Solución a los problemas del curso de Electrodinámica

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1 Project # 1

1 .1 Problem 1

One way to see how different are the two integrals is by making a change of variable to put one into the other.

We start with the formula for the second problem representation. That is,

$$V(\boldsymbol{b}) = \frac{\sigma}{2\epsilon_0}\frac{1}{\sqrt{2}}\int_0^{\sqrt{2}h}\frac{udu}{\sqrt{h^2+u^2-\sqrt{2}uh}}.$$

Make the following change of variables $\ell = u\cos\theta = u\sqrt{2}/2$ Then $d\ell = du(\sqrt{2}/2)$ and $u = 2\ell/\sqrt{2}$, $du = (2/\sqrt{2})d\ell$, When $u \in [0, \sqrt{2}h]$, $\ell \in [0, h]$, since $h = \sqrt{2}h\sqrt{2}/2 = h$. The integral becomes

$$
\begin{aligned}
V(\boldsymbol{b}) &= \frac{\sigma}{2\epsilon_0}\frac{1}{\sqrt{2}}\int_0^h \frac{[2/\sqrt{2}]\ [2/\sqrt{2}]\ell d\ell}{\sqrt{h^2+(2\ell/\sqrt{2})^2-\sqrt{2}(2\ell/\sqrt{2})h}} \\
&= \frac{2\sigma}{2\epsilon_0}\frac{1}{\sqrt{2}}\int_0^h \frac{\ell d\ell}{\sqrt{h^2+(2\ell/\sqrt{2})^2-2\ell h}} \\
&= \frac{\sigma}{2\epsilon_0}\int_0^h \frac{\sqrt{2}\ell d\ell}{\sqrt{h^2+2\ell^2-2\ell h}} \\
&= \frac{\sigma}{2\epsilon_0}\int_0^h \frac{\sqrt{2}\ell d\ell}{\sqrt{(h-\ell)^2+\ell^2}}.
\end{aligned}
$$

Now make a change of variable ℓ by $h-\ell$, the sign changes, but so the limits of the integral so at the end nothing changes, just replace ℓ by $h-\ell$, we find

$$
\begin{aligned}
V(\boldsymbol{b}) &= \frac{\sigma}{2\epsilon_0}\int_0^h \frac{\sqrt{2}(h-\ell)d\ell}{\sqrt{[h-(h-\ell)]^2+(h-\ell)^2}} \\
&= \frac{\sigma}{2\epsilon_0}\int_0^h \frac{\sqrt{2}(h-\ell)d\ell}{\sqrt{\ell^2+(h-\ell)^2}}.
\end{aligned}
$$

This is the integral proposed in part (i) of the homework except for the factor $\sqrt{2}$ here which is missing there.

The differential element is running down vertically, and the elements of surface are seen as rings of constant radius but this is not a cylinder. A surface ring element is given by

$$dS = 2\pi\rho dw$$

$$V(\boldsymbol{p}) = \frac{1}{4\pi\epsilon_0}\int_0^h \frac{\sqrt{2}\sigma[2\pi(h-\ell)]}{r'}d\ell$$

where dw is running along the generating line, and

$$dw = \sqrt{2}d\ell,$$

This explains the missing factor $\sqrt{2}$ from the first integral.

We compute r' from the Pythagoras theorem. One side is the radius $h-\ell$ (the radius), the other is ℓ, for a size

$$r' = \sqrt{(h-\ell)^2 + \ell^2}$$

and so

$$V(\boldsymbol{p}) = \frac{\sqrt{2}}{4\pi\epsilon_0} \int_0^h \frac{\sigma[2\pi(h-\ell)]d\ell}{\sqrt{(h-\ell)^2+\ell^2}} = \frac{\sqrt{2}\,\sigma}{2\epsilon_0} \int_0^h \frac{(h-\ell)d\ell}{\sqrt{(h-\ell)^2+\ell^2}}. \tag{1}$$

We focus in the indefinite integral

$$I = \int \frac{(h-\ell)d\ell}{\sqrt{(h-\ell)^2+\ell^2}} = -\int \frac{udu}{\sqrt{u^2+(h-u)^2}} = -\int \frac{udu}{\sqrt{2u^2-2hu+h^2}}$$

where $u = h - \ell$, $du = -d\ell$. We call I the symbol for the integral up to a constant.

We use the formula:

$$\int \frac{xdx}{\sqrt{ax^2+bx+c}} = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2a^{3/2}} \ln\left|2\sqrt{ax^2+bx+c} + \frac{2ax+b}{\sqrt{a}}\right|.$$

with $a = 2$, $b = -2h$, and $c = h^2$. That is

$$I = -\frac{1}{2}\sqrt{2u^2-2uh+h^2} - \frac{2h}{2\,2^{3/2}} \ln\left|2\sqrt{2u^2-2hu+h^2} + \frac{4u-2h}{\sqrt{2}}\right|.$$

and since $u = h - \ell$,

$$\begin{aligned} I \;&=\; -\frac{1}{2}\sqrt{2(h-\ell)^2 - 2(h-\ell)h + h^2} - \\ &\quad \frac{h}{2^{3/2}} \ln\left|2\sqrt{2(h-\ell)^2 - 2h(h-\ell) + h^2} + \frac{4(h-\ell)-2h}{\sqrt{2}}\right| \end{aligned}$$

and

$$
\begin{aligned}
I|_0^h &= -\frac{1}{2}h - \frac{h}{2^{3/2}}\ln|2h-\sqrt{2}h| \\
&+ \frac{1}{2}\sqrt{2h^2-2h^2+h^2} + \frac{h}{2^{3/2}}\ln\left|2\sqrt{2h^2-2h^2+h^2}+\sqrt{2}h\right| \\
&= -\frac{\cancel{h}}{\cancel{2}} - \frac{h}{2^{3/2}}\ln|2h-\sqrt{2}h| + \frac{\cancel{h}}{\cancel{2}} + \frac{h}{2^{3/2}}\ln\left|h+\sqrt{2}h\right| \\
&= \frac{h}{2\sqrt{2}}\ln\frac{\sqrt{2}h-h}{\sqrt{2}h-h} \\
&= \frac{h}{2\sqrt{2}}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1} \\
&= \frac{h}{2\sqrt{2}}\ln(1+\sqrt{2})^2 \\
&= \frac{h}{2\sqrt{2}}\ln(3+2\sqrt{2})
\end{aligned}
$$

Figure 1 shows the computation done using the *Mathematica* software *WoframAlpha* .

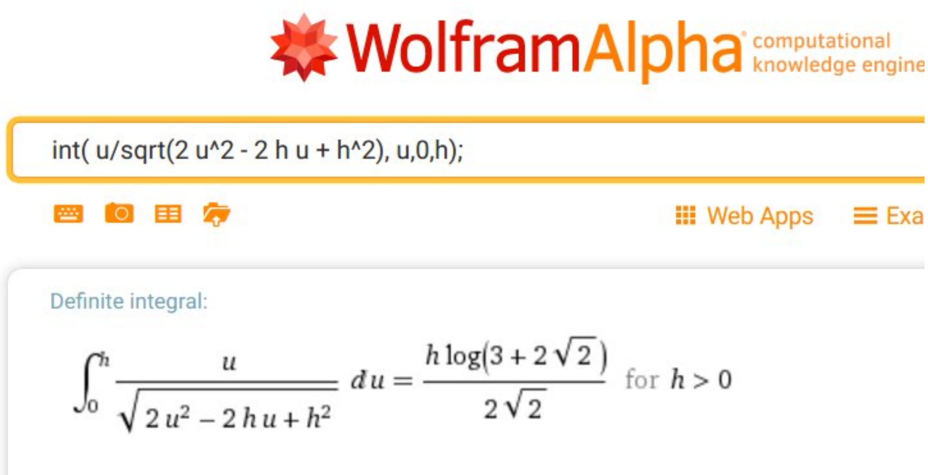

Figure 1: Verification using *WolframAlpha* of the integration computed here.

