways:

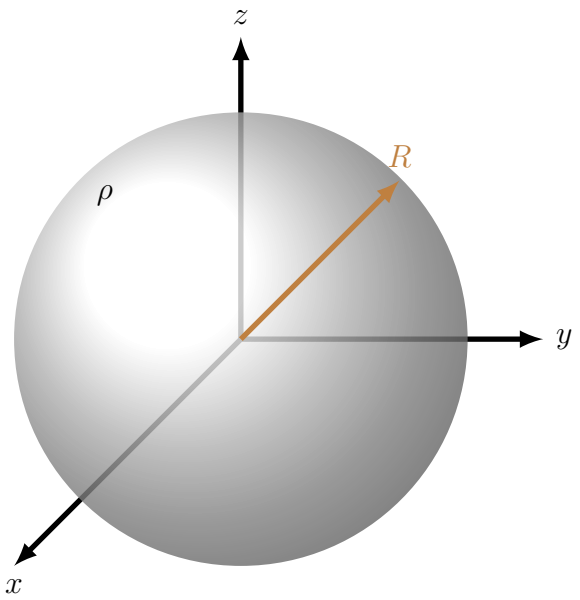

Figure 6.1: Sphere with constant charge density ρ and radius R.

(i) **By using equation 6.4** for energy computed from the charge distribution and the potential. This is,

$$W = \frac{1}{2}\int \rho V d\boldsymbol{r}.$$

We consider the integral over the sphere. Exercise 4 of chapter 2, equation 2.16 claims that the potential on a uniformly charged sphere with electrical charge density ρ is given by:

$$V(r) = \frac{q}{2R(4\pi\epsilon_0)}\left(3 - \frac{r^2}{R^2}\right) \quad , \quad r \leq R.$$

The charge density is given by

$$\rho = \frac{3q}{4\pi R^3},$$

so

$$\begin{aligned} W &= \frac{1}{2}\frac{3q}{4\pi R^3}\int_0^R \frac{q}{2R(4\pi\epsilon_0)}\left(3 - \frac{r^2}{R^2}\right) r^2 \sin\theta dr d\theta d\phi \\ &= \frac{3q^2}{64\pi^2 R^4 \epsilon_0}\left(R^3 - \frac{R^5}{5R^2}\right)(4\pi) \\ &= \frac{3q^2}{16\pi R\epsilon_0}\left(1 - \frac{1}{5}\right) \\ &= \frac{3q^2}{20\pi R\epsilon_0} \end{aligned}$$

(ii) **By Equation 6.5** ? for integration of the square of the electrical field. That is,

$$W = \frac{\epsilon_0}{2}\int_{\text{allspace}} E^2 d\boldsymbol{r}.$$

The electrical field for the uniformly charged sphere is given by

$$\boldsymbol{E} = \begin{cases} \frac{\rho r}{3\epsilon_0}\hat{\boldsymbol{r}} & r < R \\ \\ \frac{q}{4\pi r^2 \epsilon_0}\hat{\boldsymbol{r}} & r \geq R \end{cases}$$

and since $\rho = 3\,q/(4\pi R^3)$, Then

$$\boldsymbol{E} = \begin{cases} \frac{qr}{4\pi R^3 \epsilon_0}\hat{\boldsymbol{r}} & r < R \\ \\ \frac{q}{4\pi r^2 \epsilon_0}\hat{\boldsymbol{r}} & r \geq R \end{cases}$$

and

$$E^2 = \boldsymbol{E}\cdot\boldsymbol{E} = \begin{cases} \frac{q^2 r^2}{16\pi^2 R^6 \epsilon_0^2} & r < R \\ \frac{q^2}{16\pi^2 r^4 \epsilon_0^2} & r \geq R. \end{cases}$$

So

$$\begin{aligned} W &= \frac{\epsilon_0}{2}\int_0^R \frac{q^2 r^2}{16\pi^2 R^6 \epsilon_0^2}(4\pi r^2)dr + \frac{\epsilon_0}{2}\int_R^\infty \frac{q^2}{16\pi^2 r^4 \epsilon_0^2}(4\pi r^2)dr \\ &= \frac{q^2}{8\pi R^6 \epsilon_0}\int_0^R r^4 dr + \frac{q^2}{8\pi\epsilon_0}\int_R^\infty \frac{dr}{r^2} \\ &= \frac{q^2 R^5}{40\pi R^6 \epsilon_0} + \frac{q^2}{8\pi\epsilon_0 R} \\ &= \frac{q^2}{\pi R\epsilon_0}\left(\frac{1}{40}+\frac{1}{8}\right) \\ &= \frac{6q^2}{40\,\pi R\epsilon_0} \\ &= \frac{3q^2}{20\,\pi R\epsilon_0}, \end{aligned}$$

which is the same found in (i).

(iii) Here is yet another way of computing the energy of a uniformly charged sphere We build up the sphere by adding infinitesimal layers of charge (carried from infinite distance). From Gauss's law we know that for a uniformly charged sphere with charge density ρ, and radius r, the total charge is given by

$$q = q(r) = \rho\frac{4\pi r^3}{3}.$$

The field and potential outside the sphere are those of a point charge q located at the center. Then when building the sphere a new layers is spread with differential charge $dq = \rho(4\pi r^2)dr$. When bringing up this charge from infinity a potential

$$V(r) = \frac{q(r)}{4\pi\epsilon_0 r}$$

is required and the correspondng electrostatic energy is given by $dW = V(r)dq$. We integrate this to find

$$\begin{aligned} W = \int_0^R dW &= \int_0^R \frac{q(r)}{4\pi\epsilon_0 r} dq \\ &= \frac{1}{\cancel{4\pi}\epsilon_0} \int_0^R \rho \frac{\cancel{4\pi} r^3}{3r} \rho(4\pi r^2) dr \\ &= \frac{4\pi\rho^2}{3\epsilon_0} \int_0^R r^4 dr \\ &= \frac{4\pi\rho^2 R^5}{15\epsilon_0}. \end{aligned} \tag{6.8}$$

Now since

$$\rho = \frac{3q}{4\pi R^2}$$

we find that

$$W = \frac{4\pi R^5 (9q^2/(16\pi^2 R^6)}{15\epsilon_0} = \frac{3}{5} \frac{q^2}{4\pi\epsilon_0 R}.$$

This agrees with the result found in the previous exercise.

We now that since we can write the energy equation as

$$\begin{aligned} W &= \frac{1}{2} \int \rho(\boldsymbol{r}) V(r) d\boldsymbol{r} \\ &= \frac{1}{2} \int \rho(\boldsymbol{r}) \left(\int \frac{\rho(\boldsymbol{r}_i)}{4\pi\epsilon_0 r_{io}} d\boldsymbol{r}_i \right) d\boldsymbol{r} \\ &= \frac{1}{8\pi\epsilon_0} \int d\boldsymbol{r} d\boldsymbol{r}_i \frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r}_i)}{r_{io}}. \end{aligned}$$

when ρ is constant and we are integrating over a sphere of radius R we have that

$$\rho^2 \int \frac{d\boldsymbol{r} d\boldsymbol{r}_i}{r_{io}} = 8\pi\epsilon_0 W.$$

Now we use equation 6.8 and then

$$\rho^2 \int \frac{d\boldsymbol{r} d\boldsymbol{r}_i}{r_{io}} = 8\pi\epsilon_0 \frac{4\pi\rho^2 R^5}{15\epsilon_0}$$

That is

$$\int \frac{d\boldsymbol{r} d\boldsymbol{r}_i}{r_{io}} = \frac{32\pi^2 R^5}{15}$$

This provides a computation of an interesting integral of the sum of all inverse distance of each point of the sphere to any other point of the sphere.

Chapter 7

Lecture #7

7.1 Multipoles

We already know the behavior of the potential due to a point charge q in the origin, observed from a distance r to the origin. This is given by

$$V(r) = \frac{q}{4\pi\epsilon_0 r}$$

Let us now consider two equal charges of opposite sign at the points $(0, 0, d/2)$ and $(0, 0, -d/2)$. We want to estimate the potential at some distance r far from the origin. Figure 7.1 sketches the variables used in the solution of this problem. The potential due to these two charges is given by

$$v(r) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right),$$

where $q = q_+$. Since we want to evaluate this potential in terms of r we use the law of cosines. That is,

$$r_+^2 = r^2 + (d/2)^2 - 2(rd/2)\cos\theta = r^2 + (d/2)^2 - rd\cos\theta,$$

and

$$\frac{1}{r_+} = \frac{1}{\sqrt{r^2 + (d/2)^2 - rd\cos\theta}} = \frac{1}{r}\left(1 + \frac{d^2}{4r^2} - \frac{d\cos\theta}{r}\right)^{-1/2}$$

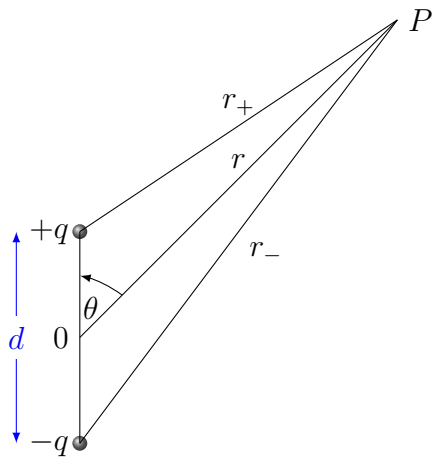

Figure 7.1: We show two equal charges of opposite signs $\pm q$, at the points $(0, 0, d/2)$ and $(0, 0, -d/2)$. The observation point is P and the distances from p to the charges are r_+ and r_-.

We only want to consider higher order terms and since $d \ll r$, we can approximate this equation by dropping the $d^2/(4r^2)$ term as

$$\frac{1}{r_+} \approx \frac{1}{r}\left(1 - \frac{d\cos\theta}{r}\right)^{-1/2}.$$

In the expansion in terms of a Taylor series in r we find

$$\frac{1}{r_+} \approx \frac{1}{r}\left(1 + \frac{d\cos\theta}{2r}\right)$$

A similar expansion for the r_- leg would give

$$\frac{1}{r_-} \approx \frac{1}{r}\left(1 - \frac{d\cos\theta}{2r}\right),$$

and so

$$\frac{1}{r_+} - \frac{1}{r_-} \approx \frac{d}{r^2}\cos\theta.$$

Then from P, a far away observation point, we see that

$$V(r) \approx \frac{qd\cos\theta}{4\pi\epsilon_0 r^2}. \tag{7.1}$$

So, while the monopole has a first order decay with distance potential, the dipole has a second order decay with distance potential.

We would like to understand the more general problem

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\boldsymbol{r}_i)\frac{d\boldsymbol{r}_i}{\|\boldsymbol{r}-\boldsymbol{r}_i\|}.$$

where $\boldsymbol{r}$ is far away from the concentration of charges. The key point here is in the function

$$\frac{1}{\|\boldsymbol{r}-\boldsymbol{r}_i\|} = \frac{1}{\sqrt{r^2+r_i^2-2rr_i\cos\theta}} = \frac{1}{r}\frac{1}{\sqrt{1+r_i^2/r^2-2(r_i/r)\cos\theta}}$$

where θ is the angle between the radial vectors $\boldsymbol{r}$ and $\boldsymbol{r}_i$. By looking at the generating functions for the Legendre polynomials C.18 we find that

$$\frac{1}{r}\frac{1}{\sqrt{1+r_i^2/r^2-2(r_i/r)\cos\theta}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r_i}{r}\right)^n P_n(\cos\theta). \tag{7.2}$$

Hence we showed that

$$\frac{1}{\|\boldsymbol{r}-\boldsymbol{r}_i\|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r_i}{r}\right)^n P_n(\cos\theta).$$

and so

$$\begin{aligned} V(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0}\int \rho(\boldsymbol{r}_i)\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r_i}{r}\right)^n P_n(\cos\theta)d\boldsymbol{r}_i \\ &= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r_i^n P_n(\cos\theta)\rho(\boldsymbol{r}_i)d\boldsymbol{r}_i. \end{aligned} \tag{7.3}$$

This is the multipole expansion of $V(\boldsymbol{r})$ in powers of $1/r$. The zero order term $n = 0$ is

$$V_0(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_0 r}$$

where Q is the total charge. This is as if the total charge is concentrated at the origin. That is the first order term is the monopole corresponding to all charge in the system. The first order term $n = 1$, is

$$V_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0 r^2} \int r_i \cos\theta \rho(\boldsymbol{r}_i) d\boldsymbol{r}_i. \tag{7.4}$$

Let us assume that

$$\rho(\boldsymbol{r}_i) = (q+)\delta(x)\delta(y)\delta(z - d/2) + (q-)\delta(x)\delta(y)\delta(z + d/2). \tag{7.5}$$

and θ is the angle in Figure 7.1. Then

$$\begin{aligned} V_1(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0 r^2} \int r_i \cos\theta [(q+)\delta(x)\delta(y)\delta(z - d/2) + (q-)\delta(x)\delta(y)\delta(z + d/2)] d\boldsymbol{r}_i \\ &= \frac{d}{2}\left(\frac{q\cos\theta}{4\pi\epsilon_0 r^2}\right) - \frac{d}{2}\left(\frac{-q\cos\theta}{4\pi\epsilon_0 r^2}\right) \quad , \quad \text{where } r_i = z \\ &= \frac{qd\cos\theta}{4\pi\epsilon_0 r^2}. \end{aligned}$$

Compare this with equation 7.1 and see that this is a dipole equation.

Since θ is the angle between the input vector $\boldsymbol{r}_i$ and the observation (output) vector $\boldsymbol{r}$, we can write

$$r_i \cos\theta = \hat{\boldsymbol{r}} \cdot \boldsymbol{r}_i,$$

and equation 7.4 turns out to be

$$V_1(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0 r^2} \int \hat{\boldsymbol{r}} \cdot \boldsymbol{r}_i \, \rho(\boldsymbol{r}_i) \, d\boldsymbol{r}_i. = \frac{1}{4\pi\epsilon_0 r^2} \hat{\boldsymbol{r}} \cdot \int \boldsymbol{r}_i \, \rho(\boldsymbol{r}_i) \, d\boldsymbol{r}_i.$$

The integral expression is the first moment of the electrical charge density function $\rho(\boldsymbol{r}_i)$ and this is known in the literature as the **dipole moment** and noted as

$$\boldsymbol{p} = \int \boldsymbol{r}_i \, \rho(\boldsymbol{r}_i) \, d\boldsymbol{r}_i. \tag{7.6}$$

The discrete version for a system of n particles is

$$\boldsymbol{p} = \sum_{i=1}^{n} \boldsymbol{r}_i \, q_i. \tag{7.7}$$

Then we can write V_1 and call it V_{dip} as

$$V_{\text{dip}}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \frac{\boldsymbol{p} \cdot \hat{\boldsymbol{r}}}{r^2}. \tag{7.8}$$

Let us assume that we only have two charges as in Figure 7.1. If we want to find the dipole for these two charges with the origin at 0, we have

$$q^+(0,0,d/2) + q^-(0,0,-d/2) = 2q(0,0,d/2) = q(0,0,d).$$

If we consider the origin at the negative charge, this is a vector from the negative charge to the positive charge. This is the convention used for moment dipoles. Moment dipoles are vectors drawn from the negative to the postive charge.

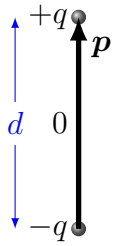

Figure 7.2: We show two equal charges of opposite signs $\pm q$, at the points $(0,0,d/2)$ and $(0,0,-d/2)$. The dipole for these two charges is represented by a vector having as origin the negative charge and ending in the positive charge.

We now go one more order on the multipole expansion 7.3. Instead of the notation r_i we will switch to r_I since the index i will be used often for different purposes. In expression 7.3 we use $n = 2$ to find

$$V_2(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int r_I^2 P_2(\cos\theta)\rho(\boldsymbol{r}_I)d\boldsymbol{r}_I = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int r_I^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\rho(\boldsymbol{r}_I)d\boldsymbol{r}_I$$

As in the dipole moment analysis we use the identity 7.6 to express the cosine as an inner product. Then

$$r_I^2\cos^2\theta = (\hat{\boldsymbol{r}}\cdot\boldsymbol{r}_I)^2$$

so

$$V_2(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\int \left(3(\hat{\boldsymbol{r}}\cdot\boldsymbol{r}_I)^2 - \boldsymbol{r}_I\cdot\boldsymbol{r}_I\right)\rho(\boldsymbol{r}_I)d\boldsymbol{r}_I$$

Let us write this equation in terms of components. To avoid confusion with too many r's let us call $\boldsymbol{r}_I = (x_1, x_2, x_3)$, and so

We have that

$$\hat{\boldsymbol{r}}\cdot\boldsymbol{r}_I = \sum_{j=1}^{3}\hat{r}_j x_j$$

$$\boldsymbol{r}_I\cdot\boldsymbol{r}_I = \sum_{j=1}^{3} x_j x_j,$$

and

$$V_2(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\int\left(3\sum_{i=1}^{3}\hat{r}_i x_i\sum_{j=1}^{3}\hat{r}_j x_j - r_I^2\right)\rho(\boldsymbol{r}_I)d\boldsymbol{r}_I.$$

We now use the identity

$$\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\delta_{ij} = 1 \quad , \quad r_I^2 = \sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\delta_{ij}r_I^2,$$

to write

$$
\begin{aligned}
V_2(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\int\left(3\sum_{i=1}^{3}\hat{r}_i x_i\sum_{j=1}^{3}\hat{r}_j x_j-\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\delta_{ij}r_I^2\right)\rho(\boldsymbol{r}_I)d\boldsymbol{r}_I\\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\int\left(\sum_{i,j=1}^{3}3\hat{r}_i\hat{r}_j x_i x_j-\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\delta_{ij}r_I^2\right)\rho(\boldsymbol{r}_I)d\boldsymbol{r}_I\\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\int(3x_i x_j-\delta_{ij}r_I^2)\rho(\boldsymbol{r}_I)d\boldsymbol{r}_I
\end{aligned}
$$

We define

$$
Q_{ij}=\int\left(3x_i x_j-\delta_{ij}r_I^2\right)\rho(\boldsymbol{r}_I)d\boldsymbol{r}_I, \tag{7.9}
$$

as the **quadrupole moment** and so

$$
V_2(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j Q_{ij}.
$$

It is easy to verify that the trace of Q is zero. That is, $Q_{11}+Q_{22}+Q_{33}=0$.

7.1.1 Change of Origin

We study the change of origin in the monopole moment Q, the dipole moment $\boldsymbol{p}$ and the quadrupole moment Q_{ij}.

If we move the origin nothing happen to Q since it quantifies the total charge and this is independent of the coordinate sytem. Let us analyze what happens to the dipole moment $\boldsymbol{p}$ under change of origin. Let us consider the discrete case and a new radius $\boldsymbol{r}_i-\boldsymbol{r}_0$,

$$
\boldsymbol{p'}=\sum_{i=1}^{n}(\boldsymbol{r}_i-\boldsymbol{r}_0)q_i=\sum_{i=1}^{n}\boldsymbol{r}_i q_i-\sum_{i=1}^{n}\boldsymbol{r}_0 q_i=\sum_{i=1}^{n}\boldsymbol{r}_i q_i-\boldsymbol{r}_0\sum_{i=1}^{n}q_i=\boldsymbol{p}-\boldsymbol{r}_0 Q.
$$

So, unless the total charge $Q = 0$ there is a change in $\boldsymbol{p}$ when we move the origin. We need to be more careful in the continuum. Let us assume that our new origin is $\boldsymbol{r}_o$ and now we shift vectors with respect to this new origin $\boldsymbol{r}_0$. That is our new vector $\boldsymbol{r}_i$ becomes $\boldsymbol{r}_i - \boldsymbol{r}_0$, and so

$$\boldsymbol{p}' = \int (\boldsymbol{r}_i - \boldsymbol{r}_o)\rho(\boldsymbol{r}_i - \boldsymbol{r}_0)d\boldsymbol{r}_i = \int \boldsymbol{r}_i\rho(\boldsymbol{r}_i - \boldsymbol{r}_0)d\boldsymbol{r}_i - \int \boldsymbol{r}_o\rho(\boldsymbol{r}_i - \boldsymbol{r}_0)d\boldsymbol{r}_i = \boldsymbol{p} - \boldsymbol{r}_0 Q.$$

Note that $\boldsymbol{p}$ should be computed this way because the charge density is shifted to account for the new origin but the old vector $\boldsymbol{r}_i$ is still the same. The last integral provides Q since we are integrating over all space that holds charges, and if required we can do a change of variables.

Let us now consider the change of origin in the quadripole moment 7.9. There $r_I^2 = x_1^2 + x_2^2 + x_3^2$, so after changing the origin

$$\begin{aligned}
Q'_{ij} &= \int \left(3(r_i - r_{oi})(r_j - r_{oj}) - \delta_{ij}\sum_{i=1}^{3}(x_i - r_{oi})^2\right)\rho(\boldsymbol{r}_i - \boldsymbol{r}_o)d\boldsymbol{r}_i \\
&= \int [(3r_i r_j - r_i^2) - \delta_{ij}r_I^2]\rho(\boldsymbol{r}_i - \boldsymbol{r}_o)d\boldsymbol{r}_i - 3r_{oi}\int r_j\rho(\boldsymbol{r}_i - \boldsymbol{r}_o)d\boldsymbol{r}_i \\
&\quad +3r_{oi}r_{oj}\int \rho(\boldsymbol{r}_i - \boldsymbol{r}_o)d\boldsymbol{r}_i + 2r_{o1}\delta_{ij}\int x_1\rho(\boldsymbol{r} - \boldsymbol{r}_0)d\boldsymbol{r}_i \\
&\quad +2r_{o2}\delta_{ij}\int x_2\rho(\boldsymbol{r} - \boldsymbol{r}_0)d\boldsymbol{r}_i + 2r_{o3}\delta_{ij}\int x_3\rho(\boldsymbol{r} - \boldsymbol{r}_0)d\boldsymbol{r}_i \\
&\quad -r_o^2\delta_{ij}\int \rho(\boldsymbol{r}_i - \boldsymbol{r}_0)d\boldsymbol{r}_i \\
&= Q_{ij} - 3r_{oi}p_j + 3r_{oi}r_{oj}Q + 2\delta_{ij}(r_{o1}p_1 + r_{o2}p_2 + r_{o3}p_3) - r_{oi}^2\delta_{ij}Q,
\end{aligned}$$

with p_i and p_j components of the dipole moment $\boldsymbol{p}$. So we see that except the first term, all other terms are dipoles and monopoles multiplied by scalars. Then, unless the dipoles and monopole moments are 0 a change of origin in the quadrupole will modified its value.

7.2 Electrical Field of a Dipole

Since the gradient of the potential $V(\boldsymbol{r})$ is the electrical field $\boldsymbol{E}(\boldsymbol{r})$, we just need to take derivatives of the expressions found above to obtain representations of the electrical fields for the dipole and quadrupole.

7.2.1 The Dipole

We start with equation 7.8 which represents the potential of a dipole in spherical coordinates. That is

$$E = -\nabla\left(\frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2}\right) = -\frac{1}{4\pi\epsilon_0}\nabla\left(\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2}\right)$$

If we assue that $\boldsymbol{p}$ is in the z direction then

$$\boldsymbol{p}\cdot\hat{\boldsymbol{r}} = p\cos\theta,$$

where p is the magnitude $\|\boldsymbol{p}\| = p$ and θ the polar angle. This makes the problem suited for spherical coordinates. The gradient of a field u in spherical coordinates is given by

$$\nabla u = \frac{1}{h_r}\frac{\partial u}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{h_\theta}\frac{\partial u}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{h_\phi}\frac{\partial u}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

In our case

$$u = -\frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2},$$

and $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$. Furthermore

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta}{r^3} \\
\frac{\partial u}{\partial \theta} &= \frac{1}{4\pi\epsilon_0}\frac{p\sin\theta}{r^2} \\
\frac{\partial u}{\partial \phi} &= 0.
\end{aligned}$$

Then we find that

$$\begin{aligned}
E_r &= \frac{1}{4\pi\epsilon_0}\left(\frac{2p\cos\theta}{r^3}\right)\hat{\boldsymbol{r}} \\
E_\theta &= \frac{1}{4\pi\epsilon_0}\frac{p\sin\theta}{r^3}\hat{\boldsymbol{\theta}} \\
E_\phi &= 0,
\end{aligned}$$

or

$$\boldsymbol{E}_{\text{dip}} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}).$$

We derive a representation that does not depends on specific coordinates. By expanding the dot product in the potential 7.8 and using repeated index (Einstein) notation we find

$$V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{r_i p_i}{r^3}.$$

Then

$$\begin{aligned}\boldsymbol{E}_{\text{dip}} = -\frac{\partial V_{\text{dip}}}{\partial r_j} &= -\frac{1}{4\pi\epsilon_0}\left(\frac{\partial r_i}{\partial r_j}\frac{p_i}{r^3} - r_i\frac{3p_i}{r^4}\frac{\partial r}{\partial r_j}\right)\\ &= -\frac{1}{4\pi\epsilon_0}\left(\frac{p_i\delta_{ij}}{r^3} - r_i\frac{3p_i r_j}{r^5}\right)\end{aligned}$$

since

$$\frac{\partial r}{\partial r_j} = \frac{r_j}{r}.$$

Recall that $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$. Hence we have that

$$(E_{\text{dip}})_j = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left(3\frac{(\boldsymbol{p}\cdot\boldsymbol{r})r_j}{r^2} - p_j\right),$$

or in vector notation

$$\boldsymbol{E}_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}\right].$$

Example 7.2.1 *To finish this lecture we illustrate the use of the dipole moment in a problem which is of great importance when formulating properties in the macro.*

Find the average field inside and outside of a sphere of radius R due to all charges in the sphere and write the answer in terms of the dipole moment $\boldsymbol{p}$.

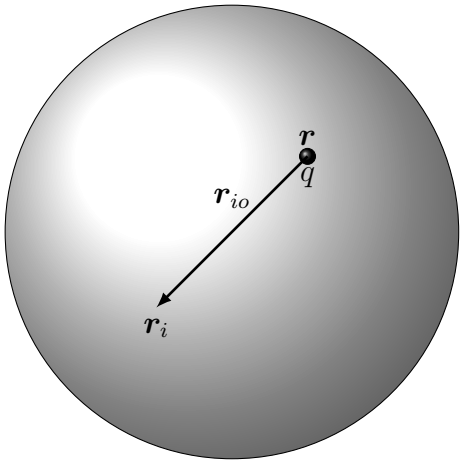

Figure 7.3: A single charge q, at a point $\boldsymbol{r}$. Variable of integration $\boldsymbol{r}_i$.

Solution: By definition the average field due to a point charge q at $\boldsymbol{r}$ is given by

$$\overline{\boldsymbol{E}} = \frac{1}{V_{\text{sphere}}} \int \boldsymbol{E} d\boldsymbol{r}_i \quad , \quad V_{\text{sphere}} = \frac{4\pi R^3}{3} \quad , \quad \boldsymbol{E} = \frac{q\,\boldsymbol{r}_{io}}{4\pi\epsilon_0 r_{io}^2}, \tag{7.10}$$

with $\boldsymbol{r}_{io} = \boldsymbol{r}_i - \boldsymbol{r}$. That is,

$$\overline{\boldsymbol{E}} = \frac{3}{4\pi R^3} \frac{q}{4\pi\epsilon_0} \int \frac{\hat{\boldsymbol{r}}_{io}\, d\boldsymbol{r}_i}{r_{io}^2}.$$

See Figure 7.3 for an illustration of the variables on this solution. If now, instead of a charge q we fill the sphere with some charges with charge density $\rho(\boldsymbol{r}_i)$, we find that the electrical field due to this density function is given by

$$\boldsymbol{E}_\rho = -\frac{1}{4\pi\epsilon_0} \int \rho(\boldsymbol{r}_i) \frac{\boldsymbol{r}_{io}}{r_{io}} d\boldsymbol{r}_i. \tag{7.11}$$

Please note the "-" sign in front due to the fact that this time the observation point is $\boldsymbol{r}$ and the charge density is defined at the point $\boldsymbol{r}_i$.

If we set

$$\rho = -\frac{3q}{4\pi R^3},$$

in equation 7.11 the two electric fields 7.11 and 7.10 are the same.

We now use Gauss's law in the integral above. We consider two cases:

(i) a sphere of radius $r < R$. That is we want to find the average in the interior of the sphere. Then

$$\begin{aligned} \int \boldsymbol{E}_\rho \cdot dS &= \frac{1}{\epsilon_0} \int \rho(\boldsymbol{r}_i) d\boldsymbol{r}_i \\ 4\pi \cancel{r^2} E_\rho &= -\frac{1}{\epsilon_0} \frac{\cancel{3}q}{\cancel{4\pi} R^3} \frac{\cancel{4\pi} r^{\cancel{3}}}{\cancel{3}} \\ &= E_\rho = -\frac{qr}{4\pi\epsilon_0 R^3}. \end{aligned}$$

Then

$$\boldsymbol{E}_\rho(\boldsymbol{r}) = -\frac{1}{4\pi\epsilon_0} \frac{q}{R^3} \boldsymbol{r},$$

and since the dipole moment is written as $\boldsymbol{p} = q\boldsymbol{r}$ and $\boldsymbol{E}_\rho = \overline{\boldsymbol{E}}$, then we find that for a single charge q

$$\overline{\boldsymbol{E}} = -\frac{1}{4\pi\epsilon_0} \frac{\boldsymbol{p}}{R^3}. \tag{7.12}$$

By supperposition we change $\boldsymbol{p} = q\boldsymbol{r}$ by the more general formula for the dipole moment

$$\boldsymbol{p} = \int \boldsymbol{r}_i \rho(\boldsymbol{r}_i) d\boldsymbol{r}_i,$$

(ii) if $\boldsymbol{r}$ is outside of the sphere still the average field $\overline{\boldsymbol{E}}$ computed above remains the same, as well as the field $\boldsymbol{E}_\rho$. The difference is that beyond R the total charge will remain the same and is not incrementing. That is, we have the following chain of operatons.

$$\begin{aligned}
\int \boldsymbol{E}_\rho \cdot d\boldsymbol{S} &= \frac{1}{\epsilon_0}\int \rho(\boldsymbol{r}_i) d\boldsymbol{r}_i \\
4\pi r^2 E_\rho &= -\frac{1}{\epsilon_0}\frac{\cancel{3}q}{\cancel{4\pi}R^3}\frac{\cancel{4\pi}R^3}{\cancel{3}} \\
E_\rho &= -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \\
\boldsymbol{E}_\rho &= -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}.
\end{aligned}$$

In terms of the dipole moment this is yields the result

$$\overline{\boldsymbol{E}} = -\frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}}{r^3}.$$

We see then that the average far field (for $r > R$) of a point charge is the same as if the charge would be located at the center of a sphere. By superposition we can change a single charge q by a general density function and conclude that the average field far away from an object with total charge Q is the same field as if the total charge Q would be located at the center of the sphere.

7.3 Problems

7.3.1 Problem 1

Use the discrete version of the dipole moment 7.7 to show that the dipole potential $V_{\text{dip}}(\boldsymbol{r})$ indeed corresponds with equation 7.1 derived in class.

7.3.2 Problem 2

Let us assume that we have two charges distributed as in equation 7.5.

$$\rho(\boldsymbol{r}_I) = (q+)\delta(x)\delta(y)\delta(z-d/2) + (q-)\delta(x)\delta(y)\delta(z+d/2).$$

Show that the monopole contribution $V_0(\boldsymbol{r})$ and the quadrapole contribution $V_2(\boldsymbol{r})$ for these two charges is 0 and so the interpretation of the expression $V_1(\boldsymbol{r})$ as a dipole is correct.

7.3.3 Problem 3

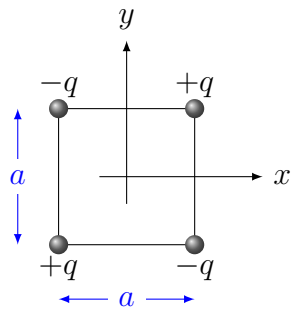

Figure 7.4: Four charges making a quadripole.

Use the following charge distribution (see Figure 7.4). All of them are located at the corners of a square $(\pm a/2, \pm a/2)$. The positive charges $+q$ are located in the vertices $\pm(a/2, a/2)$, and the negative charges in the other two vertices. Find the quadripole moment Q_{ij}. Show that the monopole as well as the dipole moments due to the four charges are 0.

Chapter 8

Lecture #8

8.1 Electric Fields in Matter

8.1.1 Polarization in The Micro

Conductors

The topic of **conductors** (as well as the topic of **capacitors**) is not included in the program for this class. Still we should have a few words about conductors.

In conductors we have a great deal of charges (electrons) that are free to move in the conductor. In conductors the charges all pile up in the surface of the conductor and arrange pushing each other away as much as possible leaving a uniform charge density distribution across the surface. This also creates a zero net charge distribution inside the volume of a conductor leaving a 0 electric field inside. We also studied the boundary conditions of the electrical field previously and stated that across a surface the tangential component of an electrical electrical field is continuous while the normal component is discontinuous. In fact it can be shown that the tangential component of the electrical field on a conductor is 0. We summarize the following four important characteristics of conductors:

(i) The electrical field inside a conductor is 0.

(ii) Any net charge must reside on the surface.

(iii) The tangential component of the electrical field is 0.

(iv) The normal component of the electrical field is σ/ϵ_0. We showed this in section 6.1.1.

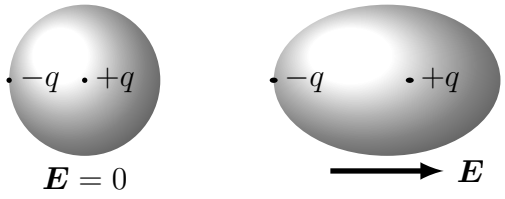

Figure 8.1: An atom on the left no external electric field applied. On the right we assume that an electric field was applied going in the direction from left to right. The negative charges are pushed to the left while the positive charges on the middle are shifted to the right.

Dielectrics

While in conductors electrons are free to move around at will, in dielectrics electrons are bound to atoms. They can displace but without leaving the atom. Hence, dielectrics are called also **insulators** . Still the cumulative displacement of electrons within an atom (or molecule) account for the behavior of dielectric materials. The two mechanisms produced by the displacement of electrons within an atom are **stretch** and **rotation.** We discuss how dipoles are induced in atoms and molecules due to electric fields, as well as permanent dipoles in molecules.

8.1.2 Induced Dipoles

Let us assume an ideal spherical neutral atom. We apply a uniform electric field $\boldsymbol{E}$. For example let us assume that the electric field goes from left to right. Then the electron cloud feel attraction of the electric field and shift a bit to the left, while the nucleus the atom, which is positively charged tends to be displaced to the right. Since the nucleus is more massive the displacement of it is relatively small compared to the displacement of the electron cloud. Figure 8.1 shows an sketch of this. This is an example of displacement of electrons by strecth. The actual shifting of the electron cloud is not as severe as in the picture; we exagerate it for illustration purposes. If the field is large enough it could push the electron out of the atom and make of it a positive ion but we will assume that this does not happen since otherwise the substance with these atoms would become a conductor. This will leave the atom **polarized** by having the plus charge shifted slightly to the right and the negative charge to the left. This effect creates an induced dipole moment which is approximately proportional to the field. This is true because the center of the electron cloud, which original coincided with the nucleus position, is displaced with respect to the

nucleus position. So we see the system as if we have two charges, the negative charged for all the electron cloud centered at some point displaced with respect to the nucleus where the center of the positive charge is located. If we remove the electric field, the system goes back to its original configuration. The proportionality constant α of this dipole moment is called the **atomic polarizability** and we write

$$\boldsymbol{p} = \alpha \boldsymbol{E}.$$

In molecules the problem could a bit more complicated and instead of a scalar we could talk about a matrix

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}.$$

and

$$\boldsymbol{p} = A\boldsymbol{E}.$$

Matrix A is known as the **polarization tensor.** While the atom we started with did not have a dipole moment but this was induced by the electric field, some molecules have built in a dipole moment.

