Selected Problems in Functional Analysis from Kreyzig's Book

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Contents

Introduction

This document solve some problems on spectral theory of operators from Erwin Kreyzig [1] Introductory Functional Analysis with Applications.

[1] ftp://ftp.sdu.edu.tr/pub/mat/serpil/[Erwin_Kreyszig]_Introductory_functional_analysis.pdf

Chapter 7

Spectral Theory of Linear Operators in Normed Spaces

1 Finite Dimensioal Case

1. Find the eigenvalues and eigenvectors of the following matrices, where a and b are real and $b \neq 0$.

$$A = \begin{bmatrix} 1 & 2 \\ -8 & 11 \end{bmatrix} \quad , \quad B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Solution:

- The characteristic equation for A is

$$(1-\lambda)(11-\lambda)+16=0 \quad \Rightarrow \quad \lambda^2 - 12\lambda + 27 = 0$$

With solutions

$$\lambda = 3 \qquad \lambda = 9$$

A simple verification is given by the fact that the trace is 12 and the determinant is $12+16=27$. Actually the eigenvalues of any 2x2 matrix can be found by solving these two equations for the trace and the determinant of A, for the unknowns λ_1 and λ_2.

- Let us solve the second problem with the method suggested above. That is

$$\begin{aligned} \lambda_1 + \lambda_2 &= 2a \\ \lambda_1 \lambda_2 &= a^2 + b^2 \end{aligned}$$

That is

$$\lambda_1(2a - \lambda_1) = a^2 + b^2.$$

which is

$$\lambda_1^2 - 2a\lambda_1 + a^2 + b^2 = 0$$

(which is the characteristic equation). The solutions are

$$\lambda_{1,2} = \frac{2a \pm \sqrt{4a^2 - 4(a^2 + b^2)}}{2} = \frac{2a \pm \sqrt{-4b^2}}{2} = \frac{2a \pm 2ib}{2} = a \pm ib$$

2. (Hermitian matrix) Show that the eigenvalues of a Hermitian matrix $A = (\alpha_{jk})$ are real.

   Solution: Let λ be a solution of $Ax = \lambda x$ for some vector x. Since A is Hermitian

$$\begin{aligned} \langle Ax, x \rangle &= \langle x, Ax \rangle \\ \lambda \|x\|^2 &= \overline{\lambda} \|x\|^2 \end{aligned}$$

   and since we assume $x \neq 0$ (for the definition of eigenvalue) then λ is real.

3. **(Skew-Hermitian matrix)** Show that the engivelaues of a skew Hermitian matrix $A = (\alpha)_{jk}$ are pure imagingary or zero.

   Solution: Let λ be a solution of $Ax = \lambda x$ for some vector x. Since A is Skew-Hermitian

$$\begin{aligned} \langle Ax, x \rangle &= -\langle x, Ax \rangle \\ \lambda \|x\|^2 &= -\overline{\lambda} \|x\|^2 \end{aligned}$$

   and since we assume $x \neq 0$ (for the definition of eigenvalue) then λ is pure imaginary or zero.

4. **(Unitary matrix)** Show that the eigenvalues of a unitary matrix have absolute value 1.

   Solution: If A is unitary, its inverse is the adjoint. So, let us assume that λ is an eigenvalue of A corresponding to eigenvector x. That is $Ax = \lambda x$. Then, since $A^*A = I$,

   $$\langle A^*Ax\,,\,x\rangle = \|x\|^2,$$

   but

   $$\langle A^*Ax\,,\,x\rangle = \langle Ax\,,\,Ax\rangle = \|Ax\|^2.$$

   So

   $$\|Ax\| = \|x\|,$$

   That is $\|\lambda x\| = \|x\|$, from which $|\lambda| = 1$.

5. Let X be a finite dimensinal inner product space and $T : X \to X$ a linear opereator. If T is self-adjoing, show that its spectrum is real. If T is unitary, show that its eigenvalues have absolute value 1.

   Solution: This problem was solved above in problems 2,3 and 4. Instead of treating the matrices A above as matrices, the solutions where developed as if A where operators T.

6. **(Trace)** Let $\lambda_1, \cdots, \lambda_n$ be the n eigenvalues of an n-rowed square matrix $A = (\alpha_{jk})$, where some or all of the λ_j's may be equal. Show that the product of the eigenvalues equals $\det A$ and their sum equals the *trace* of A, that is, the sum of the elements of the *principal diagonal:*

   $$\operatorname{trace} A = \alpha_{11} + \alpha_{22} + \cdots + \alpha_{nn}.$$

Solution:

- Let us first show that the determinant of A is the product of the eigenvalues. The characteristic polynomial for $p(\lambda)$ can be factored out as the product of the $\lambda - \lambda_i$ where λ_i is an eigenvalue of A. That is

$$p(\lambda) = \det(A - \lambda I) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n). \quad (1\ .1)$$

  The "$(-1)^n$" comes from the expansion of the determinant for the diagonal term, which is the highest degree. Hence from $p(0) = \det A$, and $p(0) = (-1)^n(-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$

$$\det A = \prod_{i=1}^{n} \lambda_i.$$

- For the trace we use similar arguments as the ones used above. However we will expand the characteristic polynomial in terms of the Levi-Civita symbol for alternating tensors. We call $B = A - \lambda I$, and

$$p(\lambda) = \det B = \varepsilon_{i_1 i_2 \cdots i_n} b_{1i_1} b_{2i_2} \cdots b_{ni_n}$$

  with Einstein summation over all i_1 indices and $= \varepsilon_{i_1 i_2 \cdots i_n}$ being 1, -1 , 0, according to $(i_1 i_2 \cdots i_n)$ being even, odd, or no permutation of the identity $(1\,2 \cdots n)$, correspondingly.

$$p(\lambda) = \det B = \varepsilon_{i_1 i_2 \cdots i_n} (a_{1i_1} - \lambda \delta_{1i_1})(a_{2i_2} - \lambda \delta_{2i_2}) \cdots (a_{ni_n} - \lambda \delta_{ni_n}).$$

  We will not expand the whole polynomial but select important terms of the polynomial. However in Appendix A we show several ways to expand the Characteristic polynomial .

  - The highest term on λ, happens for each δ_{ji_j} with index $j = i_j$, that is it comes from the diagonal. Here we have $\varepsilon_{1\,2 \cdots n} = 1$ and the term is $(-1)^n \lambda^n$, as indicated in the solution above.
  - The lowest term comes with no λ, so making $\lambda = 0$ will produces the lowest term which is $p(0) = \det A$ as already shown in the previous solution.

- Finally we want to find the coefficient of λ^{n-1}. If $n-1$ factors are already in the diagonal, the other factor has to be in the diagonal as well, since otherwise $\varepsilon_{i_1 i_2 \cdots i_n} = 0$. So we have that the $n-1$ degree term comes from the polynomial

$$(a_{11} - \lambda)(a_{22} - \lambda\delta_{22}) \cdots (a_{nn} - \lambda).$$

  and this term is $(-1)^{n-1}\lambda^{n-1}\sum_{i=1}^{n} a_{ii}$. On the other hand the characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n).$$

  and the $(n-1)^{th}$ term of this is

$$(-1)^{n-1}\sum_{i=1}^{n}\lambda_i$$

  and from comparing the previous two equations

$$\text{trace}A = \sum_{i=1}^{n}\lambda_i.$$

7. **(Inverse)** Show that the inverse A^{-1} of a square matrix A exists if and only if all the eigenvalues $\lambda_1, \cdots, \lambda_n$ of A are different from zero. If A^{-1} exits, show that it has the eivenvalues $1/\lambda_1, \cdots, 1/\lambda_n$.

   Solution: We use the property that the determinant of a product of two matrices is the product of determinants and the solution of the previous exercise. That is

   If A^{-1} exist then

$$A^{-1}A = I \qquad \Rightarrow \qquad \det A^{-1}\det A = 1.$$

   So

$$\det A^{-1} = \frac{1}{\det A}$$

and from the previous exercise $\det A$ should be different from zero and so each eigenvalue should be non-zero.

Now, let λ be an eigenvalue of A corresponding to an eigenvector x. Then

$$Ax = \lambda x \quad \Rightarrow \quad x = \lambda A^{-1}x$$

That is,

$$A^{-1}x = \frac{1}{\lambda}x,$$

so $1/\lambda$ is an eivenvalue of A^{-1} corresponding to the same eigenvector x.

8. Show that a two-rowed nonsingular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{has the inverse} \quad A^{-1} = \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

How does it follow from this formula that A^{-1} has the eivenvalues $1/\lambda_1, 1/\lambda_2$, where λ_1, λ_2 are the eivenvalues of A?

Solution: A direct multiplication produce the I_2 matrix. The inverse shown can be found by using Cramers rule [1] That is, in general

$$A^{-1} = \frac{1}{\det A}\mathrm{Adj}(A)$$

where $\mathrm{Adj}(A)$ is the adjugate matrix of A. This is the transposed of the cofactor matrix.

