Transform and Spectral Theory

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Chapter 1

Introduction

These notes were inspired by chapter 7 of Keener's [1] book. I learned from Keener's book the connection between Green's functions, Transform Theory, Dirac deltas and contour integrals. I discuss a connection between contour integrals, infinite sums, analytic continuation and Riemann surfaces. It is certainly to me wonderful and I want to set, what I see as Keener's main points on these notes.

It is understood in these notes that we are talking about linear operators L defined in a Banach space X. When a Hilbert space $\mathcal{H}$ is used, it is explicitly indicated. Basic functional analysis is assumed. Topological point set concepts such as open set, closed set, compact set, dense sets, boundary of a set, etc., are used.

If the reader is not familiar with Green functions I recommend to read first my notes on Green functions [1]. These notes show the derivations for all Green functions used to derive the transform pairs on this document.

About notation. I use *italics* upon an introduction of a new defintion.

[1]https://drive.google.com/open?id=0B4W-gdhbNpsDSHZveE9LdnhMNEk

Chapter 2

Spectrum of an Operator

It is well understood in finite dimensional spaces the connection between eigenvalues, eigenvectors, and basis of a linear operator (a matrix in this case). If the eigenvalues are distinct then the eigenvectors are linearly independent and they form a basis for the n–th dimensional space. However repeated eigenvalues could mean linearly independent (as for example in the case of the identity operator) or linearly dependent eigenvectors (think of a 2x2 upper triangular matrix such that $a_{11} = a_{22} = a_{12} = 1$. The nice thing about the case of linearly independent eigenvectors is that this will let us diagonalize the problem. That is, substitute the matrix by a similar transformation corresponding to a diagonal matrix. We say that the problem is diagonalizable. In general the closest that we can get to this is to a Jordan canonical matrix which is close to a diagonal but with some subdiagonal bands above (or below) the main diagonal.

The transformation of the problem into a diagonal problem or a Jordan canonical problem is a desired simplification because it is a coordinate transformation such that in the new coordinate system the independent variables are nicely decoupled, and the equations are the simplest. If the similar matrix is diagonal the coordinates are totally decoupled, if it is a Jordan canonical form, there is a bit of coupling between coordinates corresponding to repeated eigenvalues. Think about an ellipse centered at the coordinates (H, K) and with its principal axes not aligned with the Cartesian coordinates x, y. The equation in the $x - y$ axis representation is more complicated because it has coupled (product of the form xy) terms. The rotation of axis such that the coordinates align with the main ellipse axis will convert the equation in an equation of the canonical form

$$\frac{(X-H)^2}{a^2} + \frac{(Y-K)}{b^2} = 1,$$

where now X and Y are the new axis aligned with the main ellipse axis.

The concept of spectrum of a finite dimensional linear operator L is related to its eigenvalues, however in the functional analysis framework the concept of spectrum is more general. In the case of a linear operator L in a Banach space the spectrum is related to questions about the inverse of its extended operator $L - \lambda$. This extended operator is the same operator where its "diagonal" elements are shifted by a constant scalar λ.

In finite dimensional spaces we find the eigenvalues by solving, for example, the characteristic polynomial

$$\det(A - \lambda I) = 0.$$

for λ. That is λ is such that the operator $A - \lambda I$ has not inverse. We say that the null space of $A - \lambda I$ is non trivial (more than just the zero "0" element) and the function fails to be one–to–one and so it is not onto either since the operator domain and range are the same n–dimensional space.

The generalization to infinite dimensional spaces is not as trivial. We define as the stectrum, the set of all λ such that the operator $L - \lambda$ is not invertible.

But, what does it means, in general, that $L - \lambda$ is not intertible? from the basic functional approach it means that the function is not one–to–one or not onto. We will consider all possibilites, but first let us discuss the case where the operator $L - \lambda$ is invertible.

2.1 The Resolving Set

If $(L - \lambda)^{-1}$ exists and is bounded, we say that λ belongs to the *resolving set* for L.

Why is it called the resolving set?

If such an inverse exist, then we can solve the equation

$$(L - \lambda)u = f, \tag{2.1}$$

for u. The solution is simply

$$u = (L - \lambda)^{-1} f.$$

We can see the resolvent $(L - \lambda)^{-1}$ as a Green's function that applied to any source field f would produce a response u. Actually this problem is well studied when the operator L is an integral and the equation 2.1 is know as a Fredholm integral equation of the second kind. For example if

$$L\, u(t) = -\int_a^b k(t, s)u(s)ds,$$

then we can write equation 2.1 as

$$-\int_a^b k(t,s)u(s)ds - \lambda u(t) = f(t)$$

which usually is written as

$$u(t) = g(t) + \mu \int_a^b k(t,s)u(s)ds,$$

with $\mu = 1/\lambda$, and $g(t) = -f(t)/\lambda$, assuming $\lambda \neq 0$,

We see then why the solution of this, integral equation, for each source term $g(t)$ which has as its unknown, the function $u(s)$ inside and outside of the integral sign, can be found if the resolvent is known.

Interestingly, from operator theory, we could expand

$$(L - \lambda I)^{-1} = -\frac{1}{\lambda}(I - L)^{-1} = -\frac{1}{\lambda}\left(I + L + L^2 + \cdots L^n + \cdots\right).$$

This series is known in scattering theory as the Neumann series, and the first term of the series is known as the Born approximation.

Of course the convergence of these series is not assured, but it can be shown that if the norm of the operator L satisfies

$$\|L\| < 1$$

then the series converges. The norm of a linear operator L is defined as

$$\|L\| = \sup_{\|u\|=1} \|Lu\|. \tag{2.2}$$

What if $\lambda = 0$. If $\lambda = 0$ then equation 2.1 turns into

$$\int_a^b k(t,s)u(s)ds = f, \tag{2.3}$$

which is a Fredholm integral equation of the first kind.

In the discrete (finite dimensional) space this is a matrix equation of the type

$$Ax = y$$

with A characterized by the kernel $k(t,s)$, x is the unknown $u(t)$ and y is the source term $f(t)$.

Interestingly all transforms in the continuum are writen as integrals such as equation 2.3 where a and b extend to any number including ∞.

As a final remark; the Fredholm alternative states that if $L - \lambda$ has a closed range then $(L - \lambda)^{-1}$ exists and is unique if and only if the solutions of the equations $(L - \lambda)u = 0$ and $(L^* - \lambda)v = 0$ admit only the trivial function $u = v = 0$.

