Analytic Continuation

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Chapter 1

Introduction

In *summation theory,* series are considered, which are divergent and still carry important information. For example, asymptotic series (which in general start diverging, after a few terms are added) can yield accurate numerical approximations within the first few terms.

As an introduction to the concept of *analytic continuation,* I present a classical simple example of a series, which is not asymptotic, but illustrates, in a simple way, its meaning.

Consider first the following sum

$$S = 1 + 2 + 4 + 8 + \cdots$$

If we assume that the sum converges to a number and that we can apply the associative law, and could factor a sum, as follows

$$\begin{aligned} S - 1 &= 2\,(1 + 2 + 4 + 8 + \cdots) \\ &= 2\,S \end{aligned}$$

then

$$S - 1 = 2\,S \Rightarrow S = -1.$$

Of course this is counter–intuitive. A sum of positive numbers yields a negative number. This could only be comprehended in the complex plane.

The complex plane is an extension of the real line which provide a powerful view of functions, not seen in the real line. For example, a function which is differentiable in a complex plane in a point, is infinite differentiable in a neighborhood of the point, so it carries a great deal of smoothness. This is not the case in the real line where functions exists that are only differentiable up to certain order. The real and the imaginary parts of a complex analytical function are coupled closely through the Riemann–Cauchy conditions and these have many implications. One of those is that both, the real and the imaginary part are harmonic; in other words, they satisfy the Laplace equation. This make them very useful in the study of potential problems in electro–magnetism and fluid dynamics. The *fundamental theorem of algebra* allows us to factor a polynomial completely in the complex plane but not necessarily in the real line. Many integrals which are not solvable analytically in the real axis, are solvable analytically in the complex plane through contour integration. If f and g are differentiable in an connected open set D and $f = g$ on some neighborhood of a point $z \in$ D, then $f = g$ on D. This is not true of real–differentiable functions. The last statement is know as the *identity theorem* and it is at the hearth of analytic continuation. The technique of *analytic continuation* allows us to evaluate a function in points where its real representation would fail to converge, as in the example above.

In general, let us assume that z is a complex number, and define

$$f_1(z) = \sum_{i=0}^{\infty} z^i. \tag{1.0.1}$$

Now, if

$$\begin{aligned} \mathrm{S} &= 1 + z + z^2 + z^3 \cdots \\ &= 1 + z(1 + z + z^2 + z^3 + \cdots) \\ &= 1 + z\mathrm{S}, \end{aligned}$$

then

$$\mathrm{S} = \frac{1}{1-z}.$$

Let us define

$$f_2(z) = \frac{1}{1-z}. \tag{1.0.2}$$

It is clear that in equation 1.0.2, if $z = 2$ then $\mathrm{S} = -1$. If we expand S in a Taylor series (the geometrical series), we obtain the series 1.0.1. Both, the geometric series 1.0.1 and the rational form 1.0.2 yield the same numbers inside the unit circle $|z| < 1$, but the series 1.0.1 diverges outside the unit circle. The rational function 1.0.2 is the analytic continuation of the geometrical series 1.0.1.

Can we say that any series is convergent and we just need to use the "associative law in the right way" to find its limit in the complex plane? Let us consider

$$\mathrm{S} = 1 + 1 + 1 + \cdots,$$

if we can apply the associative law we find

$$\mathrm{S} = 1 + \mathrm{S}$$

so $\mathrm{S} = \infty$ (since $1 \neq 0$), and ∞ is a valid unique complex number. Let us now consider the $\zeta(s)$ function

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{i^s}.$$

Then

$$\zeta(0) = 1 + 1 + 1 + \cdots,$$

however we will find a representation of $\zeta(s)$ such that $\zeta(0) = -1/2$. So the associative law does not yield unique results. The branch of mathematics *summation theory* discuss these problems in more detail.

Finally let us consider the function

$$f_3(z) = \int_0^\infty e^{-t(1-z)} dt. \tag{1.0.3}$$

We evaluate the integral

$$\begin{aligned} f_3(z) &= \int_0^\infty e^{-t(1-z)} dt \\ &= -\frac{e^{-t(1-z)}}{1-z}\Big|_0^\infty \\ &= \frac{1}{1-z} \quad \text{if} \quad \operatorname{Re} z < 1 \end{aligned} \tag{1.0.4}$$

We see that the functions $f_1(z), f_2(z), f_3(z)$ (in equations 1.0.1, 1.0.2 and 1.0.3 respectively) represent the same function in the unit circle $|z| < 1$. However outside the unit circle they are different. The function $f_1(z)$ is a Taylor series with a radius of convergence (ROC) [1] equal to 1. It diverges for $|z| \geq 1$.

Figure 1.1 shows an evaluation of the phase ($\phi(z) = \operatorname{atan}\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)$) for the function

$$f_{0,19}(z) = \sum_{i=0}^{19} z^i = \frac{1-z^{20}}{1-z} \tag{1.0.5}$$

The power function being a polynomial of order 19 has 19 zeroes. The zeroes are located around zero marking the circle (of radius 1) of convergence of the function. Note that $f_{0,\infty}(z) = f_1(z)$ on the left but on the right, if $|z| < 1$, it should be equal to $f_2(z)$.

The function $f_3(z)$ has a larger domain of convergence. It converges for $\operatorname{Re} z < 1$ which is a whole half plane. Finally, the function $f_2(z)$ converges every where in the complex plane, except at the point (pole) $z = 1$.

Figure 1.2 illustrates the regions of convergence (gray shade) for the three function $f_1(z)$, $f_2(z)$ and $f_3(z)$. We see then that functions can have different representations and some rep-

[1] The radius of convergence is defined as ROC $= r$ such that

$$\frac{1}{r} = \limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$$

which comes from the ratio test. Also it is known that r satisfies the root test

$$\frac{1}{r} = \limsup_{k\to\infty} |a_k|^{1/k}.$$

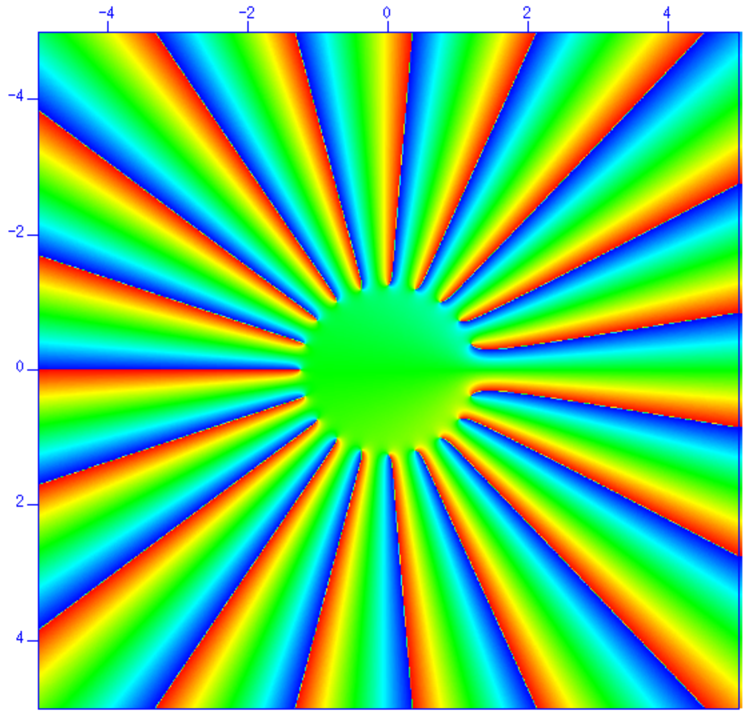

Figure 1.1: Phase plot of the first 20 terms of the power series 1.0.5

resentations are more adequate than others. Mathematicians and physicists like to consider functions where their domain of validity is the largest; such as $f_2(z)$ in 1.0.2 for this particular example. This extension is what is known as *analytic continuation.* .

An analytic continuation of a function is an extension so that the new representation is analytic in a larger domain and both, the old and new representations coincide in the original domain of definition.

The question to ask is how can we extend this domain of validity of a function so we can find an ideal representation valid in the largest subset of the complex numbers possible?

The "associative law" trick used above is not a legitimate mathematical trick. It is more of a magician trick. We want to set up valid rules that let us extend a function to larger domain of analyticity. We ask questions, for example, about the uniqueness of these extensions and the different ways to get there. That is, how can we get from representation $f_1(z)$ in equation 1.0.1 to representation $f_3(z)$ in 1.0.2? In this particular case the geometric series is well known and simple but the question is in general, how can we extend a power series such as

$$f(z) = \sum_{i=0}^{\infty} a_i z^i$$

outside of its the ROC?

There is a unique way to extend functions analytically. The problem of analytic continuation is connected with the Riemann surfaces defined in order to make a multi–valued function

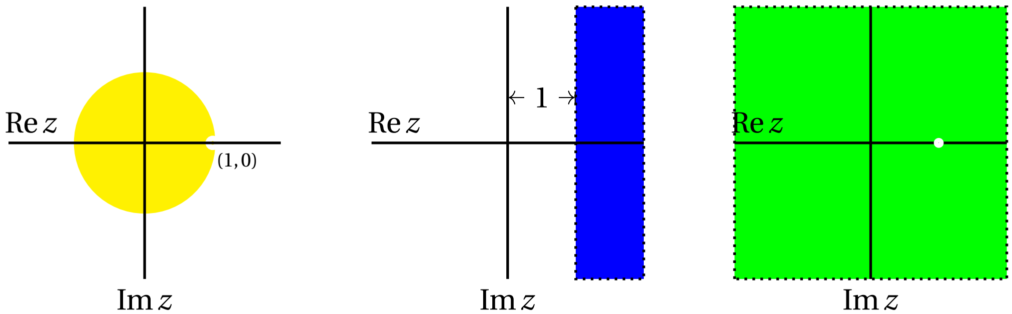

Figure 1.2: Three representations of a function which coincide inside the unit circle. The functions from left to right are $f_1(z)$, $f_2(z)$ and $f_3(z)$. The geometrical series $f_1(z)$ 1.0.1 has a radius of convergence of 1, represented by the yellow circle minus the point $(1,0)$. It is defined only in the yellow circle (including its boundary), except at the point $(1,0)$. The extension to the function $f_2(z) = 1/(1-z)$, on the right is defined in all $\mathbb{C}$ plane, except by the point $(1,0)$ where it has a simple pole. In the center is the integral representation 1.0.3 for which the blue zone represents the region of analyticity of $f_3(z)$

single–valued. I will discuss more about this later in the document.