We are ready now to introduce the coefficient left in equation 1. That is, we say that the field is given by

$$
\begin{aligned}
V(\boldsymbol{x}) &= \frac{\sqrt{2}\,\sigma}{2\epsilon_0}\frac{h}{2\sqrt{2}}\ln(3+2\sqrt{2}) \\
&= \frac{\sigma}{2\epsilon_0}\frac{h\ln(3+2\sqrt{2})}{2}
\end{aligned}
$$

We can further simplify the expression above. We see that

$$
\frac{\ln(3+2\sqrt{2})}{2} = \frac{\ln(1+2\sqrt{2}+2)}{2} = \frac{(1+\sqrt{2})^2}{2} = 1+\sqrt{2}
$$

Then

$$
V(\boldsymbol{p}) = \frac{\sigma h}{2\epsilon_0}\ln(1+\sqrt{2}).
$$

1 .2 Problem 2

Solution: This integral is hard to compute. The integral with the square root sign leads to elliptical integrals which have not analytical closed representation in terms of algebraic or trigonometrial functions.

Hence we use a different methodology. Pick any element of volume in the sphere. The element of volume in spherical coordinates is given by $dVol = r^2 \sin\theta dr d\theta d\phi$. Assume your observation point $(0,0,z)$, and the integration volume to be around the center point in spherical coordinates (r,θ,ϕ). The symmetry of the problem indicates that there is no change on the azimuthal direction. Then if now $\rho = q/(4\pi R^2)$ is the charge density we need to evaluate

$$
V(z) = \frac{3q}{16\pi^2 R^3\epsilon_0}\int \frac{1}{r'}d\hat{\boldsymbol{r}} = \frac{6\pi q}{16\pi^2 R^3\epsilon_0}\int \frac{1}{r'}r^2\sin\theta dr d\theta
$$

where r' is the distance between the observation point $(0,0,z)$ and the integrating point in the volume. We can draw a triangle from the observation point $(0,0,z)$ to the integrating point (r,θ,ϕ), and from there to the origing $(0,0,0)$. The radius of with respect to the observation point r' is given by the formula

$$(r')^2 = z^2 + r^2 - 2rz\cos\theta,$$

after using the cosine law where θ is the polar angle going from 0 (in the north pole) to π in the south pole. We then need to evaluate

$$V(z) = \frac{3q}{8\pi R^3\epsilon_0}\int_0^R dr \int_0^\pi d\theta \frac{r^2\sin\theta}{\sqrt{z^2 + r^2 - 2rz\cos\theta}}.$$

We do first the integral on θ. That is,

$$I(r) = \int d\theta \frac{r^2\sin\theta}{\sqrt{z^2 + r^2 - 2rz\cos\theta}}.$$

Call $u = z^2 + r^2 - 2rz\cos\theta$, then $du = 2rz\sin\theta d\theta$, $r^2\sin\theta d\theta = rdu/(2z)$ and so

$$I(r) = \int \frac{rdu}{2z\sqrt{u}} = \frac{2r}{2z}\sqrt{u}.$$

and replacing u back in terms of θ,

$$I(r) = \frac{r}{z}\sqrt{z^2 + r^2 - 2rz\cos\theta},$$

and then

$$I(r)|_0^\pi = \frac{r}{z}\sqrt{z^2 + r^2 - 2rz} - \frac{r}{z}\sqrt{z^2 + r^2 + 2rz} = \frac{r|r+z|}{z} - \frac{r|r-z|}{z}.$$

That is

$$I(r)|_0^\pi = \frac{r}{z}\left[r + z - |r - z|\,\right] = \begin{cases} \frac{2r^2}{z} & r < z \\ 2r & r > z \end{cases}$$

and so

$$V(z) = \frac{6q}{8\pi R^3\epsilon_0}\left[\int_0^z \frac{r^2}{z}dr + \int_z^R rdr\right] = \frac{3q}{4\pi R^3\epsilon_0}\left[\frac{z^3}{3z} + \frac{R^2}{2} - \frac{z^2}{2}\right]$$

That is

$$V(z) = \frac{3q}{4\pi R^3\epsilon_0}\left(\frac{R^2}{2} - \frac{z^2}{6}\right) = \frac{q}{8\pi R\epsilon_0}\left(3 - \frac{z^2}{3}\right).$$

1 .3 Problem 3

Find the surface area and volume of a sphere in the space of n dimensions.

Solution: We do this in parts.

(a) We first show that the volume of a n–dimensional sphere of radius R is given by

$$V_n(R) = \int_{0\leq R} r^{n-1}drd\Omega_n,$$

where Σ_n is the solid angle on n dimensions. In spherical coordinates we can choose the the following coordinates:
$r, \phi_1, \phi_2, \cdots, \phi_{n-1}$, where $\phi_1, \cdots, \phi_{n-1}$ are the angles with respect to the coordinate axes $n, n-1, \cdots, 2$ respectively. The angles range as follows: $\phi_1 \in [0, \pi)$ and $\phi_i \in [0, 2\pi)$, $i > 1$.

If $n = 2$ then there is only one angle $\phi_1 = \theta \in [0, 2\pi)$, if $n \geq 3$ then the first $n-2$ angles are in the interval $[0, 2\pi)$ (they are like azimuthal angles for the lower dimensions). We think of ϕ_1 as the polar angle (for $n \geq 3$) which goes in $[0, \pi)$, the other angles ϕ_i are like azimuthal angles which span the space in the coordinates $\phi_1, \phi_2, \cdots \phi_i$. The stretching factors for these coordinates are given by

$$
\begin{aligned}
h_r &= 1 \\
h_{\phi_1} &= r f_1(\phi_1) \\
h_{\phi_2} &= r f_2(\phi_1, \phi_2) \\
h_{\phi_3} &= r f_3(\phi_1, \phi_2, \phi_3) \\
&\vdots \\
h_{\phi_{n-1}} &= r f_{n-1}(\phi_1, \cdots, \phi_{n-1})
\end{aligned}
$$

At the moment we do not care of the particular form of the functions f_i. Besides, there is no a unique way to define this coordinates. Wikipedia [1] shows a particular definition of these coordinates. Think that ϕ_1 is the polar angle with respect to the n axis and then ϕ_2 would be the first azimuthal angle in the plane define by the axis n and $n-1$, then as we move down to axis 1, more and more angles get involved since each new angle get related to the previous angles by projecting what we have at the i-th state (angles involved $\phi_1, \phi_2, \cdots, \phi_i$), into the next $i+1$ axis. In general the function $f_i(\phi_1, \cdots, \phi_i)$ is a function of sine and cosine of the angles ϕ_j. On each case the factor r gets involved because we are looking at an arc on some projected plane and the arc always is a projection of the sphere arc (with radius r) into some subdomains. What we care, since we are interested only on the integration, is the element of volume $h_r h_{\phi_1} \cdots h_{\phi_{n-1}}$. Since we are doing integration we have that

$$
h_r h_{\phi_1} \cdots h_{\phi_{n-1}} = r^{n-1} f_1(\phi_1) f_2(\phi_1, \phi_2) \cdots f_{n-1}(\phi_1, \cdots, \phi_{n-1}),
$$

and

$$
dV = r^{n-1} dr d\Omega_n,
$$

with

$$
d\Omega_n = f_1(\phi_1) f_2(\phi_1, \phi_2) \cdots f_{n-1}(\phi_1, \cdots, \phi_{n-1}) d\phi_1 d\phi_2 \cdots d\phi_{n-1}.
$$

[1] https://en.wikipedia.org/wiki/N-sphere#Spherical_coordinates

Here $d\Omega_n$ is the solid angle in n dimensions. We then have that

$$V_n(R) = \int_{0\leq R} r^{n-1} dr d\Omega_n. \tag{2}$$

(b) Since $d\Omega_n$ does not depend on the radius r, we can write equation 2 as

$$V_n(R) = \left(\int_{0\leq R} r^{n-1} dr\right) \int d\Omega_n = \frac{\Omega_n R^n}{n} \tag{3}$$

and find that for a unit sphere

$$V_n(1) = \frac{\Omega_n}{n}. \tag{4}$$

where $V_n(1)$ is now the volume of a unit sphere.

