Bessel Funtions

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Chapter 1

Introduction

The Bessel functions [1] were first defined by Daniel Bernoulli and extended by Friedrich Bessel. Bessel functions are solutions of the differential equation.

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (x^2 - \alpha^2)u = 0.$$

These functions have many applications in solutions on problems in electrostatics, electromagnetic waves, heat flow, acoustic radiation, and some other problems where the Laplacian is involved in cylindrical coordinates.

We show here how Bessel functions appear when solving the Laplace's equation in cylindrical coordinates. Then we use the Frobenius method [2] to solve the differential equation. Later we find recursive formulas, generating functions, integral representations, asymptotic representation and special values. To end we find transforms related to the Bessel functions.

The Bessel equation is a particular case of the general Sturm-Liouville [3] differential equation which can be writen in an eigenvalue/eigenfunction form as

$$Lu = -\frac{1}{w(x)}\left[\frac{d}{dx}\left(p(x)\frac{du(x)}{dx}\right) + q(x)u(x)\right] = \lambda u(x) \tag{1.1}$$

The Sturm-Liouville operator L is self-adjoint and so its eigenvalues λ are real and its eigenfunctions $u(x)$ are orthogonal. We say more about this later.

[1] https://en.wikipedia.org/wiki/Bessel_function
[2] https://en.wikipedia.org/wiki/Frobenius_method
[3] https://en.wikipedia.org/wiki/Sturm%E2%80%93Liouville_theory

Chapter 2

Solution of Laplace equation in cylindrical coordinates

We use the well knwon method of separation of variables which provide a system of solutions known as cylindrical harmonics. [1]

The Laplacian in spherical coordinates is defined by the equation

$$\nabla^2 u = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{\rho}\frac{\partial u}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(\rho\frac{\partial u}{\partial z}\right)\right] \tag{2.1}$$

We show the general method for arbitrary boundary conditions on the cylinder and then a couple of examples with simple boundary conditions.

2.0.0.1 The General Case

We want to impose some boundary conditions on the cylinder as three conditions for its side, top and bottom. Figure 2.1 illustrates the cylinder, its dimensions and its location. The boundary conditions are specified as follow:

$$\begin{aligned} u(\rho,\theta,0) &= V_0(\rho,\theta) \\ u(\rho,\theta,L) &= V_L(\rho,\theta) \\ u(a,\theta,z) &= V_a(\theta,z) \\ \lim_{\rho\to 0} u(\rho,\theta,L) &< \infty. \end{aligned}$$

[1] https://en.wikipedia.org/wiki/Cylindrical_harmonics

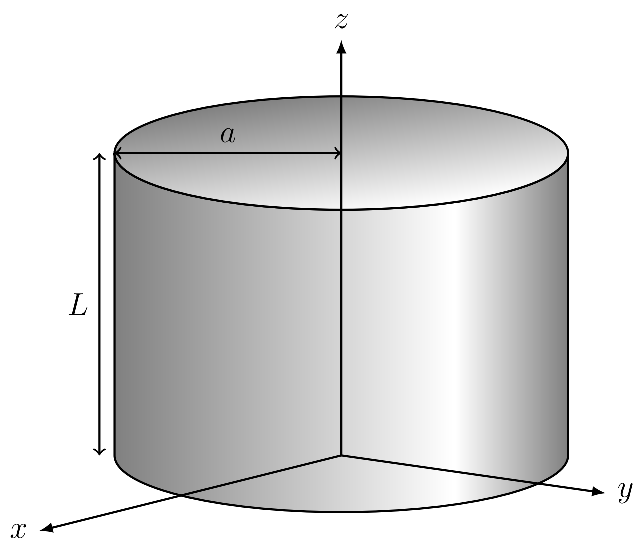

Figure 2.1: A cylinder with the potential defined on its surface.

We write equation 2.1 as

$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial^2 z} \tag{2.2}$$

We consider $u(r, \theta, z)$ as

$$u(r, \theta, z) = R(\rho)T(\theta)Z(z),$$

and plug this into equation 2.2 to find

$$\frac{TZ}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{RZ}{\rho^2}\frac{d^2T}{d\theta^2} + RT\frac{d^2Z}{dz^2} = 0.$$

We divide both sides of this equation by $R(r)T(\theta)Z(z)$ (assuming we do not have a zero solution) and move the z term to the right, to find

$$\frac{1}{R\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2 T}\frac{d^2T}{d\theta^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2}.$$

The left hand side depends on ρ, θ, while the right hand side depends only on z. The only way that this can happen is that both sides are constant. Let us call that constant $-\lambda$, and write

$$\begin{aligned} \frac{1}{R\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{1}{\rho^2 T}\frac{d^2T}{d\theta^2} &= -\lambda \\ -\frac{1}{Z}\frac{d^2Z}{dz^2} = -\lambda. \end{aligned} \tag{2.3}$$

Let us start with the last equation.

$$\frac{d^2Z}{dz^2} = \lambda Z. \tag{2.4}$$

This is an eigenvalue/eigenfunction equation for a second derivative operator. Provided some boundary conditions the operator is self-adjoint and its eigenfunctions orthogonal. This problem corresponds to one of the Sturm-Liouville operators with $p = 1$, $q = 0$, and $w = 1$. Equation 2.4 can be written as

$$\frac{d^2Z}{dz^2} - \lambda Z = 0. \tag{2.5}$$

The solutions to this equation are of the form $\mathrm{e}^{\pm\sqrt{\lambda}z}$, and a general solution can be written as

$$Z_\lambda(z) = A\mathrm{e}^{\sqrt{\lambda}z} + B\mathrm{e}^{-\sqrt{\lambda}z} \tag{2.6}$$

Since a second order equation needs two boundary conditions we can use them to solve for A and B. At the moment we have not said if $\lambda > 0$ or $\lambda < 0$. If $\lambda > 0$ the solutions are hyperbolic functions which do not oscillate, if $\lambda < 0$ then solutions are trigonometrical (sines and cosines) function which oscillate. The type of boundary conditions will determine the sign of λ.

Now, for the other part of the equation 2.3 we multiply the equation by ρ^2 and move the term on T to the right to find

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \lambda\rho^2 = -\frac{1}{T}\frac{d^2T}{d\theta^2} \tag{2.7}$$

Since the expression on the left is function only of ρ (forget about constants) and the expression on the right is only a function of θ, we find that both expressions should be equal to a constant. We call the constant μ, and find that

$$-\frac{1}{T}\frac{d^2T}{d\theta^2} = \mu$$

or

$$-\frac{d^2T}{d\theta^2} = \mu T. \tag{2.8}$$

We see that T is an eigenfunction of a self-adjoint operator. The T eigenfunctions that satisfy this should be orthogonal (depending on the boundary conditions used). As in the previous case we find that

$$T_\mu = C\mathrm{e}^{\sqrt{\mu}\mathrm{i}\theta} + D\mathrm{e}^{-\sqrt{\mu}\mathrm{i}\theta}$$

We can say, ahead of time, that $\mu > 0$, because the function should oscillate in the angle θ. Since it is a cylinder it should be periodic on θ with period at most 2π. Now, from equation 2.7 and using μ in the right hand side we see that

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \lambda\rho^2 = \mu$$

and multiplying by R,

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \lambda R\rho^2 = \mu R \tag{2.9}$$

Let us expand the differentiation to find that

$$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + (\lambda\rho^2 - \mu)R = 0 \tag{2.10}$$

We have not yet specified the sign of λ. We first assume the case of $\lambda > 0$, and then in example 2.0.0.3 we show what happens when $\lambda < 0$.

To make the association of this function with a Bessel function it is convenient to define, first the mapping

$$\begin{aligned} \lambda &\mapsto \ell^2 \\ \mu &\mapsto \alpha^2 \end{aligned} \tag{2.11}$$

then equation 2.10 turns into

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho \frac{dR}{d\rho} + (\ell^2\rho^2 - \alpha^2)R = 0 \tag{2.12}$$

This equation is known as the **parametric Bessel** equation and we show how this equation is part of the general Sturm-Liouville operator. Let us divide by ρ to find

$$\rho \frac{d^2R}{d\rho^2} + \frac{dR}{d\rho} + \left(\ell^2\rho - \frac{\alpha^2}{\rho}\right) R = 0. \tag{2.13}$$

Now, since $(xy')' = xy'' + y'$ we can collapse the first two terms to find

$$\frac{d}{d\rho}\left[\rho\frac{dR}{d\rho}\right] - \frac{\alpha^2}{\rho}R = -\ell^2\rho$$

This is the Sturm-Liouville differential equation 1.1 with

$$p(\rho) = \rho \quad , \quad q(\rho) = -\frac{\alpha^2}{\rho} \quad , \quad w(\rho) = \rho \quad , \quad \lambda = \ell^2.$$

If we provide the right boundary conditions this operator is self-adjoint under the inner product defined by the equation

$$\langle f(\rho), g(\rho)\rangle = \int_0^a f(\rho)g(\rho)\rho d\rho,$$

where in the cylinder that we consider, we know that $\rho \in [0, a]$.

If in equation 2.13 we say that $x = \rho\ell$ we find

$$x^2 \frac{d^2R}{dx^2} + x \frac{dR}{dx} + (x^2 - \alpha^2)R = 0 \tag{2.14}$$

which is Bessel [2] differential equation. The solution of the Bessel differential equation can be found using the Frobenius method [3], and we do this in the followig chapter. We should find two solutions $J_\alpha(x)$ and $Y_\alpha(x)$. They are known as Bessel function of first kind and second kind respectively. With this, a solution for R is given as

$$R_{\lambda,\mu} = E\ J_{\sqrt{\mu}}(\sqrt{\lambda}\rho) + F\ Y_{\sqrt{\mu}}(\sqrt{\lambda}\rho).$$

We are ready to do superposition and say that the general solution is a superposition of the products of the solutions found here. That is,

$$\begin{aligned} u(\rho,\theta,z) &= \int_S d\lambda\, d\mu f(\lambda,\mu) \left(A\mathrm{e}^{\sqrt{\lambda}\mathrm{i}z} + B\mathrm{e}^{-\sqrt{\lambda}\mathrm{i}z}\right)\left(C\mathrm{e}^{\sqrt{\mu}\theta} + D\mathrm{e}^{-\sqrt{\mu}\theta}\right) \\ & \left(E\ J_{\sqrt{\mu}}(\sqrt{\lambda}\rho) + F\ Y_{\sqrt{\mu}}(\sqrt{\lambda}\rho)\right). \end{aligned}$$

Observe that we do not yet know the sign of λ. In equation 2.11 we implicitly assumed that $\lambda > 0$. However λ could be negative and in this case we would obtain a different differential equation. Actually the second example 2.0.0.3 below illustrates this case.

As in the rectangular coordinates we can have the λ and μ varying over the continuum or the discrete. If the volume (domain) of the function where the boundary conditions are applied is finite then the set of λ and μ are a discrete (countable). If the dimensions of the cylindrical surface go to ∞, then the λ or μ, or both vary over the continuum. In the case of the continuum transforms need to be used to invert (for $f(\lambda,\mu)$) the integral equation. The complex exponentials are inverted with Fourier transform and the Bessel functions with Hankel transforms.

