The Euler's Integrals of the First and the Second Kind

Herman Jaramillo
www.jaramilloherman.com

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Chapter 1

Introduction

If today I google for "Gamma Function" (in quotes) I get 834.000 links. Why do I want to write more about this?

This document started as a curiosity about how the mathematicians of the past came up with the Gamma function. Of course most of the credits go to Euler but he did not work alone. The document starts with the chronological description of how the Gamma function was invented and then a few properties and new developments are included for completeness. We introduce the Beta function as well. Most of the references are in the form to Internet links to other documents which makes navigation in a computer much easier.

I use z and w as the arguments for the $\Gamma(z)$ and $\mathrm{B}(z, w)$ functions since the idea is to look at them in complex domains. The symbols z and w are well accepted notation for complex numbers. Whenever I want to emphasize that the argument is real I write x or t instead of z and y instead of w. and Beta $\mathrm{B}(z, w)$ functions.

I am very grateful with my family (Patricia, Paula Andrea and Maria Alejandra) for their patience. I spend most of my free time at home writing these material.

Chapter 2

History

For the history I will rely partly upon the article Leonhard Euler's Integral: A Historical Profile of the Gamma Function [1] by Philip J. Davis, and the article Why is the gamma function so as it is? [2] by Detlef Gronau. Another interesting source which I consulted for the history of the Gamma function is the article An Elementary [3] Exposition of the Theory of the Gamma Function, by J.L.W.V. Jensen and T.H. Gronwall. The classical book **??** A Course in Modern Analysis [4] can not escape to be referred here for important contributions in its chapter 12 on the $\Gamma(z)$ function.

The Euler Archive is an immense source with copies of the original work (publications, correspondence and translations) of Euler. I Use this often and will post this link several times along my notes.

2.1 Motivation

According to Davis, the $\Gamma(z)$ function started as an interpolation problem. Here are a few questions that motivate the discovery of the $\Gamma(z)$ function.

- Think about the sum of the number $1 + 2 + \cdots + n$. It can be seen that

$$\sigma_1(n) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

[1] http://www.maa.org/sites/default/files/pdf/upload_library/22/Chauvenet/Davis.pdf
[2] http://tmcs.math.klte.hu/Contents/2003-Vol-I-Issue-I/10.5485_TMCS.2003.0008.pdf
[3] https://www.jstor.org/stable/2007272?seq=1#page_scan_tab_contents
[4] https://books.google.com/books?id=ULVdGZmi9VcC

In fact this formula can be generalized to the sum of powers. That is, what is the result of

$$\sigma_j(n) = \sum_{k=1}^{n} k^j \ ?$$

Jacques Bernoulli (among other people) worked hard on this problem. He obtained some closed formulas. Here there is a link [5] of a copy of Bernoulli's original publication.

My notes [6] [3] show an comprehensive analysis about Bernoulli's work.

Now the interpolation question. If we want to add from 1 to 3, we find $1 + 2 + 3 = 6$. How about adding the first powers of a number from 1 to 2.5?, or adding any other powers from from 1 to 2.5? In this case Bernoulli's formulas will do the work. That is, while the symbols

$$\sum_{k=1}^{2.5} k,$$

do not make sense, the symbols

$$\left.\frac{n(n+1)}{2}\right|_{n=2.5} = \frac{(2.5)(3.5)}{2} = 4.375$$

make sense. In other words,

$$1 + 2 < 4.375 < 1 + 2 + 3, \quad \text{or} \quad 3 < 4.375 < 6.$$

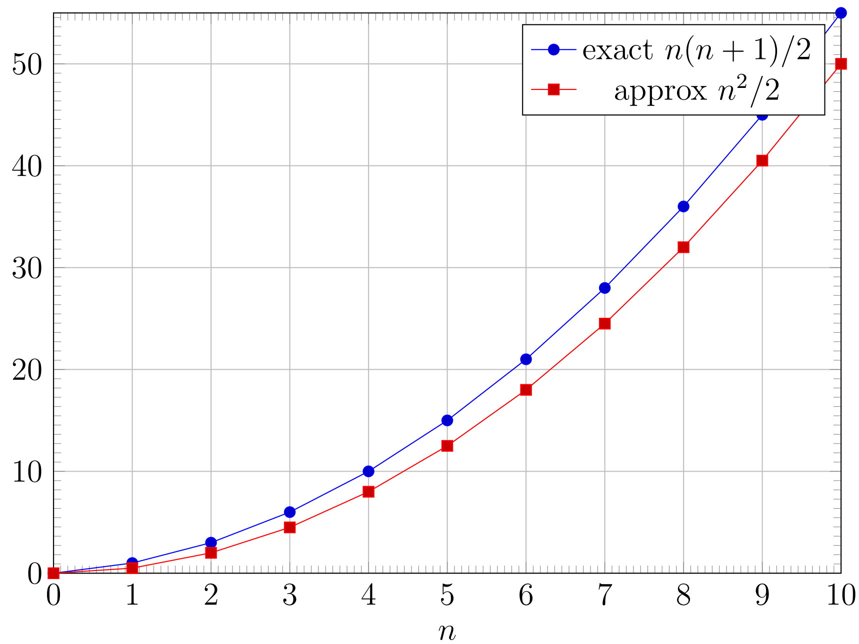
In this sense the quadratic formula $n(n + 1)/2$ is an interpolation formula for the discrete sum of n consecutive numbers starting at 1. But it is more than just having numbers sandwiched in between. The interpolation function is a convex 4.13 function.

Can we say that the integral is a better interpolator? That is, after approximating

$$\sum_{i=0}^{2.5} i \approx \int_0^{2.5} x \, dx \ ?$$

[5] http://en.wikipedia.org/wiki/File:JakobBernoulliSummaePotestatum.png
[6] https://drive.google.com/open?id=0B4W-gdhbNpsDYUxFcWY4N1ZiVzg

Figure 2.1: Triangular numbers: $n(n+1)/2$, for $n = 1, 2 \cdots 10$



Instead of 4.375, we say that the interpolation should be $2.5^2/2 = 3.12$? The answer is no. The integral is defined along the continuous axis and in this sense seems to be the ideal interpolator. However, the integral does not preserve the values of the discrete sum. It is an entire different function, although asymptotically (up to the leading order which in this case is quadratic) is a good approximation, but fails in the lower orders. This is because the truncation at each discrete space (a small rounded triangle lost for each step if the Riemann sum is considered under the curve, or a small rounded triangle is extra–added if the Riemann sum is considered above the curve). Figure 2.1 shows the problem. Here the blue curve corresponds to the discrete sum and the points hit right in the correct place. The red curve, which happens to be the integral has a drift down from the correct values.

So even though the integral is an natural extension from the discrete to the continuous, it is not an interpolation between the discrete representation. It is a good approximation and it is the nature of the particular problem in question about which to choose.

- Another problem (solved by Newton) was the following. Consider the exponentiation of a number by a natural number. The definition

$$a^3 = a \cdot a \cdot a.$$

was well understood before Newton. But what would be

$$a^{5/2} = a^{2.5} \; ?$$

Newton established that for any given natural numbers n and m

$$a^0 = 1, \quad a^{m/n} = \sqrt[n]{a^m}, \quad a^{-m} = 1/a^m. \tag{2.1}$$

This still is not a complete interpolation formula, since this only covers the rational numbers in between integers. How about a power to an irrational number? The answer here is

$$a^x = \mathrm{e}^{x \ln a}. \tag{2.2}$$

where e is the Euler (1707–1783) constant $\mathrm{e} \approx 2.7182\cdots$ and $\ln a$ is the natural logarithm of a. Who invented this? Euler? So interpolation formula 2.1 interpolates along the rational numbers and formula 2.2 is the completion of the job. That is, formula 2.2 is the interpolation of the exponentiation along the real numbers. We see that the problem starts becoming more complicated.

- To be fair, the previous example does not follow the pattern that should take us to the $\Gamma(z)$ function. The previous example is the multiplicative version of the repeated sum, which by definition is a multiplication. This is,

$$a + a + a = 3a.$$

The interpolation formula for 2.5 times a is

$$2.5 \cdot a = 2.5\, a$$

The correct path to motivate the $\Gamma(z)$ function is, what is the interpolation formula for the factorial [7]

[7] According to Gronau the notation $n!$ was first introduced in 1808 by Christian Kramp (1760–1826, professor of mathematics at Strasbourg).

$$3! = 3 \cdot 2 \cdot 1 \; ?$$

What is the meaning of 2.5! ?

In summary: The continuous interpolation of the addition operation is multiplication. The continuous interpolation of multiplication is exponentiation and the continuous interpolation of the sum of consecutive natural numbers or powers of numbers is given by the Bernoulli formulas (which start becoming more and more complex as the power increases). The question asked by mathematicians in the XVII century is what is the continuous interpolation of the factorial function?

2.2 Correspondence between C. Goldbach and D. Bernoulli

According to Gronau, and references in the linked document above, Christian Goldbach (1690–1764) posed problems such as

$$\sum_{i=1}^{n} f(i) \quad \text{with a given function} \qquad f : \mathbb{N} \to \mathbb{R}, \tag{2.3}$$

and specially the sum $1+1\cdot 2+1\cdot 2\cdot 3+\cdots$. Goldbach, considering especially the problem of interpolating the factorial, asked several mathematicians for advice. In particular in to 1722 Nikolaus Bernoulli (1687–1759) and later on in 1729 to his brother Daniel (1700–1784). This [8] letter was sent in 1729 from Daniel Bernoulli to Goldbach. In the letter Bernoulli proposes the following interpolation formula (for $x!$)

$$\left(A+\frac{x}{2}\right)^{x-1}\left(\frac{2}{1+x}\frac{3}{2+x}\frac{4}{3+x}\cdots\frac{A}{A-1+x}\right). \tag{2.4}$$

Bernoulli shows computations of his formula for $x = 3/2$ and $A = 8$, which according to him is

$$\sqrt{\frac{19}{2}}\left(\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{10}{11}\cdot\frac{12}{13}\cdot\frac{14}{15}\cdot\frac{16}{17}\right)$$

[8] http://commons.wikimedia.org/wiki/File

Bernoulli obtains $\frac{3}{2}! \approx 1.3085$, I computed the expression in *Mathematica* and found 1.38485889242621294612503979127583076081530870698088845 instead, Gronau found 1.32907. Finally Bernoulli evaluates his formula to compute 3!, with $x = 3$ and $A = 16$. He finds $6\frac{1}{204}$. According to Gronau this was the last letter between Bernoulli and and Goldbach. I will come back to Bernoulli's formula later but before, I show the correspondence between Euler and Goldback

2.3 First letter between Goldbach and Euler

Euler was friend of the Bernoulli family and Daniel asked him to write to Goldbach. A correspondence that lasted for many years. The Euler Archive website [9] has an extense list of known correspondence between Euler and many other scientists of his time and particular with Goldbach. Most of the letters can be found as PDF files.

Euler was a master of infinite products. He found an infinite product factorizations of the $\sin z$ function ([3]), which is written as

$$\sin z = z\left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{2^2\pi^2}\right)\cdots\left(1 - \frac{z^2}{k^2\pi^2}\right)\cdots \tag{2.5}$$

Another infinite factorization found by Euler was the Riemann Zeta [10] function $\zeta(s)$, in terms of elementary functions of prime numbers. The $\Gamma(z)$ function was not the exception. In a letter to Goldbach (See Figure 2.2, taken from the link shown above) dated October 13–th of 1729, he shows as its first equation

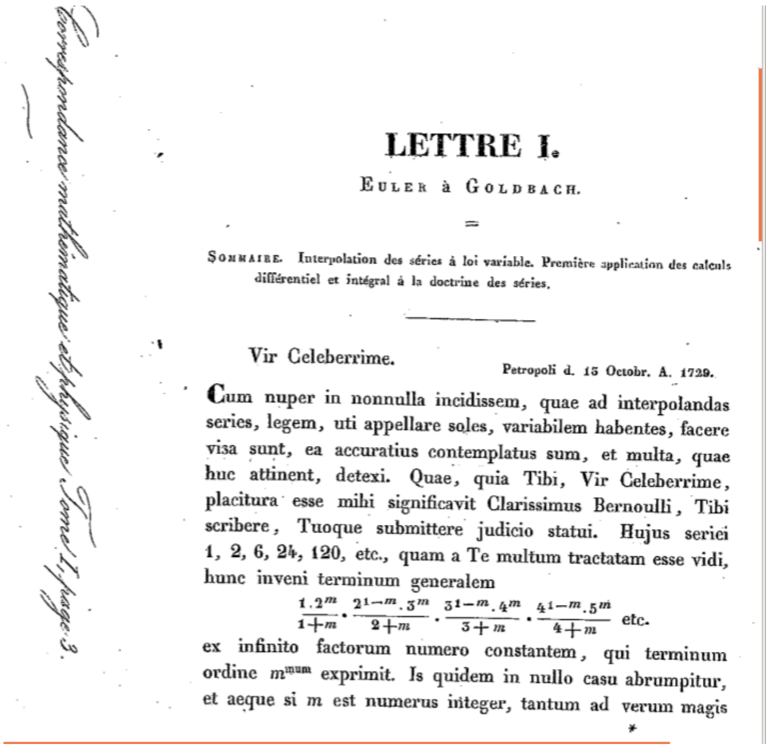
$$\frac{1.2^m}{1+m}\cdot\frac{2^{1-m}.3^m}{2+m}\cdot\frac{3^{1-m}.4^m}{3+m}\cdot\frac{4^{1-m}.5^m}{4+m}\ \text{etc.} \tag{2.6}$$

If we observe that consecutive factors of the form $k^m k^{1-m}$ cancel to k, we can write Euler's product (regrouping, and dropping factors) as

$$\frac{1\cdot 2\cdot 3\cdots}{(1+m)(2+m)(3+m)\cdots} = \frac{1\cdot 2\cdot 3\cdots m\cancel{(m+1)}\cancel{(m+2)}\cancel{(m+3)}\cdots}{\cancel{(1+m)}\cancel{(2+m)}\cancel{(3+m)}\cdots}$$

9 http://www.math.dartmouth.edu/ euler/tour/tour_27.html

10 https://en.wikipedia.org/wiki/Riemann_zeta_function



Correspondance mathématique et physique, Tome I, page 3.

LETTRE I.

EULER à GOLDBACH.

SOMMAIRE. Interpolation des séries à loi variable. Première application des calculs différentiel et intégral à la doctrine des séries.

Vir Celeberrime. Petropoli d. 13 Octobr. A. 1729.

Cum nuper in nonnulla incidissem, quae ad interpolandas series, legem, uti appellare soles, variabilem habentes, facere visa sunt, ea accuratius contemplatus sum, et multa, quae huc attinent, detexi. Quae, quia Tibi, Vir Celeberrime, placitura esse mihi significavit Clarissimus Bernoulli, Tibi scribere, Tuoque submittere judicio statui. Hujus seriei 1, 2, 6, 24, 120, etc., quam a Te multum tractatam esse vidi, hunc inveni terminum generalem

$$\frac{1.2^m}{1+m} \cdot \frac{2^{1-m}.3^m}{2+m} \cdot \frac{3^{1-m}.4^m}{3+m} \cdot \frac{4^{1-m}.5^m}{4+m} \text{ etc.}$$

ex infinito factorum numero constantem, qui terminum ordine m^{mum} exprimit. Is quidem in nullo casu abrumpitur, et aeque si m est numerus integer, tantum ad verum magis

Figure 2.2: Copy of first page of a Letter from Euler to Goldbach about his infinite produce representation of the $\Gamma(z)$ function.

which yields, of course $m!$, right? In fact we have to be more careful here. The product is either finite or not finite. If the product is finite, it should have a one–to–one correspondence between each of the two factors in the numerator with its denominator. If it is not finite, the product should converge. Euler goes into first showing the case where only two fractions are multiplied. That is,

$$\frac{1.2^m}{1+m} \cdot \frac{2^{1-m}.3^m}{2+m} = \frac{1 \cdot 2}{(1+m)(2+m)} 3^m$$

If $m = 0, 1$ this function works fine (it evaluates to 1). However what if $m = 2$ we evaluate the function as

$$\frac{2 \cdot 3^2}{3 \cdot 4} = \frac{3}{2},$$

which obviously is not 2!, so where is the catch? Euler's then moves into the general product up to n fractional factors. This is

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(1+m)(2+m)\cdots(n+m)}(n+1)^m. \tag{2.7}$$

Then he asks himself, what if $m = 2$, and list the product

$$\frac{4}{3} \cdot \frac{9}{8} \cdot \frac{16}{15} \cdot \frac{25}{24} \cdot \frac{36}{35} \text{ etc.} \tag{2.8}$$

At this point I want to make a stop and change to modern notation for better understanding. Let us define

$$a_n^m = \frac{1 \cdot 2 \cdot 3 \cdots n}{(1+m)(2+m)\cdots(n+m)}(n+1)^m = \frac{(n+1)^m}{\binom{n+m}{n}}$$

with n equal to the number of factors and the *binomial coefficient*

$$\binom{n+m}{n} = \frac{(n+m)!}{n!\,m!}. \tag{2.9}$$

Think for a_n^2 as a sequence for $m = 2$, then

$$\begin{aligned}
a_1^2 &= \frac{4}{3} \\
a_2^2 &= \frac{9}{6} \\
a_3^2 &= \frac{8}{5} \\
a_4^2 &= \frac{5}{3} \\
a_5^2 &= \frac{12}{7} \\
&\vdots
\end{aligned}$$

The last term a_5^2 agrees with Euler's accumulated product 2.8.

Next, in the sequence for $m = 3$. Euler gets

$$\frac{8}{4} \cdot \frac{27}{20} \cdot \frac{64}{54} \cdot \frac{125}{112} \quad \text{etc.}$$

which converges to 6. The partial products are given by

$$\begin{aligned} a_1^3 &= \frac{8}{4} = 2 \\ a_2^3 &= \frac{27}{10} \\ a_3^3 &= \frac{64}{20} = \frac{16}{5} \\ a_4^3 &= \frac{125}{35} = \frac{25}{7} \end{aligned}$$

which, again, should converge to $3! = 6$.

In general, let us find the limit of a_n^m and show that indeed it is $m!$

$$\begin{aligned} \lim_{n\to\infty} a_n^m = &= \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{(1+m)(2+m)\cdots(n+m)}(n+1)^m \\ &= \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots m\cancel{(m+1)}\cancel{(m+2)}\cdots \cancel{n}\cdots}{\cancel{(1+m)}\cancel{(2+m)}\cdots\cancel{n}(n+1)\cdots(n+m)}(n+1)^m \\ &= m!\frac{n+1}{n+1}\,\frac{n+1}{n+2}\cdots\frac{n+1}{n+m}\cdots \\ &= m! \end{aligned}$$

since

$$\lim \frac{n+1}{n+k} = 1,$$

for $1 \le k \le m$. Euler did not care much about convergence and limit interchanges. He trusted his great intuition

The interesting question here is, what did Euler gain with representation 2.6 ? Let us see. We think $m!$ is much simpler than representation 2.6. However what would be, for example the result of $1/2!$ in both cases?

