Notes in Complex Variable

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May 12, 2016

Contents

Chapter 1

Introduction

A brief document on complex analysis. It would be useful if the reader has some familiarity with topological terms such as closed, open, convex, simply connected, multiply connected, dense sets, etc. However, I provide the basic definitions of those concepts at the beginning of the document to make it as self-contained as possible.

I have used many sources in writing this document, including Wikipedia. Here are the a few of them:

- The class notes [1] by Todd Kapitula
- The Lecture notes [2] by Beck, Marchesi, Pixton and Sabalka, and
- The Lecture Notes [3] by Michael Taylor.
- The book The Laplace Transform: Theorey and Applications [4] by Joel L. Schiff.
- The notes on Complex Analysis [5] by Jean-Fu Kiang.
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[1] http://www.calvin.edu/ tmk5/courses/m365/S08/GraduateClassNotes.pdf
[2] http://math.sfsu.edu/beck/complex.html
[3] http://www.unc.edu/math/Faculty/met/cpx.pdf
[4] https://www.google.com/search?tbo=p&tbm=bks&q=isbn:0387227571
[5] http://cc.ee.ntu.edu.tw/ jfkiang/complex%20analysis/complex_analysis.pdf

In addition I make an effort to include new references inside the text, most of them as Internet links.

A word on notation. It is common to see a new definition in *italic* fonts. However I use instead **bold face** fonts, which are easier to spot. This is good for me as writer, but I think it could also be beneficial for the reader. I understand that bold face should be used to highlight important words or sentences, but any new definition is, in my concept, an important word. Hence new definitions as well as important words and sentences will be written in **bold face** fonts.

Chapter 2

The basics

2.1 Definition of complex numbers and basic properties

A **complex number** z is denoted by $z = x + iy$ with $x, y \in \mathbb{R}$, and $i^2 = -1$. We note

$$\operatorname{Re} z = x \qquad \text{and} \qquad \operatorname{Im} z = y.$$

By definition the **conjugate of** z is defined as

$$\overline{z} = x - i\,y.$$

The **sum** and **product** of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are defined as

$$z_1 + z_2 = (x_1 + x_2) \qquad z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

A few simple properties:

- $\operatorname{Re} z = (z + \overline{z})/2$
- $\operatorname{Im} z = (z - \overline{z})/2$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

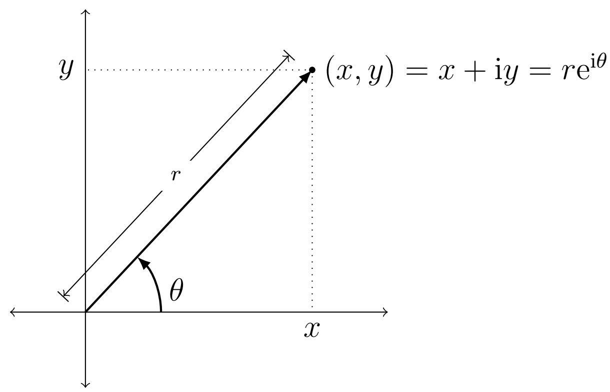


Figure 2.1: A complex number $z = x + \mathrm{i}y$, can be seen geometrically as a point (x, y) in the $\mathbb{R}^2$ plane. In polar coordinates the number can be seen also as the complex exponential $r\mathrm{e}^{\mathrm{i}\theta}$, where r is the radius $r = \sqrt{x^2 + y^2}$ and θ is the angle of the vector (x, y), or phase.

The **modulus, absolute value** or **norm** of z is defined as

$$|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}.$$

Figure 2.1 illustrates the geometry of a complex number. Note in the figure the polar representation

$$x + \mathrm{i}y = r\cos\theta + \mathrm{i}r\sin\theta = r\,\mathrm{e}^{\mathrm{i}\theta},$$

which is a convenient simplification in many cases. For example, by fixing r the set of points $r\mathrm{e}^{\mathrm{i}\theta}$ represent an arc of a circle swept by the angle θ. In many cases of contour integration, the paths are arcs of circles. Along an arc of circle only one variable changes (the angle θ) while the other (r) remains constant. A cartesian (x, y) representation has the disadvantage that both x and y change along an arc in a complicated matter.

We see that $|\mathrm{Re}\, z| \leq |z|$ and $|\mathrm{Im}\, z| \leq |z|$. This, with the product property of complex conjugate above, implies that

$$\mathrm{Re}(z\,\overline{w}) \leq |z||w|,$$

so

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z}+\overline{w}) \\ &= |z|^2+|w|^2+z.\overline{w}+\overline{z}w \\ &= |z|^2+|w|^2+2\mathrm{Re}(z\overline{w}) \\ &\leq |z|^2+|w|^2+2|z||w| \\ &= (|z|+|w|)^2, \end{aligned}$$

so we found the triangular inequality

$$|z+w| \leq |z|+|w|.$$

The Cauchy–Schwartz Inequality states that if $z_1, z_2, \cdots z_n$ and $w_1, w_2, \cdots w_n$ are in $\mathbb{C}$ then

$$\left|\sum_{i=1}^{n} z_i w_i\right|^2 \leq \sum_{i=1}^{n} |z_i|^2 \sum_{i=1}^{n} |w_i|^2.$$

The proof of this inequality is in almost any text on functional analysis, and I will omit it here.

The **multiplicative inverse** of a complex number z is

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

2.2 Limits and continuity

A review to Appendix **??** on elementary topology could be befeficial for a better understanding on limits and continuity.

As in real analysis we say that

$$\lim_{z\to z_0} f(z) = g$$

if for each $\epsilon > 0$, $\exists \delta > 0$ such that

$$\text{if} \quad |z-z_0| < \delta \quad \text{then} \quad |f(z_0)-g| < \epsilon.$$

A function f is **continuous** at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$. It is easy to show that, if the limit exists, then it is unique.

We see that the definition of limit could be stated in terms of neighborhoods. We say that the $\lim_{z\to z_0} f(z) = g$, if for each ϵ-neighborhood $N_\epsilon(f(z_0))$, in the range of f, exists a δ-neighborhood $N_\delta(z_0)$ in the domain of f such that all points inside the domain neighborhood map to points inside the range neighborhood. Continuity, of course, could also be stated in terms of neighborhoods. If for each open neighborhood $N_\epsilon(f(z_0))$ in the range $f(A)$ there is an open neighborhood $N_\delta(z_0)$ such that $f[N_\delta(z_0)] \subset N_\epsilon(f(z_0)$ then we say that the function f is continuous at z_0. In topological terms a function is continuous if it returns open neighborhoods in the domain from open neighborhoods in the range.

Limits in complex analysis have more restrictions that those in real analysis. In real analysis we only can approach a point from 2 directions (left or right). In complex analysis we can approach a point from any direction on a circle around it. As a result, some interesting properties will be observed in what follows. For example, the limit $\lim_{z\to 0} \overline{z}/z$ does not exists, since if we approach 0 through the real axis

$$\lim_{x\to 0} \frac{\overline{z}}{z} = \lim_{x\to 0} \frac{x}{x} = 1; \tag{2.2.1}$$

on the other hand, if we approach 0 through the imaginary axis

$$\lim_{y\to 0} \frac{\overline{z}}{z} = \lim_{y\to 0} \frac{-y}{y} = -1. \tag{2.2.2}$$

So the limit doesn not exist, since uniqueness is necessary.

2.3 Complex polynomial

A complex polynomial is of the form

$$f(z, \overline{z}) = \sum a_{ij} z^i \, \overline{z}^j.$$

A complex polynomial can be written as $f(x, y) = u(x, y) + \mathrm{i}(x, y)$, where u and v are of the form $\sum b_{ij} x^i y^j$, with $b_{ij} \in \mathbb{R}$ [1]

[1]From now on, if I do not put indices under and over the Σ sign, I assume that the first index is 0 and the last is some fixed natural $n > 0$.

Chapter 3

Differentiable, Holomorphic, Analytic, Entire

The concepts of holomorphic, analytic and differentiable are sometimes confused as being the same, but I will try to differentiate between them here.

Let $U \subset \mathbb{R}^2$ be open. A continuous function $f : U \to \mathbb{R}$ is C^1 (or **continuously differentiable)** on U if $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$ exist and are continuous on U. In this case we write $f \in C^1(U)$. In general, we say that $f \in C^k(U)$ if for $i = 1, 2, \cdots, k$ all partial derivatives $\partial^i f/\partial x^i$ and $\partial^i f/\partial y^i$ exist and are continuous. Here the partial derivatives are the regular partial deritatives defined for functions of real variables, applied both equally to the real and imaginary part of f. That is, if $f = u + \mathrm{i}v$ then

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x},$$

and the similarly changing x by y. Let us assume that instead of $f(x, y)$ we look at f as a function of z and $\overline{z}$. That is $f(z, \overline{z})$, and we want to find the partial derivatives of f with respect to z and $\overline{z}$.

For $f(z) = f(x + \mathrm{i}y)$,

$$\begin{aligned} df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy &= \frac{\partial f}{\partial x}\left[\frac{1}{2}(dz + d\bar{z})\right] + \frac{\partial f}{\partial y}\left[-\frac{\mathrm{i}}{2}(dz - d\bar{z})\right] \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right)d\bar{z}. \end{aligned}$$

from which we see that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right) \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right). \tag{3.0.1}$$

Then, by calling $f = z = x + \mathrm{i}y$, and then $f = \overline{z} = z - \mathrm{i}y$,

$$\begin{aligned}
\frac{\partial z}{\partial z} &= \frac{1}{2}\left(\frac{\partial x}{\partial x} + \mathrm{i}\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} - \mathrm{i}\frac{\mathrm{i}\,\partial y}{\partial y}\right) = \frac{1}{2}\Big(1 + 0 - 0 - (-1)\Big) = 1\\
\frac{\partial z}{\partial \overline{z}} &= \frac{1}{2}\Big(1 + 0 + 0 - 1\Big) = 0\\
\frac{\partial \overline{z}}{\partial z} &= \frac{1}{2}\Big(1 + 0 - 0 - 1\Big) = 0\\
\frac{\partial \overline{z}}{\partial \overline{z}} &= \frac{1}{2}\Big(1 + 0 + 0 - (-1)\Big) = 1.
\end{aligned}$$

Operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ inherit the product rule from their real functions expansions.