We observe that the 2x2 adjugate matrix of A has the same characteristic polynomial than the matrix A. Hence they both share the same eigenvalues. However in the inverse A^{-1} the division by $\det A = \lambda_1\lambda_2$ takes λ_1 into $1/\lambda_2$ and λ_2 into $1/\lambda_1$.

9. If a square matrix $A = (\alpha_{jk})$ has eigenvalues λ_j, $j = 1, \cdots, n$, show that kA has eigenvalues $k\lambda_j$, and A^m ($m \in \mathbb{N}$) has the eigenvalues λ_j^m.

[1] http://en.wikipedia.org/wiki/Cramer%27s_rule

Solution: If A has an eigenvalue λ with eigenvector x, then $Ax = \lambda x$. So, $kAx = k\lambda x$. That is $k\lambda$ is an eigenvalue of kA. Also

$$A^2x = AAx = A(\lambda x) = \lambda Ax = \lambda^2 x.$$

By induction, if $A^{k-1}x = \lambda^{k-1}x$ then

$$A^k x = AA^{k-1}x = A\lambda^{k-1}x = \lambda^{k-1}Ax = \lambda^k x.$$

So in general the eigenvalues of A^m are of the form λ^m, where λ is an eigenvalue of A.

10. If A is a square matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$ and p is any polynomial, show that the matrix $p(A)$? has the eigenvalues $p(\lambda_j)$, $j = 1, \cdots, n$.

Solution: Let us assume that $p(x) = a_0 + a_1 x + \cdots a_n x^n$, is a polynomial on x and λ is an eigenvalue of A. That is, there is an eigenvector x such that $Ax = \lambda x$. Then, using the results of the previous exercise,

$$p(A)x = a_0 x + a_1 Ax + \cdots a_n A^n x = a_0 x + a_1 \lambda x \cdots a_n \lambda^n x = p(\lambda)x.$$

That is $p(\lambda)$ is an eigenvalue of $p(A)$.

11. If x_j is an eigenvector of an n-rowed square matrix A corresponding to an eigenvalue λ_j and C is any nonsignular n-rowed square matrix, show that λ_j is an egenvalue of $\tilde{A} = C^{-1}AC$ and a corresponding eigenvectors is $y_j = C^{-1}x_j$.

Solution: We know that $Ax_j = \lambda_j x_j$ By direct substitution

$$\tilde{A}y = (C^{-1}AC)(C^{-1}x_j) = C^{-1}Ax_j = C^{-1}\lambda_j x_j = \lambda_j C^{-1}x_j = \lambda_j y.$$

So, indeed y_j is an eigenvector of $\tilde{A}$ sharing the same eigenvalue λ_j.

This is an interesting property because it says that a similarity transformation does not change the eigenvalues of a matrix. That is, they are invariant under change of basis.

12. Illustrate with a simple example that an n-rowed square matrix may not have eigenvectors which constitute a basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$). For instance, consider

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Solution: This example is a case of a Jordan Normal Form. [2].

The idea is that there are repeated eigenvalues that share the same eigenvector.

Let us find the eigenvectors of A. These are found by setting the equation

$$(A-\lambda I)(x)=0$$

Since the eigenvalues are $\lambda_1=\lambda_2=1$ we see that the equation becomes

$$A=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=0$$

which implies $y=0$. That is unit the eigenvector shared by the two eigenvalues is $e_1=(1,0)$. No more eigenvectors can be found.

A 3x3 Jordan matrix could be

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

which has three repeated eigenvalues and just one unit eigenvector, $e_1=(1,0,0)$.

13. **(Multiplicity)** The *algebraic multiplicity* of an eigenvalue λ of a matrix A is the multiplicity of λ as a root of the characteristic polynomial, and the dimension of the eigenspace of A corresponding to λ may be called the *geometric multiplicity* of λ. Find the eigenvalues and their multiplicities of the matrix corresponding to the following transformation and comment:

$$\eta_j=\xi_j+\xi_{j+1} \qquad (j=1,2,\cdots,n-1), \qquad \eta_n=\xi_n.$$

Solution: The matrix corresponding to this transformation is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots 0 & 1 \end{bmatrix}$$

[2]http://en.wikipedia.org/wiki/Jordan_normal_form

The algebraic multiplicty is n. There are n repeated eigenvalues all of them 1. The geometric multiplicy is 1. This is easy to see. By doing back substitution, in the equation $(A-\lambda I)x = 0$, starting from the previous last row equation, which produces $x_n = 0$, we find that $x_{n-1} = 0, \cdots, x_2 = 0$. The only alive component is x_1 which can take on any value. This unit is the vector $e_1 = (1,0,0,\cdots 0)$.

14. Show that the geometric multiplicity of an eigenvalue cannot exceed the algebraic multiplicity (cf. Prob. 13).

Solution: If a matrix A is diagonalizable, that is, if it can be written as

$$A = P^{-1}DP$$

where D is diagonal, then the algebraic and geometric multiplicity are the same. The difference appears with Jordan blocks.

If the geometric multiplicity equals the algebraic multiplicity, then in a Jordan block there would be as many dimensions as repeated eigenvalues and we know there is only one dimension, less probable is to find even more geometric than algebraic multiplicty.

15. Let T be the differential operator on the space X consisting of all polynomials of degree not exceeding $n-1$ and the polynomial $x = 0$ (for which the degree is not defined in the usual discussion of degree). Find all eigenvalues and eigenvectors of T and their algebraic and geometric multiplicities.

Solution: Let $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

Then $T[p(x)] = a_1 + 2a_2x + \cdots na_nx^{n-1}$

As a matrix T takes the vector $(a_0, a_1, \cdots a_n)$ into the vector $(a_1, 2a_2, \cdots na_n, 0)$. Then T is a shift and a multiplication operator

$$T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & n \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

We see that all eigenvalues are 0, so the algebraic multiplicity is n. Let us find a vector a, such that

$$Ta = \lambda a = 0a.$$

This is the space of constants a_0, which we can span with the unit vector $(1,0,\cdots 0)$.

Note: The book solves this problem in the continuum, by saying that $Tx = x' = \lambda x$, is a differential equation (assume $x = x(t)$), with solution $\exp(\lambda t)$, which is not polynomial, for $\lambda \neq 0$, hence $\lambda = 0$. Still $\exp(0t) = 1$ which is non zero. This problem is in the finite dimensional spaces section, so it should be solved with matrices.

2 Basic Concepts

1. **(Identity operator)** For the identity operator I on a normed space X, find the eigenvalues and eigenspaces as well as $\sigma(I)$ and $R_\lambda(I)$.

Solution:

- **Eigenvalues, Eigenspaces:** An eigenvalue/eigenfunction pair satisfy

$$Ix = \lambda x.$$

Since $I(x) = x$, then $\lambda = 1$ is the only eigenvalue. Any vectors x of the space satisfies the equation for the eigenvalue $\lambda = 1$. So the eigenspace is the whole space.

- The **resolving operator** is given by

$$R_\lambda(T) = (T - \lambda I)^{-1}.$$

Now, if $T = I$ then $R_\lambda(T)$ does not exists, for $\lambda = 1$, but it exists for any $\lambda \neq 1$, and it is bounded (by 1) and the X is dense in x, so the set $\{\mathbb{R} - 1\}$, (or $\{\mathbb{C} - 1\}$) is the set of regular values. That is, assuming that the field is the complex

$$\rho(T) = \{\mathbb{C} - 1\}.$$

The spectrum of T is just $\sigma(T) = 1$. This is a pure **point spectrum** . There is no continuous sepctrum and neither residual spectrum for the identity operator $T = I$.

2. Show that for a given linear operator T, the sets $\rho(T), \sigma_p(T), \sigma_c(T)$, and $\sigma_r(T)$ are mutually disjoint and their union is the complex plane.

Solution: We need to check each set against each other.

- $\rho(T)$ versus $\sigma_p(T), \sigma_c(T), \sigma_r(T)$: By definition the spectrum $\sigma(T)$ is the complement of the resolving set $\rho(T)$. That is

$$\begin{aligned} \rho(T) \cup \sigma(T) &= \mathbb{C} \\ \rho(T) \cap \sigma(T) &= \emptyset \end{aligned}$$

  This shows that $\rho(T)$ is disjoint with each $\sigma_p(T), \sigma_c(T)$, and $\sigma_r(T)$, and that the union of all of them is the whole space.

- Internal disjunction of σ_p, σ_c, and σ_r

  The point spectrum $\sigma_p(T)$ implies that $R_\lambda(T)$ does not exist, while the continuous spectrum $\sigma_c(T)$ allows for $R_\lambda(T)$ to exists, so they are disjoint. The same argument applies to the residual spectrum $\sigma_r(T)$, so this is also disjoint with respect to $\sigma_c(T)$.

  We now show that $\sigma_c(T)$ and $\sigma_r(T)$ are disjoint. The key property here is **(R3)** that is, if $\lambda \in \sigma_c(T)$, since it satisfies **(R3)** then the domain of $R_\lambda(T)$ is dense in X, while for $\sigma_r(T)$, the domain of $R_\lambda(T)$ is non dense in X. So $\sigma_c(T)$ and $\sigma_r(T)$ are disjoint.