If we define $m!$ as $m(m-1)\cdots 1$. Then we would say

$$1/2! = (1/2)(-1/2)(-3/2)(-5/2)\cdots$$

where do we go from here? First, the product sequence is oscillating in sign (it does not converge to a unique number), second, the size of the numbers grow without bounds. Of course, Euler and his friends knew this but we have to do our homework. How about using Euler's representation 2.6? Euler found that $1/2! = \sqrt{\pi}/2$ but he does not show his computation. In his own words

"*Terminus autem expnentis* $\frac{1}{2}$ *aequalis inventus est huic* $\frac{1}{2}\sqrt{(}\sqrt{-}1.l-1)$, *seu quod huic aequale est, lateri quadrati aequalis circulo, cujus diameter=1"*. The English translation (with help of google translate [11] for this would be: "But the term of the exponent equal $\sqrt{\pi}/2$ was found in this bill, that is equal to the side of a square equal to a circle, whose diameter is 1". Note that for Euler

$$\pi = \sqrt{-1}.l - 1 = \mathrm{i}\ln(-1).$$

where $\mathrm{i} = \sqrt{-1}$, $l = \ln$ and

$$\ln(-1) = \ln \mathrm{e}^{-\mathrm{i}\pi} = -\mathrm{i}\pi.$$

I will try to guess what Euler did in his own:

$$\begin{aligned}
\frac{1.2^{1/2}}{1+1/2} \cdot \frac{2^{1-1/2}.3^{1/2}}{2+1/2} \cdot \frac{3^{1-1/2}.4^{1/2}}{3+1/2} \cdot \frac{4^{1-1/2}.5^{1/2}}{4+1/2} \cdots = \frac{2\cdot 3\cdot 4\cdot \sqrt{5}}{(3/2)(5/2)(7/2)(9/2)} \cdots \\
&= \frac{2^4\cdot 2\cdot 3\cdot 4\cdot \sqrt{5}}{3\cdot 5\cdot 7\cdot 9} \cdots \\
&= \frac{2\cdot 4\cdot 6\cdot 8\cdot \sqrt{5}}{3\cdot 5\cdot 7\cdot 9} \cdots \\
&= \frac{1}{2}\cdot \sqrt{\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 20}{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9}} \cdots \\
&= \frac{1}{2}\cdot \sqrt{\left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\left(\frac{8\cdot 8}{7\cdot 9}\right)\left(\frac{10}{9}\right)\cdot 2\cdots}
\end{aligned}$$

I keep guessing and think that, at this point Euler recognized John Wallis (1616–1703) identity

$$\frac{\pi}{2} = \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\left(\frac{8\cdot 8}{7\cdot 9}\right)\left(\frac{10}{9}\right)\cdots \tag{2.10}$$

discovered in 1655. Section 2.5 shows some of the work of Wallis which inspired Euler to formulate the interpolation of the factorial formula as an integral.

Euler found $1/2! = \sqrt{\pi}/2$, and this gives him confidence on his equation which led to the development of the interpolation of the factorial function that became what we know today as the $\Gamma(z)$ function.

[11] https://translate.google.com/

As I said before, Euler was a master of infinite products (and infinite series). He found several relations between infinite products and the irrational constant π [12]. The Basel problem, posted by Pietro Mengoly (1626–1686) in 1644, was solved by Euler in 1735 and it gave him big fame because great mathematicians were trying to solve it for years without success. Among them Jacques Bernoulli (1654–1705), brother of Johann who was the father of Daniel and Nikolaus. [14]. In this document I illustrate Euler's solution of the Basel problem. Basel was the town where the Bernoulli family, and Euler were born). Wikipedia's link has good analysis and history of the Basel problem.

The statement of the Bessel problem is, what is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} =?$$

Euler found the solution

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

based on the infinite factorization of the $\sin x$ function. After Euler found the solution of the Basel problem (1734), long after the death of Jacques Bernoulli, Johan Bernoulli exclaimed, "In this way my brother's most ardent wish is satisfied ... if only my brother were still alive!" (Johann Bernoulli, *Opera* [15] , Vol. 4. p. 22.)

Before jumping into the second important letter related to the $\Gamma(z)$ function, I want to make a parallel between Bernoulli's and Euler's formulation to the interpolation of the factorial problem.

[12] According to Reeder , the symbol π, (which means "periphery" of a circle with unit diameter) was *" most probably introduced in 1706 by William Jones"* [13] just after Wallis death. Euler used it in 1736. Wikipedia also gives credit to Jones notation in his work *Synopsis Palmariorum Matheseos"* or *A new introduction of the Mathematics, of 1706 [2]*

[14] See my document [3], about the Bernoulli Numbers, after Jacques Bernoulli

[15] Opera Omnia is an incomplete collection of 72 volumes about Euler's books, papers and correspondence. The Euler Commission founded in July of 1907 by the Swiss Academy of Sciences in charge of the Opera Omnia project.

2.4 Bridge Between Bernoulli and Euler

Let us start with Bernoulli's formula 2.4 and change it look like Euler notation. Bernoulli's original formula is

$$\left(A+\frac{x}{2}\right)^{x-1}\left(\frac{2}{1+x}\cdot\frac{3}{2+x}\cdot\frac{4}{3+x}\cdot\cdots\frac{A}{A-1+x}\right).$$

Here A is one less than the number of factors. We assign $A = n+1$ and write m instead of x so that we can morf Bernoulli's formula 2.4 into Euler's equation 2.7.

$$\begin{aligned}
m! &= \left(n+1+\frac{m}{2}\right)^{m-1}\left(\frac{2}{1+m}\cdot\frac{3}{2+m}\cdot\frac{4}{3+m}\cdot\cdots\frac{n+1}{m+n}\cdots\right)\\
&= \frac{1\cdot 2\cdot 3\cdots n\cdot n+1}{(1+m)\cdot(2+m)\cdots(n+m)}\left(n+1+\frac{m}{2}\right)^{m-1}\\
&= \frac{1\cdot 2\cdot 3\cdots n}{(1+m)\cdot(2+m)\cdots(n+m)}(n+1)^m\left(\frac{n+1+\frac{m}{2}}{n+1}\right)^{m-1}\\
&= \frac{1\cdot 2\cdot 3\cdots n}{(1+m)\cdot(2+m)\cdots(n+m)}(n+1)^m.
\end{aligned}$$

We used

$$\lim_{n\to\infty}\frac{n+1+\frac{m}{2}}{n+1}=1,$$

and found a smooth transition between Bernoulli's formula 2.4 and Euler's formula 2.7. Also, we assume that we are dealing with approximations as n is a large number. Again, it seems absurd that to compute a factorial we need to compute something apparently much more complicated and have it in the limit as $n\to\infty$. The reason for this, again, is that the formula that Bernoulli, Euler, and others were looking for was not the factorial but an interpolation of the factorial. Something along these lines could have motivated Euler to write his formulation of the interpolation of the factorial. In this sense I would give more credit to Daniel Bernoulli about the invention of the interpolation of the factorial function that to Euler. Even more, according to Gronau *"Numerical experiments show that the formula of Bernoulli converges much faster to its limit than that of Euler"* . In the limit both formulas provide the same number and today that number is writen as

$$m! = \lim_{n\to\infty}\frac{n!(n+1)^m}{(m+1)(m+2)\cdots(m+n)}$$

2.5 The work of Wallis

We showed that Euler found the relation $1/2! = \sqrt{\pi}/2$, through a connection with Wallis formula 2.10. This provided fuel to Euler to discover the integral representation of the interpolation function for the factorial. Euler's derivation of the integral representation of the interpolation of the factorial function will be shown next in section 2.6. But, Before we show how Euler found the integral formulation of the $\Gamma(z)$ function, I want to make a stop and show some of Walli's work which will clearly explain the connections made by Euler.

Reeder shows the insights that Wallis had to get his formula and I found his notes ordered and clear. Therefore I follow Reeder's notes closely.

I rewrite Wallis formula 2.10 as a starting point.

$$\frac{\pi}{2} = \left(\frac{2 \cdot 2}{1 \cdot 3}\right)\left(\frac{4 \cdot 4}{3 \cdot 5}\right)\left(\frac{6 \cdot 6}{5 \cdot 7}\right)\left(\frac{8 \cdot 8}{7 \cdot 9}\right)\left(\frac{10}{9} \cdot \frac{10}{11}\right)\cdots$$

I added the last fraction to complete the symmetry. While there is not any symmetry in the decimal figures, there is a lot of symmetry on these fractions. Numerators are all ordered (in increasing values) couples with repeated even numbers while denominators are couples of ordered odd numbers of the form $(1,3);(3,5);(5,7)$ etc. The order should be respected. That is, a change of order (such as cancellation of odd multiples of 2 with the odd denominators would lead to a divergence sequence as shown by Reeder in his notes. Reeder also shows that the Wallis sequence converges numerically to $\pi/2$ but very slowly. For example calling W_k the product up to the k–th fraction Reeder found

$$\begin{aligned}
2W_3 &= 2.92571\ldots, \\
2W_4 &= 2.92215\ldots, \\
2W_5 &= 3.0028\ldots, \\
&= \vdots \\
2W_{100} &= 3.13379\ldots, \\
&= \vdots \\
2W_{1000} &= 3.14081\ldots,
\end{aligned}$$

which is certainly of slow convergence. According to Reeder, Wallis *"arrived at this formula for π by a wild and creative path, guided by guessing and* *intuition, along with lots of persistence".* Wallis was interested in computing areas under curves. Finding an area in those days was called "quadrature" from the Latin *"quadratus"*, meaning "square", in the sense that areas were built in terms of squares. Citing Reeder again, " *When Wallis showed his formula to Christian Huygens (1629–1695, inventor of the pendulum clock), the latter was highly skeptical until Wallis could demonstrate that the right side of his equation agreed with $\pi/2$ to at least nine decimal places. Although W_{1000} is nowhere close to this, William Brouckner (1620–1684, first President of the Royal Society) showed Wallis a remarkable way to approximate Wallis' product using continued fractions".* In fact Wikipedia has an extensive document about the history of π and in the continued fractions section shows several continued fraction expansions. Here is one

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{192 + \cfrac{1}{1 + \cfrac{1}{1+}}}}}}$$

This represents the sequence of fractions [16]

$$\begin{aligned}
\frac{22}{7} &= 3.142857142857143 \\
\frac{333}{106} &= 3.141509433962264 \\
\frac{355}{113} &= 3.141592920354 \\
\frac{52163}{16604} &= 3.141592387376536 \\
\frac{103993}{33102} &= 3.1415926530119025
\end{aligned}$$

[16]Each fraction is computed by truncating the fractions below it. For example, the first two fractions are $3 + 1/7 = 22/7$ and $3 + 1/(7 + 1/15) = 333/106 \cdots$. Actually Wikipedia shows an algorithm to compute the continued fractions above. $[a_i] = [3; 7; 15; 1; 292; 1; 1; \cdots]$. Take the $a_0 = \lfloor \pi \rfloor = 3$, define $u_1 = 1/(\pi - 3) \approx 7.0625$ and $a_1 = \lfloor u_1 \rfloor = 7$. $u_2 = 1/(u_1 - 7) \approx 15.9965$ and $a_2 = \lfloor u_2 \rfloor = 15$ and so on.

So the first fraction is good up to 2 decimal places and the fifth fraction is good up to 10 places. In fact, continued fractions represent the best rational approximation; that is, each is closer to π than to any other fraction with the same or smaller denominator.

Now Reeder shows how Wallis carried out the derivation of his formula. Wallis used the notation

$$\boxdot = \frac{4}{\pi},$$

the notation for π was invented just after his death as pointed above. He also note the integral representation for the area of a quarter of a unit circle as

$$\frac{1}{\boxdot} = \int_0^1 (1 - x^2)^{1/2} dx = \frac{\pi}{4}.$$

He wanted to evaluate this formula with the hope of finding a closed analytical form for π. Since he did not know how to expand the function

$$(1 - x^2)^{1/2}$$

he switched the 1/2 by the 2 and evaluate an easier formula. That is, he evaluated instead

$$\int_0^1 (1 - x^{1/2})^2 dx = \int_0^1 (1 - 2x^{1/2} + x) dx = 1 - 2 \cdot \frac{2}{3} + \frac{1}{2} = \frac{1}{6}.$$

then he looked for patterns of the form

$$\int_0^1 (1 - x^{1/p})^q dx \tag{2.11}$$

for combinations of p and q. He found that the integral above is the inverse of an integer. That is, the function

$$f(p, q) = \frac{1}{\int_0^1 (1 - x^{1/p})^q dx} = \text{integer}$$

Table 2.1 from Reeder is a small subset of the actual table computed by Wallis

q / p	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	10
2	1	3	6	10	15	21	28	36	45	55
3	1	4	10	20	35	56	84	120	165	220
4	1	5	15	35	70	126	210	330	495	715
5	1	6	21	56	126	252	462	792	1287	2002

Table 2.1: In modern notation these coefficients are $\binom{p+q}{q}$.

In fact the table is made by the binomial coefficients and so

$$f(p,q) = \left[\int_0^1 (1-x^{1/p})^q\right]^{-1} = \binom{p+q}{q}. \tag{2.12}$$

By the time Wallis made this table he did not know about Newton binomial formula (which was discovered many years later, in 1826 by Andreas von Ettingshausen. Of course, the Pascal's triangle was known for centuries before that, but no the formula of the binomial coefficients).

This is the second time we find this binomial formula(see also 2.9). I believe this formula is quite important because is related to both, the $\Gamma(z)$ and the $\mathrm{B}(z)$ functions. We will find later that

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{(x-1)!\,(y-1)!}{(x+y-1)!}$$

so

$$\begin{aligned} f(p-1,q) = \binom{p+q-1}{q} &= \frac{(p+q-1)!}{(p-1)!\,q!} \\ &= \frac{1}{q}\frac{(p+q-1)!}{(p-1)!\,(q-1)!} \\ &= \frac{1}{q\,\mathrm{B}(p,q)} \\ &= \frac{\Gamma(p+q)}{\Gamma(p)\,\Gamma(q+1)}. \end{aligned}$$

Certainly a lot of connections here, but yet to discover. That is, the $\Gamma(z)$, the $\mathrm{B}(z)$ and the values on the table 2.1 (which is an evaluation of the integral for the quadrature of the circle) are tight connected.

p \ q	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
0	1	1	1	1	1	1	1
$\frac{1}{2}$	1	$\boxdot$					
1	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$\frac{3}{2}$	1						
2	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10
$\frac{5}{2}$	1						
3	1	$\frac{35}{16}$	4	$\frac{105}{16}$	10	$\frac{231}{16}$	20

Table 2.2: In modern notation these coefficients are $\binom{p+q}{q}$.

Wallis did a larger table trying to find a continuation of the binomial coefficient formula for non–integer numbers p and q without success, and according to Reeder, Wallis wrote about this *"Although no small hope seemed to shine, what we have in hand is slippery, like Proteus, who in the same way, often escaped, and disappointed hope".* I will return equation 2.12 later, since it is the key of the connection between factorials and the $\Gamma(z)$ function, but let me keep with Wallis work on finding the expression for $\pi/2$.

Wallis wanted to compute

$$f(1/2,1/2) = \left[\int_0^1 (1-x^2)^{1/2}dx\right]^{-1} = \frac{4}{\pi} \quad \text{or}$$
$$\binom{1}{1/2} = \frac{1!}{1/2!\,1/2!} = \frac{1}{(1/2!)^2} = \frac{4}{\pi}$$

That is, he found that

$$\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2} \tag{2.13}$$

Euler was aware of this result and used it to guide him in the right direction, however Wallis did not understand this result, since he did not know what it means to have a factorial of a fraction. He keep pushing and made a new table 2.2 where he would add new p's and q's of rational form and that could be easily computed.

Just for a quick (incomplete) check Wallis table, let us evaluate $f(1, 1/2)$.

This is

$$f(1,1/2) = \left[\int_0^1 (1-x)^{1/2}\,dx\right]^{-1}$$

which we can solve by changing variables. That is,

$$u = 1-x, \quad du = -dx, \quad -u\Big|_0^1$$

$$f(1,1/2) = \left[\int_0^1 u^{1/2}du\right]^{-1} = \left[\frac{u^{3/2}}{3/2}\Big|_0^1\right]^{-1} = 3/2$$

On the other hand, computing $f(1,1/2)$ using binomial coefficients we find

$$f(1,1/2) = \binom{1+1/2}{1/2} = \frac{3/2!}{1!1/2!} = \frac{(3/2)\cancel{(1/2)!}}{1!\cancel{1/2!}} = 3/2.$$

which in both cases checks well against Wallis' table. Remember however that Wallis' did not know about binomial coefficients and this is a shortcut that we know because Newton's contribution happen after the work of Wallis and much before our days.

The next step in Wallis's work was to form a recursion formula that will carry him over the finding of the fractional missing entries in the table. The recursion formula that he found was

$$f(p,q) = \frac{p+q}{q} f(p,q-1). \tag{2.14}$$

This formula checked for all numbers in his table, but he was guessing here that it could extend to non–integer numbers. That is 2.14 he guessed that the formula made sense for any numbers p and q no necessarily integers. The first move on filling up the empty squares was

$$f\left(\frac{1}{2},\frac{3}{2}\right) = \frac{\frac{1}{2}+\frac{3}{2}}{\frac{3}{2}} f\left(\frac{1}{2},\frac{1}{2}\right) = \frac{4}{3}\,\boxdot\,.$$

Along the same lines he completes the raw for $p = 1/2$ as shown in table 2.3. At this point Wallis observes that the raws (p fixed) grows as q increases.

q / p	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$
$\frac{1}{2}$	1	$\boxdot$	$\frac{3}{2}$	$\frac{4}{3}\boxdot$	$\frac{3\cdot5}{2\cdot4}$	$\frac{2\cdot4}{5}\boxdot$	$\frac{3\cdot5\cdot7}{2\cdot4\cdot6}$	$\frac{2\cdot2\cdot4\cdot4}{5\cdot7}\boxdot$

Table 2.3: Filling up the empty boxes of Table 2.2

This can easily be proof based on the binomial function but he did not know about that at that time. He then claims (see the table) , for example that

$$\frac{2\cdot 4}{5}\boxdot < \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6} < \frac{2\cdot 2\cdot 4\cdot 4}{5\cdot 7}\boxdot$$

and using today's notation for π this is

$$\frac{2\cdot 4\cdot 4}{5\,\pi} < \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6} < \frac{2\cdot 2\cdot 4\cdot 4\cdot 4}{5\cdot 7\pi}$$

which can be written as

$$\frac{2\cdot 4\cdot 4\cdot 4\cdot 6}{3\cdot 5\cdot 5\cdot 7} < \frac{\pi}{2} < \frac{2\cdot 2\cdot 4\cdot 4\cdot 4\cdot 4\cdot 6}{3\cdot 5\cdot 5\cdot 7\cdot 7}$$

or

$$\frac{2\cdot 2\cdot 4\cdot 4\cdot 6}{3\cdot 5\cdot 5\cdot 7} < \frac{\pi}{2} < \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}{3\cdot 3\cdot 5\cdot 5\cdot 7}\cdot\frac{8}{7}.$$

Note that some re–ordering was done here and this should be prohibited, in principle, in infinite series and or products (unless some kind of convergence is guaranteed). However, this is what Wallis did, and if he accepted the last inequality and generalizing, he found:

$$\frac{2\cdot 2\cdot 4\cdot 4\cdots 2n\cdot 2n}{3\cdot 5\cdot 5\cdots 2n-1\cdot 2n+1} < \frac{\pi}{2} < \frac{2\cdot 2\cdot 4\cdot 4\cdots 2n\cdot 2n}{3\cdot 3\cdot 5\cdot 5\cdots 2n-1\cdot 2n+1}\cdot\frac{2n+2}{2n+1}.$$

Now, since

$$\lim_{n\to\infty}\frac{2n+2}{2n+1} = 1,$$

Wallis formula comes as

$$\frac{\pi}{2} = \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\cdots\left(\frac{2i\cdot 2i}{2i-1\cdot 2i+1}\right)\cdots$$

Since Wallis derived his formula in a non–rigorous way, I show in Appendix 4.8 two different proofs of Wallis formula. The first, is based on Euler's infinite product expansion of the $\sin(z)$ function 2.5 , the second derivation is based on calculus and evaluation of iterative integrals of powers of $\sin(z)$. These derivations are found in the Wikipedia website.