The polynomial

$$p(z, \overline{z}) = \sum a_{ij} z^i \, \overline{z}^j$$

contains no $\overline{z}$ terms if and only if $\partial p/\partial \overline{z} = 0$. To show this, let us first assume that $a_{ij} = 0$ for all $j > 0$ (that is, there are not terms with the factor $\overline{z}$). So

$$p(z, \overline{z}) = \sum a_{i0} z^i,$$

so

$$\frac{\partial p}{\partial \overline{z}} = \sum a_{i0} z^{i-1} \frac{\partial z}{\partial \overline{z}} = 0.$$

On the other hand, if $\partial p/\partial \overline{z} = 0$, then

$$\frac{\partial^{i+j} p}{\partial z^i \partial \, \overline{z}^j} p = 0.$$

for any $j \geq 1$. But,

$$\frac{\partial^{i+j} p}{\partial z^i \partial \, \overline{z}^j} p(0, 0) = i! \, j! a_{ij},$$

so that $a_{ij} = 0$ for any $j \geq 1$.

3.1 Holomorphic (analytic)

A function $f \in C^1(U)$ is **holomorphic (analytic)**
if

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

for every point of U. As a consequence a polyomial is holomorphic if an only if it is a function of z alone. A function is **entire** if it is holomorphic in the whole complex plane. Any polynomial $p(z)$ is an entire function.

3.2 Cauchy–Riemann conditions

From equation 3.0.1, being holomorphic is equivalent to

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + \mathrm{i}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) = 0.$$

So being holomorphic is equivalent to having

$$u_x = v_y, \qquad \text{and} \qquad u_y = -v_x \tag{3.2.2}$$

for each point in U. Here I used the notation $u_x = \partial u/\partial x$ and $u_y = \partial u/\partial y$. From now on, I will use the subscript notation, because it simplifies my LaTeX typing from the almost 30 characters "`\frac{\partial u}{\partial x}`" to just three characters "`u_x`". Still, the notation is well accepted and clear.

We see that for a function to be holomorphic is sufficient and necessary that the partial derivatives of the real and imaginary components of the complex fuction satisfy the **Cauchy–Riemann conditions** 3.2.2 in the domain U.

If $f = u + \mathrm{i}v$ is holomorphic then, from equations 3.0.1 and the Rieamman–Cauchy conditions 3.2.2

$$\begin{aligned} f_z = \frac{f_x - \mathrm{i}f_y}{2} = \frac{u_x + \mathrm{i}v_x - \mathrm{i}u_y + v_y}{2} &= u_x + \mathrm{i}v_x = f_x \\ &= -\mathrm{i}u_y + v_y = -\mathrm{i}f_y \end{aligned}$$

That is, if $f = u + \mathrm{i}v$ is holomorphic, then

$$f_z = f_x = \mathrm{i}f_y. \tag{3.2.3}$$

3.3 The Classical Approach

The classical approach to a holomorphic function is inherited from the definition of differential in real functions of a real variable. Also it can be as a special case of the most general Fréchet derivatives of Functional Analysis. That is, a function f is differentiable at a point z_0 inside the open domain U, if the limit

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \tag{3.3.4}$$

exists. We require that the derivative is the same when h approaches 0 from the horizontal

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = f_x(z_0)$$

as well as from the vertical

$$\lim_{h\to 0} \frac{f(z_0+\mathrm{i}h)-f(z_0)}{\mathrm{i}h} = \frac{f_y}{i}(z_0)$$

axis. That is

$$\mathrm{i}f_x(z_0) = f_y(z_0),$$

expand using $f = u + iv$, and these are the Cauchy-Riemann conditions.

From this approach is easy to see that most of the properties of differentiable functions (such the product rule, chain rule and others) are equally valid here. We say that a function is holomorphic in U if it is differentiable in its domain of definition U.

3.4 Examples

Next examples show why being differentiable , holomorphic and satisfying the Cauchy–Riemann conditions are not the same thing.

- Being holomorphic is more than just being differentiable at one point. We show here a function which is differentiable at just one point and not in any other point of the neighborhood. The function $f(z) = \overline{z}^2$ is differentiable at 0 and nowhere else (in particular f is not holomorphic at 0). To show this let us write $z = z_0 + re^{i\phi}$, so

$$\frac{\Delta f(z)}{\Delta z} = \frac{\overline{z}^2 - \overline{z}_0^2}{z - z_0} = \frac{(\overline{z_0 + re^{i\phi}})^2 - \overline{z}_0^2}{\cancel{z_0} + re^{i\varphi} - \cancel{z_0}}$$
$$= \frac{(\overline{z_0} + re^{-i\phi})^2 - \overline{z}_0^2}{re^{i\varphi}}$$
$$= \frac{\cancel{\overline{z}_0^2} + 2r\overline{z}_0 e^{-i\phi} + r^2 e^{-2i\phi} - \cancel{\overline{z}_0^2}}{re^{i\phi}}$$
$$= 2\overline{z}_0\, e^{-2i\phi} + re^{-3i\phi}.$$

We want to evaluate the limit of this ratio as $z \to z_0$ then $r \to 0$. However if we approach z_0 along the real axis, then $\phi = 0$ and the limit would be $2\overline{z}_0$, and in the other hand if we approach z_0 down from the vertical line $\phi = \pi/2$, and so the limit would be $-2\overline{z}_0$, so f is not differentiable at $z_0 \neq 0$. On the other hand, if $z_0 = 0$,

$$\lim_{z\to 0}\left(\frac{\overline{z}^2}{z}\right) = \lim_{z\to 0} r\, e^{-3i\phi} = \lim_{z\to 0} |z|\, e^{-3i\phi} = 0.$$

So f is differentiable at $z = 0$, and only at that point.

- The function $f(z) = \overline{z}$ is nowhere differentiable. Since

$$\lim_{z\to z_0} \frac{\overline{z - z_0}}{z - z_0} = \lim_{z\to z_0} \frac{\overline{z} - \overline{z}_0}{z - z_0} = \lim_{w\to 0} \frac{\overline{w}}{w}$$

with $w = z - z_0$. We showed in 2.2.1 and 2.2.2 that this limit does not exist.

- We ask if there is a real function over the real numbers which is differentiable only at one point. Any smooth function except but one cusp, For example $f(x) = |x|, f(x) = \sqrt{|x|}$, is differentiable everywhere but at that point. So this is easy. However here is a function that is differentiable and continouous only at one point.

$$f(x) = \begin{cases} x^2, & \text{if} \quad x \quad \text{is rational} \\ -x^2, & \text{if} \quad x \quad \text{is irrational} \end{cases}$$

- Finally here is an example of a function that satisfes the Cauchy–Riemann conditions at a point but it is not differentiable at that point. Let

$$f(z) = \begin{cases} \frac{xy(x+iy)}{x^2+y^2}, & \text{if} \quad z \neq 0 \\ 0, & \text{if} \quad z = 0 \end{cases}$$

then f satisfies the Cauchy–Riemann conditions at the origin $z = 0$, yet f is not differentiable at the origin. Let us see

$$u(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2}, & \text{if} z \neq 0 \\ 0, & \text{if} z = 0 \end{cases} \qquad v(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2}, & \text{if} \quad z \neq 0 \\ 0, & \text{if} \quad z = 0 \end{cases}$$

so, for $z \neq 0$, we evaluate the partial derivative with respect to x thinking that we are approaching $z \to 0$ along the x axis, so that $y = 0$.

$$u_x = \frac{\cancel{2(x^2} + y^2)xy - \cancel{2x^3}y}{(x^2+y^2)^2} = \frac{2xy^3}{(x^2+y^2)^2}.$$

In any case for $z \neq 0$ and $y = 0$, $u_x = 0$. Similarly

$$v_y = \frac{2(x^2 + \cancel{y^2)x}y - \cancel{2xy^3}}{(x^2+y^2)^2} = \frac{2x^3 y}{(x^2+y^2)^2}$$

and if $x = 0$, $y \neq 0$, $v_y = 0$ so so both u_x and v_y converge to 0 as $z \to 0$. The first of the Cauchy–Riemann conditions is satisfied at $z = 0$, however away from zero this is not the case. Otherwise we would have

$$2xy^3 = 2x^3 y \Rightarrow y^2 = x^2.$$

which is only valid along the two crossing lines $y = \pm x$. In the same way, for $z \neq 0$ we find

$$u_y = \frac{x^2(x^2-y^2)}{(x^2+y^2)^2} \qquad v_x = \frac{y^2(y^2-x^2)}{(x^2+y^2)^2}$$

We see that along the two lines $x^2 = y^2$ both $u_y = v_x = 0$, so the Cauchy–Rieamann conditions are satisfied along these two lines and particularly at the origin (the intersection of the two diagonal lines through the origin). The function $f(z)$ satisfies the Cauchy–Riemann conditions at the origin. However, the origin $z = 0$ is a singular point. While the limit for the partial derivatives along the coordinate axes goes to 0, this is not the case we approach the origin from different directions. As a whole, the differential does not exist. The function is not holomorphic.

Chapter 4

Integration

We introduced the concept of holomorphic by using all we could from real functions of real variables (why to re–invent the wheel?)

4.1 Complex function real interval

The idea in integration is not very different. Let us start by assuming a curve as a continuous complex valued function $\phi : [a, b] \in \mathbb{R} \subset \mathbb{R} \to \mathbb{C}$, and define

$$\int_a^b \phi(t)dt = \int_a^b \mathrm{Re}\ \phi(t)dt + \mathrm{i} \int_a^b \mathrm{Im}\ \phi(t)dt$$

4.2 Complex function along a path in the complex plane

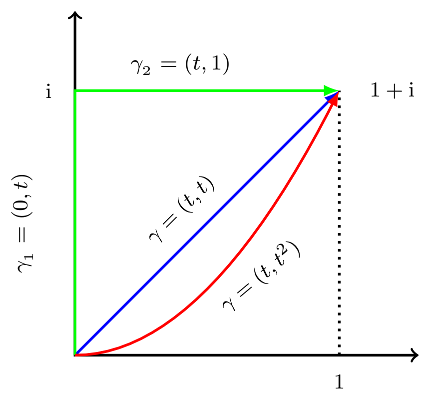
The previous section showed the integration of a function defined in a closed interval on the real $\mathbb{R}$ numbers. To extend it to an integral over a curve in the complex plane, let us assume a curve

$$\gamma : [a, b] \in \mathbb{R} \to \mathbb{C}$$

and define

$$\int_\gamma f = \int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Figure 4.1: Paths joining the point the complex number 0 with $1+i$, paths are parameterized as in $\mathbb{R}^2$.