2.0.0.2 First Example

We assume the simple case of a cylinder where there is only non-zero potential specified in the upper face $z = L$. Figure 2.2 sketches this.

The boundary conditions for this case are:

$$\begin{aligned} V(\rho,\theta,0) &= 0 \\ V(\rho,\theta,L) &= V(\rho,\theta) \\ V(a,\theta,z) &= 0 \\ \lim_{\rho\to 0} u(\rho,\theta,L) &< \infty. \end{aligned}$$

[2] https://en.wikipedia.org/wiki/Bessel_function

[3] https://en.wikipedia.org/wiki/Frobenius_method

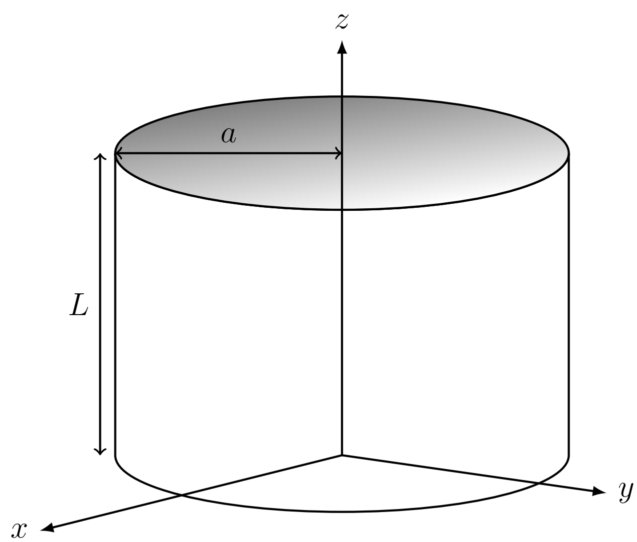

Figure 2.2: A cylinder with potential non-zero only in the upper cap $z = L$.

Let us start with equation 2.4

$$\frac{d^2Z}{dz^2} = \lambda Z.$$

along the z coordinates, and its general solution.

$$Z_\lambda(z) = A\mathrm{e}^{\sqrt{\lambda}z} + B\mathrm{e}^{-\sqrt{\lambda}z} \tag{2.15}$$

From $Z_\lambda(0) = 0$ we find that $A = -B$, and so

$$Z_\lambda(z) = A\sinh\sqrt{\lambda}z, \tag{2.16}$$

where we are assuming that $\lambda > 0$. We do not expect oscillations along the z direction.

Now, along the θ axis we choose equation 2.8 which should oscillate since the function is periodic here with period 2π.

$$\frac{d^2T}{d\theta^2} + \mu T = 0$$

The general solution is of the form

$$T_\mu(\theta) = C \sin \sqrt{\mu}\theta + D \cos \sqrt{\mu}\theta.$$

Since the function is periodic with period 2π, we have that

$$\sqrt{u} = n, \quad , \quad n \geq 0.$$

and we can write

$$T_n(\theta) = C \sin n\theta + D \cos n\theta.$$

Finally for the equation along ρ. We use equation 2.9

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho \frac{dR}{d\rho} + (\lambda\rho^2 - \mu)R = 0$$

and the solution

$$R_{\lambda,\mu} = E\ J_{\sqrt{\mu}}(\sqrt{\lambda}\rho) + F\ Y_{\sqrt{\mu}}(\sqrt{\lambda}\rho).$$

Since we assume that at $\rho = 0$ we have a finite value of the potential then $F = 0$ (the function $Y_{\sqrt{\nu}}$ diverges in 0), and so

$$R_{\lambda,\mu} = E\ J_{\sqrt{\mu}}(\sqrt{\lambda}\rho) = E\ J_n(\sqrt{\lambda}\rho).$$

From the boundary condition that the potential is 0 at $\rho = a$, we have that

$$R_{\lambda,\mu} = E\ J_n(\sqrt{\lambda}a) = 0$$

and so

$$\sqrt{\lambda}a = x_{mn}$$

where x_{mn} is the m root of $J_n(x)$, $m \geq 1$. That is, we can say

$$\lambda = \frac{x_{mn}^2}{a^2},$$

and revisit equation 2.16 to write

$$Z_{mn}(z) = A \sinh\left(\frac{x_{mn}z}{a}\right) \quad , \quad R_{mn} = E\ J_n\left(\frac{x_{mn}\rho}{a}\right)$$

We now use the superposition principle to write

$$u(\rho,\theta,z) = \sum_{n=0,m=1}^{\infty} \sinh\left(\frac{x_{mn}z}{a}\right) J_n\left(\frac{x_{mn}\rho}{a}\right)(A_{mn}\sin n\theta + B_{mn}\cos n\theta).$$

Let us normalize the orthogonal functions $\sin n\theta$ and $\cos n\theta$.

$$\langle \sin n\theta, \sin n\theta \rangle = \int_0^{2\pi} \sin^2 n\theta\ d\theta = \frac{1}{2}\int_0^{2\pi}(1-\cos 2\theta)d\theta = \pi.$$

Then $\|\sin n\theta\| = \sqrt{\pi}$, and the same norm has $\cos n\theta$. We then write instead of the equation above.

$$u(\rho,\theta,z) = \sum_{n=0,m=1}^{\infty} \sinh\left(\frac{x_{mn}z}{a}\right) J_n\left(\frac{x_{mn}\rho}{a}\right)\left(A_{mn}\frac{\sin n\theta}{\sqrt{\pi}} + B_{mn}\frac{\cos n\theta}{\sqrt{\pi}}\right). \tag{2.17}$$

Let us now apply the boundary condition on top. That is at $z = L$, we have that

$$u(\rho,\theta,L) = V(\theta,\rho) = \sum_{n=0,m=1}^{\infty} \sinh\left(\frac{x_{mn}L}{a}\right) J_n\left(\frac{x_{mn}\rho}{a}\right)\left(A_{mn}\frac{\sin n\theta}{\sqrt{\pi}} + B_{mn}\frac{\cos n\theta}{\sqrt{\pi}}\right). \tag{2.18}$$

To find A_{mn} and B_{mn} we take inner product with $\sin i\theta$ and $\cos i\theta$ respectively. That is,

$$\begin{aligned}\left\langle V(\theta,\rho), \frac{\sin i\theta}{\sqrt{\pi}}\right\rangle &= \sum_{n=0,m=1}^{\infty} \sinh\left(\frac{x_{mn}L}{a}\right) J_i\left(\frac{x_{mn}\rho}{a}\right) A_{mn}\delta_{ni} \\ &= \sum_{m=1}^{\infty} \sinh\left(\frac{x_{mi}L}{a}\right) J_i\left(\frac{x_{mi}\rho}{a}\right) A_{mi}\end{aligned}$$

or

$$\int_0^{2\pi} V(\theta,\rho)\frac{\sin i\theta}{\sqrt{\pi}}d\theta = \sum_{m=1}^{\infty} \sinh\left(\frac{x_{mn}L}{a}\right) J_i\left(\frac{x_{mi}\rho}{a}\right) A_{mi} \tag{2.19}$$

Here we use the orthogonality property of the Bessel functions. It can be shown that

$$\left\langle J_i\left(\frac{x_{mj}\rho}{a}\right), J_i\left(\frac{x_{mk}\rho}{a}\right)\right\rangle = \int_0^a \rho J_i\left(\frac{x_{mj}\rho}{a}\right) J_i\left(\frac{x_{mk}\rho}{a}\right) d\rho = \frac{a^2}{2}\left[J_{i+1}\left(\frac{x_{mk}\rho}{a}\right)\right]^2 \delta_{jk},$$

We then take the inner product of 2.19 with the orthogonal Bessel function $J_i(x_{ki}\rho/a)$. That is,

$$\begin{aligned}\int_0^a \rho \int_0^{2\pi} V(\theta,\rho)\frac{\sin i\theta}{\sqrt{\pi}} J_i\left(\frac{x_{ki}\rho}{a}\right) d\theta d\rho &= \sum_{m=1}^{\infty}\frac{a^2}{2}\left[J_{i+1}\left(\frac{x_{ki}\rho}{a}\right)\right]^2 \sinh\left(\frac{x_{mn}L}{a}\right)\delta_{km}A_{mi} \\ &= \frac{a^2}{2}\left[J_{i+1}\left(\frac{x_{ki}\rho}{a}\right)\right]^2 \sinh\left(\frac{x_{kn}L}{a}\right) A_{ki}.\end{aligned}$$

Then

$$A_{ki} = \frac{2}{\sqrt{\pi}a^2\left[J_{i+1}\left(\frac{x_{ki}\rho}{a}\right)\right]^2 \sinh\left(\frac{x_{kn}L}{a}\right)} \int_0^a \rho \int_0^{2\pi} V(\theta,\rho)\sin i\theta J_i\left(\frac{x_{ki}\rho}{a}\right) d\theta d\rho.$$

Now for the B coefficients we take the inner product in equation 2.18 with $\cos(i\theta)/\sqrt{\pi}$ to find

$$\begin{aligned}\int_0^{2\pi} V(\theta,\rho)\frac{\cos i\theta}{\sqrt{\pi}}d\theta &= \sum_{n=0,m=1}^{\infty} \sinh\left(\frac{x_{mn}L}{a}\right) J_i\left(\frac{x_{mn}\rho}{a}\right) B_{mn}\delta_{in} \\ &= \sum_{m=1}^{\infty} \sinh\left(\frac{x_{mi}L}{a}\right) J_i\left(\frac{x_{mi}\rho}{a}\right) B_{mi}.\end{aligned}$$

Now we take the inner product with $J_i(x_{ki}\rho)/a$ to find

$$\int_0^a \rho \int_0^{2\pi} V(\theta,\rho) \frac{\cos i\theta}{\sqrt{\pi}} J_i\left(\frac{x_{ki}\rho}{a}\right) d\theta d\rho = \sum_{m=1}^{\infty} \frac{a^2}{2}\left[J_{i+1}\left(\frac{x_{ki}\rho}{a}\right)\right]^2 \sinh\left(\frac{x_{mn}L}{a}\right)\delta_{km}B_{mi}$$
$$= \frac{a^2}{2}\left[J_{i+1}\left(\frac{x_{ki}\rho}{a}\right)\right]^2 \sinh\left(\frac{x_{ki}L}{a}\right) B_{ki}$$

and so

$$B_{ki} = \frac{2}{\sqrt{\pi}a^2\left[J_{i+1}\left(\frac{x_{ki}\rho}{a}\right)\right]^2 \sinh\left(\frac{x_{ki}L}{a}\right)} \int_0^a \rho \int_0^{2\pi} V(\theta,\rho)\cos i\theta J_i\left(\frac{x_{ki}\rho}{a}\right) d\theta d\rho.$$

We re-index A_{ki} and B_{ki} in terms of A_{nm} and B_{mn} and equation 2.17 is solved for the potential $u(\rho,\theta,z)$.

2.0.0.3 Second Example

We now show another example where the modified Bessel functions are involved. The hollow cylinder in Figure 2.3 with radius a and height L, has potential $V(\theta,z)$ at its surface $\rho = a$, zero potential on top and bottom. The cylinder has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$.