For example let us think of a water molecule. Figure 8.2 sketches a water molecule. Here the displacement of the electrons is by rotation.

8.1.3 Permanent Dipoles

The example of a perfect spherical cloud of electrons around a nucleus at the center of the sphere is an ideal. In general in molecules the symmetry is lost and the center of of the electron clouds for the atoms in the molecule does not align with the center of the nuclei of the atoms. Again, figure 8.2 illustrates this situation. In this sense it is that there is a dipole thought as the two net charges with a displacement which is measured as the distance between the two centers.

We see how the electric field acting on a polarization vector creates a torque. Let us refer to the figure 8.2 for this. The moment dipole $\boldsymbol{p}$ is given by

$$\boldsymbol{p} = \boldsymbol{p_1} + \boldsymbol{p}_2 = \boldsymbol{r}_1 q_1 + \boldsymbol{r}_2 q_2.$$

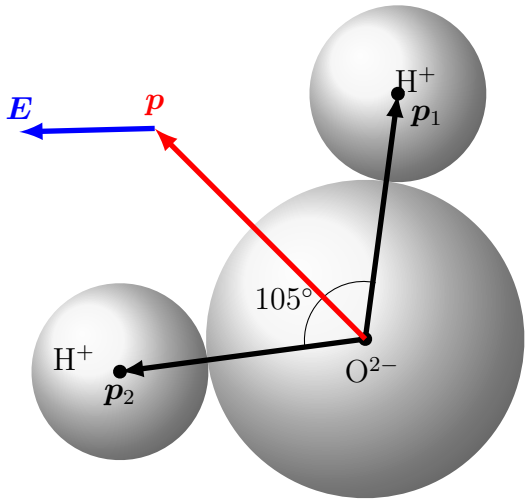

Figure 8.2: Two moment dipoles $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, and the resultant (red moment dipole) $\boldsymbol{p}$ showing the dipole moment of the water molecule. An electric field $\boldsymbol{E}$ (in blue) would generate a torque $\tau = \boldsymbol{p} \times \boldsymbol{E}$ which pushes a rotation in order to align the electrical field with the dipole moment.

Here $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the distances from the center of the oxigen atom to the center of each of the hidrogen atoms. Since $q_1 = q_2 = q$ we can write

$$\boldsymbol{p} = \boldsymbol{r}q$$

where $\boldsymbol{r} = \boldsymbol{r}_1 + \boldsymbol{r_2}$.

Now let us assume that we apply a constant field $\boldsymbol{E}$ (blue in the figure) and so we see that, since the electric force is $\boldsymbol{F}_E = q\boldsymbol{E}$,

$$\boldsymbol{\tau}_E = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times q\boldsymbol{E} = \boldsymbol{p} \times \boldsymbol{E}.$$

In general any given dipole moment $\boldsymbol{p} = q\boldsymbol{d}$ under a uniform field $\boldsymbol{E}$ experiences a torque

$$\boldsymbol{\tau}_E = \boldsymbol{p} \times \boldsymbol{E}, \tag{8.1}$$

which we call the **electric torque** due to a dipole moment $\boldsymbol{p}$.

This torque tries to align by rotating the water molecule with the electric field direction $\boldsymbol{E}$. In general in materials this happens for each molecule and we get a compound effect of all particles under the influence of the electric field $\boldsymbol{E}$.

8.1.4 Polarization in the Macro

We now can think on a macroscopic level. Assume a material composed of many (N) atoms or molecules. Each particle can have an intrinsic polarization or an induced polarization. Hence we can think of many dipole moments (each for particle). The set of all dipole moments can create an electric field as a whole and we want to be able to estimate this electric field. We define **polarization** (or better polarization density) to the total dipole created by all this particles per unit of volume. That is,

$$\boldsymbol{P} = \frac{1}{\text{Vol}} \sum_{i=1}^{n} \boldsymbol{p}_i.$$

In general a dipole moment $\Delta \boldsymbol{p}$ can change from point to point in a dielectric and we can introduce the differential form

$$\boldsymbol{P} = \frac{d\boldsymbol{p}}{dV}.$$

As a result of polarization a net charge Q_b can appear in a dielectric. This charge is called bound charge. All charges inside (if aligned properly) can cancel each other since next to a positive charge is a negative charge. Only the charges in the surface will not have a neighbour beyond the surface to cancel out. Let us take an isolated differential piece of volume dV inside the dielectric. Due to the polarization the positive bound charge $+dq_b$ is displaced a distance $\boldsymbol{d}$ with respect to the negative bound charge $-dq_b$, giving rise to the dipole moment

$$d\boldsymbol{p} = dq_b \boldsymbol{d},$$

and since $\boldsymbol{P} = d\boldsymbol{p}/dV$ then

$$\boldsymbol{P} = \frac{dq_b}{dV}\boldsymbol{d} = \rho_b \boldsymbol{d},$$

where ρ_b is the density of the bound charge in the volume we are considering. Note that we are assuming that $\boldsymbol{d}$ is constant.

8.1.5 The Field of a Polarized Object

Bound Charges

Let us suppose a polarized dielectric. That is, we have a large set of microscope aligned dipoles. We want to find out what is the electric field that these polarized (bound) charges produces. Since we know what the field of a single dipole is, we can use superposition to add up all the dipole contributions on the object. The potential is easier to handle that the electric field, then we work with the potential. The potential due to a single dipole $\boldsymbol{p}$ is given by equation 7.8 which we re-write here

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}} \cdot \boldsymbol{p},$$

where $\boldsymbol{r}$ is the radio vector from the dipole to observation point. That is if $\boldsymbol{r}_i$ is the location of particular dipole center inside the object that we are studying we need to take into account this center as a new center of coordinates and write the expression above as

$$\frac{1}{4\pi\epsilon_0 r_{io}^2}\hat{\boldsymbol{r}}_{io} \cdot \boldsymbol{p}(\boldsymbol{r}_i),$$

where

$$r_{io} = \|\boldsymbol{r} - \boldsymbol{r}_i\| \quad , \quad \hat{\boldsymbol{r}}_{io} = \frac{\boldsymbol{r} - \boldsymbol{r}_i}{\|\boldsymbol{r} - \boldsymbol{r}_i\|}.$$

Now, since $\boldsymbol{P}(\boldsymbol{r}_i)$ is the polarization density we need to find the integral

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \cdot \boldsymbol{P}(\boldsymbol{r}_i). \tag{8.2}$$

We illustrate this with an example.

Example 8.1.1 *Find the electric field due to a uniformly polarized sphere with radius R.*

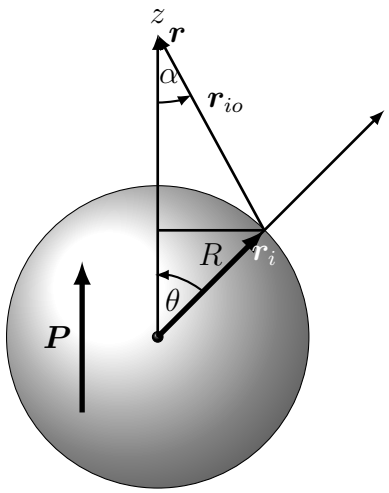

Figure 8.3: A uniformly polarized sphere.

Solution:

(i) Let us first assume $r \geq R$. Since $\boldsymbol{P}$ is constant then we can pull it out of the integral to find

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\boldsymbol{P} \cdot \int \frac{1}{r_{io}^2}\hat{\boldsymbol{r}}_{io} d\boldsymbol{r}_i$$

with $\boldsymbol{r}_{io} = \boldsymbol{r} - \boldsymbol{r}_i$. The integral depends only on $\boldsymbol{r}$ and $\boldsymbol{r}_i$, and due to the fact that the integration is along a sphere we do not loose generality if we choose $\boldsymbol{r}$ along the z axis. The angle between $\boldsymbol{r}$ and $\boldsymbol{r}_i$ is then the polar angle θ (shown in the Figure 8.3).

From symmetry we see that all components, but the z component are cancelled. We use spherical coordinates. We then have that $\|\boldsymbol{r}\| = z$, and

$$r_{io} = \sqrt{z^2 + R^2 - 2zR\cos\theta}.$$

Since we need the z component of the vector $\hat{\boldsymbol{r}}_{io}$ this can be simply $\cos\alpha$, for some angle α. That is

$$\cos\alpha = \frac{z - R\cos\theta}{r_{io}} = \frac{z - R\cos\theta}{\sqrt{z^2 + R^2 - 2zR\cos\theta}}.$$

Given that $d\boldsymbol{r}_i = r_i^2 \sin\theta \, dr_i \, d\theta d\phi$, and the problem is azimuthally symmetric we find $d\boldsymbol{r}_i = 2\pi r_i^2 \sin\theta dr_i \, d\theta$ and so

$$V(\boldsymbol{r}) = \frac{1}{2\epsilon_0}\boldsymbol{P}\cdot\hat{\boldsymbol{z}}\int_0^R\int_{-\pi/2}^{\pi/2}\frac{z - R\cos\theta}{(z^2+R^2-2zR\cos\theta)^{3/2}}\, r_i^2 \, \sin\theta dr_i d\theta \tag{8.3}$$

We can solve the r_i integral, to find

$$V(\boldsymbol{r}) = \frac{R}{3}\frac{1}{2\epsilon_0}\boldsymbol{P}\cdot\hat{\boldsymbol{z}}\int_{-\pi/2}^{\pi/2}\frac{(z - R\cos\theta)R^2}{(z^2+R^2-2zR\cos\theta)^{3/2}}\, r_i^2 \, \sin\theta dr_i d\theta$$

We already solved this integral. Observe equation 1.8 That is, we found:

$$\begin{aligned} E &= \frac{\sigma}{2\epsilon_0}\int_{-\pi/2}^{\pi/2}\frac{(o - R\sin\theta)R^2\sin\theta}{(R^2\sin^2\theta + (o - R\sin\theta)^2)^{3/2}}d\theta \\ &= \frac{\sigma}{2\epsilon_0}\int_{-\pi/2}^{\pi/2}\frac{(o - R\sin\theta)R^2\sin\theta}{(o^2+R^2-2oR\sin\theta)^{3/2}} \end{aligned}$$

where instead of o we have here z and $\sigma = 1$. The result of the evaluation of that integral (for $o = z > R$) is

$$E = \frac{\sigma R^2}{z^2\epsilon_0},$$

where we already replaced o by z. With this we find (recall the $R/3$ left outside the integral above) that

$$V(\boldsymbol{r}) = \frac{P\cos\theta R^3}{3z^2\epsilon_0} \tag{8.4}$$

(ii) Let us now assume that $r < R$. We return to integral 8.3

$$\begin{aligned} V(\boldsymbol{r}) &= \frac{1}{2\epsilon_0}\boldsymbol{P}\cdot\hat{\boldsymbol{z}}\int_0^{r=z}\int_{-\pi/2}^{\pi/2}\frac{z - R\cos\theta}{(z^2+R^2-2zR\cos\theta)^{3/2}}\, r_i^2 \, \sin\theta dr_i d\theta \\ &= \frac{z^3}{3}\frac{1}{2\epsilon_0}\boldsymbol{P}\cdot\hat{\boldsymbol{z}}\int_{-\pi/2}^{\pi/2}\frac{z - R\cos\theta}{(z^2+R^2-2zR\cos\theta)^{3/2}}\sin\theta d\theta \end{aligned}$$

We already know the evaluation of the integral on the right. The R^3 is replaced by z^3. Then a simple modification of equation 8.4 yields

$$V(\boldsymbol{r}) = \frac{P\cos\theta z}{3\epsilon_0}$$

In conclusion we find, replacing now z by r, that

$$V(r) = \begin{cases} \dfrac{P\cos\theta r}{3\epsilon_0} & \text{if } r < R \\ \dfrac{P\cos\theta R^3}{3r^2\epsilon_0} & \text{if } r \geq R \end{cases}$$

The way to solve this problem is complicated to the evaluation of the integral. Griffiths [1] (Exercise 3.9) shows an elegant way to find the potential everywhere by knowing its value in the surface of a sphere and using separation of variables on the Laplace's equation with the known solution in terms of powers of r and Legendre polynomials.

We return to equation 8.2 and show a different way to solve for the electric field potential.

$$\nabla\left(\frac{1}{r_{io}}\right) = \frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2},$$

then

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \nabla\left(\frac{1}{r_{io}}\right)\cdot\boldsymbol{P}(\boldsymbol{r}_i)d\boldsymbol{r}_i.$$

We now use multidimensional integration by parts. That is, from

$$\nabla\cdot\left(\frac{1}{r_{io}}\boldsymbol{P}(\boldsymbol{r}_i)\right) = \nabla\left(\frac{1}{r_{io}}\right)\cdot\boldsymbol{P}(\boldsymbol{r}_i) + \frac{1}{r_{io}}\nabla\cdot\boldsymbol{P}(\boldsymbol{r}_i)$$

we see that

$$\begin{aligned} V(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{r}}}{r_{io}^2}\cdot\boldsymbol{p}(\boldsymbol{r}_i) \\ &= \frac{1}{4\pi\epsilon_0}\left[\int \nabla\cdot\left(\frac{1}{r_{io}}\boldsymbol{P}(\boldsymbol{r}_i)\right)d\boldsymbol{r}_i - \int \frac{1}{r_{io}}\nabla\cdot\boldsymbol{P}(\boldsymbol{r}_i)d\boldsymbol{r}_i\right]. \end{aligned}$$

We now apply the divergence theorem to write

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_S \frac{1}{r_{io}}\boldsymbol{P}(\boldsymbol{r}_i)\cdot d\boldsymbol{S} - \int \frac{1}{r_{io}}\nabla\cdot\boldsymbol{P}(\boldsymbol{r}_i)d\boldsymbol{r}_i\right].$$

where S is a surface surrounding the volume of integration. We can interpret the first integral as an integral over surface charges with the definition

$$\sigma_b = \boldsymbol{P}\cdot\hat{\boldsymbol{n}}, \tag{8.5}$$

and the second as integral over volume charges with the definition

$$\rho_b = -\nabla\cdot\boldsymbol{P}. \tag{8.6}$$

That is, we can write the integral above as

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_S \frac{\sigma_b(\boldsymbol{r}_i)}{r_{io}}dS + \int \frac{\rho_b(\boldsymbol{r}_i)}{r_{io}}d\boldsymbol{r}_i\right]. \tag{8.7}$$

8.1.6 The Field inside a Dielectric

Equation 8.2 provides a formula to find the potential field due to some polarization density function $\boldsymbol{P}(\boldsymbol{r})$. This formula is an approximation obtained by using the principle of superposition in the continuum. It is a formula in the macro. We can not get infinitesimally close to a charged particle because this creates an infinite strength field. There is a limit of how close we can get to a particle. Due to the complexity of matter we need to do some assumptions and the quantities that we compute here are averages. We then need to set up a scale of "resolution" where we have some statistical confidence on our computations. Given some resolution length (say a distance many times the radius of an atom) we want to take an average of our field over a sphere with radius of resolution r and find approximations for the field both inside and outside of this sphere. For the inside we can use equation 7.12

$$\overline{\boldsymbol{E}}_{in} = -\frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}}{R^3}. \tag{8.8}$$

for the average field inside of the dipole moment $\boldsymbol{p}$ due to some charge density $\rho(\boldsymbol{r})$, and for the outside we use equation 8.2; that is, from the potential

$$V_{\text{out}}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2}\cdot\boldsymbol{p}(\boldsymbol{r}_i)$$

we find the field $\boldsymbol{E}_{\text{out}}$. Since the sphere is small enough we assume that there is no much variation of $\boldsymbol{P}$ inside the sphere and so it can be seen as uniformly polarized. So, from the computation of the potential found in equation 8.5

$$V(r) = \frac{Pr\cos\theta}{3\epsilon_0} = \frac{\boldsymbol{P}\cdot\boldsymbol{n}}{3\epsilon_0},$$

and the electric field corresponding is equal to

$$\boldsymbol{E}(r) = -\nabla V(r) = -\frac{\boldsymbol{P}}{3\epsilon_0}.$$

Furthermore, since, by definition, the polarization vector $\boldsymbol{P}$ is the dipole volume density we have the relation $\boldsymbol{p} = \left(\frac{4}{3}\pi R^3\right)\boldsymbol{P}$ we can write equation 8.8 as

$$\overline{\boldsymbol{E}}_{in} = -\frac{1}{3\epsilon_0}\boldsymbol{P}.$$

In any case (inside or outside of a small sphere) we find the same result. The macroscopic potential can then be computed from equation 8.2 over the whole volume of the object.

8.1.7 The Electric Displacement

Gauss's Law

Within a dialectric we can write

$$\rho = \rho_b + \rho_f$$

where ρ_b are the bound charges and ρ_f are the free charges. The free charges can be tought of as embedded ions. Gauss's law states that

$$\epsilon_0\nabla\cdot\boldsymbol{E} = \rho = \rho_b + \rho_f = -\nabla\cdot\boldsymbol{P} + \rho_f.$$

We can combine the divergence terms to produce

$$\nabla\cdot(\epsilon_0\boldsymbol{E} + \boldsymbol{P}) = \rho_f.$$

The expression in parentheis is known as the **electrical displacement** and noted as

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}. \tag{8.9}$$

Then we still can claim that

$$\nabla \cdot \boldsymbol{D} = \rho_f,$$

or in integral form

$$\oint \boldsymbol{D} \cdot d\boldsymbol{S} = Q_{f_{\text{enc}}} \tag{8.10}$$

where $Q_{f_{\text{enc}}}$ represents the total free charges enclosed in the volume.

We remark that while $\nabla \times \boldsymbol{E} = 0$, this does not happen with $\boldsymbol{D}$. From the definition of $\boldsymbol{D}$,

$$\nabla \times \boldsymbol{D} = \epsilon_0(\nabla \times \boldsymbol{E}) + (\nabla \times \boldsymbol{P}) = (\nabla \times \boldsymbol{P}).$$

Since it could happen that $\nabla \times \boldsymbol{D}$ could be non-zero, we find that there is no scalar potential for $\boldsymbol{D}$.

Boundary Conditions

We can use equation 8.10 to find the boundary conditions on a surface of a dielectric.

Let us assume a small pill box on the surface of the dielectric (see Figure 8.4) . The thickness δh of the pill box goes to zero and the integral evalues to

$$\left(D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}}\right) \Delta S = Q_{f_{\text{enc}}}$$

Now the total charge $Q_{f_{\text{enc}}}$ in the limit as the tickness of the pill box goes to zero becomes $\Delta S \sigma_f$. Then we find that

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f.$$

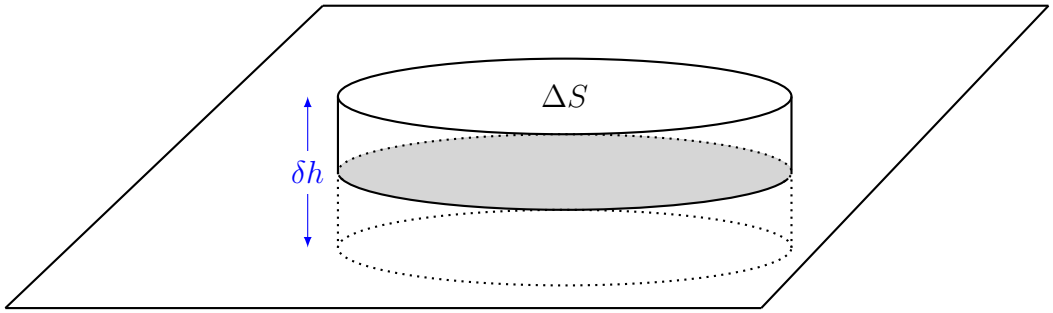

Figure 8.4: A Gaussian pillbox illustrating the evaluation of the normal component of the boundary condition of the displacement field.

For the parallel component of D, we look at the line integral around a rectangle that surounds the boundary with two small sides of length δh, and the other two sides parallel to the surface (see Figure **??**).

From Stokes theorem we know that $\nabla \times \boldsymbol{E} = 0$ and from this that the tangential components are continuous across the boundary. However, in general $\nabla \times \boldsymbol{P} \neq 0$, so we cannot conclude that the tangential comonents of $\boldsymbol{D}$ are continuous across the boundary. From Stokes theorem the circulation integral of $\boldsymbol{D}$ along the rectangle is equal to the integral of $\nabla \times \boldsymbol{D}$ over the area of the rectangle depicted in the figure. Now since $\nabla \times \boldsymbol{D} = \nabla \times \boldsymbol{P}$ all we can say is that the integral along the rectangular loop of $\boldsymbol{D}$ is equal to the integral over the rectangular loop of $\boldsymbol{P}$. Now as $\delta h \to 0$, we are left we two opposite segments of the rectangle. That is, in the evaluation of this limit we have

$$\ell(D_{\parallel}^{\text{above}} - D_{\parallel}^{\text{below}}) = \ell(P_{\parallel}^{\text{above}} - P_{\parallel}^{\text{below}}).$$

and cancelling ℓ,

$$D_{\parallel}^{\text{above}} - D_{\parallel}^{\text{below}} = P_{\parallel}^{\text{above}} - P_{\parallel}^{\text{below}}.$$

Linear Dielectrics

We disussed in the previous sections how polarization $\boldsymbol{P}$ lines up the atomic or molecular dipoles. For many materials polarization is proportional to the electric field and this is written as the equation

$$\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}. \tag{8.11}$$

The proportionality constant χ_e is known as the **electric susceptibility.** In vacuum, where there is no matter to polarize, the suceptibility is zero. The factor ϵ_0 is left to make χ_e dimensionless. This new constant χ_e depends on the microscopic structure of the substance as well as external conditions such as temperature, for example. Materials that satisfy 11.7 are known as **linear dielectrics** . Equation 11.7 is a first order (linear) approximation which is valid for some materials but in general we can think of the function $\boldsymbol{P}$ as a Taylor series in $\boldsymbol{E}$. We can now use equation 8.9 and write

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} = \epsilon_0 \boldsymbol{E} + \epsilon_0 \chi_e \boldsymbol{E} = \epsilon_0 (1 + \chi_e) \boldsymbol{E}.$$

We see then that $\boldsymbol{D}$ is also proportional to $\boldsymbol{E}$ with a constant of proportionality

$$\epsilon = \epsilon_0 (1 + \chi_e).$$

That is we can write

$$\boldsymbol{D} = \epsilon \boldsymbol{E}. \tag{8.12}$$

This new constant ϵ is called the **permitivity** of the material. Since in vacuum the suceptibility is zero we have that $\epsilon = \epsilon_0$. This is why ϵ_0 is known as the **permittivity of free space.** The factor $\epsilon_r = 1 + \chi_e$ is known as relative permitity since we can write

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0}.$$

Sometimes ϵ_r is also known as the **dielectric constant of the material.** Equation 8.12 is known as the **constitutive** relation between the displacement field and the electrical field. This is like Hook's law in elasticity where the strain τ is linearly related to stress ϵ through a similar equation where the permitivity ϵ here plays the role of compliance C there.

Homogeneity and Isotropy

Homogeneity The word **homogeneous** means that the properties do not change in space. In this case we can think of χ_e which is not a constant. Its spacial derivatives are zero. The word **inhomogeneous** means that the properties change with space and we can not ignore its spacial derivatives.

At this point we know that

$$\nabla \cdot D = \rho_f \quad , \quad \nabla \times \boldsymbol{D} = \nabla \times \boldsymbol{P}.$$

We commented out above that the curl of $\boldsymbol{D}$ could be non-zero. Let us now see when $\nabla \times D = 0$. We assume that we have a dielectric which is homogeneous then from the linear relationship between $\boldsymbol{P}$ and $\boldsymbol{E}$ (equation 8.12)

$$\nabla \times P = \nabla(\epsilon_0 \chi_e \boldsymbol{E}) = \epsilon_0 \chi_e \nabla \times \boldsymbol{E} = 0,$$

so in homogeneous media

$$\nabla \cdot \boldsymbol{D} = 0 \quad , \quad \nabla \times \boldsymbol{D} = 0.$$

On the other hand, if the medium is inhomogeneous we have, using equation A.5 that

$$\nabla \times \boldsymbol{P} = \nabla \epsilon_0 \chi_e \boldsymbol{E} = \epsilon_0 \nabla \times (\chi_e \boldsymbol{E}) = \epsilon_0 \nabla \chi_e \times \boldsymbol{E} + \chi_e \nabla \times \boldsymbol{E} = \epsilon_0 \nabla \chi_e \times \boldsymbol{E}.$$

Of course, if $\nabla \chi_e = 0$ then $\nabla \times \boldsymbol{P} = 0$ and it would be conservative as well as $\boldsymbol{D}$.

We now show that if the medium is homogeneous the bound charge density ρ_b is proportional to the free charge density ρ_f. That is,

$$\rho_b = -\nabla \cdot \boldsymbol{P} = -\nabla \cdot .(\epsilon_0 \chi_e \boldsymbol{E}) = -\nabla \cdot \left(\frac{\epsilon_0 \chi_e}{\epsilon}\right) \boldsymbol{D} = -\left(\frac{\chi_e}{1+\chi_e}\right) \rho_f$$

Isotropy An dialectric is said isotropic if its properties do not change with direction. It could happen that the direction of $\boldsymbol{D}$ and that of $\boldsymbol{E}$ do not align. This happens when χ_e is a matrix (tensor) instead of a scalar. In this case we say that the consitutive equation 8.12 equation is anisotropic where

$$\chi_e = \begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{pmatrix} \cdot$$

Here χ_e is known as the **susceptibility tensor** . it is common to see the matrix ϵ in the constitutive relation

$$\epsilon = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix}$$

where, of course $\epsilon_{ij} = \epsilon_0(1 + \chi_{ij})$. These matrices should be symmetric but we will not prove that at this time.

8.1.8 Energy in Dielectric Systems

Let us first assume that we want to move all charges (free and bound) from infinity to a final configuration. We already know the formula 6.5

$$W = \frac{\epsilon_0}{2} \int_{\text{Vol}} E^2 d\boldsymbol{r}. \tag{8.13}$$

We want to extend this formula to the case of a dialectric. When computing this formula we moved all the charges $\rho = \rho_b + \rho_f$. For dialectrics we only move free charges.

Let us assume a dielectric material where we want to bring free charges by little chunks. We have in certain time ρ_f and an instant later $\rho_f + \Delta\rho_f$. The polarization changes and so its bound charge distribution. However, at the moment we are only interested in the work done on the incremental free charge $\Delta\rho$. From equation 6.4 we find

$$\Delta W = \int \Delta\rho_f(\boldsymbol{r}_i) V(\boldsymbol{r}_i) d\boldsymbol{r}_i.$$

Since $\nabla \cdot \boldsymbol{D} = \rho_f$, then $\Delta\rho_f = \nabla \cdot (\Delta \boldsymbol{D})$. Here $\Delta\boldsymbol{D}$ is the increment on $\boldsymbol{D}$. We have then

$$\Delta W = \int [\nabla \cdot \Delta \boldsymbol{D}] V(\boldsymbol{r}_i) d\boldsymbol{r}_i$$

We want to use multi-dimensional integration by parts. That is, given that

$$\nabla \cdot [(\Delta \boldsymbol{D})V(\boldsymbol{r}_i)] = [\nabla \cdot \Delta \boldsymbol{D}]V(\boldsymbol{r}_i) + \Delta \boldsymbol{D} \cdot \nabla V(\boldsymbol{r}_i).$$

Then

$$\Delta W = \int \nabla \cdot [(\Delta \boldsymbol{D})V(\boldsymbol{r}_i)]d\boldsymbol{r}_i - \int \Delta \boldsymbol{D} \cdot \nabla V(\boldsymbol{r}_i)d\boldsymbol{r}_i$$

But, since $\nabla V = -\boldsymbol{E}$, we have that

$$\Delta W = \int \nabla \cdot [(\Delta \boldsymbol{D})V(\boldsymbol{r}_i)]d\boldsymbol{r}_i + \int \Delta \boldsymbol{D} \cdot E(\boldsymbol{r}_i)d\boldsymbol{r}_i$$

We convert the first integral, using the divergence theorem, to a surface integral and this surface integral vanishes as the surface resides to infinity. Then we find

$$\Delta W = \int \Delta \boldsymbol{D} \cdot \boldsymbol{E}(\boldsymbol{r}_i)d\boldsymbol{r}_i.$$

This equation is general for any dielectric. If we have a linear dielectric then $\boldsymbol{D} = \epsilon \boldsymbol{E}$ and so, from

$$\frac{1}{2}\Delta(\boldsymbol{D} \cdot \boldsymbol{E}) = \frac{1}{2}\Delta(\epsilon E^2) = \epsilon \Delta \boldsymbol{E} \cdot \boldsymbol{E} = \Delta \boldsymbol{D} \cdot \boldsymbol{E},$$

where we assume that $\Delta\epsilon = 0$,

$$\Delta W = \Delta \frac{1}{2} \int \boldsymbol{D}(\boldsymbol{r}_i) \cdot \boldsymbol{E}(\boldsymbol{r}_i)d\boldsymbol{r}_i.$$

The total work would be the integral of this expression which yields

$$W = \frac{1}{2} \int \boldsymbol{D}(\boldsymbol{r}_i) \cdot \boldsymbol{E}(\boldsymbol{r}_i)d\boldsymbol{r}_i.$$

We observe that in absence of dielectrics where $\epsilon = \epsilon_0$ this equation reduces to equation 8.13 above.

8.2 Problems

8.2.1 Problem 1

Let us assume that a sphere of radius R carries a polarization

$$\boldsymbol{P}(\boldsymbol{r}_i) = k\boldsymbol{r}_i,$$

where k is a constant and $\boldsymbol{r}_i$ is the vector from the center. Compute the bound charges σ_b and $\boldsymbol{\rho}_b$ and from there find the electrical field both inside and outside of the sphere.

Chapter 9

Lecture #9

9.1 Magnetostatics

Before we can explain the concept of magnetostatics we need to define what electric current means.

9.1.1 Electric Current

By **electric current** we define the flow of electric charge. This flow could be thought in a wire or through a surface. The flow of current can be though of as moving electrons or ions, or both. In a wire we have motion of electrons and in an electrolyte we have motion of ions.

When we think about charges passing through a surface we can define current as the amount of charge divided by time. We note current by the simbol I and write

$$I = \frac{Q}{t},$$

where Q is the total amount of charges passing through a surface and t is the time. At this moment we have not defined surface and we think of charges running through a wire. The standard metric unit for current is the **ampere** (noted Amp) which is Coulomb/sec. That is 1 Amp means that 1 Coulomb is passing through the cross-section of a wire in one second. By convention the direction of the electric current is that of the direction where positive charges will move. In practice the electrons are the charges that move and this make the convention opposite to common sense flow but this is the way that Benjamin Franklin assumed to start with and the one that is used today.

We can consider three type of charge flow.

(i) **Charges flowing through a wire.** Let us assume a charge density

$$\lambda = \frac{Q}{\Delta \ell}$$

where Q is the total charge in a line element $\Delta \ell$. Then from the definition of current we have that

$$I = \frac{\lambda \Delta \ell}{\Delta t} = \lambda v.$$

Since the wire is not necessarily straight we should consider this a vector equation

$$\boldsymbol{I} = \lambda \boldsymbol{v}.$$

(ii) **Charges flowing along a sheet.** In this case we consider a sheet with surface density

$$\sigma = \frac{Q}{\Delta S},$$

and then, from the definition of current

$$I = \frac{\sigma \Delta S}{t}.$$

We think of ΔS as a small rectangle. The cross section (normal to the charge flow) of the rectangle we can call $\ell_\perp$ and the side of the rectangle along the charge flow $\ell_\parallel$. That is we can write

$$I = \frac{\sigma\, \ell_\parallel \ell_\perp}{t}. = \sigma v \ell_\perp,$$

with $v = \ell_{\parallel}/t$ the velocity of propagation of the current normal to the cross segment $\ell_{\perp}$. We can introduce a new definition of charge density in the 2D case as

$$\boldsymbol{K} = \sigma \boldsymbol{v}$$

where

$$K = \frac{I}{\ell_{\perp}}.$$

(iii) **Charges flowing across a surface.** This is the 3D case considered in the next section.

9.1.2 Current Density

Related to electric current is the current density. We want to generalize the current through paths other than wires. We can think of currents crossing surfaces. We want to estimate the amount of current passing through a surface in a given amount of time. For that reason a vector field called **current density** and noted by $\boldsymbol{J}$ is defined as the current per unit area, or in terms of limits we can write

$$\boldsymbol{J} = \lim_{S \to 0} \frac{I(S)}{S}.$$

where S is the area, normal to the current displacement.

Let us assume that the charges are moving at a speed v and passing perpendicularly through little piece of surface with area S. At an interval of time Δt, the amount of charge that moves through a surface S is given by

$$Q = \rho V = Sv\Delta t\rho$$

where ρ is the charge density and the volume of charge fluid is given by $Sh = Sv\Delta t$ with $h = v\Delta t$ the total hegiht of the 3D volume element with base area S. Then we have that

$$\frac{Q}{\Delta t S} = \rho\, v.$$

The expression on the left as S is small is the current density. In this case we found the relation

$$\boldsymbol{J} = \rho \boldsymbol{v}. \tag{9.1}$$

where we already assumed that $\boldsymbol{v}$ is normal to the surface of displacement. We see then that current density is the amount of current per unit of time, per unit of area, passing through a surface. This concept is konwn in physics also as **flux** of electrical charge. If we want to find the total amount of charge passing through a surface with some velocity $\boldsymbol{v}$ given that we know the current $\boldsymbol{J}$ we need to integrate $\boldsymbol{J}$ over the surface and over time. That is

$$Q = \int_{t_0}^{t_1} \int_S \boldsymbol{J} \cdot \hat{\boldsymbol{n}} \, dS \, dt.$$

From the fundamental theorem of calculus we find there that

$$\frac{dQ}{dt} = \int_S \boldsymbol{J} \cdot \hat{\boldsymbol{n}} dS.$$

We now assume that the direction $\hat{\boldsymbol{n}}$ points outside the surface. The integral on the right is the flux through the surface S, and the rate of flux going out is opposite in sign with the change of total charge Q inside, since as the charge goes out its quantity diminshes. In other words if the vector $\hat{\boldsymbol{n}}$ points inside we will have the same sign of rate of change of charge versus flow. We can then assume $-\hat{\boldsymbol{n}}$ instead and write

$$\frac{dQ}{dt} = -\int_S \boldsymbol{J} \cdot \hat{\boldsymbol{n}} dS.$$

Now we know that $Q = \int \rho dV$, so using the divergence theorem we find

$$\frac{d}{dt} \int_V \rho \, dV = -\int_V \nabla \cdot \boldsymbol{J} dV$$

We want to take the differential inside the integral on the left hand side. Note that the integral is function only of time since the space coordinates $\boldsymbol{x}$ are dummy, in addition

the limits (surface surrounding the volume of integration) is fixed; that is, it does not depend on time). Therefore, the integration is over $\boldsymbol{x}$. If we take the differentiation inside, then we are thinking of $\rho(\boldsymbol{x}, t)$, and we need to consider only the differentiation with respect to t as a partial derivative. Please see the Leibniz integral rule [1] for an explanation of how to take the total derivative outside into a partial derivative inside the integral sign. We then find that

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} \right) dV = 0.$$

where we moved all terms to the left. Since this is done for an arbitrary volume of integration V we find that

$$\frac{d\rho}{dt} + \nabla \cdot \boldsymbol{J} = 0. \tag{9.2}$$

This equation is known as the continuity equation for the charge.

In general we can consider the three cases of current density.

