Notes on Green Functions for second order linear ODE

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Contents

Chapter 1

Introduction

Informal notes on Green functions. The focus is in those second order linear Ordinary Differential Equations (ODE) that give rise to transforms. The equations shown here are the bases for a set of transforms such as, for example, Laplace, Fourier, Mellin, Hankel, etc. This document is a pre-requiste for my notes on transform theory [1] and the document is directly made to provide all required knowledge to support the transform theory notes.

We are interested in operator of the form

$$Lu = au'' + bu' + cu \quad , \quad a \neq 0 \tag{1.1}$$

where a, b, c, u are smooth functions from the interval $[0, 1]$ to the complex numbers.

Initially we show how to solve the problem in an interval $[0, 1]$. We also use intervals of the form $[0, \infty)$ and $(-\infty, \infty)$. The extension to a general interval of the form $[a, b]$ is done by appropriate shifting and scaling factors and we will leave that as an exercise for the reader.

Let us think of the Ordinary Differential Equation (ODE)

$$Lu = f \quad , \quad u(0) = (1) = 0.$$

where $f : [0, 1] \to \mathbb{C}$ is a given continuous function. We claim that there is a solution of the form

$$u(x) = \int_0^1 G(x, y) f(y) dy \tag{1.2}$$

[1] https://drive.google.com/open?id=0B4W-gdhbNpsDQ18xSkpxdE9WSDQ

where $G : [0,1] \times [0,1] \to \mathbb{C}$ is known as a *Green's function* for the problem. The Green function satisfies the companion problem

$$L\left[G(x,y)\right] = \delta(x-y) \quad , \quad G(0,y) = G(1,y) = 0.$$

We verify this. Since $G(x,y)$ is a Green funtion it satisfies equation 1.2. If in this equation we take the operator L in both sides, and move it inside the integral, we find

$$L[u(x)] = \int_0^1 L[G(x,y)]f(y)dy = \int_0^1 \delta(x-y)f(y)dy = f(x).$$

Moreover, u satisfies the boundary conditions

$$u(0) = \int_0^1 G(0,y)f(y)dy = 0 \quad , \quad u(1) = \int_0^1 G(1,y)f(y)dy = 0$$

Before showing the formal definition of a Green function let us try to solve a simple problema where we find $G(x,y)$ by "brute force".

We use as a reference the Hilbert space $\mathcal{H}$ of the twice differentiable functions from the interval $[0,1]$ to the real numbers $\mathbb{R}$. Define the linear operator

$$\begin{aligned} L : H &\to P \\ u &\mapsto Lu = \frac{d^2 f}{dx^2}, \end{aligned} \tag{1.3}$$

with boundary conditions $u(1) = u(0) = 0$. We do not care at the moment which space is P.

That is, we want to solve the equation

$$Lu = f.$$

or

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{dg}{dx} = f \tag{1.4}$$

where g satisfies

$$\frac{du(\xi)}{d\xi} = g(\xi). \tag{1.5}$$

Integrating this last function, between 0 and ξ

$$u(\xi) - u(0) = \int_0^{\xi} g(x)\, dx. \tag{1.6}$$

On the other hand, from

$$\frac{dg(x)}{dx} = f(x) \tag{1.7}$$

and integrating between 0 and α

$$g(\alpha) - g(0) = \int_0^{\alpha} f(y)dy. \tag{1.8}$$

From $u(0) = 0$ and equations 1.6 and 1.8 we find

$$u(\xi) = \int_0^{\xi} \left(g(0) + \int_0^{x} f(y)dy \right) dx, \tag{1.9}$$

which can be simplified to

$$u(\xi) = \xi\, g(0) + \int_0^{\xi} dx \int_0^{x} f(y)dy, \tag{1.10}$$

We have first to find $g(0)$ by using the boundary condition $u(1) = 0$; we see

$$u(1) = g(0) + \int_0^{1} dx \int_0^{x} f(y)dy = 0, \tag{1.11}$$

so

$$g(0) = -\int_0^{1} dx \int_0^{x} f(y)dy. \tag{1.12}$$

The solution u is then given by

$$u(\xi) = -\xi \int_0^{1} dx \int_0^{x} f(y)dy + \int_0^{\xi} dx \int_0^{x} f(y)dy. \tag{1.13}$$

We use the Heaviside function

$$H(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0. \end{cases} \tag{1.14}$$

and rewrite 1.13 as

$$\begin{aligned} u(\xi) &= -\xi \int_0^1 dx \int_0^1 H(x-y)\, f(y)dy + \int_0^1 dx\, H(\xi - x) \int_0^1 H(x-y) f(y)dy \\ &= \int_0^1 dx \int_0^1 H(x-y)\, f(y)[H(\xi - x) - \xi]\, dy. \end{aligned} \tag{1.15}$$

That is,

$$u(\xi) = \int_0^1 G(\xi, y) f(y) dy, \tag{1.16}$$

where

$$G(\xi, y) = \int_0^1 dx\ H(x-y)\, [H(\xi - x) - \xi]. \tag{1.17}$$

Let us simplify this integral, but first I inter–change the names of ξ and x

$$G(x, y) = \int_0^1 d\xi\ H(\xi - y)\, [H(x - \xi) - x], \tag{1.18}$$

just because it looks “better”.

If $x < y$ we write

$$\begin{aligned} G(x,y) &= \int_0^x + \int_x^y + \int_y^1 d\xi\ \ H(\xi - y)\, [H(x-\xi) - x] \\ &= 0 \quad + 0 \quad + \int_y^1 d\xi\, (-x) \\ &= x\, (y-1). \end{aligned} \tag{1.19}$$

If $x > y$ we write

$$\begin{aligned} G(x,y) &= \int_0^y + \int_y^x + \int_x^1 d\xi\ \ H(\xi - y)\, [H(x-\xi) - x] \\ &= 0 \quad + \int_y^x d\xi\, (1-x) + \int_x^1 d\xi\, (-x) \\ &= (1-x)(x-y) - x(1-x) \\ &= y(x-1). \end{aligned} \tag{1.20}$$

We conclude that

$$G(x,y) = \begin{cases} x\,(y-1) & \text{if} \quad x < y \\ y\,(x-1) & \text{if} \quad x > y \end{cases} \tag{1.21}$$

We find the following characteristics for the Green function

- $G(x,y)$ is continuous, even at the line $x = y$.
- $G(x,y)$ satisfies the equation $G''(x) = 0$ where y is a parameter. We verify this

  $$G'(x,y) = \begin{cases} y-1 & \text{if} \quad x < y \\ y & \text{if} \quad x > y \end{cases}$$

  and so $G''(x,y) = 0$.
- There is jump discontinuity at $x = y$ for the first derivative of G. That is, $\lim_{x\to y^+} G'(x,y) - \lim_{x\to y^-} G'(x,y) = 1/a(x)$, where here $a(x) = 1$ is the coefficient of the second derivative term of the ODE. In this case $a(x) = 1$. Let us verify this

  $$\lim_{x\to y^+} G'(x,y) - \lim_{x\to y^-} G'(x,y) = y - (y-1) = 1.$$

This example motivates the definition shown in the next section.

Chapter 2

Operator $L = -u''$

We define a Green function, and show a technique to find Green functions with examples, all of them related to the operator $L = -u''$.

2.1 Finite Boundary

Given the linear operator L 1.1 and the equation $LG(x, y) = \delta(x - y)$, satisfying the Dirichlet boundary conditions $G(0, y) = G(1, y) = 0$, $G(x, y)$ is a Green function if

(i) $G(x, y)$ is continuous on a square $0 \le x, y \le 1$, and twice continuously differentiable on the triangles $0 \le x \le y \le 1$, $0 \le y \le x \le 1$. The left and right limits of the partial derivatives on $x = y$ are not equal. This jump discontinuity is what creates the Dirac delta source in the right hand side.