2.2 The Spectrum

As indicated above, the spectrum of a linear operator L (defined in a Banach space) is the set of all λ such that $L-\lambda$ does not have inverse. A theorem of functional analysis establishes that a bounded operator L on a Banach space is invertible, (has a bounded inverse) if an only if L is bounded below and has dense range. While I will not proof this theorem here, I believe it makes sense. In the simplest caso a linear equation in $\mathbb{R}$

$$a\,x = b,$$

the solution exists as long as $a \neq 0$. This is, a has to be bounded away from zero. In the case of finite dimensional spaces

$$Ax = b,$$

if the norm $\|A\| > 0$, then A is a non–singular matrix. In general the solution of

$$Lu = b,$$

for a linear operator L could be found as long as the norm $\|L\|$ is bounded away from zero, that is $\|L\| > 0$. The other condition is related to the function being surjective. It is only necessary that the range be dense in the space X. That is, the closure of the range is the space X ($\bar{\text{ran}}(L) = X$). If the range is closed this is the classical argument from regular functions, it is not closed then the argument is that any point in the range is as closed as we want to any point of the space X and in that sense the range is almost everywhere in X.

Let us consider all posibilities for the inverse to fail.

- $L-\lambda$ fails to be injective (one–to–one). In this case we say that there is no trivial ($u \neq 0$) solution for the equation $(L-\lambda)u = 0$. Here λ is, by definition, in the *point* or *discrete* spectrum. We call λ and *eigenvalue* and u and *eigenfunction.* The set of eigenvalues (point set spectrum) of L is noted as $\sigma_p(L)$.

- $L-\lambda$ is injective, but not surjective (onto). Here there could be two cases:

  - Is injective, not surjective but the range $\text{ran}(L)$ is dense in X. Additionally the inverse is not bounded. That is the operator $L-\lambda$ is not bounded away from zero. We say that λ is in the *continuous spectrum* . The set of points in the continuous spectrum for L is noted as $\sigma_c(L)$. The classical example of an operator with a continuous spectrum is the multiplication operator defined in the Hilbert space $\mathcal{H} = L^2([0,1)]$

$$\begin{aligned} M : \mathcal{H} &\to \mathcal{H} \\ u(x) &\mapsto x\,u(x). \end{aligned}$$

From the operator norm definition 2.2 we see that

$$\|M\| = 1.$$

That is M is bounded away from zero. However M does not have eigenvalues. Assume $Mu = \lambda u$, then

$$xu(x) = \lambda u(x).$$

So, since x can be any value in the interval $[0,1]$, and λ is fixed, $u(x) = 0$ (almost everywhere). Now, if $\lambda \notin [0,1]$, then for all $x \in [0,1]$, the operator $M - \lambda$ is invertible (since $x - \lambda \neq 0$) and then

$$(M - \lambda)^{-1} u(x) = \frac{u(x)}{x - \lambda},$$

so the set $\mathbb{R} \backslash [0,1]$ (or $\mathbb{C} \backslash [0,1]$ if the field considered is is that of the complex numbers $\mathbb{C}$) the resolvent set for the operator M. If $\lambda \in [0,1]$, then $M - \lambda$ is not onto, because $c(x-\lambda)^{-1} \notin L^2([0,1])$ for $c \neq 0$. The operator $c(x-\lambda)^{-1}$ is not bounded. So the constant functions c do not belong to the range of $M - \lambda$. However, the range of $M - \lambda$ is dense. Let us see. Pick any $f \in L^2([0,1])$, let

$$f_n(x) = \begin{cases} f(x) & \text{if} \quad |x - \lambda| \geq 1/n, \\ 0 & \text{if} \quad |x - \lambda| < 1/n, \end{cases}$$

then $f_n \to f$ in $L^2[0,1]$, and $f_n \in \operatorname{ran}(M - \lambda)$. since $(x-\lambda)^{-1} f_n(x) \in L^2([0,1])$. It follows that $[0,1]$ belongs to the continuous spectrum of M.

An interesting fact is that in the distribution theory when we consider a Dirac delta distribution $\delta(x - \lambda)$ then we see that

$$u(x)\delta(x - \lambda) = u(\lambda)\delta(x - \lambda)$$

so

$$M[\delta(x-\lambda)] = x\delta(x-\lambda) = \lambda\delta(x-\lambda).$$

so λ is an eigenvalue of M with eigenvector $\delta(x-\lambda)$. So in this sense there are eigenvalues associated with points in the continuous spectrumm of M, however they lie outside of the space $L^2([0,1])$ on wich M acts.

– The function is non surjective and the range is not dense in X. In this case the function is not surjective and we say that λ is in the *residual spectrum* . We note the residual spectrum as of the operator $\sigma_r(L)$. A classical example of an operator with a non–empty residual spectrum is the right shift operator. Given a sequence $\{x_i\} \in l^2$ we define the right shift operator L as

$$\begin{aligned} L : l^2 &\rightarrow l^2 \\ \{x_i\} &\mapsto L(\{x_i\}) = \{x_{i+1}\}. \end{aligned} \tag{2.4}$$

It takes every sample into the next. This operator leaves the first sample in 0 and hence the range has all first entry components missing (except for 0). It can be shown that the norm of the shift operator is 1. That is, $\|L\| = 1$. This is easy to see from the definition 2.2

If the space is finite dimensional, for λ being an eigenvalue of L is equivalent to say that $L-\lambda$ is not invertible, that is $(L-\lambda)^{-1}$ does not exists. In infinite dimensional spaces this is not the case. For example the right shift operator $S = S - 0$ is not invertible and 0 is not and eigenvalue, since $Sx = 0$ implies $x = 0$.