Not all functions could be extended analytically beyond its region of convergence. For example let us examine the function

$$f_\infty(z) = 1 + \sum_{i=1}^{\infty} z^{i!}.$$

Figure 1.3 shows the a phase plot for a sum up to $i = 3$, that is $f_3(z) = 1 + z + z^2 + z^6$. Since the order is only 6 we can see the 6 zeroes around the circle $|z| = 1$ marking their territory. I would like to use a high number of terms but unfortunately, the series diverges so quickly beyond $i = 1$ that even double precision would not help much. Here is what happens in the boundary of the unit circle. For each rational number $s = p/q$,

$$z = \mathrm{e}^{2\mathrm{i}\pi p/q}$$

we see that

$$z^{i!} = \mathrm{e}^{2\,\mathrm{i}\,i!\pi p/q} = 1$$

So, since $|r\mathrm{e}^{2\pi \mathrm{i} s}| = r$, then $f(z)$ diverges for $r > 1$, with s any rational number. Due to the fact that the rational numbers are dense in the interval $[0, 2\pi]$, no open set can get out of the unit circle without intersecting a point on the unit circle. The function is not analytic along any circle of radius $r > 1$ (by the same token). The case of the infinite series example 1.0.1 we have only one point trouble $z = 1$ which we can not cross but instead go around it in any way. Here there is no leakage zone. All the circle centered at 0 with radius 1 is a hard boundary that can not let the function be continued outside.

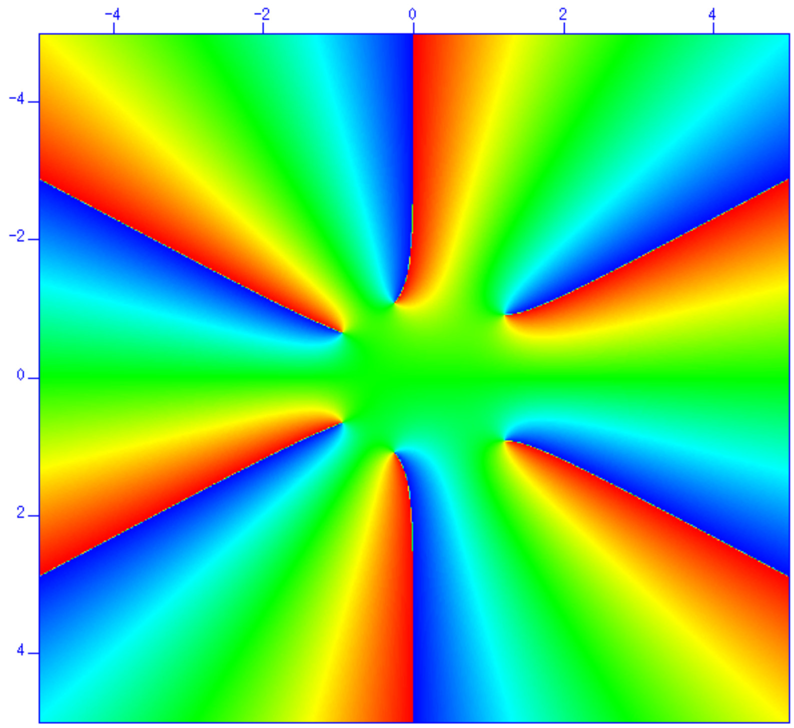

Figure 1.3: Phase plot for function $1 + z + z^2 + z^6$

Another example of a function that can not be analytically continued beyond certain boundary is

$$f(z) = \sum_{i=0}^{\infty} z^{2^n}.$$

This series converges in the unit disk $|z| < 1$ and is analytic there. The boundary of singularities that limits the domain of analyticity is called a *natural boundary*, or *natural barrier*. In general Hadamard's gap theorem [2] states that if the indices $i_1, i_2, \cdots$, of all non–zero coefficients of the power series

$$f(z) = \sum_{i=0}^{\infty} a_i z^i$$

satisfy the Hadamard's condition, $i_{k+1} > (1+\theta) i_k$, where $\theta > 0$, then the boundary of the disc of convergence series is its natural boundary. That is, the function does not have analytic continuation beyond this boundary.

The previous examples show that analytic continuation is not guaranteed. Even worse, here

[2] https://www.encyclopediaofmath.org/index.php/Hadamard_theorem

there is an example with ROC is 0,

$$f(z) = \sum_{i=0}^{\infty} i^{i+1} z^i.$$

Before showing a few methods to perform analytic continuation I want to post a few questions and provide my own answers. In physics and engeneering is common to find integrals which are hard to evaluate and are converted into infinite sums via integration by parts or series expansions (such as for example Taylor, Bremmer, orthogonal decompositions or other decompositions). The idea is that if the series is asymptotic, only a few terms would suffice to provide decent approximations. So, why do we want to convert series into analytic functions or integrals? Are we reversing course? My answer is that while series are easier to evaluate numerically, integrals or analytic functions are easier to deal with in mathematics. The theory of analysis, integral and differential calculus is powerful and its power is reflected mostly in the continuum, not in the discrete. It is the same reason for which continuum mechanics has provided great advances in physics and engineering. For example think about the evaluation of the following two problems

$$(i) \quad a = \sum_{i=0}^{n} i^{10} \qquad (ii) \quad \alpha = \int_0^n x^{10} dx.$$

The answer of the second (the continuum) is $\alpha = \frac{n^{11}}{11}$. How about the first $a =$? I provide a detailed answer of the first problem in my notes [4]. The continuum is a highway bridge that take us far fast. Once we reach a goal we can always sample (discretize) our continuum representations. We can also ask, what is first, and infinite sum (or product) or an integral? The answer is, sometimes a sum, sometimes an integral. The reason is in part historic. For example, Euler [5] found an infinite product representation of the $\Gamma(z)$ function which was not very practical for non integer evaluations. It made sense that he looked for an integral representation that he found the following year (1730) as an integral of a logarithmic function and which later Legendre formulated as the integral we know today. In many cases the integral representations are solutions of differential equations which are continuum approximations of our discrete world. There is a closed relationship between linear differential equations, Green's functions, Dirac delta representations, transforms, eigenfunction/eigenvalues, contour integrals, the Cauchy residue theorem, infinite series, analytic continuation, and Riemman surfaces that I discuss in my notes [6].

Of course we have to be aware that the real world is discrete and understand the scale of the problem we are considering so that our approximations are valid. The concepts of uncertainty principle, aliasing, resolution and accuracy (precision) should be well understood, together with questions of efficiency of implementions which, of course, should be done in finite (and hopefuly short, for economic reasons) time computations.

I used some lecture notes from professor David Simpson of the University of Colorado at Boulder at the time of writing this sections. Particularly the taxonomy of analytic continuation and a few classical examples. However, in some of the methods I have different focus. I also added a few more methods to Professor Simpson's list.

Simpson splits the methods of analytic continuation in two main branches

- Without Contour Integration
- With Contour Integration

where the deciding attribute is “contour integration”. I used these as the main key to the exposition of the methods.

Chapter 2

Analytic Continuation Without Contour Integration

2.1 The Hadamard Product

The Hadamard product appeared in J. S. Hadamard's [3] 1899 paper. A short history with plenty of references about it appears in Timo Pohlen's thesis. Hadamard proofs that if we are given two power series

$$f(z) = \sum_{i=0}^{\infty} a_i z^i \qquad g(z) = \sum_{i=0}^{\infty} b_i z^i,$$

whose radii of convergence are r_a and r_b, respectively, then the (Hadamard) product [1]defined as

$$h(z) = \sum_{i=0}^{\infty} a_i b_i z^i$$

has a radius of convergence r where

$$r \geq r_a r_b.$$

This is a very powerful result, for if one of the series is entire then the product is entire, provided the other series has a ROC larger than zero. Actually I will show an example below where the

[1]I have seen in the literature the symbol "$*$" for Hadamard product and even the word "convolution". The symbold "$*$" and the word convolution are defined in the mathematical and engeneering community through the regular product of series as follows.

$$f(z)g(z) = \sum_{i=0}^{\infty} c_i z^i, \qquad \text{where} \qquad c_i = a * b = \sum_{j=0}^{j=i} a_j b_{i-j}.$$

The Hadamard product as shown here is a point-to-point multiplication.

series diverges with ROC equal to 0 and the Hadamard product bring it up to "life". We will see how how Hadamard product is used to extend the domain of analyticity.

I start with the method of Borel summation since it provides a method to obtain the representation 1.0.2 ($f_2(z)$) from the series representation 1.0.1 ($f_1(z)$) passing through representation 1.0.3.

2.1.1 Borel summation

As in the moment generating function in statistics, Borel consider integrals of the type

$$f(z) = \int_0^\infty e^{-t}\phi(zt)dt.$$

The weight e^{-t}, which is also used to compute Laguerre [2] polynomials, is a Gaussian weight which converges faster than any power series and this make it very convenient. For example, if we know the power series

$$\phi(z) = \sum_{i=0}^{\infty} \frac{a_i z^i}{i!}, \tag{2.1.1}$$

with $|a_i| < C$ for some real postive C, which is an exponential generating function [4], and entire (from the ratio test), then

$$\begin{aligned} f(z) &= \int_0^\infty e^{-t}\phi(zt)dt \\ &= \int_0^\infty e^{-t}\sum_{i=0}^{\infty}\frac{a_i z^i}{i!}dt \\ &= \sum_{i=0}^{\infty}\frac{a_i z^i}{i!}\int_0^\infty e^{-t}\,t^i dt \quad \text{uniform convergence of the series.} \\ &= \sum_{i=0}^{\infty}\frac{\Gamma(i+1)}{i!}a_i z^i \\ &= \sum_{i=0}^{\infty} a_i z^i, \end{aligned}$$

since $\Gamma(k+1) = k!$. Therefore with the help of the exponential generating function 2.1.1 we obtain

$$f(z) = \sum_{i=0}^{\infty} a_i z^i = \int_0^\infty e^{-t}\phi(zt)dt.$$

This is a recipe to find the analytic continuation of a power series through a Gaussian type integration.

[2] http://en.wikipedia.org/wiki/Laguerre_polynomials

For example, let us consider the geometrical series 1.0.1

$$f(z) = \sum_{i=0}^{\infty} z^i = 1 + z + z^2 + z^2 + \cdots \tag{2.1.2}$$

since

$$\phi(z) = \sum_{i=0}^{\infty} \frac{z^k}{i!} = \mathrm{e}^z,$$

then

$$f(z) = \int_0^{\infty} \mathrm{e}^{-t} \phi(zt) dt = \int_0^{\infty} \mathrm{e}^{-t(1-z)} dt$$

and this is precisely equation 1.0.3 for $(f_3(z))$. Now from 1.0.4 we see that

$$g(z) = \frac{1}{1-z} = f(z), \quad z > 1. \tag{2.1.3}$$

But, then this rational function also evaluates to any complex $z \neq 1$, so the function $g(z)$ as defined in 2.1.3 is the analytic continuation of $f(z)$ as defined in 2.1.2 to the whole complex plane, except in the point $z = 1$.