(ii) $G(x, y)$ satisfies the homogeneous ordinary differential equation

$$LG = 0 \quad , \quad 0 < x < y < 1 \quad \text{and} \quad 0 < y < x < 1.$$

(iii) The jump discontinuity of $G_x(x, y)$ across the line $x = y$ is given by $1/a(y)$. That is

$$\lim_{x \to y^+} [G'(x, y)] - \lim_{x \to y^-} [G'(x, y)] = \frac{1}{a(y)}$$

We now provide an algorithm to find the Green function to solve the second order ODE $L(u) = f$.

(i) Solve the homogeneous equation $Lu = 0$. Find two linearly independent solutions u_1, u_2. These two functions should satisfy the boundary conditions, say u_1 the left boundary condition, and u_2 the right boundary condition. Form the product $Cu_1(x)u_2(y)$.

(ii) Build the Green function as

$$G(x,y) = \begin{cases} C(y)u_1(x)u_2(y) & 0 \le x \le y < 1 \\ C(y)u_1(y)u_2(x) & 0 \le y \le x < 1 \end{cases}$$

(iii) Find $C(y)$ from the jump discontinuity. That is solve the equation

$$\lim_{x \to y^+} [G'(x,y)] - \lim_{x \to y^-} [G'(x,y)] = \frac{1}{a(y)}$$

which provides

$$C(y) = \frac{1}{a(y)W(y)} \tag{2.1}$$

with

$$W(y) = u_1(y)u_2'(y) - u_2(y)u_1'(y) = \begin{vmatrix} u_1 & u_2 \\ u_2 & u_1 \end{vmatrix}. \tag{2.2}$$

The symbol W stands for *Wronskian* .

Let us illustrate the algorithm with the following example:

Solve

$$-u'' = \lambda u \quad , \quad u(0) = u(1) = 0. \tag{2.3}$$

which is a simple case of Sturm-Liouville eigenvalue problem.

(i) **Find two linearly independent solutions.** We know that the general solution is given by

$$u(x) = c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x.$$

Now, from $u(0) = 0$ we see that $c_2 = 0$, so the solution is of the form $u(x) = c_1 \sin \sqrt{\lambda} x$. We choose our first solution by picking $c_1 = 1$. That is $u_1(x) = \sin \sqrt{\lambda} x$.

To choose the second solution we return to the general solution above and use the condition $u(1) = 0$ from which we have $c_1 \sin \sqrt{\lambda} + c_2 \cos \sqrt{\lambda} = 0$. We can choose here $c_1 = \cos \sqrt{\lambda}$, and $c_2 = -\sin \sqrt{\lambda}$, (note that this selection is not unique, however there is still one more parameter to choose ahead which is $C(y)$) so that the second solution would be

$$u_2(x) = \sin \sqrt{\lambda}(x-1).$$

are two linear independent solutions for equation 2.3.

(ii) **Build a prototype of the Green function.** This is given by

$$G(x,y) = \begin{cases} C(y) \sin \sqrt{\lambda} x \ \sin \sqrt{\lambda}(x-1) & 0 \leq x \leq y < 1 \\ C(y) \sin \sqrt{\lambda} x \ \sin \sqrt{\lambda}(x-1) & 0 \leq y \leq x < 1 \end{cases}$$

(iii) **find $C(y)$.** From 2.1 we see that

$$\begin{aligned} C(y) &= \frac{1}{a(y)W(y)} \\ &= \frac{1}{u_1 u_2'(y) - u_2 u_1'(y)} \\ &= \frac{1}{\sqrt{\lambda}} \frac{1}{\sin \sqrt{\lambda} y \ \cos \sqrt{\lambda}(y-1) - \sin \sqrt{\lambda}(y-1) \cos \sqrt{\lambda} y} \\ &= \frac{1}{\sin \sqrt{\lambda}} \end{aligned}$$

with this we find that

$$G(x,y) = \begin{cases} \frac{\sin\sqrt{\lambda}x\ \sin\sqrt{\lambda}(x-1)}{\sin\sqrt{\lambda}} & 0 \le x \le y < 1 \\ \\ \frac{\sin\sqrt{\lambda}x\ \sin\sqrt{\lambda}(x-1)}{\sin\sqrt{\lambda}} & 0 \le y \le x < 1 \end{cases}.$$

2.2 Infinite Boundary with end point

2.2.1 Along the real line

Let us now consider the same operator $L = -u''$ and the eigenvalue problem $L - \lambda$:

$$-u'' = \lambda u \quad , \quad u(0) = 0 \quad , \quad \lim_{x\to\infty} u(x) = 0. \tag{2.4}$$

where this time we consider the domain in the interval $[0, \infty)$.

The general solution of this equation is of the form

$$u(x) = c_1 \mathrm{e}^{\mathrm{i}\sqrt{\lambda}x} + c_2 \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}x}$$

We apply first the boundary conditions. From $u(0) = 0$ we see $u(0) = c_1 + c_2 = 0$. We can choose $c_1 = 1/(2\mathrm{i})$, $c_2 = -1/(2\mathrm{i})$ and write

$$u_1(x) = \sin\sqrt{\lambda}x.$$

Now, from $\lim_{x\to\infty} u(x)$, we see that since $x > 0$, we need to choose either the upper or lower half complex plane for the range of $\sqrt{\lambda}$, because both terms can not converge on both half planes. Let us assume that $\sqrt{\lambda}$ is defined in the upper half plane, so its imaginary component is positive. Then we see that $c_2 = 0$ and so

$$u_2(x) = \mathrm{e}^{\mathrm{i}\sqrt{\lambda}x}.$$

Since

$$
\begin{aligned}
W(y) &= u_1(y)u_2'(y) - u_2(y)u_1'(y) \\
&= \sin\sqrt{\lambda}y\ \mathrm{i}\sqrt{\lambda}\mathrm{e}^{\mathrm{i}\sqrt{\lambda}y} - \mathrm{e}^{\mathrm{i}\sqrt{\lambda}y}\sqrt{\lambda}\cos\sqrt{\lambda}y \\
&= \sqrt{\lambda}\mathrm{e}^{\mathrm{i}\sqrt{\lambda}y}(\mathrm{i}\sin\sqrt{\lambda}y - \cos\sqrt{\lambda}y) \\
&= \sqrt{\lambda}\mathrm{e}^{\mathrm{i}\sqrt{\lambda}y}(-\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}y}) \\
&= -\sqrt{\lambda}.
\end{aligned}
$$

$$
C(y) = \frac{1}{-(-\sqrt{\lambda})} = \frac{1}{\sqrt{\lambda}}.
$$

Then the Green function is given by

$$
G(x,y) = \begin{cases} \frac{\sin\sqrt{\lambda}x\ \mathrm{e}^{\mathrm{i}\sqrt{\lambda}y}}{\sqrt{\lambda}} & 0 \le x \le y < \infty \\ \\ \frac{\sin\sqrt{\lambda}y\ \mathrm{e}^{\mathrm{i}\sqrt{\lambda}x}}{\sqrt{\lambda}} & 0 \le y \le x < \infty \end{cases}
$$

.

2.2.2 Along the Imaginary Line

Here, instead of x on the real axis, we think of x in the positive y axis. or $x \in \mathrm{i}[0,\infty)$.

Let us now consider the same operator $L = -u''$ and the eigenvalue problem $L - \lambda$:

$$
-u(\mathrm{i}x)'' = \lambda u(x) \quad , \quad u(0) = 0 \quad , \quad \lim_{x\to\infty} u(x) = 0. \tag{2.5}
$$

where this time we consider the domain in the interval $\mathrm{i}[0,\infty)$ on the positive y axis.