The boundary conditions in formal language are:

$$\begin{aligned} V(\rho,\theta,0) &= 0 \\ V(\rho,\theta,L) &= 0 \\ V(a,\theta,z) &= V(\theta,z) \\ \lim_{\rho\to 0} u(\rho,\theta,L) &< \infty. \end{aligned}$$

We start with the ordinary differential equation 2.5 along Z,

$$\frac{d^2Z}{dz^2} - \lambda Z = 0 \qquad Z(0) = 0 \quad , \quad Z(L) = 0.$$

The general solution of this equation is given by equation 2.15. This is

$$Z_\lambda(z) = A\mathrm{e}^{\sqrt{\lambda}z} + B\mathrm{e}^{-\sqrt{\lambda}z}.$$

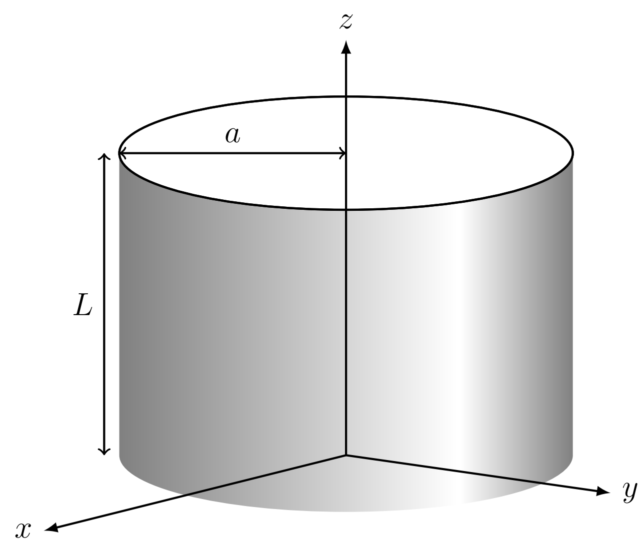

Figure 2.3: A cylinder with charges only along its side $\rho = a$.

We use the boundary conditions to find A and B. That is, from $Z_\lambda(0) = 0$, we see that $A = -B$, and then

$$Z_\lambda(z) = A(e^{\sqrt{\lambda}z} - e^{-\sqrt{\lambda}z}).$$

Since the operator L is a double derivative and we have 0 boundary conditions the functions are orthogonal under the definition of inner product and the eigenvalue λ is real (actually we always assume that the eigenvalues are real for these problems).

$$\langle f, g\rangle = \int_0^L f(x)g(x)dx.$$

We have not indicated if λ is positive or negative. If $\lambda > 0$, then the two exponentials above are real (no complex). We use the second condition $Z_\lambda(L) = 0$. That is

$$Z_\lambda(z) = A(e^{-\sqrt{\lambda}L} - e^{\sqrt{\lambda}L}) = 0.$$

Making $A = 0$ is not an option because it produces a zero solution that we want avoid. The only way that these two exponential functions cancel is by having $\lambda = 0$. However we do not want 0 eigenvalues. Then necessarily $\lambda < 0$. With this

$$Z_\lambda(z) = 2A \sin \sqrt{|\lambda|} z$$

and from $z_\lambda(L) = 0$ we find

$$Z_\lambda(L) = 2A \sin \sqrt{|\lambda|} L = 0.$$

That is,

$$\sqrt{|\lambda|} L = \pi n,$$

where $n = 1, 2, \cdots$. From here the eigenvalues have the form

$$\lambda = -\frac{\pi^2 n^2}{L^2}, \tag{2.20}$$

and

$$Z_n(z) = 2A \sin \frac{\pi n z}{L} \quad , \quad n = 1, 2, \cdots$$

where the case of $n = 0$ will not provide any contribution. At this point we can drop the value of $2A$ and instead find a coefficient that normalizes the function. That is, we find first

$$\left\langle \sin \frac{\pi n z}{L}, \sin \frac{\pi n z}{L} \right\rangle = \int_0^L \sin^2 \frac{\pi n z}{L} dz = \frac{1}{2} \int_0^L \left(1 - \cos \frac{2\pi n z}{L} \right) dz = \frac{L}{2}.$$

Then we write the normalized eigenfunctions

$$Z_n(z) = \sqrt{\frac{2}{L}} \sin \frac{\pi n z}{L}.$$

Let us now work in the θ coordinate. We write equation 2.8

$$\frac{d^2 T}{d\theta^2} + \mu T = 0.$$

Two linearly independent solutions are $\mathrm{e}^{\pm \mathrm{i}\sqrt{\mu}}$, and their combination provides

$$T_\mu(\theta) = C\mathrm{e}^{\mathrm{i}\sqrt{\mu}\theta} + D\mathrm{e}^{-\mathrm{i}\sqrt{\mu}\theta}$$

Since T_μ is periodic (in the cylinder $T_\mu(0) = T_\mu(2\pi)$ the exponentials are oscillating. That is $\mu > 0$ (otherwise, if $\mu < 0$ they would be hyperbolic functions which do not oscillate). Each exponential has the property that

$$\mathrm{e}^{\mathrm{i}\sqrt{\mu}\theta_0} = \mathrm{e}^{\mathrm{i}\sqrt{\mu}(\theta_0+2\pi)}$$

and so

$$\mathrm{e}^{2\pi\mathrm{i}\sqrt{\mu}} = 1$$

so

$$2\pi\sqrt{u} = 2m\pi,$$

and $m = \sqrt{u}$ so we have a set of eigenvalues $\mu = m^2$, and a set of solutions

$$T_n(\theta) = C\mathrm{e}^{\mathrm{i}m\theta} + D\mathrm{e}^{-\mathrm{i}m\theta} \tag{2.21}$$

Since we are assuming complex solutions we can as well assume complex coefficients. Equation 2.21 can be expanded as

$$\begin{aligned} T_n(\theta) &= C_r \cos m\theta - C_i \sin m\theta + D_r \cos m\theta - D_i \sin m\theta + \\ &\quad \mathrm{i}[C_r \sin m\theta + C_i \cos m\theta - D_r \sin m\theta + D_i \cos m\theta] \end{aligned}$$

Since we are looking for real solutions we only consider the first part of the equation above. That is,

$$T_n(\theta) = (C_r + D_r)\cos m\theta - (C_i - D_i)\sin m\theta,$$

and recycling symbols we can just say that

$$T_m(\theta) = C\sin m\theta + D\cos m\theta \quad , \quad m = 0, 1, \cdots .$$

Note that we could have said from the beginning that two linearly independent solutions of the equation $T'' + \mu T = 0$ are $\sin\sqrt{\mu}\theta$, $\cos\sqrt{\mu}\theta$, which are real functions, but we wanted to do all the work in terms of exponentials to show another approach.

Let us normalize these solutions. Define an inner product

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)dx.$$

Then compute

$$\langle \sin m\theta, \sin m\theta\rangle = \int_0^{2\pi} \sin^2 m\theta d\theta = \frac{1}{2}\int_0^{2\pi} (1 - \cos 2\theta)d\theta = \int_0^{2\pi} d\theta = \pi$$

then $\|\sin m\theta\| = \sqrt{\pi}$. The same occurs with $\cos m\theta$. Then we write

$$T_n(\theta) = C\frac{\sin m\theta}{\sqrt{\pi}} + D\frac{\cos m\theta}{\sqrt{\pi}}.$$

Finally we look at the radial component. In equation 2.10 we replace λ by $-\pi^2 n^2/L^2$ (from equation 2.20), and $\mu = m^2$ to obtain

$$\rho^2\,\frac{d^2R}{d\rho^2} + \rho\,\frac{dR}{d\rho} - \left(\frac{\pi^2 n^2 \rho^2}{L^2} + m^2\right) R = 0 \tag{2.22}$$

This is the differential equation for the modified Bessel function. Let us reduce more by defining

$$x = \frac{\pi n \rho}{L}$$
$$dx = \frac{\pi n d\rho}{L}$$

From the chain rule

$$\rho\frac{dR}{d\rho} = \rho\frac{dR}{dx}\frac{dx}{d\rho} = \rho\frac{dR}{dx}\frac{\pi n}{L} = \frac{dR}{dx}\frac{\pi n\rho}{L} = x\frac{dR}{dx}$$

We can use symbols above and say that

$$\frac{\rho}{d\rho} = \frac{x}{dx}$$

and so from here

$$\frac{\rho^2}{d\rho^2} = \frac{x^2}{dx^2}.$$

This is not correct mathematical syntax but good mnemonic. We find then that equation 2.22 reduces to

$$x^2\,\frac{d^2R}{dx^2} + x\,\frac{dR}{dx} - (x^2 + m^2)R = 0$$

which is the differential equation for the modified Bessel function [4]. As shown in the Wikipedia page the two solutions of this differential equation are the modified Bessel functions $I_m(x)$ and $K_m(x)$ and in our case, our general solution

$$R_{m,n} = EI_m\left(\frac{\pi n\rho}{L}\right) + FK_m\left(\frac{\pi n\rho}{L}\right).$$

From the condition as $\rho \to 0$ we see that the solution should be finite at the origin. Since the modified Bessel function K_m diverges at $x = 0$, we require $F = 0$, and so

$$R_{m,n} = EI_m\left(\frac{\pi n\rho}{L}\right)$$

A solution for n and m of the Laplacian equation is given by

$$u_{nm}(\rho,\theta,z) = \left[EI_m\left(\frac{\pi n\rho}{L}\right)\right]\left[C\frac{\sin m\theta}{\sqrt{\pi}} + D\frac{\cos m\theta}{\sqrt{\pi}}\right]\left[\frac{2A}{L}\sin\frac{\pi nz}{L}\right].$$

[4] https://en.wikipedia.org/wiki/Bessel_function

We are ready to do the superposition. The constants A and E can be absorbed into the coefficients C and D given them another dimension. That is, instead of C_n we now will write C_{nm}, and the same for D. We can write the solution as the superposition of the solutions above. That is, we can write:

$$u(\rho,\theta,z) = \sum_{m=0,n=1}^{\infty} \left[I_m\left(\frac{\pi n\rho}{L}\right)\right]\left[C_{mn}\frac{\sin m\theta}{\sqrt{\pi}} + D_{mn}\frac{\cos m\theta}{\sqrt{\pi}}\right]\left[\sqrt{\frac{2}{L}}\sin\frac{\pi nz}{L}\right]. \quad (2.23)$$

where C_{mn} and D_{mn} are coefficients that we need to find. For this we need to use the multi-dimensional boundary conditions. At $\rho = a$ we find

$$u(a,\theta,z) = \sum_{m=0,n=1}^{\infty} \left[I_m\left(\frac{\pi na}{L}\right)\right]\left[C_{mn}\frac{\sin m\theta}{\sqrt{\pi}} + D_{mn}\frac{\cos m\theta}{\sqrt{\pi}}\right]\left[\sqrt{\frac{2}{L}}\sin\frac{\pi nz}{L}\right] = V(\theta,z).$$