2.2 Circle-chain method

If the singularities of a function in a given region are isolated, the function could be extended beyond these singularities.

The circle–chain method is good to explain the concept of analytic continuation but is not practical and of not much use. The idea behind the circle chain method is that if the singularities (poles, branch points, essential singularities) are isolated, then open sets (particularly, open circles) can squeeze in between them to extend the domain of analyticity.

According to Abikoff [1], the circle–chain method was introduced by Weierstrass in an original paper written in 1842 and published in 1894. If a function is analytic at a given point z_0 it can be represented as a Taylor series. There could be a limit of how far from the point z_0 we can move for the function still to be analytic. The closest singularity from the point z_0 defines the radius of convergence ROC. Let us assume ROC = r. Then this means that somewhere on the boundary of the circle $\mathscr{B} = \{z : |z - z_0| = r\}$ there should be at least a singularity [3]. If the singularity is isolated, then there is a finite distance between this singularity and the closest singularity on the boundary $\mathscr{B}$ (if any), and we can squeeze a circle in between them.

Figure 2.1 shows the circle of convergence for a given Taylor series with ROC = 1, around $z_0 = 0$. We assume only two singularities at points $z_1 = (1,0)$ and $z_0 = (\cos\pi/4, \sin\pi/4)$ (red dots). We squeeze a circle with center $c = (\cos\pi/8, \sin\pi/8)$ and radius $r \approx 0.39$ so that this circle kisses the two singularities. This would be the maximal circle that extends the region of analyticity in between the two singularities. The idea behind the circle chain is to build a chain

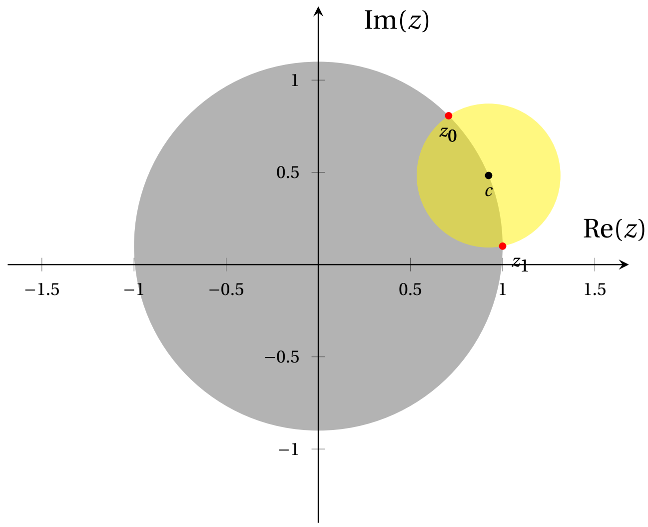

Figure 2.1: The yellow circle with center at c continues the domain of analyticity of power series. The red points are the singular points, of the given series representation.

of circles that, in this case, would cover the whole complex plain, but the two singular points z_0 and z_1.

The difficulty with the chain of circles method is that, although it is an easy concept to understand, it is difficult to implement. Given a power series around a point a, such as for example

$$f(z) = \sum_{i=0}^{\infty} a_i (z-a)^i,$$

with ROC $= r$, we can pick any point z_0 in the boundary (we want to be greedy, so we pick it in the boundary) $|z_0 - a| = r$. While the geometrical construction of the circle so that its radius is the distance between z_0 and the closest isolated singularity of the function is relatively easy to do (provided we know the location of that point, which in practice is not easy) , the algebra of the infinite series might be not.

Let us consider the general example for a power series

$$f(z) = \sum_{i=0}^{\infty} a_i (z-z_0)^i \tag{2.2.4}$$

centered at z_0 and with radius of convergence r. A point z_1 can be chosen anywhere in the domain (away from the original center z_0, but still inside the closed circle with radius r, or better, in the boundary, and away from any singularity in the boundary of the circle. A new Taylor series could be defined with center at z_1. These new Taylor series should coincide with the original Taylor series 2.2.4 in the circle $|z - z_0| < r$ (that is, in the intersection of the two circles). We want to define a function $g(z)$ which coincides with the function $f(z)$ in the common domain

[3] Otherwise we could increase the size of ROC

(inside the circle of convergence) but such that its center of definition is shifted to z_1. That is, the new function is

$$g(z) = \sum_{i=0}^{\infty} b_i (z - z_1)^i,$$

for some coefficients b_i, which we should determine. To find the coefficients b_i we have in mind that if the two functions are analytic in their domain of intersection, they share their values up to all derivative orders, so from Taylor expansions we have: (recall z_0 is not a singular point)

$$\begin{aligned}
b_0 &= g(z_1) = f(z_1) = \sum_{i=0}^{\infty} a_i (z_1 - z_0)^i \\
b_1 &= g'(z_1) = f'(z_1) = \sum_{i=1}^{\infty} a_i\, i\, (z_1 - z_0)^{i-1} \\
&\vdots \\
b_k &= \frac{g^{(k)}(z_1)}{k!} = \frac{f^{(k)}(z_1)}{k!} = \sum_{i=k}^{\infty} a_i \binom{i}{k} (z_1 - z_0)^{i-k}.
\end{aligned} \tag{2.2.5}$$

If we know the analytic expansion of the geometrical series, we would not have to make the chain of circles, because this analytic expansion itself should reveal the singularities and the complement of the set of singularities is the domain of analyticity of the function. So this method makes sense only if we know about an analytic representation of the infinite series, but this means that we would have to evaluate an infinite series for each coefficient that we want to find. It is in this sense that the method is not of practical use. Furthermore, better methods to continue analytically power series are shown. In section 2.7 we discuss in particular the continued functions methods.

The following example from professor Simpson illustrates this.

$$f(z) = \sum_{i=1}^{\infty} \frac{z^i}{i}. \tag{2.2.6}$$

It is known that $f(z)$ is a representation of logarithmic function $g(z) = -\ln(1-z)$. So, as I said above, this is an analytic representation where we know that is has a zero at $z = 0$, a singularity at $z = 1$ and a brunch cut where the function becomes multivalued (up to integer multiples of the phase angle 2π.) However it is a good example to see how the analytic continuation extends the validity of the function in a unique way, providing a different point of view of the concept of Riemann surface of multivalued function. We observe that

$$f'(z) = \sum_{i=1}^{\infty} z^{i-1} = \frac{1}{1-z}$$

and so

$$f^{(i)}(z) = \frac{(i-1)!}{(1-z)^i}$$

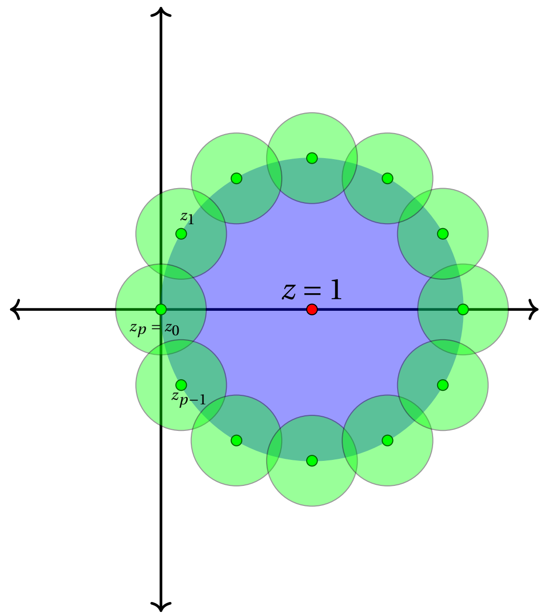

Figure 2.2: Initial circle of convergence and possible new circles for expanding the domain of analyticity of the function $f(z)$ defined in equation 2.2.6. Possibly for at least one circle could pass a branch cut, so we need to be careful here. We discuss this on the main text.

We want to extend the region of definition outside of the circle of convergence, which is a circle centered at $z = 1$ with radius $r = 1$. We could have a finite number of new circles $p > 1$, with centers at $z_k = 1 - \mathrm{e}^{-2\pi i k/p}$ with $k = 1, 2, \cdots p$, and use the coefficients 2.2.5 where $z_0 = 0$ to find a continuation of the series valid at those circles. Figure 2.2 illustrates the geometry of the problem with $p = 12$.

We have

$$b_i = \frac{f^{(i)}(z_k)}{i!} = \frac{1}{i\,(1-z_k)^i}$$

and so the extension around the center z_k can be written as

$$\begin{aligned} f_k(z) &= \sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(z_k)(z-z_k)^n \\ &= f(z_k) + \sum_{i=1}^{\infty} \frac{1}{i} \left(\frac{z-z_k}{1-z_k}\right)^i \\ &= f(z_k) + f\left(\frac{z-z_k}{1-z_k}\right). \end{aligned} \tag{2.2.7}$$

Now, since again

$$f_k'(z) = \frac{1}{1-z_k} \sum_{i=1}^{\infty} \left(\frac{z-z_k}{1-z_k}\right)^i = \frac{1}{1-z}$$

we can recursively apply equation 2.2.7, starting at $k = 1$ to obtain the sequence

$$\underbrace{g(z)}_{\text{new}} = \underbrace{f(z_1) + f\left(\frac{z_2 - z_1}{1 - z_1}\right) + \cdots + f\left(\frac{z_p - z_{p-1}}{1 - z_{p-1}}\right)}_{\text{Additive constant}} + \underbrace{f\left(\frac{z - z_p}{1 - z_p}\right)}_{\text{Old } f(z),\text{ since } z_p = 0}$$

We now evaluate the additive constant at the point.

$$\frac{z_k - z_{k-1}}{1 - z_{k-1}} = \frac{\mathrm{e}^{-2\pi\mathrm{i}(k-1)/p} - \mathrm{e}^{-2\pi\mathrm{i}k/p}}{\mathrm{e}^{-2\pi\mathrm{i}(k-1)/p}} = 1 - \mathrm{e}^{-2\pi\mathrm{i}/p} = z_1$$

for $1 < k \le p$. So the additive constant does not depend on k, but it depends on $p > 1$. The additive constant is

$$pf(z_1) = p\underbrace{(1 - e^{-2\pi\mathrm{i}/p})}_{z_1}\frac{f(z_1)}{z_1} = \underbrace{p(1 - e^{-2\pi\mathrm{i}/p})}_{\to -2\pi\mathrm{i}}\underbrace{\sum_{i=0}^{\infty}\frac{z_1^i}{i+1}}_{\to 1}$$

where the limits are taken as $p \to \infty$. Observe that

$$p(1 - e^{-2\pi\mathrm{i}/p}) \quad = \quad \frac{1 - \mathrm{e}^{-2\pi\mathrm{i}/p}}{1/p}$$

and from L'Hôpital's rule

$$\lim_{p\to\infty} p(1 - e^{-2\pi\mathrm{i}/p}) = \lim_{p\to\infty} \frac{2\pi\mathrm{i}\cancel{d(1/p)/dp}}{\cancel{d(1/p)/p}} = 2\pi\mathrm{i}.$$

That is, when going around the circle in the chain, we ended up in a phase change of $2\pi\mathrm{i}$. In the way that

$$\ln \mathrm{e}^{\mathrm{i}\theta} \ne \ln \mathrm{e}^{\mathrm{i}(\theta+2\pi)} \quad \Rightarrow \quad \mathrm{i}\theta \ne \mathrm{i}\theta + 2\pi\mathrm{i},$$

while $\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}(\theta+2\pi)}$. The exponential function is a single valued function but not injective. Then the inverse is a multi-valued function. The same happens here with the function $f(z)$ which happens to be a series representation of the $-\ln(1 - z)$ function. To make the function single valued, we need to go into different Riemann surfaces corresponding to the different branches of the same function.