The general solution of this equation is of the form

$$
u(x) = c_1\mathrm{e}^{\mathrm{i}\sqrt{\lambda}x} + c_2\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}x}
$$

We apply first the boundary conditions.

Now, from $\lim_{x\to\infty} u(x)$, we see that since $x = \mathrm{i}y$ with $y > 0$ and we are choosing $\sqrt{\lambda} > 0$, $c_2 = 0$, so that $u(x)$ is in $L^2[0,\infty)$. We then have

$$u_2(x) = \mathrm{e}^{\mathrm{i}\sqrt{\lambda}x}.$$

Since

$$\begin{aligned} W(y) &= u_1(y)u_2'(y) - u_2(y)u_1'(y) \\ &= (\mathrm{e}^{\mathrm{i}\sqrt{\lambda}y} - \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}y})\mathrm{i}\sqrt{\lambda}\mathrm{e}^{\mathrm{i}\sqrt{\lambda}y} - \mathrm{e}^{\mathrm{i}\sqrt{\lambda}y}[(\mathrm{i}\sqrt{\lambda}(\mathrm{e}^{\mathrm{i}\sqrt{\lambda}y} + \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}y})] \\ &= -2\mathrm{i}\sqrt{\lambda} \end{aligned}$$

$$C(y) = \frac{1}{-(-2\mathrm{i}\sqrt{\lambda})} = \frac{1}{2\mathrm{i}\lambda}$$

Then the Green function is given by

$$G(x,y) = \begin{cases} \frac{(\mathrm{e}^{\mathrm{i}\sqrt{\lambda}x} - \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}x})\mathrm{e}^{\mathrm{i}\sqrt{\lambda}y}}{2\mathrm{i}\sqrt{\lambda}} & 0 \le x \le y < \infty \\ \frac{(\mathrm{e}^{\mathrm{i}\sqrt{\lambda}y} - \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}y})\mathrm{e}^{\mathrm{i}\sqrt{\lambda}x}}{2\mathrm{i}\sqrt{\lambda}} & 0 \le y \le x < \infty \end{cases}$$

or

$$G(x,y) = \begin{cases} \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}(x+y)} - \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}(x-y)}}{2\mathrm{i}\sqrt{\lambda}} & 0\mathrm{i} \le x \le y < \infty\mathrm{i} \\ \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}(x+y)} - \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}(y-x)}}{2\mathrm{i}\sqrt{\lambda}} & 0\mathrm{i} \le y \le x < \infty\mathrm{i} \end{cases}$$

which can be written in a more compact way as

$$G(x,y) = \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}(x+y)} - \mathrm{e}^{\mathrm{i}\sqrt{\lambda}|x-y|}}{2\mathrm{i}\sqrt{\lambda}}.$$

for x and y in the positive imaginary axis.

2.3 Infinite Boundary with no end points

Let us now consider the same operator $L = -u''$ and the eigenvalue problem $L - \lambda$:

$$-u'' = \lambda u \quad , \quad \lim_{x \to \pm\infty} u(x) = 0. \tag{2.6}$$

where this time we consider the domain in the interval $(-\infty, \infty)$.

The general solution of this equation is of the form

$$u(x) = c_1 \mathrm{e}^{\mathrm{i}\sqrt{\lambda}x} + c_2 \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}x}$$

Given a reference point y, we consider two branches, the points $x < y$, and those $x > y$. For the points x such that $x < y$, since x can approach $-\infty$, the function converges to zero only if $c_1 = 0$. Likewise for the second branch of points $x > y$, we need $c_2 = 0$, otherwise the function $u(x)$ diverges. We then can form the Green function

$$G(x,y) = \begin{cases} c_2 \mathrm{e}^{\mathrm{i}\sqrt{\lambda}x} & x < y \\ \\ c_1 \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}x} & x > y \end{cases}$$

We need to find c_1 and c_2. This coefficients can be found using the two conditions: continuity of the function and discontinuity of the first derivative.

$$\begin{aligned} c_2 \mathrm{e}^{\mathrm{i}\sqrt{\lambda}y} &= c_1 \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}y} \quad , \quad \text{continuity at } y \\ -c_1 \mathrm{i}\sqrt{\lambda}\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}y} - c_2 \mathrm{i}\sqrt{\lambda}\mathrm{e}^{\mathrm{i}\sqrt{\lambda}y} &= 1 \quad , \quad \text{discontinuity of } G_x \text{ at } y \end{aligned}$$

From the first equation $c_1 = c_2 \mathrm{e}^{2\mathrm{i}\sqrt{\lambda}y}$, and from this in the second equation

$$-c_2 \mathrm{i}\sqrt{\lambda}\mathrm{e}^{\mathrm{i}\sqrt{\lambda}y} - c_2 \mathrm{i}\sqrt{\lambda}\mathrm{e}^{\mathrm{i}\sqrt{\lambda}y} = 1$$

That is,

$$\begin{aligned} c_2 &= -\frac{1}{2\mathrm{i}\sqrt{\lambda}\mathrm{e}^{\mathrm{i}\sqrt{\lambda}y}} \\ c_1 &= -\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}y}}{2\mathrm{i}\sqrt{\lambda}}. \end{aligned}$$

Then the Green function is

$$G(x,y) = \begin{cases} -\frac{e^{i\sqrt{\lambda}(x-y)}}{2i\sqrt{\lambda}} & x < y \\ \\ -\frac{e^{-i\sqrt{\lambda}(x-y)}}{2i\sqrt{\lambda}} & x > y \end{cases}$$

or in more compact form

$$G(x,y) = -\frac{e^{-i\sqrt{\lambda}|x-y|}}{2i\sqrt{\lambda}}$$

Chapter 3

Operator $L = -x(xu')'$

3.1 Infinite Boundary with end point

Boundary conditions are $u(0) = 0$, and $\lim_{x\to\infty} u(x) = 0$.

The spectral equation is

$$Lu - \lambda u = -x(xu')' - \lambda u = 0.$$

We can expand the previous expression to

$$Lu - \lambda u = -x^2 u'' - xu' - \lambda u = 0. \tag{3.1}$$

There is not a method for solving general second order linear ODEs. If the coefficients are constant the solution is easy by mapping the problem to the solution of a quadractic (characteristic) equation. If one solution is known then the system can be reduced to a first order ODE which can be solved using the integrating factor [1].

Before we can find the Green function we need to find the two solutions for the homogeneous equation $Lu - \lambda u = 0$. Equation 3.1 corresponds to a general equation

$$x^2 u'' + axu' + bu = 0. \tag{3.2}$$

[1]https://en.wikipedia.org/wiki/Integrating_factor

known as the Cauchy-Euler equation [2] which we solve in Appendix A. Here $a = 1$, and $b = \lambda$.