I include both derivations for the following reasons

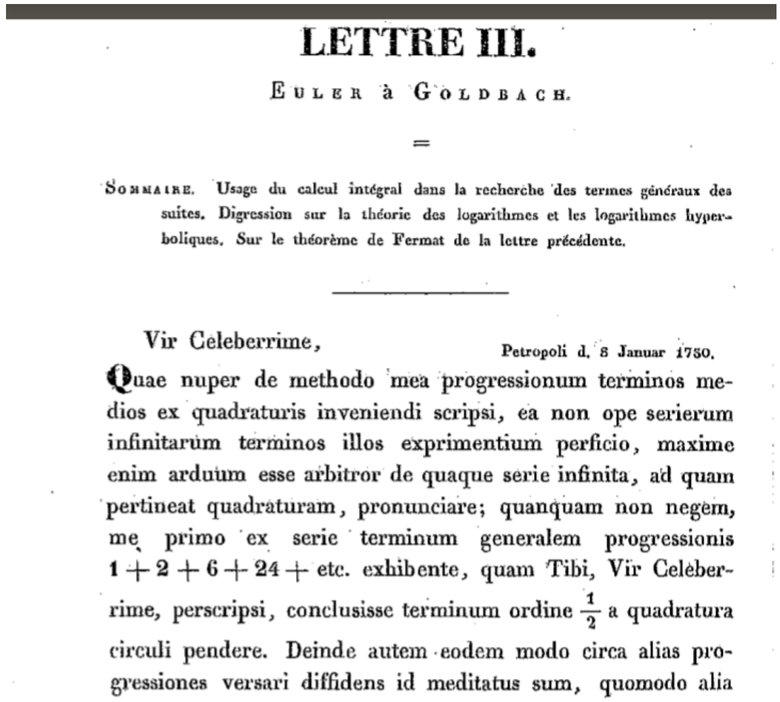
- The first derivation is based on Euler infinite product formula for the $\sin z$ function. The derivation is amazingly quick and simple. I am not sure about the origin of this derivation. Wikipedia refers to Wolfram but Wolfram does not indicates where the proof came from. However, given the present context I find quite relevant to include this derivation.

- The second derivation is based on calculus and is a derivation that Euler himself could have reached with the tools he had. I also ignore the origin of this derivation but I include it because it start showing the link between the Beta $\mathrm{B}(z, w)$ (no yet defined) function, the $\Gamma(z)$ function, integrals of circular type and factorias. I believe this is what Euler saw when he decided to go for an integral representation of the Γ function. I offer more details about these connections in the appendix.

2.6 Second letter between Euler and Goldbach and Euler's key paper on the invention of the Beta $\mathrm{B}(s, t)$ and Gamma $\Gamma(x)$ functions.

The second letter from Euler to Goldbach, dated Jan 8, 1730 takes the $\Gamma(z)$ function into a another "dimension". That is, Euler changes the infinite product representation by an integral representation. Figure 2.3 shows the a copy of the first page of such a letter, taken from the Euler Archive.

Euler expanded on his letter to Goldbach in his article [1] *De progressionibus transcendentibus seu quarum termini generales algebraic dari nequeunt* (On transcendental progressions, that is, those whose general terms cannot be given algebraically). The article can be found in the Euler Archive, where also an English translation can be found (by Stacy G. Langton), and also here.



LETTRE III.

EULER à GOLDBACH.

SOMMAIRE. Usage du calcul intégral dans la recherche des termes généraux des suites. Digression sur la théorie des logarithmes et les logarithmes hyperboliques. Sur le théorème de Fermat de la lettre précédente.

Vir Celeberrime, Petropoli d. 8 Januar 1750.

Quae nuper de methodo mea progressionum terminos medios ex quadraturis inveniendi scripsi, ea non ope serierum infinitarum terminos illos exprimentium perficio, maxime enim arduum esse arbitror de quaque serie infinita, ad quam pertineat quadraturam, pronunciare; quanquam non negem, me primo ex serie terminum generalem progressionis $1+2+6+24+$ etc. exhibente, quam Tibi, Vir Celeberrime, perscripsi, conclusisse terminum ordine $\frac{1}{2}$ a quadratura circuli pendere. Deinde autem eodem modo circa alias progressiones versari diffidens id meditatus sum, quomodo alia

Figure 2.3: Copy of first page of the second Letter from Euler to Goldbach about a formula for the interpolation of the factorial function.

Euler initially thought that the general interpolation term for $n!$ could be given algebraically or at least exponentially, but after he found that fractional input yields functions of π, he realized that neither an algebraic nor an exponential form of this could be found and that motivated the title of his article.

Euler started assuming, as we have discussed previously, that the interpolation formula for the factorial should have an integral form which should depend on n. He assumed the following ansatz [17]

$$a_{en} = \int_0^1 x^e\,(1-x)^n dx. \tag{2.15}$$

I added the left hand a_{en} expression, to make it a bit clearer. Euler did not

[17] I will discuss below what could had made Euler think about this ansatz.

introduce limits on the integral, but I added the limits (from 0 to 1) because at the end he will accommodate them this way. He knew that the limits should not be part of the resulting interpolation for the factorial function. The dummy variable used is x, the variable (index) that would carry the n–th factorial term is n, and an extra variable e that could help him massage the integral.

How did he come to this ansatz? As I said before he was inspired mainly by Wallis, but Newton and Stirling also used integrals such as 2.15. For example integral 2.11 is a particular case (for $p = 1$) of Euler integral with $e = 0$ and $q = n$. The binomial expansion of $(1-x)^n$ is given by [18]

$$(1-x)^n = 1 - \frac{n}{1}x + \frac{n(n-1)}{1 \cdot 2}\, x^2 - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \text{etc.}$$

and so equation 2.15 becomes

$$a_{en} = \frac{x^{e+1}}{e+1} - \frac{nx^{e+2}}{1 \cdot (e+2)} + \frac{n(n-1)x^{e+3}}{1 \cdot 2 \cdot (e+3)} - \frac{n(n-1)(n-2)x^{e+4}}{1 \cdot 2 \cdot 3 \cdot (e+4)} + \text{etc.}$$

Euler then proceed to evaluate the previous expression for $x = 1$ which up to a constant is fine and he does not care about this constant. That is, he finds

$$a_{en} = \frac{1}{e+1} - \frac{n}{1 \cdot (e+2)} + \frac{n(n-1)}{1 \cdot 2 \cdot (e+3)} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot (e+4)} + \text{etc.}$$

$n = 0, 1, 2 \cdots n$, and $x = 1$ That is,

$$\begin{aligned}
a_{e0} &= \frac{1}{e+1} \\
a_{e1} &= \frac{1}{(e+1)(e+2)} \\
a_{e2} &= \frac{1 \cdot 2}{(e+1)(e+2)(e+3)} \\
&\vdots \\
a_{en} &= \frac{1 \cdot 2 \cdot 3 \cdots n}{(e+1)(e+2)(e+3) \cdots (e+n+1)}
\end{aligned}$$

[18] I will use a notation closer to the one given by Euler instead of the most compact binomial form $\binom{n}{m}$ combined with the sum Σ sign as it is customary today.

This is

$$a_{en} = \int_0^1 x^e\,(1-x)^n dx = \frac{1\cdot 2\cdot 3\cdots n}{(e+1)(e+2)(e+3)\cdots(e+n+1)}$$

from which Euler arrives at the expression

$$(e+n+1)\int_0^1 x^e\,(1-x)^n dx = \frac{1\cdot 2\cdot 3\cdots n}{(e+1)(e+2)(e+3)\cdots(e+n)} \tag{2.16}$$

I want to make a stop and study this result. We can write

$$\begin{aligned} \frac{1\cdot 2\cdots n}{(e+1)(e+2)\cdots(e+n)} &= \frac{(1\cdot 2\cdots e)(1\cdot 2\cdots n)}{1\cdot 2\cdots e\cdot(e+1)(e+2)\cdots(e+n)} \\ &= \left(\frac{1\cdot 2\cdots e\cdot(e+1)(e+2)\cdots(e+n)}{(1\cdot 2\cdots n)(1\cdot 2\cdots e)}\right)^{-1} \\ &= \binom{e+n}{n}^{-1} \end{aligned} \tag{2.17}$$

Now we find the binomial coefficient for the third time (the other two are in equations 2.9 and 2.12)

We want to compare the result of Wallis (that he tabulated without knowing an analytical formula for it at that time) with this result of Euler. From 2.12

$$\int_0^1 (1-x^{1/p})^q dx = \binom{p+q}{q}^{-1}, \tag{2.18}$$

and from 2.16 and 2.17

$$(e+n+1)\int_0^1 x^e\,(1-x)^n dx = \binom{e+n}{n}^{-1}. \tag{2.19}$$

The similarity of these two results is evident. We show next why Wallis and Euler were talking about the same issue, although Wallis did not see the binomial formula and Euler did (even when Euler did not use the Newton binomial notation) see it. Wallis was really close when he wrote table 2.1 but did not have the Newton connection.

Let us assume the following change of variables in integral 2.18.

$$y = x^{1/p}, \qquad dy = \frac{1}{p} x^{1/p-1} dx = \frac{1}{p}\frac{y}{y^p} dx = \frac{y^{1-p}}{p} dx \Rightarrow dx = p\, y^{p-1}\, dy$$

to find

$$\int_0^1 (1 - x^{1/p})^q dx = p \int_0^1 y^{p-1} (1-y)^q \, dy. \tag{2.20}$$

At this point right side integral 2.20 looks close to Euler's integral 2.19. We change notation so that we can better compare the results . That is, we call $p = e$, $q = n$ and $y = x$ and write

$$\int_0^1 (1 - x^{1/e})^n dx \quad = \quad e \int_0^1 x^{e-1} (1-x)^n \, dx$$

Is it possible that Euler started at this integral (from Wallis) to get to his form? That is,

$$e \int_0^1 x^{e-1} (1-x)^n \, dx \stackrel{?}{=} (e+n+1) \int_0^1 x^e \, (1-x)^n dx = \binom{e+n}{n}^{-1}.$$

where the left integral is from Wallis and the center from Euler? The right binomial is the common result obtained by both. If this is the case, is not obvious to me. Euler's integral has the same power in $(1-x)$ but one higher power in x. Doing integration by parts will raise one power by 1 and lower the other by 1. So integration by parts does not seem to be a good way to get one from the other. For positive numbers the order of the polynomial in Wallis integrand is $e + n - 1$, while that in Euler integrand is $e + n$. In this sense I believe that Euler used Wallis integral but accommodated the power of x from x^{e-1} to x^e. This is his starting equation for a_{en} 2.15. Then after arriving at the expression 2.16 he found that the factor to match both equations was $e + n + 1$. In any case this is a historical moment because the function a_{en} as defined in 2.15 is what is know today as the beta $\mathrm{B}(z, w)$ function. That is

$$\mathrm{B}(e, n) = a_{e-1n-1} = \int_0^1 x^{e-1} \, (1-x)^{n-1} dx. \tag{2.21}$$

Why the shift of negative 1 as the more "natural" definition of a_{en}? This and other attributes of the beta function will be studied in section 3. For the moment I want to close this parenthesis and return to Euler's work on the derivation of the factorial interpolation function.

Euler shows an example when $e = 2$; where the n^{tn} order term would be given by

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{3 \cdot 4 \cdot 5 \cdots (n+2)} \qquad \text{or} \qquad \frac{1 \cdot 2}{(n+1)(n+2).}$$

which in integral form (and integrating several time by parts) is

$$\begin{aligned}
&(n+3)\int_0^1 x^2(1-x)^n dx = \\
&\qquad -\cancelto{0}{(n+3)\frac{(1-x)^{n+1}}{n+1}x^2\Big|_0^1} + (n+3)\int_0^1 2x\frac{(1-x)^{n+1}}{n+1}dx \\
&\quad = \frac{2(n+3)}{n+1}\int_0^1 x(1-x)^{n+1}dx \\
&\quad = -\cancelto{0}{\frac{2(n+3)}{(n+1)(n+2)}x(x-1)^{n+2}\Big|_0^1} + \frac{2(n+3)}{(n+1)(n+2)}\int_0^1 \frac{(1-x)^{n+2}}{n+1}dx \\
&\quad = -\frac{2\cancel{(n+3)}}{(n+1)(n+2)\cancel{(n+3)}}(1-x)^{n+3}\Big|_0^1 + \\
&\quad = \frac{2}{(n+1)(n+2)}.
\end{aligned}$$

In general, if instead of e being an integer, it is a fraction of the form f/g, the general term $(e+n+1)a_{en}$ would take the form of

$$\frac{f+(n+1)g}{g^{n+1}}\,a_{en} \quad = \quad \frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)} \tag{2.22}$$

(this is easy to see by replacing e by f/g in equation 2.16 .)

The point that Euler wants to emphasize is that the quantity $e = f/g$ is not a natural number anymore. He did not care about irrational numbers at this moment and hoped that the formula, if is analytic, could be naturally extended to all real numbers. At this moment he "almost" got the integral representation for the interpolation of the factorial function in equation 2.22. Why "almost"? If $g = 0$ and $f = 1$ the left hand side does not seem to have sense, but the right side is $n!$. Still, he was in the right track. Euler used the integral for the quadrature of the circle as a benchmark to tie his results, so far, to a well known correct formula. Here he goes and uses the particular case of $f = 1$, $g = 2$ (the fraction $e = 1/2$). Then he evaluates formula 2.22 to find

$$\frac{2n+3}{2}\, a_{\frac{1}{2}\, n} = \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots (2n+1)}$$

and since, from equation 2.15

$$a_{\frac{1}{2}\, n} = \int_0^1 (x-x^2)^n dx$$

then assuming $n = 1/2$ he finds

$$2\, a_{\frac{1}{2}\frac{1}{2}} = 2\int_0^1 dx\sqrt{x-x^2}dx = \frac{\pi}{2}$$

The integral is over a circle with center at $(h, k) = (0, 1/2)$ and radius $1/2$ [19]. which has an area of $\pi/4$. See that this matches the entry $\frac{1}{2}\frac{1}{2}$ in table 2.2 as found by Wallis ($f(1/2, 1/2) = 1/a_{\frac{1}{2}\frac{1}{2}} = 4/\pi$).

After some discussion about the binomial coefficient (that somehow looks out of sinchronization with the main ideas on the paper) Euler goes back so equation 2.22. This is, after replacing a_{en} by its integral representation 2.15,

$$\frac{f+(n+1)g}{g^{n+1}}\int_0^1 x^{f/g}(1-x)^n dx \quad = \quad \frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}$$

where, as was indicated above, the substitution of $f = 1$ and $g = 0$, generates the factorial $1\cdot 2\cdot 3\cdots n$. Euler uses the notation

$$\int_0^1 \frac{x^{\frac{f}{0}}(1-x)^n}{0^{n+1}}dx = \lim_{g\to 0}\int_0^1 \frac{x^{\frac{f}{g}}(1-x)^n}{g^{n+1}}dx,$$

In order to evaluate this limit, Euler first makes the change of variables

$$y \quad = \quad x^{(f+g)/g} \qquad y^{g/(f+g)} = x \Rightarrow dx = \frac{g}{f+g}\, y^{-f/(f+g)}dy,$$

so

$$\lim_{g\to 0}\,(f+(n+1)g)\int_0^1 \frac{x^{\frac{f}{g}}(1-x)^n}{g^{n+1}}dx = \lim_{g\to 0}\frac{f+(n+1)g}{f+g}\int_0^1 \frac{(1-y^{\frac{g}{f+g}})^n}{g^n}dy.$$

Then he assumes that the limit can go inside the integral and uses L'Hôpital's rule [20] since both numerator and denominator go to zero as g

[19] $y = \sqrt{x-x^2} \Rightarrow y^2 + x^2 - x = 0 \Rightarrow y^2 + (x-1/2)^2 = 1/4$.

[20] According to Wikipedia this rule was discovered by Johan Bernoulli.

goes to zero. From

$$y^{\frac{g}{f+g}} = \mathrm{e}^{\frac{g}{f+g}\ln y} \quad \Rightarrow \frac{d}{dg} y^{\frac{g}{f+g}} = y^{g/(f+g)} \left(\ln y \frac{(f + \cancel{g} - \cancel{g})}{(f+g)^2} \right)$$

we see

$$\begin{aligned}
\lim_{g \to 0} \frac{f+(n+1)g}{f+g} \int_0^1 \frac{(1-y^{\frac{g}{f+g}})^n}{g^n} dy &= \int_0^1 \lim_{g\to 0} \left(\frac{1-y^{\frac{g}{f+g}}}{g} \right)^n dy \\
&= \int_0^1 \left(\lim_{g\to 0} - \frac{y^{g/(f+g)} f \ln y}{(f+g)^2} \right)^n dy \\
&= \int_0^1 (-\ln y)^n dy
\end{aligned}$$

where Euler assumes $f = 1$ and uses the symbol $\ln y = ly$. This is

$$n! = \int_0^1 (-\ln x)^n dx \tag{2.23}$$

He tests his formula with $3! = 6$ as follows

$$\begin{aligned}
3! &= \int_0^1 (-\ln x)^3 dx \\
&= \int_0^\infty y^3 e^{-y} dy \quad (\text{from} \quad y = -\ln x \quad e^{-y} = x \quad dx = -e^{-y} dy \quad) \\
&= \cancel{-e^{-y} y^3 \Big|_0^\infty} + \int_0^\infty 3\, e^{-y} y^2 dy \qquad\qquad u = y^3 \quad dv = e^{-y} \\
&= \cancel{-2e^{-y} y^2 \Big|_0^\infty} + \int_0^\infty (2\cdot 3)\, e^{-y} y dy \qquad u = y^2 \quad dv = 2e^{-y} \\
&= \cancel{-2\cdot 3 e^{-y} y \Big|_0^\infty} + \int_0^\infty (2\cdot 3)\, e^{-y} dy \qquad u = y \quad dv = 2\cdot 3\, e^{-y} \\
&= 1\cdot 2\cdot 3
\end{aligned}$$

So indeed the formula checks well for $n = 3$, and the pattern is clear of integers $n > 3$ after iterative integration by parts. The next natural test is to find $(1/2)!$, however Euler gets short here and says [21] " *Nevertheless, it is*

[21] from the English translation

very useful for getting the terms of fractional index, especially since, up to this time, not even the most tedious method has been able to do this. If we set $x = 1$ *,*[22] *the corresponding term will be* $\int dx\sqrt{-\ln x}$*, the value of which is given by quadratures. But at the beginning [11] I claimed that this term is equal to the square root of a circle whose diameter is 1. Up to this point, indeed, it is not permissible to draw that conclusion, because of the shortcomings of analysis; below, however, will appear a method by which those intermediate terms can be reduced to the quadratures of algebraic curves. By putting these things together, it may be that no little improvement of analysis will result."*

Thanks to the contributions of Gauss(1777–1855) and Fubini (1879–1943) this integral is relatively easy to evaluate today.

$$\begin{aligned}
\left(\frac{1}{2}\right)! &= \int_0^1 (-\ln x)^{1/2} dx \\
&= \int_0^\infty y^{1/2} \mathrm{e}^{-y} dy \quad \text{substitution} \quad y = -\ln x \\
&= \int_0^\infty (2x^2)\, \mathrm{e}^{-x^2} dx \quad \text{substitution} \quad y^{1/2} = x \\
&= \int_0^\infty x \left[(-2x)\, \mathrm{e}^{-x^2}\right] dx \quad \text{grouping} \\
&= \cancel{-x\mathrm{e}^{-x^2}\Big|_0^\infty} + \int_0^\infty \mathrm{e}^{-x^2} dx \quad \text{by parts} \quad u = x dv = (-2x)\mathrm{e}^{-x^2} \\
&= \frac{\sqrt{\pi}}{2}. \qquad (2.24)
\end{aligned}$$

The last result is shown in appendix B.