(c) Let us now consider the problem in Cartesian coordinates. We take the n axis and slice in "horizontal" $x_n =$ constant hyperplanes it between its smaller and larger values. On each slice on the x_n axis we get a sphere in the lower n-1 dimensional space. That is we write

$$x_1^2 + \cdots + x_n^2 = R^2,$$

and from here

$$x_n = \sqrt{R^2 - x_1^2 + \cdots x_{n-1}^2}$$

call $\rho^2 = x_1^2 + \cdots + x_{n-1}^2$. Note that this equation for radius ρ corresponds to an n-1 dimensional sphere. For each x_n between $-R$ and R, and $\rho = \pm\sqrt{R^2 - x_n^2}$ we find the volume of the n-1 dimensional sphere that it cuts through. That is,

$$\begin{aligned}
V_n(R) &= \int_{-R}^{R}\left(\int_0^{\sqrt{R^2-x_n^2}} V_{n-1}(\rho')d\rho'\right)dx_n \\
&= 2\int_0^R \left(\int_0^{\rho} V_{n-1}(\rho')d\rho'\right)dx_n \\
&= 2\int_0^R \frac{\Omega_{n-1}\rho^{n-1}}{n-1}dx_n \qquad \text{from using equation 3}
\end{aligned}$$

and from $x_n = \sqrt{R^2-\rho^2}$, we see that $dx_n = -\rho d\rho/(R^2-\rho^2)^{1/2}$ so

$$\begin{aligned}
V_n(R) &= \frac{2\Omega_{n-1}}{n-1}\int_0^R \rho^n (R^2-\rho^2)^{-1/2}d\rho \\
&= \frac{2\Omega_{n-1}}{R(n-1)}\int_0^R \rho^n(1-(\rho/R)^2)^{-1/2}d\rho
\end{aligned}$$

Call $u=\rho/R$, then $du = d\rho/R$, $d\rho = Rdu$, $\rho=[0,1]$ and

$$\begin{aligned}
V_n(R) &= \frac{2\Omega_{n-1}}{R(n-1)}\int_0^1 R^n u^n(1-u^2)^{-1/2}Rdu && (5)\\
&= \frac{2R^n\,\Omega_{n-1}}{n-1}\int_0^1 u^n(1-u^2)^{-1/2}du \\
&= \frac{\cancel{2}R^n\,\Omega_{n-1}}{\cancel{2}(n-1)}\text{Beta}\left(\frac{1}{2},\frac{n+1}{2}\right) \\
&= \frac{R^n\,\Omega_{n-1}}{(n-1)}\frac{\Gamma(1/2)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma[(n/2+1]} \\
&= \frac{R^n\,\Omega_{n-1}}{\cancel{(n-1)}}\frac{\sqrt{\pi}\,\left(\frac{\cancel{n-1}}{2}\right)\Gamma[(n-1)/2]}{(n/2)\Gamma(n/2)} \\
&= R^n\,\Omega_{n-1}\frac{\sqrt{\pi}\,\Gamma[(n-1)/2]}{n\,\Gamma(n/2)} && (6)
\end{aligned}$$

(d) We now compare equations 3 and 6 and find

$$\frac{\Omega_n\, R^n}{n} = R^n\, \Omega_{n-1} \frac{\sqrt{\pi}\, \Gamma[(n-1)/2]}{n\, \Gamma(n/2)}$$

From this we get the recursion

$$\Omega_n = \frac{\sqrt{\pi}\, \Gamma[(n-1)/2]}{\Gamma(n/2)} \Omega_{n-1}.$$

We have then that

$$\begin{aligned}
\Omega_n &= \frac{\sqrt{\pi}\, \cancel{\Gamma[(n-1)/2]}}{\Gamma(n/2)} \frac{\sqrt{\pi}\, \Gamma[(n-2)/2]}{\cancel{\Gamma((n-1)/2)}} \Omega_{n-2} \\
&= \frac{\pi}{\Gamma(n/2)} \left(\Gamma(n-2)/2\right) \Omega_{n-2} \\
&= \frac{\pi}{\Gamma(n/2)} \left(\Gamma(n-2)/2\right) \frac{\sqrt{\pi}\, \Gamma[(n-3)/2]}{\Gamma((n-2)/2)} \Omega_{n-3} \\
&= \frac{\pi^{3/2}}{\Gamma(n/2)} \cancel{\left(\Gamma(n-2)/2\right)} \frac{\Gamma[(n-3)/2]}{\cancel{\Gamma((n-2)/2)}} \Omega_{n-3} \\
&= \frac{\pi^{3/2}}{\Gamma(n/2)} \Gamma[(n-3)/2] \Omega_{n-3}
\end{aligned}$$

The pattern is clear. The power of π is $n/2$ when the recursion gets to $\Omega_{n-n} = \Omega_0$. At that point $\Gamma(0) = 1$, The value of $\Gamma(n/2)$ in the denominator never cancels, it stays the same. So we get to

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma n/2} \Omega_2 = \frac{2\, \pi^{n/2}}{\Gamma(n/2)} \Omega_0.$$

We can assume $\Omega_0 = 1$ (a point has a "solid angle" of 1, and so

$$\Omega_n = \frac{2\, \pi^{n/2}}{\Gamma(n/2)}.$$

Using equation 4 we see that the volume of a unit sphere is then given by

$$V_n(1) = \frac{\Omega_n}{n} = \Omega_n = \frac{2\,\pi^{n/2}}{n\Gamma(n/2)}.$$

The area of a surface of radius R gets scaled by R^{n-1} while the volume gets scaled by R^n. Then the area of a spherical surface of radius R in an n-dimensional space is given by

$$\Omega_n(R) = \frac{2\,\pi^{n/2} R^{n-1}}{\Gamma(n/2)}.$$

while the volume is given by

$$V_n(R) = \frac{2R^n\,\pi^{n/2}}{n\Gamma(n/2)}.$$

Let us test this formula for a few values:

- **For $n = 1$ an interval,**

  $$\Omega_2 = \frac{2\cancel{\pi^{1/2}}}{\cancel{\pi^{1/2}}} = 2.$$

  These are the two end points of the interval. The length of the unit interval is 2 while the length of the interval with radius R is given by $V_2(R) = 2R$.

- **For $n = 2$ a circle in $\mathbb{R}^2$,**

  $$\Omega_2 = \frac{2\pi}{\Gamma(1)} = 2\pi.$$

  The unit circumference length.

  The arc length of the circle is

$$\Omega_2(R) = 2\pi R$$

and the area

$$V_2(R) = \frac{2\pi R^2}{2} = \pi R^2.$$

- **For $n = 3$ a sphere in $\mathbb{R}^3$.**

$$\Omega_3 = \frac{2\pi^{3/2}}{\Gamma(3/2)} = \frac{2\pi^{3/2}}{(1/2)\Gamma(1/2)} = \frac{2\pi^{3/2}}{(1/2)(\pi^{1/2})} = 4\pi^2.$$

The area of the unit sphere.
The area of a sphere of radius R is

$$\Omega_3(R) = 4\pi R^2,$$

and the volume

$$V_3(R) = \frac{4\pi^3}{3}.$$

In general note that the volume is the integral of the area along the radial direction. So we only have to learn one to know the other.