The spectrum is the disjoint union of the three spectra defined above. That is

$$\sigma = \sigma_c \cup \sigma_p \cup \sigma_r$$

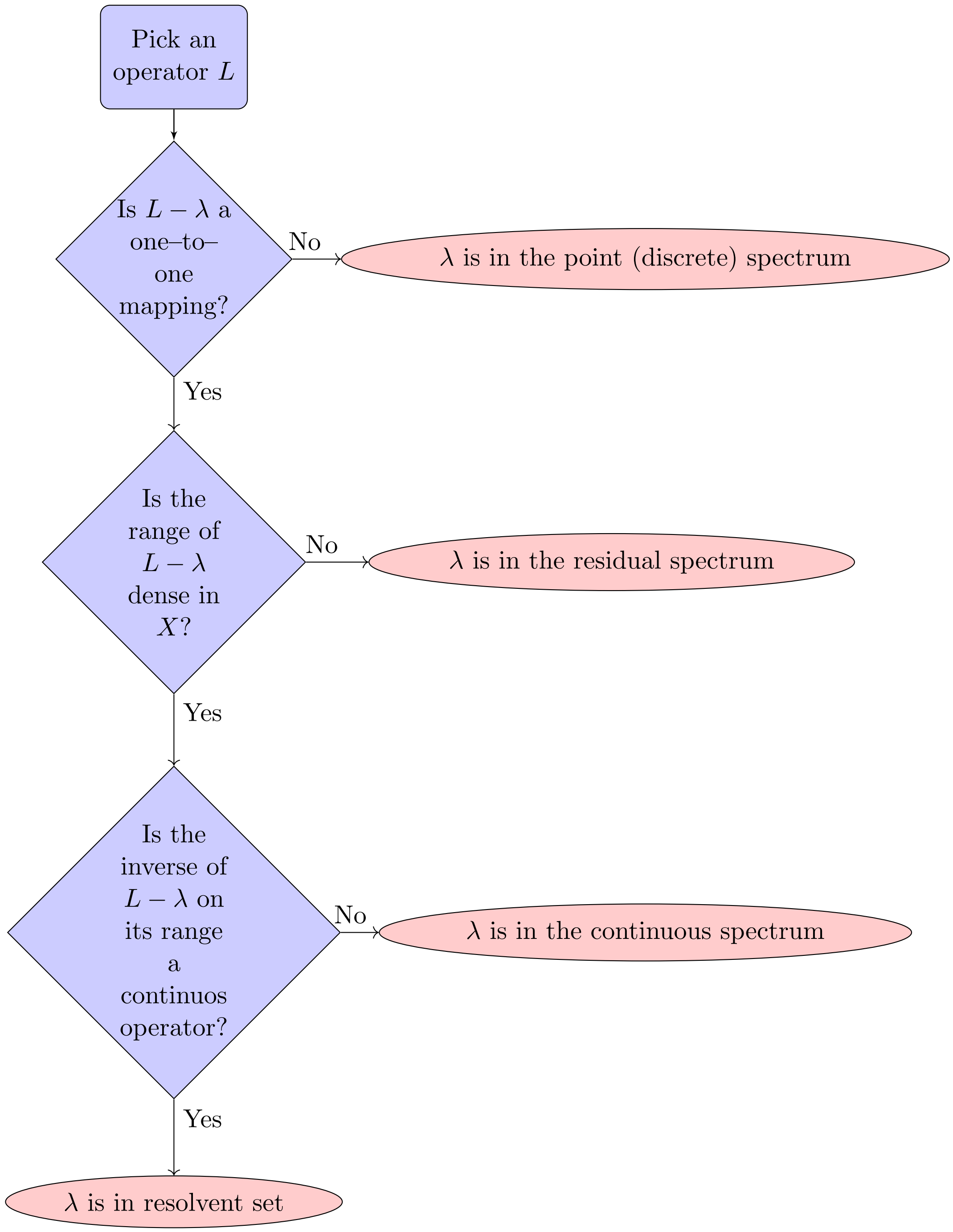
and the whole fild ($\mathbb{R}$ or $\mathbb{C}$) is the union of the spectrum and the resolvent set.

Keener proposes the flow chart 2.1 to get a better feelling for the difference among the previous definitions.

When saying "Is the inverse of $L-\lambda$ on its range a continouous operator", Keener means that it is bounded.

Finally, before moving to transform theory, Keener proofs that λ is in the residual spectrum of L if and only if $\bar{\lambda}$ is in the descrete spectrum of L^* (the adjoint of L). His arguments go like this.

Figure 2.1: Flow chart to illustrate the different definitions of spectra as well as the definition of revolvent sets



Let X be a Hilbert space (can it be a Banach space?) and L a linear operator defined in X. The Fredholm alternative states that the whole space $\mathcal{H}$ is given by the sum of the closure of the range of L and the null space of the adjoint. That is

$$\mathcal{H} = \overline{\mathrm{ran}(L)} + N(L^*),$$

that is, any vector in the Hilbert space $\mathcal{H}$ is an orthogonal sum of a component along the closure of the range of L and a component along the null space of the adjoint L^*. Hence, the closure of the range of $L - \lambda$ is a proper subset of $\mathcal{H}$ if and only the null space of $L^* - \lambda$ is not the trivial set. That is, there is a vector v such that $L^* v = \bar{\lambda} v$. In consequence if L is a self–adojoint operator it does not have a residual spectrum.

We now study the role of the discrete and continuous spectrum in the construction of bases in a Hilbert space.

Chapter 3

Transform Theory

3.1 Fundamentals

We explore the relationship between Dirac Deltas, contour integrals, transform pairs, and Green's functions in the context of the spectrum of a linear operator. Then we set up a sequence of derivations for known transform pairs.

Let us assume a linear operator L in a Hilbert space $\mathcal{H}$, such that it has a complete countable set of orthonormal eigenfunctions $\{\phi_k\}$, with $L\phi_k = \lambda_k\,\phi_k$. Since the set of eigenfunctions is complete, for any given function u in the Hilbert space $\mathcal{H}$, we can write

$$u = \sum_{k=1}^{\infty} \alpha_k \phi_k,$$

where $\alpha_k = \langle u, \phi_k \rangle$. We then have that

$$Lu = \sum_{k=1}^{\infty} \alpha_k L(\phi_k) = \sum_{k=1}^{\infty} \lambda_k\,\alpha_k \phi_k.$$

Furthermore

$$L^2 u = L(Lu) = \sum_{k=1}^{\infty} \lambda_k^2\,\alpha_k \phi_k,$$

and inductively

$$L^n u = \sum_{k=1}^{\infty} \lambda_k^n\,\alpha_k \phi_k.$$

If f is a polynomial we see that

$$f(L)u = \sum_{k=1}^{\infty} f(\lambda_k)\,\alpha_k\phi_k.$$

If f is a holomorphic (analytic) function in the complex plane, in a point z_0 then f can be expanded as a Taylor series in a disk around z_0 and in this case, if we can extend the operator mathematics so that the infinite sum of operators makes sense, we then can say that

$$f(L)u = \sum_{k=1}^{\infty} f(\lambda_k)\,\alpha_k\phi_k. \tag{3.1}$$

In particular if we want to invert the operator $L-\lambda$, we can think as $f(L) = (L-\lambda)^{-1}$. The function $f(x) = 1/(x-\lambda)$ is analytic (except at $x = \lambda$, where it as a simple pole). Then formula 3.1 for $f(x) = 1/(x-\lambda)$ becomes

$$(L-\lambda)^{-1}u = \sum_{k=1}^{\infty} \frac{\alpha_k\phi_k}{\lambda_k-\lambda}. \tag{3.2}$$

We observe that for any $\lambda \neq \lambda_k$, λ is in the resolvent set of L, while each λ_k is in the point spectrum set of L. If L is a differential operator then the inverse $(L-\lambda)^{-1}$ represents a Green's function. That is, the differential equation

$$(L-\lambda)v(x) = u(x).$$

can be solved as

$$v(x) = (L-\lambda)^{-1}u(x) = \int G(x,\xi,\lambda)u(\xi)d\xi, \tag{3.3}$$

where λ is treated as a parameter and integration is taken over the appropriate domain. The Green's function, by defintion satisfies the fundamental equation

$$(L-\lambda)\,G(x,\xi,\lambda) = -\delta_\lambda(x-\xi).$$

[1] This shows a clear connection between the Green's function and a Dirac Delta. This connection is well known in basic differential equations, however the connection that we will show

[1]The subscript λ in δ_λ is just a label. The Delta distribution is unique.

below is less common and it represents a powerful tool to characterize Dirac delta distributions as contour integration over the complex parameter λ, and a tool to generate transform pairs.