2.7 Legendre

According to Gronau, Euler also used the formula

$$\int_0^1 (-\ln x)^{t-1} dx = \int_0^1 \left(\ln \frac{1}{x}\right)^{t-1} dx \qquad (2.25)$$

to refer to the interpolation function of the factorial (here $(t-1)!$). Let us transform this formula by using the change of variables

[22] Evidently a slip for '$n = \frac{1}{2}$'. Tr.

$$y = \ln\left(\frac{1}{x}\right) \quad , \quad \mathrm{e}^{-y} = x \quad , \quad dx = -\mathrm{e}^{-y}dy.$$

Hence

$$\begin{aligned} \int_0^1 \left(\ln\frac{1}{x}\right)^{t-1} dx &= \int_0^\infty y^{t-1}\mathrm{e}^{-y}dy \\ &= \Gamma(t). \end{aligned} \tag{2.26}$$

With this definition $\Gamma(n) = (n-1)!$ instead of $n!$. This has created some controversy. According to Gronau, the reason for this definition of $\Gamma(n) = (n-1)!$ instead of one that produces directly $n!$ is due to the fact that Adrien–Marie Legendre (1752–1833), who gave the name Γ function found 2.25 formula instead of the first version 2.23.

2.8 From Euler to Gauss and Weierstrass

My main source for this section is the article: An Elementary [23] Exposition of the Theory of the Gamma Function, by J.L.W.V. Jensen and T.H. Gronwall, and the book [4] A Course in Modern Analysis [24].

2.8.1 The question of uniqueness

We illustrate how the recursive formula

$$f(z+1) = zf(z) \tag{2.27}$$

could be satisfied by a non countable infinite number of functions.

Let us assume that there is a function $F(z)$ well defined such that $F(z) = F(z+1)$. [25]

Let us define the function $p(z) = F(z)f(z)$. Then

$$p(z+1) = F(z+1)\,f(z+1) = F(z)\,[zf(z)] = z\,F(z)\,f(z) = z\,p(z).$$

[23] https://www.jstor.org/stable/2007272?seq=1#page_scan_tab_contents

[24] https://books.google.com/books?id=ULVdGZmi9VcC

[25] for example a periodic function with period 1.

So, there are at least as many functions that satisfy the recursion $f(z+1) = zf(z)$ as periodic functions with period 1.

Uniqueness is an important point and it took many years before there could be proved uniqueness of the Γ function. We will touch on this point again, after doing some additional development. We just want to be aware that, at the moment, for the analysis on recursive functions of the form 2.27, we will be consider particular solutions of a functional equation.

2.8.2 Gauss form

Let us then assume that a function F satisfies the recursion relation. Then we can build the chain of equations

$$F(z) = \frac{F(z+1)}{z} = \frac{F(z+2)}{z(z+1} = \cdots \frac{F(z+n)}{z(z+1)\cdots(z+n-1)}. \tag{2.28}$$

We assume that $\mathrm{Re}(z) > 0$. If z is a positive number then clearly

$$F(z+n) = (z+n-1)\cdot(z+n-2)\cdots n\cdot(n-1)!\cdot F(1),$$

where clearly we encounter a non uniqueness situation again since F could have been any well defined function, but recall that F is just a particular solution. From this equation it should be clear that

$$\lim_{n\to\infty} \frac{F(z+n)}{n^z(n-1)!} = F(1).$$

Then, from equation 2.28 we see that

$$F(z) = \frac{F(z+n)}{z\cdot(z+1)\cdots(z+n-1)} = \lim_{n\to\infty} \frac{F(1)(n-1)!\,n^z}{z\cdot(z+1)\cdots(z+n-1)}. \tag{2.29}$$

This is an interesting finding. Consider this equation with $n+1$ instead of n and m instead of z. That is

$$F_{n+1}(m+1) = F(1)\lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{m\cdot(m+1)\cdot(n+m)}(n+1)^m.$$

This is (removing the limit symbols) equation 2.7. We are not implying that this is what Euler thought, since the history explained above tells us otherwise. However this line of thought shown by Jensen and Gronwall conveys clearly Euler's equation by using only the idea of the recursive formula 2.27.

Let us rewrite equation 2.29 as

$$F(z) = F(1) \lim_{n\to\infty} \frac{n! n^{z-1}}{z \cdot (z+1) \cdots z+n-1}.$$

Then,

$$F(z+1) = F(1) \lim_{n\to\infty} \frac{n! n^{z}}{z \cdot (z+1) \cdots z+n} = F(1) \lim_{n\to\infty} \frac{n^z}{z} \prod_{i=1}^{n} \frac{i}{z+i}.$$

Now if $F(1) = 1$ Gauss defined

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z}{z} \prod_{i=1}^{n} \frac{i}{z+i}. \tag{2.30}$$

We found that Gauss representation is basically an algebraic transformation of Euler's formula 2.7.

It is interesting to see that Gauss representation could have been obtained directly from Legendre's representation by repeated integration by parts, and the substitution.

$$\mathrm{e}^{-t} = \lim_{n\to\infty} \left(1 + \frac{-t}{n}\right)^n.$$

That is, using uniform convergence of the integral we can interchange the limit operation with the integral operator and find

$$
\begin{aligned}
\Gamma(z) &= \int_0^\infty t^{z-1}\mathrm{e}^{-t}dt \quad , \quad \mathrm{Re}(z) > 0 \\
&= \int_0^\infty t^{z-1} \lim_{n\to\infty}\left(1+\frac{-t}{n}\right)^n dt \\
&= \lim_{n\to\infty}\int_0^n t^{z-1}\left(1+\frac{-t}{n}\right)^n dt \\
&= \cancelto{0}{\frac{t^z}{z}\left(1+\frac{-t}{n}\right)^n\Bigg|_0^n} - \int_0^n \frac{t^z}{z}\left(\frac{-1}{n}\right)\left(1+\frac{-t}{n}\right)^{n-1} \\
&= \cancelto{0}{\frac{t^{z+1}}{z(z+1)}\left(1+\frac{-t}{n}\right)^{n-1}\Bigg|_0^n} - \int_0^n \frac{t^{z+1}}{z(z+1)}\frac{n(n-1)}{n^2}\left(1+\frac{-t}{n}\right)^{n-2}.
\end{aligned}
$$

The process will end after the expression $(1+(-t/n))^n$ is differentiated n times. We would be let with the expression

$$
\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty}\frac{n(n-1)\cdots 1}{z(z+1)(z+2)\cdots(z+n-1)n^n}\int_0^n t^{z+n-1}dt \\
&= \frac{n!n^{n+z}}{z(z+1)(z+2)\cdots(z+n)n^n} \\
&= \lim_{n\to\infty}\int_0^n t^{z+n-1}\frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)} \\
&= \lim_{n\to\infty}\frac{n^z}{z}\prod_{i=1}^{n}\frac{i}{z+i}.
\end{aligned}
$$

which coincides with 2.30 above.

According to Jensen and Gronwall Gauss did not know Euler's product representation, so we believe Gauss found this formula directly from Legendre's integral as shown above. Weierstrass used a different formula which can be derived directly from Gauss formula 2.30. We show this next.

We now show another form of the Γ function which is due to Euler. Start with equation 2.29 and $F(1) = 1$, and note that since

$$
n^z = \left(\frac{2\cdot 3\cdot n}{1\cdot 2\cdot n-1}\right)^z = \prod_{i=1}^{n-1}\frac{i+1}{i} = \prod_{i=1}^{n-1}\left(1+\frac{1}{i}\right).
$$

We can write

$$
\begin{aligned}
\Gamma(z) = \frac{(n-1)!n^z}{z\cdot(z+1)\cdots(z+n-1)} &= \lim_{n\to\infty}\frac{(n-1)!\,n^z}{z\cdot(z+1)\cdots(z+n-1)} \\
&= \lim_{n\to\infty}\frac{n^z}{z}\frac{\prod_{i=1}^{n-1} i}{\prod_{i=1}^{n} z+i} \\
&= \lim_{n\to\infty}\frac{n^z}{z}\prod_{i=1}^{n}\left(1+\frac{z}{i}\right)^{-1} \\
&= \lim_{n\to\infty}\frac{1}{z}\prod_{i=1}^{n-1}\left(1+\frac{1}{i}\right)^{z}\prod_{i=1}^{n}\left(1+\frac{z}{i}\right)^{-1}.
\end{aligned}
$$

According to Whittaker and Watson [4], Euler sent this formula to Goldbach in a letter in 1729. [26].

2.8.3 Weierstrass form

As indicated above Weierstrass representation can be derived directly from Gauss representation. We have not shown what the Weierstrass representation is yet but instead we will derive it. Before we proceed we need to define the *Euler Mascheroni constant* [27] . This is given by

$$
\gamma = \lim_{n\to\infty}\left(\sum_{i=1}^{\infty}\frac{1}{n} - \ln n\right). \tag{2.31}
$$

My notes [28] [3] show that this constant actually exists, how to estimate it, and a geometrical interpretation of it.

We write

[26] printed in Fuss' Corresp. Math.

[27] https://en.wikipedia.org/wiki/Euler%E2%80%93Mascheroni_constant

[28] https://drive.google.com/open?id=0B4W-gdhbNpsDYUxFcWY4N1ZiVzg

$$\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} \frac{n^z}{z} \prod_{i=1}^{n} \frac{i}{z+i} \\
&= \lim_{n\to\infty} \frac{n^z}{z} \prod_{i=1}^{n} \left(1+\frac{z}{i}\right)^{-1} \\
&= \lim_{n\to\infty} \frac{\mathrm{e}^{z\ln n}}{z} \prod_{i=1}^{n} \left(1+\frac{z}{i}\right)^{-1} \\
&= \lim_{n\to\infty} \frac{\mathrm{e}^{\sum_{i=1}^{n} \frac{z}{i} - \sum_{i=1}^{n} \frac{z}{i} + z\ln n}}{z} \prod_{i=1}^{n} \left(1+\frac{z}{i}\right)^{-1}
\end{aligned} \tag{2.32}$$

The reason for the addition and subtraction in the exponential expression is to link the result with the Euler-Mascheroni constant 2.31.

That is, we write

$$\Gamma(z) = \lim_{n\to\infty} \frac{\mathrm{e}^{\sum_{i=1}^{n} \frac{z}{i} - z\gamma}}{z} \prod_{i=1}^{n} \left(1+\frac{z}{i}\right)^{-1}$$

where we already replaced the γ above [29]. Since

$$\mathrm{e}^{\sum_{i=1}^{n} \frac{z}{i}} = \prod_{i=1}^{n} \mathrm{e}^{\frac{z}{i}}.$$

Then

$$\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} \frac{\prod_{i=1}^{n} \mathrm{e}^{\frac{z}{i}}\, \mathrm{e}^{-z\gamma}}{z} \prod_{i=1}^{n} \left(1+\frac{z}{i}\right)^{-1} \\
&= \frac{\mathrm{e}^{-z\gamma}}{z} \prod_{i=1}^{\infty} \frac{\mathrm{e}^{\frac{z}{i}}}{\left(1+\frac{z}{i}\right)}.
\end{aligned}$$

This is Weierstrass representation of the $\Gamma(z)$ function.

[29] I could have waited for this up to the last step but want to save room and writing

According to Jensen and Gronwall, Weierstrass published the multiplicative inverse

$$\frac{1}{\Gamma(z)} = z\mathrm{e}^{z\gamma} \prod_{i=1}^{\infty} \left(1 + \frac{z}{n}\right) \mathrm{e}^{-\frac{z}{n}} \tag{2.33}$$

in *"his famous memoir on analytic factorials"* [30].

In summary: Goldbach was not perhaps a good problem solver but he knew which questions to ask. Thanks to Goldbach initiative, the die started rolling. Goldbach passed the question of the interpolation of the factorial [31] to Nikolaus Bernoulli and later to his Brother Daniel who offered a solution in terms of infinite products. The Bernoulli family were friends of Euler and convinced him to write to Goldbach. Euler in a first letter offered another infinite product solution. Infinite product solutions were not exactly the kind of objects they were looking for. They wanted to have a closed analytical form which is algebraic. That is, a finite sum of finite powers and/or quotients of n with some rational coefficients. When Euler picked $n = 1/2$ for evaluating his formula he found that the result was $\sqrt{\pi}/2$ and this matched well a result obtained by Wallis. He then realized that it was not possible to write this formula as an algebraic formula and derived the formula for the interpolation in terms of an integral of a logarithm function. After his second letter to Goldbach about this, he wrote the great paper [1] about transcendental progressions. Another hint that guided Euler was the fact that the interpolation function for 1/2 was related to the quadrature of the circle. Wallis was really closed and Euler first function (the Beta function) was a closed version of Wallis integral formula. Euler's integral formula was general and it should correspond to the interpolation of $n!$ if after defining $n = f/g$ then $g \to 0$. So he searched for the evaluation of such limit and found it to be the integral of a negative logarithm to the power n. Everyone deserves credit for this development but I believe Euler fairly gets the most. Legendre introduced the symbol Γ and gave the names for the title of these notes (Euler's integrals of first and second type). Later Gauss and then Weierstrass came up with different product representations which are seen to be quite useful

[30] Weierstrass, K., Ueber die Theorie der analytischen FacultAten. Journ. fur Math., vol. 51 (1856), pp. 1-60. Reprinted in Abhandlungen aus der Functionenlehre (Berlin, 1886), pp. 181-262 [and Mathematische Werke, vol. 1 (Berlin, 1894), pp. 153-211].

[31] actually his question was a bit more complicated, it was about certain sum of factorials which is known today as a hypergeometrical function.

for some computations. Complex variable start taking momentum after Euler's death and it will come the desire of every mathematician to extend the knowledge of any real function to the entire complex plane (if possible). This new "interpolation" technique is known today as *analytic continuation* and I will discuss more about it in the following sections.

Next, I discuss the Beta $\mathrm{B}(z, w)$ and Gamma $\Gamma(z)$ functions in more detail.

Chapter 3

The Beta B function

3.1 Definition and Domain of Convergence

According to Wikipedia the name Beta was given by Jacques Binet (1786–1856). As indicated above the Beta function is defined as

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \tag{3.1}$$

and introduced first by Euler in his searching for the interpolation of the factorial (see equation 2.21).

First, we want to evaluate the domain of convergence of the Beta function. Initially in the real numbers and then in the complex. For this let us split the interval of integration in two parts $[0,1/2]$ and $[1/2,1]$. For the interval $[0,1/2]$ we have

$$0 \leq t^{x-1}(1-t)^{y-1} \leq t^{x-1}.$$

and from the p–test C.1, we find that

$$\int_0^{1/2} t^{x-1}(1-t)^{y-1}dt < \infty$$

provided that $x > 0$. Now for $t \in [1/2,1]$ we can do the substitution in 3.1 $u = 1-t$ (restricting the integration interval to $[1/2,1]$) so,

$$\int_{1/2}^1 t^{x-1}(1-t)^{y-1}dt = \int_0^{1/2}(1-u)^{x-1}u^{y-1}du,$$

where now

$$0 \le (1-u)^{x-1}u^{y-1} \le u^{y-1}.$$

and then

$$\int_{1/2}^{1} t^{x-1}(1-t)^{y-1}dt < \infty.$$

provided that $y > 0$. So

$$\begin{aligned} \mathrm{B}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}dt \\ &= \int_0^{1/2} t^{x-1}(1-t)^{y-1}dt + \int_{1/2}^{1} t^{x-1}(1-t)^{y-1}dt < \infty \end{aligned}$$

if $x > 0$ and $y > 0$.

The analytic continuation (extension to complex plane) of the Beta function from $\mathrm{B}(x,y)$ to the (double) complex planes z and w has the same form $\mathrm{B}(z,w)$ as long as the real parts of z and w are positive. That is

$$\mathrm{Re}\, z > 0 \quad \text{and} \quad \mathrm{Re}\, w > 0.$$

3.2 Properties of the Beta B function

3.2.1 Symmetry

We show that $\mathrm{B}(x,y) = \mathrm{B}(y,x)$. That is

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \int_0^1 (1-u)^{x-1}(u)^{y-1}du = \mathrm{B}(y,x)$$

where we used the change of variables $u = 1-t$.

3.2.2 Relation with the Γ function

We first compute $\Gamma(x)\Gamma(y)$. That is,

$$\Gamma(x)\Gamma(y) = \int_0^\infty \mathrm{e}^{-u} u^{x-1} du \int_0^\infty \mathrm{e}^{-v} v^{y-1} dv = \int_0^\infty \int_0^\infty e^{-u-v} u^{x-1} v^{y-1} du dv.$$

We now perform the following change of variables.

$$u(t,z) = z\,t \quad , \quad v(z,t) = z\,(1-t) \quad , \quad du\,dv = \frac{\partial(u,v)}{\partial(t,z)} dt dz,$$

with

$$\frac{\partial(u,v)}{\partial(t,z)} = \begin{vmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} z & t \\ -z & 1-t \end{vmatrix} = z(1-t) + zt = z.$$

Let us find the new domain for the variables t and z. Both u and v vary between 0 and ∞. From the definition of $v(t,z)$ we see that $t < 1$, otherwise v could be negative, and from the definition $u(t,z)$, we see that $t \geq 0$. Hence we have that $t \in [0,1]$. Clearly $z \in [0,\infty)$. We then have that

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^\infty dz \int_0^1 dt\; e^{-z} (z\,t)^{x-1} \,(z\,(1-t))^{y-1} z \\ &= \int_0^\infty dz e^{-z} z^{x+y-1} \int_0^1 t^{x-1} (1-t)^{y-1} dt = \Gamma(x+y)\mathrm{B}(x,y). \end{aligned}$$

Hence we found the important relation

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)\mathrm{B}(x,y) \quad , \quad \mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3.2}$$

In particular the binomial coefficient $\binom{n+m}{n}$ can be written as

$$\binom{n+m}{n} = \frac{(n+m)!}{n!m!} = \frac{\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+1)} = \frac{1}{\mathrm{B}(m+1,n+1)}.$$

3.2.3 Algebraic and Trigonometric Integrands

We present a few more important representations of the Beta function.