Figure 2.3 (taken from Wikipedia, done by Yamishita Makoto) shows the phase behavior of the analytic continuation of the logarithm function. The chain of circles climbs along a spiral parking-lot-type ramp which actually is the Riemann surface that transforms the multi–valued logarithmic function into a single valued function. Abikoff [1] discusses Weierstrass chain of circles (that he called *analytic configuration*) and its relation with Riemann surfaces and how these "overlapping discs define the topology, a concept which underlines the modern notion of manifold".

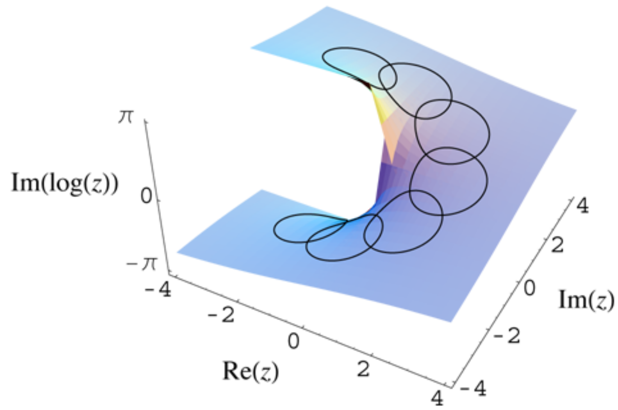

Figure 2.3: Analytic Continuation of the logarithm function $\ln(z)$. Taken from Wikipedia)

2.3 Reflection Methods

Reflection methods are based on reflection properties of functions in the following way. If we know the behavior of a function in a half plane and a reflection formula, with respect to an axis which defines the half plane, then the properties on that half plane can be readily extended to the other half plane and in particular analytic continuation.

2.3.1 Schwarz reflection

The Schwarz reflection principle applies when we know a function defined in some part of the upper half complex plane including the real line, where the function vanishes. If that function $f(z)$ is analytic we could define a new function which coincides in the upper half plane with $f(z)$, that is, a function $g(z) = f(z)$ for the upper half plane, and that extends the analytic behavior to the lower plane. This extension is given after defining $g(z) = \overline{f}(\overline{z})$, for z in the lower plane.

Let us assume that R^+ is a region in the upper complex plane, such that f is analytic in R^+. Let us call R^0 the boundary of thie region in in the real line $\mathbb{R}$, and also that f is continuous in $\mathrm{R}^+ \cup \mathrm{R}^0$. We can extend R^+ to a region R by considering the conjugation of R^+, that is $\mathrm{G} = \mathrm{R}^+ \cup \mathrm{R}^0 \cup (\mathrm{R}^+)^*$. Then $\mathrm{G} = \mathrm{G}^*$. We should prove that if we define $g(z) = \overline{f}(\overline{z})$ then g is analytic in in R and it is an extension (analytic continuation) of f defined in R^+.

2.3.2 Gamma $\Gamma(z)$ reflection

The Gamma reflection formula [4] states that

$$\Gamma(z) = \frac{\pi}{\Gamma(1-z)\,\sin\pi z} \tag{2.3.8}$$

The function $\Gamma(1-z)$ on the right is analytic for $1-\text{Re}(z) > 0$, that is, for $\text{Re}(z) < 1$, and from the $\sin\pi z$, we see that the function is analytic for $z < 1$ and $z \neq \pm1, \pm2, \cdots$. We could argue that formula 2.3.8 has a restriction on the left side because $\Gamma(z)$ is valid only for $\text{Re}(z) > 0$, and so $\Gamma(1-z)$ is defined for $0 < \text{Re}(z) < 1$, so only on the strip $0 < \text{Re}(z) < 1$ is the equation valid. However the right hand side of equation 2.3.8 is valid in the whole complex plane $\mathbb{C}$ except for the points $\pm1, \pm2, \cdots$. We then say that the right hand side is the analytic continuation of the function on the left of the equal sign beyond the strip $0 < \text{Re}(z) < 1$. The right hand side of the equation 2.3.8 is the analytic continuation of the $\Gamma(z)$ function into the left complex plane. Both functions are well defined and agree for $0 < \text{Re}(z) < 1$ but only the left side is defined for $\text{Re}(z) < 0$. The original definition of the Γ function 2.10.11 coincides with this new from in the interval $0 < \text{Re}(z) < 1$.

Zeta $\zeta(z)$ reflection

2.4 use of a functional equation

2.5 Replace factors by integrals or sums

2.6 Subtract similar series/integral

2.7 Continued Functions

Given a closed form of a function $f(z)$ we can build a continued function as

$$b_1 f(zb_2 f(zb_3 f(zb_4 \cdots))) \tag{2.7.9}$$

such that we match the Taylor series

$$\sum_{i=0}^{\infty} a_i z^i. \tag{2.7.10}$$

The idea is that the representation 2.7.9 would have a larger domain of analyticity than the corresponding Taylor series 2.7.10. If this is the case, then the sequence

$$b_1,\ b_1 f(z_b 2),\ b_1 f(zb_2 f(zb_3)),\ b_1 f(zb_2 f(zb_3 f(zb_4)))\cdots$$

converges faster than the corresponding Taylor series 2.7.10.

Two particular cases are of interest:

[4] https://en.wikipedia.org/wiki/Reflection_formula

2.7.1 The Continued Exponentials

Write

$$\sum_{i=0}^{\infty} a_i z^i = b_0 \mathrm{e}^{b_1 z \mathrm{e}^{b_2 z \mathrm{e}^{b_3 z \cdots}}}$$

Expand the second expression in Taylor series and find

$$\begin{aligned} b_0 &= a_0 \\ b_1 &= a_1 a_0 \\ b_2 &= a_0 a_1 a_2 \frac{1}{2} a_0 a_1^2 \\ b_3 &= a_0 a_1 a_2 a_3 \frac{1}{2} a_0 a_1 a_2^2 + \frac{1}{6} a_0 a_1^3 \end{aligned}$$

example:

$$e^{z e^{z e^{z e^{z \cdots}}}} = \sum_{i=0}^{\infty} \frac{(i+1)^{i-1}}{i!} z^i.$$

with radius of convergence ROC = −e. The function on the left converges to a cardioid (picture here) going all the way to $z = -2/e$. Write

$$e^z, e^{x e^z}, e^{z e^{z e^z}}, \cdots$$

2.7.2 The Continued Fractions

Here we use the following representation [5]

$$\sum_{i=0}^{\infty} a_i z^i = \cfrac{b_0 z}{1 + \cfrac{b_1 z}{1 + \cfrac{b_2 z}{1 + \cfrac{b_3 z}{1 + \cfrac{b_4 z}{1 + \cfrac{b_5 z}{1 + \cdots}}}}}}.$$

[5] This is not the only way to represent continued fractions.

2.8 Ramanujan's formula

2.9 Padé summation

2.10 Using the recursion formula

the Γ function is defined as as

$$\Gamma(z) = \int_0^{\infty} \mathrm{e}^{-t} t^{z-1} dt. \tag{2.10.11}$$

Using integration by parts, we can show [5] the recursion formula:

$$\Gamma(z) = (z-1)\Gamma(z-1)$$

By applying this recursion formula repeatedly, starting at $\Gamma(z+n)$ we find

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)(z+2)\cdots(z+n-1)}. \tag{2.10.12}$$

which continues the function up to $\mathrm{Re}(z) > -n$, leaving out the vertical asymptotes $\mathrm{Re}(z) = k$, $k = -1, -2, \cdots, -1-n$. The analytic continuation of the Gamma function is meromorphic in the entire complex plane.

If $z = n$ is a positive integer, then it is well known that $\Gamma(n) = (n-1)!$. Integral **??** diverges for $\mathrm{Re} z \le 0$. However from **??**

$$\Gamma(z) = (z-1)\Gamma(z-1)$$

With this, for example for any $0 < \mathrm{Re} z < 1$, we find

$$\Gamma(z-1) = \frac{\Gamma(z)}{z-1} \tag{2.10.13}$$

which is well defined as long as $z \neq 0$. This extends the definition of the function to the domain $\mathrm{Re} z > -1$, $z \neq 0$. Recursively, we find the formula

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)(z+2)\cdots(z+n-1)}. \tag{2.10.14}$$

which continues the function up to $\mathrm{Re}(z) > -n$, leaving out the vertical asymptotes $\mathrm{Re}(z) = k$, $k = -1, -2, \cdots, -1-n$. The analytic continuation of the Gamma function is meromorphic in the entire complex plane. Figure 2.4 shows a picture of the norm of Gamma function ($|\Gamma(z)|$) for $-5 \le \mathrm{Re} z \le 5$ and $-5 \le \mathrm{Im}(z) \le 5$. Observe the poles at the negative integers. set of notes contains methods based on contour integration. These are:

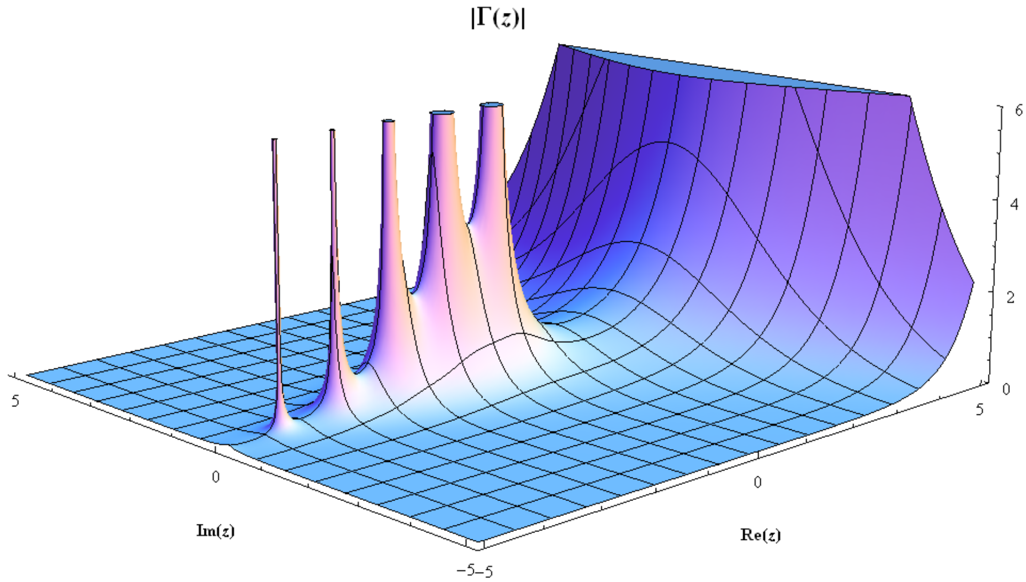

Figure 2.4: $|\Gamma(z)|$ function. The truncated spikes show clearly the singularities (simple poles) at the negative integer numbers.(taken from Wikipedia)

Chapter 3

Analytic Continuation With Contour Integration

3.1 Closed form evaluation of the defining integral

Here is an example.