Returning to equation 3.3

$$-\int G(x,\xi,\lambda)u(\xi)d\xi = \sum_{k=1}^{\infty}\frac{\alpha_k\phi_k}{\lambda-\lambda_k}.$$

We now integrate along a circular contour around 0 with radius going to ∞ (C_∞), on the complex parameter λ. Then

$$\begin{aligned}\int_{C_\infty}\left(\int G(x,\xi,\lambda)\right)u(\xi)d\xi d\lambda &= -\int_{C_\infty} d\lambda\sum_{k=1}^{\infty}\frac{\alpha_k\phi_k}{\lambda-\lambda_k}\\ &= -2\pi\mathrm{i}\sum_{k=1}^{\infty}\alpha_k\phi_k\\ &= -2\pi\mathrm{i}\,u(x),\end{aligned}$$

where we applied the Cauchy residue theorem and $\mathrm{i}=\sqrt{-1}$. We find then that under the conditions assumed

$$u(x) = -\frac{1}{2\pi\mathrm{i}}\int_{C_\infty} d\lambda\left(\int d\xi G(x,\xi,\lambda)u(\xi)\right).$$

From distribution theory we see that this occurs when

$$-\frac{1}{2\pi\mathrm{i}}\int_{C_\infty} d\lambda\, G(x,\xi,\lambda) = \delta(x-\xi). \tag{3.4}$$

We then see the connection between Green's functions and Dirac Delta distributions and contour integration in the complex domain. That is, given any Green's function corresponding to a linear operator L with a complete set of eigenfunctions, we can define a representation of a Dirac delta distribution by doing the contour integral above. Furthermore we will show one more link on these chain of events. The connection of all the above with the theory of transforms.

3.2 Sine and cosine transform pairs

We illustrate this connection with a classical example. Let us assume the operator L be the second order differentiation

$$Lu = -u''$$

with boundary conditions $u(0) = u(1) = 0$. The Green's function for the $L - \lambda$ operator is given by

$$G(x,\xi,\lambda) = \begin{cases} \frac{\sin\sqrt{\lambda}x \ \sin\sqrt{\lambda}(1-\xi)}{\sqrt{\lambda}\sin\sqrt{\lambda}}, & 0 \le x < \xi < 1 \\ \frac{\sin\sqrt{\lambda}\xi \ \sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}\sin\sqrt{\lambda}}, & 0 \le \xi < x < 1. \end{cases}$$

(see notes on Green functions) To evaluate the contour integral 3.4 we observe that $G(x,\xi,\lambda)$ has simple poles at points of the form $\lambda = k^2\pi^2$.

We also note that $\lambda = 0$ is not a pole (nor a brunch cut either) since

$$\begin{aligned} \lim_{\lambda\to 0} G(x,\xi,\lambda) &= \begin{cases} \lim_{\lambda\to 0} \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}x} \ x(1-\xi)\frac{\sin\sqrt{\lambda}(1-\xi)}{1-\xi}, & 0 \le x < \xi < 1 \\ \lim_{\lambda\to 0} \frac{\sin\sqrt{\lambda}\xi}{\sqrt{\lambda}\xi}\xi\,(1-x)\ \frac{\sin\sqrt{\lambda}(1-x)}{1-x}, & 0 \le \xi < x < 1 \end{cases} \\ &= \begin{cases} x(1-\xi), & 0 \le x < \xi < 1 \\ \xi\,(1-x), & 0 \le \xi < x < 1. \end{cases} \end{aligned}$$

We apply the Cauchy residue theorem. Assume first that $0 \le x \le \xi < 1$, then

$$\begin{aligned}
\delta(x-\xi) &= -\frac{1}{2\pi\mathrm{i}}\int_{C_\infty} d\lambda\, G(x,\xi,\lambda) \\
&= -\frac{1}{\cancel{2\pi\mathrm{i}}}\cancel{2\pi\mathrm{i}}\sum_{k=1}^{\infty}\mathrm{Res}\; G(x,\xi,\lambda) \\
&= -\sum_{k=1}^{\infty}\lim_{\lambda\to k^2\pi^2}\frac{(\lambda-k^2\pi^2)\sin\sqrt{\lambda}x\ \sin\sqrt{\lambda}(1-\xi)}{\sqrt{\lambda}\sin\sqrt{\lambda}} \\
&= -\sum_{k=1}^{\infty}\frac{\sin(k\pi x)\sin[k\pi(1-\xi)]}{k\pi}\lim_{\lambda\to k^2\pi^2}\frac{\lambda-k^2\pi^2}{\sin\sqrt{\lambda}} \\
&= -\sum_{k=1}^{\infty}\frac{\sin(k\pi x)\sin[k\pi(1-\xi)]}{k\pi}\lim_{\sqrt{\lambda}\to k\pi}\frac{2\sqrt{\lambda}}{\cos\sqrt{\lambda}} \\
&= -2\sum_{k=1}^{\infty}(-1)^k\frac{\sin(k\pi x)\sin[k\pi(1-\xi)]}{\cancel{k\pi}}\cancel{k\pi} \\
&= 2\sum_{k=1}^{\infty}(-1)^k\sin(k\pi x)(-1)^k\sin(k\pi\xi)] \\
&= \sum_{k=1}^{\infty}2\sin(k\pi x)\sin(k\pi\xi)
\end{aligned}$$

If, instead of choosing $0\le x\le\xi<1$, we choose $0\le\xi<x<1$, then we use the second branch of the Green's function above which is the same as the first branch except by swapping x and ξ. So the result in this case would

$$\delta(x-\xi) = 2\sum_{k=1}^{\infty}\sin(k\pi\xi)\sin(k\pi x) \tag{3.5}$$

which is the same result found using the first branch of the Green's function.

We then found the Dirac delta distribution corresponding to the second derivative operator $Lu=-u''$, as an infinite sum of products of sine functions. We will see now how this representation also generates a sine transform pair.

If we multiply by $U(\xi)$ and integrate along ξ in the interval $[0,1]$ we find

$$\int_0^1 d\xi u(\xi)\delta(x-\xi) = \sum_{k=1}^{\infty} \sin(k\pi x)\left(2\int_0^1 d\xi\ \sin(k\pi\xi)u(\xi)\right).$$

We can write the previous equation as the following *transform* pair.

$$\begin{aligned} U_k &= 2\int_0^1 d\xi \sin(k\pi\xi)u(\xi) \\ u(x) &= \sum_{k=1}^{\infty} U_k \sin k\pi x. \end{aligned}$$

This is the *sine transform pair.* $u(x)$ transforms to U_k and U_k to $u(x)$. It is a transform between a function u defined in the continuum interval $[0,1]$, and an infinite (discrete) sequence U_k, $k = 1, 2, \cdots$. We also see that $u(x)$ is the Fourier sine series approximation with coefficients U_k. Note that instead of Dirichlet boundary conditions ($u(0) = u(1) = 0$) we would have used Neumann boundary conditions ($u'(0) = u'(1) = 0$) we would have obtained a cosine transform pair.