If in the definition 3.1 we use the substitution $t = s/(s+1)$, then

$$1 - t = 1 - \frac{s}{s+1} = \frac{1}{s+1} \quad , \quad dt = \frac{dt}{ds}ds = \frac{(s+1)-s}{(s+1)^2}ds = \frac{1}{(s+1)^2}ds$$

and since $t \in [0,1]$ we find that $s \in [0, \infty)$, so

$$\mathrm{B}(x,y) = \int_0^\infty \frac{s^{x-1}}{(s+1)^{x-1}} \frac{1}{(s+1)^{y-1}} \frac{1}{(s+1)^2} ds = \int_0^\infty \frac{s^{x-1} ds}{(s+1)^{x+y}} \tag{3.3}$$

with $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$. This is the Beta function representation with a rational integrand. To find the trigonometric integrand representation we make the substitution $t = \cos^2 \phi$ in equation 3.1. That is,

$$t = \cos^2 \phi \quad , \quad dt = -2 \sin \phi \cos \phi \, d\phi$$

and so, since $t \in [0,1]$, we see that $\phi \in [\pi/2, 0]$. Then

$$\begin{aligned} \mathrm{B}(x,y) &= -2 \int_{\pi/2}^0 \cos^{2x-2} \phi \sin^{2y-2} \phi \cos \phi \sin \phi d\phi \\ &= 2 \int_0^{\pi/2} \cos^{2x-1} \phi \sin^{2y-1} \phi d\phi \end{aligned} \tag{3.4}$$

with $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$.

Chapter 4

The Gamma Γ function

4.1 Definition and Domain of Convergence

As indicated in 2.26, the Γ function is defined as

$$\Gamma(x) = \int_0^\infty \mathrm{e}^{-t} t^{x-1} dt. \tag{4.1}$$

where, for the moment, we consider $x \in \mathbb{R}$. We study the domain of convergence, a few properties and its analytic continuation. Initially the function 4.1 has absolute convergence for $x > 0$. Let us see why.

$$\Gamma(x) = \int_0^1 \mathrm{e}^{-t} t^{x-1} dt + \int_1^\infty \mathrm{e}^{-t} t^{x-1} dt$$

The first integral is bounded since

$$\int_0^1 \mathrm{e}^{-t} t^{x-1} dt \leq \int_0^1 t^{x-1} dt = \frac{t^x}{x}$$

which has a finite value as long as $x > 0$. If $t > 1$, then Taylor expansion for e^t has a generic term $t^n/n! > 0$. Since every term T is positive, the sum of all of them (e^t) is larger than T. That is

$$\mathrm{e}^t > \frac{t^n}{n!} \quad \Longrightarrow \quad \mathrm{e}^{-t} < \frac{n!}{t^n} \quad \Longrightarrow \quad \mathrm{e}^{-t} t^{x-1} < \frac{n!}{t^{n+1-x}}.$$

so by choosing $n > x$

$$\left|\int_1^\infty \mathrm{e}^{-t} t^{x-1} dt\right| < \left|\int_1^\infty \frac{n!}{t^{n+1-x}} dt\right| = \left.\frac{n!(n-x)}{t^{n-x}}\right|_1^\infty = n!(n-x).$$

since $n - x > 0$. That is, for each $x < \infty$ we find that the $\Gamma(x)$ function to converges.

We want to extend the Γ function analytically, but before, we need to find a few important properties.

4.2 Properties of the Gamma Function

4.2.1 The Recursion Formula

From integration by parts

$$\Gamma(x) = \cancelto{0}{-\mathrm{e}^{-t} t^{x-1}\Big|_0^\infty} + \int_0^\infty \mathrm{e}^{-t}(x-1)t^{x-2} dt = (x-1)\Gamma(x-1). \tag{4.2}$$

4.2.2 Euler's reflection formula

The formula states that

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin \pi x} \tag{4.3}$$

Note that with this formula it is inmediate that, using $x = 1/2$,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

consistent with equations 2.13 and 2.24 .

We proof the reflection formula using contour integration. Let us start with equation 3.2 with $y = 1 - x$, and $0 < x < 1$. That is

$$\Gamma(x)\Gamma(1-x) = \mathrm{B}(x, 1-x).$$

We now show, by using contour integration, that

$$\mathrm{B}(x, 1-x) = \frac{\pi}{\sin \pi x}.$$

For this, we use the Beta function representation 3.3, which for $y = 1 - x$ becomes

$$\mathrm{B}(x, 1-x) = \int_0^\infty \frac{s^{x-1} ds}{s+1} \tag{4.4}$$

Here is where we need to use contour integrals. We evaluate this integral using two different techniques:

4.2.2.1 Substitution of the independent variable for an exponential and contour integration

We first make the substitution $s = e^t$, $ds = e^t dt$, and $t \in (-\infty, \infty)$. So we need to compute

We compute

$$\mathrm{B}(x, 1-x) = \int_{-\infty}^\infty \frac{\mathrm{e}^{t(x-1)} \mathrm{e}^t dt}{\mathrm{e}^t + 1} = \int_{-\infty}^\infty \frac{\mathrm{e}^{tx} dt}{\mathrm{e}^t + 1} \quad , \quad 0 < x < 1.$$

Let us consider the contour integral

$$I = \int_C f(z) dz, \tag{4.5}$$

with

$$f(z) = \frac{\mathrm{e}^{zx}}{\mathrm{e}^z + 1} \quad , \quad 0 < x < 1.$$

and C is the contour that we need to determine. In the complex plane, the poles of the integrand are the roots of $e^z + 1$, that is $\mathrm{e}^z = -1 = \mathrm{e}^{(2k+1)\mathrm{i}\pi}$ so the roots are $z_k = (2k+1)\mathrm{i}\pi$, for $k = 0, \pm 1, \pm 2, \cdots$. Then $f(z)$ has an infinite number of poles all lying on the imaginary axis. We will select a contour that has only one pole as shown in Figure 4.1. The contour C can be seen as the union of $C = C_1 \cup C_2 \cup C_3 \cup C_4$, where C_1 and C_3 are horizontal lines from $-R$ to R with opposite orientation. We want to let R grow to ∞. The paths

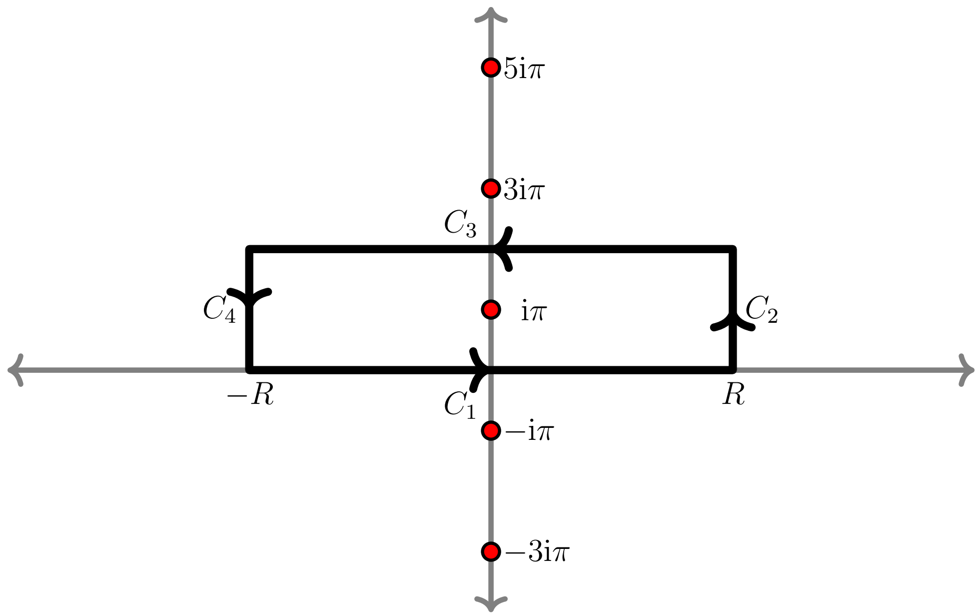


Figure 4.1: The contour C to compute integral 4.5.

C_2 and C_4 are vertical lines between 0 and $2\pi\mathrm{i}$ with opposite orientations showed in the figure. From the Cauchy Residue Theorem [1] we evaluate the integral over C. The residue corresponding to the pole $z_0 = \pi\mathrm{i}$, is computed using the expression

$$\lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0}\frac{(z-z_0)\,\mathrm{e}^{zx}}{\mathrm{e}^z+1} = \lim_{z\to z_0}\frac{\mathrm{e}^{zx}+(z-z_0)\mathrm{e}^{zx}}{e^z} = \mathrm{e}^{z_0(x-1)}.$$

where we used L'Hôpital's rule.

Hence $I = 2\pi\mathrm{i}\ \mathrm{e}^{\pi i(x-1)}$ since the only residue inside the contour is at $z = \mathrm{i}\pi$. That is,

$$2\pi\ \mathrm{i}\ \mathrm{e}^{\mathrm{i}\pi(x-1)} = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)dz.$$

We want to find $I_1 = \int_{C_1} f(z)dz$, as $R\to\infty$. Let us first find the integral along the horizontal path C_3.

[1]https://en.wikipedia.org/wiki/Residue_theorem

$$\begin{aligned} I_3 &= \int_R^{-R} \frac{\mathrm{e}^{(t+2\pi\mathrm{i})x}}{\mathrm{e}^{t+2\pi\mathrm{i}}+1}dt \\ &= \mathrm{e}^{2\pi\mathrm{i}x}\int_R^{-R} \frac{\mathrm{e}^{tx}}{\mathrm{e}^{t+2\pi\mathrm{i}}+1}dt \\ &= \mathrm{e}^{2\pi\mathrm{i}x}\int_R^{-R} \frac{\mathrm{e}^{tx}}{\mathrm{e}^{t}+1}dt \\ &= -\mathrm{e}^{2\pi\mathrm{i}x} I_1, \end{aligned}$$

where we reversed the sign since I_1 is computed from $-R$ to R instead of going in the opposite direction.

The integral I_2 along the path C_2 is evaluated as follows

$$I_2 = \int_0^{2\pi} \frac{\mathrm{e}^{(R+\mathrm{i}t)x}}{\mathrm{e}^{R+\mathrm{i}t}+1}dt = \frac{\mathrm{e}^{Rx}}{\mathrm{e}^R}\int_0^{2\pi} \frac{\mathrm{e}^{(\mathrm{i}t)x}}{\mathrm{e}^{\mathrm{i}t}+1/\mathrm{e}^R}dt = \mathrm{e}^{R(x-1)}\int_0^{2\pi} \frac{\mathrm{e}^{(\mathrm{i}t)x}}{\mathrm{e}^{\mathrm{i}t}+1/\mathrm{e}^R}dt.$$

Now, since $0 < x < 1$ (so $x-1 < 0$), and the last integral is bounded we have that $\lim_{R\to\infty} I_2 = 0$. The same argument applies for the integral I_4 along the path C_4. We then have that, from $I = I_1 + I_2 + I_3 + I_4$,

$$2\pi\mathrm{i}\mathrm{e}^{\mathrm{i}\pi(x-1)} = (1-\mathrm{e}^{2\pi\mathrm{i}x})I_1$$

and

$$I_1 = \int_0^\infty \frac{s^{x-1}ds}{s+1} = \frac{2\pi\mathrm{i}\mathrm{e}^{\mathrm{i}\pi(x-1)}}{1-\mathrm{e}^{2\pi\mathrm{i}x}} = \frac{\pi}{\frac{\mathrm{e}^{\mathrm{i}\pi x}-\mathrm{e}^{-\mathrm{i}\pi x}}{2\mathrm{i}}} = \frac{\pi}{\sin \pi x}.$$

4.2.2.2 Direct evaluation using contour integration

We use the contour shown in Figure 4.2 The singularities are a pole at $s = -1$ and a branch point at $s = 0$ where the function is multivalued. We can use the positive x axis as a branch cut. The contour encloses the pole, so the integral along the contour $C = C_1 \cup C_2 \cup C_3 \cup C_4$ is given by

$$\int_C \frac{s^{x-1}ds}{s+1} = 2\pi\mathrm{i}(-1)^{x-1} = 2\pi\mathrm{i}\,\mathrm{e}^{\mathrm{i}\pi(x-1)}.$$

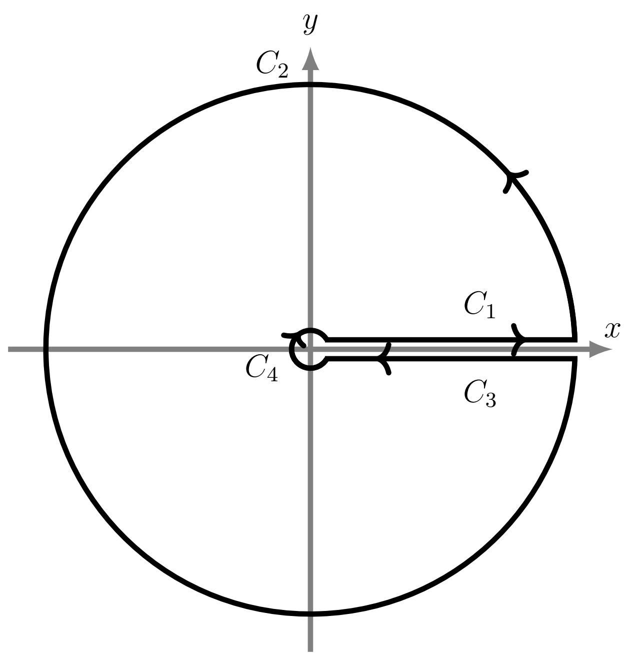


Figure 4.2: Contour for integration of equation 4.4

since $(-1)^{x-1} = \mathrm{e}^{-\mathrm{i}\pi(x-1)}$ is the only residue of the integrand. We now evaluate the four individual integrations for the four paths in the figure. Let us call $I_i = \int_{C_i} s^{x-1}/(1+s)ds$. We start with the integrals along the circular paths. For the small circle we can write $s = \epsilon \mathrm{e}^{\mathrm{i}\theta}$ where $\theta \in [\delta, 2\pi - \delta]$ with ϵ the radius of the disk, and δ the initial angle of integration. We then change variables from s to θ with $ds = \mathrm{i}\epsilon\mathrm{e}^{\mathrm{i}\theta}$. That is,

$$|I_4| = \left| \int_{2\pi-\delta_\epsilon}^{\delta_\epsilon} \frac{\mathrm{i}\epsilon^x \mathrm{e}^{\mathrm{i}\theta(x+1)} d\theta}{\epsilon\mathrm{e}^{\mathrm{i}\theta} - 1} \right| \leq \int_{\delta_\epsilon}^{2\pi-\delta_\epsilon} \frac{|\epsilon^x| d\theta}{1-\epsilon} = (2\pi - 2\delta_\epsilon)\frac{|\epsilon^x|}{1-\epsilon}$$

which goes to 0 as $\epsilon, \delta_\epsilon \to 0$, since $0 < x < 1$. Likewise along the big circle $s = R\mathrm{e}^{\mathrm{i}\theta}$ and $ds = \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}d\theta$, and so

$$|I_2| = \left| \int_{\delta_R}^{2\pi-\delta_R} \frac{\mathrm{i}R^x \mathrm{e}^{\mathrm{i}\theta(x+1)} d\theta}{R\mathrm{e}^{\mathrm{i}\theta} - 1} \right| \leq \int_{\delta_R}^{2\pi-\delta_R} \frac{|R^x| d\theta}{R-1} = (2\pi - 2\delta_R)\frac{R^x}{R-1}$$

We use the L'Hôpital rule to find that

$$\lim_{R\to\infty}|I_2| = \lim_{R\to\infty} 2(\pi-\delta_R)\frac{xR^{x-1}}{1} = \lim_{R\to\infty}\frac{x}{R^{1-x}} = 0$$

since $1 > 1-x > 0$. We are left with the integrals along C_1 and C_3. If in the integral along C_1 we take the limit as $\epsilon \to 0, R \to \infty$, is the original integral 4.4. On the other hand, the integral over C_3 has the argument shifted by 2π with respect to the original integral. That is,

$$\lim_{\epsilon\to 0, R\to\infty}\int_{C_3}\frac{s^{x-1}ds}{(s+1)^{x+1}} = \int_{\infty}^{0}\frac{\mathrm{e}^{2\pi\mathrm{i}(x-1)}s^{x-1}ds}{(\mathrm{e}^{2\pi\mathrm{i}}s+1)^{x+1}} = -\mathrm{e}^{2\pi\mathrm{i}(x-1)}\int_0^\infty\frac{s^{x-1}ds}{(s+1)^{x+1}}$$

Putting all integrals together we find that

$$(1-\mathrm{e}^{2\mathrm{i}\pi(x-1)})\int_0^\infty\frac{s^{x-1}ds}{(s+1)^{x+1}} = 2\pi\ \mathrm{i}\ \mathrm{e}^{\mathrm{i}\pi(x-1)}.$$

Hence

$$\int_0^\infty\frac{s^{x-1}ds}{(s+1)^{x+1}} = \frac{2\pi\mathrm{i}\mathrm{e}^{\mathrm{i}\pi(x-1)}}{1-\mathrm{e}^{2\mathrm{i}\pi(x-1)}} = \frac{\pi}{(-\mathrm{e}^{-\pi x}+\mathrm{e}^{\mathrm{i}\pi x})/2i} = \frac{\pi}{\sin\pi x}$$

We then showed that Euler's reflection formula 4.3 is correct.

4.2.3 Derivative of the Gamma function

We introduce the logarithmic derivative of the Gamma function.

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz}\ln\Gamma(z).$$

The logarithmic derivative of the Gamma function is also known as the *digamma function* [2]. . A second derivative would be called *trigamma* a third *tetragamma* , a fourth *pentagamma* , and so forth. A more general term is the *polygamma function* which involves multiple derivatives. .

[2] https://en.wikipedia.org/wiki/Digamma_function

Let us recall the Weierstrass represention of the multiplicative inverse of the Gamma function 2.33. If we take logarithm in both sides of the equation we find

$$-\ln\Gamma(z) = \ln z + \gamma z + \sum_{i=1}^{\infty} \ln\left(1+\frac{z}{n}\right) - \frac{z}{n}.$$

We now take the derivative with respect to z to find

$$\begin{aligned}
\psi(z) &= \frac{\Gamma'(z)}{\Gamma(z)} \\
&= -\frac{1}{z} - \gamma + \sum_{i=1}^{\infty} \frac{1}{i} - \frac{1}{x+i} \\
&= -\gamma + \sum_{i=1}^{\infty} \frac{1}{i} - \frac{1}{x+i-1} \\
&= -\gamma + \sum_{i=1}^{\infty} \frac{x-1}{i(x+i-1)}.
\end{aligned} \tag{4.6}$$

From the recursion formula $\Gamma(z+1) = z\Gamma(z)$ and taking derivatives see that, after dividing by $\Gamma(z+1)$,

$$\psi(z+1) = \frac{1}{z} + \psi(z).$$

From this recursion formula we can evaluate $\psi(x)$ when n is a positive integer, assuming we know $\psi(1)$. That is, for example

$$\begin{aligned}
\psi(2) &= \frac{1}{1} + \psi(1) = 1 + \psi(1) \\
\psi(3) &= \frac{1}{2} + \psi(2) = 1 + \frac{1}{2} + \psi(1) \\
&\vdots \\
\psi(n+1) &= \sum_{k=1}^{n} \frac{1}{k} + \psi(1).
\end{aligned}$$

Clearly, from equation 4.6 $\psi(1) = -\gamma$, so

$$\psi(n+1) = \sum_{k=1}^{n} \frac{1}{k} - \gamma.$$

4.3 Analytic continuation

This section shows a diversity of methods that could be used to continue analytically the $\Gamma(z)$ function. They are: using a functional equation, doing integration by parts, recursive formula, and finally using a contour integral (Hankel–type contour).