1. **For 1D :**

$$\text{current density} = \text{current} = \boldsymbol{I} = \lambda \boldsymbol{v}.$$

2. **For 2D :**

$$\text{current density} = \text{current/cross} - \text{line element} = \boldsymbol{K} = \frac{\boldsymbol{I}}{\ell_\perp} = \sigma \boldsymbol{v}.$$

3. **For 3D :**

$$\text{current density} = \text{current/cross} - \text{area element} = \boldsymbol{J} = \frac{\boldsymbol{I}}{\Delta S_\perp} = \rho \boldsymbol{v}.$$

[1] https://en.wikipedia.org/wiki/Leibniz_integral_rule

9.1.3 The Biot-Savart Law

There is a parallel between charges for electric fields and currents for magnetic fields. Stationary (not moving) charges produce electric fields which are time independent. We will observe that steady currents (electric currents which are time independent) produce constant magnetic fields and hence the term magnetostatics. In practice there is no such a thing as a steady current which by definition would be there for ever, which does not have a start or end point. However the concept is useful to understand the theory and as a step to get to the more realistic problem of non steady currents. The steady current (magnetostatics) approximation has proven good for some practical purposes. It is then required in magnetostatics that $\partial\rho/\partial t = 0$, and from the continuity equation we have that

$$\Delta \cdot \boldsymbol{J} = 0.$$

The Biot-Savart law states that the magnetic field for a steady line current is given by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{I}\times\hat{\boldsymbol{r}}_{io}}{r_{io}^2}d\ell = \frac{\mu_0 I}{4\pi}\int \frac{d\boldsymbol{\ell}\times\hat{\boldsymbol{r}}_{io}}{r_{io}^2}. \tag{9.3}$$

where we used $\boldsymbol{I}d\ell = Id\boldsymbol{\ell}$. Please observe the parallel of this an Coulomb's law. Here instead of charges we deal with currents and instead of a regular product between charge and input-output radio vector we use a cross-product between the current and the relative (input-output) location vector. The integration is a line integration because we are dealing with line currents. This law serves the puppose of introduceing the magnetic field. Here μ_0 is a constant known as **permeability of free space** with value

$$\mu_0 = 4\pi \times 10^{-7}\mathrm{N/A}^2. \tag{9.4}$$

With these units $\boldsymbol{B}$ comes out in newtons per ampere-meter or **teslas** (T).

Example 9.1.1 *Assume a steady current $\boldsymbol{I} = I\hat{\boldsymbol{z}}$ in a wire the direction of the positive z axis. Find the magnetic field a distance s from the wire.*

solution: We use the Biot-Savart law

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\boldsymbol{\ell}\times\hat{\boldsymbol{r}}_{io}}{r_{io}^2}.$$

Refer to Figure 9.1. The direction of the cross product is normal to both the vector $d\boldsymbol{\ell}$ and to $\boldsymbol{r}_{io}$. If in the figure we think about both $d\boldsymbol{\ell}$ and $\boldsymbol{r}_{io}$ are in the page ($x-z$ axis), then the direction of $\boldsymbol{B}$ would go inside the page, and if $\boldsymbol{r}_{io}$ goes inside the page then the direction of $\boldsymbol{B}$ goes to the left. That is, $\boldsymbol{B}$ is tangential to the circle in the figure and it suggests a rotation counter-clockwise as seen from below. If we think about cylindrical coordinates, the field $\boldsymbol{B}$ should be written as $\boldsymbol{B} = B\hat{\boldsymbol{\phi}}$, where ϕ is the azimuthal angle.

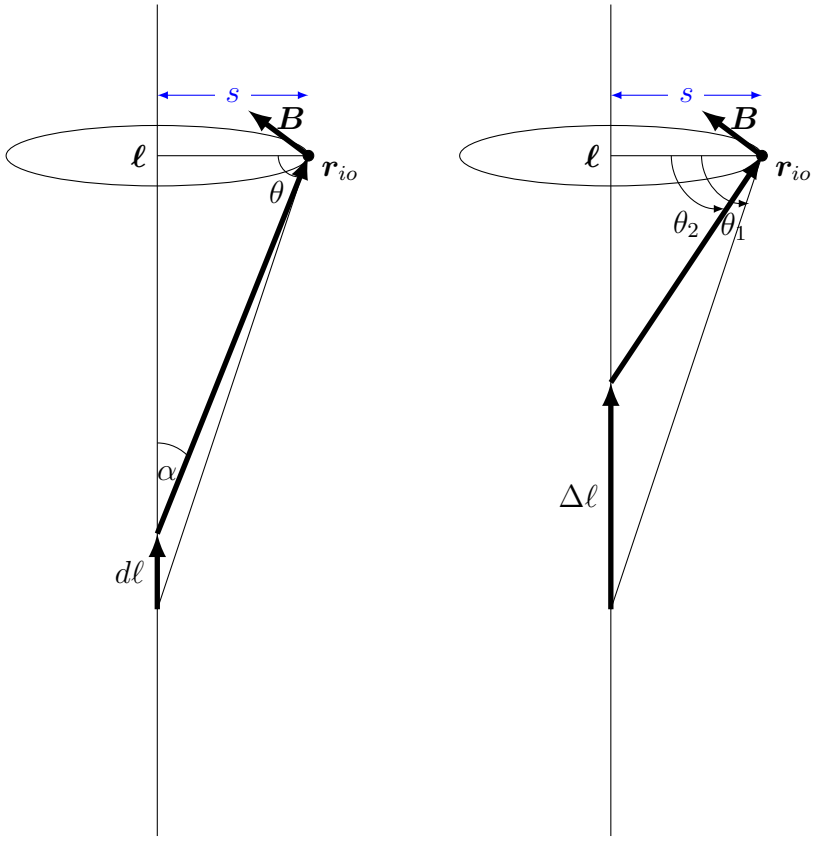

Figure 9.1: An steady electric current runs from below toward the z axis. We want to compute the magnetic field $\boldsymbol{B}$ a distance s from the axis.

Let us now find the magnitude. Let α be the angle between $d\boldsymbol{\ell}$ and $\boldsymbol{r}_{io}$. Then

$$d\boldsymbol{\ell} \times \hat{\boldsymbol{r}}_{io} = \|d\boldsymbol{\ell} \times \hat{\boldsymbol{r}}_{io}\|\hat{\boldsymbol{\phi}} = d\ell \sin\alpha\hat{\boldsymbol{\phi}} = d\ell \cos\theta\hat{\boldsymbol{\phi}}.$$

where θ is the angle that $\boldsymbol{r}_{io}$ makes with the horizontal plane (see the figure). Now we want to do a change of variables. Given that $\ell = s\tan\theta$ and $r_{io} = s/\cos\theta$ we find

$$d\ell = \frac{1}{\cos\theta^2}d\theta \quad , \quad \frac{d\boldsymbol{\ell}\times\hat{\boldsymbol{r}}_{ij}}{r_{io}^2} = \frac{\cancel{s}\cos^{\cancel{3}}\theta}{s^{\cancel{2}}\cancel{\cos^2\theta}}d\theta\hat{\boldsymbol{\phi}}$$

So if we consider the integration over the segment $\Delta\ell$. The two ends of the segment connect the tip of the vector $\boldsymbol{r}_{io}$, with θ angles θ_1, θ_2 as shown in the right frame of the figure. We find.

$$\boldsymbol{B}_{\Delta\ell} = \frac{\mu_0 I\hat{\boldsymbol{\phi}}}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta d\theta = \frac{\mu_0 I\hat{\boldsymbol{\phi}}}{4\pi s}(\sin\theta_2 - \sin\theta_1).$$

We now consider the whole line which goes from $\theta_1 = -\pi/2$ to $\theta_2 = \pi/2$, and so

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}. \tag{9.5}$$

As in the electrical field the magnetic field behaves as $1/s$ where s is the distance to the line.

9.1.4 Force due to magnetostatic field: Lorentz Force Law

Provided that we have a charge Q moving at a speed $\boldsymbol{v}$ on a constant magnetic field $\boldsymbol{B}$ then a force

$$\boldsymbol{F}_{\text{mag}} = Q(\boldsymbol{v}\times\boldsymbol{B}) \tag{9.6}$$

acts on the charge Q. This is **Lorentz force law.** This law comes from observations. If we combine this with the force due to an electric field we see that

$$\boldsymbol{F} = Q[\boldsymbol{E} + (\boldsymbol{v}\times\boldsymbol{B})]. \tag{9.7}$$

We show an example of an interesting particle trajectoriy generated by the forces in equation 9.7

Example 9.1.2 *Let us assume that a particle with positive charge Q is at rest $v_0 = 0$ and feels the presence of constant electric and magnetic fields $\boldsymbol{B} = B\hat{\boldsymbol{x}}$, and $\boldsymbol{E} = E\hat{\boldsymbol{z}}$. Find the trajectory of the particle.*

Solution: We make a qualitative description of the solution and then proceed to a formal mathematical solution. When the electric field is sensed the particle (positively charged) moves in the direction of the field, that is in the z positive direction. Since now the particle has a velocity the magnetic field $\boldsymbol{B}$ starts pushing the particle in the direction orthogonal to the plane spanned by $\boldsymbol{v}$ and $\boldsymbol{B}$, and from the right hand side rule this is toward the positive part of the y axis. As the particle accelerates the magnetic forces $\boldsymbol{F}_{\text{mag}}$ increases and it will turn the particle to go down since the force is normal to the trajectory (a centripetal force). Now, since the particle is now moving against the electric field it will start moving slower and eventually will stop. At this point a whole cycle in the trajectory is completed and the trajectory becomes periodic.

Let us now set up the differential equations for the problem. We have that

$$\boldsymbol{v} \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ v_x & v_y & v_z \\ B & 0 & 0 \end{vmatrix} = Bv_z\hat{\boldsymbol{y}} - v_xB\hat{\boldsymbol{z}},$$

and from Laurentz's equation 9.7 and $\boldsymbol{F} = md\boldsymbol{v}/dt$, we see that

$$\frac{d\boldsymbol{v}}{dt} = \frac{Q}{m}(E\hat{\boldsymbol{z}} + Bv_z\hat{\boldsymbol{y}} - v_xB\hat{\boldsymbol{z}}).$$

Since there is no component (on the right side) in the x direction we see that

$$\frac{dv_x}{dt} = 0.$$

So $v_x = C$, where C is a constant, and from the initial condition $v_x(0) = 0$, we see that $V_x = 0$. Then the particle moves in the $y - z$ plane as predicted in the qualitiative analysis. We have then two ODEs with two unknonws (for velocities). That is,

$$\begin{aligned} \frac{dv_y}{dt} &= \frac{QB}{m}v_z \\ \frac{dv_z}{dt} &= \frac{Q}{m}E - \frac{QB}{m}v_x \end{aligned} \tag{9.8}$$

We take a time derivative of the first equation and use the second,

$$\frac{d^2v_y}{dt^2} = \frac{QB}{m}\frac{dv_z}{dt} = \frac{QB}{m}\left(\frac{Q}{m}E - \frac{QB}{m}v_x\right).$$

Let us call $\omega = QB/m$, and so

$$\frac{d^2 v_y}{dt^2} + \omega^2 v_x = \frac{\omega^2 E}{B}.$$

The general solution of the homogenous equation is

$$v_{yh} = C_1 \cos \omega t + C_2 \sin \omega t.$$

Since the right hand side term is independent of t we can guess a particular solution of the form $v_x(t) = A_1 t^2 + A_2 t + A_3$ and find A_1, A_2, and A_3. That is

$$2A_1 + \omega^2 (A_1 t^2 + A_2 t + A_3) = \omega^2 \frac{E}{B}.$$

From here $A_1 = 0$, $A_2 = 0$, and $A_3 = \frac{E}{B}$. So

$$v_{yp} = \frac{E}{B},$$

is a particular solution of the equation. Then the general solution for v_x is given by

$$v_y = \frac{E}{B} + C_1 \cos \omega t + C_2 \sin \omega t$$

We now find C_1 and C_2. From the initial condition $\boldsymbol{v}(0) = 0$ we see that

$$v_y(0) = \frac{E}{B} + C_1 = 0$$

implies $C_1 = -E/B$, and

$$v_y(t) = \frac{E}{B} - \frac{E}{B} \cos \omega t + C_2 \sin \omega t$$

We still have a constant C_2 that we need to solve for. When we differentiated, we introduced one more constant that was not in the original problem. We only have

conditions for initial position and initial velocity but a second differential equation on velocity needs an initial condition in acceleration. We derive that condition from physical principles. The initial acceleration is along the z direction since at rest the particle does not feel the $\boldsymbol{B}$ field, only the $\boldsymbol{E}$ field which is along the vertical aligned with the initial force. So we can say that $dv_y/dt = 0$ at $t = 0$. That is,

$$\left.\frac{dv_y(t)}{dt}\right|_{t=0} = \left.\frac{\omega E}{B}\sin\omega t + \omega C_2 \cos\omega t\right|_{t=0} = 0.$$

This is,

$$C_2 = 0,$$

and

$$v_y(t) = \frac{E}{B} - \frac{E}{B}\cos\omega t. \tag{9.9}$$

We return to equation 9.8 with our knowledge of v_y, that is

$$\frac{\omega E}{B}\sin\omega t = \frac{QB}{m}v_z.$$

That is,

$$v_z(t) = \frac{E}{B}\sin\omega t. \tag{9.10}$$

We need to do one more integration that takes us from velocity to positions. From integrating 9.9 and 9.10 we find

$$\begin{aligned} y(t) &= \frac{Et}{B} - \frac{E}{B\omega}\sin\omega t + C_4 \\ z(t) &= -\frac{E}{B\omega}\cos\omega t + C_5. \end{aligned}$$

Now, since $y(0) = z(0) = 0$,

$$C_4 = 0 \quad , \quad C_5 = \frac{E}{B\omega},$$

and

$$\begin{aligned} y(t) &= \frac{E}{\omega B}(\omega t - \sin\omega t) \\ z(t) &= \frac{E}{\omega B}(1 - \cos\omega t). \end{aligned}$$

This is the parametric equation of a cycloid. We find the following characteristics of this trajectory:

(i) The function is periodic. The function starts at $(0, 0, 0)$. The highest point (maximum) is achieved at $\omega t = \pi$. Here

$$(x, y, z) = \left(0, \frac{\pi E}{\omega B}, \frac{2E}{\omega B}\right)$$

(ii) The next $z = 0$ point is obtained at $\omega t = 2\pi$, where we find

$$(x, y, z) = \left(0, \frac{2\pi E}{\omega B}, 0\right)$$

The time period is then $T = 2\pi/\omega$. In the y axis the period is given by $2\pi E/(\omega B)$.

(iii) If we take a snap shot (freeze) at some time $t = t_0$. We find that the particle would be located at a circle. Let us find the circle and, where in the circle, is the particle is located.

Let us write the trajectory as

$$\begin{aligned} y - \frac{Et_0}{B} &= -\frac{E}{\omega B}\sin\omega t_0 \\ z - \frac{E\omega}{B} &= -\frac{E}{\omega B}\cos\omega t_0, \end{aligned}$$

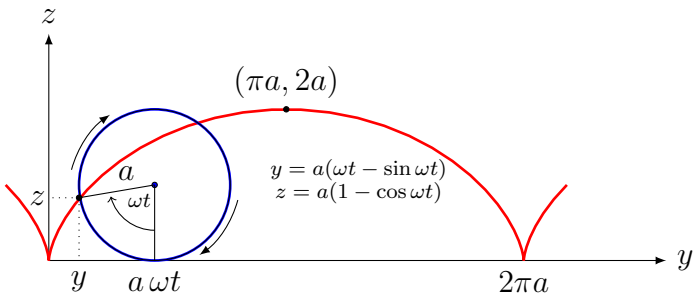

Figure 9.2: Illustration of the cycloid generated by a particle in a constant magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{x}}$ and constant electric field $\boldsymbol{E} = E\hat{\boldsymbol{z}}$. Here $a = E/(\omega B)$ is the radius of the circle.

and from here

$$\left(y - \frac{Et_0}{B}\right)^2 + \left(z - \frac{E\omega}{B}\right)^2 = \frac{E^2}{\omega^2 B^2}.$$

This is a circle of radius R and center C at

$$\begin{aligned} R &= \frac{E}{\omega B} \\ C &= \left(0, \frac{Et_0}{B}, \frac{E\omega}{B}\right). \end{aligned}$$

Since the center is moving with a constant velocity $dC/dt = (E/B)\hat{\boldsymbol{y}}$, we say that the particle is rolling along in a circle which is moving at constant speed. Think about a little pebble in a tire of a car moving at constant speed.

Figure 9.2 illustrates this. The cycloid is painted in red and the riding circle in blue.

Work of a Magnetic Field Force

We show that magnetic field forces do not do any work. By definition of work

$$dW_{\text{mag}} = \boldsymbol{F}_{\text{mag}} \cdot d\boldsymbol{\ell} = Q(\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v}dt = 0.$$

where we used the equation $d\boldsymbol{\ell} = \boldsymbol{v}dt$, with $\boldsymbol{v}$ being the velocity of the partcle along the trajectory $\boldsymbol{\ell}$. Then we see that magnetic fields do not do work.a f

Force on a wire with current

A special case of a force due to a magnetic field is that of a force in a wire with current.

Let us assume that we have a wire carrying a current $\boldsymbol{I}$ under a constant magnetic field $\boldsymbol{B}$ with charge density λ. We assume then that charges are traveling at a velocity $\boldsymbol{v}$. For each element of charge dq in the wire we have a Laurentz's force.

$$d\boldsymbol{F}_{\text{mag}} = \boldsymbol{v} \times \boldsymbol{B}\, dq = (\boldsymbol{v} \times \boldsymbol{B})\, \lambda\, d\ell.$$

So

$$\boldsymbol{F}_{\text{mag}} = \int_{\boldsymbol{\ell}} (\boldsymbol{v} \times \boldsymbol{B}) \lambda d\ell = \int_{\boldsymbol{\ell}} (\boldsymbol{I} \times \boldsymbol{B}) d\ell. \tag{9.11}$$

9.1.5 The Divergence and Curl of the Magnetic Field

We recall from equation 9.5 due to a straight line current.

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}}.$$

The magnetic field at a distance s from the z axis circulates around the axes. We compute the closed line integral

$$\oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = \frac{\mu_0 I}{2\pi s} \oint d\ell = \mu_0 I.$$

This result is independent of the path $\boldsymbol{\ell}$. That is, pick a general differential

$$d\boldsymbol{\ell} = ds\hat{\boldsymbol{s}} + sd\phi\hat{\boldsymbol{\phi}} + dz\hat{\boldsymbol{z}},$$

then

$$\oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I.$$

This assumes that the loop was circled around the z axis only once. If the loop does not contain the z axis the total angle swept is 0. See Figure 9.3. In this case the swept

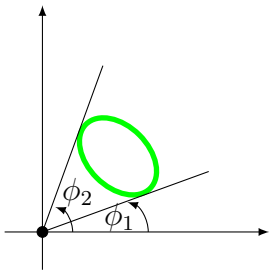

Figure 9.3: Assume the wire is at the origin running from inside the page towards the reader. Assume that the green contour is used for the integration. A complete loop will take us from the angle ϕ_1 to the angle ϕ_2 and back to ϕ_1. This is true regardless the number of turns.

angle goes from ϕ_1 to ϕ_2 and back to ϕ_1. If the z axis is enclosed by the loop and we give n turns then the result would be $n\mu_0 I$. We can infer that if I_{enc} is the total current enclosed then

$$\oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}}.$$

In addition, if $\boldsymbol{J}$ is the current density going through the are $\boldsymbol{S}$ fo the wire we know that

$$I_{\text{enc}} = \int \boldsymbol{J} \cdot d\boldsymbol{S},$$

and applying Stokes theorem

$$\int (\nabla \times \boldsymbol{B}) \cdot d\boldsymbol{S} = \oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}} = \int \mu_0 \boldsymbol{J} \cdot d\boldsymbol{S}.$$

and since the surface $\boldsymbol{S}$ was arbitrary we can say conclude that in this case

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}.$$

We show next that this equation is not only valid for straight infinite line currents but for more general cases.

We start with the Biot-Savart law 9.3. A volume element can be written as

$$d\boldsymbol{r}_i = d\boldsymbol{\ell}\Delta S,$$

then we can write

$$\boldsymbol{J}d\boldsymbol{r}_i = \frac{\boldsymbol{I}d\boldsymbol{\ell}\cancel{\Delta\mathcal{S}}}{\cancel{\Delta\mathcal{S}}},$$

and replace equation 9.3 by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}\times\hat{\boldsymbol{r}}_{io}}{r_{io}^2}d\boldsymbol{r}_i. \tag{9.12}$$

This is the volume integral representation of the Biot-Savart law. It compares to the Columb's law and the superposition principle by identifying current density with charge density and the cross product being replaced by regular scalar vector product. We want to compute the divergence and curl of $\boldsymbol{B}$ following this representation. We start with the divergence:

$$\nabla\cdot\boldsymbol{B} = \frac{\mu_0}{4\pi}\int \nabla\cdot\left(\frac{\boldsymbol{J}\times\hat{\boldsymbol{r}}_{io}}{r_{io}^2}\right)d\boldsymbol{r}_i.$$

We use identity A.7 to write

$$\nabla\cdot\left(\frac{\boldsymbol{J}\times\hat{\boldsymbol{r}}_{io}}{r_{io}^2}\right) = (\nabla\times\boldsymbol{J})\cdot\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} - \boldsymbol{J}\cdot\left(\nabla\times\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2}\right).$$

Since $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{r}_i)$, and the curl is taken with respect to the output variable $\boldsymbol{r}$ we see that $\nabla\times\boldsymbol{J} = \nabla_{\boldsymbol{r}}\times\boldsymbol{J}(\boldsymbol{r}_i) = 0$. Now, from equation A.11 with $n=-2$ we see that

$$\nabla\times\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} = 0.$$

and so

$$\nabla\cdot\boldsymbol{B} = 0.$$

That is, the divergence of the magnetic field is 0.

Let us now study the curl of the magenetic field. We apply the curl in equation 9.12.

$$\nabla \times \boldsymbol{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\frac{\boldsymbol{J} \times \hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right) d\boldsymbol{r}_i.$$

Equation A.8 provides an expansion the curl operator

$$\nabla \times \left(\frac{\boldsymbol{J} \times \hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right) = \boldsymbol{J} \left(\nabla \cdot \frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right) - \frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} (\nabla \cdot \boldsymbol{J}) + \left[\left(\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right) \cdot \nabla \right] \boldsymbol{J} - (\boldsymbol{J} \cdot \nabla) \frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2}.$$

Since $\boldsymbol{J}$ does not depend on $\boldsymbol{r}$ and the gradient is with respect to $\boldsymbol{r}$ we drop two terms here and write

$$\nabla \times \left(\frac{\boldsymbol{J} \times \hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right) = \boldsymbol{J} \left(\nabla \cdot \frac{\hat{\boldsymbol{r}}_{\boldsymbol{io}}}{r_{io}^2} \right) - (\boldsymbol{J} \cdot \nabla) \frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2}. \tag{9.13}$$

We have, from equation 2.4

$$\delta(\boldsymbol{r} - \boldsymbol{r}_i) = \frac{1}{4\pi} \left(\nabla \cdot \frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right).$$

We show that the integral over the last term in equation 9.13 is zero. Since the gradient of a vector field is a matrix we take one component j and treat the problem as the gradient of a scalar.

We do integration by parts. That is, from

$$\nabla \cdot \left[\left(\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right)_j \boldsymbol{J} \right] = \nabla \left(\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right)_j \cdot \boldsymbol{J} + \left(\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right)_j \nabla \cdot \boldsymbol{J},$$

we find

$$(\boldsymbol{J} \cdot \nabla) \left(\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right)_j = \nabla \cdot \left[\left(\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right)_j \boldsymbol{J} \right] - \left(\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2} \right)_j \nabla \cdot \boldsymbol{J},$$

The last term here is 0 since the divergence is with respect to $\boldsymbol{r}$ and $\boldsymbol{J}$ is a function only of $\boldsymbol{r}_i$. For the first term on the integration we apply the divergence theorem and then its integral is

$$\int_S \left(\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2}\right)_j \boldsymbol{J} dS.$$

We can take the surface S as large as we want to include all currents. We will not current in the surface and the integral would be zero.

We then find that

$$(\nabla \times \boldsymbol{B})(\boldsymbol{r}) = \frac{\mu_0}{\cancel{4\pi}} \int \boldsymbol{J}(\boldsymbol{r}_i)(\cancel{4\pi})\delta(\boldsymbol{r} - \boldsymbol{r}_i) d\boldsymbol{r}_i = \mu_0 J(\boldsymbol{r}).$$

That is we derived **Amperè's Law**

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}. \tag{9.14}$$

Example 9.1.3 *We consider the case of an infinite wire shown in example 9.1.1.*

We already know that the magnetic field $\boldsymbol{B}$ encircles the z axis. By Stokes theorem and Amperè's law we can write

$$\begin{aligned} \int_S \nabla \times \boldsymbol{B} \cdot d\boldsymbol{S} = B \oint d\ell &= 2\pi B s \\ \mu_0 \int_S \boldsymbol{J} \cdot d\boldsymbol{S} &= \mu_0 I. \end{aligned}$$

From here

$$B = \frac{\mu_0 I}{2\pi s},$$

or better

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}},$$

9.2 Problems

9.2.1 Problem 1

Use the Lorentz Force law 9.7 to find the trajectory of a particle Q under the following conditions. We assume that the particle is moving with constant velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$. At the instant that the particle was located at the point $(0, -1, 0)$ a constant magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ was applied. Show that the particle trajectory is circular. Find the frequency of motion in terms of the constants of the problem. Where is the particle at $t = 2$ s after the particle started to move under the influence of $\boldsymbol{B}$.

9.2.2 Problem 2

Consider the cylinder or radius a in Figure 9.4. We think of this cylinder as the piece of

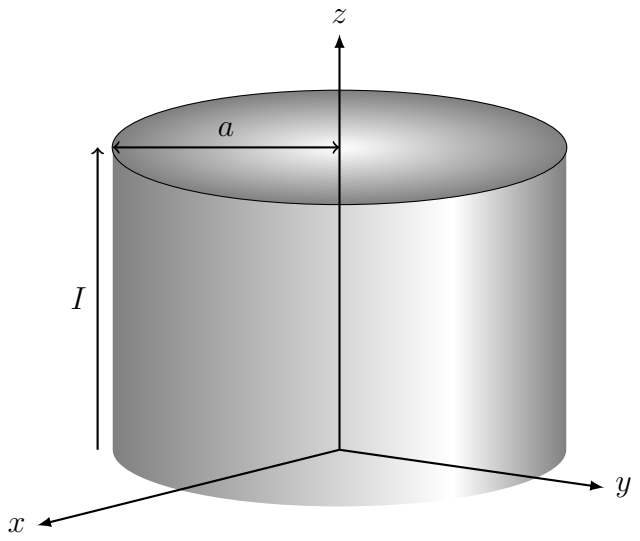

Figure 9.4: A piece of cylindrical wire with with radius a.

a wire which carries a steady current I. Consider two options

(i) The current is uniformly distributed over the outside surface of the wire.

(ii) The current is distributed such that $\boldsymbol{J}$ is proportional to the distance s from the axis.

Find the magnetic field $\boldsymbol{B}$ both inside and outside the cylinder.

Chapter 10

Lecture #10

10.1 Comparison of Electrostatics and Magnetostatics

In electrostatics we started with Coulomb's law and derived Gauss' law. In magnetostatics we started with Biot-Savart law and derived Ampère's law. In electrostatics the fundamental concept of charge allowed us to introduce the electrical field while in magnetostatics the fundamental concept of current allowed us to introduce the magnetic field. The equations for the Columb's law and Biot-Savart law are parallel. In the first charges are the fundamental new element while in the second the currents are the fundamental new element. In Coulomb's law the product is scalar and and in Biot-Savart's law the produc is vectorial (cross–product). We have the following fundamental relations:

(i) **For the electric field $\boldsymbol{E}$.**

$$\begin{cases} \nabla \cdot \boldsymbol{E} = \dfrac{1}{\epsilon_0}\rho & \text{(Gauss}'\text{s law)} \\ \\ \nabla \times \boldsymbol{E} = 0 & \text{the electric field is curl} - \text{less.} \end{cases}$$

$$\boldsymbol{F}_E = q\boldsymbol{E}.$$

(ii) **For the magnetic field $\boldsymbol{B}$.**

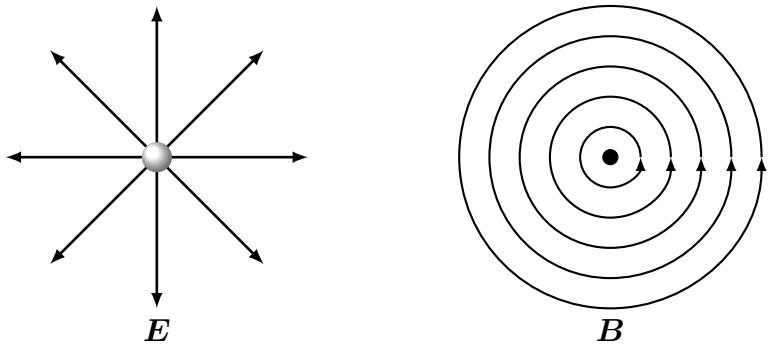

Figure 10.1: On the left there is a point charge and its electric field represented by radial diverging rays. On the right there is a current comming from below the plane of view toward the reader and the magenetic field is rotationg around the current.

$$\begin{cases} \nabla \cdot \boldsymbol{B} = 0 & \text{the magnetic field is diverence} - \text{less} \\ \nabla \times \boldsymbol{E} = \mu_0 \boldsymbol{J} & \text{Ampère's law} \end{cases}$$

$$\boldsymbol{F}_M = q(\boldsymbol{v} \times \boldsymbol{B}).$$

The combination of both forces is Lorentz's law

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}).$$

The four equations for the electric and magnetic field are the Maxwell equations for electro/magneto-statics.

The electric fields diverges radially (along $\hat{\boldsymbol{r}}$ away from a point charge while the magnetif fields curls (along $\hat{\boldsymbol{\phi}}$) around a current. Figure 10.1 illustrates this.

10.2 The Vector Potential for the Magnetic Field

We showed how $\nabla \times \boldsymbol{E} = 0$ implied that we can write $\boldsymbol{E}$ as the gradient of a scalar potential V in the form

$$\boldsymbol{E} = -\nabla V.$$

We now show that since $\nabla \cdot \boldsymbol{B} = 0$, then we can write $\boldsymbol{B}$ as the curl of a vector potential $\boldsymbol{A}$. That is, we can write

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

If we want that $\boldsymbol{F} = \nabla \times \boldsymbol{A}$ for some vector field $\boldsymbol{A}$ then we require

$$\boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

That is we require

$$F_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \quad , \quad F_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \quad , \quad F_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}.$$

There could be many $\boldsymbol{A}$ which satisfy this equation (as a simple illustration, once you find one $\boldsymbol{A}$ you can add any constant to this $\boldsymbol{A}$ and that would be another valid solution. More generally from equation A.2 the curl of the gradient is 0 and so we can add $\nabla\phi$ to $\boldsymbol{A}$ and still get a solution of the equation above). Let us assume that $A_x = 0$, then

$$F_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \quad , \quad F_y = -\frac{\partial A_z}{\partial x} \quad , \quad F_z = \frac{\partial A_y}{\partial x}.$$

If we integrate the second and third equations we find

$$\begin{aligned} A_y(x,y,z) &= \int_0^x F_z(x',y,z)dx' + C_1(y,z) \\ A_z(x,y,z) &= -\int_0^x F_y(x',y,z)dx' + C_2(y,z). \end{aligned}$$

We now plug these solutions into the first equation for F_x. That is

$$\begin{aligned} F_x &= -\int_0^x \frac{\partial F_y}{\partial y} dx' + \frac{\partial C_2(y,z)}{\partial y} - \int_0^x \frac{\partial F_z}{\partial z} dx' - \frac{\partial C_1(y,z)}{\partial z} \\ &= \int_0^x -\left(\frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx' + \frac{\partial C_2(y,z)}{\partial y} - \frac{\partial C_1(y,z)}{\partial z} \\ &= \int_0^x \frac{\partial F_x}{\partial x} dx' + \frac{\partial C_2(y,z)}{\partial y} - \frac{\partial C_1(y,z)}{\partial z} \quad , \quad \text{using } \nabla \cdot \boldsymbol{F} = 0 \end{aligned}$$

Now,

$$\int_0^x \frac{\partial F_x}{\partial x} dx' = F_x(x,y,z) - F_x(0,y,z),$$

so

$$\frac{\partial C_2(y,z)}{\partial y} - \frac{\partial C_1(y,z)}{\partial z} = -F_x(0,y,z),$$

We can pick $C_1 = 0$ and integrate C_2 as

$$C_2(y,z) = -\int_0^y F_x(0,y',z) dy'.$$

Then we find a solution

$$\begin{aligned} A_x &= 0 \\ A_y &= \int_0^x F_z(x',y,z) dx' \\ A_z &= -\int_0^x F_y(x',y,z) dx' + \int_0^y F_x(0,y',z) dy' \end{aligned} \tag{10.1}$$

We leave as an exercise to the reader to verify that $\nabla \times \boldsymbol{A} = \boldsymbol{F}$.

We can compute the vector potential. Let us see. By using Ampère's law, the vector potential identity, and equation A.4 we find

$$\nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}. \tag{10.2}$$

We have lots of freedom to pick $\boldsymbol{A}$. Since, from equation A.2 we have that $\nabla \times \nabla \phi = 0$, we can add the gradient of any function ϕ to $\boldsymbol{A}$ and still have a vector potential. Then we would like to have a divergenless vecto potential. That is, a potential such that $\nabla \cdot \boldsymbol{A} = 0$.. If we could find such a vector potential then equation 10.2 would turn out to be

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}, \tag{10.3}$$

and this is equivalent to three Poisson's equations. One for each component of $\boldsymbol{A}$. Let us then assume that we have some vector potential $\boldsymbol{A}_0$ whcih we want to modify to find a new potential wich is divergenless. We then want to change $\boldsymbol{A}_0$ by adding $\nabla \phi$ such that

$$\nabla \cdot (\boldsymbol{A}_0 + \nabla \phi) = 0.$$

That is,

$$\nabla^2 \phi = -\nabla \cdot \boldsymbol{A}_0.$$

Then we need to solve another Poisson's equation where now the source term is $-\nabla \cdot \boldsymbol{A}_0$ instead of $-\rho/\epsilon_0$.

In general then we can assume that we already have a divergenless vector potential and solve the vector Poisson's equation 10.3. If the currents are distributed in a finite volume we can use the Green's function method of the Poisson's equation to find the solution. That is we can say that

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r_i})}{r_{io}} d\boldsymbol{r}_i. \tag{10.4}$$

If instead of a current flowing through a surface we have a current in a surface flowing through a line this equation would be

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{K}(\boldsymbol{r_i})}{r_{io}} d\boldsymbol{S}, \tag{10.5}$$

where S is the surface along which the current is flowing.

If instead of a current flowing through a line element we have a current in a surface flowing in a wire the equation would be

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{I}(\boldsymbol{r_i})}{r_{io}} d\boldsymbol{\ell}, \tag{10.6}$$

where S is the surface along which the current is flowing.

We need to say that the vector potential is at the moment not very useful since it is a vector with three dimensions and it defeats the purpose of simplifying $\boldsymbol{B}$ with another three component vector equation. The vector potential is useful in other context which is out of the scope of this course.

10.3 Magnetostatic Boundary Conditions

In the same way that the electric field suffers a discontinuity across a surface of charges, the magnetic field is discontinuous across a surface of currents. We use the same ideas here as we did in the electric field. That is, since the magnetic field is divergenless and using the divergence theorem.

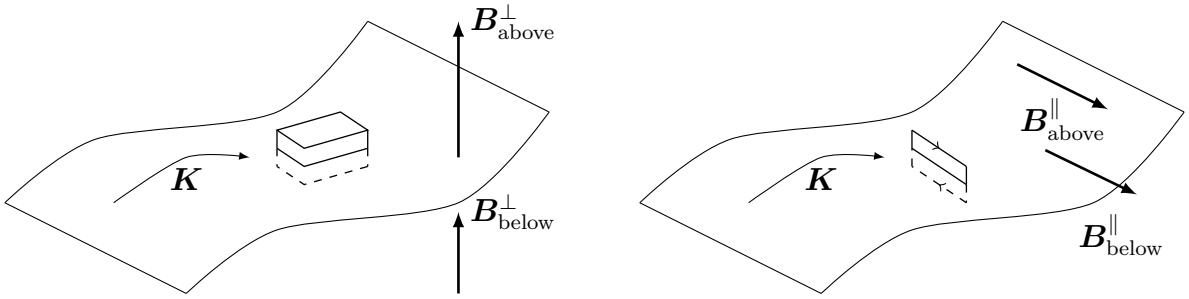

Figure 10.2: The frame on the left shows a thin box around the surface. On the right we see a path circulating around the surface.

$$\oint \boldsymbol{B} \cdot d\boldsymbol{S} = 0,$$

we can take a little box around the surface (see Figure 10.2), as we did for the electric field. The box is very thin with one face above the surface and another below. Since

the box height approaches 0 in the limit the integral (assuming that the magnetic field is constant in the small piece of box) we find

$$\Delta S(B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}}) = 0.$$

That is

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}.$$

Then the magnetic field is continuous across a surface.