3. **(Invariant subspace)** A subspace Y of a normed space X is said to be *invariant* under a linear operator $T : X \to X$ if $T(Y) \subset Y$. Show that an eigenspace of T is invariant under T. Give examples.

Solution: Let us assume that Y is an eigenspace of T. It could be that for a given λ there are several vectors y_i $i = 1, 2, \cdots k$, (geometrical multiplicity of k) such that $Ty_i = \lambda y_i$. Then the span of the vectors y_i we call Y,

$$Y = \text{span}\{y_1, y_2, \cdots, y_k\}$$

for some λ.

We prove that $T(Y) \subset Y$. Pick a $y \in T(Y)$. Then

$$y = \sum_{i=1}^{k} c_i y_i$$

for some coefficients c_i, and from the linearity of T and the definition of eigenvalues for T

$$T(y) = \sum_{i=1}^{k} c_i T(y_i) = \sum_{i=1}^{k} c_i \lambda y_i$$

so $T(y) \in Y$, since it is in the span of the vectors y_i, with coefficients λc_i.

Then the eigenspace Y is invariant of T.

Examples:

- **The identity operator** . The simplest (and most uninteresting) example. This operator maps the whole space X into the whole space X. So it is invariant. The whole space X is the eigenspace of the single eigenvalue $\lambda = 1$, with algebraic multiplicy equal to the geometric multiplicity n.
- **A rotation operator** . A finite rotation operator in 2D with an angle $0 < \theta < 2\pi$. Does it have invariant subspaces? Think about this: All vectors change directions, so it should not have an invariant direction. Let us see why this is a deceiving argument. Define T as the matrix

  $$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

  Let us find the eigenvalues of this matrix. The characteristic polynomial is computed from

  $$(\cos\theta - \lambda)(\cos\theta - \lambda) + \sin^2\theta = \sin^2\theta + \cos^2\theta - 2\lambda\cos\theta + \lambda^2$$

  That is, the characteristic polynomial is

  $$p(\lambda) = \lambda^2 - 2\lambda\cos\theta + 1$$

  which has the two complex roots

  $$\lambda_\pm = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta = \mathrm{e}^{\pm i\theta}$$

  So, there is always an eigenvalue, but it is no necessarily real.

  If $\theta = 0$, this is the identity matrix in the previous example. Let us assume that $0 < \theta < 2\pi$. What is the eigenspace for, say $\lambda = \mathrm{e}^{i\theta}$? Find $(x,y)^T$ such that

  $$\begin{bmatrix} \cos\theta - \mathrm{e}^{i\theta} & \sin\theta \\ -\sin\theta & \cos\theta - \mathrm{e}^{i\theta} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first equation is

$$x\cos\theta - xe^{i\theta} + y\sin\theta = 0,$$

form which

$$(y - ix)\sin\theta = 0.$$

So we have that setting $x = 1$, and $y = i$, the vector $(1,i)^T$ is an eigenvector of the rotation matrix. Similarly by using the second eigenvalue we could find that the vector $(1,-i)$ is another eigenvector of the rotation matrix.

How do we interpret this? This is an interesting problem. First, think that the eigenvectors are independent of the amount of rotation ($0 < \theta < 2\pi$). This provides a continuum of matrices all sharing exactly the same eigenvectors and eigenvalues. We can not think about this problem in $\mathbb{R}^2$. It does not work. The problem should thought as embedded in $\mathbb{C}^2$, which in a way seems 4 dimensional. What is invariant here?

Before we attempt to answer the question, think about 3x3 rotation matrix. Without doing any math, it should be clear that the axis of rotation does not change and is the invariant in the 3D real space. There should be 2 more axes embedded in the plane of rotation which have the same role as the invariants that we are considering here.

The invariants have to be those lines parallel to the $(1,i)$ and $(1,-i)$ vectors (which compactly we can write as the unitary complex numbers $e^{\pm\pi/4}$).

The Mathematics website [3] together with this blog [4] discuss interesting ideas about this problem.

The problem is that the plane of rotation is a plane in the complex numbers, spanned by two vectors $(1,i)$, $(1,-i)$. The matrix R should leave these two directions invariant, but for real numbers these directions are ouside the domain. The real numbers in two dimensions are spanned by $(1,1),(1,-1)$. We can not compare $\mathbb{R}^2$, with $\mathbb{C}^2$. The second one seems to be fourth-dimensional. Instead we could compare $\mathbb{C}$ with $\mathbb{R}^2$, where the (x,y) vector in $\mathbb{R}^2$ is seen as the $x+iy$ vector in $\mathbb{C}$. Here $\mathbb{C}$ is one-dimensional in $\mathbb{C}$, so it is spanned by only one vector. We know that we can rotate a complex number an angle θ by multiplying it by $e^{i\theta}$.

[3] http://math.stackexchange.com/questions/241097/geometric-interpretation-for-complex-eigenvectors-of-a-2x2-rotation-matrix

[4] http://twistedoakstudios.com/blog/Post7254_visualizing-the-eigenvectors-of-a-rotation

We can define an operator

$$\begin{aligned} T_\theta : \mathbb{C} &\rightarrow \mathbb{C} \\ z &\mapsto e^{i\theta} z. \end{aligned}$$

This operator achives a rotation in the complex plane (which is what we want). Now, because we have only one dimension in the
complex "plane", any non-zero complex number serves as a basis for the expansion of the complex "line". By the definition of T_θ the only eigenvalue of T $(0 < \theta < 2\pi)$ is $e^{i\theta}$. The eigenvector for this is given by the solution of

$$T_\theta - \lambda I = 0.$$

That is

$$e^{i\theta} z - e^{i\theta} = 0.$$

where $I = 1 + 0i$. Then $e^{i\theta}(z-1) = 0$, from which $z = 1$. This means that in the complex plane, the only invariant is the line $z = 1$ (horizontal axis). This sounds like a contradiction. How is it possible that the horizontal line stays that way under rotation. The answer is that in the complex field, the coefficients are complex and with $1 + 0i$ you can achieve any number in the plane by just multiplying this number with an scalar positive value $Ae^{i\theta}$, with a real scaling value A. We only need $0 \leq \theta < \pi$, if A could be negative. The complex plane, is not the complex plane anymore, it is the complex line, and the invariant is the whole space. The complex number 1, is as good as $1 + i$, or $1 - i$, or any non-zero complex number. Each one spans the whole complex line.
Let us now try associate $\mathbb{C}^2$ with $\mathbb{R}^4$ in the following way. Think of $\mathbb{R}^4$ as being spanned by the following vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ i \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ i \end{bmatrix}.$$

The rotation matrix would be the 4x4 matrix

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If we repeat the calculations above, with this new matrix we will find that in addition to the eigenvalues $\lambda_{\pm} = \mathrm{e}^{\pm i\pi/4}$, we have two additional (reapeated) eigenvalues of $\lambda_{3,4} = 0$.
the invariant vectors are

$$\begin{bmatrix} 1 \\ 0 \\ i \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -i \\ 0 \end{bmatrix}$$

Together with two new invariant $(0,1,0,1)^T$ and $(0,0,0,1)^T$, due to the zero second and fourth rows of zeroes.
The problem of the rotation matrix in $\mathbb{R}^2$ has no solution in the real plane. The existence of eigenvalues is only guaranteed in the complex field.

Finally, let us think about the following. The matrices

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

share the same eigenvalues, since they have the same trace and determinant (and they are 2x2). It is interesting that the eigenvalues are eactly the amount or roation $\mathrm{e}^{i\theta}$ which the matrices perform in the complex plane. So while the geometric interpretation of the eigenvalues in this problem is clear, the geometric interpretation for the eigenvectors is still non clear. Usually the eigenvalues are seen as scaling along the axis (invariants or eigendirections), but here they do not scale, they do pure rotations.

- **The differential operator**

$$T[f(t)] = \frac{df}{dt}.$$

Assume

$$T[f(t)] = \lambda f(t) \quad , \quad \frac{df}{dt} - \lambda f(t) = 0.$$

The solution to this differential equation is

$$f(t) = k\mathrm{e}^{\lambda t}.$$

for an arbitrary constant k.

The subspace corresponding to λ is the space of all functions of the type $g(t) = c\mathrm{e}^{\lambda t}$, with c a constant. The application of T to any $g(t)$ produces

$$T[g(t)] = c\lambda \mathrm{e}^{\lambda t},$$

which is in the same space of exponential functions of the form a $c\mathrm{e}^{\lambda t}$, where now the constant is λc, instead of c, but a constant anyway.

Appendix B shows a collection of examples for the various shift operators and their invariant spaces.

4. If Y is an invariant subspace under a linear operator T on an n-dimensional normed space X, what can be said about a matrix T with respect to a basis $\{e_1, \cdots, e_n\}$ for X such that $Y = \text{span}\{e_1, \cdots, e_m\}$?