We build the characteristic equation

$$p^2 + (1-a)p + b = p^2 + \lambda^2 = 0,$$

with roots $p = \pm\mathrm{i}\lambda$ and so the homogeneous solutions are

$$\begin{aligned} u_1(x) &= x^{\mathrm{i}\sqrt{\lambda}} = \mathrm{e}^{\mathrm{i}\sqrt{\lambda}\ln x} \\ u_2(x) &= x^{-\mathrm{i}\sqrt{\lambda}} = \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}\ln x}. \end{aligned}$$

To construct the Green function we need to find two linearly independent solutions that satisfy the boundary conditions. The general solution has the form

$$u(x) = c_1\mathrm{e}^{\mathrm{i}\sqrt{\lambda}\ln x} - c_2\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}\ln x}$$

We need to guarantee convergence. If we assume that $\sqrt{\lambda}$ is defined in the upper half complex plane (positive imaginary part) then as $x \to 0$, the first term goes to ∞ and the second goes to 0. Then we need $c_1 = 0$. So we choose $u_1(x) = C\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}\ln x}$. Now, for the boundary condition as $x \to \infty$. Choose a reference point $\xi > 0$, and assume $x > \xi$. As $x \to \infty$, in the general homogeneous solution we need to remove the second term so that we want $u_2(x) = C\mathrm{e}^{\mathrm{i}\sqrt{\lambda}\ln x}$. That is, we build the Green function based upon

$$G(x,\xi,\lambda) = \begin{cases} C\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}\ln x}\mathrm{e}^{\mathrm{i}\sqrt{\lambda}\ln\xi} & 0 < x < \xi < \infty \\ C\mathrm{e}^{\mathrm{i}\sqrt{\lambda}\ln\xi}\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}\ln x} & 0 < \xi < x < \infty, \end{cases}$$

and to find C we use the discontinuity condition on the first derivative. That is, from equation 2.1

$$C(y) = \frac{1}{a(y)W(y)}$$

[2] https://en.wikipedia.org/wiki/Cauchy%E2%80%93Euler_equation

with $a(y) = -y^2$, and

$$W(y) = \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}\ln y}\frac{\mathrm{i}\sqrt{\lambda}}{y}\,\mathrm{e}^{\mathrm{i}\sqrt{\lambda}\ln y} + \mathrm{e}^{\mathrm{i}\sqrt{\lambda}\ln y}\frac{\mathrm{i}\sqrt{\lambda}}{y}\,\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}\ln y} = \frac{2\mathrm{i}\sqrt{\lambda}}{y}.$$

Hence

$$C = -\frac{1}{2\mathrm{i}\sqrt{\lambda}y}\frac{\mathrm{i}}{2\sqrt{\lambda}y}$$

and then using $C = C(\xi)$,

$$G(x,\xi,\lambda) = \begin{cases} \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}\ln x}\mathrm{e}^{\mathrm{i}\sqrt{\lambda}\ln \xi}}{2\sqrt{\lambda}\xi} & 0 < x < \xi < \infty \\ \\ \frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\sqrt{\lambda}\ln \xi}\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}\ln x}}{2\sqrt{\lambda}\xi} & 0 < \xi < x < \infty, \end{cases}$$

which could be also written as

$$G(x,\xi,\lambda) = \begin{cases} \frac{\mathrm{i}}{2\sqrt{\lambda}\xi}\left(\frac{\xi}{x}\right)^{\mathrm{i}\sqrt{\lambda}} & 0 < x < \xi < \infty \\ \\ \frac{\mathrm{i}}{2\sqrt{\lambda}\xi}\left(\frac{x}{\xi}\right)^{\mathrm{i}\sqrt{\lambda}} & 0 < \xi < x < \infty. \end{cases}$$

3.2 References

Bibliography

[1] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, 1922.

Appendices

Appendix A

Solution of the Cauchy-Euler equation

In this appendix we solve the Cauchy-Euler equation 3.2

$$x^2u'' + axu' + bu = 0.$$

From the observation that the second derivative has a second order monomial x^2, the first derivative a first order monomial x, and the dependent term has a zero order monomial it make sense that a soluton of the type $u(x) = x^p$ would fit. Let us first try this solution. We find,

$$x^2(p)(p-1)x^{p-2} + axpx^{p-1} + bx^p = 0.$$

That is, we need to solve a second order polynomial equation

$$p^2 + (a-1)p + b = 0.$$

The solution to this algebraic equation is

$$p = \frac{(1-a) \pm \sqrt{(a-1)^2 - 4b}}{2}$$

There are three possibilities, according to the discriminant $d = (a-1)^2 - 4b$.

(i) $d > 0$. Then we have two real roots r_1, r_2 and the general solution is $c_1 x^{r_1} + c_2 x^{r_2}$.

(ii) $d < 0$. We have two complex solutions providing a general solution of the form $c_1 x^{\frac{1}{2}[(1-a)+\mathrm{i}\sqrt{d}]} + c_2 x^{\frac{1}{2}[(1-a)-\mathrm{i}\sqrt{d}]}$.

(iii) $d = 0$. Then we need to reduce the order of the equation from 2 to 1.

Let us assume that $u_1(x)$ is a solution of the equation 3.2. We construct a new solution $u_2(x) = v(x)u_1(x)$. If we substitute this back into equation 3.2 we find

$$\begin{aligned}
0 &= x^2[v''(x)u_1(x) + 2v'(x)u_1'(x) + v(x)u_1''(x)] + \\
&\quad ax[v'(x)u_1(x) + v(x)u_1'(x)] + bv(x)u_1(x) \\
&= v(x)[x^2u_1''(x) + axu_1'(x) + bu_1(x)] + \\
&\quad x^2v''(x)u_1(x) + 2x^2v'(x)u_1'(x) + axv'(x)u_1(x) \\
&= x^2v''(x)u_1(x) + 2x^2v'(x)u_1'(x) + axv'(x)u_1(x) \\
&= x^2v''(x)u_1(x) + v'(x)[2x^2u_1'(x) + axu_1(x)].
\end{aligned}$$

Now, since $u_1(x) = x^{\frac{1-a}{2}}$ we see that the last equation turns into

$$\begin{aligned}
0 &= x^{2+\frac{1-a}{2}}v''(x) + \left[\cancel{2}\frac{1-a}{\cancel{2}}x^{1+\frac{1-a}{2}} + ax^{1+\frac{1-a}{2}}\right]v'(x) \\
&= x^{2+\frac{1-a}{2}}v''(x) + x^{1+\frac{1-a}{2}}v'(x).
\end{aligned}$$

If we assume further that $x \neq 0$, we can divide by $x^{1+\frac{1-a}{2}}$ to find

$$0 = xv''(x) + v'(x).$$

We now see that this is a differential equation of second order on $v(x)$, but that there is no term on $v(x)$ so we can make the substitution $w(x) = v'(x)$ and write

$$0 = xw'(x) = \frac{d}{dx}[xw(x)].$$

This means that $xw(x) = C$ for a constant C. That is

$$w(x) = \frac{C}{x},$$

and since $v'(x) = w(x)$ we find that

$$v(x) = C \ln x$$

and we can write the second solution as

$$u_2(x) = \ln x x^{\frac{1-a}{2}}.$$

Then the general solution of the homogeneous equation, for this case, has the form

$$u(x) = c_1 x^{\frac{1-a}{2}} + c_2 \ln x \, x^{\frac{1-a}{2}}.$$

It is interesting that there is a very different way to solve the Cauchy-Euler equation. It can be shown that with the substitution $x = \mathrm{e}^t$, the Cauchy-Euler equation 3.2 becomes the equation

$$\frac{d^2u}{dt^2} + (a-1)\frac{dy}{dt} + by(t) = 0,$$

where now the coefficients are constant and the solution of the differential equation is obtained by solving the characteristic (algebraic) equation $r^2 + (a-1)r + b = 0$.

Appendix B

Bessel Functions

The Bessel functions [1] are solutions of the Bessel differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{B.1}$$

with $\nu \in \mathbb{C}$. This equation is known as the Bessel's differential equation. According to the Wikipedia page with the link above Daniel Bernoulli was the first person who defined them and Friedrich Bessel generalized them.

The Bessel differential equation results when doing separation of variables in the finding of the solution of differential operators where the Laplacian operator is present, in problems where cylindrical or spherical symmetry are encountered.