Now, for the tangential component we use a Amperian loop around the surface, Ampère's law and Stoke's theorem. See Figure 10.2. Here

$$\oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = (B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}})\ell = \mu_0 I_{\text{enc}} = \mu_0 K \ell.$$

That is,

$$\boldsymbol{B}^{\parallel}_{\text{above}} - \boldsymbol{B}^{\parallel}_{\text{below}} = \mu_0 \boldsymbol{K}. \tag{10.7}$$

10.4 Multipole Expansion of the Vector Potential

We use the current loop to expand its mathematical representation in terms of a series in powers of $1/r$. The starting point is formula 10.6

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{I}(\boldsymbol{r_i})}{r_{io}} d\boldsymbol{\ell}. \tag{10.8}$$

Recall that $r_{io} = \|\boldsymbol{r} - \boldsymbol{r}_i\| = \sqrt{r^2 + r_i^2 - 2rr_i\cos\theta}$ where θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}_i$. We can use the expansion 7.2

$$\frac{1}{r_{io}} = \frac{1}{\sqrt{r^2 + r_i^2 - 2rr_i\cos\theta}} = \frac{1}{r}\sum_{n=0}^{\infty} \left(\frac{r_i}{r}\right)^n P_n(\cos\theta).$$

in equation 10.8 to find

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint r_i^n P_n(\cos\theta) d\boldsymbol{\ell}_i.$$

This is the multipole expansion of a current loop. As we did in the electrical field we call the term of order $1/r$ a **magnetic monopole** , the term of order $1/r^2$ a **magnetic dipole** the term of order $1/r^3$ a **magnetic quadrupole** etc. Let us study the first few terms of this multipole expansion.

If $n = 0$ we have a monopole

$$\boldsymbol{A}_{\text{mono}}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi} \frac{1}{r} \oint d\boldsymbol{\ell}_i = 0,$$

since the line integral along a loop is zero. This is consistent with the fact that $\nabla \cdot \boldsymbol{B} = 0$. This says that there are no magnetic monopoles. Let us now assume $n = 1$, so

$$\boldsymbol{A}_{\text{dip}}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r_i \cos\theta d\boldsymbol{\ell}_i = \frac{\mu_0 I}{4\pi r^2} \oint \boldsymbol{r}_i \cdot \hat{\boldsymbol{r}} d\boldsymbol{\ell}_i.$$

We leave as an exercise to prove that

$$\oint \boldsymbol{r}_i \cdot \boldsymbol{r} d\boldsymbol{\ell}_i = -\hat{\boldsymbol{r}} \times \int d\boldsymbol{a},$$

where

$$\int d\boldsymbol{a}$$

is known as the vector area the loop. With this dipole of the vector potential can be written as

$$\boldsymbol{A}_{\text{dip}}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{r^2},$$

where $\boldsymbol{m}$ is known as the **magnetic dipole moment:**

$$\boldsymbol{m} = I \int d\boldsymbol{a} = I\boldsymbol{a},$$

10.5 Problems

10.5.1 Problem 1

(i) Veify that equations 10.1 satisfy $\nabla \times A = \boldsymbol{F}$.

(ii) Let $F = z\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}} + y\hat{\boldsymbol{z}}$. Use equatios 10.1 to find the magnetic $\boldsymbol{A}$ and verify that indeed $\nabla \times \boldsymbol{A} = \boldsymbol{F}$.

10.5.2 Problem 2

Investigate the concept of "vector area" and show that

$$\oint \boldsymbol{r}_i \cdot \boldsymbol{r} d\boldsymbol{\ell}_i = -\hat{\boldsymbol{r}} \times \int d\boldsymbol{a},$$

and so

$$\boldsymbol{A}_{\text{dip}}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{r^2},$$

where $\boldsymbol{m}$ is known as the **magnetic dipole moment:**

$$\boldsymbol{m} = I \int d\boldsymbol{a} = I\boldsymbol{a}, \tag{10.9}$$

Chapter 11

Lecture #11

11.1 Magnetization

11.1.1 Diamagnets, Paramagnets, and Ferromagnets

We recall that some materials can be polarized by the presense of an electric field $\boldsymbol{E}$. The polarization usually takes place in the direction of $\boldsymbol{E}$. In the case of magnetic fields there is also a magnetic polarization. The polarization can be aligned with the magnetic field $\boldsymbol{B}$ and in this case the material is named **paramagnet** . It could also happens that the polarization could have opposite direction to $\boldsymbol{B}$, and in this case the material is known as **diamagnet** . Diagmagnetism is a quantum mechanical effect that occurs in all materias. The justification for diamagnetism is out of the scope of these notes. Some materials can have a permanent magnetization and they are known as **ferromagnets.** While paramagnetic and ferromagnetic processes cause attraction forces diamagnetism causes repulsion forces. While we know that materials such as iron, nickel, and cobalt react to a magnetic field, we need to say that all materials should react to a magnetic field. We do not see, for example, wood being attracted by a magnet. The reason is that the paramagnetic and diamagnetic forces (while one is attractive the other should be repulsive) are very week. The forces produced by iron are 10^4 or 10^5 greater than those produced by other materials.

11.1.2 Torques and Forces on Magnetic Dipoles

We start by assuming a current on a tilted loop making and angle θ with respect to the z axis. Figure 11.1 sketches such a loop. The loop is rectangular with side lengths a and b. It is assumed to be fixed to the x axis at the points $(b/2, 0, 0)$ and $(-b/2, 0, 0)$

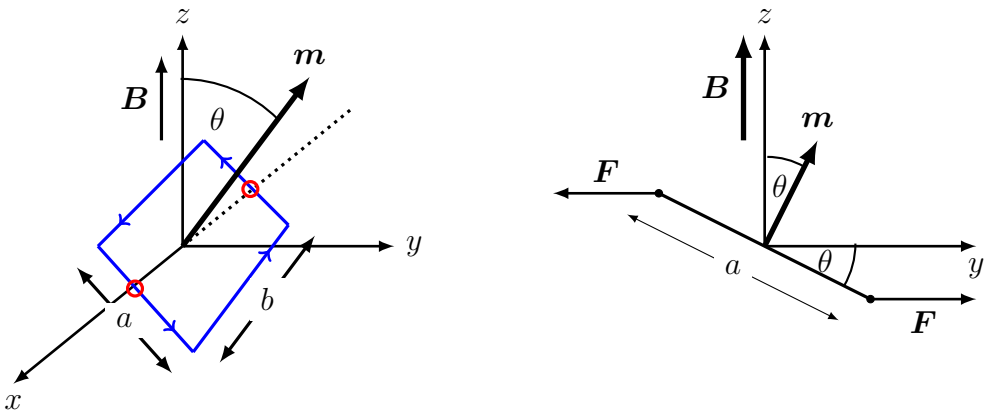

Figure 11.1: Left: the rectangular blue loop with side lengths a and b, is tilted an angle θ with respect to the z axis. A current I is circulating through it in the counter clockwise direction as seen from above. It is pinned to the x axis at the locations marked with the red circles. Right: The y-z plane view.

represented by the two small red circles in the figure.

The force in the wire can be computed using equation 9.11. That is,

$$\boldsymbol{F}_{\text{mag}} = \int_{\boldsymbol{\ell}} (\boldsymbol{I} \times \boldsymbol{B})\, d\boldsymbol{\ell}.$$

Along the segments of the wire with length a we have that for the segment in the negative side of x the force is pointing in the direction $-\hat{\boldsymbol{x}}$ and the segment in the positive side of x the force is pointing in the direction $\hat{\boldsymbol{x}}$. The force in both segments is equal so the net force along the x direction is 0. Now, for the y direction we see that the force in the segment of length b in the negative side of z is pointing in the $\hat{\boldsymbol{y}}$ direction while the for in the wire on the positive side of z is pointing in the direction $-\hat{\boldsymbol{y}}$. Both forces also cancel but they create a torque on the wire since we assume that the wire is fixed at some points $(-b/2, 0, 0,)$ and $(b/2, 0, 0)$. The directions of the forces are indicated in the right frame of the figure. The magnitude is given by

$$F_{\text{mag}} = IBb.$$

note that the angle betwen $\boldsymbol{B}$ and $\boldsymbol{I}$ is $\pi/2$. Then the torque is given by

$$\boldsymbol{\tau}_{\text{mag}} = \boldsymbol{d} \times \boldsymbol{F}_{\text{mag}} = IBba \sin\theta \hat{\boldsymbol{x}}. = m \sin\theta \hat{\boldsymbol{x}},$$

where $\boldsymbol{d} = \boldsymbol{a}$, a vector of magintude a in the direction of the segment of length a and $m = abI$ is the magnitude of the magnetic dipole moment. Since θ is the angle between the magnetic dipole moment vector $\boldsymbol{m}$ (see equation 10.9 for the definition of magnetic dipole moment) and $\boldsymbol{B}$ we then can say that the torque is given by

$$\boldsymbol{\tau}_{\text{mag}} = \boldsymbol{m} \times \boldsymbol{B}.$$

This equation provides the torque due to a constant magnetic field $\boldsymbol{B}$ due to an infinitesimal steady current in a loop $\boldsymbol{I}$. Please compare this to the electric torque due to the electric dipole $\boldsymbol{p}$ in equation 8.1. This torque tries to align the loop so that its normal direction coincides with the field $\boldsymbol{B}$. As in the case of the electric field in materials we claim that the magnetic torque happens at atomic level and we since for each atom we have electrons circling its nucleos this is a tiny current that creates a torque in the presense of a constant magnetic field $\boldsymbol{B}$. This torque is what causes **paramagnetism.** In fact think of an atom. Each electron has a spin which is a small current in a loop. It is a law of quantum mechanics that, to balance the torques in an atom, the electron spins are coupled so that one is in one direction and the other in the opposite direction. This will neutralize the torques within in atom. However some electrons are unpaired. These unpaired electrons cause paramagnetism.

11.1.3 Definition of Magnetization M

We now want to provide a measure of magnetization. In the presense of a magnetic field $\boldsymbol{B}$ materials can get some magnetic polarization in a direction or against the direction of the magnetic field $\boldsymbol{B}$. As we did with the electric field we want to define a magnetization density $\boldsymbol{M}$ which measures the the magnetic dipole moment per unit of volume. In this wave magnetization $\boldsymbol{M}$ is like polarization $\boldsymbol{P}$ for the electric field.

11.2 The Field of a Magnetized Object

11.2.1 Bound Currents

Let us assume a magnetized material with magnetization $\boldsymbol{M}$. We want to find the field produced by this magnetization. The vector potential is given by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^2}.$$

In this equation we have $\boldsymbol{r}$ relative to the origin. A more general equation is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}_{io}}{r_{io}^2}.$$

where $\boldsymbol{r}_{io} = \boldsymbol{r} - \boldsymbol{r}_i$. That is we write the equation with respect to a vector $\boldsymbol{r}_i$. We want to integrate this expression in a volume V. Instead of $\boldsymbol{m}$ we use the magnetization density (per unit volume) $\boldsymbol{M}$. That is, the vector potential due to a volume $\boldsymbol{V}$ of magnetic diapoles is given by

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int_V \boldsymbol{M}(\boldsymbol{r}_i)\times\frac{\hat{\boldsymbol{r}}_{io}}{r_{io}^2}\,d\boldsymbol{r}_i.$$

As we did in the electrical field we use the identity

$$\nabla\frac{1}{r_{io}} = \frac{\hat{\boldsymbol{r}}}{r_{io}^2}$$

and write

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_V \left[\boldsymbol{M}(\boldsymbol{r}_i)\times\nabla\frac{1}{r_{io}}d\boldsymbol{r}_i\right].$$

We want to do multidimensional integration by parts. That is, we use the identity

$$\nabla\times\frac{\boldsymbol{M}(\boldsymbol{r}_i)}{r_{io}} = \nabla\frac{1}{r_{io}}\times\boldsymbol{M}(\boldsymbol{r}_i) + \frac{1}{r_{io}}\nabla\times\boldsymbol{M}(\boldsymbol{r}_i),$$

so

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left\{\int_V \frac{1}{r_{io}}\nabla\times\boldsymbol{M}(\boldsymbol{r}_i)d\boldsymbol{r}_i - \int_V \nabla\times\frac{\boldsymbol{M}(\boldsymbol{r}_i)}{r_{io}}d\boldsymbol{r}_i\right\}.$$

We now use identity A.13 on the last integral above

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left\{\int_V \frac{1}{r_{io}}\nabla\times\boldsymbol{M}(\boldsymbol{r}_i)d\boldsymbol{r}_i + \oint_S \frac{\boldsymbol{M}(\boldsymbol{r}_i)}{r_{io}}\times\hat{\boldsymbol{n}}dS\right\}. \tag{11.1}$$

where $d\boldsymbol{S} = d\hat{\boldsymbol{n}}S$. We compare the first integral with the magnetic vector potential integral 10.4 and the second integral with the magnetic vector potential integral 10.5. Then it is apparent that the numerator of the integrand in the first term is a volume charge density while the numerator of the second integrand is a surface charge density. Then it is natural to define

$$\boldsymbol{J}_b = \nabla\times\boldsymbol{M} \tag{11.2}$$
$$\boldsymbol{K}_b = \boldsymbol{M}\times\hat{\boldsymbol{n}}, \tag{11.3}$$

With this we can write

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\oint_S \frac{\boldsymbol{K}_b(\boldsymbol{r}_i)}{r_{io}}dS + \frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{J}_b(\boldsymbol{r}_i)}{r_{io}}dV,$$

where we commuted the sum to compare this equation with its corresponding equation for the electric potential 8.7. Instead of the vector potential $\boldsymbol{A}$ here we have in equation 8.7 the scalar potential V there; intestead of the volume bound current density J_b here we have the volume bound charge density ρ_b there; and instead of the surface bound current density $\boldsymbol{K}$ here we have the surface bound charge density σ_b there.

11.3 Auxiliary Field H

11.3.1 Ampére's law in Magnetized Materials

We can define the free current density in a material from the equation

$$\boldsymbol{J}_f = \boldsymbol{J} - \boldsymbol{J}_b,$$

where $\boldsymbol{J}$ is the total current density and $\boldsymbol{J}_b$ is the bound current density defined in equation 11.2. The bound current is due to magnetization and the free current is due to moving charges. We know from Ampère's law

$$\frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) = \boldsymbol{J} = \boldsymbol{J}_b + \boldsymbol{J}_f = \nabla \times \boldsymbol{M} + \boldsymbol{J}_f.$$

That is

$$\boldsymbol{J}_f = \nabla \times \left(\frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M} \right).$$

We define a new concept

$$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}, \tag{11.4}$$

If we are dealing with a volumetric currents then we have the form of Ampère's law

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f. \tag{11.5}$$

Note that we are ignoring the surface bound currents $\boldsymbol{K}_b$. If these currents are localized we can make the surface in the first integral of equation 11.1 go to infinity and then the integral will vanish leaving us only with the volumetric currents. Still we need surface currents to be able to derive tangential boundary conditions. We will address this point in the boundary conditions sections that follows.

We can integrate equation 11.5 over an Amperian loop $\boldsymbol{\ell}$, and use Stoke's theorem to find

$$\oint_{\boldsymbol{\ell}} \boldsymbol{H} \cdot d\boldsymbol{\ell} = I_{f_{\text{enc}}},$$

where $I_{f_{\text{enc}}}$ is the total *free* current through the Amperian loop $\boldsymbol{\ell}$. At this point I want to make a parallel of between the electric and magnetic field. This is shown in table 11.1. For any given equation corresponding to the electric field we can "almost" replace in the equation the symbol appearing on the first column by that symbol in the second column and we would get the corresponding equation in the magnetic field. This table should grow as we introduce new definitions. We have not gave any name to the field $\boldsymbol{H}$. There is an interesting discussion about the names of $\boldsymbol{B}$ and $\boldsymbol{H}$ in [1]. In the same way that $\boldsymbol{D}$ allows us to write Gauss's law in terms of free charge only, $\boldsymbol{H}$ let us express

Table 11.1: A parallel between Electric and Magnetic Fields.

Electric	Magnetic
$\boldsymbol{E}$	$\boldsymbol{B}$
q	I
$\cdot$	$\times$
ϵ_0	$\frac{1}{\mu_0}$
V	$\boldsymbol{A}$
λ	$\boldsymbol{I}$
σ	$\boldsymbol{K}$
ρ	$\boldsymbol{J}$
$\boldsymbol{p}$	$\boldsymbol{m}$
$\boldsymbol{P}$	$\boldsymbol{M}$
σ_f	$\boldsymbol{K}_f$
ρ_f	$\boldsymbol{J}_f$
σ_b	$\boldsymbol{K}_b$
ρ_b	$\boldsymbol{J}_b$
$\boldsymbol{D}$	$\boldsymbol{H}$
Culoumb	Biot-Savart
Gauss	Ampère

Ampère's law based only on the free current. The advantage of $\boldsymbol{H}$ is that we can control free current. We do not have much control on bound currents.

When we were dealing with the electric field we found that $\nabla \times \boldsymbol{E} = 0$ while $\nabla \times \boldsymbol{D} = \nabla \times \boldsymbol{P}$. In the case of the magnetic field we know that $\nabla \cdot \boldsymbol{B} = 0$; however

$$\nabla \cdot \boldsymbol{H} = \nabla \cdot \frac{\boldsymbol{B}}{\mu_0} - \nabla \cdot \boldsymbol{M} = -\nabla \cdot \boldsymbol{M}. \tag{11.6}$$

Then if $\nabla \cdot \boldsymbol{M} \neq 0$ we have that $\nabla \cdot \boldsymbol{H} \neq 0$.

11.3.2 Boundary Conditions in H

We start with equation 11.6 by integrating over a small box in the surface as we have done with the electric and magnetic field before. Then after applying the divergence theorem we find

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}).$$

For the tangential boundary conditions we write

$$\nabla \times \boldsymbol{H} = \frac{1}{\mu_0} \nabla \times \boldsymbol{B} - \nabla \times \boldsymbol{M}.$$

We then integrate this along a volume to find

$$\int_V \nabla \times \boldsymbol{H} d\boldsymbol{x} = \int_V \frac{1}{\mu_0} \left(\nabla \times \boldsymbol{B} - \nabla \times \boldsymbol{M} \right) d\boldsymbol{x}.$$

Now we use identity A.13 to convert the volume integrals into surface integrals finding

$$\oint_{\boldsymbol{S}} \boldsymbol{H} \times \hat{\boldsymbol{n}} dS = \frac{1}{\mu_0} \oint_{\boldsymbol{S}} \boldsymbol{B} \times \hat{\boldsymbol{n}} dS - \oint_{\boldsymbol{S}} \boldsymbol{M} \times \hat{\boldsymbol{n}} \, dS.$$

Now we defined in equation 11.3 $\boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}}$ so we can write this equation as

$$\oint_{\boldsymbol{S}} \boldsymbol{H} \times \hat{\boldsymbol{n}} dS = \frac{1}{\mu_0} \oint_{\boldsymbol{S}} \boldsymbol{B} \times \hat{\boldsymbol{n}} dS - \oint_{\boldsymbol{S}} \boldsymbol{K}_b dS.$$

We now can pick a small box with very tiny edge normal to the surface and integrate over it. Since the sides do not contribute we only get the top and bottom contributions. Also since the box is small we can consider the fields to be constant on it. That we get from integrating the previous equation

$$(\boldsymbol{H}\times\hat{\boldsymbol{n}})_{\text{above}}-(\boldsymbol{H}\times\hat{\boldsymbol{n}})_{\text{below}}=\frac{1}{\mu_0}\left((\boldsymbol{B}\times\hat{\boldsymbol{n}})_{\text{above}}-(\boldsymbol{B}\times\hat{\boldsymbol{n}})_{\text{below}}\right)-(\boldsymbol{K}_{b_{\text{above}}}-\boldsymbol{K}_{b_{\text{below}}}).$$

Note that each of the terms above should be multiplied by the area S of the top (or bottom) of the box but we already simplified them. We know that the tangential component of a vector field $\boldsymbol{a}$ is extracted by taking the cross-product with the normal vector. That is

$$\boldsymbol{B}\times\hat{\boldsymbol{n}}=\boldsymbol{B}^{\parallel}.$$

We now use the magnetic field $\boldsymbol{B}$ tangential boundary conditions 10.7 to find that

$$\boldsymbol{H}^{\parallel}_{\text{above}}-\boldsymbol{H}^{\parallel}_{\text{below}}=(\boldsymbol{K}^{\parallel}_{\text{above}}-\boldsymbol{K}^{\parallel}_{\text{below}})-(\boldsymbol{K}^{\parallel}_{b_{\text{above}}}-\boldsymbol{K}^{\parallel}_{b_{\text{below}}})$$

But we can define

$$\boldsymbol{K}_{\text{free}}=\boldsymbol{K}-\boldsymbol{K}_b$$

and write instead

$$\boldsymbol{H}^{\parallel}_{\text{above}}-\boldsymbol{H}^{\parallel}_{\text{below}}=K^{\parallel}_{f_{\text{above}}}-K^{\parallel}_{f_{\text{below}}}$$

or in terms of the normal vector $\hat{\boldsymbol{n}}$,

$$\boldsymbol{H}^{\parallel}_{\text{above}}-\boldsymbol{H}^{\parallel}_{\text{below}}=\boldsymbol{K}_{f_{\text{above}}}\times\hat{\boldsymbol{n}}-\boldsymbol{K}_{f_{\text{below}}}\times\hat{\boldsymbol{n}}.$$

We can just call

$$\boldsymbol{K}_{\text{free}}=\boldsymbol{K}_{f_{\text{above}}}-\boldsymbol{K}_{f_{\text{below}}}$$

and write

$$\boldsymbol{H}^{\parallel}_{\text{above}}-\boldsymbol{H}^{\parallel}_{\text{below}}=\boldsymbol{K}_{\text{free}}\times\hat{\boldsymbol{n}}.$$

11.4 Relations between B and H

11.4.1 Magnetic Susceptibility and Permeability

In the section on linear dialectrics 8.1.7 we showed an approximation relationship between the polarization density vector $\boldsymbol{P}$ and the electric field $\boldsymbol{E}$ which is linear. That is, equation 11.7 that we write again here

$$\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}.$$

This linear relationship is a first order (linear) approximation valid if the field $\boldsymbol{E}$ is not too strong. We are tempted to extend this relationship to the magnetic field by saying that the magnetization density vector $\boldsymbol{M}$ is proportional to the magnetic field $\boldsymbol{B}$ and write

$$\boldsymbol{M} = \frac{1}{\mu_0}\chi_m \boldsymbol{B},$$

where χ_m is a constant. However this time the parallel between electric and magnetic field illustrated in table 11.1 fails. Table 11.1 is a good guide but it is not bullet proof. For historical reasons physicists decided that instead of relating M with B the proper thing was to stablish a linear relationship between $\boldsymbol{M}$ and $\boldsymbol{H}$. This breaks the parallel shown in table 11.1 at the expense of adding some confusion in the literature. This relation is

$$\boldsymbol{M} = \chi_m \boldsymbol{H}, \tag{11.7}$$

where χ_m is known as the **magnetic susceptibility** . Note that $\boldsymbol{M}$ is still linear with respect to $\boldsymbol{B}$. viTo see this observe that from equation 11.4

$$\boldsymbol{M} = \chi_m \frac{1}{\mu_0}\boldsymbol{B} - \chi_m \boldsymbol{M},$$

and so

$$\boldsymbol{M} = \chi_m \frac{1}{\mu_0(1+\chi_m)}\boldsymbol{B}, \tag{11.8}$$

so there is a linear relationship between $\boldsymbol{M}$ and $\boldsymbol{B}$ but the coefficient is not χ_m/μ_0 as we would like. I guess that the inclination to relate $\boldsymbol{M}$ directly to $\boldsymbol{H}$ and not to $\boldsymbol{B}$ is due to the fact that $\boldsymbol{H}$ is more popular among experimentalist since measuring free currents is easier than measuring the combined current due to a different in potentials and the bound current $\boldsymbol{J}_b$ current embedded in the material in the form of some magnetization phenomena. My argument is speculative but still I think it is unfortunate to break the symmetry so far established between the electric and magnetic fields. The magnetic susceptibility is a dimensionless quantity, its value changes according to material properties. The this website [1] shows the magnetic susceptibility values of a few substances. We observe that the magnetic susceptibility coefficients vary somewhere between 10^{-6} and 10^{-4}. Since equation 11.7 is a linear relationship between the field $\boldsymbol{H}$ and the field $\boldsymbol{M}$ we say that a substance which satisfies this relationship is **linear** .

If we combine equations 11.7 and 11.8 we find that

$$\boldsymbol{H} = \frac{1}{\mu_0(1+\chi_m)}\boldsymbol{B},$$

or

$$\boldsymbol{B} = \mu_0(1+\chi)\boldsymbol{H}.$$

We define a new quantity

$$\mu = \mu_0(1+\chi),$$

and rewrite the equation above as

$$\boldsymbol{B} = \mu\boldsymbol{H}.$$

The coefficient μ is known as the **permeability** of the material. In the vacuum $\chi_m = 0$ and so $\mu = \mu_0$ the permeability of free space.

11.4.2 Homogeneity and Isotropy

If $\boldsymbol{X}_m$ is independent of the position, that is if its spatial derivatives vanish we say that the material is homogeneous. Then we find

[1] http://www.kshitij-iitjee.com/Classification-of-magnetic-substances

$$\boldsymbol{J}_b = \nabla \times \boldsymbol{M} = \nabla \times \chi_m \boldsymbol{H} = \chi_m \nabla \times \boldsymbol{H} = \chi_m \boldsymbol{J}_f.$$

This interesting relationship shows a link between free and bounded currents and that the bound currents are much smaller (about 10^{-5}) than the free currents.

If the material is not homogeneous it is heterogeneous and then instead of the equation above we have

$$\boldsymbol{J}_b = \nabla \times \boldsymbol{M} = \nabla \times \chi_m \boldsymbol{H} = \chi_m \nabla \times \boldsymbol{H} + \nabla \chi_m \times \boldsymbol{H} = \chi_m \boldsymbol{J}_f + \nabla \chi_m \times \boldsymbol{H}.$$

We see now that the vectors J_b and J_b are not necessarily collinear.

It could happen that the material is anisotropic. That is the relation between $\boldsymbol{B}$ and $\boldsymbol{H}$ although linear does not preserve the direction. That is, while $\boldsymbol{H}$ could point in some direction $\boldsymbol{B}$ could point in a different relation. The relation is through a matrix μ of permeability coefficients. That is, it could happen that

$$\mu = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33}. \end{pmatrix}$$

It happens that this matrix is symmetric (the prove of this is outside of the scope of these notes). Together with the matrix $\epsilon = (\epsilon_{ij})$ we need 12 coefficients (6 from the matrix ϵ and 6 from the matrix μ) to completely characterize an isotropic material. The two equations

$$D_i = \sum_{j=1}^{3} \epsilon_{ij} E_j \quad , \quad B_i = \sum_{j=1}^{3} \mu_{ij} H_j$$

are known the electric and magnetic **constitutive relations** for matter. Please refer to the Wikipedia [2] website for more information about constitutive equations.

11.5 Ferromagnetism

The magnetic properties of a material depend on how the electrons are configured in the atom. If electrons are paired up (that is in pairs of electrons where one electron of the

[2] https://en.wikipedia.org/wiki/Constitutive_equation

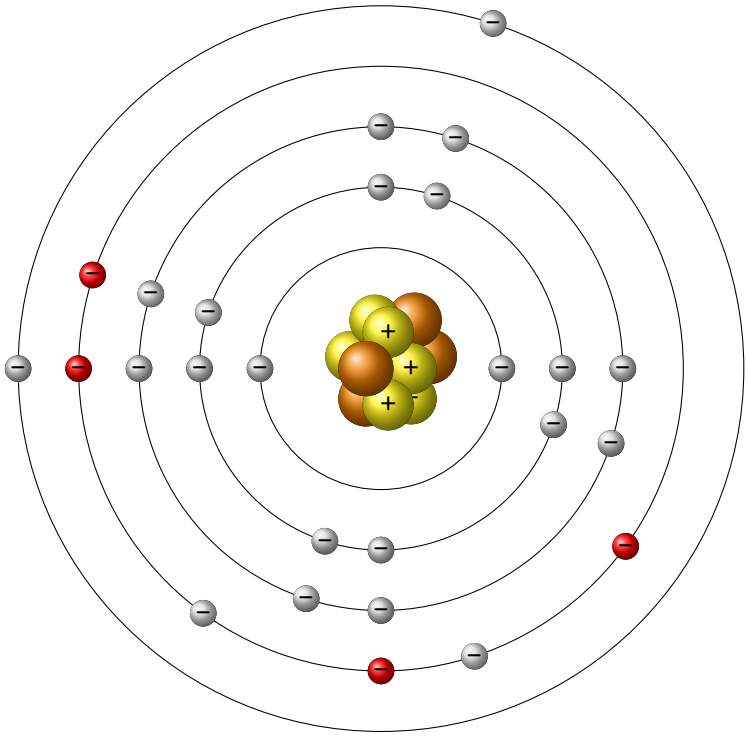

Figure 11.2: A sketh of an iron atom. All electrons but 4 (red) electrons are unpaired. The unpaired electrons create magnetic dipoles that contribute to the ferromagnetic character of the iron material.

pair is spinning in one direction and the other in an opposite direction) the material is a diamagnet. Electrons unpaired in an atom cause paramagentism or ferromagetism in the material. Figure 11.2 shows an sketch of an iron atom where 4 electrons (red) are unpaired.

Ferromagnets present some interesting properties. For example they are divided in some zones known as **domains.** In each domain the atoms are polarized in the same direction but between domains the polarization direction is random. When subjected to a magnetic field the ferromagnets move the walls of the domains in a way as to align the magnetic dipoles along the external magnetic field. The Wikepedia [3] website show an excellent treatment of domains (also known as Weiss domains). Photographs as well as movies of microcrystalline grains are displayed on this website. A movie recorded

[3] https://en.wikipedia.org/wiki/Magnetic_domain

from a Kerr microscope which illustrates how the walls of the domains in a silicon steel grain are shifted so that the magnetic dipoles align with the magnetic field which is directed downward. When paramagnets and diamagnets are subjected to magnetic fields their magnetic dipoles tend to align with the magnetic field. The stronger the magnetic field, the stronger the alignment. The lower the temperature the stronger the alignment. Once the magnetic field disappears these paramagnets and diamagnets return to the original state of no magnetic polarization. Ferromagnets present a stronger dipole magnet alignment under strong magnetic field and low temperatures, however they do not return completely to an unpolarized material. Some remaining magnetization stays after the field is suspended.

In summary, a ferromagnet is a material such that

- It has a few electrons in the atom unpaired.
- It is divided in some regions known as domains. On each domain the magnetic dipole moments are aligned in the same direction but between moments there is no a preferred alignment. Domains can have a number of atoms of the order of 10^{20}.
- When the ferromagnet is subjected to some external magnetic field the domains move their walls in a way that there is a preferred direction for magnetic polarization along the whole volume. This direction is that of the exterior magnetic field.
- After the magnetic field is turned off some remaining magnetization stays in the ferromagnet.

Here is how we can magnetize a piece of iron. Think of an iron bar. Let us assume that we can wrap the bar with some coil. We can create an external field by injecting some current through the coil in some direction. As we increase the current the magnetization increases, that is the walls spread and the magnetic dipoles inside the iron bar align in the direction of the magnetic field. There is a saturation point such that no further alignment is possible. At this point we can slow down the current until we can stop it completely. We should observe that even though there is no current, there is some magnetic polarization remaining in the iron bar. We would like to take this polarization back to zero where the dipole moments are randomly oriented. This can be possible by reversing the current so that the magnetic field now goes in the opposite direction. If we increase the magnitude of the current eventually the magnetization goes in the opposite direction of the first magnetization and will reach a point where no further alignment could be achieved. That is, the magnetization reached a saturation point. If

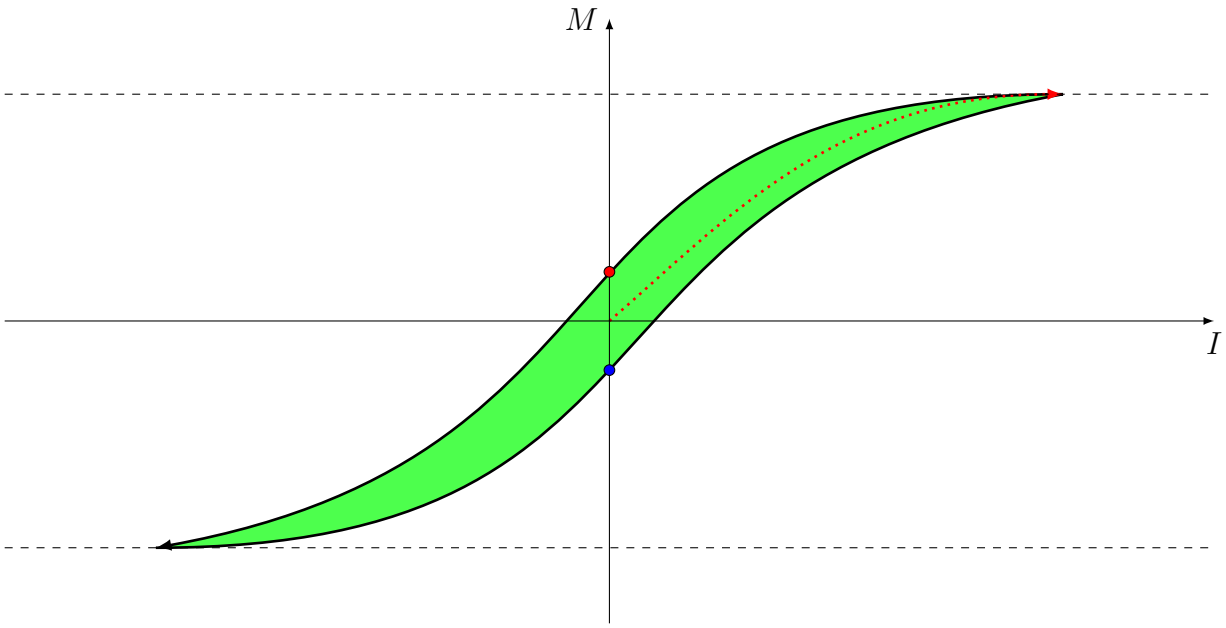

Figure 11.3: A hysteresis loop. Let us assume that we start where there is no current and no magnetization. That is at $(0, 0)$. We turn the current on and move through the red line until the magnetization reachs a saturation point (upper right). Then turn the current off and the magnetization reduces but not to zero (the red point in the positive y axis). At this point we are at the point $(0, \boldsymbol{M})$. Then we turn the current in the different direction and move to negative critial point (lower left). Then turn the current back to zero (the blue point in the negative y axis)

we slow down the current magnitude until it is turned off completely we observe that the magnetization do not get to zero but some remaining (opposite to the first) magnetization stays in the iron bar. If, again, we turn the current back in the original direction the magnetization increases until it reaches the saturation point of the first face. The whole cycle explained here is known as hysteresis. Figure 11.3 shows an sketch of this process.