Since the functions $\sqrt{2/L}\sin(\pi nz/L)$ are orthonormal we take the inner product with these functions and find

$$\begin{aligned}\left\langle V(\theta,z),\sqrt{\frac{2}{L}}\sin\frac{\pi iz}{L}\right\rangle &= \sum_{m=0,n=1}^{\infty} \left[I_m\left(\frac{\pi na}{L}\right)\right]\left[C_{mn}\frac{\sin m\theta}{\sqrt{\pi}} + D_{mn}\frac{\cos m\theta}{\sqrt{\pi}}\right]\delta_{ni} \\ &= \sum_{m=0}^{\infty}\left[I_m\left(\frac{\pi ia}{L}\right)\right]\left[C_{mi}\frac{\sin m\theta}{\sqrt{\pi}} + D_{mi}\frac{\cos m\theta}{\sqrt{\pi}}\right].\end{aligned}$$

That is

$$\sqrt{\frac{2}{L}}\int_0^L V(\theta,z)\,\sin\frac{\pi iz}{L}dz = \sum_{m=0}^{\infty}\left[I_m\left(\frac{\pi ia}{L}\right)\right]\left[C_{mi}\frac{\sin m\theta}{\sqrt{\pi}} + D_{mi}\frac{\cos m\theta}{\sqrt{\pi}}\right]. \quad (2.24)$$

We now take inner product with $\sin(j\theta)/\sqrt{\pi}$ and from the orthogonality conditions find

$$\sqrt{\frac{2}{\pi L}}\int_0^{2\pi} d\theta \int_0^L V(\theta,z)\,\sin\left(\frac{\pi iz}{L}\right)\sin j\theta\,dz = \sum_{m=0}^{\infty} I_m\left(\frac{\pi ia}{L}\right)C_{mi}\delta_{mj} = I_j\left(\frac{\pi ia}{L}\right)C_{ij},$$

and from here

$$C_{ij}=\sqrt{\frac{2}{\pi\,L}}\frac{1}{I_j(\pi ia/L)}\int_0^{2\pi}d\theta\int_0^L V(\theta,z)\,\sin\left(\frac{\pi iz}{L}\right)\sin j\theta dz$$

Finally in equation 2.24 we take the orthogonal product with $\cos(j\theta)/\sqrt{\pi}$ to find

$$\begin{aligned}\sqrt{\frac{2}{\pi\,L}}\int_0^{2\pi}d\theta\int_0^L V(\theta,z)\,\sin\left(\frac{\pi iz}{L}\right)\cos j\theta dz &= \sum_{m=0}^{\infty}\left[I_m\left(\frac{\pi ia}{L}\right)\right]D_{mi}\delta_{mj}\\ &= I_j\left(\frac{\pi ia}{L}\right)D_{ji}\end{aligned}$$

and from here

$$D_{ji}=\sqrt{\frac{2}{\pi\,L}}\frac{1}{I_j(\pi ia/L)}\int_0^{2\pi}d\theta\int_0^L V(\theta,z)\,\sin\left(\frac{\pi iz}{L}\right)\cos j\theta dz$$

The final solution is given by 2.23 where C_{mn} and D_{mn} above are inserted. Let us, for example assume that $u(a,\theta,z)=V(\theta,z)=V_0$, so

$$\begin{aligned}C_{ij} &= \sqrt{\frac{2}{\pi\,L}}\frac{1}{I_j(\pi ia/L)}\int_0^{2\pi}d\theta\int_0^L V_0\sin\left(\frac{\pi iz}{L}\right)\sin j\;\theta dz\\ &= \sqrt{\frac{2}{\pi L}}\frac{V_0}{I_j(\pi ia/L)}\int_0^{2\pi}\sin j\theta\;d\theta\int_0^L\sin\left(\frac{\pi iz}{L}\right)dz\\ &= \sqrt{\frac{2}{\pi L}}\frac{V_0}{I_j(\pi ia/L)}\left.\frac{\cos j\theta}{j}\right|_0^{2\pi}\left.\frac{L}{\pi i}\cos\left(\frac{\pi iz}{L}\right)\right|_0^L\\ &= \frac{\sqrt{2L}V_0}{ij\pi^{3/2}}(1-1)[(-1)^i-1]\\ &= 0.\end{aligned}$$

Note that if $j=0$ from the first line of the chain of equations above, then $C_{ij}=0$, so we are dividing by j with a good reason. so $C_{ij}=0$. Let us find now D_{ji},

$$
\begin{aligned}
D_{ji} &= \sqrt{\frac{2}{\pi\,L}}\frac{1}{I_j(\pi i a/L)}\int_0^{2\pi} d\theta \int_0^L V_0 \sin\left(\frac{\pi i z}{L}\right)\cos j\theta\; dz \\
&= \sqrt{\frac{2}{\pi L}}\frac{V_0}{I_j(\pi i a/L)}\int_0^{2\pi}\cos j\theta d\theta \int_0^L \sin\left(\frac{\pi i z}{L}\right) dz
\end{aligned}
$$

We see that the only non zero value is produced by $j = 0$. That is

$$
D_{0i} = -\sqrt{\frac{2}{\pi L}}\frac{2\pi V_0}{I_0(\pi i a/L)}\frac{L}{\pi i}\cos\left(\frac{\pi i z}{L}\right)\Bigg|_0^L = -\frac{2\sqrt{2L}V_0}{i\sqrt{\pi}I_0(\pi i a/L)}[(-1)^i - 1]
$$

So only for i odd $D_{0i} \neq 0$, and we have from the final solution 2.23

$$
u(\rho,\theta,z) \;=\; \sum_{n=1}^{\infty} I_m(0)\left[D_{0n}\frac{1}{\sqrt{\pi}}\right]\sqrt{\frac{2}{L}}\,\sin\left(\frac{\pi n z}{L}\right) = \frac{8V_0}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{1}{m}\frac{I_0(\pi m\rho/L)\sin(\pi n z/L)}{I_0(\pi i\rho/L)}
$$

Chapter 3

Solution of the Bessel differential equation

The Bessel functions [1] are solutions of the Bessel differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{3.1}$$

with $\nu \in \mathbb{C}$. This equation is known as the Bessel's differential equation. According to the Wikipedia page with the link above Daniel Bernoulli was the first person who defined them and Friedrich Bessel generalized them.

The Bessel differential equation results when doing separation of variables in the finding of the solution of differential operators where the Laplacian operator is present, in problems where cylindrical or spherical symmetry are encountered.

There are many representations found in the literature for Bessel functions. For example as integrals, asymptotic, recursive, etc. There are also many properties of the Bessel functions. Here we will only find series representations since they are the most relevant to evaluate Green functions for Bessel type equations.

We observe that the equation is singular at $x = 0$. The most common method to solve this differential equation is the Frobenius method [2] where we assume that the solution is analytic and then it can be expanded into an infinite series. This is not the only way to solve this equation. J. B. McLeod shows some disadvantages of the Frobenius method and proposes a different way to solve the Bessel differential equation. Here we will follow the tradition and use the Frobenius method.

[1] https://en.wikipedia.org/wiki/Bessel_function

[2] https://en.wikipedia.org/wiki/Frobenius_method

Since the ODE is of second order it should have two linearly independent solutions. The solutions are known as the Bessel function of first kind $J_\nu(x)$ and the Bessel function of the second kind $Y_\nu(x)$. The general solution is then given by

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x).$$

The Frobenius method assumes that the singular points (where the coefficients are undefined or infinite) are regular. That is regular singular [3] points. This means that after normalizing the coefficient of y'', the coefficients of the y' and y terms have at most poles of order 1 and 0 respectively.

The proof of the Frobenius method is beyond the scope of these notes. We observe that the Bessel equation has analytic coefficients in the whole $\mathbb{C}$ plane except at $x = 0$, which is a regular singular point.

The Frobenius method assumes that a solution can be written as an finite series in the form

$$y(x) = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{r+k}. \tag{3.2}$$

where r is a real number. The idea here is to plug this expression into the differential equation and find some recursion formula to compute the coefficients a_k, which will render the solution. We have then that

[3] https://en.wikipedia.org/wiki/Regular_singular_point

$$
\begin{aligned}
0 &= x^2 \sum_{k=0}^{\infty} (r+k)(r+k-1)a_k x^{r+k-2} + x \sum_{k=0}^{\infty} a_k (r+k) x^{r+k-1} \\
&+ (x^2 - \nu^2) \sum_{k=0}^{\infty} a_k x^{r+k} \\
&= \sum_{k=0}^{\infty} (r+k)(r+k-1)a_k x^{r+k} + \sum_{k=0}^{\infty} a_k (r+k) x^{r+k} \\
&+ \sum_{k=0}^{\infty} a_k x^{r+k+2} - \nu^2 \sum_{k=0}^{\infty} a_k x^{r+k} \\
&= \sum_{k=0}^{\infty} (r+k)(r+k-1)a_k x^{r+k} + \sum_{k=0}^{\infty} a_k (r+k) x^{r+k} \\
&+ \sum_{k=2}^{\infty} a_{k-2} x^{r+k} - \nu^2 \sum_{k=0}^{\infty} a_k x^{r+k} \\
&= a_0 x^r [r(r-1) + r - \nu^2] + a_1 x^{r+1} [r(r+1) + (r+1) - \nu^2] \\
&+ \sum_{k=2}^{\infty} [a_k[(r+k)^2 - \nu^2] + a_{k-2}] x^{r+k} \\
&= a_0 x^r (r^2 - \nu^2) + a_1 x^{r+1} [(r+1)^2 - \nu^2] + \sum_{k=2}^{\infty} [a_k[(r+k)^2 - \nu^2] + a_{k-2}] x^{r+k}.
\end{aligned}
$$

We assume that $a_0 \neq 0$. We consider two cases: $\nu \neq 1/2$ and $\nu = 1/2$.

3.1 $\nu \neq 1/2$:

We soon will see why $\nu = 1/2$ is a special value.

- **From the coefficient of x^r:** $r = \pm\nu$. One solution chooses $r = \nu$ and the other $r = -\nu$.

- **From the coefficient of x^{r+1}:** We find that $a_1((r+1)^2 - \nu^2) = 0$, and since $r = \pm\nu$ we see that $a_1(\nu^2 \pm 2\nu + 1 - \nu^2) = 0$. Or $a_1(\pm 2\nu + 1) = 0$. So either $a_1 = 0$, or $\nu = 1/2$. But here we assume $\nu \neq 1/2$ and consider later the case $\nu = 1/2$.

- **From the coefficient of x^{r+k}:**

$$a_k = -\frac{a_{k-2}}{(r+k)^2 - \nu^2} \quad , \quad k \geq 2.$$

We observe that, since $a_1 = 0$, all odd terms $a_{2n+1} = 0$.

Let us now consider the two cases of $r = \pm\lambda$.