Solution: Let us assume that Y is invariant for T. That is $T(Y) \subset Y$. We know that for each $y \in Y$, $y = \sum_{i=1}^{m} c_i e_i$. Also, by linearity of T, as well as the invariance property,

$$T(y) = \sum_{bi=1}^{m} c_i T(e_i) = \sum_{i=1}^{m} c_i \sum_{j=1}^{m} d_j e_j$$

Then T is uniquely defined by the product of the coefficients $c_i d_j$, on the basis e_j. So we can represent T by the matrix $k \times k$ (a_{ij}), where $a_{ij} = c_i d_j$, in the basis $\{e_1, \cdots, e_m\}$.

5. Let (e_k) be a total orthonormal sequence in a separable Hilbert space H and let $T : H \to H$ be defined at e_k by

$$Te_k = e_{k+1} \qquad (k = 1, 2, \cdots)$$

and then linearly and continuously extended to H. Find invariant subspaces. Show that T has no eigenvalues.

Solution: Section 2 .1.1.2 shows not only this operator which is the example 7.2-2 in the textbook, corresponding to the right (causal) shift, but we include the left (causal) shift, as well as the bilateral left and right shifts. This provide some interesting geometrical insights into the spectrum of operators in the infinite dimensional spaces.

- For any $k \geq 1$, since $Te_k = e_{k+1}$, then the span of $S_k = \{e_k, e_{k+1}, \cdots\}$ is such that $T(S_k) \subset S_k$. There are an infinite number of these invariant subspaces. Still no eigenvalues.
- Let us find an explicit representation of this operator in ℓ_2. Pick $\xi_2 = (\xi_1, \xi_2, \cdots \xi_k, \cdots) = \sum \xi_i e_k$. The operator T is

  $T(\xi) = T\left(\sum \xi_i e_k\right) = \sum \xi_k T e_k = \sum \xi_k e_{k+1} = (0, \xi_1, \xi_2, \cdots \xi_{k+1}, \cdots).$

  This is the right shift operator in ℓ_2. Let us assume that there is λ such that $T(\xi) = \lambda \xi$, That is

  $$(0, \xi_1, \xi_2, \cdots, \xi_k, \cdots) = (\lambda \xi_1, \lambda \xi_2, \cdots, \lambda \xi_k, \cdots)$$

  If $\lambda \neq 0$, then $\xi_1 = \xi_2 = \cdots \xi_k = \cdots = 0$, so λ is not an eigenvalue. However if $\lambda = 0$ we see that also $\xi_i = 0$, $i = 1, 2, \cdots$. So in any case there are no eigenvalues of this operator.

  Besides serving as an example, this makes a good counter example for some important cases.

  - In infinite dimensional space it could happens that an operator does not have an eigenvalue.
  - While an eigenvector will have an invariant subspace the opposite is no true.
  - This is an example of an spectral value which is not an eigenvalue. Here $\lambda = 0$ belongs to the residual spectrum σ_r, since the range of T will have a zero on the first component, it could not be dense on the space $X = \ell_2$.
  - This is an example of an operator that preserves the norm. That is $\|T\xi\| = \|x\|$. That is, it is a bounded linear operator T, with $\|T\| = 1$ in an infinite dimensional space. Still it is not unitary.

6. **(Extension)** The behavior of the various parts of the stpectrum under extension of an operator is of practical interest. If T is a bounded linear operator and T_1 is a linear extension of T, show that we have $\sigma_p(T_1) \supset \sigma_p(T)$ and for any $\lambda \in \sigma_p(T)$ the eigenspace of T is contained in the eigenspace of T_1.

Solution: Let us assume that $\lambda \in \sigma_p(T)$. This means that $(T-\lambda I)^{-1}$ does not exist. That is, the function $T-\lambda I$ is no one-to-one, and so the extension for T_1, $T_1-\lambda I$ is not either one-to-one. So $(T_1-\lambda I)^{-1}$, does not exist. Then $\lambda \in \sigma_p(T_1)$.

7. Show that $\sigma_r(T_1) \subset \sigma_r(T)$ in Prob. 6.

Solution: Let us assume $\lambda \in \sigma_r(T_1)$. This means that $R_\lambda(T) = (I-\lambda T_1)^{-1}$ exists but the domain is not dense in X. That is, the range of $T_1-\lambda I$ is not dense in X . Since T_1 is an extension of T, then it is not dense in X, the range of $T-\lambda_I$, which is a subset of the range of $T_1-\lambda X$ could not be dense in X. Hence $\lambda \in \sigma_r(T)$, and $\sigma_r(T_1) \subset \sigma_r(T)$.

8. Show that $\sigma_c(T) \subset \sigma_c(T_1) \cup \sigma_p(T_1)$ in Prob. 6.

Solution: Let us assume $\lambda \in \sigma_c(T)$. So $R_\lambda(T)$ exits and it is defined in a set which is dense in X, but it is no bounded. Let us assume that $\lambda \notin \sigma_c(T_1)$. This means that $R_\lambda(T_1)$ does not exists, or is not defined in a set that is dense in X. But if it is not defined in a set that is dense in X, this means that $T_1-\lambda I$ does not map into a set that is dense in X. Hence $T-\lambda I$ does not map either into a set that is dense in X (being this a subset of the range of $T_1-\lambda I$). This is a contradiction, since $\lambda \in \sigma_c(T)$. Therefore $R_\lambda(T_1)$ does not exists, and so $\lambda \in \sigma_p(T)$. From problem 6. $\lambda \in \sigma_p(T_1)$. This proofs that $\sigma_c(T) \subset \sigma_c(T_1) \cup \sigma_p(T_1)$.

9. Show directly (without using Probs. 6 and 8) that $\rho(T_1) \subset \rho(T) \cup \sigma_r(T)$ in Prob. 6

Solution: Pick $\lambda \in \rho(T_1)$. This means that the resolving operator $R_\lambda = (T_1-\lambda I)^{-1}$ exists, is bounded, and is defined in a set that is dense in X. Let us assume that $\lambda \notin \rho(T)$. This could only mean that the constrain from T_1 to T reduces the range of $T-\lambda I$ to a set that is non-dense in X. That is $\lambda \in \sigma_r(T)$. This shows that $\rho(T_1) \subset \rho(T) \cup \sigma_r(T)$.

10. How does the statement in Prob. 9 follow from Probs. 6 and 8?

Solution: From Probs. 6 and 8

$$\sigma_c(T) \cup \sigma_p(T) \subset \sigma_c(T_1) \cup \sigma_p(T_1).$$

The complements reverse the inclusion property. That is

$$\sigma_r(T_1) \cup \rho(T_1) \subset \sigma_r(T) \cup \rho(T).$$

but $\rho(T_1) \subset \sigma_r(T) \cup \sigma(T_1)$, so

$$\rho(T_1) \subset \rho(T) \cup \sigma_r(T).$$

as desired.

3 Spectral Properties of Bounded Linear Operators

1. Let $X = C[0,1]$ and define $T : X \to X$ by $Tx = vx$, where $v \in X$ is fixed. Find $\sigma(T)$. Note that $\sigma(T)$ is closed.

Solution: By definition λ is in the *point spectrum* if the operator $T - \lambda I$ does not exist. That is if $T - \lambda I$ is no one-to-one.

$$(T - \lambda I)x = vx - \lambda x = (v - \lambda)x$$

One-to-one means that the null space of the operator is $\{0\}$. The null space is the set of x such that

$$(v - \lambda)x = 0.$$

We have two cases here:

(i) Call t the independent variable for v. That is $v = v(t)$. If v is non-constant, then there is an interval in $[0,1]$ where v is either increasing or decreasing and does not contain in its range, the value of λ. This means that there is a neighborhood where $(v - \lambda)x = 0$, and $v - \lambda \neq 0$. This means that in that neighborhood $x(t) = 0$ identically. The function fails to be one-to-one in that neighborhood.

Appendices

Appendix A

Coefficients of the Characteristic Polynomial

We show two methods to compute the coefficients of the characteristic polynomials.

(i) **The Newton's identity.** From this identity we show how a recursion is built based on the trace of powers of the matrix A.

(ii) **In terms of minors. Here we refer to computation of minor determinants.**

1 Newton's Identity

Let $\lambda_1, \lambda_2, \cdots \lambda_n$ be the roots of the a characteristic polynomial

$$p(\lambda) = \lambda^n + c_1 \lambda^{n-1} + \cdots + c_n$$

[1] if

$$s_k = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k,$$

then the coefficients c_k are

$$c_k = -\frac{1}{k}(s_k + s_{k-1}c_1 + s_{k-2}c_2 + \cdots + s_2 c_{k-2} + s_1 c_{k-1}).$$

[1]Note that this polynomial corresponds to the expansion of $\det(\lambda I - A)$. The expansion of $\det(A - \lambda I)$ will have a factor of $(-1)^n$ in front, and we will take this into account at the end of this section.