Let us assume that were given the following Fourier series function:

$$\sum_{i=1}^{\infty} \frac{\cos nx}{n}.$$

Which is valid in some region of the complex domain, say for example the interval $(0, 2\pi)$ where for obvious reasons we do not include the points $x = 0$ and $x = 2\pi$, where the sum is the divergente harmonic series.

We would like to extend this function beyond this zone to the largest possible zone of convergence in the complex plain. Define

$$f_n(\gamma) = \int_{\gamma = C(t)} e^{inz} dz$$

where γ is a continuous path not defined yet but fixed. Each f_n is analytic in the upper half plane and we require $\gamma \subset C^+(z)$, where if $z \in \gamma$, $\mathrm{Re}(z) > 0$.

Due to the uniform continuity of the integral in the upper half plane we assert that

$$f(\gamma) := \sum_{n=1}^{\infty} f_n = \int_{\gamma} \sum e^{inz} dz$$

but, using the geometrical series

$$\sum_{n=1}^{\infty} e^{inz} = \frac{e^{iz}}{1 - e^{iz}}$$

and so

$$\sum_{n=1}^{\infty} f_n(\gamma) = \int_{\gamma} \frac{e^{iz}}{1 - e^{iz}} dz \tag{3.1.1}$$

with the antiderivative:

$$\int \frac{e^{iz}}{1-e^{iz}} = i\log(1-e^{iz}) + \text{constant}$$

We now worry about γ. Let γ the straight segment $x+i\epsilon$ that joins the points $x+i\epsilon$ to $\pi/2+i\epsilon$, with $\epsilon > 0$, where $0 < x \le \pi/2$ are real numbers, then

$$\begin{aligned} f_n(\gamma) &= \left.\frac{e^{inz}}{i\,n}\right|_{z=x+i\epsilon}^{\pi/2+i\epsilon} \\ &= e^{-\epsilon n}\frac{e^{in\pi/2}-e^{inx}}{i\,n} \\ &= e^{-n\epsilon}\frac{\cos(n\pi/2)+i\sin(n\pi/2)-\cos nx + i\sin nx}{i\,n} \end{aligned}$$

Then take the imaginary part of $f_n(\gamma)$ and get

$$\text{Im}[f_n(\gamma)] = e^{-n\epsilon}\frac{\cos nx + \cos n\pi/2}{n}$$

On the other hand

$$\begin{aligned} \int_\gamma \frac{e^{iz}}{1-e^{iz}} &= i\log(1-e^{iz})\Big|_{z=x+i\epsilon}^{\pi/2+i\epsilon} \\ &= i[\log(1-e^{-\epsilon}e^{i\pi/2}) - \log(1-e^{-\epsilon}e^{ix})] \end{aligned}$$

We now express this into its real and imaginary parts, but we are only interested on the imaginary part since that is what matches the series of cosines above. This is, we first find the modulus

$$\left|\frac{1-e^{-\epsilon i}e^{i\pi/2}}{1-e^{-\epsilon}e^{ix}}\right| = \frac{\sqrt{1+e^{-2\epsilon}}}{\sqrt{e^{-2\epsilon}\sin^2 x + (1-e^{-\epsilon}\cos x)^2}}$$

From which

$$\text{Im}\int_\gamma \frac{e^{iz}}{1-e^{iz}} = \log\frac{\sqrt{1+e^{-2\epsilon}}}{\sqrt{e^{-2\epsilon}\sin^2 x + (1-e^{-\epsilon}\cos x)^2}}$$

So from equation 3.1.1 and the results shown where we match the imaginary components

$$\sum_{i=1}^{\infty} e^{-n\epsilon}\frac{\cos nx + \cos n\pi/2}{n} = \log\frac{\sqrt{1+e^{-2\epsilon}}}{\sqrt{e^{-2\epsilon}\sin^2 x + (1-e^{-\epsilon}\cos x)^2}}$$

Since the sum in the left converges uniformly for any $\epsilon > 0$, we can take the limit as $\epsilon \to 0$, inside the sum.

$$\sum_{i=1}^{\infty}\frac{\cos nx}{n} + \sum_{i=1}^{\infty}\frac{\cos n\pi/2}{n} = -\frac{\log(1-\cos x)}{2} \quad , \quad 0 < x \le \frac{\pi}{2}.$$

Now,

$$\sum_{i=1}^{\infty} \frac{\cos n\pi/2}{n} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -\log 2$$

Hence

$$\sum_{i=1}^{\infty} \frac{\cos nx}{n} = -\frac{\log(2-2\cos x)}{2} \quad , \quad 0 < x \leq \frac{\pi}{2}.$$

At this point we converted this in an exercise of analytic continuation. The function

$$f(z) = -\frac{\log(2-2\cos z)}{2}$$

Is analytic in the complex plane except for singularities at $z = 2kpi$, where k is any integer.

3.2 Modification of the integration path to a Hankel–type contour

3.2.1 The Gamma function

Hermann Hankel [1] in 1884 derived the analytic continuation of the $\Gamma(z)$ function by using a contour integral on the complex plane. Here we illustrates Hankel's method.

The problem with the $\Gamma(z)$ integral comes from the factor t^{z-1}. At $t = 0$ the function has a branch point since it is multivalued for non integer z. We could think of the positive x-axis as a branch cut and draw a contour that does not include the singularity at $t = 0$.

Figure 3.1 shows the Hankel's contour C to evaluate the integral

$$\mathrm{I} = \int_{\mathrm{C}} \mathrm{e}^{-t} t^{z-1} dt \tag{3.2.2}$$

where now t is a complex variable. The integrand is analytic as a function of t since the contour C does not goes through the singularity $t = 0$. The integral is bound for all z since now there is no singularity and the exponential absorbes most of the amplitude on the integral. That is, the decaying exponential overpowers the possible raising of t^{z-1} for $\mathrm{Re}(z) \gg 0$. We see that the negative real part of z is a small non-zero number. The contour C, oriented in a counter-clockwise direction, is the union of three contours. That is,

$$\mathrm{C} = \mathrm{C}_1 \cup \mathrm{C}_2 \cup \mathrm{C}_3.$$

The contour C_1 is the set of $z = x + iy$, with $y = \epsilon$, and x going from R to ϵ. The path C_2 is a tiny circle of radius ϵ going counter-clockwise and centered at zero. That is, here $t = \epsilon \mathrm{e}^{\mathrm{i}\theta}$ for

[1] https://en.wikipedia.org/wiki/Hermann_Hankel

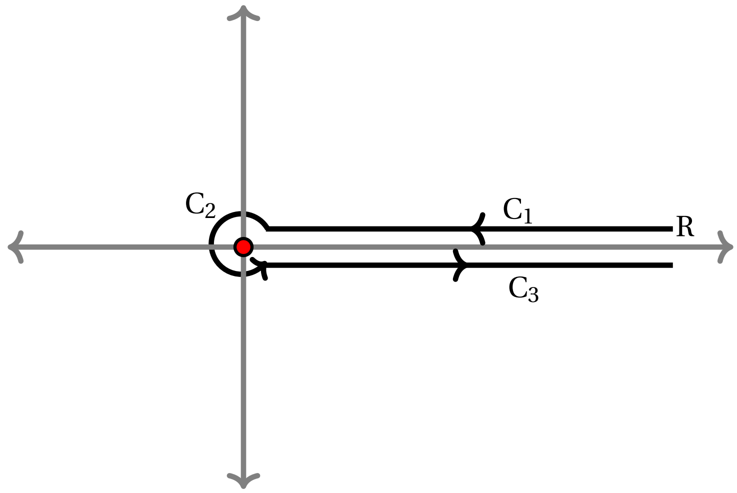

Figure 3.1: Hankel contour C to compute integral 3.2.2.

$\theta \in [2\delta, 2\pi - \delta]$, for a small angle δ. Finally, the contour C_3 is the reverse of contour C_1. It is $z = x + iy$ with $x \in [\epsilon, R]$ and $y = -\epsilon$. We do the integral in three steps

$$I_i = \int_{C_i} f(t)dt$$

for $f(z) = e^{-t}t^{z-1}$. Then take the limit as $\epsilon, \delta \to 0$, and $R \to \infty$. Let us start with the integral around C_2. This is, $dt = i\epsilon e^{i\theta}d\theta$ and

$$I_2 = i\epsilon \int_{\delta}^{2\pi-\delta} e^{-i\epsilon e^{i\theta}} (e^{-i\theta})^{z-2} d\theta.$$

Since the integrand is bounded then we have that $\lim_{\epsilon \to 0} I_2 = 0$. We are left with two integrals. The integral over the top path which is

$$I_1 = -\int_0^R e^{-x-i\epsilon}(x + i\epsilon)^{z-1}dx.$$

Clearly, as $\epsilon \to 0$, $\delta \to 0$, and $R \to \infty$ we have that

$$\lim_{\epsilon \to 0, R \to \infty} I_1 = -\Gamma(z).$$

For the third contour C_3 we have that the argument of the complex variable t is $2\pi i - \delta$ since the contour already rotated that much. Then t along this contour has a phase shifted by $2\pi i$, or we understand here that $t = 2\pi i(x - iy)$. Then

$$I_3 = \int_0^R e^{2\pi i - x - i\epsilon}[e^{2\pi i}(x + i\epsilon y)]^{z-1} = e^{2\pi z i}\int_0^R e^{-x-i\epsilon}(x+\epsilon y)^{z-1},$$

and so

$$\lim_{\epsilon\to 0, R\to\infty} I_3 = e^{2\pi i z}\Gamma(z).$$

This is

$$\lim_{\epsilon\to 0,\delta\to 0, R\to\infty} I = (e^{2\pi i}z - 1)\Gamma(z),$$

and we found that the Hankel contour integral for the $\Gamma(z)$ function is

$$H(z) = \frac{1}{e^{2\pi z i} - 1}\int_C e^{-t} t^{z-1} dt. \tag{3.2.3}$$

is such that as the contour C deforms taking the limit $R\to\infty$, $\delta\to 0$, $\epsilon\to 0$, we find

$$\lim_{\epsilon\to 0,\delta\to 0, R\to\infty} H = \Gamma(z),$$

and $H(z)$ is the analytic continuation of the $\Gamma(z)$ function.