(i) $r = \lambda$:

$$a_{2k} = -\frac{a_{2k-2}}{(\nu+2k)^2 - \nu^2} = -\frac{a_{2k-2}}{4k\nu + 4k^2} = -\frac{a_{2k-2}}{4k(\nu+k)}.$$

Starting from the bottom we see that

$$\begin{aligned}
a_2 &= -\frac{a_0}{4\nu+4} = -\frac{a_0}{4(\nu+1)} \\
a_4 &= -\frac{a_2}{4\cdot 2(\nu+1)} = \frac{(-1)^2\, a_0}{4^2\cdot 2(\nu+2)(\nu+1)} \\
a_6 &= -\frac{a_4}{4\cdot 3(\nu+3)} = \frac{(-1)^3 a_0}{4^3\; 3\cdot 2(\nu+3)(\nu+2)(\nu+1)} \\
&\vdots \\
a_{2k} &= \frac{(-1)^k a_0\; \Gamma(\nu+1)}{2^{2k}\; k!\; \Gamma(\nu+k+1)}.
\end{aligned}$$

We are free to give a_0 any value we want since it is a constant. Then we assign $a_0 = 1/\Gamma(\nu+1)$, and write the series 3.2 as

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}. \tag{3.3}$$

This important function is known as the *Bessel function of first kind*. [4]

[4] http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html

(ii) $r = -\lambda$: The same development above would lead to the formula

$$J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu}, \tag{3.4}$$

which is the same as 3.3 after replacing ν by $-\nu$.

We will find soon that for $\nu = n \in \mathbb{Z}$ the functions J_n and J_{-n} are not linearly independent, but before, let us attack the case of $\mu = 1/2$. s Figure 3.1 shows the first 3 Bessel functions of first kind for $\nu = 0, 1, 2$.

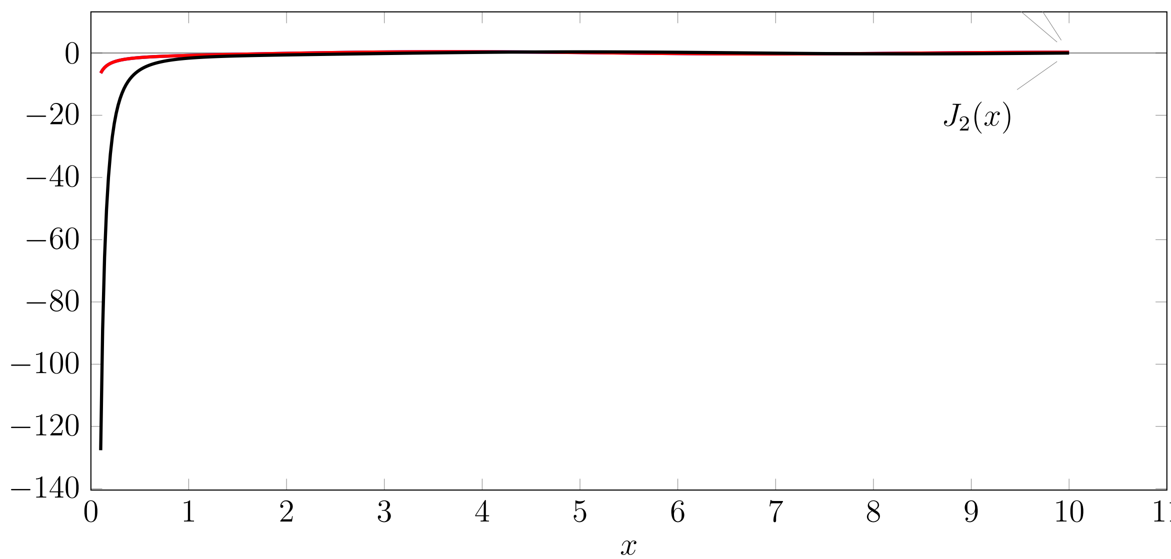

Figure 3.1: Bessel functions J_i, for $i = 0, 1, 2$.

3.2 $\nu = 1/2$:

If $\nu = 1/2$ then we have the ODE

$$Ly = x^2 y'' + x\,y' + \left(x^2 - \frac{1}{4}\right) y = 0.$$

We start again from scratch by assuming a solution of the form

$$y = \sum_{k=0}^{\infty} a_k x^{k+r},$$

which after being inserted back into the equation yields

$$Ly = \sum_{k=0}^{\infty} a_k \left((k+r)(k+r-1) + (k+r) - \frac{1}{4} \right) x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r+2} = 0.$$

That is,

$$\begin{aligned} a_0 \left(r^2 - \frac{1}{4} \right) x^r + a_1 \left((1+r)^2 - \frac{1}{4} \right) x^{r+1} & \\ + \sum_{k=2}^{\infty} \left[a_k \left((k+r)^2 - \frac{1}{4} \right) + a_{k-2} \right] x^{k+r} &= 0 \end{aligned} \tag{3.5}$$

with $k \geq 2$. From the first term of this sum we find that (since $a_0 \neq 0$) $r = \pm 1/2$. Let us consider these two cases.

(i) $r = 1/2$:

Here we have that the first term evaluates to 0 and the second term to $2a_1$ so that $a_1 = 0$ and all odd powers are zero as before. We now find the recursion formula to evaluate all a_k, for $k \geq 2$. This is

$$\begin{aligned} a_k \left((k+1/2)^2 - \frac{1}{4} \right) + a_{k-2} &= 0 \quad \text{now, multiply by 4} \\ a_k[(2k+1)^2 - 1] + 4a_{k-2} &= 0 \end{aligned}$$

or

$$a_k = -\frac{4a_{k-2}}{(2k+1)^2 - 1} = -\frac{4a_{k-2}}{4k(k+1)} = -\frac{a_{k-2}}{k(k+1)} \quad , \quad k \geq 2.$$

Starting from the bottom we have that

$$\begin{aligned} a_2 &= -\frac{a_0}{3\cdot 2} \\ a_4 &= -\frac{a_2}{4\cdot 4} = \frac{(-1)^2\,a_0}{5!} \\ &\vdots \\ a_{2k} &= \frac{(-1)^k\,a_0}{(2k+1)!}. \end{aligned}$$

Then

$$y_1(x) = x^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} = x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \tag{3.6}$$

where we recognize the last sum as the Taylor representation for the $\sin x$ function. That is

$$y_{1/2}(x) = x^{-\frac{1}{2}} \sin x.$$

(ii) $r = -1/2$. We return to the sum 3.5 where, after replacing r by $-1/2$, the first two terms are 0 and a_0, a_1 are free parameters. Considering the other terms for $k \geq 2$ we see that

$$a_k\left((k-1/2)^2 - \frac{1}{4}\right) + a_{k-2} = 0$$

That is, after multiplying by 4 and simplifying:

$$4\,a_k\,k(k-1) + 4a_{k-2} = 0,$$

so we find the recursion

$$a_k = -\frac{a_{k-2}}{k(k-1)}.$$

Let us find the sequence a_{2k}, $k = 0, 1, 2, \ldots$.

$$\begin{aligned}
a_2 &= -\frac{a_0}{2!} \\
a_4 &= -\frac{a_2}{4 \cdot 3} = (-1)^2 \frac{a_0}{4!} \\
&\vdots \\
a_{2k} &= (-1)^k \frac{a_0}{(2k)!},
\end{aligned}$$

$$\begin{aligned}
a_3 &= -\frac{a_1}{3!} \\
a_5 &= -\frac{a_3}{5 \cdot 4} = (-1)^2 \frac{a_1}{5!} \\
&\vdots \\
a_{2k+1} &= (-1)^k \frac{a_0}{(2k+1)!},
\end{aligned}$$

Then we find

$$y_{-1/2}(x) = a_0 x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + a_1 x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1},$$

or

$$y_{-1/2}(x) = x^{-\frac{1}{2}} (a_0 \sin x + a_1 \cos x)$$

We observe that $y_{1/2}(x)$ and $y_{-1/2}(x)$ are linearly independent solutions, so they span the whole space of solutions for $\nu = 1/2$ in the ODE. Why didn't we call $y_{1/2} = J_{1/2}$ and the like for $-1/2$? We observe that $y_{-1/2}$ is a combination of a sine and a cosine function. The sine function $sin(x)$ was already counted in $y_{1/2}(x)$ so we do not need that here.

Having $a_0 = 0$ will provide a good solution which is linearly independent with $y_{1/2}(x)$. Then it seems natural, assuming $a_0 = 1$ in the first case and $a_0 = 0$ in the second with $a_1 = 1$ to have

$$\begin{aligned} J_{1/2}(x) &\stackrel{?}{=} x^{\frac{1}{2}} \sin x \\ J_{-1/2}(x) &\stackrel{?}{=} x^{-\frac{1}{2}} \cos x. \end{aligned}$$

These two functions are linearly independent and satisfy the Bessel equation 3.1. However still the names above are not the ones used in the literature. We explain this next.

Please observe equation 3.3. It is intersting that while in the derivation of this equaton, we assumed that $\nu \neq 1/2$, there is nothing that stop us from inserting $\nu = 1/2$ into this equation. That is, we write

$$J_{1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\ \Gamma(1/2 + k + 1)} \left(\frac{x}{2}\right)^{2k+1/2}. \tag{3.7}$$

Then use properties of the $\Gamma(z)$ function to reduce this equation. That is, we observe that

$$\Gamma(1/2 + k + 1) = \left(k + \frac{1}{2}\right) \cdot \left(k - \frac{1}{2}\right) \cdot \cdots \cdot \frac{3}{2} \cdot \Gamma\left(\frac{3}{2}\right).$$

However we know that $\Gamma(3/2) = \sqrt{\pi}/2$ (please see my notes on the Gamma and Beta functions.) [5] Then we have that

$$\begin{aligned} \Gamma(1/2 + k + 1) &= \frac{(2k+1) \cdot (2k-1) \cdot \cdots 3 \cdot 1}{2^k} \frac{\sqrt{\pi}}{2} \\ &= \frac{(2k+1)!\sqrt{\pi}}{2^k 2^k k!\, 2} \\ &= \frac{(2k+1)!\sqrt{\pi}}{2^{2k+1} k!}. \end{aligned}$$

Then we substitute this equation into equation 3.7. To find

[5] https://drive.google.com/open?id=0B4W-gdhbNpsDaTNvbF9VcGgyR1E

$$J_{1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} \cancel{k!}}{\cancel{k!}\,(2k+1)!\sqrt{\pi}} \left(\frac{x}{2}\right)^{2k+1/2} = \sqrt{\frac{2}{\pi}} x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

now compare this last equation with equation 3.6 and observe that $J_{1/2}(x) = \sqrt{2/\pi} y_1(x)$; so up to a scaling factor, the two solutions are the same. We have then, by definition that

$$J_{1/2}(x) \equiv \sqrt{\frac{2}{\pi x}} \sin x.$$

In the same fashion, if $\lambda = -1/2$ we would find that the solution above after changing the since function for $\cos x$ would satisfy the Bessel equation 3.1 wiht $\lambda = -1/2$. That is, we have

$$J_{-1/2}(x) \equiv \sqrt{\frac{2}{\pi x}} \cos x.$$

Figure 3.2 shows the functions $J_{1/2}$ and $J_{-1/2}$ together with their envelopes.

The functions $J_{1/2}(x)$ and $J_{-1/2}(x)$ are linearly independent. We show that if ν is not an integer $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent. In the next section we will see that if ν is an integer then J_ν and $J_{-\mu}$ are linearly dependent.

The proof shown here is taken from [1], 1922 book on Theory of Bessel functions. [6]Theory of Bessel functions.