We first observe, from the recursion formula 4.2, that if n is a positive integer $\Gamma(n) = (n-1)!$. Then the $\Gamma(x)$ function is an extension of the factorial to the positive real numbers. We do not say yet that this is an analytic continuation. Analytic continuation is achieved form a region of the complex plane to another region of the complex plane, not from a simple set of discrete points. The extension from n to x when moving from the factorial to the Γ function is known as an interpolation since we fill the wholes in between positive integers preserving the values of the the function and factorial in the positive integers.

The $\Gamma(x)$ function is define for $x > 0$. We can continue this function to the complex numbers z such that $\mathrm{Re}(z) > 0$, since if $z = x + \mathrm{i}y$

$$|t^{z-1}| = |\mathrm{e}^{(z-1)\ln t}| = |\mathrm{e}^{x\ln t - \mathrm{i}y\ln t}| = |\mathrm{e}^{x\ln t - \ln t}\mathrm{e}^{-\mathrm{i}y\ln t}| = |\mathrm{e}^{(x-1)\ln t}| = |t^{x-1}|,$$

where we used $|\mathrm{e}^{-\mathrm{i}y\ln t}| = 1$. That is, by moving up or down from the real axis we are not increasing the size of the integrand, just the phase. We then say that the $\Gamma(z)$ function is convergent for $\mathrm{Re}(z) \geq 0$.

4.3.1 From the reflection formula

From Euler's reflection formula 4.3 we see that

$$\Gamma(z) = \frac{\pi}{\Gamma(1-z)\,\sin\pi z} \tag{4.7}$$

The function $\Gamma(1-z)$ on the right is analytic for $1 - \text{Re}(z) > 0$, that is, for $\text{Re}(z) < 1$, and from the $\sin \pi z$, we see that the function is analytic for $z < 1$ and $z \neq \pm 1, \pm 2, \cdots$. We could argue that formula 4.3 has a restriction on the left side because $\Gamma(z)$ is valid only for $\text{Re}(z) > 0$, and so $\Gamma(1-z)$ is defined for $0 < \text{Re}(z) < 1$, so only on the strip $0 < \text{Re}(z) < 1$ is the equation valid. However the right hand side of equation 4.3 is valid in the whole complex plane $\mathbb{C}$ except for the points $\pm 1, \pm 2, \cdots$. We then say that the right hand side is the analytic continuation of the function on the left of the equal sign beyond the strip $0 < \text{Re}(z) < 1$. The right hand side of the equation 4.7 is the analytic continuation of the $\Gamma(z)$ function into the left complex plane. Both functions are well defined and agree for $0 < \text{Re}(z) < 1$ but only the left side is defined for $\text{Re}(z) < 0$. The original definition of the Γ function 4.1 coincides with this new from in the interval $0 < \text{Re}(z) < 1$.

4.3.2 Using Integration by parts

Let us start with the definition of the $\Gamma(z)$ function:

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1} dt.$$

The integral has two factors t^{z-1} and e^{-t}. For finite negative and all positive t values, the exponential function has a good behavior. On the other hand the function t^{z-1} requires $\text{Re}(z) - 1 > 0$ to avoid poles at $t = 0$. If we push the exponent to a higher value, we could increase the domain of analyticity. This is done by integration by parts. That is, we define $dv = t^{z-1}dt$, $u = \mathrm{e}^{-t}$ and

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty \mathrm{e}^{-t} t^{z-1} dt \quad , \quad \text{Re}(z) > 0 \\
&= \cancelto{0}{\left.\frac{e^{-t}t^z}{z}\right|_{t=0}^{\infty}} + \frac{1}{z}\int_0^\infty t^z \mathrm{e}^{-t} dt \quad , \quad \text{Re}(z) > -1 \quad , \quad z \neq 0.
\end{aligned}$$

We observe that the last integral has a wider domain of convergence. We could do one more integration by parts to find

$$\Gamma(z) = \frac{1}{z(z+1)} \int_0^\infty t^{z+1} \mathrm{e}^{-t} dt \quad \text{Re}(z) > -2, \quad , \quad z \neq 0, -1.$$

and extend this to any finite negative number n as

$$\Gamma(z) = \frac{1}{z(z+1)\cdots(z+n)}\int_0^\infty t^{z+n}\mathrm{e}^{-t}dt \quad \mathrm{Re}(z) > n-1\ ,\ z \neq 0, -1, \cdots n.$$

We see then that the $\Gamma(z)$ function can be analytically continued to the plane $\mathrm{Re}(z) < 0$ and has simple poles at $z = 0, -1, -2, \cdots$.

4.3.3 Using the Recursion Formula

This is no different from the integration by parts above. After all, the recursion formula was obtained by integration by parts. From the recursion formula 4.2 that

$$\Gamma(z+1) = z\Gamma(z).$$

This means that if $z \neq 0$, we can write

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1). \tag{4.8}$$

where the function on the right is valid for $\mathrm{Re}(z+1) > 0$, and $z \neq 0$. That is $\mathrm{Re}(z) > -1, z \neq 0$. Then the symbol $\Gamma(z)$ on the left is understood as the new extended Γ function, or analytically continued to the complex numbers $\mathrm{Re}(z) > -1$, $z \neq 1$. In general for any finite negative number n we can write

$$\Gamma(z) = \frac{1}{z(z+1)\cdots(z+n)}\Gamma(z+n+1).$$

Figure 4.3 illustrates a piece of the $\Gamma(z)$ function.

Again, from the equation above, it is clear that the $\Gamma(z)$ function has simple poles at $z = 0, -1, -2, \cdots$. Let us find the residues of those poles. We write

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)}$$

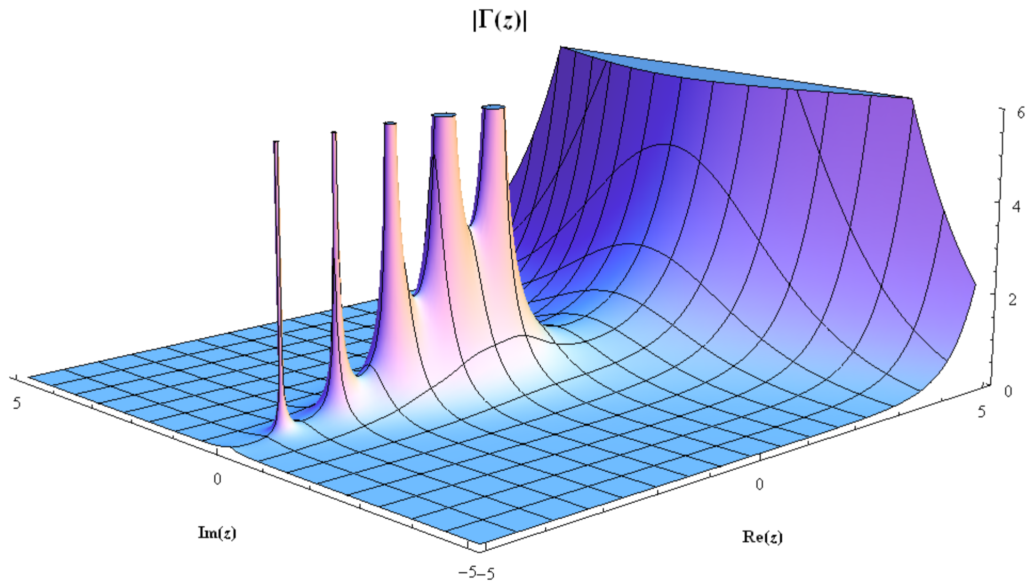


Figure 4.3: $|\Gamma(z)|$ function. The truncated spikes show clearly the singularities (simple poles) at the negative integer numbers.(taken from Wikipedia)

Let us start with the simple case of $n = 0$. That is

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

where clearly the function $\Gamma(z)$ has a simple pole at $z = 0$. with residue

$$\mathrm{Res}_{z=0}\Gamma(z) = \lim_{z\to 0}\Gamma(z+1) = 1.$$

In general for the residue at $z = -n$ we compute

$$\lim_{z\to -n}(-1)^n \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)} = \frac{(-1)^n}{n!}.$$

4.3.4 Replacing Factors by Sums or Integrals

From definition 4.1

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1} dt.$$

Since e^{-t} goes to zero much faster than t^{z-1} as $t \to \infty$ then the problem happens at $t \to 0$ where t^{z-1} diverges. We split the integral in two parts:

$$\Gamma(z) = \underbrace{\int_0^1 \mathrm{e}^{-t} t^{z-1} dt}_{\text{Analytic for Rez>0}} + \underbrace{\int_1^\infty \mathrm{e}^{-t} t^{z-1} dt}_{\text{Entire}}$$

and rewrite the first part as

$$\begin{aligned}
\int_0^1 t^{z-1}\, \mathrm{e}^{-t} dt &= \int_0^1 t^{z-1} \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 t^{z-1+n}\, dt \quad \text{The sum is abs. convergent} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{t^{z+n}}{z+n}\bigg|_0^1 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n} < \infty, \quad \forall\, z \neq -n, \quad n \geq 0.
\end{aligned}$$

So

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n} + \int_1^\infty \mathrm{e}^{-t} t^{z-1} dt.$$

is the analytic continuation of the $\Gamma(z)$ function to the complex plane except at points $n = 0, -1, -2 \cdots$.

4.3.5 Using a Hankel Countour Integral

Hermann Hankel [3] in 1884 derived the analytic continuation of the $\Gamma(z)$ function by using a contour integral on the complex plane. Here we illustrates Hankel's method.

The problem with the $\Gamma(z)$ integral comes from the factor t^{z-1}. At $t = 0$ the function has a branch point since it is multivalued for non integer z. We could think of the positive x-axis as a branch cut and draw a contour that does not include the singularity at $t = 0$.

[3] https://en.wikipedia.org/wiki/Hermann_Hankel

Figure 4.4 shows the Hankel's contour C to evaluate the integral

$$I = \int_C \mathrm{e}^{-t} t^{z-1} dt \tag{4.9}$$

where now t is a complex variable. The integrand is analytic as a function of t since the contour C does not goes through the singularity $t = 0$. The

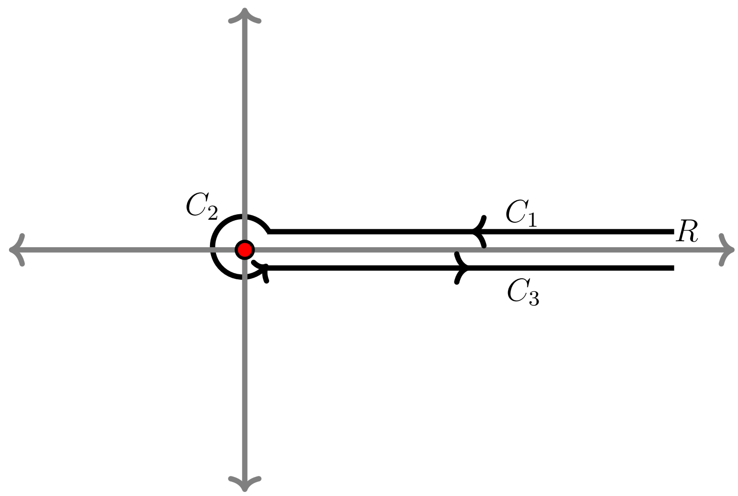


Figure 4.4: Hankel contour C to compute integral 4.9.

integral is bound for all z since now there is no singularity and the exponential absorbes most of the amplitude on the integral. That is, the decaying exponential overpowers the possible raising of t^{z-1} for $\mathrm{Re}(z) \gg 0$. We see that the negative real part of z is a small non-zero number. The contour C, oriented in a counter-clockwise direction, is the union of three contours. That is,

$$C = C_1 \cup C_2 \cup C_3.$$

The contour C_1 is the set of $z = x + iy$, with $y = \epsilon$, and x going from R to ϵ. The path C_2 is a tiny circle of radius ϵ going counter-clockwise and centered at zero. That is, here $t = \epsilon \mathrm{e}^{\mathrm{i}\theta}$ for $\theta \in [2\delta, 2\pi - \delta]$, for a small angle δ. Finally, the contour C_3 is the reverse of contour C_1. It is $z = x + iy$ with $x \in [\epsilon, R]$ and $y = -\epsilon$. We do the integral in three steps

$$I_i = \int_{C_i} f(t)dt$$

for $f(z) = \mathrm{e}^{-t}t^{z-1}$. Then take the limit as $\epsilon, \delta \to 0$, and $R \to \infty$. Let us start with the integral around C_2. This is, $dt = \mathrm{i}\epsilon\mathrm{e}^{\mathrm{i}\theta}d\theta$ and

$$I_2 = \mathrm{i}\,\epsilon \int_{\delta}^{2\pi-\delta} \mathrm{e}^{-\mathrm{i}\epsilon\mathrm{e}^{\mathrm{i}\theta}}(\mathrm{e}^{-\mathrm{i}\theta})^{z-2}d\theta.$$

Since the integrand is bounded then we have that $\lim_{\epsilon\to 0} I_2 = 0$. We are left with two integrals. The integral over the top path which is

$$I_1 = -\int_0^R \mathrm{e}^{-x-\mathrm{i}\epsilon}(x + \mathrm{i}\epsilon)^{z-1}dx.$$

Clearly, as $\epsilon \to 0$, $\delta \to 0$, and $R \to \infty$ we have that

$$\lim_{\epsilon\to 0, R\to\infty} I_1 = -\Gamma(z).$$

For the third contour C_3 we have that the argument of the complex variable t is $2\pi i - \delta$ since the contour already rotated that much. Then t along this contour has a phase shifted by 2πi, or we understand here that $t = 2\pi\mathrm{i}(x-\mathrm{i}y)$. Then

$$I_3 = \int_0^R e^{2\pi\mathrm{i}-x-\mathrm{i}\epsilon}[\mathrm{e}^{2\pi\mathrm{i}}(x + \mathrm{i}\epsilon y)]^{z-1} = \mathrm{e}^{2\pi zi}\int_0^R \mathrm{e}^{-x-\mathrm{i}\epsilon}(x + \epsilon y)^{z-1},$$

and so

$$\lim_{\epsilon\to 0, R\to\infty} I_3 = \mathrm{e}^{2\pi\mathrm{i}z}\Gamma(z).$$

This is

$$\lim_{\epsilon\to 0,\delta\to 0,R\to\infty} I = (\mathrm{e}^{2\pi\mathrm{i}}\mathrm{z} - 1)\Gamma(z),$$

and we found that the Hankel contour integral for the $\Gamma(z)$ function is

$$H(z) = \frac{1}{\mathrm{e}^{2\pi z\mathrm{i}} - 1}\int_C \mathrm{e}^{-t}t^{z-1}dt. \tag{4.10}$$

is such that as the contour C deforms taking the limit $R \to \infty$, $\delta \to 0$, $\epsilon \to 0$, we find

$$\lim_{\epsilon\to 0,\delta\to 0,R\to\infty} H = \Gamma(z),$$

and $H(z)$ is the analytic continuation of the $\Gamma(z)$ function.

We can write equation 4.10 differently by knowing that

$$\mathrm{e}^{2\pi z\mathrm{i}} - 1 = \frac{\mathrm{e}^{\pi z\mathrm{i}} - \mathrm{e}^{-\pi z\mathrm{i}}}{\mathrm{e}^{-\pi z\mathrm{i}}} = \frac{2\mathrm{i}\sin\pi z}{\mathrm{e}^{-\pi z\mathrm{i}}} = \frac{2\mathrm{i}\sin\pi z}{(-1)^z}$$

Then

$$H(z) = -\frac{1}{2\mathrm{i}\sin\pi z}\int_C \mathrm{e}^{-t}(-t)^{z-1}dt, \tag{4.11}$$

which as an analytic continuation of the $\Gamma(z)$ function has simple poles $z = 0, -1, -2, \cdots$.

4.4 Relationship between the Laplace transform and the Gamma function

The Gamma function is related to the Laplace transform of t^{z-1}. Let us see,

$$\begin{aligned} \Gamma(z) &= \int_0^\infty \mathrm{e}^{-t}t^{z-1}dt \qquad (4.12) \\ &= \int_0^\infty \mathrm{e}^{-st}(st)^{z-1}sdt \\ &= s^z\mathcal{L}(t^{z-1}) \end{aligned}$$

We can write

4.5 Logarithmic Convexity

This session illustrates some important inequalities related to the $\Gamma(x)$ function. For the moment, we focus on x as a real number. We use the concept of convexity to help on the establishment of such inequalities.

A function $g : \mathbb{R} \to \mathbb{R}$ is called *convex* on an interval $[\alpha, \beta] \subseteq \mathbb{R}$ if and only if

$$\begin{aligned} &\forall a, b \in [0,1] : \exists : a + b = 1 : \forall x_1, x_2 \in [\alpha, \beta] : \\ &g(ax_1 + bx_2) \leq ag(x_1) + bg(x_2). \end{aligned} \tag{4.13}$$

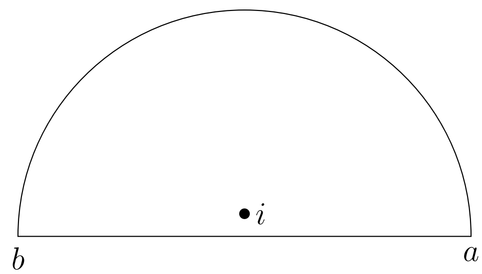
Convexity, means that the segment that joints two points in the graph of the function, lies under the curve. The opposite term is *concavity* . In this case the segment that joints two points of the curve lies above the curve. From calculus, convexity means that the first derivative, if it exists is increasing, and the second derivative (if it exists) is positive in the observation interval. If we multiply a function by -1 it changes convexity. That is, if it is convex it will become concave and vice-versa. The $\Gamma(x)$ function is convex for $x > 0$, but for $x < 0$ it switches between convex and concave at each integer interval of the form $[n, n+1]$ where $n < 0$, is integer. This switching is due to the flip on signs implied by equation 4.8. A function $g(x)$ is called logarithmically convex if its logarithm is convex. If $g'(x)$ exists then it is monotonically increasing. If g is twice differentiable then $g''(x) > 0$. In this case

$$\begin{aligned} (\ln g(x))' &= \frac{g'}{g} \qquad \text{and} \\ (\ln g(x))' = &= \frac{g''g - g'^2}{g^2} \qquad \text{and} \end{aligned}$$

The $\Gamma(x)$ function is logarithmically convex.

We prove this.

Work In Progress ...



On the other hand,

$$
\begin{aligned}
\Gamma(z+1) &= \lim_{n\to\infty} \frac{n!n^{z+1}}{(z+1)(z+2)\cdots(z+1+n)} \\
&= \lim_{n\to\infty} \left(z\frac{n!n^{z}}{(z+1)(z+2)\cdots(z+n)} \frac{n}{z+1+n} \right) \\
&= z\Gamma(z) \lim_{n\to\infty} \frac{n}{z+1+n} \\
&= z\Gamma(z).
\end{aligned}
$$

the following replacement by a contour integral. Let the contour γ be

4.6 Relation Between the Γ function and the Hurwitz-Riemann Zeta function

The definition of the Hurwitz-Riemann Zeta function is

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \quad , \quad x > 0 \tag{4.14}$$

with $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$. If $x = 1$ this equation represents the Riemann Zeta function

$$\zeta(s) = \zeta(s,1) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad , \quad \mathrm{Re}(s) > 1.$$

By default we assume $\zeta(s,1) = \zeta(s)$.

Here we use a method which uses the Euler $\Gamma(z)$ function to find a formula for the Hurwitz-Riemann Zeta function in terms of the original Riemann Zeta function. Then we find a representation of the Hurwitz-Riemann Zeta function in terms of the $\Gamma(z)$ function.