We now wonder how can we get back to the point $(0, 0)$? Of course we need to turn the current off, but, to remove the remaining magnetization in the iron bar we could heat it until it reaches some temperature (the **Curie point**). This point shaply removes the remaining magnetization. Different materials present different Curie points. The Wikepedia [4] website presents good illustrations of the phenomena observed when

[4] https://en.wikipedia.org/wiki/Curie_temperature

a magnetized material reaches the Curie temperature. It also shows a table (in Kevin scale) with different Curie points for ferromagents such as iron, cobalt, nickel and others. Another way to remove the permanent magnetization is by hitting the material to shake its dipole moments into random directions.

Chapter 12

Lecture #12

12.1 Electromotive Force

12.1.1 Ohm's Law

We now wonder what makes charges to move. There should be a force pushing the charges so that they get displaced along a wire or some material. A first approximation model is to propose that the current is directly proportional to a force which pushes the charges. That is, we establish a linear relationship between current density $\boldsymbol{J}$ and force $\boldsymbol{f}$. This relation in the literature is expressed by the equation

$$\boldsymbol{J} = \sigma \boldsymbol{f},$$

where σ is a constant known as the **conductivity** of the material. In the literature it is common to see the inverse of this constant $\rho = 1/\sigma$ and call it **resistivity.** The resistivity and conductivity [1] Wikipedia website has a complete treatment of conductivity and presents a table with the conductivity (and resistivity) of a few materials such as carbon, silver, copper, gold, etc.

It is important, at this point, to note that we already used the symbols σ and ρ for surface and volume charge densities. Still, I will stick to this notation since it is commonly used in the literature.

We now wonder what kind of force $\boldsymbol{f}$ could be pushing charges. We can think of gravity, chemical, or other forces; however in our case we are interested in electromagnetic

[1] https://en.wikipedia.org/wiki/Electrical_resistivity_and_conductivity

forces which would have the greatest impact in the charge displacement. In this case the forces are those due to Laurentz law. That is, we can write

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}),$$

where $\boldsymbol{v}$ is the velocity of the moving particle. While the propagation speed (and we will have more to say about this later) is very high the particle velocity or drift velocity [2] of charges is quite small and usually the second term is considered negligible. A copper electron could take up to 40 hours to travel 1 meter. In the context of these notes we will approximate the current density with only the contribution due to the electric field. That is we write

$$\boldsymbol{J} = \sigma \boldsymbol{E}. \tag{12.1}$$

This equation is known as **Ohm's law** .

If charges are stationary then $\boldsymbol{J} = 0$ and since $\sigma \neq 0$ then $\boldsymbol{E} = 0$ which is consistent with the fact that charges inside a conductor produce a null electric field. If charges are moving equation 12.1 applies. In metals where σ is conductivity is very high we find that $\boldsymbol{E} = \boldsymbol{J}/\sigma \approx 0$. If the conductivity is perfect (∞) then $\boldsymbol{E}$ would be actually 0. A **conductor** is a material with high conductivity while a **resistor** is a material with low conductivity. There is no a sharp division between a conductor and a resistor. All conductors have some resistance. Insulators have a high resistance while conductors have very low resistance. The dynamic range between conductors and insulators is usually large. From the resistivity and conductivity [3] Wikepedia we can observe conductivity values from the order of 10^8 to order 10^{-8}.

Metals are good conductors and non-metals are good insulators so according to the purpose and material being used at the moment in the laboratory it could be useful to use one (conductor) or the other (resistor) term.

According to Wikipedia, [4] the formulation in equation 12.1 is due to Gustav Kirchhoff. [5] However the original formulation of Ohm's law, due to Ohm himself is different. Ohm's law was given by Georg Ohm [6] in a 1827 publication as.

[2] https://en.wikipedia.org/wiki/Drift_velocity
[3] https://en.wikipedia.org/wiki/Electrical_resistivity_and_conductivity
[4] https://en.wikipedia.org/wiki/Ohm%27s_law
[5] https://en.wikipedia.org/wiki/Gustav_Kirchhoff
[6] https://en.wikipedia.org/wiki/Georg_Ohm

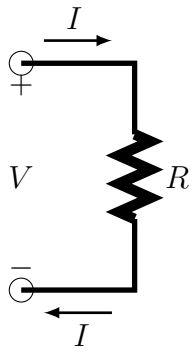

Figure 12.1: Illustration of the symbols I for current, V for voltage and R for resistance used in the equation for Ohm's law.

$$I = \frac{V}{R} \tag{12.2}$$

where I is the current through a conductor in amperes, V is the voltage measure across the conductor in volts and R is the **resistance** in ohms. Figure 12.1 illustrates the symbols involved in the formulation of Ohm's law 12.2.

To show that Ohm's law in equation 12.2 is a consequence of the first formulation presented in equation 12.1 let us think of a little piece of cylindrical wire with cross sectional area A, length L, and conductivity σ. We apply a potential difference between the two ends of the wire V which causes a current I to flow. On the cylindrical surface $\boldsymbol{J} \cdot \hat{\boldsymbol{n}} = 0$, otherwise there would be currents flowing through the surface and current should follow the direction of the axis of the cylindrical wire. Since $\boldsymbol{J} = \sigma \boldsymbol{E}$ we see that $\boldsymbol{E} \cdot \hat{\boldsymbol{n}} = 0$. Then the potential V satisfies the equation $\partial V/\partial n = 0$, where $\partial/\partial n$ is the normal derivative in the radial direction of the cylinder. At this point we know the potential in the two end sections of the cylinder and its normal derivative in the surface of the wire. From the uniqueness theorem for the Laplace equation (see Problem 3.4.1) we know that there is a unique solution to the differential equation $\nabla^2 V = 0$. Since the radial derivative of the potential is 0, the potential should only change along the axis (let us call it z) of the cylinder. That is we have $V = V(z)$ and any linear function of z satisfies the Laplace's equation. In particular

$$V(z) = az$$

is a solution of the Laplace's equation. We still need to determine a. Let us assume that the first end of the wire is at $z = 0$ and that the potential there is 0 and the second end is L and the potential there is V_0. That is $V(L) = aL = V_0$, then

$$a = \frac{V_0}{L},$$

and the potential is given by

$$V(z) = \frac{V_0 z}{L},$$

so the electric field is given by

$$\boldsymbol{E} = -\nabla V = -\frac{V_0}{L}\hat{\boldsymbol{z}}.$$

Since $\boldsymbol{E}$ is uniform and $\boldsymbol{J} = \sigma \boldsymbol{E}$ then $\boldsymbol{J}$ is also uniform we then can compute the current I from the current density as follows:

$$I = JA = \sigma EA = \frac{\sigma A}{L} V.$$

We then can write this equation as

$$I = \frac{V}{R} \quad , \quad R = \frac{L}{\sigma A},$$

and so the form of Ohm's law 12.2 is explicit on the example. Here the resistance R is inversely proportional to A and this makes sense, the thinner the wire the harder is for the particles to flow through it. Also if the section of cable is longer we expect more resistance, and the resistance is inversely proportional to the conductivity σ.

12.1.2 Electromotive Force

The electromotive force (emf) is an amount of energy per unit charge transferred from some type of source such as chemical, thermal, piezoelectric, etc. to electrical energy. The electromagnetic force is the potential difference that forces current to flow in a

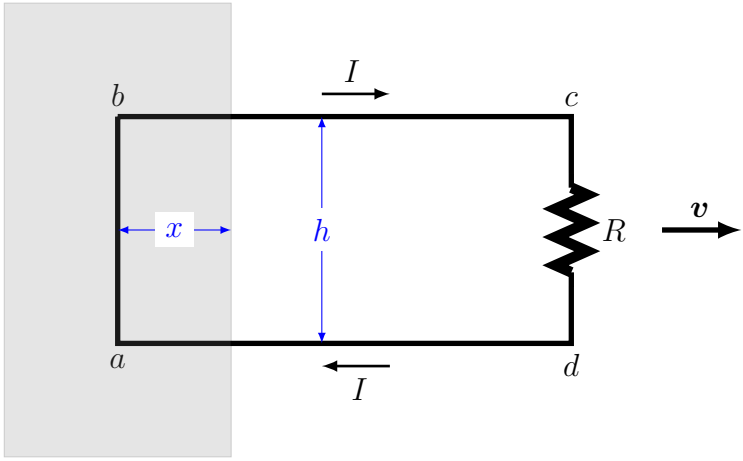

Figure 12.2: A wire with a resistor in a magnetic field pointing inside the page.

circuit. The symbol for emf is $\mathcal{E}$ and its units are volts. Please observe that that the electromotive force is not actually a force.

We represent the emf with the integral‘

$$\mathcal{E} = \oint \boldsymbol{f} \cdot d\boldsymbol{\ell}.$$

where $\boldsymbol{f}$ is the force per unit charge. Typically the loop $\boldsymbol{\ell}$ is an electric circuit. This integral measures the potential difference between the two ends of a circuit.

12.1.3 Motional emf

We said above that the emf is generated by a difference in potential created by some sources. An important source to generate currents comes from the force created by a moving magnetic field. That is, from the Laurentz law we know that a moving magnetic field can create a force and this force can push electrons in a loop.

Let us assume that we have a wire loop with a resistance in a magnetic field which points inside the page. Figure 12.2 illustrates this. The magnetic field pointing inside the page is described by the gray shade. Since we have free charges on the wire and they are moving to the right at speed $\boldsymbol{v}$ a Laurentz force is created. The magnetic Laurentz's force per unit charge is $\boldsymbol{F}_{\text{mag}} = \boldsymbol{v} \times \boldsymbol{B}$. The force, being the cross product of $\boldsymbol{v}$ and B which are along the x-z axes (assuming the z axis into the page) is along the y direction

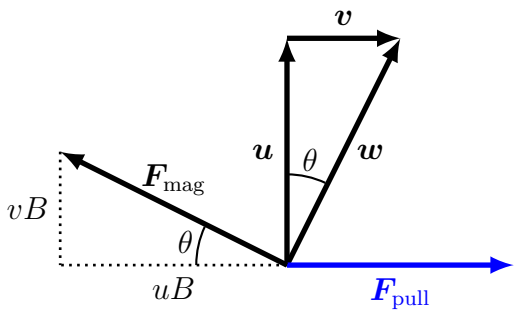

Figure 12.3: The horizontal velocity $\boldsymbol{v}$ creates a current along the vertical with velocity $\boldsymbol{u}$. The resultant velocity is now $\boldsymbol{w}$. This will create a magnetic force per unit charge $\boldsymbol{F}_{\text{mag}}$ toward the north-east direction. The horizontal component uB requires some $\boldsymbol{F}_{\text{pull}}$ (blue)force to the right to compensate and be able to move the loop to the right. That force makes the work of moving the loop to the right.

and so $\boldsymbol{F}_{\text{mag}} \cdot d\boldsymbol{\ell} = 0$ along the two segments bc and ad which are perpendicular to the force. No work is done here. We end up only with the integral along the vertical segments. However the edge cd is outside of the field $\boldsymbol{B}$ and the force is due only to the edge ab. That is, if $\boldsymbol{v}$ and $\boldsymbol{B}$ are constants, then along the segment ab and

$$\mathcal{E} = \oint \boldsymbol{F}_{\text{mag}} \cdot d\boldsymbol{\ell} = hBv. \tag{12.3}$$

If the magnetic field does not make work who is making the work here? The answer is: whoever is moving the loop to the right. Once the current start flowing upward at a velocity $\boldsymbol{u}$, we then have a combination of two velocities. The horizontal velocity $\boldsymbol{v}$ and the vertical velocity $\boldsymbol{u}$. The total velocity is now $\boldsymbol{w}$ and the force, being normal to velocity is directed now on the north-west direction. This creates a horizontal magnetic force component per unit charge uB to the left and a magnetic force vertical component per unit charge vB up. Figure 12.3 illustrates this. To be able to pull the loop at a constant velocity $\boldsymbol{v}$ the horizontal component $\boldsymbol{F}_{\text{pull}}$ in the figure should cancel out the horizontal component created by the magnetic field uB. That is we need $\|\boldsymbol{F}_{\text{pull}}\| = uB$. The particle displacement is along the direction of the particle velocity $\boldsymbol{w}$. The total displacement is then given by $\ell = h/\cos\theta$, and the work done here is

$$\mathcal{E}' = \int_{\ell} \boldsymbol{F}_{\text{pull}} \cdot d\boldsymbol{\ell} = uB \left(\frac{h}{\cos\theta} \right) \sin\theta.$$

where $\mathcal{E}'$ means here just work per unit of charge. But $\tan\theta = v/u$, so we find

$$\mathcal{E}' = hBv,$$

which is what we found in equation 12.3. We found then that $\mathcal{E} = \mathcal{E}'$. That is, the work per unit charge along the loop is the same work per unit charge done by the particle going along the tilted trajectory in Figure 12.3. Note that these computations are done at a snapshot, an instant of time. The force, which initially is pointing up, its tilted counter clockwise to the angle shown in the figure and this creates a different trajectory which changes continuously. Something like the magnetic force $\boldsymbol{F}_{\text{mag}}$ rotating counterclockwise.

We now express the emf in terms of the magnetic flux. The magnetic flux through a loop $\boldsymbol{\ell}$ enclosing an area $\boldsymbol{S}$ is defined by the surface integral

$$\Phi \equiv \int_{\boldsymbol{S}} \boldsymbol{B} \cdot d\boldsymbol{S},$$

and for the constant magnetic field in our example this is

$$\Phi = Bhx,$$

where the area of the loop is Bhx (note that we are only interested on the area under the field $\boldsymbol{B}$.) We now take into account time t. That is we define $x = x(t)$, so

$$\frac{d\Phi}{dt} = hB\frac{dx}{dt} = -hBv \tag{12.4}$$

with $v = \|\boldsymbol{v}\|$. The negative sign is due the fact that the velocity is pointing to the right and the x function (please observe Figure 12.2 again) decreases in the direction of the vector $\boldsymbol{v}$. We then find that equation 12.4 produces the same result as equation 12.3, and in this particular case we can write

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

We show that in fact this property, named as the **flux rule** , is general and it does not depend of the form of the loop. Let us assume that we have the flux computed at two near times t and $t + dt$. At t we have a surface S, and at $t + dt$ a surface S', the two surfaces have some overlapping region and the non-overlapping region we call Σ. Then

$$d\Phi = \Phi(t + dt) - \Phi(t) = \Phi_\Sigma = \int_\Sigma \boldsymbol{B} \cdot d\boldsymbol{S}.$$

We want to find the element of surface $d\boldsymbol{S}$ between the two snapshots. A piece of the wire moves a distance $\boldsymbol{v}dt$, the piece of wire has a length element $d\boldsymbol{\ell}$. The size and orientation of the small element of surface (a parallelogram) is

$$d\boldsymbol{S} = (\boldsymbol{v} \times d\boldsymbol{\ell})dt.$$

We then can say that

$$\frac{d\Phi}{dt} = \oint \boldsymbol{B} \cdot (\boldsymbol{v} \times d\boldsymbol{\ell}). \tag{12.5}$$

We know that the actual velocity $\boldsymbol{w}$ of a particle in the wire is

$$\boldsymbol{w} = \boldsymbol{v} + \boldsymbol{u}$$

where $\boldsymbol{u}$ is the component of $\boldsymbol{w}$ along the direction normal to $\boldsymbol{v}$, and so we can write equation 12.5 as

$$\frac{d\Phi}{dt} = \oint \boldsymbol{B} \cdot (\boldsymbol{w} \times d\boldsymbol{\ell}).$$

We can commute the triple product (choosing the appropriate sign) to find

$$\frac{d\Phi}{dt} = -\oint (\boldsymbol{w} \times \boldsymbol{B}) \cdot d\boldsymbol{\ell}.$$

However $\boldsymbol{w} \times \boldsymbol{B} = \boldsymbol{F}_{\text{mag}}$ the magnetic force caused by $\boldsymbol{B}$ on the wire moving with a velocity vector $\boldsymbol{w}$. Then

$$\frac{d\Phi}{dt} = -\oint \boldsymbol{F}_{\text{mag}} \cdot d\boldsymbol{\ell}.$$

The last integral is precisely the negative of the emf E. That is,

$$\mathcal{E} = -\frac{d\Phi}{dt},$$

as claimed. The minus "-" sign in front of $d\Phi/dt$ is an statement of **Lenz law.** It says that the induced flux (current) due to the changing magnetic field opposes the emf. Let us see it this way. A changing magnetic field induces some current. The current itself (by Ampere's law) induces a new magnetic field. That new magnetic field is created in a direction as to reduce or oppose the original magnetic field that created the currents. It is like an inertia principle.

12.2 Electromagnetic Induction

12.2.1 Faraday's Law

The Faraday's law of induction relates the magnetic field in an electric circuit to the emf. According to Wikipedia [7] Michael Faraday in 1831 and Joseph Henry in 1832 discovered what we know today as the law of electromagnetic induction. Faraday was looking for different sources of emf other than batteries. He knew Ampere's law and the fact that a steady electric field (current) created a magnetic field. Then he conjectured that a steady magnetic field could create an electric field but he failed in its experiments. He later set up three basic experiments and recorded some observations. The three basic experiments were as follow. He took a loop of wire and a magnet. Then,

(i) when moving the wire through the magnetic field he observed some current.

(ii) When moving the magnet with respect to the wire he observed some current.

(iii) While keeping the wire and magnet fixed he changed the strength on the magnetic field and observed some current.

We already studied the first experiment above in the motional emf section and found

$$\mathcal{E} = -\frac{d\Phi}{dt}. \tag{12.6}$$

[7] https://en.wikipedia.org/wiki/Faraday%27s_law_of_induction

There is no surprise on finding the second experiment provides the same result. What really matters here is the relative motion between the circuit and the magnet. Faraday claimed that a changing magnetic field induces an electric field. However he did not have the mathematical tools to formulate the law in mathematical terms. The mathematical formulation was provided by James Clerk Maxwell [8] and Oliver Heaviside [9]. We state initially that the Faraday's law is given by equation 12.6. Since $\mathcal{E} = \oint \boldsymbol{E} \cdot d\boldsymbol{\ell}$ and $\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{S}$ we find the Faraday law in integral form

$$\oint \boldsymbol{E} \cdot \boldsymbol{\ell} = -\int \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{S}.$$

The differentiation of $\boldsymbol{B}$ inside is a partial derivative since $\boldsymbol{B}$ depends also on spatial coordinates $\boldsymbol{x}$. We can convert the first integral into a surface integral by using Stoke's theorem. Then since the surfaces of integration could be arbitrary we find the differential form of the Faraday's law

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{12.7}$$

The first observation is that if the magnetic field $\boldsymbol{B}$ is time independent, that is if its derivative with respect to time is 0, then $\int_{\boldsymbol{\ell}} \boldsymbol{E} \cdot d\boldsymbol{\ell} = 0$, or $\nabla \times \boldsymbol{E} = 0$ as was proven already in electrostatics.

The induction law gave rise to the invention of the electric motor which is a big part of the industrial revolution and in a way has run the economy of the world.

12.2.2 Induced Electrical Field

With Faraday's law we now have two different ways to generate electric fields: those responding to electric charges and those associated with a changing magnetic field. We computed the first using Coulomb's law; the second one can be found by using Faraday's law 12.7

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}.$$

We showed that Ampère's law is given by

[8] https://en.wikipedia.org/wiki/James_Clerk_Maxwell
[9] https://en.wikipedia.org/wiki/Oliver_Heaviside

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}.$$

We can see a parallel between these two equations and in the same way that we computed $\boldsymbol{B}$ from the knowledge of the current $\boldsymbol{J}$ we can compute the electric field $\boldsymbol{E}$ with the knowledge of the time derivative of the magnetic field. In integral form we have a parallel between Ampère's law

$$\oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_o I_{\text{enc}}$$

and Faraday's law

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi}{dt}.$$

Then the tricks used to find $\boldsymbol{B}$ around Ampèrian loops can be recycle here using Faradayan loops.

It is important to recognize that the use of Ampère's law is not valid unless we are in a magnetostatic context. In principle the magnetostatic approximation is good unless the field fluctuates too rapidly.

12.2.3 Inductance

Let us assume that we have two loops of wire. We run a steady current I_1 around loop 1. This produces a magnetic field $\boldsymbol{B}_1$. Some field lines for $\boldsymbol{B}_1$ go through the second loop. This means that there will be a flux Φ_2 through the second loop. Since the current I_1 is steady we could compute Biot-Savart law

$$\boldsymbol{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\boldsymbol{\ell}_1 \times \hat{\boldsymbol{r}}_{io}}{r_{io}^2}.$$

Since the we consider the magnitude I_1 of the current constant we see that $\boldsymbol{B}_1$ is proportional to I_1 and so the flux

$$\phi_2 = \int \boldsymbol{B}_1 \cdot d\boldsymbol{S}$$

is proportional to I_1 as well. That is,

$$\Phi_2 = K_{21} I_1, \tag{12.8}$$

where K_{21} is a constant of proportionality that we will find below.

Let us now assume that $\boldsymbol{A}_1$ is the vector potential for $\boldsymbol{B}_1$

$$\boldsymbol{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\boldsymbol{\ell}_1}{r_{io}}$$

Then

$$\Phi_2 = \int_{\boldsymbol{S}_2} \boldsymbol{B}_1 \cdot d\boldsymbol{S}_2 = \int_{\boldsymbol{S}_2} (\nabla \times \boldsymbol{A}_1) \cdot d\boldsymbol{S}_2 = \oint_{\boldsymbol{\ell}_2} \boldsymbol{A}_1 \cdot d\boldsymbol{\ell}_2 = \frac{\mu_0 I_1}{4\pi} \oint_{\boldsymbol{\ell}_2} \left(\oint_{\boldsymbol{\ell}_1} \frac{d\boldsymbol{\ell}_1}{r_{io}} \right) \cdot d\boldsymbol{\ell}_2,$$

and matching this equation with equation 12.8 we find that

$$K_{21} = \frac{\mu_0}{4\pi} \oint_{\boldsymbol{\ell_2}} \left(\oint_{\boldsymbol{\ell}_1} \frac{d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2}{r_{io}} \right)$$

This formula is known as **Neumann formula** . While it is not useful for practical calculations it shows two important things.

(i) K_{12} depends only on the geometry of the loops.

(ii) The integral is the same if we interchange the two loops. That is we find that

$$K_{21} = K_{12}.$$

This is and interesting finding; it means reciprocity. The flux through loop 2 when running a current I through loop 1 is identical to to the flux through loop 1 when we run the same current I through loop 2. We can drop the subscripts and simply use K for the constant of proportionality and write

$$\Phi_2 = K I_1.$$

We now can use Faraday's law and write

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -K\frac{dI_1}{dt}.$$

Please observe that when taking the derivative of I_1 with respect to t we are assuming that the current changes with time and this violates the use of the Biot-Savart law. We assume that the change is slow and the equation is an approximation. This type of approximations are known as **quasistatic** approximations. The negative sign in from of the derivative above implies that that the emf produced in the second loop due to a change of current in the first loop resists the change on the current I_1.

We see how a current on one loop induces a current in a second loop. Let us now think of only one loop. A current on the loop induces a current on itself. In this case the constant of proportionality is called **self-inductance** or **inductance** L and we write

$$\Phi = LI. \tag{12.9}$$

In this case we write

$$\mathcal{E} = -L\frac{dI}{dt}. \tag{12.10}$$

Inductance is measured in **henries** (H). A henry is a volt-second per ampere. An inductor in a circuit regulates current. If the current is increasing then the voltage opposing the change is created by the magnetic field of the coil. Figure 12.4 illustrates a simple circuit with an inductor and a source of emf.

12.2.4 Energy in Magnetic Fields

Energy Stored in Inductors

We know that $\mathcal{E}$ is the magnetic work done by one charge in the whole circuit. If Q is the total amount of charges in the being moved in the circuit then

$$W = -\mathcal{E}Q$$

the minus "-" sign is convention, if we choose a negative sign we would get a negative energy. That is, we choose the work done against the emf.

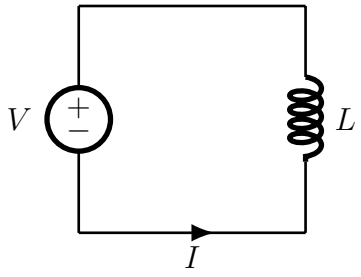

Figure 12.4: The inductor labeled with L regulates current so that if the current I is increasing a voltage opposing the change is created by the magnetic field of the coil.

Taking the derivative of this equation, assuming that the emf is time invariant, and using equation 12.10 we find that

$$\frac{dW}{dt} = -\mathcal{E}\frac{dQ}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}.$$

If we integrate this equation in time we find

$$W = \frac{1}{2}LI^2.$$

That is, the energy stored in an inductor depends only on the geometry of the inductor and on the total current through it. We want to generalize this circuit (1D manifold) energy concept to the energy going through surfaces and volumes. We do this next using the concept of magnetic flux.

Energy Stored in Magnetic Fields

From the definition of magnetic flux, the vector potential of $\boldsymbol{B}$ and the Stoke's theorem we have that

$$\Phi = \int_{\boldsymbol{S}} \boldsymbol{B} \cdot d\boldsymbol{S} = \int_{\boldsymbol{S}} (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{S} = \oint_{\boldsymbol{\ell}} \boldsymbol{A} \cdot d\boldsymbol{\ell}.$$

Since flux as given by equation 12.9 is $\Phi = LI$, then we can write

$$LI = \oint_{\boldsymbol{\ell}} \boldsymbol{A} \cdot d\boldsymbol{\ell},$$

and so

$$W = \frac{1}{2} LI^2 = \frac{1}{2} I \oint_{\boldsymbol{\ell}} \boldsymbol{A} \cdot d\boldsymbol{\ell}.$$

We assume that the current is spatially constant along the loop and take the I inside the integral to find

$$W = \frac{1}{2} \oint_{\boldsymbol{\ell}} (\boldsymbol{A} \cdot \boldsymbol{I}) dl.$$

Since $\boldsymbol{I} = \boldsymbol{J} dS$ we can integrate along the surface to get a new form of energy (which we call also W) this time along a volume. That is

$$W = \frac{1}{2} \int_V (\boldsymbol{A} \cdot \boldsymbol{J}) d\boldsymbol{r}_i.$$

where $\boldsymbol{r}_i = \boldsymbol{x}$ is an integration variable running in the volume V. We now use Ampere's law $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ (again here we assume a quasistatic approximation) to write the previous integral as

$$W = \frac{1}{2\mu_0} \int \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B}) d\boldsymbol{r}_i.$$

We now do multi-dimensional integration by parts using the identify A.7. That is

$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B}),$$

or using the definition of potential

$$\boldsymbol{A} \cdot (\nabla \times \boldsymbol{B}) = \boldsymbol{B} \cdot \boldsymbol{B} - \nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}),$$

then, using the divergence theorem

$$\begin{aligned} W &= \frac{1}{2\mu_0}\left[\int_V B^2 d\boldsymbol{r}_i - \int_V \nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})d\boldsymbol{r}_i\right] \\ &= \frac{1}{2\mu_0}\left[\int_V B^2 d\boldsymbol{r}_i - \oint_{\boldsymbol{S}} (\boldsymbol{A}\times\boldsymbol{B})d\boldsymbol{S}\right] \end{aligned}$$

If we allow the integration to go to infinity and providing that the decay of the product of the potential $\boldsymbol{A}$ and the field $\boldsymbol{B}$ is larger than $1/r^2$ we can set the second integral to zero and write

$$W = \frac{1}{2\mu_0}\int_{\mathbb{R}^3} B^2 d\boldsymbol{r}_i. \tag{12.11}$$

We see the parallel of this equation and its counterpart for the electric field 6.5

$$W_E = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} E^2 d\boldsymbol{r}.$$

12.3 Maxwell's Equations

12.3.1 Introduction

Maxwell's equations form the foundations of classical electromagnetism, classical optics, and electric circuits. They have a central role in classic and modern technology such as for example optical and radio technologies, power generation, motors, wireless communications, television, computers, etc. Maxwell equations present a summary of the theory so far covered in these notes based on electrical charges, electrical currents and their variation.

According to Wikipedia [10] Maxwell proposed originally 20 equations and the four equations that we will present below are due to Oliver Heaviside [11].

The four Maxwell equations in differential form are

[10] https://en.wikipedia.org/wiki/Maxwell%27s_equations
[11] https://en.wikipedia.org/wiki/Oliver_Heaviside

$$\begin{aligned}
\nabla \cdot \boldsymbol{E} &= \frac{1}{\epsilon_0}\rho \qquad \text{(Gauss's law)} \\
\nabla \cdot \boldsymbol{B} &= 0 \qquad \text{(no name)} \\
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{(Faraday's law)} \\
\nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{J} \qquad \text{(Ampére's law).}
\end{aligned}$$

We find that there is an inconsistency on these equations. We know that the divergence of the curl is 0 (see equation A.3). However, from Ampére's law we find

$$\nabla \cdot (\nabla \times \boldsymbol{B}) = \mu_0 (\nabla \cdot \boldsymbol{J}).$$

While for steady currents the divergence of the current density is 0, in general the right hand side of this equation is non-zero. For example charges pile up in a capacitor and there the current density has non-zero divergence. In fact, the continuity equation 9.2 we see that if the charge density ρ is not stationary we get a non-zero divergence. The context of Ampère's law is magnetostatics and we should not expect to extend to the electrodynamic domain. Next we show how Maxwell fixed this problem.

12.3.2 How Maxwell Fixed Ampére's Law

Maxwell used the continuity equation 9.2 to fix the inconsistency above. That is

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial (\epsilon_0 \nabla \cdot \boldsymbol{E})}{\partial t} = -\nabla \cdot \left(\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right)$$

We now move the last term to the left to find

$$\nabla \cdot \mu_0 \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) = 0.$$

So instead of having Ampère's law applied to $\boldsymbol{J}$ only we define a new Ampère's law

$$\nabla \times \boldsymbol{B} = \mu_0 \, \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}.$$

Of course in magnetostatics when we consider that the electric field $\boldsymbol{E}$ is not changing with time the traditional Ampère's law is obtained from the previous equation. The addition to the last term in the equation above provides more symmetry to Maxwell's equation. Note that the change of magnetic field with time induces an electric field. Likewise the change of electric field with time induces some magnetic field $\boldsymbol{B}$.

12.3.3 Maxwell's Equations

After the correction for Ampère's law we now have the final Maxwell equations

$$\begin{aligned}
\nabla \cdot \boldsymbol{E} &= \frac{1}{\epsilon_0}\rho && \text{(Gauss's law)} \\
\nabla \cdot \boldsymbol{B} &= 0 && \text{(no name)} \\
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} && \text{(Faraday's law)} \\
\nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} && \text{(Extended Ampére's law).}
\end{aligned}$$

We add to these equations the Laurentz law formula

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}).$$

Together, the Maxwell's equation and the Laurent'z law above form the core of classical electrodynamics. We observe that, for example, the continuity equation is obtained from the extended Ampère's law by taking the divergence in both sides of the equation and using Gauss's law. That is, we find

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}.$$

12.3.4 Magnetic Charge

Let us assume that the source of charges ρ and of currents $\boldsymbol{J}$ vanish. We can write the resulting four Maxwell equations as

$$\begin{aligned}
\nabla \cdot \boldsymbol{E} &= 0 \quad , \quad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \\
\nabla \cdot \boldsymbol{B} &= 0 \quad , \quad \nabla \times \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}.
\end{aligned} \tag{12.12}$$

We see some symmetry in these equations. They are not perfectly symmetric because we can not interchange $\boldsymbol{E}$ by $\boldsymbol{B}$ and get the same equations but they are almost symmetric. The factor $\mu_0\epsilon_0$ destroy the symmetry but this factor is due to the units used in these

notes. If instead of using the SI units a Gaussian system is chosen then the equations would be symmetric.

We can artificially add symmetry to Maxwell's equation by writing them as

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= \frac{1}{\epsilon_0}\rho_e \quad , \quad \nabla \times \boldsymbol{E} = -\mu_0 \boldsymbol{J}_m - \frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} &= \mu_0 \rho_m \quad , \quad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}_e + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}. \end{aligned}$$

where we added the source term $-\mu_0 J_m$ to the right of the equation for $\nabla \times \boldsymbol{E}$ and we added the source term $\mu_0 \rho_m$. Again there is no perfect symmetry due to the constants but it is quite closed to a perfect symmetry. The equations above would be possible if there exists ρ_m magnetic charges and currents $\boldsymbol{J}_m$ due to magnetic charges. Maxwell as well as physicist all over the world have been looking for magnetic charges without success. The fact that $\nabla \cdot \boldsymbol{B}$, and that the expansion in terms of multipoles for the magnetic vector potential do not show monopoles stress the non existence of magnetic charges.

12.3.5 Maxwell Equations in Matter

The Maxwell equations above are valid in free space. We know that in materials there exist bound charges and bound currents related to electric and magnetic polarization effects, for which we do not have much control. It would be nice if we can write Maxwell equations in terms of sources related only to free charges and free currents. In this section we re-derive the Maxwell equations as they apply to materials. We already learned that electric polarization $\boldsymbol{P}$ is associated with some bound charge ρ_b through equation 8.6, and surface charge through equation 8.5. That is,

$$\rho_b = -\nabla \cdot \boldsymbol{P} \quad , \quad \sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}},$$

Likewise bound currents in a volume $\boldsymbol{J}_b$ or bound charges in a surface $\boldsymbol{K}$ are associated with magnetic polarization $\boldsymbol{M}$, through equations 11.2 and 11.3 as shown next

$$\boldsymbol{J}_b = \nabla \times \boldsymbol{M} \quad , \quad \boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}}.$$

In these equations we assume that there is no change in time. Let us assume that the polarization $\boldsymbol{P}$ is a function of time. That is $\boldsymbol{P} = \boldsymbol{P}(t)$. Let us assume that we have

a small piece of cylinder with the polarization vector $\boldsymbol{P}$ running along the axis of the cylinder. From equation $\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}$ above we see that $\sigma_b = P$ at one end of the cylinder and $\sigma_b = -P$ at the other end of the cylinder. If P changes by ΔP in some time Δt then the surface charge density σ_b also changes. This change of density must be due to a current density $\boldsymbol{J}_p$ parallel to $\boldsymbol{P}$ of the form

$$\boldsymbol{J}_p = \frac{\partial \sigma_b}{\partial t}\hat{\boldsymbol{n}} = \frac{\partial \boldsymbol{P}}{\partial t}.$$

This current is called the **polarization current.** This current is not present at any static state.