Here we used a Dirac delta representation to build a transform pair. Keener [1] shows an example where L is the Sturm-Livoullille [2] operator

$$Lu = -\frac{1}{w}(pu')' - qu$$

with $u(x)$, a function with second continuous derivatives in $(0,1)$ with some boundary conditions at $x = 0, 1$. The function $w(x) > 0$, is known as weight or density function and assumed continuous in $[a,b]$, p is continuously differentiable, and q is continuous in $[a,b]$.

The importance of the operator L cannot be underestimated, it encompasses a wealth of number of problems appearing from the method of separation of variables in partial differential equations. They have the nice property of self-adjointness which provides real (physical) distinc eigenvalues with orthogonal eigenfunctions. W.N. Everitt [3] shows a catalog of Sturm-Liouville differential equations. Table 3.1 (from [2]) illustrates a set of functions and properties associated to the Sturm-Liuvielle operator. The example shown above with $Lu = -u''$ is a Sturm-Liouville, operator with $w = 1$, $p = 1$, $q = 0$, and $[a,b] = [0,1]$. The example in Keener's

[2] https://en.wikipedia.org/wiki/Sturm%E2%80%93Liouville_theory

[3] http://www.math.niu.edu/SL2/papers/birk0.pdf

Name associated with function	$p(x)$	$q(x)$	$w(x)$	interval	Boundary conditions	eigenvalues n=1,2...,	Normalized eigenvectors
Fourier	1	0	1	$-[\pi,\pi]$	$y(\pi)=y(-\pi)$ $y'(\pi)=y'(-\pi)$	$0, n^2$	$1, \sin nx, \cos nx$
Bessel	x	k^2x^{-1}	x	$[0,1]$	$y(0)=$ finite $y(1)=0$	$\mu_n, J_k(\mu_n)=0$	$J_k(\mu_n(x)$
Legendre polynomials	$1-x^2$	0	1	[-1,1]	$y(-1)$ and $y(1)$ finite	$n(n+1)$	$P_n(x)$
Tchebycheff polynomials	$(1-x^2)^{1/2}$	0	$(1-x^2)^{-1/2}$	[-1,1]	$y(-1)$ and $y(1)$ finite	n^2	$T_n(x)$
Hermite polynomials	$\exp(-x^2)$	0	$\exp(-x^2)$	$(-\infty,\infty)$	As $\|x\|\to\infty$, $y=\mathcal{O}(x^k)$ for some $k>0$	$0, 2n$	$H_0(x), H_n(x)$
Laguerre	$x\exp(-x)$	0	$x\exp(-x)$	$[0,\infty)$	$y(0)$ finite; as $x\to\infty$, $y=\mathcal{O}(x^k)$ for some $k>0$	n	$L_n(x)$

Table 3.1: A few examples of the Sturm–Liouville operator.

book uses $p = x$, $w = x$, $q = 0$, and the interval $[0, 1]$. The development of the transform pair with these parameters follows step-by-step the development done here but, of course, the details are more complicated . The Green function satisfies the equation

$$-\frac{1}{x}(xG')' - \lambda G = \delta(x-\xi),$$

Instead of dealing with sine functions that example deals with Bessel functions. The resulting transform pair is

$$\begin{aligned} U_k &= \frac{2}{[J_0'(\mu_k)]^2}\int_0^1 d\xi\, J_0(\mu_k\xi)\, u(\xi) \\ u(x) &= \sum_{k=1}^{\infty} U_k J_0(\mu_k x). \end{aligned}$$

with $u_k = \sqrt{\lambda}$ and $J_0(x)$ is the Bessel function [4] of the first kind.

The sums shown in the transform pairs are originated from the fact that we have a countable number of eigenvalues. We show in the next example, what happens when the spectrum of the operator L is not the point spectrum and instead it is the continuous spectrum.

[4] http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html

3.3 Fourier sine and cosine integral pairs

Let us consider

$$L(u) = -u''$$

for continuous differentiable functions u in $[0, \infty)$, which are square integrable. That is they are in $L^2[0, \infty)$. The Green function $G(x, \xi, \lambda)$ satisfies the equation

$$-G''(x, \xi, \lambda) - \lambda G(x, \xi, \lambda) = \delta(x - \xi).$$

The Green function (see my notes on Green functions) is given by

$$G(x, \xi, \lambda) = \begin{cases} \frac{\sin \sqrt{\lambda} x \ \mathrm{e}^{\mathrm{i}\sqrt{\lambda}\xi}}{\sqrt{\lambda}} & 0 \leq x < \xi < \infty \\ \\ \frac{\sin \sqrt{\lambda} \xi \ \mathrm{e}^{\mathrm{i}\sqrt{\lambda}x}}{\sqrt{\lambda}} & 0 \leq \xi < x < \infty. \end{cases}$$

.

We show that there are no eigenvalues/eigenfunctions for the operator L defined here.

The only possible eigenvalues satisfy the relation $Lu = \lambda u$, that is $-u'' - \lambda u = 0$. The general solution of this equation is $u(x) = c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x$. If we force the boundary condition $u(0) = 0$, then $c_2 = 0$ and the solution is $u(x) = c_1 \sin \sqrt{\lambda} x$. However this function is not square integrable in $[0, \infty)$. We see that

$$\|u\|^2 = \int_0^\infty |u|^2 dx = \int_0^\infty |\sin \sqrt{\lambda} x|^2 dx = \infty.$$

So there are no eigenfunctions and hence no eigenvalues. The only case in which the norm of u is finite is for $\lambda = 0$, which produces the trivial function $u = 0$, which can not be an eigenfunction, by definition. We see that the inverse $(L - \lambda)^{-1}$ in equation 3.2 is not bounded since we can obtain λ_k as close as zero as we want. Hence the set of λ which satisfies the equation $(L - \lambda)u = 0$, is in the continuous spectrum. That is the real line $[0, \infty)$ is the continuous spectrum of the operator L here.