There are many representations found in the literature for Bessel functions. For example as integrals, asymptotic, recursive, etc. There are also many properties of the Bessel functions. Here we will only find series representations since they are the most relevant to evaluate Green functions for Bessel type equations.

We observe that the equation is singular at $x = 0$. The most common method to solve this differential equation is the Frobenius method [2] where we assume that the solution is analytic and then it can be expanded into an infinite series. This is not the only way to solve this equation. J. B. McLeod shows some disadvantages of the Frobenius method and proposes a different way to solve the Bessel differential equation. Here we will follow the tradition and use the Frobenius method.

[1] https://en.wikipedia.org/wiki/Bessel_function

[2] https://en.wikipedia.org/wiki/Frobenius_method

Since the ODE is of second order it should have two linearly independent solutions. The solutions are known as the Bessel function of first kind $J_\nu(x)$ and the Bessel function of the second kind $Y_\nu(x)$. The general solution is then given by

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x).$$

The Frobenius method assumes that the singular points (where the coefficients are undefined or infinite) are regular. That is regular singular [3] points. This means that after normalizing the coefficient of y'', the coefficients of the y' and y terms have at most poles of order 1 and 0 respectively.

The proof of the Frobenius method is beyond the scope of these notes. We observe that the Bessel equation has analytic coefficients in the whole $\mathbb{C}$ plane except at $x = 0$, which is a regular singular point.

The Frobenius method assumes that a solution can be written as an finite series in the form

$$y(x) = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{r+k}. \tag{B.2}$$

where r is a real number. The idea here is to plug this expression into the differential equation and find some recursion formula to compute the coefficients a_k, which will render the solution. We have then that

[3] https://en.wikipedia.org/wiki/Regular_singular_point

$$\begin{aligned}
0 &= x^2\sum_{k=0}^{\infty}(r+k)(r+k-1)a_kx^{r+k-2} + x\sum_{k=0}^{\infty}a_k(r+k)x^{r+k-1} \\
&+ (x^2-\nu^2)\sum_{k=0}^{\infty}a_kx^{r+k} \\
&= \sum_{k=0}^{\infty}(r+k)(r+k-1)a_kx^{r+k} + \sum_{k=0}^{\infty}a_k(r+k)x^{r+k} \\
&+ \sum_{k=0}^{\infty}a_kx^{r+k+2} - \nu^2\sum_{k=0}^{\infty}a_kx^{r+k} \\
&= \sum_{k=0}^{\infty}(r+k)(r+k-1)a_kx^{r+k} + \sum_{k=0}^{\infty}a_k(r+k)x^{r+k} \\
&+ \sum_{k=2}^{\infty}a_{k-2}x^{r+k} - \nu^2\sum_{k=0}^{\infty}a_kx^{r+k} \\
&= a_0x^r[r(r-1)+r-\nu^2] + a_1x^{r+1}[r(r+1)+(r+1)-\nu^2] \\
&+ \sum_{k=2}^{\infty}[a_k[(r+k)^2-\nu^2]+a_{k-2}]x^{r+k} \\
&= a_0x^r(r^2-\nu^2) + a_1x^{r+1}[(r+1)^2-\nu^2] + \sum_{k=2}^{\infty}[a_k[(r+k)^2-\nu^2]+a_{k-2}]x^{r+k}.
\end{aligned}$$

We assume that $a_0 \neq 0$. We consider two cases: $\nu \neq 1/2$ and $\nu = 1/2$.

B.1 $\nu \neq 1/2$:

We soon will see why $\nu = 1/2$ is a special value.

- **From the coefficient of x^r:** $r = \pm\nu$. One solution chooses $r = \nu$ and the other $r = -\nu$.

- **From the coefficient of x^{r+1}:** We find that $a_1((r+1)^2-\nu^2) = 0$, and since $r = \pm\nu$ we see that $a_1(\nu^2 \pm 2\nu + 1 - \nu^2) = 0$. Or $a_1(\pm 2\nu + 1) = 0$. So either $a_1 = 0$, or $\nu = 1/2$. But here we assume $\nu \neq 1/2$ and consider later the case $\nu = 1/2$.

- **From the coefficient of x^{r+k}:**

$$a_k = -\frac{a_{k-2}}{(r+k)^2 - \nu^2} \quad , \quad k \geq 2.$$

We observe that, since $a_1 = 0$, all odd terms $a_{2n+1} = 0$.

Let us now consider the two cases of $r = \pm\lambda$.

(i) $r = \lambda$:

$$a_{2k} = -\frac{a_{2k-2}}{(\nu+2k)^2 - \nu^2} = -\frac{a_{2k-2}}{4k\nu + 4k^2} = -\frac{a_{2k-2}}{4k(\nu+k)}.$$

Starting from the bottom we see that

$$\begin{aligned}
a_2 &= -\frac{a_0}{4\nu+4} = -\frac{a_0}{4(\nu+1)} \\
a_4 &= -\frac{a_2}{4\cdot 2(\nu+1)} = \frac{(-1)^2\, a_0}{4^2\cdot 2(\nu+2)(\nu+1)} \\
a_6 &= -\frac{a_4}{4\cdot 3(\nu+3)} = \frac{(-1)^3 a_0}{4^3\ 3\cdot 2(\nu+3)(\nu+2)(\nu+1)} \\
&\vdots \\
a_{2k} &= \frac{(-1)^k a_0\, \Gamma(\nu+1)}{2^{2k}\ k!\, \Gamma(\nu+k+1)}.
\end{aligned}$$

We are free to give a_0 any value we want since it is a constant. Then we assign $a_0 = 1/\Gamma(\nu+1)$, and write the series B.2 as

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu} . \tag{B.3}$$

This important function is known as the *Bessel function of first kind.* [4]

[4] http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html

(ii) $r = -\lambda$: The same development above would lead to the formula

$$J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-\nu + k + 1)} \left(\frac{x}{2}\right)^{2k-\nu}, \tag{B.4}$$

which is the same as B.3 after replacing ν by $-\nu$.

We will find soon that for $\nu = n \in \mathbb{Z}$ the functions J_n and J_{-n} are not linearly independent, but before, let us attack the case of $\mu = 1/2$. s Figure B.1 shows the first 3 Bessel functions for $\nu = 0, 1, 2$.

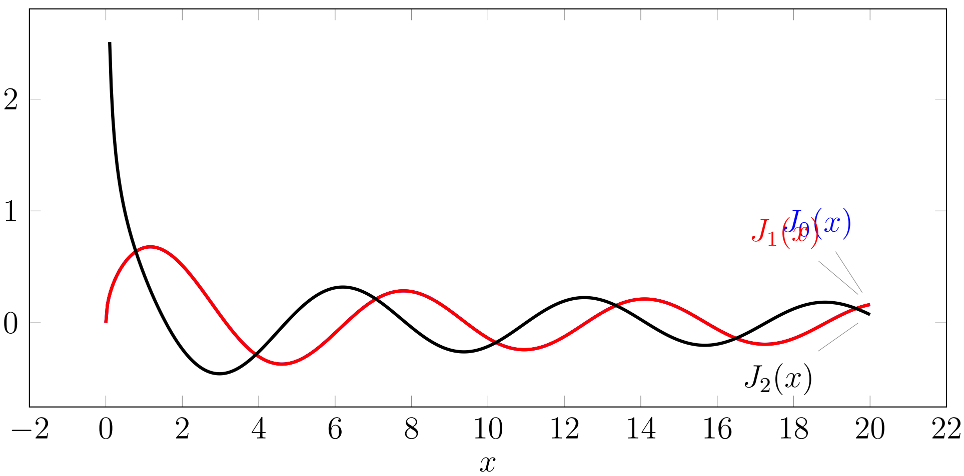

Figure B.1: Bessel functions J_i, for $i = 0, 1, 2$.