Recall the definition of Wronskian in equation **??**. We show that if the wronskian of $u_1(x)$, $u_2(x)$ is non-zero at any point of the domain, the functions $u_1(x)$ and $u_2(x)$ are linearly independent.

Let us assume that there are two coefficients c_1, c_2 such that

$$c_1 u_1(x) + c_2 u_2(x) = 0.$$

Let us take the derivative of this equation. That is,

$$c_1 u_1'(x) + c_2 u_2'(x) = 0.$$

[6]http://tinyurl.com/gs2oc3e

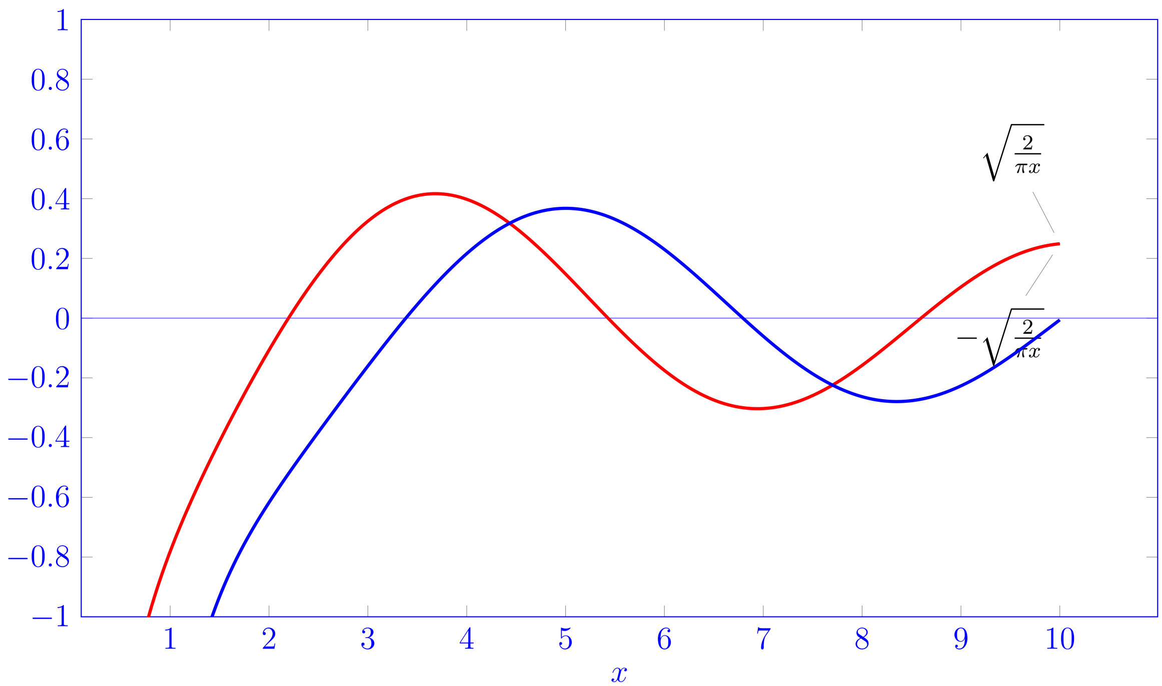

Figure 3.2: Bessel functions $J_{1/2}$ and $J_{-1/2}$ and their envelope curves $\pm\sqrt{\frac{2}{\pi x}}$.

The two equations above can be written in a matrix form as

$$\begin{pmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The Wronskian happens to be the determinant of this matrix. If that determinant is non-zero, this means that $c_1 = c_2 = 0$, and this means that the functions $u_1(x)$ and $u_2(x)$ are linearly independent.

We now compute the Wronskian of $J_\nu(x)$ and $J_{-\nu}(x)$. Let us call $L_\mu = L - \mu$ the Bessel equation. We can multply the equations $L_\mu J_\mu(x) = 0$, and $L_\mu J_{-\mu} = 0$ by $J_{-\mu}$ and J_μ respectively and subtract the results. Then we find

$$
\begin{aligned}
J_{-\mu}(x)L_\mu[J_\mu(x)] - J_\mu(x)\,L_\mu[J_{-\mu}(x)] &= x^2 J_{-\mu}(x)J''_\mu(x) + xJ_{-\mu}(x)J'_\mu(x) + \\
&\quad \cancel{(x^2-\nu^2)J_{-\mu}(x)J_\mu(x)} - \\
&\quad x^2 J_\mu(x)J''_{-\mu}(x) - xJ_\mu(x)J'_{-\mu}(x) \\
&- \cancel{(x^2-\nu^2)J_\mu(x)J_{-\mu}(x)} \\
&= 0.
\end{aligned}
$$

That is, assuming $x \neq 0$,

$$xJ_{-\mu}(x)J''_\mu(x) + J_{-\mu}(x)J'_\mu(x) - xJ_\mu(x)J''_{-\mu}(x) - J_\mu(x)J'_{-\mu}(x) = 0,$$

or

$$\frac{d}{dx}\,x[J_\mu(x)J'_{-\mu}(x) - J_{-\mu}(x)J'_\mu(x)] = 0.$$

Now, from the definition of Wronskian **??** we see that

$$\frac{d}{dx}\,x\,W(J_\mu(x), J_{-\mu}(x)] = 0,$$

and so $x\,W(J_\mu(x), J_{-\mu}(x)] = C$ for a constant C. We want to evaluate the constant C, such that

$$W(J_\mu(x), J_{-\mu}(x)] = \frac{C}{x}. \tag{3.8}$$

First, from equations 3.3 and 3.4 we have that

$$J_\nu(x) = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu [1+\mathcal{O}(x^2)] \quad , \quad J'_\nu(x) = \frac{1}{2\,\Gamma(\nu)}\left(\frac{x}{2}\right)^{\nu-1}[1+\mathcal{O}(x^2)]$$

and similarly, by changing ν by $-\nu$. Then

$$
\begin{aligned}
J_\nu(x)J'_{-\nu}(x) &= \left(\frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu [1+\mathcal{O}(x^2)]\right)\left(\frac{1}{2\,\Gamma(-\nu)}\left(\frac{x}{2}\right)^{-\nu-1}[1+\mathcal{O}(x^2)]\right) \\
&= \frac{1}{\Gamma(\nu+1)}\frac{1}{\Gamma(-\nu)}\frac{1}{x}[1+\mathcal{O}(x^2)].
\end{aligned}
$$

Likewise

$$\begin{aligned}J_{-\nu}(x)J'_{\nu}(x) &= \left(\frac{1}{\Gamma(-\nu+1)}\left(\frac{x}{2}\right)^{-\nu}[1+\mathcal{O}(x^2)]\right)\left(\frac{1}{2\,\Gamma(\nu)}\left(\frac{x}{2}\right)^{\nu-1}[1+\mathcal{O}(x^2)]\right)\\ &= \frac{1}{\Gamma(-\nu+1)}\frac{1}{\Gamma(\nu)}\frac{1}{x}[1+\mathcal{O}(x^2)).\end{aligned}$$

Hence the Wroskian is

$$W[J_\nu(x), J_{-\nu}(x))] = \left(\frac{1}{\Gamma(\nu+1)}\frac{1}{\Gamma(-\nu)} - \frac{1}{\Gamma(-\nu+1)}\frac{1}{\Gamma(\nu)}\right)\frac{1}{x} + \mathcal{O}(x).$$

By matching coefficients of equal powers on this expression and equation 3.8 we observe that

$$C = \left(\frac{1}{\Gamma(\nu+1)}\frac{1}{\Gamma(-\nu)} - \frac{1}{\Gamma(-\nu+1)}\frac{1}{\Gamma(\nu)}\right)$$

We now use the reflection formula [7] for the $\Gamma(z)$ function (please see my notes on the Gamma and Beta functions for a proof of this formula) [8]

That is, we have

$$C = -\frac{\sin \pi\nu}{\pi} - \frac{\sin \pi\nu}{\pi} = -\frac{2\,\sin \pi\nu}{\pi}.$$

Hence, if $\nu \notin \mathbb{Z}$ then $C \neq 0$, and the Wronskian does not vanish. These means that the two solutions are linearly independent for $\nu \notin \mathbb{Z}$.

3.3 The case of $\nu = n \in \mathbb{Z}$

If ν is an integer n, we show next, that the solutions $J_n(x)$ and $J_{-n}(x)$ solutions are linearly dependent and so we need to look for a second solution to be able to get the general solution of the homogeneous ODE. Let us assume $\nu > 0$, $\nu \in \mathbb{Z}$. Then we have that the Gamma function $\Gamma(-\nu+k+1)$ diverges any argument equal or smaller than

[7]https://en.wikipedia.org/wiki/Reflection_formula

[8]https://drive.google.com/open?id=0B4W-gdhbNpsDaTNvbF9VcGgyR1E

0, and since this Gamma function is dividing, then the sum 3.4 evaluates all terms with negative arguments of the Γ function in 0. More clearly, we have that

$$\frac{1}{\Gamma(k-n+1)} = 0 \quad , \quad \text{for} \quad k = 0, 1, 2, n-1.$$

Hence we can write 3.4 as

$$\begin{aligned}
J_{-n}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{2k-n} \\
&= \sum_{k=n}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{2k-n} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{(k+n)!\,\Gamma(\cancel{-n}+k+\cancel{n}+1)} \left(\frac{x}{2}\right)^{2(k+n)-n} \\
&= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+n)!\,\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+n} \\
&= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+n+1)} \left(\frac{x}{2}\right)^{2k+n} \\
&= (-1)^n J_n(x).
\end{aligned}$$

That is, we showed that

$$J_{-n}(x) = (-1)^n J_n(x), \tag{3.9}$$

and so the functions $J_n(x)$ and $J_{-n}(x)$ for $n \in \mathbb{Z}$ are not linearly independent. This creates an inconvenient situation and mathematicians looked for a linear independent couple of functions which does not depend on the

One idea is to rewrite equation 3.9 as a difference. That is,

$$J_{-\nu}(x) - (-1)^{\nu} J_{\nu}(x) = 0.$$

Since both, J_ν, and $J_{-\nu}$ are solutions of the Bessel equation, their difference is solution as well. Of course this difference is not interesting since for $\nu = n \in \mathbb{Z}$ the difference is the trivial 0. Hankel's idea was to form a quotient so that the function is "regularized". That is Hankel thought about the function

$$\frac{J_\nu(x) - (-1)^n J_{-\nu}(x)}{\nu - n}. \tag{3.10}$$

which looks undefined but if, with the use of the L'Hôpital [9] rule, we can show that the limit as $\nu \to n$ exists and is well defined, we can adopt this new function as a linearly independent solution which works even for $\nu \in \mathbb{Z}$. Since for $\nu \neq n$, the above equation is a solution for the Bessel equation we could think that the limit as $\nu \to n$ is a solution as well. We can not assume that yet. Instead, we will find the limit and then prove that the limit is actually a solution of the Bessel function which is linearly independent of the solution above known as the Bessel function of the first kind 3.3. This is the origin of the Bessel function of the second kind.