2 Project # 2

2 .1 Problema 1

We show uniqueness of solution for the Poisson's equation with mixed boundary conditions. Let us assume that there are two solutions u_1 and u_2 and define $u = u_1 - u_2$. As we did with Direchlet boundary conditions, let us define the auxiliary function.

$$J = \int_V (\nabla u \cdot \nabla u)\, dV = \int_V \|\nabla u\|^2 dV.$$

Since both u_1 and u_2 satisfy the same equation and boundary conditions u is zero in some part of the boundary and $\partial u/\partial n = 0$ in the other part of the boundary. We showed that, since $\nabla^2 u = 0$ and after using an identity for the divergence of the product of u and ∇u, $\|\nabla u\|^2 = \nabla \cdot (u \nabla u)$, and from here that

$$\int_V \|\nabla u\|^2 dV = \int_S (u \nabla u)\, d\boldsymbol{S} = \int_S u \frac{\partial u}{\partial n} dS \tag{7}$$

Then since u is zero in some part of the boundary S and $\partial u/\partial n$ in the rest of the boundary, $J = 0$, and the integrand $\|\nabla u\|^2 = 0$. This means that either u is constant or 0. Since $u = 0$ in part of the boundary but u is constant inside the volume surrounded by S, then $u = 0$ everywhere inside the volume and so $u_1 = u_2$ inside the volume. From here the solution is unique.

2 .2 Problema 2

The solution of this problem is in the class notes. If we imposed both Dirchlet and Neumann boundary conditions the problem is ill-posed.

The solution of the Poisson's equation with Neumann boundary conditions (see the notes) is given by the equation:

$$u(\boldsymbol{r}) = \int_S u(\boldsymbol{r_i}) \frac{\partial G(\boldsymbol{r}, \boldsymbol{r}_i)}{\partial n} d\boldsymbol{r}_i - \int_V G(\boldsymbol{r}, \boldsymbol{r}_i) \rho(\boldsymbol{r}_i) d\boldsymbol{r}_i.$$

If we apply the Dirichlet boundary condition then we find

$$u(\boldsymbol{r}) = -\int_V G(\boldsymbol{r}, \boldsymbol{r}_i) \rho(\boldsymbol{r}_i) d\boldsymbol{r}_i.$$

From here we can compute $\partial u/\partial \boldsymbol{n}$ anywhere inside the volume V, and take the limit to reach the boundary S. How can we guarantee that this limit is 0? See that we can have an arbitry source distribution $\rho(\boldsymbol{r}_i)$, which can violate the zero derivative in the boundary.

2 .3 Problema 3

We use Gram-Schmidth to orthogonalize $1, x, x^2, x^3$ subjected to the inner product definition:

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

We start with the function $f(x) = 1$. Its square norm is

$$\int_{-1}^{1} dx = 2$$

Then, the **first** normalized polynomial is $1/\sqrt{2}$ or $\sqrt{2}/2$. We now project the second function x onto this vector. That is

$$\left\langle x\ ,\ \frac{1}{\sqrt{2}}\right\rangle = \int_{-1}^{1} \frac{x}{\sqrt{2}}dx = \frac{1}{\sqrt{2}}\frac{x^2}{2}\bigg|_{-1}^{1} = \frac{1}{\sqrt{2}}0 = 0.$$

Then x is already orthogonal to $1/\sqrt{2}$. Let us normalize x. We have

$$\langle x, x\rangle = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\bigg|_{-1}^{1} = \frac{2}{3}.$$

Then the **second** normalized vector is $\sqrt{3/2}\,x$. Let us now project x^2 into the previous two orthonormal vectors. The projection coefficient is given by

$$\left\langle x^2\ ,\ \sqrt{\frac{3}{2}}x\right\rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}}x^3 dx = 0.$$

The projection of x^2 on $1/\sqrt{2}$ is given by

$$\left\langle x^2\ ,\ \frac{1}{\sqrt{2}}\right\rangle = \int_{-1}^{1} \frac{x^2}{\sqrt{2}}dx = \frac{2}{\sqrt{2}}\frac{1}{3} = \frac{\sqrt{2}}{3}.$$

Then the projection of x^2 into the two previous orthonormal vector is

$$x^2 - \frac{\sqrt{2}}{3}\frac{1}{\sqrt{2}} = x^2 - \frac{1}{3}.$$

We find the norm of this vector.

$$\begin{aligned}\left\langle x^2 - \frac{1}{3}\,,\, x^2 - \frac{1}{3}\right\rangle &= \int_{-1}^{1}\left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)dx \\ &= \frac{2}{5} - \frac{4}{9} + \frac{2}{9} \\ &= \frac{18 - 20 + 10}{45} \\ &= \frac{8}{45},\end{aligned}$$

and

$$\sqrt{\frac{45}{8}} = \frac{3\sqrt{5}}{2\sqrt{2}}$$

Then the **third** normalized vector is

$$\frac{3\sqrt{5}}{2\sqrt{2}}x^2 - \frac{\sqrt{5}}{2\sqrt{2}} = \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right).$$

Let us now find the projection coefficient of x^3 in the previous vector

$$\left\langle x^3\,,\, \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\right\rangle = \sqrt{\frac{5}{2}}\int_{-1}^{1}\left(\frac{3}{2}x^5 - \frac{1}{2}x^3\right)dx = 0$$

Now, the projection of the vector in $\sqrt{3/2}x$ is

$$\left\langle x^3\,,\, \sqrt{\frac{3}{2}}x\right\rangle = \sqrt{\frac{3}{2}}\int_{-1}^{1}x^4 dx = 2\sqrt{\frac{3}{2}}\frac{1}{5} = \frac{\sqrt{6}}{5}.$$

The projection of x^3 into the constant function is 0. Then we find the orthogonal vector

$$x^3 - \frac{\sqrt{6}}{5}\sqrt{\frac{3}{2}}x = x^3 - 3x = x^3 - \frac{3}{5}x.$$

We now normalize this function

$$\begin{aligned} \left\langle x^3 - \frac{3x}{5}, x^3 - \frac{3x}{5} \right\rangle &= \int_{-1}^{1} \left(x^6 - \frac{6x^4}{5} + \frac{9x^2}{25} \right) dx \\ &= \frac{2}{7} - \frac{12}{5}\frac{1}{5} + \frac{18}{25}\frac{1}{3} \\ &= \frac{2}{7} - \frac{12}{25} + \frac{6}{25} \\ &= \frac{50 - 84 + 42}{175} \\ &= \frac{8}{175}. \end{aligned}$$

Now

$$\frac{\sqrt{175}}{\sqrt{8}} = \frac{5\sqrt{7}}{2\sqrt{2}}.$$

Then the **fourth** function is

$$\sqrt{\frac{7}{2}}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)$$

In summary we have the collection

zero	$\frac{1}{\sqrt{2}}1$
first	$\frac{3}{\sqrt{2}}x$
second	$\sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$
third	$\sqrt{\frac{7}{2}}\left(\frac{5}{2}x^3 - \frac{3}{2}\right)x$
$\vdots$	
nth	$\sqrt{\frac{2n+1}{2}}P_n.$

where P_n is the n-th Legendre polynomial.

This makes sense since P_n are orthogonal polynomials of degree n and their norm is $\sqrt{2/(2n+1)}$. If two sets of mutually orthogonal vectors share one direction in common they should share all directions. If they both have the same orientation the corresponding orthonormal transformation should make them into the same set.