We can write equation 3.2.3 differently by knowing that

$$e^{2\pi z i} - 1 = \frac{e^{\pi z i} - e^{-\pi z i}}{e^{-\pi z i}} = \frac{2i\sin\pi z}{e^{-\pi z i}} = \frac{2i\sin\pi z}{(-1)^z}$$

Then

$$H(z) = -\frac{1}{2i\sin\pi z}\int_C e^{-t}(-t)^{z-1} dt, \tag{3.2.4}$$

which as an analytic continuation of the $\Gamma(z)$ function has simple poles $z = 0, -1, -2, \cdots$.

3.2.2 The Hurwitz Zeta function

The definition of the Hurwitz Zeta [2] function is given by the equation

[2]https://en.wikipedia.org/wiki/Hurwitz_zeta_function

$$\zeta(s,x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^s} \quad , \quad x>0 \tag{3.2.5}$$

with $s\in\mathbb{C}$, $\mathrm{Re}(s)>1$. If $x=1$, Hurwitz Zeta-function is reduced the original Riemann Zeta function. That is, by default we assume $\zeta(s,1)=\zeta(s)$. This function could be analytically continued. There are several methods to prove that a function is analytic. For example, Apostol [**?**] in his book [3]book Introduction to Analytic Number Theory derives the analytic continuation of the Hurwitz Zeta function by finding complex integral representation along a contour and work with this alternative representation. Cohen [**?**] Volume II on Number Theory. [4] uses a recursion formulas to perform analytic continuation of the Hurwitz Zeta function $\zeta(s,x)$.

Here we use a method which employes the Euler $\Gamma(z)$ function to find a formula for the Hurwitz Zeta function in terms of the original Riemann Zeta function. Then we find a representation of the Hurwitz Zeta function in terms of the $\Gamma(z)$ function.

We relate the Hurwitz Zeta function with the $\Gamma(z)$ function. From the definition of the $\Gamma(s)$ function, and making the change of variables $u=(n+x)t$, $du=(n+x)dt$, we have that

$$\Gamma(s)=\int_0^\infty u^{s-1}\mathrm{e}^{-u}dt=(n+x)^s\int_0^\infty t^{s-1}\mathrm{e}^{-(n+x)t}dt.$$

We now sum up over n from $n=0$ to ∞ and find

$$\Gamma(s)\zeta(s,x)=\sum_{n=0}^{\infty}\int_0^\infty t^{s-1}\mathrm{e}^{-(n+x)t}\,dt=\int_0^\infty t^{s-1}\frac{\mathrm{e}^{-xt}}{1-\mathrm{e}^{-t}}dt,$$

where we used the geometric series $1+z+\cdots+z^n+\cdots=1/(1-z)$, with $z=\mathrm{e}^{-t}$. We then have that

$$\zeta(s,x)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{e}^{-xt}}{1-\mathrm{e}^{-t}}dt \quad , \quad \mathrm{Re}(s)>1. \tag{3.2.6}$$

This establishes a first relation between the Hurwitz Zeta function $\zeta(s,x)$ and the $\Gamma(s)$ function. In particular by choosing $x=1$ we find the relation between the Riemann Zeta function and the $\Gamma(s)$ function

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}dt \quad , \quad \mathrm{Re}(s)>1.$$

Hurwitz performed an analytic continuation of the Hurwitz Zeta function much like Riemann did on the Riemann Zeta function, by using contour integration along a Hankel contour.

[3]https://books.google.com/books?isbn=1475755791
[4]https://books.google.com/books?isbn=0387498931.

We start with the integral representation of the Hurwitz Zeta function 3.2.6 , which we write as a function of z instead of s and such that $\mathrm{Re}(s) > 1$.

$$\zeta(t,x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-xt}}{1-e^{-t}} dt$$

The idea is to modify the integral in equation 3.2.6 by reversing the sign of t in its power factor and choose an appropriate (Hankel) contour. That is,

$$I = \int_C (-t)^{s-1} \frac{e^{-xt}}{1-e^{-t}} dt = \int_\infty^{(0+)} (-t)^{s-1} \frac{e^{-xt}}{1-e^{-t}} dt \tag{3.2.7}$$

where the contour C is shown in figure 3.1. The notation $\int_\infty^{(0+)}$ is used in Whittaker and Watson, [**?**] to indicate that the contour starts at ∞ along the positive axis then goes over zero counter-clockwise returning to ∞. We look this integration as the integral over three contours $C = C_1 \cup c_2 \cup C_3$. The contour C_2 is a small circle with radius ϵ. Along this small circle we can make the change of variables $t = \epsilon e^{i\theta}$, $dt = i\epsilon e^{i\theta} d\theta$, and write

$$I_2 = \int_{C_2} (-t)^{s-1} \frac{e^{-xt}}{1-e^{-t}} dt = \int_\delta^{2\pi-\delta} e^{\pi i(s-1)}\, \epsilon^{s-1} e^{i\theta(s-1)} \frac{e^{-x\epsilon e^{i\theta}}}{1-e^{-\epsilon e^{i\theta}}} i\epsilon d\theta = i\epsilon^s e^{\pi i(s-1)} \int_\delta^{2\pi-\delta} \frac{e^{i\theta(s-1)} e^{-x\epsilon e^{i\theta}}}{1-e^{-\epsilon e^{i\theta}}},$$

where δ is a small angle. It is easy to show that $|ie^{i(\pi+\theta)(s-1) - x\epsilon e^{i\theta}}| < C$ for some positive constant C, and so

$$|I_2| \le C\epsilon^s \int_\delta^{2\pi-\delta} \frac{1}{1-e^{-\epsilon e^{i\theta}}}$$

For the limit as $\epsilon \to 0$ we can use L'Hôpital's rule, that is

$$|I_2| \le C \lim_{\epsilon\to 0} \int_\delta^{2\pi-\delta} \frac{\epsilon^{s-1}}{e^{-\epsilon e^{i\theta}} e^{i\theta}} = 0,$$

since $\mathrm{Re}(s) > 1$.

For the C_1 path we consider the argument of t equal to $-\pi$, and for the C_2 path we add 2π to the argument, so it will be considered as π. Let us now evaluate the integral along the path C_1, going from some large number R to ϵ That is,

$$\int_{C_1} (-t)^{s-1} \frac{e^{-xt}}{1-e^{-t}} dt = \int_R^\epsilon e^{-i\pi(s-1)} (t)^{s-1} \frac{e^{-xt}}{1-e^{-t}} dt.$$

If we take the limit as $\epsilon \to 0$ and $R \to \infty$ we find

$$\lim_{\epsilon\to 0, R\to\infty} I_1 = e^{-i\pi(s-1)}\int_\infty^0 t^{s-1}\frac{e^{-xt}}{1-e^{-t}}dt = -e^{-i\pi(s-1)}\int_0^\infty t^{s-1}\frac{e^{-xt}}{1-e^{-t}}dt.$$

Now, for the integral on the path C_3 we have

$$\lim_{\epsilon\to 0, R\to\infty} I_3 = e^{i\pi(s-1)}\int_0^\infty t^{s-1}\frac{e^{-xt}}{1-e^{-t}}dt$$

and combining the three integrals, and taking the limit as $\epsilon\to 0$ and $R\to\infty$ we find that

$$\begin{aligned}\int_\infty^{(0+)}(-t)^{s-1}\frac{e^{-xt}}{1-e^{-t}}dt &= \left(e^{i\pi(s-1)} - e^{-i\pi(s-1)}\right)\int_0^\infty t^{s-1}\frac{e^{-xt}}{1-e^{-t}}dt \\ &= 2i\sin(\pi(s-1))\int_0^\infty t^{s-1}\frac{e^{-xt}}{1-e^{-t}}dt \\ &= -2i\sin(\pi s)\int_0^\infty t^{s-1}\frac{e^{-xt}}{1-e^{-t}}dt\end{aligned}$$

Now, from equation 3.2.6 we find that

$$\zeta(s,x) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{e^{-xt}}{1-e^{-t}}dt = -\frac{1}{2i\Gamma(s)\sin(\pi s)}\int_\infty^{(0+)}(-t)^{s-1}\frac{e^{-xt}}{1-e^{-t}}dt$$

We now use the reflection formula [5] for the $\Gamma(s)$ function. That is

$$\Gamma(s)\sin(\pi s) = \frac{\pi}{\Gamma(1-s)},$$

to find

$$\zeta(s,x) = -\frac{\Gamma(1-s)}{2\pi i}\int_\infty^{(0+)}(-t)^{s-1}\frac{e^{-xt}}{1-e^{-t}}dt \tag{3.2.8}$$

Note that $t=0$ is not part of the contour of integration, so the function is not multivalued. The function is analytic everywhere except in those points where the $\Gamma(1-s)$ has poles at $s = 1,2,3,\cdots$. We also know that $\zeta(s,x)$ is analytic for $\mathrm{Re}(s) > 1$, so the only region to question is the vertical line with $\mathrm{Re}(s) = 1$ in the complex plane.

Note that

[5] https://en.wikipedia.org/wiki/Reflection_formula

$$\lim_{s\to 1}\frac{\zeta(s,x)}{\Gamma(1-s)} = \frac{1}{2\pi i}\int_{\infty}^{(0+)} \frac{e^{-xt}}{1-e^{-t}}dt = -1$$

using the Cauchy residue theorem with $z = 0$. Now, since $\Gamma(1-s)$ has a simple pole at $s = 1$, then we find that $\zeta(s,x)$ as defined above has a simple pole at $s = 1$. We found then that 3.2.8 is the analytic continuation of the Hurwitz Zeta function to the whole complex plane with a simple pole singularity only at $s = 1$.

3.3 Modification of the integration path to a Pochhammer–type contour

Chapter 4

Formal Treatment

The book in Complex Analysis: Selected Topics [1] by Mario Gonzalez [2], presents a formal discussion on the topic of analytic continuation. We follow Gonzalez discussion in this chapter.

4.1 Basic Definitions and Theorems

Definition 4.1.1 *An ordered pair* $\{f, \mathrm{R}\}$ *where* R *is a region in* $\mathbb{C}$ *and* f *is an analytic function in* R *is defined as a* **function element** .

This definition attaches a function to its region of analyticity. Equality of function elements means equality on the domains of defintion and values. That is $\{f_1, \mathrm{R}_1\} = \{f_2, \mathrm{R}_2\}$ if and only if $\mathrm{R}_1 = \mathrm{R}_2$ and $f_1 = f_2$.