We now show that the polarization current satisfies the continuity equation. That is

$$\nabla \cdot \boldsymbol{J}_p = \nabla \cdot \frac{\partial \boldsymbol{P}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{P}) = -\frac{\partial \rho_b}{\partial t}.$$

We conclude that the charge density can be separated into two different parts $\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \boldsymbol{P}$ while the current density can be separated into three parts. That is,

$$\boldsymbol{J} = \boldsymbol{J}_f + \boldsymbol{J}_b + \boldsymbol{J}_p = \boldsymbol{J}_f + \nabla \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t}.$$

We can rewrite Gauss's law as

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}(\rho_f - \nabla \cdot \boldsymbol{P}),$$

and since $\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$ we find that

$$\nabla \cdot \boldsymbol{D} = \rho_f.$$

This is Gauss's law where only free charges are involved. On the other hand from the extended Ampère's law

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J}_f + \nabla \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t},$$

and using $\boldsymbol{H} = (1/\mu_0)\boldsymbol{B} - \boldsymbol{M}$ from equation 11.4 we find

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t}$$

Since Faraday's law and $\nabla \cdot \boldsymbol{B}$ do not involve charges ρ or currents $\boldsymbol{J}$ they stay the same. We find then the new four Maxwell's equations in materials

$$\begin{aligned} \nabla \cdot \boldsymbol{D} &= \rho_f \quad , \quad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} &= 0 \quad , \quad \nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t} \end{aligned} \tag{12.13}$$

Note that in these equations we have involved the four fields $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, and $\boldsymbol{H}$. We can write the equations in terms of $\boldsymbol{D}$ and $\boldsymbol{H}$ (or $\boldsymbol{E}$ and $\boldsymbol{B}$ only) by using the constitutive relations

$$\boldsymbol{D} = \epsilon \boldsymbol{E} \quad , \quad \boldsymbol{B} = \mu \boldsymbol{H}.$$

12.3.6 Maxwell's equations in integral form

The Maxwell's equations in integral form are quite useful for many purposes but in the context of these chapter they will be used to find the boundary conditions on the differential equations that they represent. We start with equations 12.13 and do volume integration. Since the volume integration of the free charge ρ_f is the total charge free Q_f, we see, after applying the divergence theorem that

$$\int_V \nabla \cdot \boldsymbol{D} d\boldsymbol{r}_i = \int_{\boldsymbol{S}} \boldsymbol{D} \cdot d\boldsymbol{S} = Q_f.$$

Along the same lines it is clear that

$$\int_{\boldsymbol{S}} \boldsymbol{B} \cdot d\boldsymbol{S} = 0.$$

Now for the equations with curl we think of a closed loop $\boldsymbol{\ell}$ and integrate along the area of the closed loop

$$\int_{\boldsymbol{S}} \nabla \times \boldsymbol{E} \cdot d\boldsymbol{S} = -\frac{d}{dt} \int_{\boldsymbol{S}} \boldsymbol{B} \cdot d\boldsymbol{S}.$$

then we apply Stoke's theorem. We find

$$\oint_{\boldsymbol{\ell}} \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_{\boldsymbol{S}} \boldsymbol{B} \cdot d\boldsymbol{S}.$$

and, again, taking the integration around a loop and using Stoke's theorem

$$\oint_{\boldsymbol{\ell}} \boldsymbol{H} \cdot d\boldsymbol{\ell} = \boldsymbol{I}_f + \frac{d}{dt} \int_{\boldsymbol{S}} \boldsymbol{D} \cdot d\boldsymbol{S}.$$

That is, the Maxwell's equation in integral form are

$$\begin{aligned} \int_{\boldsymbol{S}} \boldsymbol{D} \cdot d\boldsymbol{S} &= Q_f & \int_{\boldsymbol{S}} \boldsymbol{B} \cdot d\boldsymbol{S} = 0. \\ \oint_{\boldsymbol{\ell}} \boldsymbol{E} \cdot d\boldsymbol{\ell} &= -\frac{d}{dt} \int_{\boldsymbol{S}} \boldsymbol{B} \cdot d\boldsymbol{S} & \oint_{\boldsymbol{\ell}} \boldsymbol{H} \cdot d\boldsymbol{\ell} = \boldsymbol{I}_f + \frac{d}{dt} \int_{\boldsymbol{S}} \boldsymbol{D} \cdot d\boldsymbol{S}. \end{aligned}$$

12.3.7 Boundary Conditions

We use the Maxwell's equations in integral form to derive the boundary conditions. As we have done several times already we are concern about normal and tangential boundary conditions. For the normal boundary conditions we take a piece of surface and place a small box with tiny thickness such that the center of the box is tangent to the surface. In this case we use the surface integrals. From the first surface integral for the field $\boldsymbol{D}$ across the surface we find that

$$\boldsymbol{D}^{\perp}_{\text{above}} S - \boldsymbol{D}^{\perp}_{\text{below}} S = \sigma_f \hat{\boldsymbol{n}} S$$

where S is the area of the top (or bottom) of the box. Since $S \neq 0$ we can cancel it and find

$$\boldsymbol{D}^{\perp}_{\text{above}} - \boldsymbol{D}^{\perp}_{\text{below}} = \sigma_f \hat{\boldsymbol{n}}.$$

The same reasoning yields

$$\boldsymbol{B}^{\perp}_{\text{above}} - \boldsymbol{B}^{\perp}_{\text{below}} = 0.$$

Now, for the tangential conditions we use the line integrals. In this case we construct a small rectangular loop with tiny sides perpendicular to the surface. We start with the integral of the field $\boldsymbol{E}$ along the loop. On the left we find

$$\ell \boldsymbol{E}^{\parallel}_{\text{above}} - \ell \boldsymbol{E}^{\parallel}_{\text{below}},$$

where we dropped the sides perpendicular to the surface since in the limit they go to zero. On the right side we have the integral

$$-\frac{d}{dt}\int_{\boldsymbol{S}} \boldsymbol{B} \cdot d\boldsymbol{S}.$$

However, as the sides normal to the surface shrink to zero the surface area goes to zero and since the field $\boldsymbol{B}$ is bounded, in the limit we find that

$$\boldsymbol{E}^{\parallel}_{\text{above}} - \boldsymbol{E}^{\parallel}_{\text{below}} = 0.$$

We now pick the same loop but use line integral along $\boldsymbol{H}$. Figure 12.5 sketches the loop and the current density through the surface $\boldsymbol{K}$.

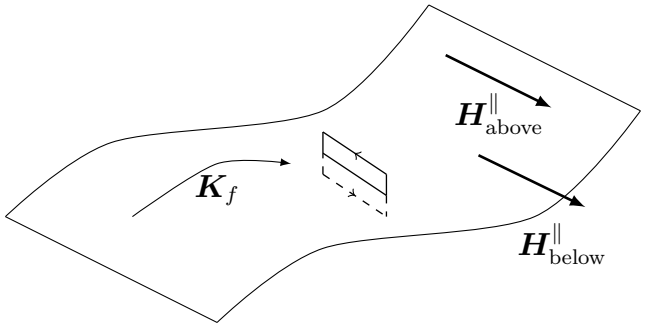

Figure 12.5: We see a path circulating around the surface. The path side which is perpendicular to the surface should approach 0 in the limit.

$$\ell \boldsymbol{H}^{\parallel}_{\text{above}} - \ell \boldsymbol{H}^{\parallel}_{\text{below}} = \boldsymbol{I}_{f_{\text{enc}}}.$$

Here $\boldsymbol{I}_{f_{\rm enc}}$ is the current enclosed by the loop. There is no volume current density here but there is surface current density $\boldsymbol{K}$. A vector normal to the loop can be constructed by taking the cross product of the normal to the surface $\hat{\boldsymbol{n}}$ and the loop element parallel to the surface $d\ell$. The flowing through the segment is the dot product of the charge density $\boldsymbol{K}$ and this vector that is

$$I_{f_{\rm enc}} = \boldsymbol{K}_f \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{\ell}) = (\boldsymbol{K}_f \times \hat{\boldsymbol{n}}) \cdot \boldsymbol{\ell}.$$

We find then that discontinuity of the parallel component of the field $\boldsymbol{H}$ is given by

$$\boldsymbol{H}^{\parallel}_{\rm above} - \boldsymbol{H}^{\parallel}_{\rm below} = \boldsymbol{K}_f \times \hat{\boldsymbol{n}}.$$

Note that we chose the amperial loop so that the current is perpendicular to it.

In summary we have the four boundary conditions

$$\begin{aligned} \boldsymbol{D}^{\perp}_{\rm above} - \boldsymbol{D}^{\perp}_{\rm below} &= \sigma_f \hat{\boldsymbol{n}} & \qquad \boldsymbol{B}^{\perp}_{\rm above} - \boldsymbol{B}^{\perp}_{\rm below} &= 0 \\ \boldsymbol{E}^{\parallel}_{\rm above} - \boldsymbol{E}^{\parallel}_{\rm below} &= 0 & \qquad \boldsymbol{H}^{\parallel}_{\rm above} - \boldsymbol{H}^{\parallel}_{\rm below} &= \boldsymbol{K}_f \times \hat{\boldsymbol{n}}. \end{aligned}$$

We can write these conditions in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$ by using the constitutive relation $\boldsymbol{D} = \mathrm{e}\boldsymbol{E}$, and $\boldsymbol{B} = \mu \boldsymbol{H}$. That is we find

$$\begin{aligned} \epsilon_{\rm above}\boldsymbol{E}^{\perp}_{\rm above} - \epsilon_{\rm below}\boldsymbol{E}^{\perp}_{\rm below} &= \sigma_f \hat{\boldsymbol{n}} & \qquad \boldsymbol{B}^{\perp}_{\rm above} - \boldsymbol{B}^{\perp}_{\rm below} &= 0 \\ \boldsymbol{E}^{\parallel}_{\rm above} - \boldsymbol{E}^{\parallel}_{\rm below} &= 0 & \qquad \frac{1}{\mu_{\rm above}}\boldsymbol{B}^{\parallel}_{\rm above} - \frac{1}{\mu_{\rm below}}\boldsymbol{B}^{\parallel}_{\rm below} &= \boldsymbol{K}_f \times \hat{\boldsymbol{n}}. \end{aligned}$$

If there are no charges or free currents we have continuity in all fields both in the tangential and normal directions. That is, we find

$$\begin{aligned} \epsilon_{\rm above}\boldsymbol{E}^{\perp}_{\rm above} - \epsilon_{\rm below}\boldsymbol{E}^{\perp}_{\rm below} &= 0 & \qquad \boldsymbol{B}^{\perp}_{\rm above} - \boldsymbol{B}^{\perp}_{\rm below} &= 0 \\ \boldsymbol{E}^{\parallel}_{\rm above} - \boldsymbol{E}^{\parallel}_{\rm below} &= 0 & \qquad \frac{1}{\mu_{\rm above}}\boldsymbol{B}^{\parallel}_{\rm above} - \frac{1}{\mu_{\rm below}}\boldsymbol{B}^{\parallel}_{\rm below} &= 0. \end{aligned}$$

These equations play an important roll in electromagnetic wave propagation and it dictates how waves reflect and refract. Particularly in optics Snell's law and Fermat's law are consequences of the boundary conditions shown above.

12.4 Wave Equations and Poynting's theorem

With Maxwell's equations we have the machinery needed to derive the electromagnetic wave equations.

12.4.1 Wave Equations

Electromagnetic Waves in Free Space

We start with the electromagnetic waves in the vacuum by using the source and current free Maxwell equations 12.12

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= 0 & \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} &= 0 & \nabla \times \boldsymbol{B} &= \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \end{aligned}$$

These equations are coupled in a way that $\boldsymbol{E}$ depends on the time derivative of $\boldsymbol{E}$ and vice versa. That is, we observe that $\nabla \times \boldsymbol{E}$ is a function of $\boldsymbol{B}$, and that $\nabla \times \boldsymbol{B}$ depends on the time derivative of $\boldsymbol{E}$. There is a reciprocity between $\boldsymbol{E}$ and $\boldsymbol{B}$ and we can say the same of the $\boldsymbol{B}$ field with respect to the $\boldsymbol{E}$ field. We cascade the two dependencies which will provide a second derivative with respect to time. That is, we write

$$\begin{aligned} \nabla \times (\nabla \times \boldsymbol{E}) &= -\frac{\partial}{\partial t} \nabla \times \boldsymbol{B} = -\mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \\ \nabla \times (\nabla \times \boldsymbol{B}) &= \mu_0 \epsilon_0 \frac{\partial}{\partial t} \nabla \times \boldsymbol{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} \end{aligned}$$

We observe now the second time-derivative operator on the right. On the left we should simplify the expression using equation A.4. That is, we write

$$\begin{aligned} \nabla(\nabla \cdot \boldsymbol{E}) - \nabla \cdot \nabla \boldsymbol{E} &= -\mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \\ \nabla(\nabla \cdot \boldsymbol{B}) - \nabla \cdot \nabla \boldsymbol{B} &= -\mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}. \end{aligned}$$

Using $\nabla \cdot \nabla = \nabla^2$ and since we using a free charge expression $\nabla \cdot \boldsymbol{E} = 0$ and $\nabla \cdot \boldsymbol{B} = 0$. Then we simplify the above equation to

$$\begin{aligned}\nabla^2 \boldsymbol{E} &= \mu_0\epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \\ \nabla^2 \boldsymbol{B} &= \mu_0\epsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}.\end{aligned}$$

These are six wave partial differential equations for the three components of $\boldsymbol{E}$ and the three components of $\boldsymbol{B}$ wave speed (from the theory of the wave equation) is given by

$$c_0 = \frac{1}{\sqrt{\mu_0\epsilon_0}}.$$

The computation of the wave speed on the equations above provides $c_0 = 2.99792458 \times 10^8$ m/s which is the speed of light. At this moment Maxwell showed that light is an electromagnetic wave. This idea was proposed by the Dutch physicist Christiaan Huygens in 1678 but his ideas was rejected since Newton who had much more influence in the time proposed that light was made of particles (the corpuscular theory of light). More than a century passed and Huygens died being ignored. In 1803 Thomas Young designed an experiment with two slits equally separated where he showed that light obeyed the phenomena of interference. This was the first experimental evidence that Huygens was right and the attention was turned into light being a wave phenomena. The theoretical support was completed by James Maxwell with the wave equations derived above. In 1900 Max Planck proposed the idea of a light quantum, a finite package of energy which depends on frequency and velocity of radiation. In 1905 Albert Einstein proposed the solution of light having the dual phenomena of being a corpuscular as well as a wave phenomena giving credit to both Newton and Huygens.

We could use the more general equations of Maxwell in materials to find wave equations of light propagating in bodies but we will leave this for later study.

12.4.2 Poynting's theorem

The Poynting's theorem is due to the Brithish physicist John Henry Poynting [12]. We found that the energy required to assemble a static charge distribution is given by 6.5

$$W = \frac{\epsilon_0}{2} \int_{\text{Vol}} E^2 d\boldsymbol{r}_i.$$

[12] https://en.wikipedia.org/wiki/John_Henry_Poynting

Likewise, the energy required to get some currents move against the back emf is given by equation 12.11

$$W = \frac{1}{2\mu_0}\int_{\mathbb{R}^3} B^2 d\boldsymbol{r}_i.$$

We can infer that the total energy for the electromagnetic fields is given by

$$U_{\text{em}} = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\boldsymbol{r}_i. \tag{12.14}$$

We derive this equation in the context of conservation of energy for electrodynamics. We want to evaluate the amount of dW work done by some charges and currents which are time dependent, in an interval of time dt. That is,

$$dW = \boldsymbol{F}\cdot d\boldsymbol{\ell} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{v}dt = q\boldsymbol{E}\cdot\boldsymbol{v}dt.$$

We consider instead of q a differential dq to integrate over all charges in a volume. Now we know that $dq = \rho d\boldsymbol{r}_i$, and $\rho\boldsymbol{v} = \boldsymbol{J}$, so

$$dW = (\rho d\boldsymbol{r}_i)\boldsymbol{E}\cdot\boldsymbol{v}dt = \boldsymbol{E}\cdot\boldsymbol{J}d\boldsymbol{r}_i dt.$$

From here we have

$$\frac{dW}{dt d\boldsymbol{r}_i} = \boldsymbol{E}\cdot\boldsymbol{J}.$$

Then, from Maxwell's equation

$$\nabla\times\boldsymbol{H} = \boldsymbol{J} + \frac{\partial\boldsymbol{D}}{\partial t},$$

(assuming $\boldsymbol{J} = \boldsymbol{J}_f$), and after integrating over a volume V with differential volume $d\boldsymbol{r}_i$,

$$\frac{dW}{dt} = \int_V \boldsymbol{E}\cdot\boldsymbol{J}d\boldsymbol{r}_i = \int_V \boldsymbol{E}\cdot\left(\nabla\times\boldsymbol{H} - \frac{\partial\boldsymbol{D}}{\partial t}\right)d\boldsymbol{r}_i =$$

We transform this integral using the identity A.7

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{H})$$

to find

$$\int_V \boldsymbol{E} \cdot \boldsymbol{J} d\boldsymbol{r}_i = \int_V \left(\boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) - \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) - \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} \right) \boldsymbol{r}_i.$$

We now use Maxwell's equation $\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$ to find

$$\int_V \boldsymbol{E} \cdot \boldsymbol{J} d\boldsymbol{r}_i = \int_V \left(-\boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) - \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} \right) \boldsymbol{r}_i. \tag{12.15}$$

Using the following identities

$$\begin{aligned} \epsilon \frac{\partial \boldsymbol{E} \cdot \boldsymbol{E}}{\partial t} &= 2\boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} \\ \frac{1}{\mu} \frac{\partial \boldsymbol{B} \cdot \boldsymbol{B}}{\partial t} &= 2\boldsymbol{B} \cdot \frac{\partial \boldsymbol{H}}{\partial t} \end{aligned}$$

homogeneous media and write equation 12.15 as

$$\begin{aligned} \int_V \boldsymbol{E} \cdot \boldsymbol{J} d\boldsymbol{r}_i &= -\frac{1}{2} \int_V \left[2\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) + \frac{\partial}{\partial t} \left(\epsilon \boldsymbol{E} \cdot \boldsymbol{E} + \frac{1}{\mu} \boldsymbol{B} \cdot \boldsymbol{B} \right) \right] d\boldsymbol{r}_i \\ &= -\int_V \frac{1}{\mu_0} \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) d\boldsymbol{r}_i - \frac{d}{dt} \int_V \frac{1}{2} \left(\epsilon E^2 + \frac{1}{\mu} B^2 \right) d\boldsymbol{r}_i \\ &= -\oint_S \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B} \, d\boldsymbol{S} - \frac{d}{dt} \int_V \frac{1}{2} \left(\epsilon E^2 + \frac{1}{\mu} B^2 \right) d\boldsymbol{r}_i \end{aligned}$$

where we applied the divergence theorem with S being the surface bounding V. This is the **Poynting's theorem.** The first integral is the energy going across the surface S. The second integral is time derivative of the total energy U_{em}, in Equation 12.14. The energy per unit of time, per unit area in the first term is called the **Poynting vector** and we write it as

$$\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B}).$$

We used the symbol $\boldsymbol{S}$ for an area element. In order to avoid confusion with the Poynting vector let us use the symbol $\boldsymbol{a}$ for area element and so the quantity $\boldsymbol{S} \cdot \boldsymbol{a}$ is the energy per unit time crossing the infinitesimal surface $d\boldsymbol{a}$. That is, we can identify $\boldsymbol{S}$ with the energy flux density. We now use the definition of Poynting vector and equation 12.14 to write

$$\frac{dW}{dt} = -\frac{dU_{\text{em}}}{dt} - \oint \boldsymbol{S} \cdot d\boldsymbol{a}$$

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Appendices

Appendix A

Some identities from Multidimensional Calculus

A.1 Basic Definitions and Conventions

A.1.1 Levi-Civita

The **Levi-Civita** symbol ϵ_{ijk} is defined as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk) \text{ is an even permutation of } (123) \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (123) \\ 0 & \text{otherwise.} \end{cases}$$

For example, for any two repeated indices the Levi-Citita symbol is 0. That is

$$\epsilon_{IIj} = \epsilon_{jII} = \epsilon_{IjI} = \epsilon_{III} = 0.$$

Even permutations are cyclic. Figure A.1 sketches the concept of perfmutation in 3D. The symbol can be generalized to any dimensions but in the context of these notes we only need 3 dimensions.

A.1.2 Kronecker Delta

The **Kronecker delta** δ_{ij} is defined by the equation

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

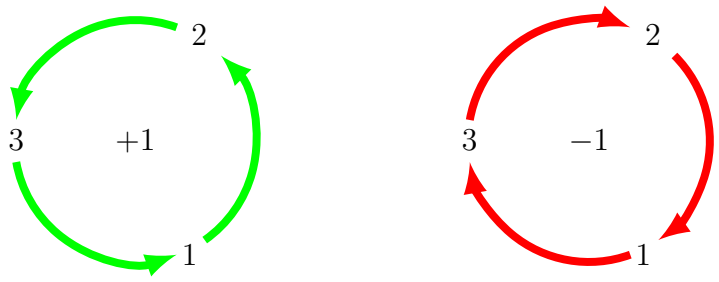

Figure A.1: Even permutations (123),(231),(312) are cycles in the green circle, odd perfmutations are (132),(321),(213), cycles in the red circle.

A.1.3 The $\epsilon - \delta$ relationships

There are a few interesting relations between the Levi-Civita symbol ϵ_{ijk} and the Kronecker delta δ_{ij}.

We introduce the **Einstein-Summation** convention since it will be needed for the following developments. Indices are either repeated (only two instances of an index on an term) or not (one instance only). If there is a repeated index in an expression then there is summation along that index. For example (in 3 dimensions)

$$a_{ii} \quad \text{means} \quad \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}.$$

The repeated indices are called **dummy** indices, the single indices are called **free** indices. To appreciate the power of the notation we could verify that the determinant of a 3x3 matrix $A = (a_{ij})$ is given by

$$|A| = \epsilon_{ijk} a_{1i} a_{2j} a_{3k}.$$

There are 27 terms implied in this equation since i, j, and k (three repeated indices) vary from 1 to 3. From those 27 terms at most 6 are non-zero (the permutaions of the numbers 1,2,3).

The Levi-Civita symbol is tightly related to the determinant. Think of a matrix with vectors $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$. The determinant of the matrix is either 1 or -1 according to the permutation of the vectors in the matrix. Even more, there could be two repeated vectors and then the determinant is 0. This coincides exactly with the definition of the Levi-Civita tensor. Such a matrix can be written as

$$A = (a_{ij}) = \begin{pmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3}. \end{pmatrix}$$

If $i = 1$, $j = 2$, $k = 3$ we have the identity matrix, if $i = j$, or $i = k$ or $j = k$ then we have two repeated rows and the determinant is zero. The matrix has three "1" and six "0" numbers. If (ijk) is an even permutation the determinant of the matrix is 1, if (ijk) is an odd permutation the determinant of the matrix is -1. Then a formal definition of ϵ_{ijk} in terms the matrix above is

$$\begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3}. \end{vmatrix} = \epsilon_{ijk}$$

We can rewrite this equation as

$$\begin{vmatrix} \delta_{l1} & \delta_{l2} & \delta_{l3} \\ \delta_{m1} & \delta_{m2} & \delta_{m3} \\ \delta_{n1} & \delta_{n2} & \delta_{n3}. \end{vmatrix} = \epsilon_{lmn}$$

and multiply both equations above.

$$\begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3}. \end{vmatrix} \begin{vmatrix} \delta_{l1} & \delta_{l2} & \delta_{l3} \\ \delta_{m1} & \delta_{m2} & \delta_{m3} \\ \delta_{n1} & \delta_{n2} & \delta_{n3}. \end{vmatrix} = \epsilon_{ijk}\epsilon_{lmn}$$

Since in the matrix spaces, the product of the determinants is the determinant of the product we multiply the two matrices and take the determinant. Furthermore the determinant of a matrix is the same as the determinant of the transposed. For example, assuming the second matrix was transposed, we can multiply the second raw with the third row (which after transposing is the third column) to find the (2,3) element of the resulting matrix. That is,

$$a_{23} = \delta_{jp}\delta_{np} = \delta_{jn}.$$

Hence we find the more general relationship

$$\begin{vmatrix} \delta_{li} & \delta_{lj} & \delta_{lk} \\ \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ni} & \delta_{nj} & \delta_{nk}. \end{vmatrix} = \epsilon_{ijk}\epsilon_{lmn}$$

There is an interesting case which is used commonly on deriving equations with double cross products or expressions with curls of cross products. Let us assume in the equation above that $i = l$, Then

$$\begin{vmatrix} 1 & \delta_{lj} & \delta_{lk} \\ \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ni} & \delta_{nj} & \delta_{nk}. \end{vmatrix} = \epsilon_{imn}\epsilon_{ijk}$$

and expanding the determinant [1] we see that

$$\epsilon_{ijk}\,\epsilon_{imn} = \delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}. \tag{A.1}$$

This is known as the $\epsilon - \delta$ identity.

A.1.4 Einstein convention: Determinant, dot product, divergence, Laplacain, Cross product and Curl in short hand notation

The Einstein repeated index rule indicates that if in a term an index is repeated there is sum along that index. For example

$$\sum_{i=1}^{3} a_{ii} \quad \text{, Einstein notation} \quad a_{ii}$$

We use this rule combined with the definition of the Kronecker delta and the Levi-Civita symbol to simplify the writing of some important symbols in what it is known as **short hand notation.** We can not have more than two instances of the same index. Repeated indices are called **dummy** and not repeated indices are free indices. In any expression, if

[1] We already know that on each raw there is onle one "1" so the other two elements in the same raw are 0

written as components the same number of free indices on the left should balance those in the right. For example the expression

$$a_{ii}b_{jj}c_{klm} = d_k e_{lm},$$

has a correct syntax in short hand notation. Repeated indices are found by pairs and they do not need to balance with other indices. Free index balance. Here the free indices are k, l and m.

We start with the determinant.

$$\det(A) = \epsilon_{ijk}a_{i1}a_{j2}a_{k3}.$$

Note that this is a big compression. There are 27 terms here (at most 9 are non -zero).

With this notation for example the dot product can be written

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$$

More symbols in the right but we display components which show the detail of the expression. The short hand derivative. We define

$$\frac{\partial f}{\partial x_i} = f_{,i}.$$

Six simbols on the left replaced by three symbols in the right. Note tha the comman "," is very important here. The divergence in short hand notation is

$$\nabla \cdot \boldsymbol{f} = f_{i,i}.$$

The cross product is

$$(\nabla \times \boldsymbol{f})_i = \epsilon_{ijk} f_{j,i}.$$

The Laplacian is written as

$$\nabla^2 \boldsymbol{f} = f_{,ii}.$$

We are now ready to derive the equations that start this lecture.

A.2 Differential Identities

Let us assume that $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$, and $\boldsymbol{i}$ are vector functions of $\boldsymbol{x}$ (or constants) in the three-dimensional space. Let us assume that ϕ is a scalar function of $\boldsymbol{x}$. We have the following list of identities:

$$\begin{aligned}
\nabla^2\boldsymbol{\phi} &= \nabla\cdot\nabla\phi \quad \text{by definition} \\
\boldsymbol{f}.(\boldsymbol{g}\times\boldsymbol{h}) &= \boldsymbol{g}\cdot(\boldsymbol{h}\times\boldsymbol{f}) = \boldsymbol{h}\cdot(\boldsymbol{f}\times\boldsymbol{g}) \\
\boldsymbol{f}\times(\boldsymbol{g}\times\boldsymbol{h}) &= \boldsymbol{g}(\boldsymbol{h}\cdot\boldsymbol{f}) - \boldsymbol{h}(\boldsymbol{f}\cdot\boldsymbol{g}) \\
(\boldsymbol{f}\times\boldsymbol{g})\cdot(\boldsymbol{h}\times\boldsymbol{i}) &= (\boldsymbol{f}\cdot\boldsymbol{h})(\boldsymbol{g}\cdot\boldsymbol{i}) - (\boldsymbol{f}\cdot\boldsymbol{i})(\boldsymbol{g}\cdot\boldsymbol{h}) \\
\nabla\times\nabla\phi &= 0 \qquad (A.2) \\
\nabla\cdot(\nabla\times\boldsymbol{f}) &= 0 \qquad (A.3) \\
\nabla\times(\nabla\times\boldsymbol{f}) &= \nabla(\nabla\cdot\boldsymbol{f}) - \nabla\cdot\nabla\boldsymbol{f} \qquad (A.4) \\
\nabla\times\phi\boldsymbol{f} &= \nabla\phi\times\boldsymbol{f} + \phi\nabla\times\boldsymbol{f} \qquad (A.5) \\
\nabla\cdot(\phi\,\boldsymbol{f}) &= \phi\nabla\cdot\boldsymbol{f} + \boldsymbol{f}\cdot\nabla\phi \\
\nabla(\boldsymbol{f}\cdot\boldsymbol{g}) &= (\boldsymbol{f}\cdot\nabla)\boldsymbol{g} + (\boldsymbol{g}\cdot\nabla)\boldsymbol{f} + \boldsymbol{f}\times(\nabla\times\boldsymbol{g}) + \boldsymbol{g}\times(\nabla\times\boldsymbol{f}) \qquad (A.6) \\
\nabla\cdot(\boldsymbol{g}\times\boldsymbol{h}) &= \boldsymbol{h}\cdot(\nabla\times\boldsymbol{g}) - \boldsymbol{g}\cdot(\nabla\times\boldsymbol{h}) \qquad (A.7) \\
\nabla\times(\boldsymbol{f}\times\boldsymbol{g}) &= \boldsymbol{f}(\nabla\cdot\boldsymbol{g}) - \boldsymbol{g}(\nabla\cdot\boldsymbol{f}) + (\boldsymbol{g}\cdot\nabla)\boldsymbol{f} - (\boldsymbol{f}\cdot\nabla)\boldsymbol{g} \qquad (A.8)
\end{aligned}$$

We show here (using index notation and the Levi-Civita symbol) equations A.2, A.4, A.6, A.7 and A.8. That is,

$$(\nabla\times\nabla\phi)_i = \epsilon_{ijk}(\nabla\phi)_{k,j} = \epsilon_{ijk}\phi_{,kj} = 0.$$

Since the tensor $\phi_{,jk}$ is symmetric and the Levi-Civita tensor ϵ_{ijk} is anti-symmetric.

Now,

$$\begin{aligned}
[\nabla\times(\nabla\times\boldsymbol{f})]_i &= \epsilon_{ijk}[\nabla\times\boldsymbol{f}]_{k,j} \\
&= \epsilon_{ijk}(\epsilon_{kmn}f_{n,m})_{,j} \\
&= \epsilon_{ijk}\epsilon_{kmn}f_{n,mj} \\
&= \epsilon_{kij}\epsilon_{kmn}f_{n,mj} \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})f_{n,mj} \\
&= f_{j,ij} - f_{i,jj} \\
&= (f_{j,j})_{,i} - f_{i,jj} \\
&= (\nabla\cdot\boldsymbol{f})_{,i} - \nabla^2 f_i
\end{aligned}$$

Then we find that

$$\nabla \times (\nabla \times \boldsymbol{f}) \;=\; \nabla(\nabla \cdot \boldsymbol{f}) - \nabla \cdot \nabla \boldsymbol{f}.$$

Let us now prove A.6

$$\nabla(\boldsymbol{f} \cdot \boldsymbol{g})_i = (f_j g_j)_{,i} = f_{j,i} g_j + f_j g_{j,i} = (\boldsymbol{g} \cdot \nabla) f_i + (\boldsymbol{f} \cdot \nabla) g_i$$

We move to the proof of equation A.7. That is,

$$\begin{aligned} \nabla \cdot (\boldsymbol{g} \times \boldsymbol{h}) &= [\epsilon_{ijk} g_j h_k]_{,i} \\ &= \epsilon_{ijk} g_{j,i} h_k + \epsilon_{ijk} g_j h_{k,i} \\ &= h_k(\epsilon_{kij} g_{j,i}) - g_j(\epsilon_{jik} h_{k,i}) \\ &= \boldsymbol{h} \cdot (\nabla \times \boldsymbol{g}) - \boldsymbol{g} \cdot (\nabla \times \boldsymbol{h}) \end{aligned}$$

where the minus "-" sign appears since $\epsilon_{ijk} = -\epsilon_{jik}$.

Now for identity A.8. The quantity is a vector and the component i is given by

$$\begin{aligned} [\nabla \times (\boldsymbol{f} \times \boldsymbol{g})]_i &= \epsilon_{ijk} (\boldsymbol{f} \times \boldsymbol{g})_{k,j} \\ &= \epsilon_{ijk} (\epsilon_{klm} f_l g_m)_{,j} \\ &= \epsilon_{ijk} \epsilon_{klm} (f_l g_m)_{,j} \\ &= \epsilon_{ijk} \epsilon_{klm} (f_{l,j} g_m + f_l g_{m,j}). \end{aligned}$$

We now use the $\epsilon - \delta$ identity A.1

$$\epsilon_{ijk} \epsilon_{klm} = \epsilon_{kij} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}.$$

Then

$$[\nabla \times (\boldsymbol{f} \times \boldsymbol{g})]_i \;=\; (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl})(f_{l,j} g_m + f_l g_{m,j}) = f_{i,j} g_j + f_i g_{j,j} - f_{j,j} g_i - f_j g_{i,j}.$$

In nabla symbols this is

$$\nabla \times (\boldsymbol{f} \times \boldsymbol{g}) = (\boldsymbol{g} \cdot \nabla)\boldsymbol{f} + (\nabla \cdot \boldsymbol{g})\boldsymbol{f} - (\nabla \cdot \boldsymbol{f})\boldsymbol{g} - (\boldsymbol{f} \cdot \nabla)\boldsymbol{g},$$

which up to the summation order is equation A.8. We need to clarify the first and last terms after the equal sign. We have not defined the gradient on a vector field. The meaning of, for example $(\boldsymbol{f} \cdot \nabla)\boldsymbol{g}$ is that each component g_i can have a gradient and we will dot $\boldsymbol{f}$ with each gradient of g_i. We can think of this operation as a vector matrix multiplication. The raw vector $\boldsymbol{f}$ with the matrix $\nabla \boldsymbol{g}$.

A.2.1 Some important identities with the ∇ symbol.

Let $\boldsymbol{r_{io}}$ be the separation vector from a fixed point $\boldsymbol{r}_i = (x_i, y_i, z_i)$ to the point $\boldsymbol{r}_0 = (x, y, z) = \boldsymbol{r}$, and let r_{io} be its length. We show

(i) $\nabla r_{io}^2 = 2\boldsymbol{r}_{io} = 2r_{io}\hat{\boldsymbol{r}}_{io}$.

(ii) $\nabla(1/r_{io}) = -1/{r_{io}}^2\hat{\boldsymbol{r}}_{\boldsymbol{io}}$.

(iii) $\nabla r_{io}^n = n r_{io}^{n-1}\hat{\boldsymbol{r}}_{\boldsymbol{io}}$

Solution: I do this in two different ways: using spherical coordinates and the chain rule and using Cartesian coordinates and brute force derivatives.

(i) **Using spherical coordinates and the chain rule:**
Let us solve the problem of finding Let us consider the problem ∇r^n. In spherical coordinates we have

$$\nabla r^n = \frac{1}{h_r}\frac{\partial r^n}{\partial r}\hat{\boldsymbol{r}} + \cancel{\frac{1}{h_\theta}\frac{\partial r^n}{\partial \theta}} + \frac{1}{h_\phi}\frac{\cancel{\partial r^n}}{\partial \phi}\cancel{\to 0} = n r^{n-1}\hat{\boldsymbol{r}}.$$

since $h_r = 1$, and r is independent of θ, ϕ.

What if instead of r we have $r_{io} = \|\boldsymbol{r} - \boldsymbol{r}_i\|$ with $\boldsymbol{r} = (x, y, z)$ and $\boldsymbol{r}_i = (x', y', z')$.

Then we can use the chain rule as follows. Let $f(\boldsymbol{x})$ be a scalar field and $\boldsymbol{g}(\boldsymbol{x})$ a vector field. Then we show that

$$\nabla_{\boldsymbol{x}} f(\boldsymbol{g}(\boldsymbol{x})) = A^T \nabla_{\boldsymbol{g}} f$$

where A is the matrix with components $\partial g_i/\partial x_j$. Let us take an i-th component of the gradient of $f(\boldsymbol{g}(\boldsymbol{x}))$. Using the chain rule we have

$$\frac{\partial f}{\partial x_i} = \sum \frac{\partial f}{\partial g_j}\,\frac{\partial g_j}{\partial x_i}$$

which is the multiplication of the transposed of the matrix A with the $\nabla_g f$.

Now let us consider the more general problem of

$$r_{io}^n = \|\boldsymbol{r} - \boldsymbol{r}_0\|^n.$$

We make a change of variables $\boldsymbol{r}_{io} = \boldsymbol{r} - \boldsymbol{r}_0$, and consider the function $f(\boldsymbol{x}) = \|\boldsymbol{x}\|$. Then to apply the chain rule we need to find A, the matrix of derivatives of the j component of the vector $\boldsymbol{r}_{io}$ with respect to the i component of $\boldsymbol{x}$. Clearly $A = I$, so we have that

$$\nabla_{\boldsymbol{r}} r_{io}^n = \nabla_{\boldsymbol{r}_{io}} r_{io}^n = n r_{io}^{n-1} \hat{\boldsymbol{r}}_{io}.$$

(ii) **Using Cartesian coordinates:**
By definition

$$r_{io}^n = \left(\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}\right)^n.$$

Then

$$\frac{\partial r_{io}^n}{\partial x} = n r_{io}^{n-1}[2(x-x_0)/(2r_{io})] = n r_{io}^{n-2}(x-x_0)$$

and doing the same for all three components,

$$\nabla r_{io}^n = n r_{io}^{n-1}[2(x-x_0)/(2r_{io})] = n r_{io}^{n-2}\boldsymbol{r}_{io} = n r_{io}^{n-1}\hat{\boldsymbol{r}}_{io}.$$

That is

$$\nabla r_{io}^n = n r_{io}^{n-1}\hat{\boldsymbol{r}}_{io}. \tag{A.9}$$

A.2.2 Verification of the Green's function for the Laplacian.