If $\gamma(t)$ is only piece–wise differentiable in the subintervals

$$[a, c_1], [c_1, c_2], \cdots, [c_{n-1}, c_n], [c_n, b],$$

[1] then we write

$$\int_\gamma f = \int_a^{c_1} f(\gamma(t))\gamma'(t)dt + \int_{c_1}^{c_2} f(\gamma(t))\gamma'(t)dt + \cdots + \int_{c_n}^{b} f(\gamma(t))\gamma'(t)dt.$$

For example, let us assume $f(z) = \overline{z}$. We showed above that this function is nowhere differentiable.

Let us use three different paths that connect the origin with the point $1 + \mathrm{i}$, shown in Figure 4.1

- $\gamma = t + \mathrm{i}t$ is the line segment (blue in Figure 4.1) from $z = 0$ to $1 + \mathrm{i}$, $\gamma'(t) = 1 + \mathrm{i}$, so $\overline{z} = t - \mathrm{i}t$ and

$$\int_\gamma f = \int_0^1 (t - \mathrm{i}t)(1 + \mathrm{i})dt = \int_0^1 2\,tdt = 2.$$

[1] Mathematicians are good at finding counter-examples. In this document γ is a rectifiable path. That is, a curve for which is length can be computed and is finite. It should be "sufficiently" smooth but we will not stop in details such as what "sufficiently" means. For example a finite number of joined smooth (differentiable) segments would be a good curve for our curve.

- $\gamma = t + \mathrm{i}t^2$ is the arc of parabola (red in Figure 4.1) from $z = 0$ to $1 + \mathrm{i}$, $\gamma'(t) = 1 + 2\mathrm{i}t$
$$\int_\gamma f = \int_0^1 (t - \mathrm{i}t^2)(1 + 2\mathrm{i}t)dt = \int_0^1 (t + \mathrm{i}t^2 + 2t^3)dt = 1 + \frac{\mathrm{i}}{3}.$$
- γ consists of the two segments (green in Figure 4.1) from the origin to i and from i to 1+i. Parameterizations are it for the first with $\gamma'(t) = \mathrm{i}dt$ and $t + \mathrm{i}$ for the second with $\gamma'(t) = dt$. So
$$\int_\gamma f = \int_0^1 (-\mathrm{i}t)\mathrm{i}dt + \int_0^1 (t - \mathrm{i})dt = 1 - \mathrm{i}.$$

So we see that the integral depends on the path that joints the end points $z_0 = 0$ and $z_1 = 1 + \mathrm{i}$. This is known in physics as a non conservative field.

4.2.1 Path parametrization independence. Arc length

While the path integral could depend on the selected path between two end points, it should not depend on the parametrization of the path of integration. If the path integral depends on the path parametrization it would not be well defined. We show that the path integral is well defined.

Assume that $z_1(t)$, $z_2(t)$ are equivalent parametrization of a smooth curve γ, then both yield the same path integral of any function $f(z)$ along γ. To prove this, let us assume $t(s) : [c, d] \to [a, b]$ re-parameterizes $z_2(t), c \le t \le d$, to $z_1(t)$, $a \le t \le b$, so that $z_1(t(s)) = z_2(s)$, Then using the change of variables $t = t(s)$ on integrals over a real variable, we get

$$\int_a^b f[z_1(t)]z_1'(t)dt = \int_c^d f[z_1(t(s))]z_1'[t(s)]t'(s)ds,$$

but, from the chain rule

$$z_1[t(s)] = z_2(s) \Rightarrow z_1'(t(s))t'(s) = z_2'(s),$$

so

$$\int_a^b f[z_1(t)]z_1'(t)dt = \int_c^d f[z_2(s)]z_2'(s)ds.$$

This theorem is true for general path integrals along a path $\gamma : [a, b] \subset \mathbb{R} \to \mathbb{R}^n$ and the proof follows the same lines.

From calculus , the length, if it exists, of a smooth curve $\gamma : [a, b] \subset \mathbb{R} \to \mathbb{R}^n$ is given by

$$s = \int_a^b |\gamma'(t)| dt$$

for any parametrization $\gamma(t), a \le t \le b$ of γ. This definition is valid also in the complex plane.

Two interesting examples are

- Pythagoras Theorem: Let us find the length of the segment from the origin $(0, 0)$ to the point $z = x + \mathrm{i}y$, $x > 0$. We can parametrize $z(t) = t + \mathrm{i}\, t(y/x)$, $0 \le t \le x$, so $\gamma'(t) = 1 + \mathrm{i}(y/x)$

  $$s = \int_0^x |\gamma'(t)| dt = \int_0^x \sqrt{1 + y^2/x^2} dt = x\sqrt{1 + y^2/x^2} = \sqrt{x^2 + y^2}.$$

  Actually to find the magniture of the vector $\gamma'(t)$, the Pythagoras theorem (which is the Euclidean norm) is used. This is circular reasoning. It is more a verification than a proof of anything.

- Length of the circumference. The path $[0, 2\pi] \subset \mathbb{R} \to \mathbb{C} : \gamma(t) = r\mathrm{e}^{\mathrm{i}t}$ represents a circumference with radius r. We have

  $$\gamma'(t) = r\mathrm{i}\mathrm{e}^{\mathrm{i}t}.$$

  so

  $$s = \int_0^{2\pi} |\gamma'(t)| dt = \int_0^{2\pi} r|\mathrm{e}^{\mathrm{i}t}| dt = 2\pi r.$$

Chapter 5

The Cauchy's Theorems

5.1 Introduction

The Cauchy's integral theorems are at the hearth of complex variables. Before stating and proving them, let us make a small introduction.

Following the line of borrowing whatever we can from what we know about real variables let us review of of the fundamental theorems of calculus.

Let $F(x)$ be a function $F : [a, b] \in \mathbb{R} \to \mathbb{R}$. If $F'(x)$ exists and is continuous in a $[a, b]$ then

$$\int_a^b F'(x)dx = F(b) - F(a). \tag{5.1.1}$$

What this fundamental theorem of calculus is telling us, is that the evaluation of the integral of a differential over a "volume" is the contribution of the "surface" boundary points. In the real line, the interval has only two end points, and the normals to this boundary point to opposite directions, that is the reason of the "minus" sign. The generalization of this theorem of n dimensions is the divergence theorem. That is, if $\boldsymbol{F} \in \mathbb{R}^n \to \mathbb{R}^n$ is a vector field with continuous divergence, then

$$\oint_V \nabla \cdot \boldsymbol{F} dV = \oint_{\partial V} \boldsymbol{F} \cdot d\boldsymbol{S} \tag{5.1.2}$$

Again, the integral of a differential (the divergence) of a field over a volume is the sum of all contributions in the surface of that volume for that field. Integration cancels one dimension along the normal to the surface. It is easier to understand by considering the volume element as the product of the surface element and a normal to the surface element, then in this new parametrization the derivative will act along that normal and could be seen as the one–dimensional problem along the normal to the surface. Equation 5.1.2 is known in the mathematical physics literature as the divergence theorem or the Gauss theorem.

This problem is a natural problem of differential geometry and more especifically in the theory of manifolds where mathematicias write

$$\int_M d\omega = \int_{\partial M} \omega.$$

where M is a manifold (a piece of $\mathbb{R}^n$) and ∂M is an orientable boundary of that manifold, ω is a differential form, and $d\omega$ is known as the exterior derivative. This is the generalized version of Stoke's theorem.

In $\mathbb{R}^2$ a piece of $\mathbb{R}^2$ is a closed region R. Along that boundary of the region the tangent vector elements are

$$(dx, dy)$$

so the outside normal vector follows the direction

$$(dy, -dx),$$

and the Gauss theorem 5.1.2 becomes in 2D

$$\int_R [(F_1)_x + (F_2)_y]\, dx\, dy = \int_{\partial R} F_1 dy - F_2 dx.$$

This theorem is usually written as [1]

$$\oint_C (Ldx + Mdy) = \int_D (M_x - L_y) dxdy, \tag{5.1.3}$$

where $L = F_1$ and $M = -F_2$. Observe that this is also Stoke's theorem when the surface is a piece of the $\mathbb{R}^2$ plane.

Recall the examples illustrated in Figure 4.1, where by selecting different paths we can get different integration values. Wouldn't it be nice if we could say that the integral is path independent? (conservative). That is, that the integral depends only on the end points of the curve, in the same way as in the equation 5.1.1? and if this possible what conditions we should put to the function F? This is the extension from the real, to the complex fundamental theory of calculus. Here is the statement of the theorem:

[1] http://en.wikipedia.org/wiki/Green's_theorem

5.2 Extension of the Fundamental Theorem of Calculus

Let $f(z) = F'(z)$ be the derivative of a single-valued complex function $F(z)$ defined on a domain $U \subset \mathbb{C}$. Let γ be any curve with initial point a and final point b. Then

$$\int_C f(z)dz = \int_C F'(z)dz = F(b) - f(a).$$

This is a natural extension of the real case to the complex. From the definition of integral over a path (contour integral), we have that

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b F'(z(t))z'(t)dt$$

and from chain rule and the real fundamental theorem of calculus

$$\int_C f(z)dz = \int_a^b F'(t)dt = F(b) - F(a).$$

if $F = u + \mathrm{i}v$, use the fundamental theorem of calculus in both u and v.

This sounds good. However if finding the primitive (or antiderivative) of a real function of real value could be hard, doing the same in a complex function of a complex variable is not easier.

The Fundamental Theorem of Complex Analysis suggest that if we take two different paths to go from a point a to a point b, we can reverse, one of the paths and then get a loop , and since reversing the path, reverses the sign of the integral, we would find that the integral from a to b, is the same as that from b to a with the sign reversed. Then, the integral around the loop should zero. In this case we say that

$$\int_C f(z)dz = 0 \quad \text{where } C \text{ is a closed contour.}$$

We would like to know under which assumptions on f is this true. This is precisely the Cauchy's Integral theorem. [2].

[2]http://en.wikipedia.org/wiki/Cauchy%27s_integral_theorem

5.3 Cauchy's Integral Theorems

5.3.1 Introduction

We present a series of results derived from Cauchy's integral formulas which not only are useful from the computational point of view, but present a framework for theoretical development on analytic functions.