We now define *direct* analytic continuation and *trivial* analytic continuation.

Definition 4.1.2 *Given a function element* $\{f_1, \mathrm{R}_1\}$ *we say that a function element* $\{f_2, \mathrm{R}_2\}$ *is a* **direct analytic continuation** of $\{f_1, \mathrm{R}_1\}$ if $\mathrm{R} = \mathrm{R}_1 \cap \mathrm{R}_2 \neq \emptyset$ and $f_1(z) = f_2(z), \forall z \in \mathrm{R}$, or at least on an infinite subset of R having an accumulation point in R. Note that an accumulation point is a limit point, and since the function is continuous, then the limit point z is reached by the sequence of $\{z_n\}$, points. Since the function is analytic at one point, it is analytic in all the neighborhood R. This is one of the powerful tools of analytic functions. Due to the symmetry on the definition we can also say that $\{f_1, \mathrm{R}_1\}$ is also a direct analytic continuation of $\{f_2, \mathrm{R}_2\}$.

Figure 4.1 illustrates the concepts on this definition. If $\mathrm{R}_1 \subset \mathrm{R}_2$ we say that f_2 the analytic continuation is **trivial** . Note that the **trivial** analytic continuation could be seen as a special case of the more general **direct** analytic continuation.

We now discuss the issues of existence and uniqueness. The existence of analytic continuation is not guaranteed. We will show this later. The uniqueness is, provided existence. That is, if the pairs $\{f_1, \mathrm{R}_1\}$ and $\{f_2, \mathrm{R}_2\}$, are direct analytic continuations of $\{f, \mathrm{R}\}$, then they owe to be equal. This is

[1] https://books.google.com/books/about/Complex_Analysis.html?id=0z0NjTrJCkIC&hl=en

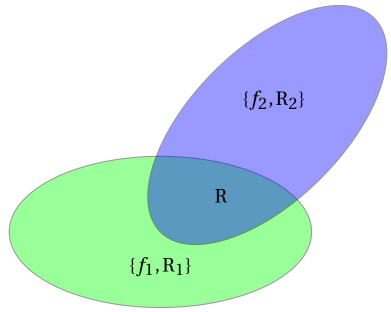

Figure 4.1: Illustration of **direct** analytic continuation. Start in any of the two function element pairs. For example $\{f_1, R_1\}$, then find the second pair $\{f_2, R_2\}$. Such that $R = R_1 \cap R_2 \neq \emptyset$ and such that $f_1(z) = f_2(z)$ in the intersection R. Then in this case the pair $\{f_2, R_2\}$ is the **direct** analytic continuation of the pair $\{f_1, R_1\}$. There is reciprocity. So we can also say that $\{f_1, R_1\}$ is a **direct** analytic continuation of $\{f_2, R_2\}$.

Theorem 4.1.3 *If* $\{f_1, R_1\}$ *and* $\{f_2, R_2\}$ *are direct analytic continuations of* $\{f, R\}$ *, and we assume* $R_1 = R_2$*, then* $\{f_1, R_1\} = \{f_2, R_2\}$.

Proof: Since $f_1(z) = f_2(z)$ in $R \neq \emptyset$, they coincide in at least one point and so since they both are analytic in R, they coincide everywhere in R. So they are the same analytic continuation $\{f_1, R_1\} = \{f_2, R_2\}$ of $\{f, R\}$. This proofs that **direct** analytic continuation is a well defined term.

We then say that f_1 and f_2 are **partial local representations** of a function $F(z)$ analytic in $R_1 \cup R_2$, which is defined by

$$F(z) = \begin{cases} f_1(z) & \text{for } z \in R_1 \\ f_2(z) & \text{for } z \in R_2. \end{cases}$$

It is precisely in this way that we understood analytic continuation from the beginning of this document. There is no necessary a unique closed form expression for $F(z)$. We justify the need of the triple intersection $R_1 \cap R_2 \cap R_3$ to be non-empty with an example. That is, there could happen that $R_1 \cap R_2 \neq \emptyset$ and $R_2 \cap R_3 \neq \emptyset$, and still $f_1(z) \neq f_3(z)$. This would happens when the triple intersection is empty. The pool of samples to look for counter-examples is the set of multivalued functions. Here is the example in Gonzalez's book. Consider the function element pairs $\{f_1, R_1\}$, $\{f_2, R_2\}$, and $\{f_3, R_3\}$, where

$$\begin{aligned} R_1 &= \{z : |z - e^{i\pi/6}| < 1\} \\ R_2 &= \{z : |z - e^{i5\pi/6}| < 1\} \\ R_3 &= \{z : |z + i| < 1\} \end{aligned}$$

The function to consider is $\log z$ which has infinite many different values, according to the do-

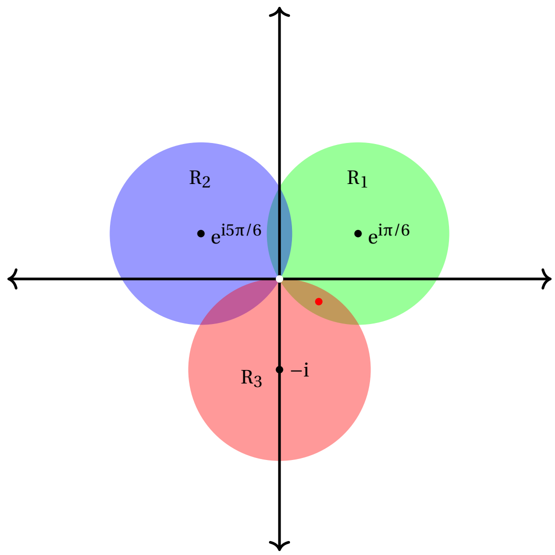

Figure 4.2: Regions of domain of for the functions. The red point provides an example where $f_1(z) \neq f_3(z)$.

main where it is defined. For example consider:

$$\begin{aligned} f_1(z) &= \log_1 z|R_1 \quad , \quad \log_1 z = \ln|z| + i\theta_1 \quad , \quad -\pi < \theta_1 \leq \pi \\ f_2(z) &= \log_2 z|R_2 \quad , \quad \log_2 z = \ln|z| + i\theta_2 \quad , \quad -\frac{\pi}{2} < \theta_2 \leq \frac{3\pi}{2} \\ f_3(z) &= \log_3 z|R_3 \quad , \quad \log_3 z = \ln|z| + i\theta_3 \quad , \quad 0 < \theta_3 \leq 2\pi \end{aligned}$$

where $\log_i z|R_i$ means that the function $\log_i z$ is restricted to the region R_i, for $i = 1,2,3$. Figure 4.2 illustrates the domains for the three f_i functions. The regions are three unitary circles with centers at $e^{i\pi/6}, e^{i\pi/6}$, and $-i$ respectively. The three circles have a common point at $(0,0)$. However, because the inequalities are strict, the circles do not have boundary, and the point $(0,0)$ is excluded. The midpoint in the intersection $R_1 \cap R_3$, will help to prove the statement. The intersection of the two circles is 30 degrees under the axis, that is at $e^{-i\pi/6}$, on the region R_1, and in $e^{i11\pi/6}$ in region R_3. The midpoint, and its logarithm is then

$$\begin{aligned} \frac{1}{2}e^{-i\pi/6} \quad &\Rightarrow \quad f_1(z) = \ln\frac{1}{2} - \frac{\pi}{6} \\ \frac{1}{2}e^{-i11\pi/6} \quad &\Rightarrow \quad f_3(z) = \ln\frac{1}{2} + \frac{11\pi}{6}. \end{aligned}$$

In general analytic functions are defined as series, definite integrals, infinite products, etc. Other than that we find polynomials, exponentials, logarithms, trigonometrical functions, rational functions, and fractional powers.

The greatest zone of analyticity of a function is called the **region of existence,** or **region of regularity** of the function, and its boundary is known as the **natural boundary** of the function.

The fundamental question is how to extend the domain of the function by adding adjacent regions until it reaches its region of existence? For example this is achieved by considering the power series $\sum z^i$, with a region of convergence corresponding to a unit circle with center at 0, and reaching the function $1/(1-z)$ which is analytic in the whole complex plane $\mathbb{C}$ except at the point $z = 1$.

4.2 Methods of Analytic Continuation

In the following sections we discuss a set of methods extend the domain of analyticity of functions. Some of those methods have been discussed already above, but we provide additional information.

4.2.1 The Weirstrass Method

We already discussed this method in section about the circle chain method 2.2. We provide a more formal approach. The method is based on a chain of Taylor series expansions around a sequence of points. Suppose $f(t)$ is analytic in a region R, and let $z_0 \in$ R. We write

$$f_0(z) = \sum_{i=0}^{\infty} a_i (z - z_0)^i \quad , \quad |z - z_0| < r$$

If the radius of convergence $r = \infty$ then the function is entire and there is no need to do any analytic continuation. Let us then assume that $r < \infty$. We want to find another point z_1 inside the region R and a circle around z_1 which will exit the region R. If it is possible to get part of the circle outside of the region R, we could then extend the analyticity of the function outside of R, otherwise if no circle can be found to go beyond the boundary analytically, the boundary of R is a “hard wall”. Such a “hard wall” is known as a **natural boundary** .

The idea behind the Weierstrass method is to find a new center of expansion of the series above. That is, for a new center point z_1 we we want to find coefficients b_i, such

$$f_1(z) = \sum_{i=0}^{\infty} b_i (z - z_1)^i.$$

We found that the coefficients b_k provided by equation 2.2.5 are required for the new representation $f_1(z)$. To derive b_k we assumed a Taylor series expansion where each $b_k = f^{(k)}(z_1)/k!$. Here is a different way to obtain the coefficients, suggested in Gonzalez’s textbook, using the binomial expansion of $(u+v)^n$, with $u = z - z_1$ and $v = z_1 - z_0$,

$$\begin{aligned}
\sum_{i=0}^{\infty} a_i (z-z_0)^n &= \sum_{i=0}^{\infty} a_i (z - z_1 + z_1 - z_0)^n \\
&= \sum_{i=0}^{\infty} a_i \sum_{k=0}^{n} \binom{i}{k} (z_1 - z_0)^{i-k} (z_1 - z_1)^k \\
&= \sum_{k=0}^{\infty} \underbrace{\left(\sum_{i=k}^{\infty} \binom{i}{k} a_i (z_1 - z_0)^{i-k} \right)}_{b_k} (z_1 - z_1)^k \\
&= \sum_{k=0}^{\infty} b_k (z - z_1)^k.
\end{aligned}$$

where again

$$b_k = \sum_{i=k}^{\infty} a_i \binom{i}{k} (z_1 - z_0)^{i-k},$$

as in equation 2.2.5. We still need to justify the change on the summation above. This is provided by the fact that

$$\sum_{i=0}^{\infty} |a_i| \sum_{k=0}^{\infty} \binom{i}{k} |z_1 - z_0|^{n-k} |z - z_1|^k = \sum_{i=0}^{\infty} |a_n| (|z - z_1| + |z_1 - z_0|)^n,$$

and this last sum converges for $|z - z_1| + |z_1 - z_0| < r$. Again, the idea here is to find z_1 such that the disc of convergence crosses out the region R. The discussion for the example in Figure 2.1 illustrates well how the method works.