As before we want to compute the integral 3.4. It is interesting that in the case of the point spectrum the integral included all poles of the form $k^2\pi^2$ in the positive real axis and now, because of the branch point at $\lambda = 0$, we need to include all the real axis as a branch cut. We

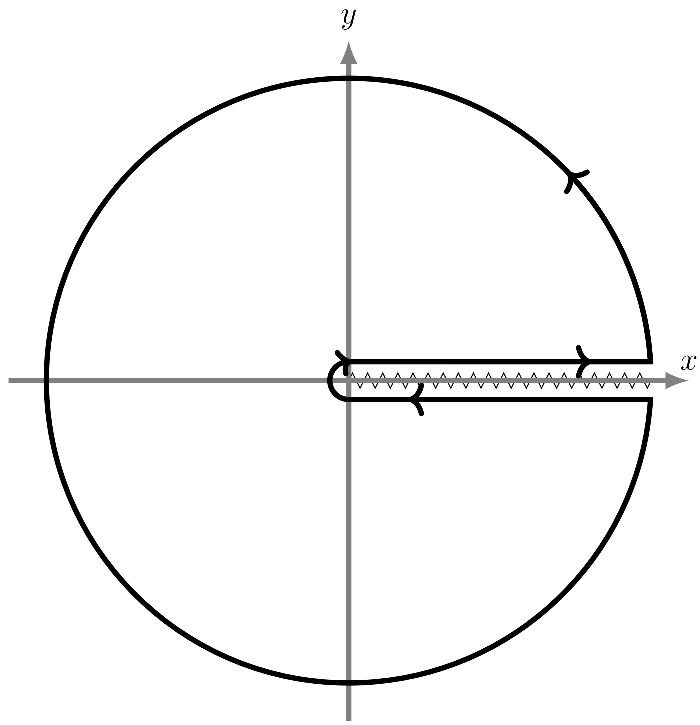


Figure 3.1: Contour C_R for integration in equation 3.6

use a Hankel type contour. Figure 3.1 shows the contour of integration used here. Since there are no poles or singularities inside the contour we have that

$$\int_{C_R} d\lambda\, G(x,\xi,\lambda) = 0 \tag{3.6}$$

We show that, in the limit as the radius of the circle $R \to \infty$, the contribution of the integral along the circle is 0, so that we end up with only the integrals along the two horizontal segments.

Let us evaluate the integral along the arcs at the end of the contour. We can consider λ in polar coordinates. That is, $\lambda = R\mathrm{e}^{\mathrm{i}\theta}$ with $d\lambda = R\mathrm{i}\mathrm{e}^{\mathrm{i}\theta} d\theta$.

If $0 \leq x \leq y < \infty$ we consider the first branch of the Green function as

$$
\begin{aligned}
|G_1(x,\xi,\lambda)|d\lambda &= \left| \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta/2} \frac{\sin(\sqrt{R}\mathrm{e}^{\mathrm{i}\theta/2}x)\ \mathrm{e}^{\mathrm{i}\sqrt{R}\mathrm{e}^{\mathrm{i}\theta/2}\xi}}{\sqrt{R}\mathrm{e}^{\mathrm{i}\theta/2}} \right| d\theta \\
&= \left| \frac{\sqrt{R}}{2} \left(\mathrm{e}^{\mathrm{i}\sqrt{R}\mathrm{e}^{\mathrm{i}\theta/2}x} - \mathrm{e}^{-\mathrm{i}\sqrt{R}\mathrm{e}^{\mathrm{i}\theta/2}x} \right) \mathrm{e}^{\mathrm{i}\sqrt{R}\mathrm{e}^{\mathrm{i}\theta/2}\xi} \right| d\theta \\
&\leq \frac{\sqrt{R}}{2} \left| \mathrm{e}^{\mathrm{i}\sqrt{R}\mathrm{e}^{\mathrm{i}\theta/2}(x+\xi)} \right| d\theta + \frac{\sqrt{R}}{2} \left| \mathrm{e}^{\mathrm{i}\sqrt{R}\mathrm{e}^{\mathrm{i}\theta/2}(-x+\xi)} \right| d\theta
\end{aligned}
$$

Now, since $0 < x < \xi$, then the argument of the exponential functions is in the upper half region of the complex plane. Then as $R \to \infty$, the exponential functions decay faster than $\sqrt{R}$ increases, and since the region of integration (θ ranges in a set of small angles) is finite then in the limit the integral along the circular segments on the left and right sides of the contour is 0. For the second branch of the Green function, where $0 \leq \xi < x < \infty$, we find a similar result where the argument of the second exponential function is $(x - \xi)$ and since in the second branch $x > \xi$, then again the functions inside the absolute value bars decay exponentially which is much faster than $\sqrt{R}$, as $R \to \infty$. We then claim that the only contributions to the integral are along the horizontal part of the contours.

If we make the change of variable $\lambda = \nu^2$, $d\lambda = 2\nu d\nu$, so in the ν range there is no brunch cut and the path unfolds. Since $\nu = \pm\sqrt{\lambda}$. For $+\lambda$ we start at $+\infty$, and move toward zero under the branch cut. Then the -2π rotation around $\lambda = 0$, becomes a $-\pi$ rotation around $\nu = 0$, making the contour over the brunch cut point to the left and going from 0 to ∞. Figure 3.2 shows the contour in terms of the new variable ν.

Let us pick up the first branch of the Green function with $0 \leq x < \xi \leq 1$. Then

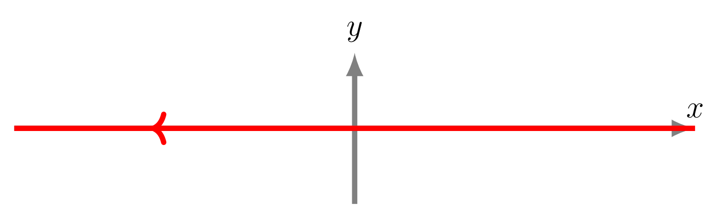


Figure 3.2: Contour (in red) after the transformation $\lambda = \mu^2$.

We have then

$$\begin{aligned}
\frac{1}{2\pi \mathrm{i}}\int_{C_\infty} \frac{\sin\sqrt{\lambda}x\ \mathrm{e}^{\mathrm{i}\sqrt{\lambda}\xi}}{\sqrt{\lambda}} &= \frac{1}{\not{2}\pi \mathrm{i}}\not{2}\int_{\infty}^{-\infty} \not{\mu}\frac{\sin\mu x\ \mathrm{e}^{\mathrm{i}\mu\xi}}{\not{\mu}} d\mu \\
&= -\frac{1}{\pi \mathrm{i}}\int_{-\infty}^{\infty} \sin\mu x\ \mathrm{e}^{\mathrm{i}\mu\xi} d\mu \\
&= -\frac{1}{\pi}\int_{-\infty}^{\infty} \sin\mu x\ \sin\mu\xi\ d\mu
\end{aligned}$$

We find , from equation 3.4 that

$$\delta(x-\xi) = \frac{1}{\pi}\int_{-\infty}^{\infty} \sin\mu x\ \sin\mu\xi\ d\mu = \frac{2}{\pi}\int_{0}^{\infty} \sin\mu x\ \sin\mu\xi\ d\mu \tag{3.7}$$

We find that if, instead of applying the first branch of the Green function, we assume $0 \le y < x \le \infty$ the result does not change.