Initially we show that the integral of a holomorphic function along a closed loop is zero. Then we use this result to find a representation of the function $f(z)$ which depends only on the points along the boundary of the region of analicity considered. Then with this representation comes its derivatives which can be extended up to infinity. In the real numbers derivatives can stop soon due to discontinuities, in complex analysis, as long as a function is holomorphic, it has an infinite number of derivatives. With these derivatives, a power series can be developed (Taylor series) which shows the equivalence between power series expansions and holomorphism. We use the results so far obtained to to find the inverse of a holomorphic function, and prove the important maximum principle, Liouville's theorem, and the fundamental theorem of algebra.

5.3.2 The Integral of a Holomorphic Function in a Closed Loop

Let U be an open subset of the complex numbers $\mathbb{C}$ which is simply connected [3]. If $f : U \to \mathbb{C}$ is holomorphic, and γ is a rectifiable closed path (see footnote in page 20), then

$$\oint_\gamma f(z)dz = 0.$$

where the symbol $\oint$ indicates that the path γ is a single closed loop.

As indicated in the Wikipedia page above, Cauchy used Green's theorem from vector calculus, but Gousart provided a proof that did not require vector calculus, neither continuity of partial derivatives [4] Still Wikipedia uses Green's theorem to prove this theorem. The idea here is to split $f = u + iv$

[3] we do not want here regions with holes. There are generalizations of this theorem for regions with holes but this is outside the scope of this document

[4] Hence sometimes this is known as the Cauchy-Gosart's theorem.

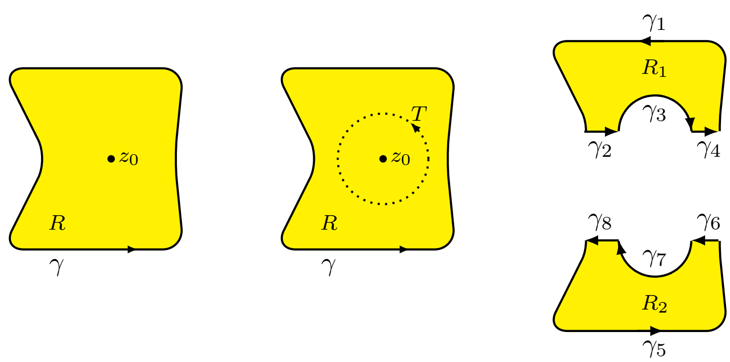


Figure 5.1: On the left we see the original region R with its boundary γ and the point inside z_0. On the center we think about a small circle around point z_0 with boundary T. On the right we see the region R minus the disc with boundary T, splitted into two regions R_1 and R_2. The γ_i curves, $i = 1, \ldots, 8$ describe the countours of integration to prove the result.

and $z = x + \mathrm{i}y$. Then expand the product into the four terms, callect real an imaginary parts. Each part (real and imaginary) looks like the Green's theorem formula 5.1.3. Apply Green's theorem changing the functions into their partial derivatives and then apply the Cauchy–Riemann conditions. Each part (real and imaginary) will turn to be identically 0.

5.3.3 The Cauchy's Integral Formula

Together with Cauchy's integral theorem comes Cauchy's Integral Formula [5]. This is, provided that f is holomorphic in a region R, such that $z_0 \in R$,

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial R} \frac{f(z)}{z - z_0} dz.$$

The proof is simple and it can be followed with the help of Figure 5.1.

(i) Start with the region R in yellow with the point z_0 inside.

(ii) Make a small circle with center at z_0 inside the region. Center frame.

[5] http://en.wikipedia.org/wiki/Cauchy%27s_integral_formula

(iii) Remove the circle and split what is left into two regions as shown in the right side.

The details are in the third step which we expand here. We call the boundary of the disk T. The curve γ is the boundary of the region R. Note that

- $\gamma = \gamma_1 \cup \gamma_5$.
- $-T = \gamma_3 \cup \gamma_7$ where we write the minus "-" sign on $-T$ to indicate that T is defined as clockwise while the curve $\gamma_3 \cup \gamma_7$ is going counter-clockwise.
- $\gamma_2 = -\gamma_8 \qquad \gamma_4 = -\gamma_6$ where the minus "-" sign means that the curve is integrated along the reverse direction.

Then

$$\int_{\partial R_1} \frac{f(z)}{z-z_0} dz + \int_{\partial R_2} \frac{f(z)}{z-z_0} dz = \int_{\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4} \frac{f(z)}{z-z_0} dz + \int_{\gamma_5 \cup \gamma_6 \cup \gamma_7 \cup \gamma_8} \frac{f(z)}{z-z_0} dz$$

and since the function $f(z)/(z-z_0)$ is holomorphic in both R_1 and R_2 (note that the pole z_0 is outside of these regions), the integral is zero. That is

$$\int_{\bigcup_{i=1}^{8} \gamma_i} \frac{f(z)}{z-z_0} dz = 0.$$

We note that since the integral along the region γ_2 cancels that along the region γ_8, and so the integral over the region γ_4 cancels that along the region γ_6, we reduce this integral to the integral along the regions

$$\gamma_1 \cup \gamma_5 \cup \gamma_3 \cup \gamma_7 = \gamma - T$$

That is, we can write

$$\int_\gamma \frac{f(z)}{z-z_0} dz = \int_T \frac{f(z)}{z-z_0} dz.$$

This means that we can deform the contour of integration by shrinking it to a small circle, as long as we do not go over any singular points. In this way we could also compute integrals with several singularities.

The integral along T is easier to compute form a change of variables. It is natural to think about the small circle as $|z - z_0| = \epsilon e^{i\theta}$ for ϵ being the radius of the circle (which fortunatelly is constant) and the angle θ going between 0 and 2π (and this is where the 2π comes from). So for $z = z_0 + \epsilon e^{i\theta}$, $dz = i\epsilon e^{i\theta}$, and

$$\int_T \frac{f(z)}{z - z_0} dz = \int_0^{2\pi} \frac{f(z_0 + \epsilon e^{i\theta})}{\cancel{\epsilon e^{i\theta}}} i\cancel{\epsilon e^{i\theta}} d\theta = i \int_0^{2\pi} f(z_0 + e^{i\theta}) d\theta.$$

Finally, since the function f is holomorphic we can take the limit as $\epsilon \to 0$ and find that that

$$\int_\gamma \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0),$$

or in other words

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz. \tag{5.3.4}$$

It is common to recur to the mean value theorem, and say that the integral is given by some value $2\pi f(z_1)$, where z_1 is a value inside the circle having z_0 as center, and since that circle can shrink as much as we can, the only value that is always in that circle is its center z_0.

Note that the representation 5.3.4 is telling us that any point inside an holomorphic region can be expressed as an integral along the boundary of the region. So, knowing the values of the function along the boundary is enough to know any point inside the region.

5.3.4 The Power Series of a Holomorphic Function

We discuss more about power series in chapter 7. Here we introduce both the Taylor first and then the Laurent series, derived from the Cauchy Integral formula 5.3.4, and the geometrical series expansion of $1/(1 - z)$.

We can rewrite the representation function 5.3.4 as

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{\xi - z} d\xi. \tag{5.3.5}$$

from which we can take its derivative with respect to z as follows,

$$f'(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(z)}{(\xi - z)^2} d\xi.$$

and in general, after i derivatives

$$f^{(i)}(z) = \frac{i!}{2\pi \mathrm{i}} \int_\gamma \frac{f(z)}{(\xi - z)^{i+1}} d\xi. \tag{5.3.6}$$

This says that if f is holomorphic in a region R, then we can take as many derivatives as we want. That is $f \in C^\infty(R)$.

Here is an interesting result. Let us assume that f is holomorphic in some region and z_0 is an interior point over which we can enclose a circle inside the region. We write $\xi - z = (\xi - z_0) + (z - z_0)$, and

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\xi)}{(\xi - z_0) - (z - z_0)} d\xi.$$

with the purpose of expand f as a power series in $z - z_0$. That is, we can write

$$\frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0} \sum_{i=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^i, \tag{5.3.7}$$

which is a power series which converts as long as

$$\left| \frac{z - z_0}{\xi - z_0} \right| < 1$$

Since power series are uniformly convergent we can then write

$$f(z) = \frac{1}{2\pi \mathrm{i}} \sum_{i=0}^{\infty} \frac{f(\xi)}{\xi - z_0} \left(\frac{z - z_0}{\xi - z_0} \right)^n d\xi.$$

What this is telling us, is that if f is holomorphic in a region R and z_0 is a point on that region, it can be expanded as an infinite (power) series

$$f(z) = \sum_{i=0}^{\infty} a_i (z - z_0)^i$$

with

$$a_i = \frac{f^{(i)}(z_0)}{i!} = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{i+1}} d\xi.$$

Note that we came to this expression by using the geometrical series expansion in equation 5.3.7. We could have also arrived to this expression using Taylor series expansion with coefficients $a_i = f^{(i)}(z_0)/i!$ since we know from equation 5.3.6 we how how to take the i^{th} derivative of f at the point z_0. The interesting thing about this result is that we showed that a holomorphic function can be expanded as a power series with a radius of convergence. Of course power series are holomorphic on their radius of convergence, so we have the equivalence of holomorphic and power series expansion. The holomorphic concept was developed using the Cauchy-Riemann identities from the definition of differentiation of a complex function. The power series was taken by Weirstrass as the point of the departure for analytic functions.

5.3.5 Laurent Series

While the Taylor series works in a disk $|z - z_0| < r$, there is an extension with negative powers terms which reduces the circular disc to a circular ring. That is, let us consider the annulus

$$\mathcal{A} = \{z \in \mathcal{C} : r_0 < |z - z_0| < r_1\}.$$

with $0 < r_1 < r_2 < \infty$. We will perform contour integration as follows. Consider the boundary of the annulus $\gamma = \partial A$ as given by the two circles $\gamma = \{z : |z - z_0| = r_j, j = 0, 1\}$ and we consider each circle in the counter-clockwise direction. Figure 5.2 illustrates this. We then assume that f is analytic in the ring $\mathcal{A}$, including its boundary $\gamma = \gamma_1 - \gamma_0$. Then from Cauchy's integral theorem we can represent

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi \mathrm{i}} \int_{\gamma_0} \frac{f(\xi)}{\xi - z} d\xi \qquad (5.3.8)$$

[6]

[6] To understand why the integral over the ring is the subtraction of the two integrals over the circumferences, see Figure 5.1 and imagine that the region R is a disk with center at z_0. The path that cut from the internal to the external circumferences integrates to 0.