B.2 $\nu = 1/2$:

If $\nu = 1/2$ then we have the ODE

$$Ly = x^2 y'' + x\,y' + \left(x^2 - \frac{1}{4}\right) y = 0.$$

We start again from scratch by assuming a solution of the form

$$y = \sum_{k=0}^{\infty} a_k x^{k+r},$$

which after being inserted back into the equation yields

$$Ly = \sum_{k=0}^{\infty} a_k \left((k+r)(k+r-1) + (k+r) - \frac{1}{4} \right) x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r+2} = 0.$$

That is,

$$\begin{aligned} a_0 \left(r^2 - \frac{1}{4} \right) x^r + a_1 \left((1+r)^2 - \frac{1}{4} \right) x^{r+1} & \\ + \sum_{k=2}^{\infty} \left[a_k \left((k+r)^2 - \frac{1}{4} \right) + a_{k-2} \right] x^{k+r} &= 0 \end{aligned} \tag{B.5}$$

with $k \geq 2$. From the first term of this sum we find that (since $a_0 \neq 0$) $r = \pm 1/2$. Let us consider these two cases.

(i) $r = 1/2$:

Here we have that the first term evaluates to 0 and the second term to $2a_1$ so that $a_1 = 0$ and all odd powers are zero as before. We now find the recursion formula to evaluate all a_k, for $k \geq 2$. This is

$$\begin{aligned} a_k \left((k+1/2)^2 - \frac{1}{4} \right) + a_{k-2} &= 0 \quad \text{now, multiply by 4} \\ a_k[(2k+1)^2 - 1] + 4a_{k-2} &= 0 \end{aligned}$$

or

$$a_k = -\frac{4a_{k-2}}{(2k+1)^2 - 1} = -\frac{4a_{k-2}}{4k(k+1)} = -\frac{a_{k-2}}{k(k+1)} \quad , \quad k \geq 2.$$

Starting from the bottom we have that

$$\begin{aligned} a_2 &= -\frac{a_0}{3 \cdot 2} \\ a_4 &= -\frac{a_2}{4 \cdot 4} = \frac{(-1)^2\, a_0}{5!} \\ &\vdots \\ a_{2k} &= \frac{(-1)^k\, a_0}{(2k+1)!}. \end{aligned}$$

Then

$$y_1(x) = x^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} = x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \tag{B.6}$$

where we recognize the last sum as the Taylor representation for the $\sin x$ function. That is

$$y_{1/2}(x) = x^{-\frac{1}{2}} \sin x.$$

(ii) $r = -1/2$. We return to the sum B.5 where, after replacing r by $-1/2$, the first two terms are 0 and a_0, a_1 are free parameters. Considering the other terms for $k \geq 2$ we see that

$$a_k \left((k - 1/2)^2 - \frac{1}{4} \right) + a_{k-2} = 0$$

That is, after multiplying by 4 and simplifying:

$$4\, a_k\, k(k-1) + 4a_{k-2} = 0,$$

so we find the recursion

$$a_k = -\frac{a_{k-2}}{k(k-1)}.$$

Let us find the sequence a_{2k}, $k = 0, 1, 2, \ldots$.

$$\begin{aligned} a_2 &= -\frac{a_0}{2!} \\ a_4 &= -\frac{a_2}{4 \cdot 3} = (-1)^2 \frac{a_0}{4!} \\ &\vdots \\ a_{2k} &= (-1)^k \frac{a_0}{(2k)!}, \end{aligned}$$

$$\begin{aligned} a_3 &= -\frac{a_1}{3!} \\ a_5 &= -\frac{a_3}{5 \cdot 4} = (-1)^2 \frac{a_1}{5!} \\ &\vdots \\ a_{2k+1} &= (-1)^k \frac{a_0}{(2k+1)!}, \end{aligned}$$

Then we find

$$y_{-1/2}(x) = a_0 x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + a_1 x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1},$$

or

$$y_{-1/2}(x) = x^{-\frac{1}{2}} (a_0 \sin x + a_1 \cos x)$$

We observe that $y_{1/2}(x)$ and $y_{-1/2}(x)$ are linearly independent solutions, so they span the whole space of solutions for $\nu = 1/2$ in the ODE. Why didn't we call $y_{1/2} = J_{1/2}$ and the like for $-1/2$? We observe that $y_{-1/2}$ is a combination of a sine and a cosine function. The sine function $sin(x)$ was already counted in $y_{1/2}(x)$ so we do not need that here. Having $a_0 = 0$ will provide a good solution which is linearly independent with $y_{1/2}(x)$. Then it seems natural, assuming $a_0 = 1$ in the first case and $a_0 = 0$ in the second with $a_1 = 1$ to have

$$\begin{aligned} J_{1/2}(x) &\stackrel{?}{=} x^{\frac{1}{2}} \sin x \\ J_{-1/2}(x) &\stackrel{?}{=} x^{-\frac{1}{2}} \cos x. \end{aligned}$$

These two functions are linearly independent and satisfy the Bessel equation B.1. However still the names above are not the ones used in the literature. We explain this next.

Please observe equation B.3. It is intersting that while in the derivation of this equaton, we assumed that $\nu \neq 1/2$, there is nothing that stop us from inserting $\nu = 1/2$ into this equation. That is, we write

$$J_{1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(1/2 + k + 1)} \left(\frac{x}{2}\right)^{2k+1/2}. \tag{B.7}$$

Then use properties of the $\Gamma(z)$ function to reduce this equation. That is, we observe that

$$\Gamma(1/2 + k + 1) = \left(k + \frac{1}{2}\right) \cdot \left(k - \frac{1}{2}\right) \cdot \dots \cdot \frac{3}{2} \cdot \Gamma\left(\frac{3}{2}\right).$$

However we know that $\Gamma(3/2) = \sqrt{\pi}/2$ (please see my notes on the Gamma and Beta functions.) [5] Then we have that

$$\begin{aligned} \Gamma(1/2 + k + 1) &= \frac{(2k+1) \cdot (2k-1) \cdot \dots \cdot 3 \cdot 1}{2^k} \frac{\sqrt{\pi}}{2} \\ &= \frac{(2k+1)!\sqrt{\pi}}{2^k 2^k k!\, 2} \\ &= \frac{(2k+1)!\sqrt{\pi}}{2^{2k+1} k!}. \end{aligned}$$

Then we substitute this equation into equation B.7. To find

[5] https://drive.google.com/open?id=0B4W-gdhbNpsDaTNvbF9VcGgyR1E

$$J_{1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} \cancel{k!}}{\cancel{k!}\,(2k+1)!\sqrt{\pi}} \left(\frac{x}{2}\right)^{2k+1/2} = \sqrt{\frac{2}{\pi}} x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

now compare this last equation with equation B.6 and observe that $J_{1/2}(x) = \sqrt{2/\pi} y_1(x)$; so up to a scaling factor, the two solutions are the same. We have then, by definition that

$$J_{1/2}(x) \equiv \sqrt{\frac{2}{\pi x}} \sin x.$$

In the same fashion, if $\lambda = -1/2$ we would find that the solution above after changing the since function for $\cos x$ would satisfy the Bessel equation B.1 wiht $\lambda = -1/2$. That is, we have

$$J_{-1/2}(x) \equiv \sqrt{\frac{2}{\pi x}} \cos x.$$

Figure B.2 shows the functions $J_{1/2}$ and $J_{-1/2}$ together with their envelopes.

The functions $J_{1/2}(x)$ and $J_{-1/2}(x)$ are linearly independent. We show that if ν is not an integer $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent. In the next section we will see that if ν is an integer then J_ν and $J_{-\mu}$ are linearly dependent.