We show that

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2} \right) = 4\pi\delta(\boldsymbol{r}).$$

Solution We make use of orthogonal coordinates. We think of a vector field $\boldsymbol{A} = (A_1, A_2, A_3)$ with coordinates (u_1, u_2, u_3) along the unit vectors $\hat{\boldsymbol{u_1}}$, $\hat{\boldsymbol{u_2}}$, $\hat{\boldsymbol{u_3}}$.

$$\nabla \cdot \boldsymbol{A} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial A_1 h_2 h_3}{\partial u_1} + \frac{\partial A_2 h_1 h_3}{\partial u_2} + \frac{\partial A_3 h_1 h_2}{\partial u_3} \right).$$

In spherical coordinates we have that $u_1 = r, u_2 = \theta, u_3 = \phi$, $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$. Then, if

$$\boldsymbol{A} = \frac{\hat{\boldsymbol{r}}}{r^2},$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r^2 \sin\theta} \frac{\partial (1/\cancel{r^2})\cancel{r^2}\sin\theta}{\partial r} = 0.$$

Now, let us integrate $\nabla \cdot \boldsymbol{A}$ on a closed volume having 0.

$$\int_V \nabla \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2} \right) = \int_A \frac{\hat{\boldsymbol{r}}}{r^2} \cdot \hat{\boldsymbol{r}} dA = \int_A \frac{1}{\cancel{r^2}} \cancel{r^2} \sin\theta d\theta d\phi = 4\pi.$$

That is

$$\begin{cases} \nabla \cdot \boldsymbol{A} = 0 & \boldsymbol{r} \neq 0 \\ \int_V \nabla \cdot \boldsymbol{A} d\boldsymbol{r} = 4\pi & \boldsymbol{r} = 0. \end{cases}$$

Hence , the result

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2} \right) = 4\pi\delta(\boldsymbol{r}). \tag{A.10}$$

We now use equation A.9 with $n = -1$, to find that $\nabla(1/r) = -\frac{\hat{\boldsymbol{r}}}{r^2}$. Hence,

$$\nabla \cdot (\nabla(1/r)) = -4\pi\delta(\boldsymbol{r}).$$

but the gradient of the divergence is the Laplacian. That is

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta(\boldsymbol{r}).$$

Here are a few more identities:

(a) Find the divergence of the function

$$\boldsymbol{v} = \frac{\hat{\boldsymbol{r}}}{r}.$$

What is the general formula of $\nabla \cdot (r^n \hat{\boldsymbol{r}})$?

(b) Find the curl of $r^n \hat{\boldsymbol{r}}$.

Solution

(a) We use Einstein repeated index notation. We write

$$v_i = \frac{x_i}{x_j x_j}.$$

By definition

$$\nabla \cdot \boldsymbol{v} = v_{i,i} = \frac{r^2\delta_{ii} - 2\, x_i x_j \delta_{ij}}{r^4} = \frac{3r^2 - 2\, x_i x_i}{r^4} = \frac{r^2}{r^4} = \frac{1}{r^2}$$

If $n = 2$ we solved the problem in detail in the previous section. Let us assume $n \neq 2$.

We write

$$\boldsymbol{v} = r^n \hat{\boldsymbol{r}} = r^{n-1} \boldsymbol{r},$$

and

$$v_i = r^{n-1}x_i.$$

Now

$$r_{,i} = \frac{x_i}{r}$$

so

$$v_{i,i} = (n-1)r^{n-2}x_i\frac{x_i}{r} + r^{n-1}\delta_{ii} = (n-1)r^{n-3}r^2 + 3r^{n-1} = (n+2)r^{n-1}.$$

(b) We find $\nabla \times (r^n\hat{\boldsymbol{r}})$. From

$$v_i = r^{n-1}x_i$$

we find

$$\begin{aligned}(\nabla \times \boldsymbol{v})_i &= \epsilon_{ijk}v_{k,j} \\ &= \epsilon_{ijk}(r^{n-1}x_k)_{,j} \\ &= \epsilon_{ijk}[(n-1)r^{n-2}x_jx_k/r + r^{n-1}\delta_{jk}] \\ &= \epsilon_{ijk}[(n-1)r^{n-3}x_jx_k + r^{n-1}\delta_{jk}]\end{aligned}$$

Both x_jx_k and δ_{jk} are symmetric tensors (matrices) and since ϵ_{ijk} is anti-symmetric the product is 0. See that for each jk we can reverse this couple and have $\epsilon_{ijk} = -\epsilon_{ikj}$, while the symmetric tensor keeps its sign and value.

Then

$$\nabla \times (r^n\hat{\boldsymbol{r}}) = 0. \tag{A.11}$$

A.3 Multidimensional Integrals: From Stokes, to Gauss, to Green

A.3.1 The Stokes Theorem

I take parts of the good document shown in Wikipedia [2] about the subject. I will extract interesting facts from that document.

- **The theorem in general :**
  The subject is part of differential geometry. In its more general form Stoke's theorem say that "the integral of a differential form ω over the boundary of some orientable manifold Ω is equal to the integral of its exterior derivative $d\omega$ over the whole Ω" In symbols:

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega. \tag{A.12}$$

  Written in this form is pretty abstract and it needs the study of differential geometry. The exterior derivative is what carries the "curl" .

  In one dimension the external derivative of $f(x)$ is just $f'(x) = df/dx$ and The right side of this equation is $\int_a^b f'(x)$, while the left side is $f(b) - f(a)$, the evaluation of the function in the boundary which consists only of 2 points.

  The historical notes in the Wikepedia are interesting:

  –

  – The equation and generalization above was discovered by E. Cartan and published in 1945.

  – Lord Kelvin (William Thomson) communicated to George Stokes in a letter dated July,2 1850. Stokes passed it as a question on the 1854 Smith's Prize exam that led to the result that has his name and was actually first published by Hermann Hankel in 1861.

- **The traditional Kelvin-Stokes theorem**
  In the three dimensional space the exterior derivative of a vector field $\boldsymbol{f}$ is $\nabla \times \boldsymbol{f}$,

[2] https://en.wikipedia.org/wiki/Stokes'_theorem

we consider a surface Σ and its path enclosing the surface as $\partial\Sigma$, and so A.12 formula takes the well known form

$$\int_{\Sigma} \nabla \times \boldsymbol{f}\, d\Sigma = \int_{\partial\Sigma} \boldsymbol{f} \cdot d\boldsymbol{r}.$$

If we call $\boldsymbol{f}(\boldsymbol{x}) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ and expand the equation above we find

$$\begin{aligned} & \int_{\Sigma} \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \right] \\ = & \int_{\partial\Sigma} Pdx + Qdy + Rdz \end{aligned}$$

- **An important identity:**
  We show that

$$\int_{V} \nabla \times \boldsymbol{v} d\boldsymbol{x} = - \oint_{S} \boldsymbol{v} \times d\boldsymbol{S}.$$

Let us use the identity

$$\nabla \cdot (\boldsymbol{v} \times \boldsymbol{c}) = \boldsymbol{c} \cdot (\nabla \times \boldsymbol{v}) - \boldsymbol{v} \cdot (\nabla \times \boldsymbol{c}) = \boldsymbol{c} \cdot (\nabla \times \boldsymbol{v})$$

Then we find from the divergence theorem that

$$\int_{V} \nabla \cdot (\boldsymbol{v} \times \boldsymbol{c}) d\boldsymbol{x} = \oint_{S} (\boldsymbol{v} \times \boldsymbol{c}) \cdot d\boldsymbol{S} = \oint_{S} (\boldsymbol{v} \times \boldsymbol{c}) \cdot \hat{\boldsymbol{n}} d\boldsymbol{s}$$

and from the identity above we see that

$$\int_{V} \nabla \cdot (\boldsymbol{v} \times \boldsymbol{c}) d\boldsymbol{x} = \int_{V} \boldsymbol{c} \cdot (\nabla \times \boldsymbol{v}) d\boldsymbol{x},$$

and since $(\boldsymbol{v} \times \boldsymbol{c}) \cdot \hat{\boldsymbol{n}} = (\hat{\boldsymbol{n}} \times \boldsymbol{v}) \cdot \boldsymbol{c}$,

$$\boldsymbol{c} \cdot \int_V \nabla \times \boldsymbol{v}\, d\boldsymbol{x} = \boldsymbol{c} \cdot \oint_S (\boldsymbol{n} \times \boldsymbol{v}) dS = -\boldsymbol{c} \cdot \oint_S \boldsymbol{v} \times d\boldsymbol{S}$$

and since $\boldsymbol{c}$ is arbitrary we can say that

$$\int_V \nabla \times \boldsymbol{v}\, d\boldsymbol{x} = -\oint_S \boldsymbol{v} \times d\boldsymbol{S}. \tag{A.13}$$

- **Green's Theorem**
  In the case of the 2D plane, we can think of $R = 0$ and that we only have the functons $P(x, y)$ and $Q(x, y)$ so we find

$$\int_\Sigma \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_{\partial\Sigma} P dx + Q dy$$

  which is Green's theorem.

Ilya Kapovich [3] presents derivations of the Green's and Gauss' theorems from the Stoke's theorems. The idea is to define the form ω for the appropriate problem in equation A.12.

In the case of the Green's theorem we are in the two dimensional space with some function $(\alpha(x, y), \beta(x, y))$, and we define the form along the curve $\partial\Omega$ as $\omega = \alpha dx + \beta dy$, and the exterior derivative is

$$\begin{aligned} d\omega &= \left(\frac{\partial \alpha}{\partial x} dx + \frac{\partial \alpha}{\partial y} dy \right) \wedge dx + \left(\frac{\partial \beta}{\partial x} dx + \frac{\partial \beta}{\partial y} dy \right) \wedge dy \\ &= \frac{\partial \alpha}{\partial y} dy \wedge dx + \frac{\partial \beta}{\partial x} dx \wedge dy \\ &= \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) dx \wedge dy. \end{aligned}$$

Then we apply equation A.12 to find

[3] http://www.math.uiuc.edu/ kapovich/481-14/stokes.pdf

$$\int_{\partial\Omega} \alpha dx + \beta dy = \int_{\Omega} \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) dx \wedge dy.$$

(note that in the calculus world $dx \wedge dy$ is $dxdy$. This is Green's theorem.

For the case of Gauss (divergence) theorem we assume a field

$$\boldsymbol{F} = (\alpha(x, y, z), \beta(x, y, z), \gamma(x, y, z)),$$

and define the two form (for a surface) as $w = \alpha dy \wedge dz + \beta dz \wedge dx + \gamma dx \wedge dy$, then the external derivative is

$$d\omega = \left(\frac{\partial \alpha}{\partial x} dx + \frac{\partial \alpha}{\partial y} dy + \frac{\partial \alpha}{\partial z} dz \right) \wedge dy \wedge dz +$$
$$\left(\frac{\partial \beta}{\partial x} dx + \frac{\partial \beta}{\partial y} dy + \frac{\partial \beta}{\partial z} dz \right) \wedge dz \wedge dx + \left(\frac{\partial \gamma}{\partial x} dx + \frac{\partial \gamma}{\partial y} dy + \frac{\partial \gamma}{\partial z} dz \right) \wedge dx \wedge dy$$

From the first term we get $(\partial\alpha/\partial x)dx \wedge dy \wedge dz$, from the second we get $(\partial\beta/\partial y)dy \wedge dz \wedge dx$, and from the third $(\partial\gamma/\partial z)dz \wedge dx \wedge dy$. Then

$$d\omega = \left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} \right) dx \wedge dy \wedge dz.$$

and the Stokes theorem becomes

$$\int_V \left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} \right) dx \wedge dy \wedge dz. = \int_{\partial V} \alpha dy \wedge dz + \beta dz \wedge dx + \gamma dx \wedge dy$$

but $dy \wedge dz$ is in the direction $\boldsymbol{n}_x$, $dz \wedge dx$ in the direction $\boldsymbol{n}_y$ and $dx \wedge dz$ in the direction $\boldsymbol{n}_z$, and so

$$\int_V \nabla \cdot \boldsymbol{F} dV = \int_{\partial V} \boldsymbol{F} \cdot \boldsymbol{n} dA.$$

which is Gauss' theorem.

A.3.2 Green's Identities

We can derive the Green Identities directly from the Gauss' theorem above. In the Gauss' theorem let us assume that $\boldsymbol{F} = u(\boldsymbol{x})\nabla v(\boldsymbol{x})$ Then, since

$$\nabla \cdot (u\nabla v) = u\nabla^2 v + \nabla v \cdot \nabla u,$$

we find

$$\int_V (u\nabla^2 v + \nabla v \cdot \nabla u)\, dV = \int_S u\nabla v \cdot d\boldsymbol{S}. = \int_S u\frac{dv}{d\boldsymbol{n}} dS.$$

This is Green's first identity. We rewrite this equation interchanging u and v and find

$$\int_V (v\nabla^2 u + \nabla u \cdot \nabla v)\, dV = \int_S v\frac{du}{d\boldsymbol{n}} dS.$$

and from a subtraction in the two previous equations we find

$$\int_V (u\nabla^2 v - v\nabla^2 u) dV \int_S \left(u\frac{dv}{d\boldsymbol{n}} - v\frac{du}{d\boldsymbol{n}} \right) dS.$$

which is Green's second identity.

Appendix B

Euler Equations

We deal here with the second order ordinary differential equation

$$r^2R'' + \alpha rR' + \beta R = 0, \tag{B.1}$$

which appears when solving the Laplacian equation using separation of variables in spherical coordinates. Sometimes this equation is also known as the Euler-Cauchy [1] equation. This is a version of the general equation

$$a_n x^n y^{(n)}(x) + a_{n-1}x^{(n-1)}(x) + \cdots + a_0 y(x) = 0.$$

We have the following observation. Let us assume $x = \mathrm{e}^u$, $u = \ln x$, and calling $y' = dy/du$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = y'\frac{1}{x} = \frac{y'}{\mathrm{e}^u} \\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\frac{dy}{dx} = \frac{d(dy/dx)}{du}\frac{du}{dx} = \frac{d(y'/\mathrm{e}^u)}{du}\frac{1}{\mathrm{e}^u} \end{aligned}$$

Now

$$\frac{d(y'/\mathrm{e}^u)}{du} = \frac{\mathrm{e}^u y'' - y'\mathrm{e}^u}{\mathrm{e}^{2u}} = \frac{y'' - y'}{\mathrm{e}^u},$$

[1] https://en.wikipedia.org/wiki/Cauchy%E2%80%93Euler_equation

and so

$$\frac{d^2y}{dx} = \frac{y''}{\mathrm{e}^{2u}} - \frac{y'}{\mathrm{e}^{2u}}.$$

We do one more trial

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{d^2y}{dx} = \frac{d}{du}\left[\frac{y''-y'}{\mathrm{e}^{2u}}\right]\frac{du}{dx} = \frac{\mathrm{e}^{2u}(y'''-y'') - 2\mathrm{e}^{2u}(y''-y')}{\mathrm{e}^{4u}}\frac{1}{\mathrm{e}^{u}}.$$

That is,

$$\frac{d^3y}{dx^3} = \frac{y''' - 3y'' + 2y'}{\mathrm{e}^{3u}}.$$

In general d^ny/dx^n is a polynomial in $y^{(i)}$ where all its coefficients have a factor e^{-iu}, which will cancel the factor $x^i = \mathrm{e}^{iu}$ at the front of each $y^{(i)}$ derivative. So the substitution makes a differential equation with variable coefficients into one with constant coefficients.

We return to our particular case of equation B.1. Let us assume that $R(r) = \mathrm{e}^u$. That is $u = \ln R$, and so using the same equations above (changing y by R, and x by r) we find

$$\begin{aligned}\frac{dR}{dr} &= \frac{R'}{\mathrm{e}^u}\\ \frac{d^2R}{dr^2} &= \frac{R''-R'}{\mathrm{e}^{2u}},\end{aligned}$$

and so B.1 equation becomes

$$\frac{d^2R}{du^2} + (\alpha-1)\frac{dR}{du} + \beta R = 0.$$

This equation is an ordinary second order differential equation with constant coefficients and has solutions as follows. We solve the characteristic polynomial

$$m^2 + (\alpha-1)m + \beta = 0.$$

There could be three cases:

(i) That the roots $m_1 \neq m_2$ are and different (real or complex). Then a general solution is given by

$$R(u) = A\mathrm{e}^{m_1 u} + B\mathrm{e}^{m_2 u}.$$

Let us assume that the roots are of the form $m_1 = \mu_1 + \mathrm{i}\nu_1$, and $m_2 = \mu_2 + \mathrm{i}\nu_2$, where $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathbb{R}$, and $\mathrm{i} = \sqrt{-1}$. Then

$$\begin{aligned} R(u) &= A\,\mathrm{e}^{\mu_1 u + \mathrm{i}\nu_1 u} + B\,\mathrm{e}^{\mu_2 u + \mathrm{i}\nu_2 u} \\ R(r) &= A\,r^{\mu_1}\mathrm{e}^{\mathrm{i}\nu_1 \ln(r)} + B\,r^{\mu_2}\mathrm{e}^{\mathrm{i}\nu_2 \ln(r)}. \end{aligned}$$

Instead of exponentials we can use trigonometrical functions and redefine A and B to write

$$R(r) = A\,r^{\mu_1}\cos(\nu_1 \ln(r)) + B\,r^{\mu_2}\sin(\nu_2 \ln(r)).$$

Of course, if the roots are real $\nu_1 = \nu_2 = 0$, and

$$R(r) = A\,r^{\mu_1} + B\,r^{\mu_2}.$$

(ii) The two roots are real and equal $m_1 = m_2$ (zero discriminant) Then we have as a general solution

$$R(u) = A\mathrm{e}^{m_1 u} + Bu\mathrm{e}^{m_2 u}.$$

and in terms of r this is

$$R(r) = Ar^{m_1} + Br^{m_2}\ln r.$$

Appendix C

Solution of the Legendre differential equations

We consider the two cases of Legendre differential equations obtained by the method of separation of variables on the Laplace equation in spherical coordinates. We start with the simpler equation obtained when azimuthal simmetry is considered and then move to the more general equation considering azimuthal dependence.

C.1 The Legendre Differential Equation

We start with equation 5.7. That is

$$\frac{d}{dx}\left[(1-x^2)\frac{dT}{dx}\right] + n(n+1)T = 0. \tag{C.1}$$

This equation can be solved by the Frobenius method (power series solution). We do this next

C.1.1 The Frobenius Method Series Solution

We assume that the solution can be written in form of a power series

$$T(x) = \sum_{j=0}^{\infty} a_j x^j.$$

We use the notation P when it is a function of x and T when it is a function of θ. We substitute this equation in equation 5.7.

$$(1-x^2)\frac{dP}{dx} = \sum_{j=0}^{\infty} a_j j x^{j-1} - \sum_{j=0}^{\infty} a_j j x^{j+1}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] = \sum_{j=0}^{\infty} a_j j(j-1)x^{j-2} - \sum_{j=0}^{\infty} a_j j(j+1)x^{j}$$

$$\lambda P = \sum_{j=0}^{\infty} \lambda a_j x^j,$$

and adding the previous two expressions

$$\sum_{j=0}^{\infty} a_j j(j-1)x^{j-2} - a_j[j(j+1) - \lambda x^j = 0.$$

Each coefficient should vanish. For $j = 0, 1$

Choose some $j > 1$, and

$$a_{j+2}(j+2)(j+1) - a_j[j(j+1) - \lambda] = 0.$$

or

$$a_{j+2} = \frac{j(j+1) - \lambda}{(j+1)(j+2)}\, a_j \tag{C.2}$$

Note that if $a_0 = 0$ all even coefficients are 0, if $a_1 = 0$ all odd coefficients are zero. If both a_0 and a_1 are zero then all the terms in the sum is zero and this is just the trivial solution. Let us obtain two independent solutions. We make first $a_0 \neq 0$ and $a_1 = 0$. The recursion above produces:

$$a_1 = a_3 = \cdots = 0,$$

all odd terms become zero. On the other hand

$$a_2 = -\frac{\lambda}{2}a_0 \quad , \quad a_4 = \frac{2\cdot 3-\lambda}{3\cdot 4}a_2 = \frac{2\cdot 3-\lambda}{4!}(-\lambda)a_0$$

Let us do one more iteration to find the pattern.

$$a_6 = \frac{4\cdot 5-\lambda}{5\cdot 6}a_4 = \frac{(4\cdot 5-\lambda)(2\cdot 3-\lambda)(-\lambda)}{6!}a_0.$$

The pattern is that

$$a_{2k} = \frac{[(2k-2)\cdot(2k-1)-\lambda][(2k-4)\cdot(2k-3)-\lambda]\cdots[2\cdot 3-\lambda])(-\lambda)}{(2k)!}a_0.$$

This can be shown by induction. We write this expression in compact form as

$$a_{2k} = \frac{a_0}{(2k)!}\prod_{i=1}^{k-1}[(2i)\cdot(2i+1)-\lambda]. \quad , \quad k\geq 1, \quad , \quad a_0 = 0.$$

and so we have the first solution

$$T_1(x) = a_0\sum_{k=0}^{\infty}\left(\prod_{i=1}^{k-1}[(2i)\cdot(2i+1)-\lambda]\right)\frac{x^{2k}}{(2k)!}. \tag{C.3}$$

A second series solution independent of the first is obtained by choosing $a_0 = 0$, and $a_1 \neq 0$. In this case all even terms are 0 and only the odd terms survive. Here are a few of them

$$a_3 = \frac{1\cdot 2-\lambda}{2\cdot 3}a_1 \quad ,$$
$$a_5 = \frac{3\cdot 4-\lambda}{4\cdot 5}a_3 = \frac{(3\cdot 4-\lambda)(1\cdot 2-\lambda)}{5!}a_1$$

The pattern for a_{2k+1} is

$$a_{2k+1} = \frac{a_1}{(2k+1)!}\prod_{i=1}^{k}[(2i-1)2i-\lambda]$$

and the second solution is

$$T_2(x) = a_1 \sum_{k=0}^{\infty} \left(\prod_{i=1}^{k} [(2i-1)2i - \lambda] \right) \frac{x^{2k+1}}{(2k+1)!}.$$

They are linearly independent since one only has odd powers of x and the other even powers of x.

Let us use the ratio test to find where the solutions converge (in the interval $[-1, 1]$ which is the interval of interest). Let us, for the solution $T_1(x)$ call $c_k = a_{2k}x^{2k}$, then

$$\frac{c_{k+1}}{c_k} = \frac{\frac{a_0}{(2k+2)!} \prod_{i=1}^{k} [(2i) \cdot (2i+1) - \lambda] x^{2k+2}}{\frac{a_0}{(2k)!} \prod_{i=1}^{k-1} [(2i) \cdot (2i+1) - \lambda] x^{2k}} = \frac{(2k) \cdot (2k+1) - \lambda}{(2k+2)(2k+1)} x^2$$

and

$$\lim_{k \to \infty} \frac{c_{k+1}}{c_k} = x^2$$

so we need for convergence $x \in (-1, 1)$. Likewise we define in $T_2(x)$, $c_k = a_{2k-1}x^{2k-1}$, and then

$$\frac{c_{k+1}}{c_k} = \frac{a_{2k+1}x^{2k+1}}{a_{2k-1}x^{2k-1}} = \frac{\frac{a_1}{(2k+1)!} \prod_{i=1}^{k} [(2i-1)2i - \lambda]}{\frac{a_1}{(2k-1)!} \prod_{i=1}^{k-1} [(2i-1)2i - \lambda]} = \frac{(2k-1)(2k) - \lambda}{(2k+1)(2k)} x^2 \tag{C.4}$$

$$\lim_{k \to \infty} \frac{c_{k+1}}{c_k} = x^2$$

from which we find that the second solution $T_2(x)$ converges in the interval $(-1, 1)$.

We now show that at least one of the two functions $T_1(x)$, or $T_2(x)$ diverges in at least one of the two edges (1 or -1) for any λ. We are considering $\lambda \in \mathbb{R}$. Let us consider $\lambda < 0$. Then in the function $T_1(x)$, with $x = 1$, every term can be compared as follow.

$$\frac{a_0}{(2k)!} \prod_{i=1}^{k-1} [(2i) \cdot (2i+1) - \lambda] \geq \frac{a_0}{(2k)!} \prod_{i=1}^{k-1} [(2i) \cdot (2i+1)] = \frac{a_0}{2k} = \frac{a_0}{2} \frac{1}{k}.$$

and since the harmonic series $\sum_{k=1}^{\infty}(1/k)$ diverges then the series for $T_1(x)$ diverges. The same idea applies for $T_2(x)$ but choosing $x = -1$ where since all the powers are odd, the series has all its terms negative, its absolute value makes all the terms positive and it diverges as well. If $\lambda = 0$ the series are harmonic (up to a multiplication by a constant), and so they diverge.

If $\lambda > 0$, integer, the series terminate at some finite number where $\lambda = n(n+1)$. Here then the reason why many texts identify λ with $n(n+1)$ without explanation and write the equation directly in terms of $n(n+1)$. If $\lambda \neq 0$, then the series are infinite. We use the limit comparison test [1]. We choose the k coefficient of the T_1 series $c_k = a_{2k}x^{2k}$ with $x = 1$ and compare with the coefficient b_k of the even harmonic series $b_k = 1/(2k)$. [2]. Then we evaluate the limit

$$\begin{aligned}
\lim_{k\to\infty}\frac{c_k}{b_k} &= \frac{\frac{a_0}{(2k)!}\prod_{i=1}^{k-1}[(2i)\cdot(2i+1)-\lambda]}{1/(2k)} \\
&= a_0 \lim_{k\to\infty}\frac{\prod_{i=1}^{k-1}[(2i)\cdot(2i+1)]}{((2k)!/(2k)} \\
&= a_0 \lim_{k\to\infty}\frac{(2k-1)!}{(2k)!/(2k)} \\
&= a_0 \lim_{k\to\infty}\frac{(2k)!}{(2k)!} \\
&= 1
\end{aligned}$$

So, since the harmonic series diverges then the function $T_1(x)$ for $\lambda > 0$, and not equal to $n(n+1)$ diverges. This proof could also be done in the same way for $T_2(x)$, and it is valid even for $\lambda < 0$.

At this moment we drop the use of λ for that of $n(n+1)$. We revisit recursive equation C.2 and observe that when $j = n$, a_{n+2} =0, and since from there each coefficient is a function of this, $a_k = 0$, for $k \geq n+2$.

We have the following observations:

1. If $n = 2k$ (even), then the series for $T_1(x)$ terminates at a_{2k} and $T_1(x)$ is a polynomial of degree $2k$. The series for $T_2(x)$ is infinite (it does not terminate) and has radius of convergence equal to 1. T_2 is unbounded.

[1] https://en.wikipedia.org/wiki/Limit_comparison_test

[2] Note that the even harmonic series since $\sum \frac{1}{2k} = \frac{1}{2}\sum \frac{1}{k}$

2. If $n = 2k + 1$ (odd), then the series T_2 terminates at a_{2k+1} and T_2 is a polynomial of degree $2k + 1$ while the solution y_1 is unbounded.

We consider next the two different case which are call Legendre functions of first (the convergent) and second (the divergent) kind

C.1.2 Legendre Polynomials: Functions of the first kind

We focus in the case where the series is finite (and then converges). That is we assume that is the case of $n = 2k$ for a series T_1 and $n = 2k + 1$ for the sries T_2.

It is easier to buld the polynomial in terms of the a_n instead of a_0. That is we build the recursion going up in the indices. From equation C.2 we find that

$$a_j = \frac{(j+1)(j+2)}{j(j+1) - n(n+1)} a_{j+2} = -\frac{(j+1)(j+2)}{(n-j)(n+j+1)}$$

$j < n$. That is $j = n-2, n-4, \cdots 1$, or it could end in $j = 0$ if j is even. Since we want to reverse the order, with respect n by $2k$ we can replace $j = n - 2k$ to find

$$a_{n-2k} = -\frac{(n-2k+1)(n-2k+2)}{(n-n+2k)(n+n-2k+1)} a_{n-2k+2} = -\frac{(n-2k+1)(n-2k+2)}{2k(2n-2k+1)} a_{n-2k+2}.$$

The maximum value that k can take is $k = [n/2]$. If n is even then this will go on the left as far as c_0, but if n is odd it will go on the left as far as c_1. The smallest value of $k = 1$ providing c_n on the right. We can not go further than n which is the order of the polynomial. Hence we have $k = 1, 2, \cdots, [n/2]$. We now want to propagate the recursion so that on the right hand side we have a_n. Let us do a few steps.

$$\begin{aligned}
a_{n-2k} &= -\frac{(n-2k+1)(n-2k+2)}{2k(2n-2k+1)} a_{n-2k+2} \\
&= (-1)^2 \frac{(n-2k+1)(n-2k+2)}{2k(2n-2k+1)} \frac{(n-2k+3)(n-2k+4)}{(2k-2)(n+n-2k+3)} a_{n-2k+4} \\
&= (-1)^2 \frac{(n-2k+1)(n-2k+2)(n-2k+3)(n-2k+4)}{2^2(k)(k-1)(2n-2k+1)(2n-2k+3)} a_{n-2k+4}
\end{aligned}$$

We take this until we reach on the right hand side a_n. That is unitl $-2k + 2k = 0$. The expression is

$$a_{n-2k} = (-1)^k \frac{n(n-1)\cdots(n-2k+1)}{2^k(k!)(2n-2k-1)(2n-2k+1)\cdots(2n-1)} a_n$$

We then have the polynomial

$$\sum_{k=0}^{[n/2]} a_{n-2k} x^{n-2k} = \sum_{k=0}^{[n/2]} (-1)^k \frac{n(n-1)\cdots(n-2k+1)}{2^k(k!)(2n-2k-1)(2n-2k+1)\cdots(2n-1)} a_n x^{n-2k}$$

Let us further reduce this equation. We focus on the coefficient ignoring a_n. The denominator can be written as

$$\begin{aligned} 2^k(k!)(2n-1)(2n-3)\cdots(2n-2k-1) &= \frac{2^k k!(2n-1)(2n-2)\cdots(2n-2k-1)}{(2n-2)(2n-4)\cdots(2n-2k-2)} \\ &= \frac{2^k\, k!(2n-1)!}{2^{k-1}(2n-2k)!(n-1)(n-2)\cdots(n-k-1)} \\ &= \frac{2\, k!(2n-1)!\,(n-k)!}{(2n-2k)!(n-1)!} \end{aligned}$$

The numerator can be written as

$$\frac{n!}{(n-2k)!}.$$

Now, the numerator over the denominator

$$\begin{aligned} \frac{n!(2n-2k)!(n-1)!}{2\, k!(2n-1)!(n-k)!(n-2k)!} &= \frac{2\,(n!)^2\,(2n-2k)!}{n\, k!\,(2n-1)!(n-k)!(n-2k)!} \\ &= \frac{\cancel{2n}\,(n!)^2(2n-2k)!}{\cancel{2n}\, k!(2n)!(n-k)!(n-2k)!} \end{aligned} \tag{C.5}$$

We now are ready to pick a value for c_n. Since c_n does not involve k and we want to simplify the expression above, we would like c_n to have in its numerator $(2n)!$ and its denominator $(n)!^2$. The chosen value in practice is given by

$$c_n = \frac{(2n)!}{2^n (n!)^2}.$$

where the extra 2^n in the denominator is to fix $P_n(1) = 1$ (this is not obvious yet and we could just ignore this factor and fix it after we expand a few polynomials). Expression C.5 simplifies to.

$$\frac{\cancel{(n!)^2}(2n-2k)!}{k!\cancel{(2n)!}(n-k)!(n-2k)!}\frac{\cancel{(2n)!}}{2^n\cancel{(n!)^2}} = \frac{(2n-2k)!}{2^n\, k!(n-k)!(n-2k)!}$$

At this point we change the notation from T_1 to P_n since we have polynomials of order n, and write the polynomial solutions as

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}x^{n-2k}. \tag{C.6}$$

Observe that if n is even/odd then only even/odd powers are involved. See that before we have T_1 only on even powers and T_2 only on odd powers. Here we have a set of solutions P_n which will have even or odd powers depending on the oddness of n.

Here are the first six Legendre polynomials

$$\begin{aligned} P_0(x) &= 1 & P_1(x) &= x \\ P_2(x) &= \tfrac{1}{2}(3x^2-1) & P_3(x) &= \tfrac{1}{2}(5x^3-3x) \\ P_4(x) &= \tfrac{1}{8}(35x^4-30x^2+3) & P_6(x) &= \tfrac{1}{8}(63x^5-70x^3+15x). \end{aligned}$$

Figure C.1 shows a graph of these functions.

C.2 Legendre Functions of the second kind

Here we are interested in the case where the series is infinite (and does not converge either at 1 or -1). That is, we assume that is the case of $n = 2k$ for a series T_2 and $n = 2k$ for the sries T_1.

We can work these series to reduce their representation to simpler forms or we can use the method or reduction of order to come up with a representation of this second solution.