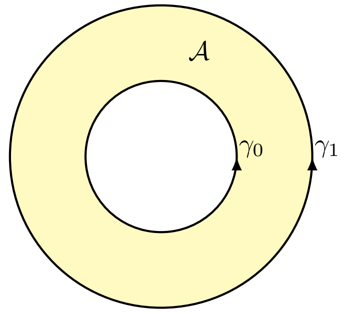


Figure 5.2: Ring of validity for Laurent Series

As with the Taylor series above we use the geometric series expansion for the fraction terms inside the integral signs. That is,

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^k \quad , \quad \xi \in \gamma_1,$$

and

$$\frac{1}{\xi - z} = -\frac{1}{(z - z_0) - (\xi - z_0)} = -\frac{1}{z - z_0} \sum_{k=0}^{\infty} \left(\frac{\xi - z_0}{z - z_0} \right)^k \quad , \quad \xi \in \gamma_0.$$

We replace these series into the integral representation 5.3.8 to find

$$\begin{aligned} f(z) &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1} d\xi f(\xi) \left[\frac{1}{\xi - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^k + \frac{1}{z - z_0} \sum_{k=0}^{\infty} \left(\frac{\xi - z_0}{z - z_0} \right)^k \right] \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1} d\xi f(\xi) \left[\sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\xi - z_0)^{k+1}} + \sum_{k=0}^{\infty} (\xi - z_0)^k (z - z_0)^{-k-1} \right] \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1} d\xi f(\xi) \left[\sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\xi - z_0)^{k+1}} + \sum_{k=-1}^{-\infty} (\xi - z_0)^{-k-1} (z - z_0)^k \right]. \end{aligned}$$

Hence we write

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k. \tag{5.3.9}$$

with

$$c_k = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1} \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi \quad , \quad k \in \mathbb{Z}. \tag{5.3.10}$$

5.3.6 Residues

Let us assume the Laurent series representation 5.3.9

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k.$$

for $0 < |z - z_0| < r$. Let us assume a pole of order i on z_0. That is, the sum above is

$$f(z) = \sum_{k=-i}^{\infty} c_k (z - z_0)^k. \tag{5.3.11}$$

The coefficient c_{-1}, is termed the **residue of** $f(z)$ **at** z_0 and it is abbreviated as $\mathrm{Res}_{z_0}(f)$. From equation 5.3.10, this coefficient has the integral representation

$$\mathrm{Res}_{z_0}(f) = c_{-1} = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} f(\xi) d\xi.$$

This coefficient is very important because the integral of f in the closed curve γ can be computed by the formula

$$\int_{\gamma} f(\xi) d\xi = 2\pi \mathrm{i} \mathrm{Res}_{z_0}(f).$$

In other words the integral of a function can be computed from its expansion as a Laurent series. In the same way that the coefficients of a Taylor series represent derivatives (with some factorial divisors), the negative terms of the Laurent series (the mirror coefficients) are integrals with some product factorials. We will visit this statement later.

The analysis done so far was done for a single pole z_0. Let us assume a function with, for example, three isolated poles

$$\frac{f(z)}{(z - z_0)(z - z_1)(z - z_2)}$$

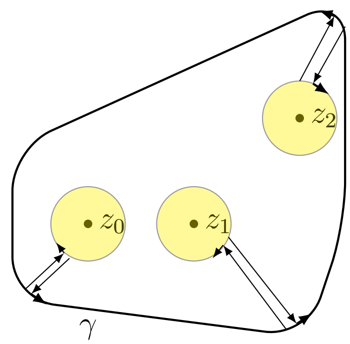


Figure 5.3: Three poles at z_0, z_1, z_2. Small circles around the poles together with the segments joint the circles with the outside contour define, following the arrows a contour to compute the integral of the function $1/(z - z_0)(z - z_1)(z - z_2)]$ along the outside contour γ.

Figure 5.3 shows an example where the three poles z_0, z_1, and z_3 are sketched with three small circles around them. The integral around the outside contour can be put in terms of the integrals around the three circles as follows. Pick any point in the outside contour (boundary of the region of integration) and follow the arrows. The region enclosed by the contour labeled by the arrows has no singularities and so the integral along that contour is 0 by the Cauchy theorem. The integral along the 6 straight segments is 0 because they come in equal pairs with opposite directions. Then the integral along the contour labeled by the arrows which is the integral over the outside border minus (because the three circles are oriented counter-clockwise) the integral along the three circles is 0. That is, the integral along the outside border is the same as the integral along the three circles. The idea works the same for any finite number of poles. That is, if there is a region with a finite number of poles, the integral is given by the contribution of the integrals along circles surrounding those poles.

Let us put this into our specific example. For the pole z_0, the function

$$f(z) = \frac{1}{(z - z_0)(z - z_1)(z - z_2)} = \frac{1}{z - z_0}\left[\frac{1}{(z - z_1)(z - z_2)}\right] = \frac{1}{z - z_0} g(z).$$

For the circle C_0 around z_0 the expression in square brackets is analytic, and we call it $g(z)$, so

$$\int_{C_0} \frac{g(z)}{z - z_0} = 2\pi \mathrm{i} g(z_0) = \frac{2\pi \mathrm{i}}{(z_0 - z_1)(z_0 - z_2)}$$

where $g(z_0)$ is the residue for pole z_0. Similarly we have

$$\begin{aligned} \mathrm{Res}_{z_1}(f) &= \frac{1}{(z_1 - z_0)(z_1 - z_2)} \\ \mathrm{Res}_{z_2}(f) &= \frac{1}{(z_2 - z_0)(z_2 - z_1)}, \end{aligned}$$

and so the integral along the whole region enclosing the three points is given by

$$\int_\gamma f(z)dz = \sum_{i=0}^{2} \mathrm{Res}_{z_i}(f) = 2\pi \mathrm{i} \sum_{i=0}^{2} \frac{1}{\prod_{j \neq i}(z_i - z_j)}.$$

In general for any finite number of simple poles (say k, for example), the formula

$$\int_\gamma f(z)dz = \sum_{i=0}^{k} \mathrm{Res}_{z_i}(f)$$

is valid. It is common to see integration over semicircles which radius approaches infinity and an infinite number of poles generating an infinite series of residues. This problem can be seen more formally as a limit of finite semicircles having a finite number of residues. For example think about the function $f(z) = \pi \cos \pi z / \sin \pi z$, which has an infinite number of zeroes at $z = k$, $k = 0, \pm 1, \pm 2 \cdots$. This example is discussed here [7]. The book The Laplace Transform: Theorey and Applications [8] by Joel L. Schiff, shows an example of the computation of the inverse Laplace using an infinite number of poles.

The discussion above assumes that we have a simple pole or a collection of simple poles. Equation 5.3.11 shows the case of $f(z)$ with a pole of multiplicity i, at z_0. To extract the coefficient c_{-1} from this equation we multiply

[7] http://www.math.odu.edu/ jhh/ch95.PDF
[8] https://www.google.com/search?tbo=p&tbm=bks&q=isbn:0387227571

$f(z)$ by $(z - z_0)^i$ to find

$$f(z)(z - z_0)^i = \sum_{k=-i}^{\infty} c_k (z - z_0)^{k+i} = \sum_{k=0}^{\infty} c_{k-i} (z - z_0)^k$$

Call $b_k = c_{k-i}$, and this is a regular Taylor series. From regular Taylor series we know that

$$b_k = \frac{1}{k!} \frac{d^{(k)}}{dz^k} [f(z)(z - z_0)^i] \bigg|_{z=z_0}$$

The coefficient c_{-1} implies $k - i = -1$, $k = i - 1$, so

$$c_{-1} = b_{i-1} = \frac{1}{(i-1)!} \frac{d^{(i-1)}}{dz^{i-1}} [f(z)(z - z_0)^i] \bigg|_{z=z_0} .$$

5.3.7 The Inverse of a Holomorphic Function

A further application of the Cauchy's integral formula can be used to invert for a function. That is, let $f : R \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic and one-to-one function. We can think of a function $g : f(R) \to R$ such that $g \circ f(z) = z$, $\forall z \in R$. Let us use w for the domain of g and z for the domain of f. From the Cauchy's theorem, assuming that there is such a g, it should satisfy

$$g(w) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_f} \frac{g(u)}{u - w} du.$$

where γ_f is a path around $f(R)$. We then make the substitution $u = f(z)$, then $du = f'(z)dz$, $f^{-1}(u) = g(u) = z$ and

$$g(w) = \frac{1}{2\pi i} \int_{\gamma_f} \frac{z f'(z)}{f(z) - w} dz$$

So this should be the inverse of $f(z)$. We can further verify this as follows: Since f is holomorphic and invertible, for each $w \in \gamma_f$, $f(z) - w$ has a unique zero $f^{-1}(w) := z_0$. Hence, $f(z) = w + (z - z_0)h(z)$ on R, with $h(z)$ holomorphic in R. That $f(z) - w$ has a unique zero, is because $f(z)$ is a one-to-one function (at least locally around z_0). Hence

$$h(z) = \frac{f(z) - w}{z - z_0}$$

is holomorphic. The only point which could make the function h non holomorphic is $z = z_0$, but $f(z_0) - w = 0$, and since the zero is unique,

$$h(z_0) = \lim_{z \to 0} \frac{f(z) - w}{z - z_0} \neq 0$$

and finite.

So $f'(z) = h'(z)$, and $f(z) - w = h(z)(z - z_0)$, from which

$$f'(z) = h'(z)(z - z_0) + h(z),$$

and

$$\begin{aligned} g(w) &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_f} \frac{z f'(z)}{f(z) - w} dz \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_f} \frac{z(h'(z)(z - z_0) + h(z))}{h(z)(z - z_0)} dz \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_f} z \left(\frac{1}{z - z_0} + \frac{h'(z)}{h(z)} \right) dz \\ &= z_0 = f^{-1}(w). \end{aligned}$$

5.4 The Maximum Principle, Liouville's Theorem, the Fundamental Theorem of Algebra, and Complete Characterization of Holomorphic Functions

5.4.1 The Maximum Principle

5.4.2 Liouville's Thereom

5.4.3 The Fundamental Theorem of Algebra

5.4.4 The Complete Characterization of Holomorphic Functions

Up to this point we have seen several concepts that connect holomorphic functions. We formalize the connection of them in the following:

Theorem 5.4.5 *Assume that U is an open connected set $U \subset \mathbb{C}$, and $f : U \to \mathbb{C}$ be $C^1(U)$. We show that the following statements are equivalent:*

(i) *f satisfies the Cauchy-Rieamann equations on U.*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

with $f(z) = u(z) + \mathrm{i}v(z) \quad , \quad z \in U$.