2 .4 Problem 4

Solution: We find the norm $s = \|\sin nx\|$.

$$\langle \sin nx, \sin nx \rangle = \int_{-\pi}^{\pi} \sin^2 nx dx = \int_{-\pi}^{\pi} \frac{1}{2}(1 - \cos 2x) dx = \frac{1}{2}(x - \sin 2x)\Big|_{-\pi}^{\pi}.$$

Then $s = \sqrt{\pi}$.

It is no true tha Dirac delta can be represented as an infinite sum of functions $\sin nx$. The reason is that the set $\{\sin nx\}$ is not complete. This means that in the space of functions spanned by trigonometrical polynomials not all of them can be represented as sums of sines. All sine functions are assymetric ($sin[-nx] = -\sin[nx]$). The even functions are required to **complete** the set. That is the set of functions $\{\cos nx\}$. In fact it is more compact to say that the set of exponential functios $\{\mathrm{e}^{inx}\}$ is complete. The complex exponential functions have both sine and cosine components. The representation is given by

$$\delta(x-y)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\mathrm{e}^{in(x-y)}.$$

2 .5 Problem 5

Solution: The problem with Jackson's formulation is that when he makes the separation of variables and gets to a system of ordinary differential equations. This new system should come together with new boundary conditions inherited from the boundary conditions of the global problem. That is, if for example we have the equation

$$\frac{1}{Z}\frac{d^2Z}{dz}=-\gamma^2,$$

it should come with boundary conditions for the function Z only in the extreme points $Z(0)$ and $Z(c)$ and not in the whole interval.

2 .6 Problem 6

Find the eigenvalues and the eigenfunctions of the operator L defined as

$$Lu=\frac{d^2u}{dx^2}$$

subject to the boundary conditions $u(-1)=u(1)=0$. Verify that the eigenfunctions are orthogonal since the operator L in the interval $[-1,1]$ is self-adjoint.

Solution: We need to find λ and the $u(x)$ such that

$$Lu=\lambda u.$$

The solution of this differential equation is given by $\mathrm{e}^{\pm\sqrt{\lambda}x}$. The general solution has the form

$$u(x) = A\mathrm{e}^{\sqrt{\lambda}x} + B\mathrm{e}^{-\sqrt{\lambda}x}$$

We now impose the boundary conditions to solve for A and B. That is

$$\begin{aligned} A\mathrm{e}^{-\sqrt{\lambda}} + B\mathrm{e}^{\sqrt{\lambda}} &= 0 \quad , \quad x = -1 \\ A\mathrm{e}^{\sqrt{\lambda}} + B\mathrm{e}^{-\sqrt{\lambda}} &= 0 \quad , \quad x = 1 \end{aligned}$$

We multipy the first equation by $\mathrm{e}^{\sqrt{\lambda}}$, and the second by $\mathrm{e}^{-\sqrt{\lambda}}$, to find

$$\begin{aligned} A + B\mathrm{e}^{2\sqrt{\lambda}} &= 0 \\ A + B\mathrm{e}^{-2\sqrt{\lambda}} &= 0 \end{aligned}$$

we subtract the two equations to find

$$B(\mathrm{e}^{2\sqrt{\lambda}} - \mathrm{e}^{-2\sqrt{\lambda}}) = 0.$$

The case $B = 0$ leaves us with a zero solution which is not an eigenfunction. We assume $B \neq 0$, and find

$$\mathrm{e}^{2\sqrt{\lambda}} - \mathrm{e}^{-2\sqrt{\lambda}} = 0 \tag{8}$$

Since $\lambda = 0$ is not an eigenvalue let us consider two cases.

(i) $\lambda > 0$. In this case equation 8 does not have a solution (the only real solution of that equation is $\lambda = 0$ which is not an eigenvalue).

(ii) $\lambda < 0$. Then we write $\lambda = -(-\lambda)$, and $\sqrt{\lambda} = \mathrm{i}\sqrt{-\lambda}$. Equation 8 turns out to be

$$2\mathrm{i} \sin 2\sqrt{-\lambda} = 0.$$

This means that

$$2\sqrt{-\lambda} = n\pi,$$

for $n = 1, 2, \cdots$, and

$$-\lambda = \frac{n^2\pi^2}{4},$$

Then the eigenvalues are all negative of the form

$$\lambda_n = -\frac{n^2\pi^2}{4},$$

and the eigenfunctions are of the form

$$u_n(x) = \sin 2\sqrt{-\lambda_n}x = \sin n\pi x.$$

We claim that the set $\{u_n(x)\}$ is a set of orthogonal functions, since the opeartor L under the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx = 0$$

is self-adjoint. Let us verify the orthogonality.

Choose $n \neq m$. Then

$$\begin{aligned}
\langle \sin n\pi x, \sin m\pi x\rangle &= \frac{1}{2}\int_{-1}^{1} dx \cos[(n-m)\pi x] - \cos[(n+m)\pi x] \\
&= \frac{1}{2}\,\frac{\sin[(n-m)\pi x]}{(n-m)\pi} - \frac{\sin[(n+m)\pi x]}{(n+m)\pi}\Bigg|_{-1}^{1} \\
&= 0.
\end{aligned}$$

3 Project # 3

3 .1 Problem 1

We do this in steps

- **Step 1:** Set up the problem in the corresponding coordinate system with the given boundary conditions.

  The Laplace's equation in 2D cylindrical coordinates is given by

$$\nabla^2 u(r,\theta) = \frac{1}{h_r h_\theta}\left[\frac{\partial}{\partial r}\frac{(h_\theta/h_r)u(r,\theta)}{\partial r} + \frac{\partial}{\partial \theta}\frac{\partial (h_r/h_\theta)u(r,\theta)}{\partial \theta}\right] = 0.$$

  We use that $h_r = 1$, and $h_\theta = r$ to write (multiplying previously by r),

$$\begin{aligned}\nabla^2 u(r,\theta) &= \frac{1}{r}\left[\frac{\partial}{\partial r}\frac{ru(r,\theta)}{\partial r} + \frac{\partial}{\partial \theta}\frac{\partial (1/r)u(r,\theta)}{\partial \theta}\right] \\ &= \frac{1}{r}\left[\frac{\partial u(r,\theta)}{\partial r} + r\frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u(r,\theta)}{\partial \theta^2}\right] \\ &= \frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2}.\end{aligned}$$

  We then need to solve the equation

$$\frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2} = 0\ ,\ 1 < r < 2\ ,\ 0 \le \theta \le 2\pi, \quad (9)$$

  subjected to the boundary conditions.

$$u(1,\theta) = \cos\theta \quad , \quad \frac{\partial u(2,\theta)}{\partial r} = \sin 2\theta \quad , \quad 0 \le \theta \le 2\pi.$$

- **Step 2:** Use separation of variables and find two ODEs. We assume that the solution is $u(r,\theta) = R(r)\Theta(\theta)$, and replace it in equation 9 to find

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0.$$

where we use the $'$ symbol to denote total derivative with respect to the argument of the function.