The proof shown here is taken from [1], 1922 book on Theory of Bessel functions. [6]Theory of Bessel functions.

Recall the definition of Wronskian in equation 2.2. We show that if the wronskian of $u_1(x)$, $u_2(x)$ is non-zero at any point of the domain, the functions $u_1(x)$ and $u_2(x)$ are linearly independent.

Let us assume that there are two coefficients c_1, c_2 such that

$$c_1 u_1(x) + c_2 u_2(x) = 0.$$

Let us take the derivative of this equation. That is,

[6]http://tinyurl.com/gs2oc3e

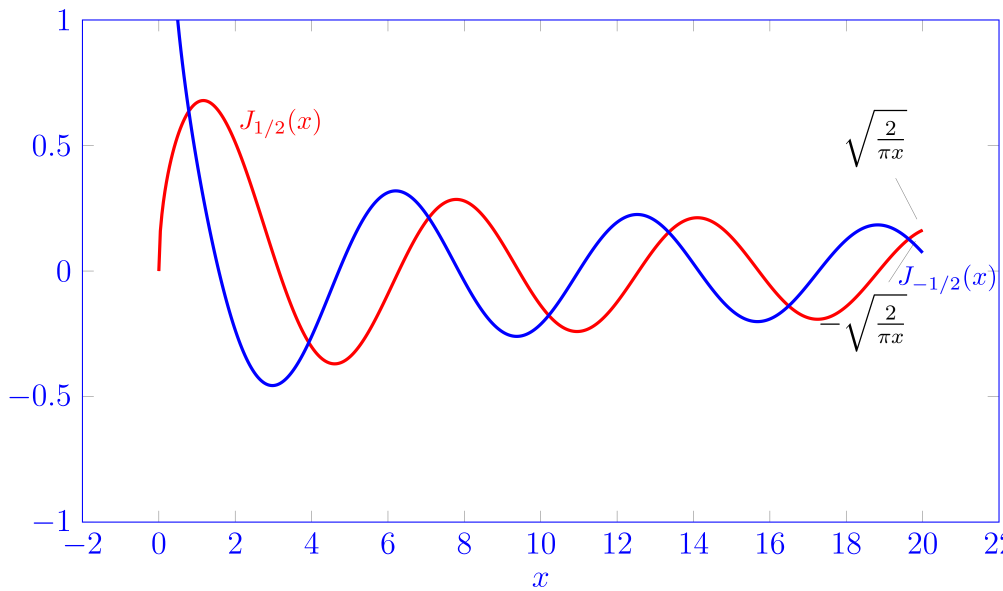

Figure B.2: Bessel functions $J_{1/2}$ and $J_{-1/2}$ and their envelope curves $\pm\sqrt{\frac{2}{\pi x}}$.

$$c_1 u_1'(x) + c_2 u_2'(x) = 0.$$

The two equations above can be written in a matrix form as

$$\begin{pmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The Wronskian happens to be the determinant of this matrix. If that determinant is non-zero, this means that $c_1 = c_2 = 0$, and this means that the functions $u_1(x)$ and $u_2(x)$ are linearly independent.

We now compute the Wronskian of $J_\nu(x)$ and $J_{-\nu}(x)$. Let us call $L_\mu = L - \mu$ the Bessel equation. We can multply the equations $L_\mu J_\mu(x) = 0$, and $L_\mu J_{-\mu} = 0$ by $J_{-\mu}$ and J_μ respectively and subtract the results. Then we find

$$\begin{aligned}
J_{-\mu}(x)L_\mu[J_\mu(x)] - J_\mu(x)\,L_\mu[J_{-\mu}(x)] &= x^2 J_{-\mu}(x)J''_\mu(x) + xJ_{-\mu}(x)J'_\mu(x) + \\
&\quad \cancel{(x^2-\nu^2)J_{-\mu}(x)J_\mu(x)} - \\
&\quad x^2 J_\mu(x)J''_{-\mu}(x) - xJ_\mu(x)J'_{-\mu}(x) \\
&- \cancel{(x^2-\nu^2)J_\mu(x)J_{-\mu}(x)} \\
&= 0.
\end{aligned}$$

That is, assuming $x \neq 0$,

$$xJ_{-\mu}(x)J''_\mu(x) + J_{-\mu}(x)J'_\mu(x) - xJ_\mu(x)J''_{-\mu}(x) - J_\mu(x)J'_{-\mu}(x) = 0,$$

or

$$\frac{d}{dx}\,x[J_\mu(x)J'_{-\mu}(x) - J_{-\mu}(x)J'_\mu(x)] = 0.$$

Now, from the definition of Wronskian 2.2 we see that

$$\frac{d}{dx}\,x\,W(J_\mu(x), J_{-\mu}(x)] = 0,$$

and so $x\,W(J_\mu(x), J_{-\mu}(x)] = C$ for a constant C. We want to evaluate the constant C, such that

$$W(J_\mu(x), J_{-\mu}(x)] = \frac{C}{x}. \tag{B.8}$$

First, from equations B.3 and B.4 we have that

$$J_\nu(x) = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu [1+\mathcal{O}(x^2)] \quad , \quad J'_\nu(x) = \frac{1}{2\,\Gamma(\nu)}\left(\frac{x}{2}\right)^{\nu-1} [1+\mathcal{O}(x^2)]$$

and similarly, by changing ν by $-\nu$. Then

$$\begin{aligned}
J_\nu(x)J'_{-\nu}(x) &= \left(\frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu [1+\mathcal{O}(x^2)]\right)\left(\frac{1}{2\,\Gamma(-\nu)}\left(\frac{x}{2}\right)^{-\nu-1}[1+\mathcal{O}(x^2)]\right) \\
&= \frac{1}{\Gamma(\nu+1)}\frac{1}{\Gamma(-\nu)}\frac{1}{x}[1+\mathcal{O}(x^2)].
\end{aligned}$$

Likewise

$$J_{-\nu}(x)J'_{\nu}(x) = \left(\frac{1}{\Gamma(-\nu+1)}\left(\frac{x}{2}\right)^{-\nu}[1+\mathcal{O}(x^2)]\right)\left(\frac{1}{2\,\Gamma(\nu)}\left(\frac{x}{2}\right)^{\nu-1}[1+\mathcal{O}(x^2)]\right)$$
$$= \frac{1}{\Gamma(-\nu+1)}\frac{1}{\Gamma(\nu)}\frac{1}{x}[1+\mathcal{O}(x^2)).$$

Hence the Wroskian is

$$W[J_\nu(x), J_{-\nu}(x))] = \left(\frac{1}{\Gamma(\nu+1)}\frac{1}{\Gamma(-\nu)} - \frac{1}{\Gamma(-\nu+1)}\frac{1}{\Gamma(\nu)}\right)\frac{1}{x} + \mathcal{O}(x).$$

By matching coefficients of equal powers on this expression and equation B.8 we observe that

$$C = \left(\frac{1}{\Gamma(\nu+1)}\frac{1}{\Gamma(-\nu)} - \frac{1}{\Gamma(-\nu+1)}\frac{1}{\Gamma(\nu)}\right)$$

We now use the reflection formula [7] for the $\Gamma(z)$ function (please see my notes on the Gamma and Beta functions. for a proof of this formula) [8]

That is, we have

$$C = -\frac{\sin\pi\nu}{\pi} - \frac{\sin\pi\nu}{\pi} = -\frac{2\,\sin\pi\nu}{\pi}.$$

Hence, if $\nu \notin \mathbb{Z}$ then $C \neq 0$, and the Wronskian does not vanish. These means that the two solutions are linearly independent for $\nu \notin \mathbb{Z}$.