(ii) $\frac{\partial f}{\partial \bar{z}} = 0 \quad z \in U$,

(iii) *$f'(z)$ exist $\forall z \in U$. That is, f is complex differentiable on every point $z \in U$.*

(iv) *f can be expanded locally as a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ at every point $z_0 \in U$.*

(v) *For every piecewise C^1 curve $\gamma : [0, 1] \to U$ with $\gamma(0) = \gamma(1)$ (that is γ is closed)*

$$\int_\gamma f(z)dz = 0.$$

Proof: We show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i)$.

Chapter 6

Harmonic Functions

Let us assume that f is C^2, holomorphic. From the Cauchy–Riemann conditions 3.2.2

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

where we used the continuity of the v_{xy} and v_{yx}. We see that the u satisfy the Laplace equation $\Delta u = 0$, that is:

$$u_{xx} + u_{yy} = \Delta u = 0.$$

Along the same lines it is easy to see that

$$v_{xx} + v_{yy} = \Delta v = 0.$$

We rewrite equation 3.0.1 in subindex notation.

$$f_z = \frac{1}{2}\left(f_x - \mathrm{i} f_y\right) \qquad f_{\bar{z}} = \frac{1}{2}\left(f_x + \mathrm{i} f_y\right)$$

so

$$\begin{aligned} f_{\overline{z}z} &= \frac{1}{2}[(f_{\overline{z}})_x - \mathrm{i}(f_{\overline{z}})_y] \\ &= \frac{1}{2}\left[\frac{1}{2}(f_{xx} + \mathrm{i} f_{yx}) - \mathrm{i}\,\frac{1}{2}(f_{xy} + \mathrm{i} f_{yy})\right] \\ &= \frac{1}{4}(f_{xx} + f_{yy}) \quad \text{if} \quad f_{xy} = f_{yx} \end{aligned}$$

and since $\Delta u = \Delta v = 0$ and partial differentiation is a linear operator, then

$$f_{xx} + f_{yy} = 0.$$

That is, f, u and v satify all the Laplace equation

$$f_{z\bar{z}} = \Delta f = \Delta u = \Delta v = 0.$$

They are all **harmonic.** The equation for f is complex with right hand side $0 = 0 + \mathrm{i}0$ and those for u and v are real. We also write

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{\partial^2 f}{\partial \bar{z} \partial z} = \frac{1}{4} \Delta f = 0. \tag{6.0.1}$$

We found that if a function $f = u + \mathrm{i}v$ is C^2 holomorphic then its real (u) and imaginary (v) components are harmonic. We ask if the opposite is true. That is, can we say that if u and v are harmonic then $f = u + \mathrm{i}v$ is holomorphic? This is not the case. For example, let us define

$$f(z) = u(z) + \mathrm{i}v(z) = \ln(x^2 + y^2).$$

for $z = x + \mathrm{i}y \in C \backslash \{0\}$. Since $v(z) = 0$, v is harmonic. Now let us show that $u(z)$ is harmonic in $C \backslash \{0\}$.

$$\begin{aligned} u_x &= \frac{2x}{x^2+y^2} \quad u_y = \frac{2y}{x^2+y^2} \\ u_{xx} &= \frac{2\,(x^2+y^2) - 4x^2}{(x^2+y^2)^2} = \frac{-2(x^2-y^2)}{(x^2+y^2)^2} \\ u_{yy} &= \frac{2(x^2+y^2) - 4y^2}{(x^2+y^2)^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2} \end{aligned}$$

so

$$u_{xx} + u_{yy} = 0.$$

So both u and v are harmonic. However f is not holomorphic since

$$u_x = \frac{2x}{x^2+y^2} \neq 0 = v_y = 0,$$

for $x \neq 0$. To fix this an additional constraint should be added. What breaks the holomorphic attribute is the singularity at $z = 0$. If the region of interest does not contains singularities then we can show that the harmonic behaviors are equivalent to the holomorphic attribute.

Assume that the domain of definition U is a **simply connected** region. Then we can say that if u is harmonic on a simply connected region U, there exists a harmonic function v such that $f = u + \mathrm{i}v$ which is holomophic in U. The function v is called a **harmonic conjugate** of u. So having u and v harmonic, does not help much. The combination $u + \mathrm{i}v$ should be holomorphic,

Here is an example taken from Wikipedia [1] which provides the $u(x, y)$ function (harmonic). We want to find $v(x, y)$, such that $f(x, y) = u(x, y) + \mathrm{i}v(x, y)$. Consdier $u(x, y) = \mathrm{e}^x \sin y$. That $u(x, y)$ can be verified by taking double derivatives. That is,

$$\begin{aligned} u_x &= \mathrm{e}^x \sin y \qquad u_y = \mathrm{e}^x \cos y \\ u_{xx} &= \mathrm{e}^x \sin y \qquad u_{yy} = -\mathrm{e}^x \sin y \end{aligned}$$

from which $u_{xx} + u_{yy} = \Delta u = \nabla^2 u = 0$. Now, to find $v(x, y)$ we use the Cauchy-Riemann equations. That is,

$$\begin{aligned} u_x &= v_y = \mathrm{e}^x \sin y \\ u_y &= -v_x = \mathrm{e}^x \cos y. \end{aligned}$$

We should solve first order coupled differential equation system

$$\begin{aligned} v_y &= \mathrm{e}^x \sin y \\ v_x &= -\mathrm{e}^x \cos y. \end{aligned}$$

for the conjugate function v. Integrating the first equation along y we see that $v = -\mathrm{e}^x \cos y + C$, we observe that this solution also satisfies the second equation. So any function of the type

$$f(z) = u(z) + \mathrm{i}v(z) = \mathrm{e}^x \sin y - \mathrm{i}\mathrm{e}^x \cos y$$

(add a constant if you wish) is holomophic and satisfies, together with its real and imaginary parts the Laplace equation (is harmonic).

Let us now turn to real polynomials and compare them with complex polynomials in the context of harmonic functions. Let $P(x, y)$ be a real–valued harmonic polynomial. There is a holomorphic polynomial Q such

[1] http://en.wikipedia.org/wiki/Harmonic_conjugate

that $P = \operatorname{Re} Q$. To prove this let us recall that

$$x = \operatorname{Re} z = \frac{z + \overline{z}}{2} \qquad y = \operatorname{Im} z = \frac{z - \overline{z}}{2\mathrm{i}}$$

so

$$P(x, y) = \sum a_{ij} x^i y^j = \sum b_{ij} z^i \overline{z}^j$$

for some complex coefficients b_{ij} which are linear combinations of the coefficients a_{ij}. From equation 6.0.1 we see that all factors with no zero coefficient b_{ij} for $i > 0$ and $j > 0$ get cancelled out. We should have

$$P(x, y) = b_{00} + \sum b_{0j} \overline{z}^j + \sum b_{i0} z^i \qquad i > 0, j > 0.$$

Since P is real–valued, $P = \overline{P}$, so

$$b_{00} = \overline{b_{00}} \quad (\text{so} \quad b_{00} \in \mathbb{R}) \qquad b_{i0} = \overline{b_{0i}}$$

so we can regroup

$$\begin{aligned} P(x, y) &= b_{00} + \sum_{i \geq 0} b_{i0} z^i + \sum_{i \geq 0} \overline{b_{i0}} \overline{z}^i \\ &= b_{00} + 2 \operatorname{Re} \sum_{i \geq 0} a_{i0} z^i \\ &= \operatorname{Re} Q(z). \end{aligned}$$

This rule allows us to characterize real–valued harmonic polynomials. For example, all real–valued harmonic polynomials of second degree should be of the form

$$\begin{aligned} P(x, y) &= \operatorname{Re} Q(z) = \operatorname{Re}(b_0 + b_1 z + b_2 z^2), \quad b_0 \in \mathbb{R}, \quad b_1, b_2 \in \mathbb{C} \\ &= b_0 + \operatorname{Re}\left[b_{1r} x + b_{2r}(x^2 - y^2) - 2b_{2i} xy - b_{1i} y) + \mathrm{i}(b_{1i} y + 2b_{2r} xy) \right] \\ &= b_0 + b_{1r} x - b_{1i} y + b_{2r}(x^2 - y^2) - 2b_{2i} xy. \end{aligned}$$

We will have much more to say about harmonic functions in the Hardy spaces section **??**

Chapter 7

Power Series

Some texts introduce analytic functions in a different way. For example: A function is said analytic [1] in a point z_0 if we can expand it as a Taylor series around z_0. That is,

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k. \tag{7.0.1}$$

The coefficient c_k is given by

$$c_k = \frac{f^{(k)}(z_0)}{k!},$$

with $f^{(k)}(z_0)$ being the k^{th} derivative of f at z_0. The radius of the largest circle around the point z_0 such that $|f(z)| < \infty$ is called the **Radius Of Convergence,** ROC . A function that is analytic in each point of its domain is said to be **holomorphic.** A function which is analytic for all complex numbers (infinite ROC) is defined as **entire.**

More specifically, we want to know that after certain N all but a finite number of powers $c_k^{1/k}(z - z_0)^n$ are bounded in size by 1. We just use the simplest power series and from it we learn about the radius of convergence. This is the power series

$$f(z) = \sum_{k=0}^{\infty} z^k, \tag{7.0.2}$$

[1] This definition can be weakened by just saying that an analytic function at a point has a derivative. Then it can be shown that if it has one derivative it has infinite number of derivatives.

from which we know that it converges uniformly if $|z| < 1$. We will have much to say about this power of series in the analytic continuation section 8.