Please compare this equation with 3.5, for the case of the discrete or the point spectrum expansion. Let us multiply both sides of the previous equation by $u(x)$ and integrate over x, from 0 to ∞

$$u(\xi) = \int_0^\infty dx u(x)\left(\frac{2}{\pi}\int_0^u \infty \sin\mu x \sin\mu\xi d\mu\right) = \int_0^\infty d\mu \left(\frac{2}{\pi}\int_0^\infty dx\ u(x)\sin\mu x\right)\sin\mu\xi$$

This suggest the pair

$$\begin{aligned}
U(\mu) &= \frac{2}{\pi}\int_0^\infty dx u(x)\sin\mu x \\
u(\xi) &= \int_0^\infty d\mu U(\mu)\sin\mu x
\end{aligned}$$

This is the *Fourier sine integral transform pair*

If instead of using the boundary condition $u(0) = 0$, we use the boundary condition $u'(0) = 0$ we would have obtained the following pair:

$$\begin{aligned} U(\mu) &= \frac{2}{\pi}\int_0^\infty dx u(x)\cos\mu x \\ u(\xi) &= \int_0^\infty d\mu U(\mu)\cos\mu x \end{aligned}$$

which corresponds to the *Fourier cosine integral transform pair.* It is interesting to see that the factor $2/\pi$ could be in either side or for symmetry reasons it could be split as $\sqrt{2/\pi}$ in both sides. As long as there is consistency on the use of the inverse with respect to the forward it does not matter. Note that in this abstract definition either one could be the inverse while the other would be the forward. In the context of signal processing the inverse (synthesis) is a function of time t while the forward (analysis) is a function of frequency ω.

3.4 The Fourier transform

Now we derive the Fourier transform [5] pair. Here the operator is the same. That is, $Lu = -u''$, but the boundary conditions are both at infinity. That is, we assume that the domain of functions is in $L^2(-\infty,\infty)$. The Green's function is defined by the equation

$$-G'' - \lambda G = \delta(x-\xi).$$

It can be shown (see my notes on Green functions) that the Green function for this operator under the given boundary conditions is

$$G(x,\xi,\lambda) = -\frac{\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}|x-\xi|}}{2\mathrm{i}\sqrt{\lambda}}$$

We want to compute

$$-\frac{1}{2\pi\mathrm{i}}\int_{C_\infty} G(x,\xi,\lambda)d\lambda.$$

The contour C_∞ is the same shown in Figure 3.1, since again, here the positive x axis is a branch cut. The same procedure done in the sine Fourier integral transform with the substitution of $\lambda = \mu^2$, $d\lambda = 2\mu d\mu$, produces

[5] Perhaps the most important of all transforms

$$-\frac{1}{2\pi \mathrm{i}}\int_{C_\infty} G(x,\xi,\lambda)d\lambda = -\frac{1}{2\pi \mathrm{i}}\int_{\infty}^{-\infty} \cancel{2}\cancel{\mu}\frac{\mathrm{e}^{\mathrm{i}\mu|x-\xi|}}{\cancel{2}\mathrm{i}\cancel{\mu}}d\mu = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu|x-\xi|}d\mu.$$

We then found the Dirac delta representation

$$\delta(x-\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu|x-\xi|}d\mu,$$

Let us assume $x > \xi$, then

$$\delta(x-\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu(x-\xi)}d\mu. \tag{3.8}$$

If, on the other hand, $x < \xi$, then we have

$$\delta(x-\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\mu(x-\xi)}d\mu.$$

We now make the substitution $\mu = -\nu$, $d\mu = -d\nu$, and find

$$\delta(x-\xi) = -\frac{1}{2\pi}\int_{\infty}^{-\infty} \mathrm{e}^{\mathrm{i}\nu(x-\xi)}d\nu = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu(x-\xi)}d\nu.$$

So equation 3.8, which has no absolute bars, is valid in any case of $x > \xi$ or $x < \xi$.

From equation 3.8 we will build a transform pair. We multiply both sides of the previous equation by $u(x)$ and integrate over x between $-\infty$ and ∞ to find

$$u(\xi) = \int_{-\infty}^{\infty} dx u(x)\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu(x-\xi)}d\mu = \int_{-\infty}^{\infty} dx\ u(x)\ \mathrm{e}^{\mathrm{i}\mu x}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu\ \mathrm{e}^{-\mathrm{i}\mu\xi}$$

From which we observe the *Fourier transform pair*

$$\begin{aligned} U(\mu) &= \int_{-\infty}^{\infty} dx\ u(x)\ \mathrm{e}^{\mathrm{i}\mu\xi} \\ u(\xi) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\mu)\mathrm{e}^{-\mathrm{i}\mu\xi}d\mu. \end{aligned}$$

As with the sine and cosine integral Fourier transforms we can have the $1/2\pi$ scaling in any of the two operators or have instead $1/\sqrt{2\pi}$ be present in both operators.

3.5 The Laplace Transform

We choose x to be along the imaginary axis. That is, or $x \in \mathrm{i}(-\infty, \infty)$. Instead of x we want to think of ix for x real, and the substitution $s = \mathrm{i}x$.

The operator (ODE) is $Lu = -u''$, with boundary conditions that $\lim_{x\to\pm\infty} u(x) = 0$. Then the spectral problem has as a Green function, the solution to the equation

$$-G''(s,\xi,\lambda) - \lambda G(s,\xi,\lambda) = \delta(s-\xi).$$

The Green function, in terms of x and y, is given by (see my notes on Green functions) [6]

$$G(x,y,\lambda) = -\frac{\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}|x-y|}}{2\mathrm{i}\sqrt{\lambda}}.$$

where x, y are in the positive real numbers, and in terms of $s = \mathrm{i}x$, and $t = \mathrm{i}y$,

$$G(x,y,\lambda) = -\frac{\mathrm{e}^{\sqrt{\lambda}|s-t|}}{2\mathrm{i}\sqrt{\lambda}}.$$

We want to compute the right hand side of the following equation:

$$\delta(s-t) = -\frac{1}{2\pi\mathrm{i}}\int_{C_\infty} G(s,t,\lambda)d\lambda$$

where the contour C_∞ contains all the spectrum (which is located in the imaginary line). As it is, this function as has a brunch cut at 0, so we make the change of variables $\lambda = \mu^2$, $d\lambda = 2\mu d\mu$, and unfold the integral where now μ runs from $-\infty$ to ∞ along the imaginary axis. We have then

$$\begin{aligned} -\frac{1}{2\pi\mathrm{i}}\int_{C_\infty} G(s,t,\lambda)d\lambda &= -\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \cancel{2\mu}\frac{\mathrm{e}^{\mu|s-t|}}{\cancel{2}\mathrm{i}\cancel{\mu}}d\mu \\ &= \frac{1}{2\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{\mu|s-t|}d\mu. \end{aligned}$$

[6] https://drive.google.com/open?id=0B4W-gdhbNpsDSHZveE9LdnhMNEk

We observe that that there is no need for the absolute bars in the exponent (check this by assuming $|s| > |t|$ and then $|t| > |s|$) so we write

$$\delta(s-t) = \frac{1}{2\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{\mu}(s-t)d\mu.$$