1 .0.0.1 Proof. We can write the characteristic polynomial as

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = (\lambda^n + c_1\lambda^{n-1} + \cdots + c_n).$$

By taking the derivative of p we see that

$$\frac{p'(\lambda)}{p(\lambda)} = \frac{n\lambda^{n-1} + (n-1)c_1\lambda^{n-2} + \cdots + c_{n-1}}{\lambda^n + c_1\lambda^{n-1} + \cdots + \lambda + c_n} = \sum_{i=1}^{n} \frac{1}{\lambda - \lambda_i}$$

where the first fraction was computed from the expanded $p(\lambda)$ and the second from the factored form.

We use the geometric series. That is,

$$\begin{aligned} \frac{1}{\lambda - \lambda_1} &= \frac{1}{\lambda(1 - \lambda_1/\lambda)} = \frac{1}{\lambda}\left(1 + \frac{\lambda_1}{\lambda} + \cdots + \left(\frac{\lambda_1}{\lambda}\right)^j + \cdots\right) \\ &= \frac{1}{\lambda} + \frac{\lambda_1}{\lambda^2} + \cdots + \frac{\lambda_1^j}{\lambda^{j+1}} + \cdots \end{aligned}$$

and so

$$\sum_{i=1}^{n} \frac{1}{\lambda - \lambda_i} = \sum_{i=1}^{n} \frac{1}{\lambda} + \frac{\lambda_i}{\lambda^2} + \cdots + \frac{\lambda_i^j}{\lambda^{j+1}} + \cdots$$

where we assume $|\lambda| > \max|\lambda_i|$ to assure convergence of each geometrical series. We then write

$$\sum_{i=1}^{n} \frac{1}{\lambda - \lambda_i} = \frac{n}{\lambda} + \frac{s_1}{\lambda^2} + \cdots \frac{s_j}{\lambda^{j+1}} + \cdots$$

Then

$$\begin{aligned} n\lambda^{n-1} + (n-1)c_1\lambda^{n-2} + \cdots + c_{n-1}\lambda &= (\lambda^n + c_1\lambda^{n-1} + \cdots + \lambda + c_n) \\ &\quad \left(\frac{n}{\lambda} + \frac{s_1}{\lambda^2} + \cdots \frac{s_j}{\lambda^{j+1}} \cdots\right) \end{aligned}$$

We now match coefficients. That is

$$\begin{aligned} &\lambda^{n-2} && \Rightarrow \quad c_1(n-1) = s_1 + nc_1 \quad \Rightarrow \quad c_1 = -s_1 \\ &\lambda^{n-3} && \Rightarrow \quad c_2(n-2) = s_2 + c_1 s_1 + c_2 n \quad \Rightarrow \quad c_2 = \frac{-s_2 - c_1 s_1}{2} \\ &\quad\vdots \\ &\lambda^{n-k+1} && \Rightarrow c_k(n-k) = s_k + c_1 s_{k-1} + \cdots c_k n \end{aligned}$$

So,

$$c_k = -\frac{s_k + c_1 s_{k-1} + \cdots + c_{k-1} s_1}{k}$$

So, if we have a matrix A which characteristic polyonomial is $p(\lambda)$. Then from the first coefficient

$$c_1 = -s_1 = -\sum_{i=1}^{n} \lambda_i$$

we find that this is the trace of the matrix.

The second coefficient

$$c_2 = -\frac{s_2 - c_1 s_1}{2}$$

where $s_2 = \sum_{i=1}^{n} \lambda_i^2$, which is the trace of A^2, and so on.

If instead, of using the definition $p(\lambda) = \det(\lambda I - A)$, we use $p(\lambda) = \det(A - \lambda I)$, we need to reverse the signs of the coefficients for n odd. We will take this into account in the examples.

Let us illustrate this with one example. Let

$$A = \begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix}$$

The characteristic polynomial is

$$p(\lambda) = -x^3 + 4x^2 + 3x - 18.$$

The eigenvalues are

$$\lambda_1 = 3 \quad \lambda_2 = 3 \quad \lambda_3 = -2.$$

Let us find the coefficients if $p(\lambda)$, from the Newton recursion formula.

$$c_1 = -s_1 = -\sum_{i=1}^{3} \lambda_i = -4$$

however we should reverse the sign, since now we are using $p(\lambda) = \det(A - \lambda I)$, so $c_1 = 4$ and this is also the trace of the matrix.

To compute c_2 we will need s_2 which is the sum of the three eigenvalue squared. However, let us assume that we do not know the eigenvalues (since we are trying to find the characteristic polynomial to find from it the eigenvalues).

We know that the eigenvalues of A^2 are the squares of the eigenvalues of A, and so the traceA^2 should be s_2. We find

$$A^2 = \begin{pmatrix} 23 & 6 & 7 \\ 20 & 9 & 10 \\ -38 & -12 & -10 \end{pmatrix}$$

and so $s_2 = 23 + 9 - 10 = 22$, then

$$c_2 = \frac{-s_2 - c_1 s_1}{2} = \frac{-22 + 16}{2} = -3$$

and reversing the sign we have

$$c_2 = 3.$$

Finally, to find c_3 we need to compute A^3 which is

$$A^3 = \begin{pmatrix} 65 & 27 & 19 \\ 140 & 27 & 70 \\ -146 & -54 & -46 \end{pmatrix}$$

from which $s_3 = 65 + 27 - 46 = 46$,

$$c_3 = -\frac{s_3 + c_1 s_2 + c_2 s_1}{3} = -\frac{46 + (-4)(22) + (-3)(4)}{3} = 18$$

and reversing the sign

$$c_3 = -18.$$

We observe that the coefficient c_k can be, recursively computed from the traces of the matrices $A, A^2, \cdots A^k$, and previous coefficients. $c_1, \cdots c_{k-1}$.

Another way to compute the coefficients is by using some minor determinants. Next we show this method.

2 The Characteristic Polynomial from some Minor Determinants

We want to find the coefficients of the characteristic polynomial,

$$p(\lambda) = \det(A - \lambda I) = (-1)^n(\lambda^n + c_1\lambda^{n-1} + \cdots + c_n).$$

Instead, let us use an auxiliary function $q(\lambda) = \det(A + \lambda I)$ and think of λ as a vector of entires $(\lambda_1, \cdots, \lambda_n)$, and define a function

$$f(\lambda_1, \lambda_2, \cdots, \lambda_n) = \det[A + \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)]$$

we observe that $p(\lambda) = f(-\lambda, -\lambda, -\cdots, -\lambda)$. If we expand f in terms of Levi-Civita symbols and using Einstein notation

$$f(\lambda_1, \lambda_2, \cdots, \lambda_n) = \varepsilon_{i_1 i_2 \cdots i_n}(a_{1i_1} + \delta_{i_1 1}\lambda_1)(a_{2i_2} + \delta_{i_2 2}\lambda_2)\cdots(a_{ni_n} + \delta_{i_n n}\lambda_n)$$

We expand the product, after the Levi-Civita symbol as the following sum

$$\begin{aligned}
&a_{1i_1}a_{2i_2}\cdots a_{ni_n} + \\
&\lambda_{j_1}\delta_{i_{j_1}j_1}a_{1i_1}a_{2i_2}\cdots \hat{a}_{j_1 i_{j_1}}\cdots a_{ni_n} + \\
&\lambda_{j_1}\lambda_{j_2}\delta_{i_{j_1}j_1}\delta_{i_{j_2}j_2}a_{1i_1}a_{2i_2}\cdots \hat{a}_{j_1 i_{j_1}}\cdots \hat{a}_{j_2 i_{j_2}}\cdots a_{ni_n} + \\
&\vdots \\
&\lambda_{j_1}\cdots\lambda_{j_k}\delta_{i_{j_1}j_1}\cdots\delta_{i_{j_k}j_k}a_{1i_1}\cdots \hat{a}_{j_1 i_{j_1}}\cdots \hat{a}_{j_k i_{j_K}}\cdots a_{ni_n} + \\
&\vdots \\
&\lambda_{j_1}\cdots\lambda_{j_{n-1}}\delta_{i_{j_1}j_1}\cdots\delta_{i_{j_{n-1}}j_{n-1}}a_{1i_1}\cdots \hat{a}_{j_1 i_{j_1}}\cdots \hat{a}_{j_{n-1} i_{j_{n-1}}}\cdots a_{ni_n} + \\
&\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_n}\delta_{i_{j_1}j_1}\delta_{i_{j_2}j_2}\cdots\delta_{i_{j_n}j_n}a_{1i_1}a_{2i_2}\cdots \hat{a}_{j_1 i_{j_1}}\hat{a}_{j_2 i_{j_2}}\cdots \hat{a}_{j_n i_{j_n}}\cdots a_{ni_n}
\end{aligned}$$