Here are two ways to find the radius of convergence:

(i) **The Cauchy-Hadamard criterion:** Define $w = c_k^{1/k}(z - z_0)$, so we can rewrite 7.0.1 as 7.0.2 with w instead of z, where we know that the radius of convergence is 1. We write

$$S = \lim_{k\to\infty} \sup |w_k| = \lim_{k\to\infty} \sup \sqrt[k]{|c_k(z-z_0)^k|} = \lim_{k\to\infty} \sup \sqrt[k]{|c_k|}|z - z_0|$$

where S is the upper limit, or limit of the supremum or least upper bound of the tail. What the symbol $\lim_{k\to\infty} \sup |w_k| = S$, named the upper limit, and noted as $\overline{\lim}$ where the word "sup" is replaced by the line above the word "lim", is telling us is that except for a finite number of terms in the sum (which we do not care because they add to a finite number) the limit of the largest element in the shrinking tail is S. Note that is an upper bound so by controlling the upper bound we control the grow, and by picking the least we are narrowing the radius to get the least possible. We require that $S < 1$ and so we need

$$|z - z_0| < \frac{1}{\lim_{k\to\infty} \sup \sqrt[n]{|c_k|}},$$

or in the upper limit notation:

$$|z - z_0| < \frac{1}{\overline{\lim}_{k\to\infty} \sqrt[n]{|c_k|}}.$$

This is how we find the radius of convergence. That the radius of convergence is

$$r = \frac{1}{\overline{\lim}_{k\to\infty} \sqrt[n]{|c_k|}}.$$

is known as the Cauchy-Hadamard theorem [2] The proof presented in Wikipedia is as follows:

The radius of convergence is the distance from the center of the boundary of of the disk of convergence and a translation of the coordinate

[2] http://en.wikipedia.org/wiki/Cauchy%E2%80%93Hadamard_theorem

system will not change this radius, so without loosing generality we can assume after translating the coordinate system by z_0, $z_0 = 0$, to simplify writing. The meaning of radius of convergence is that $f(z) = \sum c_k z^k$ diverges of $|z| > r$ and converges for $|z| < r$. [3] The proof assumes first that $|z| < r$. By definition $S = \lim_{n\to\infty} \sup A = \overline{\lim}_{n\to\infty} a_k$, where $A = \{a_k : k \in \mathbb{Z}\}$ is a set of real numbers, if $\forall\epsilon > 0$, $S + \epsilon > a_k, \forall a_k \in A = \{a_k\}$, except by a finite number of points in C. That is for all but a finite number of coefficients a_k in the series, $a_k > S + \epsilon$. In our case $a_k = \sqrt[k]{|c_k|}$. We do not care about the finite number of terms. They add to a finite number. For the infinite number of terms remaining, their values are $\sqrt[k]{|c_k|} < S + \epsilon$, where

$$\frac{1}{S+\epsilon} < \frac{1}{S} = r,$$

and we know that in this region the series converges. This verifies the convergence case. Now by the same definition of upper limit as S, if we subtract ϵ, we get an infinite number of terms $\sqrt[k]{|c_k|}$ such that

$$\sqrt[k]{|c_k|} > S - \epsilon,$$

but

$$\frac{1}{S-\epsilon} > \frac{1}{S} = r,$$

so there is an infinite number of terms outside of the circle of convergence, for which the sum diverges.

[4].

[3]Note that nothing is said about $|z| = r$. It could converge or diverge. Again the simplest example $f(z) = \sum z^k$, diverges for $z = 1$, but converges in the unit circle for $z \neq 1$, to $1/(1-z)$.

[4]The concept of sup, (as well as the concept of inf) is needed because some sets do not reach out to their maximum. Whenever a set has a maximum, the maximum is the sup. For example the real interval $(0, 1)$ does not have a maximum, but $\sup(0, 1) = S = 1$. In this case $\forall\epsilon > 0$, 0 elements are larger that $1 + \epsilon$ in $(0, 1)$. You only need a tiny $\epsilon > 0$, to exit the set once you are standing on the sup of the set. The combination of the "sup" with the "upper limit" are needed because sequences and series (which are sequences) sometimes are iterative between various values (say for example $(-1)^k a_k, a_k > 0$. This could converge to two different limits and we want to be sure that we are picking the safest choice.

(ii) **The ratio test:** Let us now assume a power series

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k,$$

Let us assume that each term is smaller than the previous by a factor S, at least after some possible large number K. That is $|c_{k+1}(z-z_0)^{k+1}| \leq S|c_k(z-z_0)^k|, \forall k \geq K$. Then we can say that

$$\begin{aligned} |f(z)| &= \left| \sum_{k=0}^{\infty} c_k (z - z_0)^k \right| \\ &\leq \sum_{k=0}^{K} |c_k (z - z_0)^k| + \sum_{k=0}^{\infty} c_K (z - z_0)^K (1 + S + S^2 + \cdots S), \end{aligned}$$

and then the series converges if $S < 1$ [5] So it makes sense to say that if

$$\lim_{k \to \infty} \frac{|c_k (z - z_0)^k|}{|c_{k+1} (z - z_0)^{k+1}|} = S,$$

with $S < 1$ the series converge, that is if

$$\lim_{k \to \infty} \frac{|c_k|}{|c_{k+1}|} = S|z - z_0| \leq |z - z_0|$$

We want $|z - z_0| \leq r$, by definition of convergence radius, so

$$r \geq \lim_{k \to \infty} \frac{|c_k|}{|c_{k+1}|}.$$

On the other hand is $S \geq 1$, the series diverges, and also

$$\lim_{k \to \infty} \frac{|c_k|}{|c_{k+1}|} = S|z - z_0| \geq |z - z_0|$$

[5] We do not care about the first finite sum at the beginning. It is interesting that in convergent series we are only interested on the tail, while on asymptotic (no convergent) we are only interested about on head (first few terms) of the series.

and since r is the smallest possible distance of convergence

$$r \leq \lim_{k\to\infty} \frac{|c_k|}{|c_{k+1}|},$$

so

$$r = \lim_{k\to\infty} \frac{|c_k|}{|c_{k+1}|},$$

The second method is easier to implement that the first. However observe that both methods require the knowledge of the coefficients c_k.

The relationship between power series and holomorphic (analytic) functions is proven in section 5.3.4.

Here are a few examples of with a direct computation of their radius of convergence

- Let $m \in \mathbb{Z}^+$

$$\sum_{k=0}^{\infty} k^m z^k \quad , \quad r = \lim_{k\to\infty} \frac{k^m}{(k+1)^m} = 1.$$

-

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} \quad , \quad r = \lim_{k\to\infty} k+1 = \infty.$$

-

$$\sum_{k=0}^{\infty} k! z^k \quad , \quad r = \lim_{k\to\infty} \frac{1}{k+1} = 0.$$

7.1 Singularities and Zeroes

Singularities and zeroes are better understood in the context of the Laurent 5.3.9 expansion which is a bilateral Taylor expansion. That is, at a point of interest z_0 we assume

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k. \tag{7.1.3}$$

We categorize singularities as follows:

- If $c_k = 0$ for all $k < 0$ we say that f has a **removable singularity** at z_0. This type of singularity is not seen in the Taylor series expansion. We show a couple of examples:

  (i) $f(z) = z/z$. The Taylor expansion is $f_T(z) = 1$. In the original form the function is not defined at $z = 0$, but defining $f(0) = 1$, make it regular. It "removes" the singularity.

  (ii) $f(z) = \mathrm{sinc}(z) = \sin z/z$. Taylor series expansion at $z_0 = 0$ is

  $$\begin{aligned} f_T(z) &= \frac{1}{z}(z - z^3/3! + z^5/5! + \cdots + (-1)^k z^{2k+1}(2k+1)! + \cdots \\ &= 1 - z^2/3! + z^4/5! + \cdots + (-1)^k z^{2k}/(2k+1)! + \cdots \end{aligned}$$

  It makes sense to define $f(0) = 1$, and this will "remove" the singularity. A point z_0 is a removable singularity if we can assign a complex number to $f(z_0)$ that makes f analytic at z_0. Where all coefficients corresponding to the negative powers are non-zero.

More formally we say that z_0 is a removable singularity if it is in a neighborhood $U \subset \mathbb{C}$, such that f is holomorphic in $U \backslash \{z_0\}$. It is interesting that if f is holomorphic in $U \backslash \{z_0\}$ and bounded, then z_0 is a removable singularity as we show now. For this consider an auxiliary function

$$\begin{aligned} g : U &\to \mathbb{C} \\ z &\mapsto \begin{cases} (z - z_0)^2 f(z) & z \in U \backslash z_0 \\ 0 & z = z_0 \end{cases} \end{aligned}$$

Since f is bound in U, g is continuous in U and so it is

$$\begin{aligned} g'(z) &= 2(z - z_0) f(z) + (z - z_0)^2 f'(z) \quad z \in U \backslash z_0 \\ g'(0) &= 0. \end{aligned}$$

which implies that g is holomorphic and it can be expanded as a power series

$$g(z) = \sum_{k=0}^{\infty} c_n (z - z_0)^k \quad , \quad z \in U.$$

and since $g(z_0) = g'(z_0) = 0$, then $c_0 = c_1 = 0$ and

$$g(z) = (z - z_0)^2 h(z) \quad , \quad h(z) = \sum_{k=0}^{\infty} c_{2+k}(z - z_0)^k \quad , \quad z \in U.$$

But

$$f(z) = \frac{g(z)}{(z - z_0)^2} \quad , \quad z \neq z_0,$$

and so

$$f(z) = h(z) = \sum_{k=0}^{\infty} c_{2+k}(z - z_0)^k \quad z \neq z_0.$$

For $z = z_0$ we have $h(z_0) = c_2$, so we can define $f(z_0) = c_2$ and this will remove the singularity. Hence removing singularities are related to neighborhoods mapped into bounded sets.

- If $c_i \neq 0$, for some $i < 0$, $c_k = 0$ for all $k < i$, then we say that f has a **pole of order** i at z_0. A pole of order 1 is called a **simple pole** . The simplest example of a pole of order $i \geq 1$, is

$$f(z) = \frac{1}{(z - z_0)^i}.$$

- If $c_j \neq 0$ for an infinite number of negative integers we say that f has an **essential singularity** at z_0. The classical example for this is

$$f(z) = \mathrm{e}^{1/(z - z_0)}.$$

The Taylor series expansion of e^w is

$$e^w = 1 + \frac{w}{2!} + \cdots + \frac{w^k}{k!} + \cdots$$

Replace w by $1/(z - z_0)$ and then the Taylor series maps to a Laurent series

$$f(z) = \mathrm{e}^{1/(1 - z_0)} = 1 + \frac{1}{z - z_0} + \frac{1}{2!(z - z_0)} + \cdots + \frac{1}{k!(z - z_0)^k} + \cdots$$

Essential singularities have interesting properties. For example, for any neighborhood U of an essential singularity, the mapping $f(U)$ is everywhere in the complex plane $\mathbb{C}$. No matter how small the neighborhood U. It is almost like saying that we can map a point into the whole complex plane. The formal statement of this theorem is

Theorem 7.1.1 *If $f : R\backslash z_0 \to \mathbb{C}$ has an essential singularity at z_0, then for any neighborhood U of z_0 in R, the image $f(U\backslash z_0)$ is dense in $\mathbb{C}$.*

Proof: Assume that for some neighborhood U of z_0, the image $f(U\backslash z_0)$ miss a neighborhood $V \subset \mathbb{C}$ with $\omega_0 \in V$. We show that this produces a contradiction. Since $f(z) - w_0$ is non zero in V, then $g(z) = 1/(f(z) - w)$ is holomorphic and bounded in $U\backslash z_0$, so z_0 is a removable singularity for g. We can remove the singularity and call the extension g_{ext}. If $g_{ext}(z_0) \neq 0$, then z_0 is removable for f, otherwise it z_0 is a pole for f. In any case this contradicts that z_0 is an essential singularity for f. So, $f(U)$ is all over the complex plane.