Let us expand $(1-z)^{-s}$ in Taylor series around $z_0 = 0$. If $f(z) = (1-z)^{-s}$,

$$
\begin{aligned}
f(0) &= 1 \\
f'(0) &= -s(1-z)^{-s-1}\Big|_{z=0} = s \\
f''(0) &= (-1)^2 s(s+1)(1-z)^{s-1}\Big|_{z=0} = s(s-1) \\
&\vdots \\
f^{(n)}(0) &= (-1)^n s(s+1)(s+2)\cdots(s+n-1) = \frac{\Gamma(s+n)}{\Gamma(s)}.
\end{aligned}
$$

That is,

$$
(1-z)^{-s} = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)n!} z^n. \tag{4.15}
$$

With this we find that

$$
\begin{aligned}
\zeta(s,x) &= \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \\
&= \frac{1}{x^s} + \sum_{n=1}^{\infty} \frac{1}{n^s} \frac{1}{\left(1+\frac{x}{n}\right)^s} \\
&= \frac{1}{x^s} + \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(s+k)}{\Gamma(s)k!} \left(\frac{x}{n}\right)^k \\
&= \frac{1}{x^s} + \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(s+k)}{\Gamma(s)k!} x^k \sum_{n=0}^{\infty} \frac{1}{n^{s+k}} \\
&= \frac{1}{x^s} + \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(s+k)}{\Gamma(s)k!} x^k \zeta(s+k).
\end{aligned}
$$

We relate the Hurwitz-Riemann Zeta function with the $\Gamma(z)$ function. From the definition of the $\Gamma(s)$ function, and making the change of variables $u = (n+x)t$, $du = (n+x)dt$, we have that

$$\Gamma(s) = \int_0^\infty u^{s-1} \mathrm{e}^{-u} dt = (n+x)^s \int_0^\infty t^{s-1} \mathrm{e}^{-(n+x)t} dt.$$

We now sum up over n from $n = 0$ to ∞ and find

$$\Gamma(s)\zeta(s,x) = \sum_{n=0}^\infty \int_0^\infty t^{s-1} \mathrm{e}^{-(n+x)t}\, dt = \int_0^\infty t^{s-1} \frac{\mathrm{e}^{-xt}}{1-\mathrm{e}^{-t}} dt,$$

where we used the geometric series $1 + x + \cdots + x^n + \cdots = 1/(1-x)$. We then have that

$$\zeta(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{e}^{-xt}}{1-\mathrm{e}^{-t}} dt \quad , \quad \mathrm{Re}(s) > 1. \tag{4.16}$$

This establishes a new relation betwen the Hurwitz-Riemann Zeta function $\zeta(s,x)$ and the $\Gamma(s)$ function. In particular by choosing $x = 1$ we find the relation between the Riemann Zeta function and the $\Gamma(s)$ function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} dt \quad , \quad \mathrm{Re}s > 1.$$

We can write

$$\Gamma(s)\zeta(s) = \int_0^\infty t^{s-1} \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} dt = \int_0^1 t^{s-1} \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} dt + \int_1^\infty t^{s-1} \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} dt \tag{4.17}$$

the second integral is entire since e^{-t} dominates over t^{s-1} for large $\mathrm{Re}(t)$. The function $\mathrm{e}^{-t}/(1-\mathrm{e}^{t})$ has a simple pole at $t = 0$. We can expand this function in a Taylor series around $t = 0$ as

$$\frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + \mathcal{O}(t^3).$$

That is, we can write the integrand as an infinite series of the form

$$\sum_{i=0}^{\infty} c_i t^{s+i-2} = \frac{1}{t} + \sum_{i=0}^{\infty} c_i t^{s+i-1}.$$

Since $\mathrm{Re}(s) > 1$, we have that the series converges uniformly for $|t| < 1$, and we can integrate term by term to find

$$\Gamma(s)\zeta(s) = \int_0^1 t^{s-2} dt + \sum_{i=0}^{\infty} \int_0^1 c_i t^{s+i-1} dt = \frac{1}{s-1} + \sum_{i=0}^{\infty} \frac{c_i}{s+i}.$$

So the right hand side of the equation 4.17 is meromorfic (has only isolated singularities). That is the product $\Gamma(s)\zeta(s)$ has a simple pole at $s = 1$ and removable singularities at $z = 0, -1, -2, \cdots$ depending if $c_n = 0$, or $c_n \neq 0$ (the residue coefficients). This implies that the Zeta function $\zeta(s)$ extends to a meromorphic function in the complex plane, since $\Gamma(s)$ has no zeroes and simple poles at $s = 0, -1, -2, \cdots$. Since $\Gamma(1) = 1$ $\zeta(s)$ has a simple pole at $s = 1$ with residue 1 (from the fraction $1/(s-1)$). This is, we continued the Zeta function to the complex plane.

We now derive the reflection formula for the Zeta function.

4.7 Reflection Formula for the Zeta Function

The reflection formula for the Zeta function is given by

$$\zeta(z) = \pi^{z-1} 2^z \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z).$$

We use again a Hankel contour on evaluating the integral

$$I = \int_C \frac{w^{z-1}}{\mathrm{e}^w - 1} dw \tag{4.18}$$

Let us use the convention I_{C_i} for the integral along the contour C_i, and $I_{[0,\infty)}$ as

$$I_{[0,\infty)} \int_0^{\infty} \frac{t^{z-1}}{\mathrm{e}^t - 1} dt \tag{4.19}$$

Figure 4.5 shows the contour C. The idea is the same idea used to analitically continue the $\Gamma(z)$ function by using a Hankel contour but this time we use the representation of the product of the $\Gamma(z)$ and the Zeta $\zeta(z)$ functions in equation 4.17. It is like doing the analytic continuation of the $\Gamma(z)\zeta(z)$ using a Hankel contour integration. That is, we show that the integral I in equation 4.18 is the analytic continuation of the integral $I_{[0,\infty)}$ in equation 4.19. We avoid the branch cut singularity (where the function is

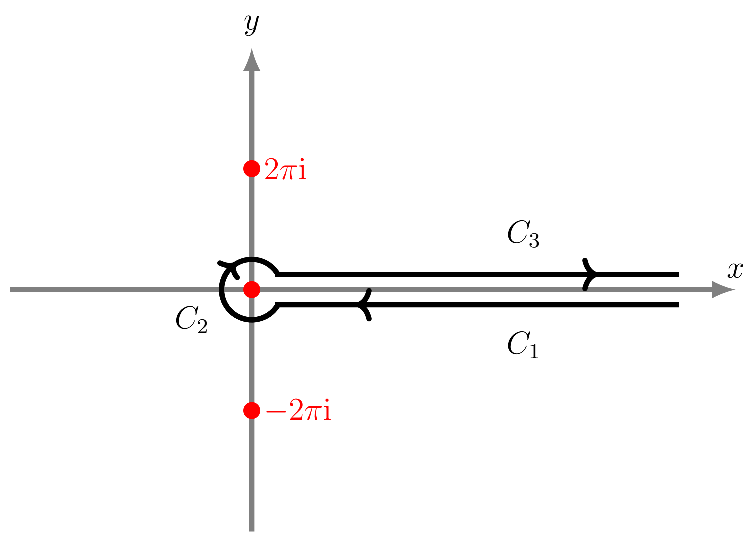


Figure 4.5: Contour for integration of equation 4.18. The poles are shown in large red dots

multivalued) by going around $z = 0$. The argument above the x axis for w is 0, while the argument under the x axis is considered to be 2π. Let us first evaluate the integral along the small circle.

First we observe that by using the Maclaurin series for e^w we have that

$$\mathrm{e}^w - 1 = w + \frac{1}{2}w^2 + \mathcal{O}(w^3)$$

and so if $|w| \ll 1$ we can say that

$$\frac{|\mathrm{e}^w - 1|}{|w|} \geq \frac{1}{2}$$

or better, that

$$\frac{|w|}{|\mathrm{e}^w - 1|} \leq 2$$

So, in the estimation of the integral over the small circle we can say that

$$|I_{C_2}| = \left| \int_{C_2} \frac{w^{z-1}}{\mathrm{e}^w - 1} dw \right| \leq \int_{C_2} \left| \frac{w^{z-1}}{\mathrm{e}^w - 1} \right| dw \leq 2 \int_{C_2} |w^{z-2}| dw$$

Let us now make the change of variables $w = \epsilon \mathrm{e}^{\mathrm{i}\theta}$ and $dw = \mathrm{i}\epsilon \mathrm{e}^{\mathrm{i}\theta} d\theta$. Then, if $\delta > 0$ is the small aperture angle where the branch cut meets the small circle. Then

$$|I_{C_2}| \leq 2 \int_{\delta}^{2\pi - \delta} |\epsilon^{z-1}| d\theta \leq (2\pi - 2\delta)|\epsilon^{z-1}.$$

and for $\mathrm{Re}(z) > 1$ we see that

$$\lim_{\epsilon \to 0, \delta \to 0} |I_{C_2}| = 0.$$

Clearly the integral over the path C_3 approaches $I_{[0,\infty)}$ as $\epsilon \to 0$, and to evaluate the integral along C_1 we recall that here the argument of the complex variable is advanced 2π. So we replace w by $w\mathrm{e}^{2\pi\mathrm{i}}$, and using t instead of w as the dummy integration variable

$$I_3 = \int_{\infty}^{0} \frac{\mathrm{e}^{2\pi i(z-1)} t^{z-1}}{\mathrm{e}^{2\pi \mathrm{i} w} \mathrm{e}^t - 1} = -\mathrm{e}^{2\pi\mathrm{i}(z-1)} \int_0^{\infty} \frac{t^{z-1} dt}{\mathrm{e}^{2\pi i} \mathrm{e}^t - 1} = -\mathrm{e}^{2\pi\mathrm{i}(z-1)} \int_0^{\infty} \frac{t^{z-1} dt}{\mathrm{e}^t - 1}$$

We then find that $I = (1 - \mathrm{e}^{2\pi\mathrm{i}(z-1)}) I_{[0,\infty]}$, and since

$$1 - \mathrm{e}^{2\pi\mathrm{i}(z-1)} = 1 - \mathrm{e}^{2\pi\mathrm{i}z} = -2\mathrm{i} \frac{\mathrm{e}^{\pi\mathrm{i}z} - \mathrm{e}^{-\pi\mathrm{i}z}}{2\mathrm{i}} = -2\mathrm{i} \sin \pi z,$$

we find that

$$\int_0^\infty \frac{t^{z-1}dt}{\mathrm{e}^t - 1} = \lim_{\epsilon,\delta\to 0} \frac{1}{2\mathrm{i}\sin\pi z} \int_C \frac{w^{z-1}dw}{\mathrm{e}^w - 1}.$$

We then see from equation 4.17 that the right hand side of the equation

$$\Gamma(z)\zeta(z) = \frac{1}{2\mathrm{i}\sin\pi z} \int_C \frac{w^{z-1}dw}{\mathrm{e}^w - 1}. \tag{4.20}$$

is an analytic continuation of of the product $\Gamma(z)\zeta(z)$. Please compare this result with equation 4.11. The idea is exactly the same except for a few details.

We now evaluate the integral in equation 4.20 using a different contour. We name

$$I = \int_C \frac{w^{z-1}}{\mathrm{e}^w - 1},$$

where C is the contour in Figure 4.6

As usual we note I_{C_i} the in2tegration over the path C_i . We can write the Hankel contour integral as $I_H = I_{C_1\cup C_2\cup C_3}$ and so from $I_C = I_H + I_{C_2}$ we see that

$$I_H = I_C - I_{C_2} \tag{4.21}$$

with

$$I_H = \int_{C_1\cup C_2\cup C_3} \frac{w^{z-1}dw}{\mathrm{e}^w - 1} = 2\mathrm{i}\sin(\pi z)\Gamma(z)\zeta(z). \tag{4.22}$$

So we need to estimate two new integrals to get the integral over the Hankel contour I_H.

First, we compute I_C using the Cauchy-Residue theorem. That is, the function $f(z) = w^{z-1}/(\mathrm{e}^w - 1)$ has simple poles at $w = \pm 2\pi m\mathrm{i}$. for $m = 1, 2, \cdots$. To compute the residues need to evaluate

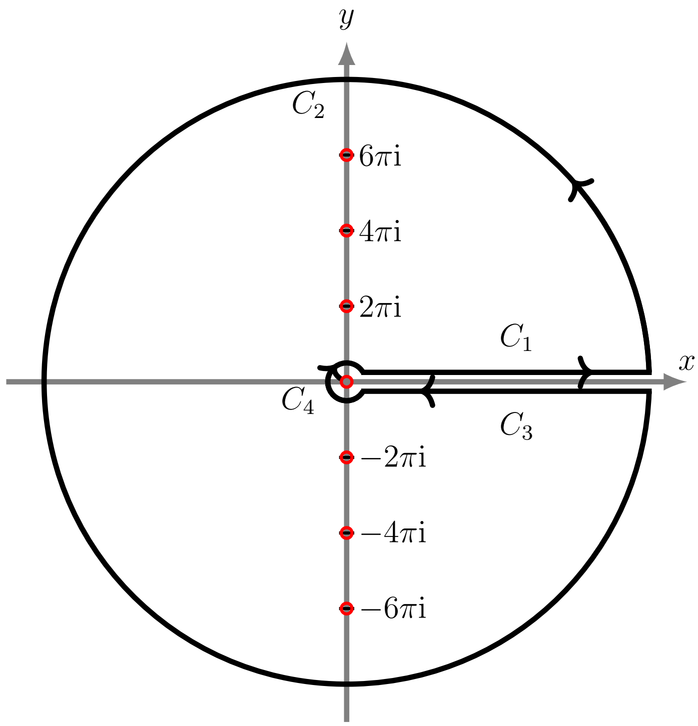


Figure 4.6: Contour for integration of equation 4.18. The poles are shown in small red circles.

$$\begin{aligned}
\mathrm{Res}(w = 2m\pi) &= \lim_{w \to 2mi\pi} \frac{(w)(w^{z-1})}{\mathrm{e}^w - 1} \\
&= \lim_{w \to 2mi\pi} \frac{zw^{z-1}}{\mathrm{e}^w} \\
&= \frac{z(2m\pi\mathrm{i})^{z-1}}{\mathrm{e}^{2\pi mi}} \\
&= z(2m\pi\mathrm{i})^{z-1}
\end{aligned}$$

where we used L'Hôpital's rule. We want to let the radius R of the circle go to ∞ so this will create a series

$$\begin{aligned}
\lim_{R\to\infty} I_C &= \sum_{m=-\infty, m\neq 0}^{\infty} 2\pi \mathrm{i} z(2m\pi \mathrm{i})^{z-1} \\
&= 2^z\pi^z \mathrm{i} z \sum_{m=-\infty, m\neq 0}^{\infty} (\mathrm{i}m)^{z-1} \\
&= 2^z\pi^z \mathrm{i} z \sum_{m=1}^{\infty} (\mathrm{i}m)^{z-1} + (-\mathrm{i}m)^{z-1} \\
&= 2^z\pi^z \mathrm{i} z \sum_{m=1}^{\infty} m^{z-1}(\mathrm{e}^{\mathrm{i}\pi(z-1)/2} + \mathrm{e}^{-\mathrm{i}\pi(z-1)/2}) \\
&= 2^z\pi^z \mathrm{i} z \sum_{m=1}^{\infty} m^{z-1} 2\cos\left(\frac{\pi(z-1)}{2}\right) \\
&= -2^{z+1}\pi^z z \sum_{m=1}^{\infty} m^{z-1}\cos\left(\frac{\pi z}{2} - \frac{\pi}{2}\right) \\
&= 2^{z+1}\pi^z z \mathrm{i}\sin\left(\frac{\pi z}{2}\right) \sum_{m=1}^{\infty} m^{z-1} \\
&= 2^{z+1}\pi^z z \mathrm{i}\sin\left(\frac{\pi z}{2}\right) \zeta(1-z).
\end{aligned}$$

Here we observed at the end how the Zeta function was reflected with respect to the vertical axis $x = 1$. We are left to compute the integral along the great circle C_2 with radius R. Let us consider the variable $w = R\mathrm{e}^{\mathrm{i}\theta}$, with $dw = R\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}$. So

$$I_{C_2} = \int_{C_2} \frac{w^{z-1}}{\mathrm{e}^w - 1} = \int_{\delta}^{2\pi-\delta} \frac{R^{z-1}\mathrm{e}^{(z-1)\mathrm{i}\theta} d\theta}{\mathrm{e}^{R\mathrm{e}^{\mathrm{i}\theta}} - 1} = R^{z-1} \int_{\delta}^{2\pi-\delta} \frac{\mathrm{e}^{(z-1)\mathrm{i}\theta} d\theta}{\mathrm{e}^{R\mathrm{e}^{\mathrm{i}\theta}} - 1}$$

where δ is a small angle that will approach to zero in the limit. That is we find

$$|I_2| \leq R^{z-1} \int_{\delta}^{2\pi-\delta} \frac{d\theta}{\mathrm{e}^{R\cos\delta} - 1} = R^{z-1} \frac{2(\pi-\delta)}{\mathrm{e}^{R\cos\delta} - 1}.$$

Now, since $\mathrm{Re}(z) \geq 1$, we have that the last expression goes to 0 as $R \to \infty$. Hence we have that $\lim_{R\to\infty,\delta\to 0} I_2 = 0$.

We then found, in equation 4.21, that

$$I_H = I_C - I_{C_2} = I_C$$

and since, from equation 4.22, we have that $I_H = \Gamma(z)\zeta(z)2\mathrm{i}\sin\pi z$, then we find

$$\Gamma(z)\zeta(z)2\mathrm{i}\sin\pi z = 2^{z+1}\pi^z z\mathrm{i}\sin\left(\frac{\pi z}{2}\right)\zeta(1-z).$$

This is

$$\zeta(z) = \frac{2^{z+1}\pi^z z\sin\left(\frac{\pi z}{2}\right)\zeta(1-z)}{\Gamma(z)2\sin\pi z}$$

We now use the Euler reflection formula for the $\Gamma(z)$ function 4.7 to find that

$$\zeta(z) = \frac{2^{z+1}\pi^z z\sin\left(\frac{\pi z}{2}\right)\zeta(1-z)\Gamma(1-z)\sin\pi z}{\pi 2\sin\pi z}.$$

We conclude that

$$\zeta(z) = \pi^{z-1}2^z\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta(1-z).$$

as claimed. This is the $\zeta(z)$ reflection formula. This reflection formula can be used to analytically continue the Riemann Zeta function to the negative side of the complex plane where zeros are found at even negative numbers.

I found article on the Zeta function by Dragan Miličič [4] useful for the development of this section.

4.8 Asymptotic Behavior of the Gamma Function

I present a formal derivation of the Gamma asymptotic series (Stirling) in my notes on Bernoulli numbers [5] [3] . Here I show a heuristic explanation of why the Stirling approximation should do as good as it does.

[4] http://www.math.utah.edu/ milicic/zeta.pdf

[5] https://drive.google.com/open?id=0B4W-gdhbNpsDYUxFcWY4N1ZiVzg

The Stirling formula estimates an approximation of the Γ function for large values of its argument. That is, let us assume that we want to approximate the integral

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} dt.$$

This integral could be written as

$$\Gamma(x) = \int_0^\infty \mathrm{e}^{(x-1)\ln t - t} dt.$$

Then we can expand the exponent $\phi(t) = (x-1)\ln t - t$ in a Taylor series around some point t_0. We want t_0 to be located in a place such that we can obtain the best approximation of the integral. Figure 4.7 illustrates the integrand of the function.