The symbol $\hat{a}$ means that this factor is excluded from the sum, or in other words it is 1, and this make sense since look at the δ symbol at the beginning of the factor with the same indices. That is, the term on each binomial factor that is picked up instead of the a coefficient. Each of these terms should be accompanied with the Levi-Civita symbol $\varepsilon_{i_1 i_2 \cdots i_n}$. Hence we recognize that the first term (corresponding to c_n) is the determinant of A. The second term (corresponding to c_{n-1}) is the sum of all minor A_{n-1n-1} determinants along the diagonal , and so forth, until the c_1 coefficient is the sum of all minor determinants A_{11} which are just the diagonal entries and that is the trace$A$. In the last term all the a's are suppressed and only the δ's stay providing a coefficient of 1 for the λ^n term. We found an expansion for $f(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Now we want to compute $p(\lambda)$ which is $f(-\lambda, -\lambda, \cdots, -\lambda)$. Then since each λ^{n-i} has as coefficient c_i, then the sign carried by c_i is $(-1)^{n-i}$, so

In equations we find

$$\begin{aligned}
c_1 &= (-1)^{n-1}\text{trace}A \\
c_2 &= (-1)^{n-2}\sum A_{22} \qquad \text{(diagonal two – order minors)} \\
c_3 &= (-1)^{n-3}\sum A_{33} \qquad \text{(diagonal three – order minors)} \\
&\vdots \\
c_k &= (-1)^{k-1}\sum A_{kk} \qquad \text{(diagonal k – order minors)} \\
&\vdots \\
c_n &= (-1)^0 \det A = \det A
\end{aligned}$$

Let us use the same example above to test these coefficients. That is, let

$$A = \begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix}$$

- $c_1 = 4$.
- c_2. The three minors of second order along the diagonal are

$$\begin{aligned}
\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} &= \begin{vmatrix} 3 & 10 \\ -2 & 4 \end{vmatrix} = 12 + 20 = 32 \\
\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} &= \begin{vmatrix} -3 & -3 \\ 2 & 4 \end{vmatrix} = -12 + 6 = -6 \\
\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= \begin{vmatrix} -3 & 1 \\ 20 & 3 \end{vmatrix} = -9 - 20 = -29
\end{aligned}$$

and $c_2 = -(32 - 6 - 29) = 3$,

- and finally, $c_3 = \det A = 18$.

Appendix B

Shift, Shift, ..., and more Shift

This appendix shows a collections of examples for the shift (left and right) operators in various spaces. Finite dimensional, countable infinite dimensional, and continuum infinite dimensional spaces.

1 Finite Dimensional Space

1 .1 The cyclic right shift operator

This operator is defined as

$$\begin{array}{rcl} T:\mathbb{R}^n & \to & \mathbb{R}^n \\ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} & \mapsto & \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_1 \end{bmatrix} \end{array}$$

It can be represented by the matrix A

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

This is an interesting operator. It is unitary. That is $AA^T = A^T A = I$.
There are several ways to check on this.

- By direct computation. Just check it.
- The transposed of A, that is A^T , will reverse the process. While A is a right shift, then the transposed will be a left shift. Think about a round table and moving people places along a circle clockwise one position, and then moving them back (counter-clockwise) one position. The round table comes back to the initial configuration. Note that the 1 in the last row is very important. We will see in the next example the no-cycling shift and the big difference it makes.
- A permutation matrix, which moves the last row into the first row will make of A the identity. A permutation matrix is the identity with some rows interchanged. (Here only one row)
- The vector coordinates do not change. Just the order, so $\|Ax\| = \|x\|$. It does not change the size of the vector.

Let us find the eigenvalues. The characteristic polynomial is given by

$$p(\lambda) = \det \begin{bmatrix} -\lambda & 1 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & -\lambda \end{bmatrix} = (-1)^{n+1} + (-1)^{2n}(-\lambda)^n$$

which can be computed by expanding the determinant along the last row in two terms. The first entry on the last row "1" and the last $-\lambda$. Setting the characteristic equation to zero means

$$\lambda^n - 1 = 0.$$

So the $k^t h$ eigenvalue is $\lambda_k = \mathrm{e}^{2ik\pi/n}$, which are n roots of 1 along the unit circle.

It is interesting to think again about the round table example. If n people are sitting around a round table and they move one place toward their right (say counter-clockwise), the inverse operator is to move one place toward thier left (say clockwise). The location of the people around the table is like the location of the n roots of 1. Uniformly distributed.

Let us now find the eigenvectors.

Assume,

$$Ax = \lambda x.$$

This is, for $x = (x_1, x_2, \cdots, x_n)^T$,

$$\begin{aligned} x_2 &= \lambda x_1, \\ x_3 &= \lambda x_2, \\ &\vdots \\ x_1 &= \lambda x_{n-1}. \end{aligned}$$

We can say that $x_1 = 1$ without loss of generality, since there is at least one degree of fredom or a null space on the $Ax - \lambda x = 0$ equation. Then

$$\begin{aligned} x_1 &= 1 \\ x_2 &= \lambda, \\ x_3 &= \lambda^2, \\ &\vdots \\ x_{n-1} &= \lambda^{n-2} \\ x_n &= \lambda^{n-1} \\ 1 &= \lambda^n \end{aligned}$$

The last equation is not new. It is the characteristic equation

$$p(\lambda) \equiv \lambda^n - 1 = 0$$

So an eigenvector x for an eigenvalue λ is given by

$$x = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix}$$

In the round table example, the eigenvector represents all the people sitting in the table, by picking one as the starting order.

The invariant spaces are the multiples of these eigenvectors. Think about they this way. Pick an eigenvector as shown above. This is a set of n points along the unit circle equally (uniformly) distributed. A scaling α of this vector will contract or expand the circle with a new radius α. That is how the invariants work. They are all circles with points which where are rotated by and angle $2k\pi/n$, and they

We do not include the left cyclic operator because its analysis does not provide new insights over the right cyclic operator developed above. The left cyclic is the adjoint, and the inverse of the right cyclic unitary opeator.

1 .2 The right shift operator

This operator is defined as

$$\begin{array}{rcl} T:\mathbb{R}^n & \to & \mathbb{R}^n \\ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} & \mapsto & \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ 0 \end{bmatrix} \end{array}$$

It can be represented by the matrix A

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

There is a big difference between the right shift and the cyclic right shift operators. It has all eigenvalues 0 (roots of the characteristic polynomial $\lambda^n = 0$) with algebraic multiplicy n and geometric multiplicity 1. The operator does not have inverse (see the last row). It looses the last element x_n and will not recover it by doing a left shift. The determinant and trace of this matrix are 0.

If we set the equation $Ax = \lambda x$, then we have

$$\begin{aligned} x_2 &= \lambda x_1 = 0 \\ x_3 &= \lambda x_2 = 0 \\ &\vdots \\ x_n &= \lambda x_{n-1} = 0 \\ 0 &= \lambda x_n = 0 \end{aligned}$$

So $x_2 = x_3 = \cdots = x_n = 0$. We have no information of x_1 which becomes the free parameter. That is the vector $x = (1, 0, \cdots, 0)$ is an eigenvector of this matrix. That is $Ax = \lambda x = 0$. The geometrical multiplicity is 1, so there are no more (non-zero) eigenvectors.

We observe that the space spanned by $x = (1, 0, \cdots, 0)^T$ is the invariant space. If $S = \{cx, c \in \mathbb{C}\}$, then $T(S) = 0 \subset S$.

2 The discrete infinite dimensional ℓ_2

2 .1 One side (causal) shifts

2 .1.1 The causal left and right shifts

2 .1.1.1 The causal (one-sided) left shift Let us denote L as the operator $L[(\xi_1, \xi_2, \cdots)] = (\xi_2, \xi_3, \cdots)$. In ℓ_2 it is clear that $\|L(v)\| \leq \|v\|$ and the equality is only achieved when $\xi_1 = 0$. This means that $\|L\| \leq 1$. Now, from Theorem 7.3.4,

$$|\lambda| \leq \|L\| \leq 1,$$

for each $\lambda \in \sigma(T)$. So, the spectrum of T is all inside the unit complex circle $|\lambda| \leq 1$. We show that if $|\lambda| < 1$, then λ is in the point spectrum (eigenvalue) of T. That is $\lambda \in \sigma_p(T)$. The operator $L - \lambda I = 0$, estabishes that

$$(\xi_2, \xi_1, \cdots) = (\lambda \xi_1, \lambda \xi_2, \cdots).$$

From which

$$\begin{aligned}\xi_2 &= \lambda\xi_1\\ \xi_3 &= \lambda\xi_2 = \lambda^2\xi_1\\ &\vdots\\ \xi_k &= \lambda^{k-1}\xi_1.\end{aligned}$$

Hence

$$Lv = \lambda v$$

has a solution for each $|\lambda| < 1$, since the sequence $\xi_1(\lambda, \lambda^2, \cdots, \lambda^k, \cdots)$ is in ℓ_2. This means that all points inside the unit circle, that is $|\lambda| < 1$ are eigenvalues of L. Hence:

- The one–sided left shift operator has a point spectrum of $\sigma_p(L) = \{\lambda : |\lambda| < 1\}$.