B.3 The case of $\nu = n \in \mathbb{Z}$

If ν is an integer n, we show next, that the solutions $J_n(x)$ and $J_{-n}(x)$ solutions are linearly dependent and so we need to look for a second solution to be able to get the general solution of the homogeneous ODE. Let us assume $\nu > 0$, $\nu \in \mathbb{Z}$. Then we have that the Gamma function $\Gamma(-\nu+k+1)$ diverges

[7] https://en.wikipedia.org/wiki/Reflection_formula

[8] https://drive.google.com/open?id=0B4W-gdhbNpsDaTNvbF9VcGgyR1E

any argument equal or smaller than 0, and since this Gamma function is dividing, then the sum B.4 evaluates all terms with negative arguments of the Γ function in 0. More clearly, we have that

$$\frac{1}{\Gamma(k-n+1)} = 0 \quad , \quad \text{for} \quad k = 0, 1, 2, n-1.$$

Hence we can write B.4 as

$$\begin{aligned}
J_{-n}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{2k-n} \\
&= \sum_{k=n}^{\infty} \frac{(-1)^k}{k!\,\Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{2k-n} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{(k+n)!\,\Gamma(\cancel{-n}+k+\cancel{n}+1)} \left(\frac{x}{2}\right)^{2(k+n)-n} \\
&= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+n)!\,\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+n} \\
&= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+n+1)} \left(\frac{x}{2}\right)^{2k+n} \\
&= (-1)^n J_n(x).
\end{aligned}$$

That is, we showed that

$$J_{-n}(x) = (-1)^n J_n(x). \tag{B.9}$$

and so the functions $J_n(x)$ and $J_{-n}(x)$ for $n \in \mathbb{Z}$ are not linearly independent.

This creates an inconvenient situation and mathematicians looked for a linear independent couple of functions which does not depend on the order ν.

One idea is to rewrite equation B.9 as a difference. That is,

$$J_{-\nu}(x) - (-1)^{\nu} J_{\nu}(x) = 0.$$

Since both, J_ν, and $J_{-\nu}$ are solutions of the Bessel equation, their difference is solution as well. Of course this difference is not interesting since for $\nu = n \in \mathbb{Z}$ the difference is the trivial 0. Hankel's idea was to form a quotient so that the function is "regularized". That is Hankel thought about the function

$$\frac{J_\nu(x) - (-1)^n J_{-\nu}(x)}{\nu - n}. \tag{B.10}$$

which looks undefined but if, with the use of the L'Hôpital [9] rule, we can show that the limit as $\nu \to n$ exists and is well defined, we can adopt this new function as a linearly independent solution which works even for $\nu \in \mathbb{Z}$. Since for $\nu \neq n$, the above equation is a solution for the Bessel equation we could think that the limit as $\nu \to n$ is a solution as well. We can not assume that yet. Instead, we will find the limit and then prove that the limit is actually a solution of the Bessel function which is linearly independent of the solution above known as the Bessel function of the first kind B.3. This is the origin of the Bessel function of the second kind.

We take the limit in B.10 as ν approaches n. That is,

$$\begin{aligned}
\lim_{\nu\to n} \frac{J_\nu(x) - (-1)^n J_{-\nu}(x)}{\nu - n} &= \lim_{\nu\to n} \frac{J_\nu(x) - J_\nu(x) + J_\nu(x) - (-1)^n J_{-\nu}(x)}{\nu - n} \\
&= \lim_{\nu\to n} \frac{J_\nu(x) - J_\nu(x) - (-1)^n [J_{-\nu}(x) - J_{-\nu}(x)]}{\nu - n} \\
&= \left[\frac{\partial J_\nu(x)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial \nu}\right]\Bigg|_{\nu=n}.
\end{aligned}$$

Hankel noted this function as

$$Y_n(x) = \left[\frac{\partial J_\nu(x)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial \nu}\right]\Bigg|_{\nu=n}. \tag{B.11}$$

We could suggest the heuristic argument that if $J_\nu(x)$ and $J_{-\nu(x)}$ are not linearly independent because $J_\nu(x) - (-1)^n J_{-\nu}(x) = 0$, then how about differentiating this with respect to ν, and using that as a new solution. We will show that Y_n is a solution of the Bessel equation which is not identically

[9] https://en.wikipedia.org/wiki/L%27H%C3%B4pital%27s_rule

zero. Since $J_{\pm}(\nu)(x)$ are analytic functions of both x and ν we will change the order of differentiations with respect to these two parameters without error. That is, starting with the Bessel equation B.1, and since $J_{\pm\nu}(x)$ are solutions for it, we can differentiate with respect to ν for both $J_{\pm\nu}$, and interchange the order of derivatives. That is, we can say that

$$x^2 \frac{d^2}{dx^2}\frac{\partial J_{\pm\nu}}{\partial\nu} + x\frac{d}{dx}\frac{\partial J_{\pm\nu}}{\partial\nu} + (x^2-\nu^2)\frac{\partial J_{\pm\nu}}{\partial\nu} - 2\nu J_{\pm\nu}(x) = 0.$$

That is, we have the following chain of events:

$$\begin{aligned}
&L\left[\frac{\partial J_\nu(x)}{\partial x}\right] - 2\nu J_{\nu(x)} = 0 \\
&L\left[\frac{\partial J_{-\nu}(x)}{\partial x}\right] - 2\nu J_{-\nu(x)} = 0 \quad , \quad L\left[(-1)^n\frac{\partial J_{-\nu}(x)}{\partial x}\right] - (-1)^n 2\nu J_{-\nu(x)} = 0 \\
&L\left[\frac{\partial J_\nu(x)}{\partial x} - (-1)^n\frac{\partial J_{-\nu}(x)}{\partial x}\right] - 2\nu[J_\nu(x) - (-1)^n J_{-\nu}(x)] = 0.
\end{aligned}$$

We now take the limit as $\nu \to n$, and since the expressions are analytic and $\lim_{\nu\to n} J_\nu(x) - (-1)^n J_{-\nu}(x) = 0$ we find that

$$L\left[\frac{\partial J_n(x)}{\partial x} - (-1)^n\frac{\partial J_{-n}(x)}{\partial x}\right] = Y_n(x) = 0.$$

So, indeed $Y_n(x)$ is a solution of the Bessel equation. We want to find a better characterization of $Y_n(x)$ so that it is not in terms of derivatives.

For example, let us go back to the linear combination

$$J_\nu(x) - (-1)^\nu J_{-\nu}(x). \tag{B.12}$$

We know that each of the two terms $J_{\pm\nu}(x)$ is a solution of Bessel equation and they are linearly independent as long as $\nu \notin \mathbb{Z}$. In the limit as $\nu \to n \in \mathbb{Z}$ we have

$$J_n(x) - (-1)^n J_{-n}(x) = 0.$$

Of course we do not want this trivial solution so we must “regularize” it. First, we observe that $(-1)^n = \cos n\pi$ so we might rewrite expression B.12 as

$$\cos \nu\pi J_\nu(x) - (-1)^\nu J_{-\nu}(x).$$

Then, we need to divide the equation by something that goes to 0 as $\nu \to n$. Why $\sin \nu\pi$ and not just ν? The reason would be clear soon.

If we pick to divide by $\sin \nu\pi$, and use L”Höpital rule to evaluate

$$\lim_{\nu \to n} \cos \nu\pi J_\nu(x) - (-1)^\nu J_{-\nu}(x).$$

we will inmediately find equation B.11, which is what Hankel called $Y_n(x)$.

My notes in Bessel functions [10] show the expansion of the Bessel function of second kind as power series.

[10] https://drive.google.com/open?id=0B4W-gdhbNpsDZzdrUUZzU1F2aU0

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