We take the limit in 3.10 as ν approaches n. That is,

$$\begin{aligned}
\lim_{\nu \to n} \frac{J_\nu(x) - (-1)^n J_{-\nu}(x)}{\nu - n} &= \lim_{\nu \to n} \frac{J_\nu(x) - J_\nu(x) + J_\nu(x) - (-1)^n J_{-\nu}(x)}{\nu - n} \\
&= \lim_{\nu \to n} \frac{J_\nu(x) - J_\nu(x) - (-1)^n [J_{-\nu}(x) - J_{-\nu}(x)]}{\nu - n} \\
&= \left[\frac{\partial J_\nu(x)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial \nu} \right] \Bigg|_{\nu = n}.
\end{aligned}$$

Hankel noted this function as

$$Y_n(x) = \left[\frac{\partial J_\nu(x)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial \nu} \right] \Bigg|_{\nu = n}. \tag{3.11}$$

We could suggest the heuristic argument that if $J_\nu(x)$ and $J_{-\nu(x)}$ are not linearly independent because $J_\nu(x) - (-1)^n J_{-\nu}(x) = 0$, then how about differentiating this with respect to ν, and using that as a new solution. We will show that Y_n is a solution of the Bessel equation which is not identically zero. Since $J_\pm(\nu)(x)$ are analytic functions of both x and ν we will change the order of differentiations with respect to these two parameters without error. That is, starting with the Bessel equation 3.1, and since $J_{\pm\nu}(x)$ are solutions for it, we can differentiate with respect to ν for both $J_{\pm\nu}$, and interchange the order of derivatives. That is, we can say that

[9] https://en.wikipedia.org/wiki/L%27H%C3%B4pital%27s_rule

$$x^2\frac{d^2}{dx^2}\frac{\partial J_{\pm\nu}}{\partial\nu}+x\frac{d}{dx}\frac{\partial J_{\pm\nu}}{\partial\nu}+(x^2-\nu^2)\frac{\partial J_{\pm\nu}}{\partial\nu}-2\nu J_{\pm\nu}(x)=0.$$

That is, we have the following chain of events:

$$L\left[\frac{\partial J_\nu(x)}{\partial x}\right]-2\nu J_{\nu(x)}=0$$
$$L\left[\frac{\partial J_{-\nu}(x)}{\partial x}\right]-2\nu J_{-\nu(x)}=0 \quad , \quad L\left[(-1)^n\frac{\partial J_{-\nu}(x)}{\partial x}\right]-(-1)^n 2\nu J_{-\nu(x)}=0$$
$$L\left[\frac{\partial J_\nu(x)}{\partial x}-(-1)^n\frac{\partial J_{-\nu}(x)}{\partial x}\right]-2\nu[J_\nu(x)-(-1)^n J_{-\nu}(x)]=0.$$

We now take the limit as $\nu\to n$, and since the expressions are analytic and $\lim_{\nu\to n} J_\nu(x)-(-1)^n J_{-\nu}(x)=0$ we find that

$$L\left[\frac{\partial J_n(x)}{\partial x}-(-1)^n\frac{\partial J_{-n}(x)}{\partial x}\right]=Y_n(x)=0.$$

So, indeed $Y_n(x)$ is a solution of the Bessel equation. We want to find a better characterization of $Y_n(x)$ so that it is not in terms of derivatives. We do this next.

3.4 The Bessel function of second kind

3.4.1 Motivation

Let us consider the linear combination

$$J_\nu(x)-(-1)^\nu J_{-\nu}(x). \tag{3.12}$$

We know that each of the two terms $J_{\pm\nu}(x)$ is a solution of Bessel equation and they are linearly independent as long as $\nu\notin\mathbb{Z}$. In the limit as $\nu\to n\in\mathbb{Z}$ we have

$$J_n(x)-(-1)^n J_{-n}(x)=0.$$

Of course we do not want this trivial solution so we must "regularize" it. First, we observe that $(-1)^n = \cos n\pi$ so we might rewrite expression 3.12 as

$$\cos \nu\pi J_\nu(x) - J_{-\nu}(x).$$

Then, we need to divide the equation by something that goes to 0 as $\nu \to n$. Why $\sin \nu\pi$ and not just ν? The reason would be clear soon.

If we pick to divide by $\sin \nu\pi$, and use L"Höpital rule to evaluate

$$\lim_{\nu \to n} \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu x}.$$

We will inmediately find equation 3.11, which is what Hankel called $Y_n(x)$.

The function

$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi} \tag{3.13}$$

is the *Bessel function of second kind* .

Figure 3.3 shows the first 3 Bessel functions of second kind for $\nu = 0, 1, 2$.

We now find new representations of the Bessel function of second kind as infinite series. Let us start with the case of $n = 0$ and generalize this $n \neq 0$ later.

3.4.2 The case of $Y_0(x)$

First, changing n by $-n$ we see that

$$\begin{aligned}
Y_{-n}(x) &= \lim_{\mu \to -n} \frac{\partial J_\nu(x)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial \nu} \\
&= \lim_{\mu \to n} \frac{\partial J_{-\nu}(x)}{\partial \nu} - (-1)^n \frac{\partial J_\nu(x)}{\partial \nu} \\
&= \lim_{\mu \to n} (-1)^n \frac{\partial J_\nu(x)}{\partial \nu} - \frac{\partial J_\nu(x)}{\partial \nu} \\
&= (-1)^n \lim_{\mu \to n} \left[\frac{\partial J_\nu(x)}{\partial \nu} - (-1)^n \frac{\partial J_\nu(x)}{\partial \nu} \right] \\
&= (-1)^n Y_n(x).
\end{aligned}$$

Now

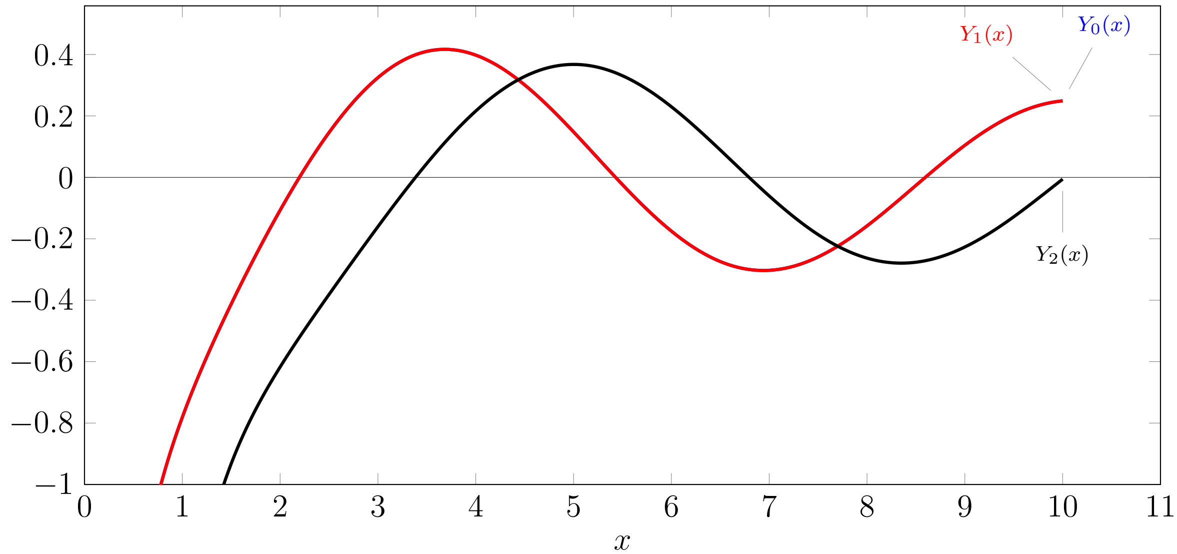

Figure 3.3: Bessel functions of second kind Y_i, for $i = 0, 1, 2$.

$$Y_0(x) = \left[\frac{\partial J_\nu(x)}{\partial \nu}\right]_{\nu=0} - \left[\frac{\partial J_{-\nu}(x)}{\partial \nu}\right]_{\nu=0}$$

and from

$$\frac{\partial J_{-\nu}(x)}{d\nu} = \frac{\partial J_\nu(x)}{d(-\nu)} = -\frac{\partial J_\nu(x)}{d(\nu)},$$

we see that

$$Y_0(x) = 2\left[\frac{\partial J_\nu(x)}{\partial \nu}\right]_{\nu=0}.$$

Now, from using the definition of the Bessel function of the first kind 3.3. we se that

$$Y_0(x) \quad = \quad 2\left[\frac{\partial}{\partial \nu}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k+\nu}\right]_{\nu=0}.$$

To find the derivatives we need to evaluate the derivative of the fraction $(x/2)^{2k+\nu}/\Gamma(\nu+k+1)$.

We use the *digamma* function. That is, the logarithmic derivative of the Gamma function,

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

and so

$$\begin{aligned}\frac{\partial}{\partial\nu}\frac{(x/2)^{2k+\nu}}{\Gamma(\nu+k+1)} &= \frac{(x/2)^{2k+\nu}\ln(x/2)}{\Gamma(\nu+k+1)} - \frac{\psi(\nu+k+1)\Gamma(\nu+k+1)(x/2)^{2k+\nu}}{\Gamma^2(\nu+k+1)} \\ &= \frac{(x/2)^{2k+\nu}\ln(x/2)}{\Gamma(\nu+k+1)} - \frac{\psi(\nu+k+1)(x/2)^{2k+\nu}}{\Gamma(\nu+k+1)} \\ &= \left(\frac{x}{2}\right)^{2k+\nu}\frac{1}{\Gamma(\nu+k+1)}\left[\ln\left(\frac{x}{2}\right) - \psi(\nu+k+1)\right].\end{aligned}$$

We then find that

$$Y_0(x) \quad = \quad 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}\left[\ln\left(\frac{x}{2}\right) - \psi(k+1)\right].$$

Since, for n integer we have the relation [10].

$$\psi(n+1) = \sum_{k=1}^{n}\frac{1}{k} - \gamma \tag{3.14}$$

where $\gamma \approx$ is the Euler-Mascheroni constant [11] then $|\psi(n+1)| \leq n+1$ and so the series above converges uniformly.

3.4.3 The case of $Y_n(x)$, $n > 0$, n integer

We start by differentiating the Bessel functions of first kind 3.3. Most of the job for this differentiation is done in the previous derivation.