4.2.2 The Circle-Chain Method Revisited

In section 2.2 we described the circle-chain method as a direct consequence of the Weierstrass but in a heuristic way. Here we formalize this method.

Definition 4.2.3 *A finite collection*

$$\{\{f_0, \mathrm{R}_0\}, \{f_1, \mathrm{R}_1\}, \cdots \{f_i, \mathrm{R}_i\}\} \tag{4.2.1}$$

of elements such that $\{f_{k+1}, \mathrm{R}_{k+1}\}$ *is a direct analytic continuation of* $\{f_k, \mathrm{R}_k\}$ *for each* $k = 0, 1, \cdots, i-1\}$. *Then the collection 4.2.1 is called a* **chain of elements** joining $\{f_0, \mathrm{R}_0\}$ and $\{f_i, \mathrm{R}_i\}$. The elements of a chain are called **links** .

We can think about the regions on the final collection above, as circles since this is the shape of the regions of convergence. This is formally proven in Theorem 4.2.6. We can also think that those circles follow some continuous path in such a way that each circle has a piece (arc) of the

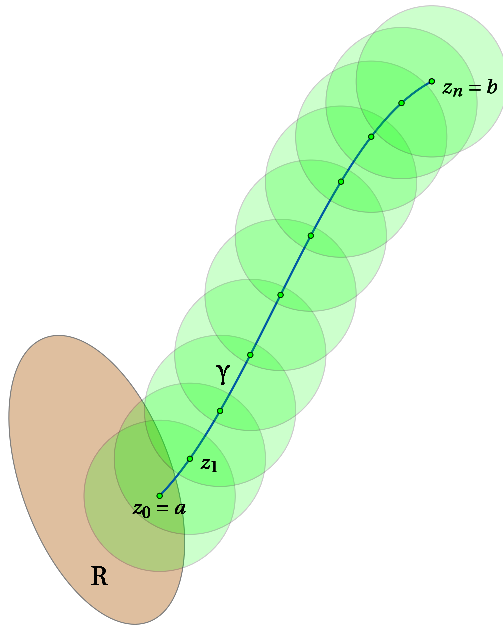

Figure 4.3: A chain of circles along a path γ.

path, and the intersection of two consecutive circles also contains a piece of the path. This is illustrated in Figure 4.3

Another example is shown in Figure 2.2, where this time the path γ is the closed (circumference) $\gamma = \{z : |z-1| = 1\}$. We say that any pair of elements $\{f_j, R_j\}$, $\{f_k, R_k\}$ are **analytic continuations of each other.** The relation between the elements $\{f_i, R_i\}$ in a chain is an equivalence relation. That is, it is reflexive, symmetric, and transitive.

In the chain of circles we can trace the path γ as we go. That is we decide which direction to choose for the next center of a circle in a way that we are exiting the original region of analyticity R and moving away from it. A different concept is provided in the opposite direction. That is, we can assume that we know the path in advance and we want to find a chain of circles that guarantees ad direct analytic continuation. We provide a formal definition of a continuation along a path γ.

Definition 4.2.4 *Let* $\{f, R\}$ *be a given function element and consider the path* $\gamma = \{z : z = \phi(t), 0 \leq t \leq 1\}$. *We want to find a continuous chain of circles along the path* γ *that satisfies the direct analytic continuation condition. That is, we can put at the head of the chain* $\{f_0, R_0\} = \{f, R\}$, *the tail at* $\{f_1, R_1\}$, *and using a real index* $t \in [0,1]$, *we can find a non-countable list of elements* $\{f_t, R_t\}$, *in between, such that* $\phi(t) \in D_t$ *(here* D_t *is a disc representing a region* R_t*), and to assure the direct analytic continuation we require that for each* $t \in [0,1]$ *there is a* $\delta > 0$, *such that if* $|t - t'| < \delta$, *then* $R_t \cap R_{t'} \neq \emptyset$ *and* $f_t(z) = f_{t'}(z)$ *for all* $z \in R_t \cap R'_t$. *We then say that* $\{f_1, R_1\}$ *is an* **analytic continuation of** $\{f_0, R_0\}$ **along** γ and that the elements $\{f_t, R_t\}$, $0 \leq t \leq 1$, form a **continuous chain** .

If γ is rectifiable (finite length) then we may choose a finite sequence of points $\{z_k\}$ which satisfies the conditions above.

Following Gonzalez's text book we use the notation ω,

$$\omega = \Big\{\{f_0, R_0\}, \{f_1, R_1\}, \cdots \{f_n, R_n\}\Big\}$$

for a collection of direct analytic continuation objects. Any non-empty collection ω is named a **general analytic function.** If the set ω has only one element we say that the collection is trivial. To each ω we can think about an analytic function $f_\omega(z)$, $z \in R = \cup_{i=0}^{n} R_i$, defined as follows

$$f_\omega(z) = \begin{cases} f_0(z) & z \in R_0 \\ f_1(z) & z \in R_1 \\ \vdots & \\ f_n(z) & z \in R_n \end{cases}$$

Note that if $z \in R_i \cap R_{i+1}$, we have that $f_i(z) = f_{i+1}(z)$, so this is a well defined function in the whole region R. However bear in mind that the functions $f_i(z)$ could be multi-valued, and so $f_\omega(z)$ can be one of many branched representations of ω. Actually the Poincaré-Volterra Theorem 4.2.8 proofs that there could be up to a countable number of values that an analytic continuation could take for a given point z_0.

We want to find the largest region R possible and so the largest chain which will contain all the direct analytic continuations of a function f. This largest chain is called a **complete (or global) analytic function,** we name it F and we claim that it exists and is unique in the following

Theorem 4.2.5 *Let* ω *be a general analytic function. Then there is a unique complete analytic function* F *such that* $\omega \subset F$.

Proof: Call $F = \{\omega_i : i \in I\}$, where I is some index set, the set of all general analytic functions which contains ω. Clearly F is not empty since $\omega \subset F$. We need to show

- **Existence:** The existence is guaranteed by Zorn's Lemma [2] That is given that the $\subset$ operation is partially ordered, it has an upper bound. However we want to show that this upper bound F is a complete analytic function. That is, that provided two elements on F there is a chain that joins them. Consider the two function elements $\{f_1, R_1\} \in F$ and $\{f_2, R_2\} \in F$. We show that there is chain that joins them. Given that F is the union of all chains containing ω, there are two indices i_1, i_2 such that both $\{f_1, R_1\} \in \omega_{i_1}$ and $\{f_2, R_2\} \in \omega_{i_2}$. Choose any particular element $\{f, R\}$ of ω. Since $\omega \subset \omega_i, \forall i \in I$, we have that $\{f_1, R_1\} \in \omega_{i_1}$ and $\{f_1, R_1\} \in \omega_{i_2}$. Then there is a chain in ω_{i_1} connecting $\{f_1, R_1\}$ to $\{f, R\}$, and also a chain in ω_{i_2} connecting $\{f_2, R_2\}$ to $\{f, R\}$. It follows that there is a chain in $\omega_{i_1} \cup \omega_{i_2} \subset F$ connecting $\{f_1, R_1\}$ to $\{f_2, R_2\}$. This shows that F is a general analytic function.

[2] http://en.wikibooks.org/wiki/Set_Theory/Zorn%27s_Lemma_and_the_Axiom_of_Choice

- **Uniqueness:** Let us assume that there is a general complete analytic function G containing ω. Since F contains all the analytic continuations that are supersets of ω, in particular $G \subset F$. Choose any $\{f_1, R_1\} \in F$, then $\{f_1, R_1\}$ is an analytic continuation of some element $\{f_0, R_0\} \in \omega$. But $\omega \subset G$, so $\{f_1, R_1\}$ is an analytic continuation of some element $\{f_0, R_0\} \in G$, and since G is a complete analytic function, $\{f_1, R_1\} \in G$. That is $F \subset G$. So $F = G$ as needed.

While the regions R do not have to be circles, it is easier to think about them as circles. That is each region R can be thought of as a

$$D(a, r) = \{z : |z - a| < r\}$$

which is an open circle centered in a and radius r.

Theorem 4.2.6 *Let* F *be a complete analytic function and suppose that* $\{f_0, R_0\}$ *and* $\{f_1, R_1\}$ *are any two function elements in* F. *If* $D_0 \subset R_0$ *and* $D_1 \subset R_1$ *there is a chain of function elements of* F *of the form* $\{f_k, D_k\}$ *connecting* $\{f_0, D_0\}$ *with* $\{f_1, D_1\}$

Proof: The elements $\{f_0, R_0\}$ and $\{f_1, R_1\}$ are in F and since F is a complete analytic function. Hence $\{f_0, R_0\}$ and $\{f_1, R_1\}$ can be linked by a chain of function elements of F. Let this chain to be

$$\omega = \Big\{\{f_{i_0}, R_{i_0}\}, \{f_{i_1}, R_{i_1}\}, \cdots \{f_{i_n}, R_{i_n}\}\Big\}.$$

where $i_0 = 0$ and $i_n = 1$. Let us pick two discs $D_0 = D(a, r_0) \subset R_0$, and $D_1 = D(b, r_1) \subset R_1$, with centers at a and b, and radii r_0 and r_1 respectively. We can create a polygonal line joining a with b and along that polygonal line construct a chain of analytic continuation functions joining a with b. This is a finite chain since the distance $|a - b|$ is finite. We can make the circles as small (or large) as possible to squeeze them on the regions R_{i_k} so that the intersection of each circle with each region is non-empty. These regions have a finite thickness, so there is a minimum radius δ for the discs in the chain, and so the chain could be constructed in a finite number of steps.

We now deal with the values that the function can take along the different regions.

Definition 4.2.7 *Let* F *be a complete analytic function. We say that* w_0 *is a* **value of** F at a point z_0, if there is a function element $\{f, R\}$ of F such that $z_0 \in R$ and $f(z_0) = w_0$.

Have in mind that functions could be multivalued. That is, for example, the square root has two values, the cubic root three values, and the $f(z) = \ln(z)$ function has an infinite number of values. The Poincaré-Volterra theorem indicates that a complete analytic function has at most a countable number of values. That is

Theorem 4.2.8 *(Poincaré-Volterra theorem). The set of values of a complete analytic function at a given point* z_0 *is at most countable.*

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