At this moment we assume that $u(x)$ is a causal function, or simple a regular (no causal function) multiplied by the Heaviside function $H(x)$. Let us now multiply both sides of this equation by $u(x)$ and integrate over s between 0 and ∞. Then

$$u(t) = \int_0^\infty ds\, u(s)\frac{1}{2\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{\mu(s-t)}d\mu. = \frac{1}{2\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu \left(\int_0^\infty ds\, u(s)\mathrm{e}^{\mu s}\right) \mathrm{e}^{-\mu t}$$

We define the expression inside parenthesis as $U(\mu)$, so that we have the Laplace transform pair:

$$\begin{aligned} U(\mu) &= \int_0^\infty ds\, u(s)\mathrm{e}^{\mu s} \\ u(t) &= \frac{1}{2\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\mu U(t)\mathrm{e}^{-t\mu}. \end{aligned}$$

The Laplace transform can be obtained directly from the Fourier transform by doing a change of variable from x to $s = \mathrm{i}x$, and making the function $u(x)$ causal. Keener uses this way to find the Laplace trransform pair.

3.6 The Mellin Transform

The Mellin transform is derived from the operator $Lu = -x(xu')'$. That is, the spectral equation is $Lu - \lambda u = -x(xu')' - \lambda u$. The domain is the interval $(0, \infty)$, with the conditions $u(0) = 0$, and $\lim_{x\to\infty} u(x) = 0$.

The Green function is given by

$$G(x,\xi,\lambda) = \begin{cases} \frac{\mathrm{i}}{2\sqrt{\lambda}\xi}\left(\frac{\xi}{x}\right)^{\mathrm{i}\sqrt{\lambda}} & 0 < x < \xi < \infty \\ \\ \frac{\mathrm{i}}{2\sqrt{\lambda}\xi}\left(\frac{x}{\xi}\right)^{\mathrm{i}\sqrt{\lambda}} & 0 < \xi < x < \infty. \end{cases}$$

(see my notes on Green functions) [7]

We now want to evaluate the integral

$$I = -\frac{1}{2\pi \mathrm{i}} \int_{C_\infty} G(x, \xi, \lambda)\, d\lambda,$$

where C_∞ is a contour having all spectral values of the operator $L - \lambda$. Let us assume that $0 < x < \xi < \infty$. Then we have

$$I = -\frac{1}{2\mathrm{i}\pi} \int_{C_\infty} \frac{\mathrm{i}}{2\sqrt{\lambda}\xi} \left(\frac{\xi}{x}\right)^{\mathrm{i}\sqrt{\lambda}} d\lambda = -\frac{1}{4\,\pi\xi} \int_{C_\infty} \frac{1}{\sqrt{\lambda}} \left(\frac{\xi}{x}\right)^{\mathrm{i}\sqrt{\lambda}} d\lambda$$

We make the change of variable $\lambda = \mu^2$, $d\lambda = 2\mu d\mu$, so that

$$I = -\frac{1}{2\,\pi\xi} \int_{-\infty}^{\infty} \left(\frac{\xi}{x}\right)^{\mathrm{i}\mu} d\mu,$$

Then

$$\delta(x - \xi) = -\frac{1}{2\,\pi\xi} \int_{-\infty}^{\infty} \left(\frac{\xi}{x}\right)^{\mathrm{i}\mu} d\mu,$$

We make a change of variable $\nu = \mathrm{i}\mu$, $d\nu = \mathrm{i}d\nu$, and write

$$\delta(x - \xi) = -\frac{\mathrm{i}}{2\,\pi\xi} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \left(\frac{\xi}{x}\right)^{\nu} d\nu = \frac{1}{2\,\pi\xi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \left(\frac{\xi}{x}\right)^{\nu} d\nu.$$

We now multiply both sides of this equation by $u(\xi)$ and integrate between 0 and ∞ to find

$$u(x) = \frac{1}{2\,\pi\mathrm{i}} \int_0^\infty d\xi u(\xi) \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\nu x^{-\nu} \xi^{\mu-1} = \frac{1}{2\,\pi\mathrm{i}} \int_{\mathrm{i}\infty}^{\mathrm{i}\infty} d\nu x^{-\nu} \left(\int_0^\infty d\xi\, \xi^{\mu-1} u(\xi)\right)$$

[7] https://drive.google.com/open?id=0B4W-gdhbNpsDSHZveE9LdnhMNEk

and after calling the expression inside the parenthesis $U(\mu)$, we find the Mellin transform pair

$$\begin{aligned} u(x) &= \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\nu F(\nu) x^{-\nu} \\ U(\nu) &= \int_0^{\infty} d\xi \, \xi^{\nu-1} u(\xi). \end{aligned} \tag{3.9}$$

There is an interesting relation between the Mellin and Fourier transforms. By expanding the operator $L - \lambda$ we find

$$(L-\lambda)u(x) = -x^2 u'' - xu' - \lambda u = 0.$$

This equation follows in the group of equations known as Cauchy-Euler equation [8].

The general Cauchy-Euler equation is

$$x^2 u'' + axu' + bu = 0.$$

It is easy to verify by direct substitution (the Wikipedia link above points to this) and evaluation that if we define

$$x = \mathrm{e}^t$$

the equation is reduced to the second order ODE with constant coefficients

$$\phi'' + (a-1)\phi + b\phi = 0.$$

which can be as easily solved using the characteristic polynomial

$$\lambda^2 + (a-1)\lambda + b = 0.$$

[8] https://en.wikipedia.org/wiki/Cauchy%E2%80%93Euler_equation

In our case, for the Mellin transform derivation, we have that $a = 1$, and $b = \lambda$, so the resulting equation is

$$\phi'' + \lambda = 0.$$

and we are back at the operator $L = u''$ used for the Fourier transform. We just need to indicate that here the function ϕ is causal.

In fact, as Keener points out, the change of variables $x = \mathrm{e}^t$ in the Mellin transform pair 3.9 produces

$$\begin{aligned} u(\mathrm{e}^t) &= \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\nu F(\nu) \mathrm{e}^{-\nu t} \\ U(\nu) &= \int_{-\infty}^{\infty} dt \mathrm{e}^{\nu t}\, \mathrm{e}^{(\nu-1)t} u(\mathrm{e}^t). \end{aligned}$$

Note that while $\mathrm{e}^t > 0$ for all finite t values, t can be used in the whole interval $(-\infty, \infty)$. The integration along the imaginary line can be changed to a real line by changing the variable $\mu = \mathrm{i}\omega$. Also, instead of saying $u(\mathrm{e}^t)$ we can just say $u(t)$ at the expense of redefining the function u so that we can write.

$$u(\mathrm{e}^t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega F(\nu) \mathrm{e}^{-\omega t}$$

(3.11)

$$U(\nu) = \int_{-\infty}^{\infty} dt\ \mathrm{e}^{\omega t} u(\mathrm{e}^t).$$

3.7 The Hankel Transform

Bibliography

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