- Since the resolvent exists for $|\lambda| > 1$ (see Theorem 7.3.4) the other part of the spectrum lies exactly along the unit circle (in the circumference). We show that $\sigma_r(L) = \{\lambda : |\lambda| = 1\}$. That is, that the unit circle represents the set of point in the continuous spectrum. All we have to do is to show that the image of $L-\lambda$, for $\lambda = 1$ is dense in ℓ_2. From Theorem 1. in Appendix C we see that a way to show that the range of L is dense in $\mathscr{H}$ is by showing that the null space of the adjoint L^* is the set $\{0\}$. The L^* operator is the right shift R operator shown in the next section (2 .1.1.2). Let $v = (\xi_1, \xi_2, \cdots)$ such that $(R-\overline{\lambda})v = 0$. This means that $\overline{\lambda}$ is an eigenvalue of the right shift operator. We show next that the right shift operator does not have eigenvalues. This means that $v = 0$, and then the range of $(R-\lambda I)^* = L-\overline{\lambda}I$ is dense in H, so the unit circle $|\lambda| = 1$ corresponds to the continuous spectrum $\sigma_c(L)$.

- The invariant subspace is the set of all vectors v,

  $$S = \left\{\sum_{i=0}^{\infty} c_i\lambda^i e_i = (c_0\,, c_1\lambda\,, c_2\lambda^2\,, \cdots\,, c_k\lambda^k\,, \cdots) : c_k \in \mathbb{C}\right\}$$

  Let us verify. Take $v \in S$, so

$$v=\sum_{i=0}^{\infty}c_i\lambda^i e_i=(c_0\,,c_1\lambda\,,c_2\lambda^2\,,\cdots,c_k\lambda^k\,,\cdots)$$

The left shift Lv yields

$$Lv=(c_1\lambda\,,c_2\lambda^2\,,\cdots,c_k\lambda^k\,,\cdots)=(d_1,d_2\lambda\,,\cdots,\,d_k\lambda_k)$$

with $d_i=\lambda c_i$, so indeed $Lv\in S$.

Note that if, instead of being on $\mathscr{H}=\ell_2$ we are in ℓ_∞, where the norm is the largest absolute value of the components of the vector, then all vectors of the form $(c,c,\cdots c)$, are invariant under any shift. So the invariance property depends on the type of Hilbert space we are on.

2 .1.1.2 The causal (one-sided) right shift This is problem 7.4.5. Let us denote R as the operator $R[(\xi_1,\xi_2,\cdots)]=(0,\xi_1,\xi_2,\cdots)$. By using the same arguments above, $\|R\|=1$. That is R is a unitary operator. Let us show that this operator does not have any eigenvalues inside the unit circle. Assume, $R-\lambda I=0$. This means $(0,\xi_1,\xi_2,\cdots)=\lambda(\xi_1,\xi_2,\cdots)$, or

$$\begin{aligned}0&=\lambda\xi_1\\ \xi_1&=\lambda\xi_2\\ &\vdots\\ \xi_k&=\lambda\xi_{k+1}.\end{aligned}$$

If $\lambda\neq 0$, then $\xi_1=0$, and by domino effect $v=0$. So if $\lambda\neq 0$, there is no eigenvector, so the only possible eigenvalue is $\lambda=0$. However, if $\lambda=0$, we see that $v=0$. So, in any case there are no eigenvalues. Still, since $\|R\|=1$, we see that the spectrum has to be inside the unit circle, and the spectrum is either the continuous spectrum or the residual spectrum.

We show that the range of $R-\lambda$ can not be dense in $\mathscr{H}$. Let us pick w an eigenvector for L corresponding to the eigenvalue $\overline{\lambda}$ (with $|\overline{\lambda}|<1$). Then, for any arbitrary vector v

$$\langle(R-\lambda)v\,,w\rangle=\langle v\,,(R^*-\overline{\lambda})w\rangle=\langle v\,,(L-\overline{\lambda})w\rangle=\langle v\,,0\rangle=0$$

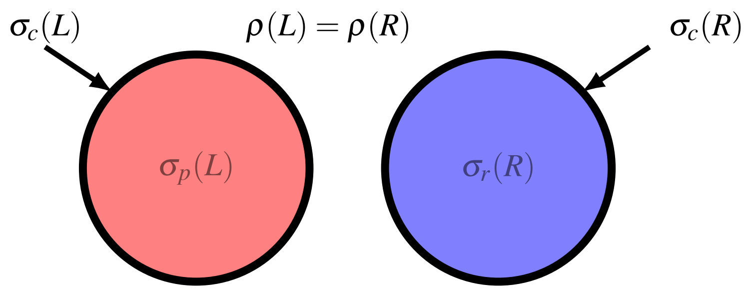

Figure B.1: We represent unit circles centered at 0. On the left we see the spectrum of the unilateral left shift L operator, On the right we illustrate the spectrum of the unilateral right shift R operator. The arrows point to the boundaries of the circles, which are circumpherences. The resolvent is the outside of the unit circle.

That is, the range of $R-\lambda$ is orthogonal to the eigenvector w (which is non-zero). Then it can not be dense in $\mathscr{H}$. So the inside of the unit circle $|\lambda| < 1$ corresponds to the residual spectrum $\sigma_r(R)$.

Finally, let us take $|\lambda| = 1$. That is, any λ in the unit circle. We show that $R-\lambda$ is dense in $\mathscr{H}$ and so $\lambda \in \sigma_c(R)$. We show that the null space of $(R-\lambda)^*$ is $\{0\}$.

Pick v in the null space of $(R-\lambda)^*$, that is, for any $w \in \mathscr{H}$,

$$0 = \langle (R-\lambda)^* v\,,\, w\rangle = \langle v\,,\,(L-\overline{\lambda})w\rangle$$

That is, null $(R-\lambda)^*$ is orthogonal to range$L-\overline{\lambda}$. From the previous problem, we now that the range of $L-\overline{\lambda}$ is dense in $\mathscr{H}$, hence null$(R-\lambda)^* = \{0\}$, and the circle $|\lambda| = 1$ is the set $\lambda_c(R)$.

As shown in problem 7.2.5 the right shift operator, which does not have eigenvalues, has many invariant subspaces. Any subspace of the form

$$S_k = \sum \alpha_{i=k}^{\infty} e_i,$$

for $k = 1, 2, \cdots$ with e_i a vectors with zeroes in all places except at the position k where there is a 1. This is a good counterexample of an operator that does not have eigenvalues and has many invariant subspaces.

Figure 2 .1.1.2 shows the summary of the results on this section.

2 .2 Two-sided shifts

We consider a vectors of the form $x = (\cdots, \xi_{-n}, \xi_{-n+1}, \cdots \xi_{-1}, \xi_0, \xi_1, \cdots \xi_n, \xi_{n+1}, \cdots)$. The left shift L operator take this vector to the vector

$$(\cdots, \xi_{-n+1}, \xi_{-n}, \cdots \xi_{-2}, \xi_{-1}, \xi_0, \cdots \xi_{n-1}, \xi_n, \cdots).$$

Clearly the left shift operator does not change the norm of the vector. That is $\|Lx\| = \|x\|$, and so the spectrum is inside the unit circle.

The equation $(L - \lambda I)x = 0$, means

$$\begin{aligned} \xi_{-n+1} &= \lambda \xi_{-n} \\ \xi_{-n} &= \lambda \xi_{-n+1} \\ &\vdots \\ \xi_{-2} &= \lambda \xi_{-1} \\ \xi_{-1} &= \lambda \xi_0 \\ \xi_0 &= \lambda \xi_1 \\ &\vdots \\ \xi_{n-1} &= \lambda \xi_n \\ \xi_n &= \lambda \xi_{n+1} \end{aligned}$$

Let us write this in terms of ξ_0. Assume $i > 0$, then

$$\xi_i = \lambda^{-1} \xi_{i-1} = \cdots = \lambda^{-i} \xi_0.$$

If $i < 0$, then

$$\xi_i = \lambda \xi_{i+1} = \cdots = \lambda^{i} \xi_0.$$

Then $Lx = \lambda x$ implies, that

$$x = \xi_0(\cdots, \lambda^i, \lambda^{i-1}, \cdots 1, \lambda^{-1}, \cdots \lambda^{-i}, \cdots)$$

which never converges for $\lambda \neq 0$, but if $\lambda = 0$, then $x = 0$. So there are no eigenvalues. This means that the spectrum is shared between the continuous $\sigma_c(L)$ and residual $\sigma_r(L)$ spectra.

3 The continuum

3 .1 The multiplication by $e^{i\theta}$

4 Operators where the spectrum is the entire real line

Appendix C

Important Theorems I could not find in Kreyzig's Book

This does not mean that the theorems are not there. I just could not find them.

We first proof an important Lemma, which is problem 3.3.10 of the Book.

1 Twice the orthogonal complement is the closure

Problem 3.3.10 of Kreyzig's book says: If $M \neq$ is any subset of a Hilbert space H, show that $M^{\perp\perp}$ is the smallest closed subspace of H which containes M, that is, $M^{\perp\perp}$ is contained in any closed subspace $Y \subset H$, such that $Y \supset M$. What the problem does not says explicitly is that

$$M^{\perp\perp} = \overline{M}.$$

since $\overline{M}$ is the smallest closed subspace on H containing M.