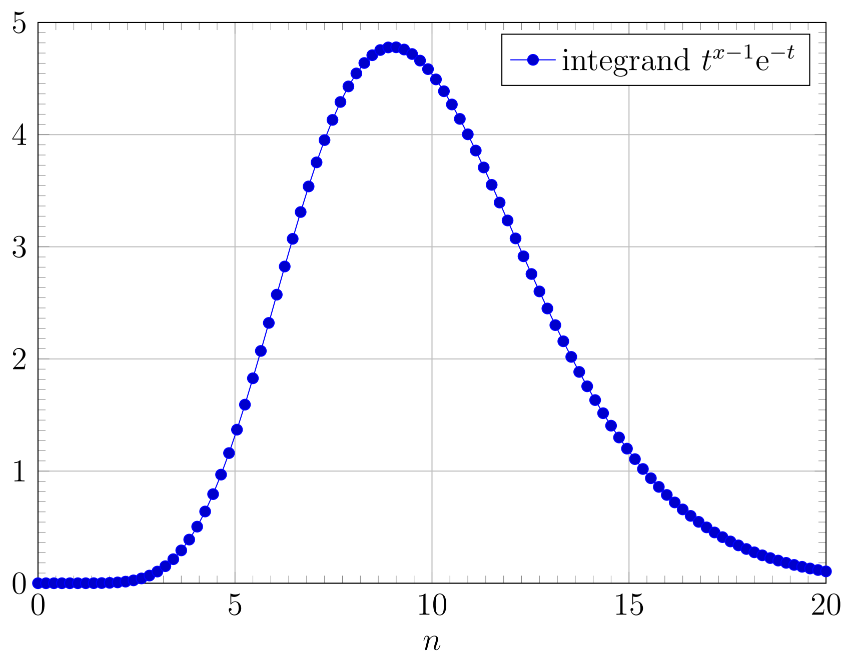


Figure 4.7: Integrand $t^{x-1}\mathrm{e}^{-t}$, for $x = 10$.

We observe that the maximum is located at $t = 9$. This could be verified by taking the derivative of the integrand $f(t) = t^{x-1}\mathrm{e}^{-t}$ and solving for $f'(t) = 0$. That is by solving

$$f'(t) = (x-1)t^{x-2}\mathrm{e}^{-t} - t^{x-1}\mathrm{e}^{-t} = \mathrm{e}^{-t}t^{x-2}[(x-1)-t] = 0,$$

for t. The solution is obviously $t = x-1$, and if $x = 10$, then $x - 1 = 9$ as asserted. We verify that this is a maximum by taking the second derivative and evaluating it there. This is:

$$\begin{aligned} f''(t) &= \frac{\mathrm{e}^{-t}[t^x x^2 - (2t+3)xt^x + t^x(t^2+2t+2)]}{t^3} \\ f''(t_0) &= \frac{(x-1)^x \mathrm{e}^{1-x}}{(x-1)^2} > 0, \end{aligned}$$

so indeed the point is a maximum for $x > 1$.

The method of stationary phase [6] for integrals of oscillatory functions rests on the idea that the main contribution of the integral happens when the phase (the argument of the exponential function) is stationary. That is, when its derivative is zero. The integrand above is not oscillatory but the function has two interesting characteristics. It is always positive, it is maximum and $t = x-1$, and it flattens there. So it is expected here as well that the major contribution to the integral is around the point $t = x - 1$. Since the maxima for the integrand is located at $t = x-1$, it rolls along with x. We would like to use the maxima as the new origin by doing the change of variables $s = t - x + 1$, $ds = dt$, $s \in [-x+1, \infty)$ and write

$$\Gamma(x) = \int_{-(x-1)}^{\infty} (s+x-1)^{x-1}\mathrm{e}^{-s-x+1}ds = \mathrm{e}^{-x+1} \int_{-(x-1)}^{\infty} \mathrm{e}^{(x-1)\ln(s+x-1)-s}ds$$

We want to expand the argument $\phi(s) = (x-1)\ln(s+x-1) - s$ of the exponential function in a Taylor series around the maximum point $s = 0$.

Let us compute a few derivatives:

[6] https://en.wikipedia.org/wiki/Stationary_phase_approximation

$$
\begin{aligned}
\phi'(t) &= \frac{x-1}{x+s-1} - 1 \\
\phi''(t) &= -\frac{x-1}{(x+s-1)^2} \\
\phi'''(t) &= 2\frac{x-1}{(x+s-1)^3} \\
&\vdots \\
\phi^{(n)}(t) &= (-1)^{n-1}(n-1)\frac{x-1}{(x+s-1)^n}
\end{aligned}
$$

Then, by evaluating these derivatives on $s=0$, we find

$$
\begin{aligned}
\phi(0) &= (x-1)\ln(x-1) \\
\phi'(0) &= 0 \\
\phi''(0) &= -\frac{1}{x-1} \\
\phi'''(0) &= \frac{2}{(x-1)^2} \\
&\vdots \\
\phi^{(n)}(0) &= (-1)^{n-1}(n-1)\frac{x-1}{(x-1)^n} \quad , \quad n>1.
\end{aligned}
$$

Then, from using Taylor series expansion on the exponent

$$
\begin{aligned}
\Gamma(x) &= \int_{-(x-1)}^{\infty} \mathrm{e}^{(x-1)\ln(x-1)-(x-1)]-\frac{1}{x-1}\frac{t^2}{2!}+\frac{2}{(x-1)^2}\frac{t^3}{3!}+\cdots+(-1)^{n-1}\frac{(n-1)t^n}{n!(x-1)^n}+\cdots} dt \\
&= \mathrm{e}^{(x-1)\ln(x-1)-(x-1)} \int_{-(x-1)}^{\infty} \mathrm{e}^{-\frac{1}{x-1}\frac{t^2}{2!}+\frac{2}{(x-1)^2}\frac{t^3}{3!}+\cdots+(-1)^{n-1}\frac{(n-1)t^n}{n!(x-1)^n}+\cdots} dt.
\end{aligned}
$$

We recognize that the leading factor in the integrand for the infinite product yields a Gaussian integral. For $x \gg 1$, we can say that

$$
\int_{-(x-1)}^{\infty} \mathrm{e}^{-\frac{1}{x-1}\frac{t^2}{2!}} \approx \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{x-1}\frac{t^2}{2!}} = \sqrt{2\pi}\sqrt{x-1},
$$

then we estimate that

$$\Gamma(x) \sim \mathrm{e}^{(x-1)\ln(x-1)-(x-1)}\sqrt{2\pi}\sqrt{x-1} = \sqrt{2\pi(x-1)}(x-1)^{x-1}\mathrm{e}^{-(x-1)}. \quad (4.23)$$

This result is known as the Stirling's approximation. While it is the correct leading order term of the asymptotic approximation of the Γ function the method generates questions. Taylor series are approximation for local values. That is, the next term not considered here is $t^3/[3(x-1)^2]$. This exponent alone forces the integrand to grow without bounds and the integral $\int_{-\infty}^{\infty} \exp(t^3/[3(x-1)^2]dt$ diverges. In fact, since the function is shifted so that 0 is at the peak of the function and near the peak the area is larger the Taylor "approximations is valid where we want it to be. For values very far from the peak the function is close to zero and the Taylor approximation being in error does not represent much risk. In fact, as $x \to \infty$, the "Gaussian-like" shape becomes spike (like a Dirac Delta) and the support shrinks to zero making the Taylor approximation perfectly valid.

Appendix A

Two Derivations of Wallis formula

This is more an exercise on the different ways to see how a classical problem solved in the XVII century, and which solution was shown above, is solved today.

A.0.1 Derivation of Wallis formula based on Euler's expansion of the $\sin(z)$ function

I start by re–writing equation 2.5

$$\sin z = z\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{2^2\pi^2}\right)\cdots\left(1-\frac{z^2}{k^2\pi^2}\right)\cdots$$

Let $z = \frac{\pi}{2}$; then

$$\begin{aligned}
\frac{2}{\pi} &= \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\cdots\left(1-\frac{1}{4k^2}\right)\cdots \\
&= \left(1-\frac{1}{4}\right)\left(1-\frac{1}{4\cdot 2^2}\right)\left(1-\frac{1}{4\cdot 3^2}\right)\cdots\left(1-\frac{1}{4k^2}\right)\cdots \\
&= \left(\frac{1\cdot 3}{2\cdot 2}\right)\left(\frac{3\cdot 5}{4\cdot 4}\right)\left(\frac{5\cdot 7}{6\cdot 6}\right)\cdots\left(\frac{4K^2-1}{4k^2}\right)
\end{aligned}$$

so

$$\frac{\pi}{2} = \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{3\cdot 5}{4\cdot 4}\right)\left(\frac{5\cdot 7}{6\cdot 6}\right)\cdots$$

A.0.2 derivation of Wallis formula based on calculus

Let us look at integrals of the form

$$I_m = f(p,q) = f^{-1}\left(\frac{1}{2},\frac{m}{2}\right) = \int_0^1 (1-x^2)^{m/2} dx.$$

By the obvious trigonometrical substitution

$$u = \cos x, \quad (1-x^2)^{m/2} = \sin^m x, \quad du = \sin x dx, \quad 0 \le u \le \frac{\pi}{2}.$$

we can return to x notation and write

$$I_m = f\left(\frac{1}{2},\frac{m}{2}\right) = \int_0^{\pi/2} \sin^{m+1} x\, dx. \tag{A.1}$$

The way to solve these type of integrals is by generating a recursive formula using integration by parts. That is if we set

$$\begin{array}{rcl} u = \sin^m x & \Rightarrow & du = m\sin^{m-1} x \cos x dx \\ dv = \sin x dx & \Rightarrow & v = -\cos x, \qquad 0 \le u \le \dfrac{\pi}{2}. \end{array}$$

So

$$\begin{aligned} I_m &= uv\Big|_0^{\pi/2} - \int_0^{\pi/2} vdu \\ &= 0 + m\int_0^{\pi/2} \sin^{m-1} x \cos^2 x dx \\ &= m\int_0^{\pi/2} \sin^{m-1} x(1-\sin^2 x) dx \\ &= m\int_0^{\pi/2} \sin^{m-1} x dx - m\int_0^{\pi/2} \sin^{m+1} x dx \\ &= mI_{m-2} - mI_m. \end{aligned}$$

Then we obtain the recursion formula:

$$(m+1)I_m = mI_{m-2} \quad \Rightarrow \quad I_m = \frac{m}{m+1} I_{m-2}.$$

Starting with the two known initial conditions, and iterating, we see that,

$$
\begin{aligned}
I_0 &= \int_0^{\pi/2} \sin x \, dx = 1 \\
I_1 &= \int_0^{\pi/2} \sin^2 x \, dx = \frac{\pi}{4} \\
I_2 &= \int_0^{\pi/2} \sin^3 x \, dx = \left(\frac{2}{3}\right) I_1 = \left(\frac{2}{3}\right) \\
I_3 &= \int_0^{\pi/2} \sin^4 x \, dx = \frac{3}{4} I_1 = \left(\frac{3}{4}\right)\left(\frac{\pi}{4}\right) \\
I_4 &= \int_0^{\pi/2} \sin^5 x \, dx = \left(\frac{4}{5}\right) I_2 = \left(\frac{4}{5}\right)\left(\frac{2}{3}\right)
\end{aligned}
$$

That is, we consider two sequences. If $k = 2\,i$ is even

$$I_{2i} = \left(\frac{2i}{2i+1}\right)\left(\frac{2i-2}{2i-1}\right)\cdots\left(\frac{2}{3}\right)$$

if $k = 2i+1$ is odd

$$I_{2i+1} = \left(\frac{2i+1}{2i+2}\right)\left(\frac{2i-1}{2i}\right)\cdots\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$$

Now, I_m is obviously a decreasing function of m since at every step we multiply by $m/(m+1)$ which is strictly smaller than 1. Then $I_{2i} > I_{2i+1} > I_{2i+2}$, that is

$$\frac{2i}{2i+1}\frac{2i-2}{2i-1}\cdots\frac{2}{3} > \frac{2i+1}{2i+2}\frac{2i-1}{2i}\cdots\frac{1}{2}\,\frac{\pi}{2} > \frac{2i+2}{2i+3}\frac{2i}{2i+1}\cdots\frac{2}{3}.$$

This means (reordering, and since this product is finite, there is not issue here)

$$\frac{2\cdot 2\cdot 4\cdot 4\cdots 2i\cdot 2i\cdot 2i+2}{1\cdot 3\cdot 3\cdot 5\cdots 2i+1\cdot 2i+1} > \frac{\pi}{2} > \frac{2\cdot 2\cdot 4\cdot 4\cdots 2i+2\cdot 2i+2}{1\cdot 3\cdot 3\cdot 5\cdots 2i+1\cdot 2i+3}$$

and since

$$\lim_{i\to\infty}\frac{2i+2}{2i+1} = 1, \quad \lim_{i\to\infty}\frac{2i+2}{2i+3} = 1,$$

then, at the limit

$$\frac{\pi}{2} = \left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\cdots\left(\frac{2i\cdot 2i}{2i-1\cdot 2i+1}\right)\cdots$$

which is Wallis formula.

Appendix B

Computation of the area of a Gaussian function

We use two methods. The first is by integration and change of variables, the other is by using the properties of the Γ function at $z = 1/2$.

B.1 Method I

Gauss who was born 70 years after Euler worked extensively on functions of the type e^{-x^2}, and today functions like this are known as *"Gaussians"*. The following derivation is a classical derivation for the integral of a Gaussian function.

Call $I = \int_0^\infty \mathrm{e}^{-x^2} dx$, so

$$\begin{aligned}
I^2 &= \int_0^\infty \mathrm{e}^{-x^2} dx \int_0^\infty \mathrm{e}^{-y^2} dy \\
&= \int_0^\infty dx\, dy\, \mathrm{e}^{-(x^2+y^2)} \quad \text{Fubinni's Rule} \\
&= \frac{1}{4} \int_{-\infty}^\infty dx\, dy\, \mathrm{e}^{-(x^2+y^2)} \quad \text{Double both intervals of integration} \\
&= \frac{1}{4} \int_0^\infty \int_0^{2\pi} dr d\theta\, r\, \mathrm{e}^{-r^2} \quad \text{Polar Coordinates} \quad x = r\cos\theta \quad y = r\sin\theta \\
&= \frac{1}{4} \int_0^\infty dr (2\pi)\, r\, \mathrm{e}^{-r^2} \\
&= -\frac{\pi}{4} \int_0^\infty dr\ [-(2r)]\ \mathrm{e}^{-r^2} \\
&= -\frac{\pi}{4} \mathrm{e}^{-r^2} \bigg|_0^\infty \\
&= \frac{\pi}{4}.
\end{aligned}$$

So

$$\int_0^\infty \mathrm{e}^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

B.2 Method II

We first show that $\Gamma(1/2) = \sqrt{\pi}$. From the representation of the Beta function 3.2 for $x = y = 1/2$ we see that

$$\mathrm{B}(1/2, 1/2) = \Gamma^2(1/2).$$

On the other hand by using the trigonometrical integrand representation 3.4 we see that

$$\mathrm{B}(1/2, 1/2) = 2 \int_0^{\pi/2} \phi d\phi = 2\, \frac{\pi}{2} = \pi.$$

Hence $\Gamma(1/2) = \sqrt{\pi}$. Now

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} dx = 2\int_0^{\infty} \mathrm{e}^{-x^2} = \int_0^{\infty} e^{-t} t^{-1/2} dt = \Gamma(1/2) = \sqrt{\pi}.$$

where we made the change of variables $t = x^2$, $dt = 2\,x\,dx$, $dx = 1/(2\sqrt{t})dt$,

Appendix C

Convergence of Integrals

This appendix shows an informal review of the convergence of integrals theory studied in elementary calculus.

To be able to understand the domain of influence of Euler's integrals we should be able to estimate their growing behavior. We start with some simple integrals known as the p–integrals.

C.1 p–integrals

p–integrals are of the form

$$\int_0^b x^p dx,$$

for p real. We know from elementary calculus

$$\int x^p dx = \begin{cases} \ln x + C & \text{if p} = -1 \\ \frac{x^{p+1}}{p+1} + C & \text{if p} \neq -1 \end{cases}$$

When $p > 0$ the integral converges as along as $b < \infty$. The interesting case is when $p \in [-\infty, 0]$. If $b = \infty$ we can split the integral into two integrals

$$\int_0^1 x^p dx \quad \text{and} \quad \int_1^\infty x^p dx.$$

We consider the case of $p < -1$, $p = -1$, $p > -1$, for either of the two intervals ($[0, 1]$ and $[1, \infty]$).

- If $p < -1$
  - Interval $[0, 1]$

$$\int_0^1 x^p dx = \left.\frac{x^{p+1}}{p+1}\right|_0^1 = \lim_{c \to 0} \frac{1}{p+1}\left(1 - c^{p+1}\right) = +\infty$$

  - Interval $[1, \infty]$

$$\int_1^\infty x^p dx = \left.\frac{x^{p+1}}{p+1}\right|_1^\infty = \lim_{c \to \infty} \frac{1}{p+1}\left(c^{p+1} - 1\right) = -\frac{1}{p+1}.$$

- If $p = -1$
  - Interval $[0, 1]$

$$\int_0^1 x^p dx = \ln x\Big|_0^1 = \lim_{c \to 0} \ln\, x = -\infty$$

  - Interval $[1, \infty]$

$$\int_1^\infty x^p dx = \ln x\Big|_1^\infty = \lim_{c \to \infty} \ln\, x = +\infty$$

- If $p > -1$ ($p < 0$)
  - Interval $[0, 1]$

$$\int_0^1 x^p dx = \left.\frac{x^{p+1}}{p+1}\right|_0^1 = \lim_{c \to 0} \frac{1}{p+1}\left(1 - c^{p+1}\right) = \frac{1}{p+1}. \tag{C.1}$$

  - Interval $[1, \infty]$

$$\int_1^\infty x^p dx = \left.\frac{x^{p+1}}{p+1}\right|_1^\infty = \lim_{c \to \infty} \frac{1}{p+1}\left(c^{p+1} - 1\right) = +\infty$$

That is, for all possible 6 combinations between intervals and values of p (if $p < 0$) there is convergence only when either $p < -1$ and the interval of integration is $1, \infty$ or when $p > 1$ and the interval of integration is $[0, 1]$. The integral

$$\int_0^\infty x^p dx,$$

is always divergent for $p < 0$.

C.2 The Comparison Test

We use the convergence (divergence) rules of the p–integrals, together with the comparison test. That is, Assume $0 \leq f(x) \leq g(x)$, for all $x \in [a, b]$, then

- If

$$\int_a^b g(x)dx \quad \text{is convergent, then} \quad \int_a^b f(x)dx \quad \text{is convergent}$$

- If

$$\int_a^b f(x)dx \quad \text{is divergent, then} \quad \int_a^b g(x)dx \quad \text{is divergent}$$

Finally we write the statement of the limit test for convergence of integrals.

C.3 The Limit Test

Let us assume that there is a point $c \in [a, b]$ such that the positive functions f and g either go to zero or to ∞ as we approach to c. Then if

$$\lim_{x \to c} \frac{f(x)}{g(x)} = 1$$

then

$$\int_a^b f(x)dx \quad \text{is convergent if only if} \quad \int_a^b g(x)dx \quad \text{is convergent.}$$

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