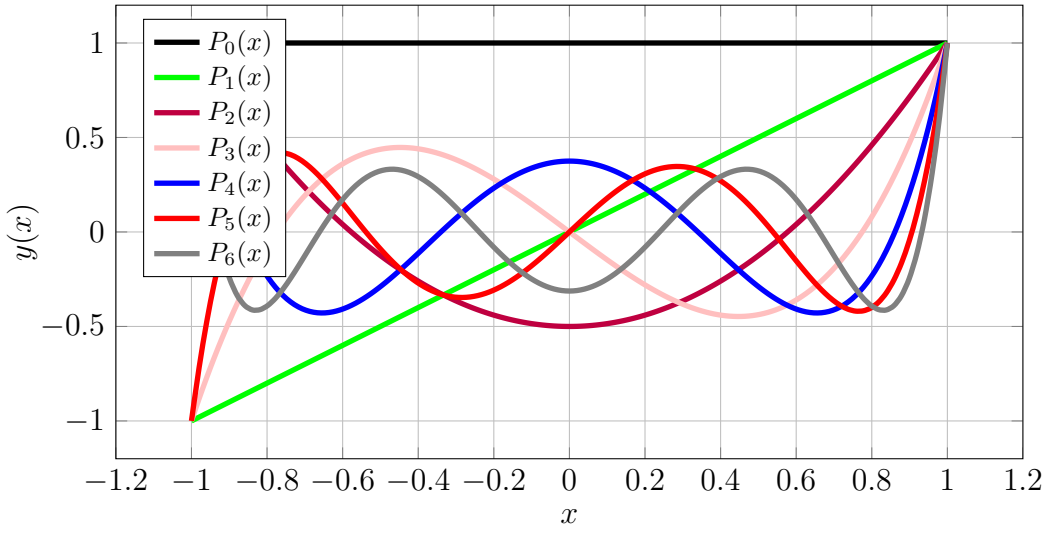

Figure C.1: First 6 Legendre Polynomials

C.2.1 From the Series Representation to Simpler Forms

C.2.2 Reduction of Order

Since a second order differential equation has two independent solutions we need to find the other solution. We use the method of reduction of order [3] to find a second. linearly independent solution to Legendre equation

$$(1-x)^2\frac{d^2T}{dx} - 2x\frac{dT}{dx} + n(n+1)T = 0.$$

That is, let $Q_n(x)$ another (linearly independent) solution for the Legendre equation above. We assume that there is a factor $v(x)$ such that

$$Q_n(x) = v(x)P_n(x).$$

We insert this into the Legendre equation to find

[3] https://en.wikipedia.org/wiki/Reduction_of_order

$$\begin{aligned} & (1-x^2)[v''(x)P_n(x) + 2v'(x)P_n'(x) + v(x)P_n''(x)] \\ - \quad & 2x[v'(x)P(x) + v(x)P'(x)] + n(n+1)v(x)P_n(x) = 0. \end{aligned}$$

Since $P_n(x)$ is a solution we can reduce the previous equation to

$$(1-x^2)[v''(x)P_n(x) + 2v'(x)P_n'(x)] - 2x\,v'(x)P(x) = 0.$$

This is a first order ODE for $v'(x)$. We divide by $(1-x^2)$ to find

$$v''(x)P_n(x) + 2v'(x)P_n'(x) - \frac{2x}{1-x^2}\,v'(x)P(x) = 0.$$

We group terms

$$v''(x)P_n(x) + v'(x)\left[2P_n'(x) - \frac{2x}{1-x^2}P(x)\right] = 0.$$

and divide by $P_n(x)$,

$$v''(x) + v'(x)\left[\frac{2P_n'(x)}{P_n(x)} - \frac{2x}{1-x^2}\right] = 0.$$

Le us call $u(x) = v'(x)$ and write

$$u'(x) + u(x)\left[\frac{2P_n'(x)}{P_n(x)} - \frac{2x}{1-x^2}\right] = 0.$$

From here

$$\frac{u'(x)}{u(x)} = \frac{2x}{1-x^2} - 2\frac{P_n'(x)}{P_n(x)},$$

and integrating

$$\begin{aligned} \ln u(x) &= \int^x \frac{2\,t}{1-t^2}dt - 2\ln P_n(x) + c_1 \\ \ln u(x) &= -\ln(1-x^2) - 2\ln P_n(x) + c_1 \\ \ln u(x) &= -\ln[(1-x^2)P_n^2(x)] + c_1. \end{aligned}$$

Then taking exponential to both sides, and calling $c_2 = \mathrm{e}^{c_1}$ (another constant),

$$u(x) = c_2 \frac{1}{(1-x^2)P_n^2(x)}.$$

This is, since $u(x) = v'(x)$, we integrate once more to get

$$v(x) = c_2 \int^x \frac{dt}{(1-t^2)P_n^2(t)} + c_3$$

Then

$$Q_n(x) = c_2 P_n(x) \int^x \frac{dt}{(1-t^2)P_n^2(t)} + c_3 P_n(x) \tag{C.7}$$

This integral is really hard to evaluate for high orders $n > 2$. We show how to evaluate it for simple case $n = 0, 1$. If $n = 0$ then

Since we are looking for a particular solution we will ignore the constants, that is we assume $c_1 = 1$ and $c_3 = 0$ and since $P_0(x) = 1$ write

$$Q_0(x) = \int^x \frac{dt}{(1-t^2)} = \frac{1}{2}\left[\int^x \frac{dt}{1+t} + \int^x \frac{dt}{1-t}\right] == \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|.$$

Now for $n = 1$, $P_1(x) = x$, and

$$Q_1(x) = x \int^x \frac{dt}{(1-t^2)t^2}$$

To Evaluate this integral we do partial fraction expansion. That is,

$$\frac{1}{(1-t^2)t^2} = \frac{At+B}{t^2} + \frac{C}{1+t} + \frac{D}{1-t} = \frac{(1-t^2)(At+B) + (t^2-t^3)C + (t^3+t^2)D}{(1-t^2)t^2}$$

From the t^3 term $-A - C + D = 0$, from the t^2 term $-B + C + D = 0$, from the t term $A = 0$. So $D = C$, and $B = 2C$. From the t^0 term $B = 1$. So we have $A = 0, B = 1, C = 1/2, D = 1/2$ and

$$\frac{1}{(1-t^2)t^2} = \frac{1}{t^2} + \frac{1}{2(1+t)} + \frac{1}{2(1-t)}.$$

and so

$$Q_1(x) = x\left[-\frac{1}{x} + \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|\right] = \frac{P_1(x)}{2}\ln\left|\frac{x+1}{x-1}\right| - 1.$$

In general, it happens that

$$Q_n(x) = \frac{P_n(x)}{2}\ln\left|\frac{x+1}{x-1}\right| + R_{n-1}(x). \tag{C.8}$$

with $R_{n-1}(x)$ a polynomial of order $n-1$. However the proof is not done by evaluating the integral C.7 in a direct way.

I need to show the identity

$$Q_n(x) = \frac{1}{2}\int_{-1}^{1}\frac{P_n(t)}{x-t}dt,$$

and from here C.8. This integral is known as the **Neumann expression** . This needs to be done.

C.2.3 The Rodriguez's Formula

The Rodriguez Formula [4] states that

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{C.9}$$

We prove this formula.

If n is integer

$$\begin{aligned}\frac{d^n}{dx^n}(x^2-1)^n &= \frac{d^n}{dx^n}\left[\sum_{k=0}^{n}(-1)^k\frac{n!}{k!(n-k)!}x^{2n-2k}\right] \\ &= \sum_{k=0}^{n}(-1)^k\frac{n!}{k!(n-k)!}\frac{(2n-2k)!}{(n-2k)!}x^{n-2k}.\end{aligned}$$

[4] https://en.wikipedia.org/wiki/Rodrigues%27_formula

The sum above does not go up to $k = n$, since after $k = [n/2]$, the derivatives are 0 then we write

$$\frac{d^n}{dx^n}(x^2-1)^n = \sum_{k=0}^{[n/2]}(-1)^k \frac{n!}{k!(n-k)!}\frac{(2n-2k)!}{(n-2k)!}x^{n-2k}.$$

It follows from C.6 that

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

The approach followed here is in reverse order. We started with Rodriguez's formula and showed that it corresponds to a Legendre polynomial. A more intuitive approach is to start at the polynomials

$$y(x) = (1-x^2)^n.$$

and take derivatives, and verify that the derivatives taken n times will get you to the Legendre differential equation. That is, we have that

$$y' = -2nx(1-x^2)^{n-1}$$

which we can write as

$$(1-x^2)y' + 2nxy = 0. \tag{C.10}$$

and starts looking a bit like a Legendre differential equation.

We want to differentiate equation C.10 k times and use the Leibniz rule. That is, if we call $u = 1-x^2$,

$$\frac{d^k}{dx^k}[uy'] = \sum_{j=0}^{k}\binom{k}{j}u^{(j)}y^{(k-j+1)}$$

Given that u is a second order polynomial only three terms of this sum will survive. That is

$$\begin{aligned}\frac{d^k}{dx^k}[uy'] &= uy^{(k+1)} + ku'y^{(k)} + k(k-1)u^{(2)}y^{(k-1)} \\ &= (1-x^2)y^{(k+1)} - 2kxy^{(k)} - \cancel{2}\frac{k(k-1)}{\cancel{2}}y^{(k-1)} = 0\end{aligned}$$

Likewise we use the Leibniz rule for the product $2nxy$ where only two terms will survive. That is

$$\frac{d^k}{dx^k}[2nxy] = 2nxy^{(k)} + 2nky^{(k-1)},$$

we combine the two results above to find

$$(1-x^2)y^{(k+1)} - 2kxy^{(k)} - k(k-1)y^{(k-1)} + 2nxy^{(k)} + 2nky^{(k-1)} = 0$$

At this point we observe that if $k = n+1$, we find

$$(1-x^2)y^{(n+2)} - 2(n+1)xy^{(n+1)} - n(n+1)y^{(n)} + 2nxy^{(n+1)} + 2n(n+1)y^{(n)} = 0$$

which simplifies to

$$(1-x^2)y^{(n+2)} - 2xy^{(n+1)} + n(n+1)y^{(n)} = 0,$$

and this is the Lagrange differential equation with $y^n = P_n$. We then showed that

$$\frac{d^n}{dx^n}(1-x^2)^n$$

satisfies the Lagrange differential equation. The factor $1/(2^n n!)$ is included to make $P(1) = 1$.

C.2.4 Orthogonality condition

Provided the norm,

$$\|P_i(x)\| = \sqrt{\langle P_i(x), P_i(x)\rangle} = \sqrt{\int_{-1}^{1} P_i^2(x)dx}.$$

the Legendre polynomials satisfy the orthogonality condition

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{nm}. \tag{C.11}$$

Since the Legendre equation is of a Sturm-Liouville type it should be self-adjoint and as such its eigenfunctions P_n, P_m comming from different eigenvalues $n(n+1)$, $m(m+1)$ should be orthogonal. Still we need to verify that the boundaries should evaluate to zero. We show that here. The operator is defined as $L = d/dx(1-x^2)dx$. Then we take the inner prodcut of LP_n with P_m. That is

$$\begin{aligned}
\langle LP_n, P_m\rangle &= \int_{-1}^{1} \frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right] P_m(x)dx \\
&= \cancelto{0}{(1-x^2)\frac{dP_n}{dx}P_m(x)\Big|_{-1}^{1}} - \int_{-1}^{1}(1-x^2)\frac{dP_n}{dx}\frac{dP_m}{dx}dx \\
&= -\int_{-1}^{2}(1-x^2)\frac{dP_n}{dx}\frac{dP_m}{dx}dx \\
&= \cancelto{0}{-(1-x^2)P_n\frac{dP_m}{dx}\Big|_{-1}^{1}} + \int_{-1}^{1} P_n(x)\frac{d}{dx}(1-x^2)P_n dx \\
&= \langle P_n, LP_m\rangle.
\end{aligned}$$

So indeed L is self-adjoint and if $n \neq m$, then $\langle P_n, P_m\rangle = 0$. The self-adjoint theorem shown in section 3.3.2 indicates orthogonality but it does not provide a way to compute the norm of a function. We still need to compute the norm of the Legendre functions.

To compute the norm of the Legendre polynomial we need to use the Rodriguez's formula. C.9 $P_n(x) = \frac{1}{2^n n!}\frac{d^n u}{dx^2}$ with $u = (x^2-1)^n$. We do this by repeated integration by parts.

$$\begin{aligned}
(2^n n!)^2 \int_{-1}^{1} P_n^2(x)dx &= \int_{-1}^{1} u^{(n)}u^{(n)}dx \\
&= u^{(n)}u^{(n-1)}\Big|_{-1}^{1} - \int_{-1}^{1} u^{(n+1)}u^{(n-1)}dx \\
&= \cdots
\end{aligned}$$

This process ends after n iterations to

$$(-1)^n \int_{-1}^{1} u^{(2n)} u^{(0)} dx = (-1)^n \int_{-1}^{1} (x^2-1)^k \frac{d^{2n}}{dx^{2n}} (x^2-1)^k dx$$

We can think of (x^2-1) as expanded in its binomail series. The highest order is x^{2n}, and after $2n$ derivatives we find $(2n)!$. Then we get to

$$(2^n n!)^2 \int_{-1}^{1} P_n^2(x) dx = (-1)^n (2n)! \int_{-1}^{1} (x^2-1)^n dx. = (2n)! \int_{-1}^{1} (1-x^2)^n dx. \tag{C.12}$$

To evaluate this integral we do the substitution $x = \sin\theta$, then $dx = \cos\theta d\theta$, and $1 - x^2 = \cos^2\theta$, then

$$\int_{-1}^{1} (1-x^2)^n dx = \int_{-\pi/2}^{\pi/2} \cos^{2n+1}\theta d\theta = 2\int_{0}^{\pi/2} \cos^{2n+1}\theta d\theta.$$

Note that $\cos^{2n+1}\theta$ is even in the interval $[-\pi/2, \pi/2]$. We find a recursion formula to integrate $\cos^k x dx$. The idea is to split this in a power of $k-1$ and a power of 1 and do integration by parts. Let us see: We call $I_k = \int \cos^k x dx$, then

$$\begin{aligned}
I_k = \int \cos^{k-1} x d(\sin x) &= \cos^{k-1} x \sin x - \int \sin x d(\cos^{k-1} x) \\
&= \cos^{k-1} x \sin x + (k-1) \int \sin x \cos^{k-2} x \sin x dx \\
&= \cos^{k-1} x \sin x + (k-1) \int \cos^{k-2} x \sin^2 x dx \\
&= \cos^{k-1} x \sin x + (k-1) \int \cos^{k-2} x (1-\cos^2 x) dx \\
&= \cos^{k-1} x \sin x + (k-1) \int \cos^{k-2} x dx - (k-1) \int \cos^k x) dx \\
&= \cos^{k-1} x \sin x + (k-1) I_{k-2} - (k-1) I_k.
\end{aligned}$$

From here

$$I_k + (k-1) I_k = \cos^{k-1} x \, \sin x + (k-1) I_{k-2}.$$

We solve for I_k

$$I_k = \frac{1}{k}\cos^{k-1} x \sin x + \frac{k-1}{k} I_{k-2}$$

If we include the end points of the integral and have in mind that for $k > 1$, $\cos^{k-1}\sin x\big|_0^{\pi/2} = 0$, and

$$\begin{aligned} I_{2n+1} &= \frac{2n}{2n+1} I_{2n-1} \\ &= \left(\frac{2n}{2n+1}\right)\left(\frac{2n-2}{2n-1}\right)\cdots\left(\frac{2}{1}\right) I_1 \\ &= \frac{[2^n n(n-1)(n-2)\cdots 1]^2}{(2n+1)(2n)(2n-1)\cdots 2\cdot 1} \\ &= \frac{(2^n n!)^2}{(2n+1)!} I_1 \end{aligned}$$

and

$$I_1 = \int_0^{\pi/2} \cos x dx = \sin x|_{x=0}^{\pi/2} = 1$$

we then showed that

$$\int_{-1}^{1} (1-x^2)^n dx = 2\frac{(2^n n!)^2}{(2n+1)!} I_1 \tag{C.13}$$

Then from using this in equation C.12 we find

$$(2^n n!)^2 \int_{-1}^{1} P_n^2(x) dx = 2(2n)! \frac{(2^n n!)^2}{(2n+1)!}$$

and

$$\int_{-1}^{1} P_n^2(x) dx = \frac{2(2n)!}{(2n+1)!} = \frac{2}{2n+1}.$$

Then we found that

$$\|P_n\| = \sqrt{\frac{2}{2n+1}}$$

and

$$\langle P_n, P_m \rangle = \frac{2}{2n+1}\delta_{nm}.$$

Which is what we claimed in equation C.11.

The Legendre polynomials. are complete [5] set in the space of piece-wise smooth functions in the interval $[-1, 1]$. That is, any piecewise smooth function in the interval $[-1, 1]$ can be expanded using Legendre polynomials. The coefficients are obtained from the equation pair of (Fourier-Legendre) equations:

$$f(x) = \sum_{n=0}^{\infty} c_i P_i(x) \tag{C.14}$$

$$c_i = \left\langle f(x), \frac{P_i(x)}{(2/(2i+1)} \right\rangle = \frac{2i+1}{2} \int_{-1}^{1} f(x) P_i(x) dx. \tag{C.15}$$

C.2.5 The Generating Function

In general, a generating function for a sequence of functions $P_n(x)$, is a function $G(x, t)$, such that

$$G(x, t) = \sum_{n=0}^{\infty} P_n(x) t^n,$$

where, by matching equal powers of t, the Taylor series expansion of $G(x, t)$ provides the functions $P_n(x)$. In particular we find $G(x, t)$ when the $P_n(x)$ are Legendre polynomials. Historically, Legendre [6] defined the polyomials $P_n(x)$ knowing that $G(x, t)$ was the Newtonian potential (gravitational potential) but here we assumed that P_n were found

[5] we do not prove this here. This requires material not covered here

[6] https://en.wikipedia.org/wiki/Legendre_polynomials

from solving the Laplace equation with azimuthal symmetry and some data given on the surface of a sphere. That is, we set up the problem of solving

$$\nabla^2 u(\boldsymbol{r}) = 0$$

subjected to some boundary conditions. Let us assume the Dirichlet boundary condition at a sphere S, $\|\boldsymbol{r}\| = r_1$,

$$u(\boldsymbol{r} \in S) = \frac{1}{\|\boldsymbol{r}_1 - \boldsymbol{r}_0\|} \tag{C.16}$$

with $\boldsymbol{r}_0$ some fixed vector in $\mathbb{R}^3$. Then the solution of the Laplace's equation

$$u(\boldsymbol{r}) = \frac{1}{\|\boldsymbol{r} - \boldsymbol{r}_0\|} \tag{C.17}$$

satisfies the Dirichlet boundary condition C.16. Since the solution of the Laplace's equation is unique with Dirichlet boundary conditions we claim that equation C.17 is THE solution of the Laplace's equation with the given Dirichlet boundary conditions. Let us further assume that $\boldsymbol{r}_0$ is aligned with the positive z axis.

The solution of the Laplace's equation using separation of variables provides

$$u_{\boldsymbol{r}_0}(r, \theta, \phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta).$$

(see equation 5.8). Here r is $\|\boldsymbol{r}\|$, θ is the polar angle that $\boldsymbol{r}$ makes with the positive axis z, and ϕ is the azimutal angle; the angle of the projection of $\boldsymbol{r}$ on the plane $z = 0$ and the x axis. It is then implicit here that we are assuming azimuthal symmetry. If there is no azimuthal symmetry we are forced to use spherical harmonic functions $Y_{nm}(\theta, \phi)$ instead of Legendre polynomials $P_n(\cos\theta)$ for this representation. Still, at the end of this derivation we show how to get around this assumption. When solving the Laplace's equation using separation of variables the radial functions r^n are solutions of the resulting Euler ODE and the polar functions $P_n(\cos\theta)$ are solutions of the Legendre differential equation. We have that

$$\|\boldsymbol{r} - \boldsymbol{r}_0\|^2 = \boldsymbol{r} \cdot \boldsymbol{r} - 2\boldsymbol{r} \cdot \boldsymbol{r}_0 + \boldsymbol{r}_0 \boldsymbol{r}_0 = r^2 - 2r\cos\alpha + 1,$$

where α is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}_0$. However, since we assume that $\boldsymbol{r}_0$ is aligned with the positive z axis $\alpha = \theta$ and the function

$$\frac{1}{\sqrt{r^2 - 2r\cos\theta + 1}}.$$

satisfies the Laplace's equation with Dirichlet boundary conditions, so we can write

$$\frac{1}{\sqrt{r^2 - 2r\cos\theta + 1}} = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta).$$

We should now find A_n to finish the problem. We show $A_n = 1$. Since the identity should be valid for all angles, let us assume $\theta = 0$, so $\cos\theta = 1$ and we know already (see discussion leading to equation C.6) that $P_n(1) = 1$, so

$$\frac{1}{|r-1|} = \frac{1}{1-r} = \sum_{n=0}^{\infty} A_n r^n,$$

where we are assuming that $r < 1$.

Since

$$\frac{1}{1-r} = 1 + r + \cdots + r^n + \cdots,$$

$A_n = 1$ and

$$\frac{1}{\sqrt{r^2 - 2r\cos\theta + 1}} = \sum_{n=0}^{\infty} r^n P_n(\cos\theta). \tag{C.18}$$

Hence

$$G(x, r) = \frac{1}{\sqrt{r^2 - 2r\cos\theta + 1}} = \frac{1}{\sqrt{r^2 - 2rx + 1}} \tag{C.19}$$

with $x = \cos\theta$ is the generating function for the Legendre polynomials. That is we have the identity

$$G(x,r) = \frac{1}{\sqrt{r^2 - 2rx + 1}} = \sum_{n=0}^{\infty} r^n P_n(x) \quad , \quad r < 1$$

At this point x could represent any value between -1 and 1. In particular if $\boldsymbol{r}$ and $\boldsymbol{r}_0$ are any two vectors with $\|\boldsymbol{r}_0\| = 1$, then $\boldsymbol{r} \cdot \boldsymbol{r}_0 = r\cos\alpha$, and we can write

$$\frac{1}{\sqrt{r^2 - 2r\cos\alpha + 1}} = \sum_{n=0}^{\infty} r^n P_n(\cos\alpha),$$

where now α is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}_0$.

It is common in the literature to find the expansion of the generating function $G(x,r)$ in Taylor series in r and from their coefficients verify that they correspond to $P_n(x)$. However this method, eventhough valid, implies more cumbersome algebra (infinite binomial expansions with fractional coefficients, Gamma functions of fractional arguments) and less physical insight.

C.2.6 Recursive Relations

We start with the first two Legendre polynomials

$$P_0(x) = 1 \quad , \quad P_1(x) = x.$$

In equation C.19 we take the derivative with respect to x (on the left hand side) and find

$$\frac{\partial G(x,r)}{\partial x} = \frac{x - r}{(1 - 2xr + r^2)^{3/2}}.$$

Now on the right hand side we have that

$$\frac{\partial G(x,r)}{\partial x} = \sum_{n=1}^{\infty} n r^{n-1} P_n(x).$$

Then, after combining the two previous equations

$$\frac{x-r}{\sqrt{1-2xr+r^2}} = (1-2xr+r^2)\sum_{n=1}^{\infty} nr^{n-1}P_n(x).$$

We replace the square root on the left hand side by its definition in terms of Legendre polynomials. That is

$$(x-r)\sum_{n=1}^{\infty} r^n P_n(x) = (1-2xr+r^2)\sum_{n=1}^{\infty} nr^{n-1}P_n(x).$$

This is

$$\begin{aligned}
x\sum_{n=1}^{\infty} r^n P_n(x) - \sum_{n=1}^{\infty} r^{n+1}P_n(x) &= (1-2xr+r^2)\sum_{n=0}^{\infty}(n+1)r^n P_{n+1}(x) \\
x\sum_{n=1}^{\infty} r^n P_n(x) - \sum_{n=2}^{\infty} r^n P_{n-1}(x) &= \sum_{n=0}^{\infty}(n+1)r^n P_{n+1}(x) - 2x\sum_{n=0}^{\infty}(n+1)r^{n+1}P_{n+1}(x) + \\
&\quad \sum_{n=0}^{\infty}(n+1)r^{n+2}P_{n+1}(x) \\
&= \sum_{n=0}^{\infty}(n+1)r^n P_{n+1}(x) - 2x\sum_{n=1}^{\infty} nr^n P_n(x) + \\
&\quad \sum_{n=2}^{\infty}(n-1)r^n P_{n-1}(x)
\end{aligned}$$

For $n \geq 2$ we match coefficients of r^n. That is,

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x),$$

and rearranging

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

This is know as the Bonnet's recursion formula [7].

[7] https://en.wikipedia.org/wiki/Legendre_polynomials#math_2

C.2.7 Integral Representations

C.3 The Generalized Legendre Differential Equation

C.3.1 The associated Legendre Functions

This is equation 5.6, with $\mu = m^2$ that we rewrite here for clarity.

$$\frac{d}{dx}\left[(1-x^2)\frac{dT}{dx}\right] + \left[n(n+1) - \frac{m^2}{1-x^2}\right]T = 0.$$

To solve this equation we start by expanding the first derivative. Since T depends both on n and m we call it T_n^m

$$(1-x^2)\frac{d^2T_n^m}{dx} - 2x\frac{dT_n^m}{dx} + n(n+1)T_n^m - \frac{m^2}{1-x^2}T_n^m = 0. \tag{C.20}$$

and assuming $m = 0$, and defining the solution of the regular Legendre equation as T_n.

$$(1-x^2)\frac{d^2T_n}{dx} - 2x\frac{dT_n}{dx} + n(n+1)T_n = 0$$

which is the ordinary Legendre equation. We now use the Leibniz rule and take m derivatives. Call $u(x) = (1-x^2)$. For the first term we have

$$\frac{d^m}{dx^m}\left[(1-x^2)\frac{d^2T_n}{dx}\right] = \sum_{k=0}^{m}\binom{m}{k}u^k {T_n}^{(m-k+2)}$$

In this sum we do not go very far since $u^(k) = 0$, for $k > 2$. Then we have the three terms

$$\frac{d^m}{dx^m}\left[(1-x^2)\frac{d^2T_n}{dx}\right] = (1-x^2)\,{T_n}^{(m+2)} + m(-2x){T_n}^{(m+1)} - m(m-1){T_n}^{(m)}$$

For the second term we use again the Leibniz rule but with $u = -2x$. Only the first two terms survive. That is

$$\frac{d^m}{dx^m}\left[-2x\frac{dT_n}{dx}\right] = (-2x){T_n}^{(m+1)} - 2m{T_n}^{(m)}$$

and for the last term we have

$$\frac{d^m n(n+1)T_n}{dx^m} = n(n+1){T_n}^{(m)}.$$

We now add the three results above to find

$$(1-x^2)\,{T_n}^{(m+2)} - 2x(m+1){T_n}^{(m+1)} + [n(n+1) - m(m+1)]{T_n}^{(m)} = 0$$

We now call $u = {T_n}^{(m)}$ and write

$$(1-x^2)\,u^{(2)} - 2x(m+1)u^{(1)} + [n(n+1) - m(m+1)]u = 0 \tag{C.21}$$

At this point we can observe the symilarity with eqution C.20, but still we are not there yet. By taking m derivatives we are introducing an $m(m+1)u$ but we want a m^2u. Besides this we are missing the divisor $1-x^2$. We then need to do a change of variables which works but it is not natural. We expect the missing $(1-x^2)$ denominator on that change of variables. This change of variables is given by

$$u = v(1-x^2)^{-m/2},$$

and so

$$\begin{aligned} u' &= v'(1-x^2)^{-m/2} + mxv(1-x^2)^{-(m+2)/2} \\ &= \left(v' + \frac{mxv}{1-x^2}\right)(1-x^2)^{-m/2}. \end{aligned}$$

Now, for the second derivative

$$\begin{aligned} u'' &= \left(v" + \frac{mxv'}{1-x^2} + \frac{mv}{1-x^2} + \frac{2mx^2v}{(1-x^2)^2}\right) \\ &\quad (1-x^2)^{-m/2} + \left(v' + \frac{mxv}{1-x^2}\right) xm(1-x^2)^{-m/2-1} \\ &= \left(v'' + \frac{2mxv'}{1-x^2} + \frac{mv}{1-x^2} + \frac{m(m+2)x^2v}{(1-x^2)^2}\right)(1-x^2)^{-m/2}. \end{aligned}$$

We now substitute back these last expressions in equation C.21

$$(1-x^2)^{1-m/2}\left(v''+\frac{2mxv'}{1-x^2}+\frac{mv}{1-x^2}+\frac{m(m+2)x^2v}{(1-x^2)^2}\right)$$
$$-2x(m+1)\left(v'+\frac{mxv}{1-x^2}\right)(1-x^2)^{-m/2}.+[n(n+1)-m(m+1)]v(1-x^2)^{-m/2}=0$$

We multiply all by $(1-x^2)^{m/2}$ and find

$$\begin{aligned}
& (1-x^2)v''+2mxv'+mv+\frac{m(m+2)x^2v}{(1-x^2)} \\
& -2x(m+1)\left(v'+\frac{mxv}{1-x^2}\right)+[n(n+1)-m(m+1)]v \\
= \;& (1-x^2)v''+mv-\frac{m^2x^2v}{1-x^2}-2xv'+[n(n+1)-m(m+1)]v \\
= \;& (1-x^2)v''-\frac{m^2x^2v}{1-x^2}-2xv'+n(n+1)v-m^2v= \\
= \;& (1-x^2)v''-2xv'-\frac{m^2v}{1-x^2}+n(n+1)v \\
= \;& (1-x^2)v''-2xv'+n(n+1)v-\frac{m^2v}{1-x^2}=0
\end{aligned}$$

and this is precisely equation C.20 with v instead of T_n. Now doing back substitution we have that

$$\begin{aligned}
v(x) &= u(x)(1-x^2)^{m/2} \\
&= (1-x^2)^{m/2}\frac{d^mT_n(x)}{dx^m} \\
&= T_n^m(x).
\end{aligned}$$

where now we used the upper index m since this last expression represents the solution of the generalized Legendre equation.

Then to build a solution of the generalized Legendre equation we use the solution of the regular Legendre equation, take its m derivative, and multiply the result by $(1-x^2)^{m/2}$.

Now we use the explicit representation C.6 of Legendre polynomials to find the associated Legendre function T_n^m that we call now P_n^m. We take m derivatives of T_n^m. Since the highest power of T_n^m is n we see that for $m > n$, $d^m P_n/dx^m = 0$, so the $P_n^m(x)$ functions terminates. Now, since the highest derivative of x^{n-2k} is $(n-2k)!/(l-2k-m)!x^{n-2k-m}$ the highest non zero value of k in the sum is $[(l-m)/2]$ and we have, up to a normalization factor

$$\begin{aligned} P_n^m(x) &= (1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m} \\ &= (1-x^2)^{m/2}\frac{1}{2^n}\sum_{k=0}^{[(l-m)/2]}\frac{(-1)^k(2n-2k)!}{(n-k)!k!(n-2k-m)!}x^{n-m-2k}. \end{aligned} \tag{C.22}$$

We can use Rodriguez's formula C.9 to express this differently. That is we write

$$\begin{aligned} P_n^m(x) &= (1-x^2)^{m/2}\frac{d^m}{dx^m}\left[\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\right] \\ &= (1-x^2)^{m/2}\frac{1}{2^n n!}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n. \end{aligned}$$

Since the generalized Legendre equation is a function of m^2 it should not matter in the equation to replace m by $-m$. In that case we could get another solution up to a multiplicative constant that we want to determine. That is we equate the equations with plus and minus m.

$$(1-x^2)^{-m/2}\frac{1}{2^n n!}\frac{d^{-m+n}}{dx^{-m+n}}(x^2-1)^n = c_{mn}(1-x^2)^{m/2}\frac{1}{2^n n!}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n.$$

with $0 \leq m \leq n$. Now this is

$$\frac{d^{-m+n}}{dx^{-m+n}}(x^2-1)^n = c_{mn}(1-x^2)^m\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n.$$

We have a $(-1)^n$ power that need to fix. The other part of the constant can be fixed by taking the expansion of derivatives in both sides and take the highest order. We expand the binomial and have

$$\frac{d^{-m+n}}{dx^{-m+n}}\sum_{j=0}^{n}\binom{n}{j}(x^2)^j(-1)^{n-j} = c_{mn}\sum_{j=0}^{m}(1)^j\binom{m}{j}(-x^2)^{m-j}\frac{d^{m+n}}{dx^{m+n}}\sum_{j=0}^{n}\binom{n}{j}(x^2)^j(-1)^{n-j}$$

The maximun power before differentition on the left is x^{2n} and after $n-m$ differentiations $(2n)(2n-1)\cdots(2n-(n-m-1))x^{2n-(n-m)} = (2n)(2n-1)\cdots(2n-(n-m-1))x^{n+m}$

The maximun power before differentition on the right is achieved by picking $j=0$ on the first sum and $j=n$ in the second. That is, before differentiation

$$(-1)^m x^{2m}\frac{d^{m+n}}{dx^{m+n}}x^{2n}$$

then differentiating this gives

$$(-1)^m x^{2m}(2n)(2n-1)\cdots(2n-(m+n-1))x^{2n-(m+n)} = (-1)^m x^{2m}(2n)(2n-1)\cdots(n-m+1)x^{m+n}.$$

We have then that

$$(2n)(2n-1)\cdots(n+m) = c_{nm}(-1)^m(2n)(2n-1)\cdots(n-m)$$
$$\frac{(2n)!}{(n+m)!} = c_{nm}(-1)^m\frac{(2n)!}{(n-m)!}.$$

and

$$c_{nm} = (-1)^n\,\frac{(n-m)!}{(n+m)!}.$$

and we conclude that

$$P_n^{-m}(x) = (-1)^n\,\frac{(n-m)!}{(n+m)!}P_n^m(x). \tag{C.23}$$

We now show the orthogonality of the associated Legendre functions.

C.3.2 Orthogonality of the associated Legendre functions

Here we consider m fix and want to evaluate the integral

$$I^m_{kn} = \int_{-1}^{1} P^m_k(x) P^m_n(x) dx,$$

Since k and n are interchangable let us assume without loss of generality that $k \geq n$.

Let us use first the Rodriguez's formula C.9 to write

$$\begin{aligned} P^m_k(x)P^m_n(x) &= (1-x^2)^{m/2}\frac{d^m P_k(x)}{dx^m}(1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m} \\ &= (1-x^2)^m \frac{d^m}{dx^m}\left[\frac{1}{2^k n!}\frac{d^k}{dx^k}(x^2-1)^k\right]\frac{d^m}{dx^m}\left[\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\right] \\ &= (1-x^2)^m \frac{1}{2^{n+k} n!\, k!}\frac{d^{m+k}}{dx^{m+k}}(x^2-1)^k \frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n. \end{aligned}$$

We now evaluate the integral I^m_{kn} by doing $m+n$ integration by parts with

$$\begin{aligned} u(x) &= (1-x^2)^m \frac{d^{m+k}}{dx^{m+k}}(x^2-1)^k \\ v'(x) &= \frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n. \end{aligned}$$

Every time we do one of the m integration by parts, the term on the left evaluated in the end points $-1, 1$ becomes zero since $(1-x^2)$ will always be there and it vanishes at the end points. Likewise, when do the n integration by parts the end point term vanishes since x^2-1 also vanishes at 1 and -1. In all this integration by parts we uncover the $(x^2-1)^k$ and cover the $(1-x^2)^m$ with $m+n$ derivatives. That is we find

$$I^m_{kn} = (-1)^{n+m}\frac{1}{2^{n+k} n! k!}\int_{-1}^{1}(x^2-1)^n \frac{d^{n+m}}{dx^{n+m}}\left[(1-x^2)^m \frac{d^{m+k}}{dx^{m+k}}(x^2-1)^k\right] dx.$$

We use the Leibniz rule to expand the factor in the brakets.

$$\frac{d^{n+m}}{dx^{n+m}}\left[(1-x^2)^m \frac{d^{m+k}}{dx^{m+k}}(x^2-1)^k\right] = \sum_{j=0}^{n+m}\binom{n+m}{j}\frac{d^j}{dx^j}(1-x^2)^m \frac{d^{2m+n+k-j}}{x^{2m+n+k-j}}(x^2-1)^k.$$

The derivative after the binomial expression is non-zero only if $j \leq 2m$, and the last derivative is non-zero only when $2m + n + k - j \leq 2k$, That is when $2m + n - j \leq k$, or when $j \geq 2m + n - k$. Since $k \geq n$ we find that $j \geq m$ and the only possibility for j is to be $2m$ and so $k \leq n$, from which the only possibility for I^m_{kn} not to be zero is for $k = n$. That is

$$I^m_{kn} = (-1)^{n+m}\delta_{kn}\frac{1}{2^{n+k}n!k!}\int_{-1}^{1}(x^2-1)^n\binom{n+m}{2m}\frac{d^{2m}}{dx^{2m}}(1-x^2)^m\frac{d^{n+k}}{x^{n+k}}(x^2-1)^k dx$$

From here we see that the functions are orthogonal. The norm square is obtained by choosing $n = k$, that is,

$$\|P^m_n(x)\|^2 = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\binom{n+m}{2m}\frac{d^{2m}}{dx^{2m}}(1-x^2)^m\frac{d^{2n}}{dx^{2n}}(-1)^n(1-x^2)^n dx$$

From the binomial expansion

$$(1-x^2)^m = \sum_{j=0}^{m}\binom{m}{j}1^j(-x^2)^{m-j} = \sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}x^{2(m-j)}.$$

Since we are differentiating $2m$ ties the only term that survives the differentiation is that with $j = 0$ and the differentiation is $(-1)^m(2m!)$. Likewise for the last derivative under the integral sign we obtain the value $(-1)^n(-1)^n(2n!)$, so we end up with

$$\begin{aligned}
\|P^m_n(x)\|^2 &= \frac{(-1)^m}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\binom{n+m}{2m}(2m!)(2n!)\,dx \\
&= \frac{(-1)^m\cancel{(2m)!}(2n)!(n+m)!}{(n-m)!\cancel{(2m)!}2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n dx \\
&= \frac{(-1)^m(2n)!(n+m)!}{(n-m)!\,2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n dx
\end{aligned}$$

We now use the result in equation C.13. That is,

$$\|P^m_n(x)\|^2 = \frac{(-1)^m(2n)!(n+m)!}{(n-m)!\,2^{2n}(n!)^2}\left[(-1)^n 2\frac{(2^n n!)^2}{(2n+1)!}\right] = \frac{2(n+m)!}{(2n+1)(n-m)!}$$

or writing differently

$$\int_{-1}^{1} P_n^m(x) P_l^m(x) = \delta_{nl} \frac{2}{2l+1} \frac{(n+m)!}{(n-m)!}. \tag{C.24}$$