As usual we divide by the product $R(r)\Theta(\theta)$ to find

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = 0,$$

We separate first the θ components. That is we write, after multiplying my r^2,

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -r^2\frac{R''(r)}{R(r)} - r\frac{R'(r)}{R(r)} = -\lambda.$$

This yields the following 2 ODEs:

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0 \quad , \quad r^2R''(r) + rR'(r) - \lambda R(r) = 0. \tag{10}$$

Since $\Theta(\theta)$ is periodic with period 2π, then

$$\Theta(0) = \Theta(2\pi) \quad , \quad \Theta'(\theta) = \Theta'(2\pi).$$

- **Step 3:** Find the eigenvalues and eigenfunctions of for in the θ domain. The equation $\Theta''(\theta) + \lambda\Theta(\theta) = 0$ has several solutions according to the value of λ. The simplest case is $\lambda = 0$ where we see that

$$\Theta''(\theta) = 0 \implies \Theta(\theta) = A + B\theta,$$

a linear function in θ. Since $\Theta(0) = A = \Theta(2\pi) = A + (2\pi)B$, then $B = 0$. A could have any arbitrary value and the eigenfunction of Θ is just 1.

We now assume that $\lambda \neq 0$, and so

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0 \implies \Theta(\theta) = A\mathrm{e}^{\mathrm{i}\sqrt{\lambda}\theta} + B\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}\theta}.$$

if $\lambda < 0$, then the functions are exponentials (the solution is in terms of hyperbolic sine and cosine functions). Since the function is periodic we see that $A = B = 0$, since no periodic function can be exponential (decreasing or increasing). If you do not believe this argument, evaluate the functions in 0 and 2π and solve for A and B. Hence the solution here is $\Theta(\theta) = 0$ and there are no egivenfunctions for $\lambda < 0$.

If $\lambda > 0$ we can write the solution in terms of sines and cosines. That is we can say that

$$\Theta(\theta) = A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta).$$

We now apply the boundary conditions. That is,

$$\begin{aligned}
\Theta(0) = \Theta(2\pi) \quad &\implies \quad A = A\cos(2\pi\sqrt{\lambda}) + B\sin(\sqrt{2\pi}\lambda) \\
\Theta'(0) = \Theta'(2\pi) \quad &\implies \quad \sqrt{\lambda}B = -A\sqrt{\lambda}\sin(2\pi\sqrt{\lambda}) + B\sqrt{\lambda}\cos(\sqrt{2\pi}\lambda).
\end{aligned}$$

We can write this in standard form

$$\begin{aligned}
(1 - \cos(2\pi\sqrt{\lambda})A - \sin(\sqrt{2\pi}\lambda)B = 0 \\
\sin(2\pi\sqrt{\lambda})A + (1 - \cos(2\pi\sqrt{\lambda})B = 0.
\end{aligned}$$

The derminant of the coefficients is

$$D = [1 - \cos(2\pi\sqrt{\lambda})]^2 + \sin^2(2\pi\sqrt{\lambda}) = 2 - 2\cos(2\pi\sqrt{\lambda}).$$

For non-trivial solutions we need $D = 0$, and this means

$$\cos(2\pi\sqrt{\lambda}) = 1.$$

This happens if $\sqrt{\lambda} = n$, or $\lambda = n^2$. Note that $\lambda > 0$. We have then that the following sequence of eigenfunctions

$$\{\cos n\theta, \sin n\theta\},$$

which solve the ordinary differential equation for Θ with the cyclic boundary conditions with $\lambda = n^2$, $n = 0, 1, 2, \cdots$.

- **Step 4:** Solve the equation 10 for R, and integrate the solution with $\Theta(\theta)$. Find the components for the supperposition solution.

  The equation for $R(r)$ is

  $$r^2 R''(r) + rR'(r) - \lambda R(r) = 0.$$

  which corresponds to an Euler [2] differential equation (its solution is in the appendix of the class notes).

  If $\lambda = 0$, then the solution can be found easily .

  $$r^2 R''(r) + rR'(r) = 0 \implies \frac{R''(r)}{R'(r)} = -\frac{1}{r}$$

  and integrating

  $$\ln R'(r) = -\ln r + \ln B$$

  for some constant B. We take exponentials and find

  $$R'(r) = \mathrm{e}^{-\ln r + \ln B} = \frac{B}{r}.$$

  and from integrating this

[2] https://en.wikipedia.org/wiki/Cauchy%E2%80%93Euler_equation

$$R(r) = A + B \ln r.$$

For $\lambda = 0$ we found the the solution in terms of $\Theta(\theta)$ is $\Theta(\theta) = 1$, so we define a particular solution

$$u_0(r, \theta) = R(r)\Theta(\theta) = A_0 + B_0 \ln r.$$

Now, for $\lambda = n^2$, the equation 10 turns out to be

$$r^2 R''(r) + rR'(r) - n^2 R(r) = 0.$$

with solutions r^n and r^{-n} so we have

$$R(r) = Ar^n + \frac{B}{r^n}.$$

We compose sine and cosine solutions as follows:

$$u_n^c(r, \theta) = \left(A_n^c r^n + \frac{B_n^c}{r^n} \right) \cos n\theta \quad , \quad u_n^s(r, \theta) = \left(A_n^s r^n + \frac{B_n^s}{r^n} \right) \sin n\theta$$

- **Step 5.** Do superposition and solve for the coefficients. The superposition solution is

$$\begin{aligned} u(r, \theta) &= u_0(r, \theta) + \sum_{n=1}^{\infty} u_n^c(r, \theta) + u_n^s(r, \theta) \\ &= A_0 + B_0 \ln r + \sum_{n=1}^{\infty} \left(A_n^c r^n + \frac{B_n^c}{r^n} \right) \cos n\theta + \left(A_n^s r^n + \frac{B_n^s}{r^n} \right) \sin n\theta, \end{aligned}$$

and

$$u_r(r,\theta) = \frac{B_0}{r} + \sum_{n=1}^{\infty} \left(nA_n^c r^{n-1} - n\frac{B_n^c}{r^{n+1}} \right) \cos n\theta \left(nA_n^c r^{n-1} - n\frac{B_n^c}{r^{n+1}} \right) \sin n\theta.$$

Here we apply the boundary conditions to find the coefficients. From $u(1,\theta) = \cos\theta$, we find

$$\cos\theta = A_0 + \sum_{n=1}^{\infty} (A_n^c + B_n^c) \cos n\theta + (A_n^s + B_n^s) \sin n\theta.$$

Since the $\cos n\theta$, and $\sin n\theta$ are linearly independent eigenfunctions, we require $A_0 = 0$, $A_n^s + B_n^s = 0, n \geq 1$, $A_1^c + B_1^c = 1$ and $A_n^c + B_n^c = 0, n > 1$. From $u_r(2,\theta) = \sin 2\theta$, we find

$$\begin{aligned} \sin 2\theta \quad &= \quad \frac{B_0}{\ln 2} + \sum_{n=1}^{\infty} \left(nA_n^c (2)^{n-1} - \frac{nB_n^c}{2^{n+1}} \right) \cos n\theta + \\ &\quad \left(nA_n^s 2^{n-1} - \frac{nB_n^s}{2^{n+1}} \right) \sin n\theta. \end{aligned}$$

Then by matching coefficients (due to the linearly independence of the functions involved), we find that $B_0 = 0$, and

$$\begin{aligned} \cancel{n}A_n^c\, 2^{n-1} - \frac{\cancel{n}B_n^c}{2^{n+1}} &= 0 \quad n \geq 1 \\ \cancel{n}A_n^s\, 2^{n-1} - \frac{\cancel{n}B_n^s}{2^{n+1}} &= 0 \quad n \neq 2 \\ 4A_2^s - \frac{B_2^s}{4} &= 1 \end{aligned}$$

Let us solve for A_1^c and B_1^c. We have two equations

$$\begin{aligned} A_1^c + B_1^c &= 1 \\ 2A_1^c - \frac{B_1^c}{2} &= 0 \end{aligned}$$

From the second equation $A_1^c = B_1^c/4$, and plugging this in the first equation

$$\frac{B_1^c}{4} + B_1^c = 1 \quad \Longrightarrow \quad \frac{5}{4}B_1^c = 1 \quad \Longrightarrow \quad B_1^c = \frac{4}{5},$$

and so $A_1^c = \frac{1}{5}$. Now we solve for A_2^s and B_2^s. We have the couple of equations

$$\begin{aligned} A_2^s + B_2^s &= 0 \\ 4A_2^s - \frac{B_2^s}{4} &= 1. \end{aligned}$$

From the first equation we see that $A_2^s = -B_2^s$, and plugging this in the second

$$-4B_2^s - \frac{B_2^s}{4} = 1 \quad \Longrightarrow \quad -\frac{17}{4}B_2^s = 1 \quad \Longrightarrow \quad B_2^s = -\frac{4}{17}$$

and $A_2^s = \frac{4}{17}$.