[10] see my notes on the Gamma and Beta functions

[11] https://en.wikipedia.org/wiki/Euler%E2%80%93Mascheroni_constant

$$
\begin{aligned}
\frac{\partial J_\nu(x)}{dx} &= \frac{\partial}{\partial \nu}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k+\nu} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{x}{2}\right)^{2k+\nu}\frac{1}{\Gamma(\nu+k+1)}\left[\ln\left(\frac{x}{2}\right)-\psi(\nu+k+1)\right].
\end{aligned}
$$

Since this function is continuous in ν for $\nu > 0$, we can take the limit as $\nu \to n$, where n is some positive integer and find

$$
\begin{aligned}
\lim_{\nu\to n}\frac{\partial J_\nu(x)}{dx} &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{2k+n}\left[\ln\left(\frac{x}{2}\right)-\psi(n+k+1)\right] \\
&= \ln\left(\frac{x}{2}\right)J_n(x)-\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k+n}\frac{(-1)^k}{k!(n+k)!}\psi(n+k+1) \\
&= \left[\ln\left(\frac{x}{2}\right)+\gamma\right]J_n(x)-\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k+n}\frac{(-1)^k}{k!(n+k)!}\sum_{i=1}^{n+k}\frac{1}{i}.
\end{aligned}
\tag{3.15}
$$

We are now set to evaluate $J_{-\nu}(x)$. A representation of the digamma function is given by

$$
\phi(\nu) = \gamma + \sum_{i=1}^{\infty}\frac{\nu-1}{i(\nu+i-1)}.
$$

This is shown in my notes on the Gamma and Beta functions. We see that the digamma function $\phi(\nu)$ has simple poles whenever $\nu+i-1=0$, and this happens if ν is an integer. If $\nu = -n$ then the argument of the digamma function in the series for the limit of the derivative of the Bessel function, turns out to be $\nu+k+1 = -n+k+1$. Now, in the series expansion for the digamma function the denominators of the form $\nu+i-1$, replacing ν by $-n+k+1$, will become $(-n+k+1)+i-1 = -n+k+i$, and since $i \geq 1$, we need to have $k \geq n$. We would have poles as long as $k < n$, so we need to separate the infinite sum for the $J_{-\nu}$ function into two parts as follows:

$$
J_{-\nu}(x) = \sum_{k=0}^{n-1}\frac{(-1)^k}{k!\,\Gamma(-\nu+k+1)}\left(\frac{x}{2}\right)^{2k-\nu}+\sum_{k=n}^{\infty}\frac{(-1)^k}{k!\,\Gamma(-\nu+k+1)}\left(\frac{x}{2}\right)^{2k-\nu}.
$$

The first sum is the trouble sum and we need to evaluate it using a technique other than the used for the evaluation of the derivative of the function J_ν, at $\nu = n > 0$. Let us proceed.

For the first sum we use the reflection formula [12] for the $\Gamma(z)$ function. That is, we write

$$\frac{1}{\Gamma(\nu+k+1)} = \frac{\Gamma(\nu-k)\,\sin[(\nu-k)\pi]}{\pi},$$

and $0 \le k < n$,

$$\begin{aligned}
&\frac{\partial}{\partial\nu}\Gamma(\nu-k)\,\sin[(\nu-k)\pi]\left(\frac{x}{2}\right)^{2k-\nu} = \\
&\quad \Gamma(\nu-k)\psi(\nu-k)\sin[(\nu-k)\pi]\left(\frac{x}{2}\right)^{2k-\nu} + \pi\Gamma(\nu-k)\cos[(\nu-k)\pi]\left(\frac{x}{2}\right)^{2k-\nu} \\
&\qquad - \Gamma(\nu-k)\sin[(\nu-k)\pi]\left(\frac{x}{2}\right)^{2k-\nu}\ln\nu \\
&\quad = \Gamma(\nu-k)\left(\frac{x}{2}\right)^{2k-\nu}\Bigg[\psi(\nu-k)\sin[(\nu-k)\pi] + \pi\cos[(\nu-k)\pi] \\
&\quad - \ln\left(\frac{x}{2}\right)\sin[(\nu-k)\pi]\Bigg].
\end{aligned}$$

and taking the limit as $\nu \to k$ we find that

$$\lim_{\nu\to k}\frac{\partial}{\partial\nu}\Gamma(\nu-k)\,\sin[(n-k)\pi]\left(\frac{x}{2}\right)^{2k-n} = \pi\Gamma(n-k)\left(\frac{x}{2}\right)^{2k-n}\cos[(n-k)\pi].$$

We then find that, for the first sum, and since $\cos[(n-k)\pi] = (-1)^{n-k}$,

$$\begin{aligned}
&\lim_{\nu\to n}\frac{\partial}{\partial\nu}\sum_{k=0}^{n-1}\frac{(-1)^k}{k!\,\Gamma(-\nu+k+1)}\left(\frac{x}{2}\right)^{2k-\nu} = \\
&\qquad\qquad \lim_{\nu\to n}\frac{\partial}{\partial\nu}\sum_{k=0}^{n-1}\frac{(-1)^k}{\pi k!}\Gamma(\nu-k)\sin[(\nu-k)\pi] \\
&\qquad\qquad = \sum_{k=0}^{n-1}\frac{(-1)^n\Gamma(n-k)}{k!}\left(\frac{x}{2}\right)^{2k-n}.
\end{aligned}$$

[12] https://en.wikipedia.org/wiki/Reflection_formula

Now, for the second sum we employ the same technique used to evaluate the derivative of J_ν, only that the sum starts at $k = n$ instead of at $k = 0$. We find,

$$\begin{aligned}
\lim_{\nu\to n}\frac{\partial}{\partial\nu}\sum_{k=n}^{\infty}\frac{(-1)^k}{k!\,\Gamma(-\nu+k+1)}\left(\frac{x}{2}\right)^{2k-\nu} &= \\
\lim_{\nu\to n}\frac{\partial}{\partial\nu}\sum_{k=0}^{\infty}\frac{(-1)^{k+n}}{(k+n)!\,\Gamma(-\nu+n+k+1)}\left(\frac{x}{2}\right)^{2(k+n)-\nu} \\
=\sum_{k=0}^{\infty}\frac{(-1)^{k+n}}{(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\frac{1}{\Gamma(k+1)}\left[-\ln\left(\frac{x}{2}\right)+\psi(k+1)\right] \\
=\sum_{k=0}^{\infty}\frac{(-1)^{k+n-1}}{k!\,(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\left[\ln\left(\frac{x}{2}\right)-\psi(k+1)\right].
\end{aligned}$$

Then, joining the result for both sums we see that

$$\begin{aligned}
\left.\frac{\partial}{\partial\nu}J_{-\nu}(x)\right|_{\nu=n} &= \sum_{k=0}^{n-1}\frac{(-1)^n(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n} \\
&+ \sum_{k=0}^{\infty}\frac{(-1)^{k+n-1}}{k!\,(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\left[\ln\left(\frac{x}{2}\right)-\psi(k+1)\right]
\end{aligned} \tag{3.16}$$

We now combine the two contributions to the Hankel's formula 3.11. That is, we add equations 3.15 and 3.16.

$$
\begin{aligned}
Y_n(x) &= \left[\frac{\partial J_\nu(x)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial \nu}\right]\Bigg|_{\nu=n} \\
&= \left[\ln\left(\frac{x}{2}\right) + \gamma\right] J_n(x) - \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^k}{k!(n+k)!} \sum_{i=1}^{n+k} \frac{1}{i} \\
&\quad - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-n} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \\
&\quad \left(\frac{x}{2}\right)^{2k+n} \left[\ln\left(\frac{x}{2}\right) - \psi(k+1)\right] \\
&= \left[\ln\left(\frac{x}{2}\right) + \gamma\right] J_n(x) - \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^k}{k!(n+k)!} \sum_{i=1}^{n+k} \frac{1}{i} \\
&\quad - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-n} + J_n(x) \left(\ln \frac{x}{2}\right) \\
&\quad - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \left(\frac{x}{2}\right)^{2k+n} \left[\sum_{i=1}^{k} \frac{1}{i} - \gamma\right] \\
&= 2\left[\ln\left(\frac{x}{2}\right) + \gamma\right] J_n(x) - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-n} \\
&\quad - \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k+n} \frac{(-1)^k}{k!(n+k)!} \left[\sum_{i=1}^{k} \frac{1}{i} + \sum_{i=1}^{n+k} \frac{1}{i}\right].
\end{aligned}
$$

Chapter 4

Green functions for selected Bessel equations

We will consider the Bessel operator $Lu = x^u + xu' + x^u$, and the eigenvalue problem $Lu = \lambda u$, with $\lambda = \nu^2$, $\nu \neq 0$.

4.1 Domain $[0, 1]$

We assume the boundary conditions.

$$u(0) = u(1) = 0.$$

The solutions of the homogeneous equation are given by the Bessel functions of first kind $J_\nu(x)$ 3.3 and the Bessel function of second kind $Y_\nu(x)$ 3.13.

The general solution is of the form

$$u(x) = c_1 J_\nu(x) + c_2 Y_\nu(x). \tag{4.1}$$

We consider the interval $[0, 1]$ divided in two pieces with a source located at some point ξ, $0 < \xi < 1$. Let us, for the moment, assume that there are two solutions $u_1(x)$, and $u_2(x)$ such that $u_1(x)$ satisfies the left boundary condition and $u_2(x)$ the right boundary condition. Then we claim that the Green function has the form

$$G(x, \xi, \mu) = \begin{cases} C(\xi) u_1(x) u_2(\xi) & 0 < x < \xi < 1 \\ C(\xi) u_1(\xi) u_2(x) & 0 < \xi < x < 1 \end{cases}$$

Where $C(\xi)$ must found by assuring the jump discontinuity of the first derivative of $G(x, \xi, \mu)$ at $x = \xi$.

To find $u_i(x)$ $(i = 1, 2)$ we start with the general solution 4.1.

First, since at $x = 0$, the Bessel function of the second kind diverges we see that on the first part of the interval $0 < x < \xi$, $c_2 = 0$. We then found a solution $u_1(x) = J_\nu(x)$, which satisfies the left boundary condition, since $J_\nu(0) = 0$.

Now for right edge, $x = 1$, we have that, from $u(1) = 0$,

$$u(1) = c_1 J_\nu(1) + c_2 Y_\nu(1) = 0. \tag{4.2}$$

We are free to choose $c_1 = Y_\nu(1)$ and $c_2 = -J_\nu(1)$, so

$$u_2(x) = Y_\nu(1)J_\nu(x) - J_\nu(1)Y_\nu(x).$$

The jump discontinuity provides $C(\xi)$ as

$$C(\xi) = \frac{1}{\xi^2 W(xi)}$$

where

$$\begin{aligned} W(\xi) &= u_1(\xi)u_2'(\xi) - u_2(\xi)u_1'(\xi) \\ &= J_\nu(\xi)[Y_\nu(1)J_\nu'(\xi) - J_\nu(1)Y_\nu'(\xi) - J_\nu'(\xi)[Y_\nu(1)J_\nu(x) - J_\nu(1)Y_\nu(x)] \end{aligned}$$

Then the Green function is given by

$$G(x, \xi, \mu) = \begin{cases} \dfrac{J_\nu(x)}{\xi^2 W(\xi)} \left[Y_\nu(1)J_\nu(\xi) - J_\nu(1)Y_\nu(\xi)\right] & 0 < x < \xi < 1 \\ \\ \dfrac{J_\nu(\xi)}{\xi^2 W(\xi)} \left[Y_\nu(1)J_\nu(x) - J_\nu(1)Y_\nu(x)\right] & 0 < \xi < x < 1 \end{cases}$$

Chapter 5

Properties of Bessel functions

5.1 Recursive formulas

5.2 Generating function

5.3 Integral representation

5.4 Asymptotic representation

5.5 Especial values

Chapter 6

Transforms derivaded from the Bessel differential equation

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