1 .0.0.3 Solution:

If M is closed, then this is Lemma 3.3-6 of the book. This is, f M is closed $M = M^{\perp\perp}$. But if M is closed then $M = \overline{M}$. The interesting case is when M is non-closed. The book at the top of page 149 shows that $M \subset M^{\perp\perp}$ as follows:

$$x \in M \implies x \perp M^{\perp} \implies x \in (M^{\perp})^{\perp}.$$

This is true regardless M is closed or not. Since $\overline{M}$ is the smallest closed set containing M, and $M^{\perp\perp}$ is closed, then

$$\overline{M} \subset M^{\perp\perp}.$$

We now show the second inclusion part. That is, pick $x \in M^{\perp\perp}$, then $\langle x\, , \, y\rangle = 0$, $\forall y \in M^{\perp}$. Then by definition $x \in M$, but $M \subset \overline{M}$, so $M^{\perp\perp} = \overline{M}$.

2 Set equalities including null and range of operators and their adjoints

There are some symmetries between operators and their adjoints, their null spaces, and their image (range) spaces, in finite dimensional spaces, which can be extended to infinite dimensional operators. This symmetries are inherited from the idempotent property of both the adjoint symbol and the orthogonal complement symbol. We first show the finite dimensional operator (matrices) version. For example let us assume an operator $\mathscr{L}(V,W)$. Then

(i) $\text{null}\,\mathscr{L} = (\text{range}\,\mathscr{L}^*)^{\perp}$

(ii) $\text{null}\,\mathscr{L}^* = (\text{range}\,\mathscr{L})^{\perp}$

(iii) $(\text{null}\,\mathscr{L})^{\perp} = \text{range}\,\mathscr{L}^*$

(iv) $(\text{null}\,\mathscr{L}^*)^{\perp} = \text{range}\,\mathscr{L}$

where the three last expressions are found by switching between adjoint and no adjoint, and range and null, from the first. A mnemonic rule here. There are two symbols. The adjoint * and the orthogonal complement $^{\perp}$. The first, if used, is a superindex acting on an operator $\mathscr{L}$. The second, if used, is a superindex acting on a set. Think that the word "null" is always on the left, since this is in the domain of the operator, and "range" on the right, since it is in the image (range) of the operator. The two symbols can be combined in four different forms, on the left ("null") side as follows

(i) No symbols at all

(ii) Just adjoint

(iii) Just orthogonal complement

(iv) both adjoint and normal complement

Then to guess the right side, (which is always "range") think that the two symbols should be present on each expression. So, for example, if they are not present in the left side, both of them should be on the right side. If one is on the left, the other is on the right. If both are on the left none of them is on the right. The third and four equalities follow directly from the first by taking orthogonal complements on both sides of those (first and second) equalities.

This is easy to prove.

$$\begin{aligned} v \in \mathscr{L} \quad &\Longleftrightarrow \quad Lv = 0 \\ &\Longleftrightarrow \quad \langle Lv\,, w\rangle = 0 \quad \forall w \in W \\ &\Longleftrightarrow \quad \langle v\,, L^* w\rangle = 0 \quad \forall w \in W \end{aligned}$$

So $v \in \mathscr{L}$ is equivalent to $v \in (\operatorname{range} L^*)^{\perp}$. this proofs the first equality. Since $(\mathscr{L}^*)^* = \mathscr{L}$. Changing $\mathscr{L}$ by $\mathscr{L}^*$ on the first equality produces the second equality.

Let us formulate the following extension of this property to infinite dimensional spaces as follows:

2 .0.0.4 Theorem 1: If $\mathscr{L} : \mathscr{H} \to \mathscr{H}$ is a bounded linear operator, then

(i) $\operatorname{null} \mathscr{L} = (\operatorname{range} \mathscr{L}^*)^{\perp}$

(ii) $\operatorname{null} \mathscr{L}^* = (\operatorname{range} \mathscr{L})^{\perp}$

(iii) $(\operatorname{null} \mathscr{L})^{\perp} = \overline{\operatorname{range} \mathscr{L}^*}$

(iv) $(\operatorname{null} \mathscr{L}^*)^{\perp} = \overline{\operatorname{range} \mathscr{L}}$

We observe that this is almost identical to the four equalities shown above. The only difference is that instead of using the ranges when the orthogonal of the "null" spaces are chosen, we use the closure of the ranges. In finite dimensional spaces the range of an operator is always closed. In infinite dimensional Hilbert spaces we have that the orthogonal complement of a non empty set is always closed (see Problem 3.4.8 of the book) [1] However the range of an operator could be open. We need the closure whenevever this set should be equal to any orthogonal complement set.

[1] Here is the proof: Pick a sequence $x_n \in M^{\perp}$. Hence for all $y \in M$, $\langle x_n\,, y\rangle = 0$. Let $x = \lim_{n\to\infty} x_n$. Since the inner product is continuous, $\langle x\,, y\rangle = \langle \lim_{n\to\infty} x_n\,, y\rangle = \lim_{n\to\infty}\langle x_n\,, y\rangle = 0$. So $x \in M^{\perp}$.

Before proving this let us remember from Theorem 3.3-4 in the book that for any closed subspace Y of a Hilbert space $\mathscr{H}$. We can write $\mathscr{H} = Y \oplus Y^{\perp}$. In particular since $\text{null}\,\mathscr{L}$ is closed, we can write $\mathscr{H} = \text{null}\,\mathscr{L} \oplus \text{null}\,(\mathscr{L})^{\perp}$. Let us prove property (iii). From Lemma 3.3-6 and since the set $\text{null}\,\mathscr{L}$ is closed we know that

$$(\text{null}\,\mathscr{L})^{\perp\perp} = \text{null}\,\mathscr{L}$$

So by taking the orthogonal complements to both sides of (iii) we find

$$\text{null}\,\mathscr{L} = \overline{\text{range}\,\mathscr{L}^*}^{\perp}.$$

We show this equality.

$$\begin{aligned} x \in \text{null}\,\mathscr{L} &\iff \langle y\,, \mathscr{L}x\rangle = 0 \quad \forall y \in \mathscr{H} \\ &\iff \langle \mathscr{L}^* y\,, x\rangle = 0 \\ &\iff x \in (\text{range}\,\mathscr{L}^*)^{\perp} \end{aligned}$$

If $\text{range}\,\mathscr{L}^*$ is closed then the equality falls in place, but how if the $\text{range}\,\mathscr{L}^*$ is no closed? That is, since $(\text{range}\,\mathscr{L}^*)^{\perp} \supset (\overline{\text{range}\,\mathscr{L}^*})^{\perp}$ we can only assure that $(\overline{\text{range}\,\mathscr{L}^*})^{\perp} \subset \text{null}\,\mathscr{L}$.

We now prove the second inclusion. That is, let us take $x \in (\text{rank}\,\mathscr{L})^{\perp}$. then for all $y \in \mathscr{H}$ we have

$$0 = \langle \mathscr{L}y\,, x\rangle = \langle y\,, \mathscr{L}^* x\rangle.$$

Therefore $\mathscr{L}^* x = 0$, so $(\text{ran}\,\mathscr{L})^{\perp} \subset \text{null}\,\mathscr{L}^*$. We take orthogonal complements and find

$$(\text{null}\,\mathscr{L}^*)^{\perp} \subset [\text{range}\,(\mathscr{L})]^{\perp\perp} = \overline{\text{range}\,\mathscr{L}}.$$

where we use Lemma 1 .

To prove (iv) we note that we can change $\mathscr{L}$ by $\mathscr{L}^*$ in (iii) , and since the adjoint is idempotent, then (iv) will result. The proof of (i) and (ii) is much like to proof done for finite dimensional spaces above.

There are several important consequences of this theorem that we list

- From Theorem 3.3-4 of the book (Direct sum) we see that the Hilbert space $\mathscr{H}$ can be written as

$$\mathscr{H} = \overline{\text{range}\,\mathscr{L}} \oplus \text{null} L^*.$$

- If $(\text{null}\,\mathscr{L}^*)^\perp = \mathscr{H}$, then the range of $\mathscr{L}$ is dense in $\mathscr{H}$.

  This is in words, what the numeral (iv) of Theorem 1 is telling us, since a set is dense in another if the closure of the first is the second.

- if $L^* = \{0\}$ then again $H = \overline{\text{range}\,\mathscr{L}}$, so the range is dense in $\mathscr{H}$.

- If the range of $\mathscr{L}$ is closed, then equation $\mathscr{L}x = y$ has solution if and only if y is orthogonal to to null$\mathscr{L}^*$. This is also known as the *Fredholm Alternative* according to James P. Keener book. Theorem 1.4 [2]

- The second part of the Fredholm alternative is the following. The Equation $\mathscr{L}x = y$ has a unique solution x if and only if the space null$\mathscr{L} = \{0\}$. From (iv) above, this is equivalent to say that range$\mathscr{L}^* = \mathscr{H}$.

[2]https://books.google.com/books/about/Principles_Of_Applied_Mathematics.html?id=5nlQAAAAMAAJ