We show that all the other coefficients not yet found are zero.

$$\begin{aligned} A_1^s + B_1^s = 0 \\ A_1^s - \frac{B_1^s}{2} = 0 \end{aligned}$$

so $A_1^s = B_1^s = 0$ (the determinant of the coefficients is non-zero).

If $n > 2$,

$$\begin{aligned} A_n^2 + B_n^s &= 0 \\ A_n^s\, 2^{n-1} - \frac{B_n^s}{2^{n+1}} &= 0 \quad n \neq 2 \end{aligned}$$

since the determinant of the coefficients is non-zero $A_n^s = B_n^s = 0$, $n \geq 2$. Now, for $n \geq 2$

$$\begin{aligned} A_n^c + B_n^c &= 0 \\ A_n^c \, 2^{n-1} - \frac{B_n^c}{2^{n+1}} &= 0, \end{aligned}$$

then, again since the determinant of the coefficients is non-zero we find that $A_n^c = B_n^c = 0$, $n \geq 2$. Then we have that

$$u(r,\theta) = \frac{1}{5}\left(r + \frac{4}{r}\right)\cos\theta + \frac{4}{17}\left(r^2 - \frac{1}{r^2}\right)\sin 2\theta.$$

3 .2 Problem 2

We first show the normal component analysis and then the tangential component.

(i) **Normal component analysis :** Let us assume a hollow cylinder. Pick a point in the surface of the cylinder with coordinates (r_0, θ_0, z_0). Make a small Gaussian piece of volume around it as follows. Take all $r_0 - \Delta r \leq r \leq r_0 + \Delta r$ is the volume between two cylinders. Then in this volume constrain $\theta_0 - \Delta\theta \leq \theta \leq \theta + \Delta\theta$, where now we have a piece of azimuth (a cake piece), and finally limit $z - \Delta z_0 \leq z \leq Z + \delta z$. The piece of cylinder with radius r and angles $\theta \in [\theta_0 - \Delta\theta, \theta_0 + \Delta\theta]$, and $z \in [z_0 - \Delta z, z_0 + \Delta z]$, is in the middle of this piece of volume. We will make $\Delta r \to 0$. From the Gauss' law

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}.$$

where ρ is the volume density. Let us now compute the field through the volume V above.

$$\int_V \nabla \cdot E d\boldsymbol{r} = \int_S \boldsymbol{E} d\boldsymbol{S}. \tag{11}$$

The cylindrical box has six surfaces. We want to make radius thickness Δr very small, that is we want $\Delta r \to 0$. This collapses the lateral surfaces and we end up only with two surfaces. The surface inside the cylinder which in the limit is a piece of surface of the cylinder, and the surface outside which also in the limit converges with the same piece of surface in the cylinder with area $r\Delta\theta\Delta z$. From the first integral in the equation above, and Gauss' law, we have that

$$\int_V \nabla \cdot E d\boldsymbol{r} = \frac{\rho}{\epsilon_0}(\Delta r)(r\Delta\theta)(\Delta z).$$

From the second integral

$$\int_S \boldsymbol{E}\, d\boldsymbol{S} = (E_\perp^+ - E_\perp^-)(r\Delta\theta)(\Delta z).$$

where since the surface is small we consider $\boldsymbol{E}$ constant. Here $E_\perp^+$ and $E_\perp^-$ are the normal components outside and inside of the cylinder of the electric field. We now apply equation 11 to find

$$\frac{\rho}{\epsilon_0}(\Delta r)(r\Delta\theta)(\Delta z) = (E_\perp^+ - E_\perp^-)(r\Delta\theta)(\Delta z),$$

and so

$$E_\perp^+ - E_\perp^- = \frac{\rho}{\epsilon_0 \Delta r}$$

but

$$\sigma = \frac{\rho}{\Delta r},$$

is the surface charge density, then

$$E_\perp^+ - E_\perp^- = \frac{\sigma}{\epsilon_0},$$

which is the normal discontinuity of the electrical field across the surface.

(ii) **Tangential component analysis:** We assume that the cylinder has a radius r_0. We pick a small loop around the cylindrical surface. In this loop let us keep the azimuth θ_0 fixed and make the loop by going from $(r_0 - \Delta r, \theta_0, z_0 - \Delta z)$ to $(r_0 - \Delta r, \theta_0, z_0 + \Delta z)$, to $(r_0 + \Delta r, \theta_0, z_0 + \Delta z)$, to $(r_0 + \Delta r, \theta_0, z_0 - \Delta z)$, and finishing at the starting point.

The are four straight segments $\ell_1, \ell_2, \ell_3, \ell_4$. We want to make the radial segments

$$
\begin{aligned}
\ell_2 &= (r_0 - \Delta r, \theta_0, z_0 + \Delta z) \quad \text{to} \quad (r_0 + \Delta r, \theta_0, z_0 + \Delta z) \\
\ell_4 &= (r_0 + \Delta r, \theta_0, z_0 - \Delta z) \quad \text{to} \quad (r_0 - \Delta r, \theta_0, z_0 \Delta z).
\end{aligned}
$$

Infinitesimally small and so the integral, in the limit, is only along the paths ℓ_1 and ℓ_2, which have different orientation. One goes up and the other goes down. Then we have

$$
\lim_{|\ell_2|,|\ell_4| \to 0} \int_{\ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4} \boldsymbol{E} \cdot d\ell = \int_{\ell_1} \boldsymbol{E} \cdot d\ell_1 - \int_{\ell_3} \boldsymbol{E} \cdot d\ell_3 = 0.
$$

We say that the integral is zero because the field $\boldsymbol{E}$ is conservative and the integral of the field around a loop is zero.

From here we have that since $|\ell_1| = |\ell_3|$, and since $\boldsymbol{E}$ is constant for the small loop that we are choosing,

$$
E_{\parallel}^{+} = E_{\parallel}^{-}
$$

This shows the continuity of the tangential component of the electrical field across a charged surface.

This proof is not general since we have a particular orientation of the loop by having θ fixed. We should not assume $r = r_0$ fixed since then we are at the surface of the cylinder. We could assume z fixed and have θ sweeping an angle. The proof is similar to this. Then at any point in the cylinder we have three components of the electrical field. The normal component and two tangential components. The two tangential components shown here are continuous when going across the surface. Any tangential component should be a linear combination of the two (vertical and horizontal) components.

4 Project # 4

4 .1 Problem 1

Let us assume that we have two charges: $+q$ at $(0, 0, d/2)$ and $-q$ at $(0, 0, -d/2)$. The dipole moment of these two charges is given by

$$\boldsymbol{p} = q(0, 0, d/2) - q(0, 0, -d/2) = q(0, 0, d).$$

The potential of the dipole is defined as

$$V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}.$$

This is precisely the potential derived in class.

4 .2 Problem 2

(i) **The monopole.** The equation is given by

$$V_0(\boldsymbol{r}) = \frac{Q}{4\pi r\epsilon_0},$$

where

$$Q = \int \rho(\boldsymbol{r}_i)d\boldsymbol{r}_i.$$

Then,

$$\begin{aligned} Q &= \int (q^+)\delta(x)\delta(y)\delta(z - d/2) + (q^-)\delta(x)\delta(y)\delta(z + d/2)d\boldsymbol{r}_i \\ &= q - q = 0. \end{aligned}$$

Since the q charges can be pulled out of the integral with their respective sign and the integral on the deltas (over the whole space or at least far from $d/2$) is 1.

(ii) **The quadrupole** : The moment for the quadrupole is defined as

$$Q_{ij} = \int (3x_i x_j - \delta_{ij} r_I^2) \rho(\boldsymbol{r}_I) d\boldsymbol{r}_I$$

where $\boldsymbol{r_I} = (x_1, x_2, x_3)$.

Since in the density function the charges are located along the z axis, that is the support of the delta is along the z axis, only components related to the z axis are non zero (you can verify this). That is, we only want to evaluate Q_{33}. That is,

$$\begin{aligned} Q_{33} &= \int (3z^2 - z^2) \rho(x, y, z) dxdydz \\ &= q \int 2z^2 [\delta(x)\delta(y)\delta(z - d/2) - \delta(x)\delta(y)\delta(z + d/2)] dxdydz \\ &= 2q[(d/2)^2 - (-d/2)^2] = 0. \end{aligned}$$

4 .3 Problem 3

(i) **The quadrupole** . The density of the four charges can be written as

$$\begin{aligned} \rho(x, y, z) &= q\,\delta(x - a/2)\delta(y - a/2)\delta(z) \\ &- q\,\delta(x - a/2)\delta(y + a/2)\delta(z) \\ &- q\,\delta(x + a/2)\delta(y - a/2)\delta(z) \\ &+ q\,\delta(x + a/2)\delta(y + a/2)\delta(z) \end{aligned}$$

where the sign of the charge is incorporated in the equation. Since all charges are in the $x-y$ plane (the support of the deltas is in this plane, the integral for Q_{ij} will be zero for the all $i, j = 3$ components. Let us evaluate the other components. We note that since

$$3x^2 - r_I^2 = 3x^2 - (x^2 + y^2 + z^2) = 2x^2 - y^2 - z^2,$$

$$
\begin{aligned}
Q_{11} &= q\int [2x^2 - y^2 - z^2]\,\delta(x - a/2)\delta(y - a/2)\delta(z)dxdydz \\
&- q\int [2x^2 - y^2 - z^2]\delta(x - a/2)\delta(y + a/2)\delta(z)dxdzdz \\
&- q\int [2x^2 - y^2 - z^2]\delta(x + a/2)\delta(y - a/2)\delta(z)dxdydz \\
&+ q\int [2x^2 - y^2 - z^2]\delta(x + a/2)\delta(y + a/2)\delta(z)\; dxdydz
\end{aligned}
$$

Since the expression $2x^2 - y^2 - z^2$ is even in x, y, and z, all the integrals evaluate to the same value (for example $\delta(y \pm a/2)$ makes a y^2 into $(a/2)^2$ regardless the sign of $a/2$. Then there are two plus and two minus all equally weighted. The sum is 0. Clearly for Q_{22} it happens the same situation.

We need to evaluate the crossing terms Q_{ij}. For the crossing terms $\delta_{ij} = 0$ and so the quadrupole formula becomes

$$
Q_{ij} = \int 3x_i x_j \rho(\boldsymbol{r}_I) d\boldsymbol{r}_I.
$$

Since $Q_{ij} = Q_{ji}$ we only evaluate Q_{12}.

$$
\begin{aligned}
Q_{12} &= 3q\int xy\,\delta(x - a/2)\delta(y - a/2)\delta(z)dxdydz \\
&- 3q\int xy\delta(x - a/2)\delta(y + a/2)\delta(z)dxdydz \\
&- 3q\int xy\delta(x + a/2)\delta(y - a/2)\delta(z)dxdydz \\
&+ 3q\int xy\delta(x + a/2)\delta(y + a/2)\delta(z)dxdydz \\
&= 3q[(a/2)^2 + (a/2)^2 + (a/2)^2 + (a/2)^2]
\end{aligned}
$$

Then we find that

$$Q_{12} = Q_{21} = 3qa^2.$$

and the quadrupole moment tensor is

$$Q_{ij} = 3qa^2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can infer that if instead of the four charges being in the $x-y$ plane they would be in the $x-z$ plane the non-zero entries would be in the 13, 31 index locations. If the four charges are in the $y-z$ plane the non-zero entries are in the 23, 32 index locations.

(ii) **The monopole** . The formula for the monopole moment is

$$Q = \int \rho(\boldsymbol{r}_i) d\boldsymbol{r}_i.$$

Since the density function $\rho(\boldsymbol{r}_i)$ consists of 4 deltas, two positive and two negative and each delta integrates to 1, we have that

$$Q = q - q - q + q = 0.$$

(iii) **The dipole** . We use the equation for the dipole moment

$$\begin{aligned} \boldsymbol{p} &= \sum_{i=1}^{4} \boldsymbol{r}_i \, q_i \\ &= q(-a/2, -a/2, 0) - q(-a/2, a/2, 0) - q(a/2, a/2, 0) + q(a/2, a/2, 0) \\ &= q(0, 0, 0) = 0. \end{aligned}$$

4 .4 Problem 4

A sphere of radius R carries a polarization

$$\boldsymbol{P} = k\boldsymbol{r}_i$$

where k is a constant and $\boldsymbol{r}_i$ is the vector from the center.

(a) Calculate the bound charges σ_a and σ_b.

(b) Find the field inside and outside the sphere.

Solution

(a) Let us find $\rho_s(\boldsymbol{r}_i)$. Since the normal to the sphere is given by $\hat{\boldsymbol{r}}_{\boldsymbol{i}}$ then

$$\sigma_b(\boldsymbol{r}_i) = \boldsymbol{P}(\boldsymbol{r}_i) \cdot \boldsymbol{r}_i \delta(R - r_i) = kr_i\delta(R - r_i).$$

The surface contribution to the potential is given by

Now for the volume bound charge density

$$\rho_b(\boldsymbol{r}_i) = -\nabla \cdot \boldsymbol{P} = -\nabla \cdot (k\boldsymbol{r}) = -3k.$$

Note that the total charge on the shell is

$$Q_S = \int_{r=R} kr_i d\boldsymbol{r}_i = 4\pi k R^3,$$

is equal to the total charge in the volume

$$Q_{\text{vol}} = -\int_{r \leq R} 3k d\boldsymbol{r} = -3k\frac{4\pi R^3}{3} = -4\pi k R^3.$$

(b) To find the electric field we use Gauss's law with the charge densities above. We have that

$$\int \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{1}{\epsilon_0} \int \rho(\boldsymbol{r}) d\boldsymbol{r}.$$

Let us assume $r < R$ and pick a Gaussian sphere of radius r Then we see that

$$E(4\pi r^2) = \frac{1}{\epsilon_0}(-4\pi k r^3) \quad , \quad E = -\frac{kr}{\epsilon_0} \quad , \quad \boldsymbol{E} = -\frac{k\boldsymbol{r}}{\epsilon_0}.$$

If $r > R$ pick a Gaussian sphere with radius r, then since the total charge $Q_S + Q_{\text{vol}} = 0$, we see that $\boldsymbol{E} = 0$. In summary:

$$\boldsymbol{E}(\boldsymbol{r}) = \begin{cases} -\dfrac{k\boldsymbol{r}}{\epsilon_0} & r < R \\ \\ 0 & r > R \end{cases}$$

What happens at $r = R$?