- If $c_j = 0$, for all $j > i > 0$, we say that f has a **zero of order** i at z_0. A zero of order $i = 1$ is called a **simple zero.** This is, $f^{(k)}(z_0) = 0$, for $i < k$ and $f^{(i)}(z_0) \neq 0$.
  - example 1: Let us assume $f(z) = z^2$. Since $f^{(0)}(z) = f^{(1)}(z) = 0$, and $f^{(2)}(z) = 2$ then f has a zero of order 2 at $z = 0$.
  - example 2: Let $f(z) = \sin z$. It is clear that $z_j = j\pi$ for any $j \in \mathbb{Z}$. Now since $f'(z) = \cos z$ and $\cos j\pi = (-1)^j$, we see that f has a simple zero at each $j\pi$, for $j \in \mathbb{Z}$.

Other points (or paths) in the complex plane are **branch points and brunch cuts,** [6], not explained here.

An alternative set of definitions (see Wikipedia) is the following

- If $\lim_{z\to z_0} f(z)$ and if $\lim_{z\to z_0} \frac{1}{f(z)}$ and exist, then z_0 is a **removable singularity** for both f and $1/f$.

[6] points when the forward function is multivalued and should be restricted so the function can have an inverse. Classical examples involve square roots and logarithmic functions.

- If $\lim_{z\to z_0} f(z)$ but $\lim_{z\to z_0} \frac{1}{f(z)}$ does not exist, then z_0 is a **zero** of f and a **pole** of $1/f$.
- If $\lim_{z\to z_0} f(z)$ does not exist but $\lim_{z\to z_0} \frac{1}{f(z)}$ exists, then z_0 is a **pole** of f and a **zero** of $1/f$.
- If neither $\lim_{z\to z_0} f(z)$ nor $\lim_{z\to z_0} \frac{1}{f(z)}$ exists, then z_0 is an essential singularity of both f and $1/f$.

It is clear, by simple set theory [7], in the previous set of definitions; that under the prescribed classificatios any complex number is one of the above.

Yet, another definition of essential singularity at z_0 is the point such that $f(z)(z-a)^n$ is not differentialble for any integer $n > 0$. It can be shown that the corresponding definions above are equivalent.

A **meromorphic** function is analytic everywhere in the finite plane except for a set of isolated singularities.

Another way to classify singularities is as

- **Isolated singularities** . This means that we can find an open set about the point z_0, where the function is holomorphic when subtracting the point z_0. An example is the function $1/z$. Here $z = 0$ is an isolated singularity.
- **Cluster singularities.** This means that no matter how small, we can not find a disc around z_0 such that z_0 is the only singularity. An example is $\tan \frac{1}{z}$. Around $z = 0$ there is no way to isolated this singularity. There are always points z_0 as close as $z = 0$ as we want, for which f is singular at z_0. Another example:

$$f(z) = \sum_{k=-\infty}^{\infty} \frac{a^k z}{(z-a^k)^2},$$

where a is a complex constant, such that $|a| > 1$. This function has double poles at the points a^k, for each $k \in \mathbb{Z}$. The points $1, a, a^2, \cdots$, accumulate around ∞, while the poles $a^{-1}, a^{-2}, \cdots$, accumulate around 0. So ∞ and 0 are cluster singular points for the function f.

[7] There are only four possibilities for a function and its inverse to behave –diverge or not diverge– as $z \to z_0$

Here is a list of important theorems related to analytic and entire functions.

- A function which is analytic for all finite values of z, and is bounded, is constant. This is known as the **Liouville's Theorem.** .
- If a function $f(z)$ is analytic for all finite values of z and as $|z| \to \infty$, $f(z) = \mathcal{O}(|z|^k)$, then $f(z)$ is a polynomial of degree $< k$. This is an extension of Liouville's theorem.
- The zeroes of an analytic function in a region are isolated points.

Let us assume the Laurent series representation 5.3.9

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k.$$

for $0 < |z - z_0| < r$. Let us assume a pole of order i on z_0. That is, the sum above is

$$f(z) = \sum_{k=-i}^{\infty} c_k (z - z_0)^k.$$

The coefficient c_{-1}, is termed the **residue of** $f(z)$ **at** z_0 and it is abbreviated as $\text{Res}(z_0)$. This coefficient has the integral representation

$$c_{-1} = \frac{1}{2\pi \mathrm{i}} \int_\gamma f(\xi) d\xi.$$

Chapter 8

Analytic Continuation

A comprehensive study of analytic continuation is presented in my notes (not a reference yet). Here is a brief summary of important bullets.

In complex analysis analytic continuation is a technique to extend the domain of a given analytical function. We say that a function g with open domain D_g is an **analytic continuation** of a function f with open domain $D_f \subset D_g$, if

$$g(z) = f(z) \quad , \quad \forall z \in D_f,$$

and $g(z)$ is analytic in D_g. It is like leaking bubbles through holes. A chain of disks can be linked into a larger set of an extended domain of analyticity.

A classical example is the function

$$f(z) = \sum_{k=0}^{\infty} z^k. \tag{8.0.1}$$

We know that the function diverges for $|z| \geq 1$. The function

$$g(z) = \frac{1}{1-z}$$

agrees with $f(x)$ in the unit disk $|z| < 1$, and it is analytic everywhere, except at the point $z = 1$. Hence g is an analytic continuation of f, which is meromorphic in the whole complex plane.

Figure 8.1 illustrates the domains of analyticity of both f and g.

An interesting example is that of the Laplace Transform. The Laplace transform is the analytic continuation of the Fourier transform, defined in

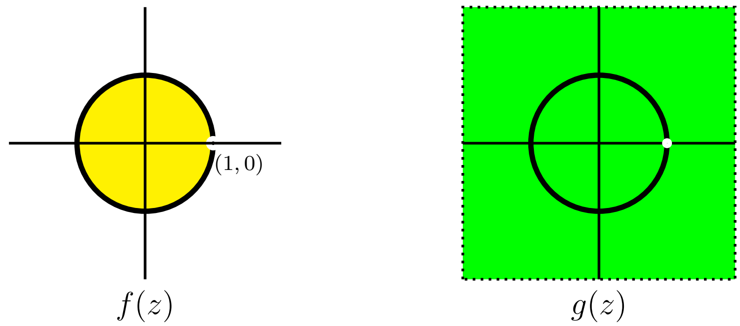


Figure 8.1: The geometrical series $f(z)$ 8.0.1 has a radius of convergence of 1. It is defined only in the yellow circle (including its boundary), except at the point $(1, 0)$. The extension to the function $g(z) = 1/(1 - z)$, on the right is defined in all $\mathbb{C}$ plane, except by the point $(1, 0)$ where it has a simple pole.

the half line $[0, \infty)$, over the complex plane, in such a way that the Laplace transform converges. This is an important step where frequencies are moved from the real to the complex plane so that causality could be achieved. The analytic continuation of the Fourier transform in the complex plane (above or below the real axis, according to the sign convention used in the defintion) guarantees causality as well as convergence. When dealing with multidimensional data in time and space it is recommended to use the Laplace transform for the temporal variable and the Fourier transform for the spatial variables, for the reasons just explained. We devote the next chapter to the treatment of the Fourier, Laplace, and Z transforms.

The Gamma function Γ is another interesting example of analytic continuation. In this case we can extend a function defined only on the integer numbers (factorial $n!$) to the complex domain, with a limited range, and from here to the complex plane except for a few holes. Section 8.1 expands on the analytic continuation of the Gamma function.

The final example was made Famous by Riemann and it was part of its doctoral dissertation. This example is shown in a section 8.3

8.1 Analytic continuation of Euler's Gamma Γ function.

The Gamma function is related to the Laplace transform of t^{z-1}. Let us see,

$$\begin{aligned}\Gamma(z) &= \int_0^\infty \mathrm{e}^{-t}t^{z-1}dt \qquad (8.1.2)\\ &= \int_0^\infty \mathrm{e}^{-st}(st)^{z-1}sdt\\ &= s^z\mathcal{L}(t^{z-1})\end{aligned}$$

If $z = n$ is a positive integer, then it is well known that $\Gamma(n) = (n-1)!$. Actually, according to the history books, it was Daniel Bernoulli in a letter [1]to Goldbach who searching for an interpolation formula for the factorial started the chain of equations that ended up in Euler's integral formulation 8.1.2. Integral 8.1.2 diverges for Re$z \leq 0$. However from integration by parts it is easy to find the recursion relation

$$\Gamma(z) = (z-1)\Gamma(z-1)$$

With this, for example for any $0 < \mathrm{Re}z < 1$, we find

$$\Gamma(z-1) = \frac{\Gamma(z)}{z-1} \qquad (8.1.3)$$

which is well defined as long long as $z \neq 0$. This extends the definition of the function to the domain Re$z > -1$, $z \neq 0$. Recursively, we find the formula

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)(z+2)\cdots(z+n-1)}. \qquad (8.1.4)$$

which continues the function up to $\mathrm{Re}(z) > -n$, leaving out the vertical asymptotes $\mathrm{Re}(z) = k$, $k = -1, -2, \cdots, -1-n$. The analytic continuation of the Gamma function is meromorphic in the entire complex plane. Figure 8.2 shows a picture of the norm of Gamma function ($|\Gamma(z)|$) for $-5 \leq \mathrm{Re}z \leq 5$ and $-5 \leq \mathrm{Im}(z) \leq 5$. Observe the poles at the negative integers.

[1]http://commons.wikimedia.org/wiki/File

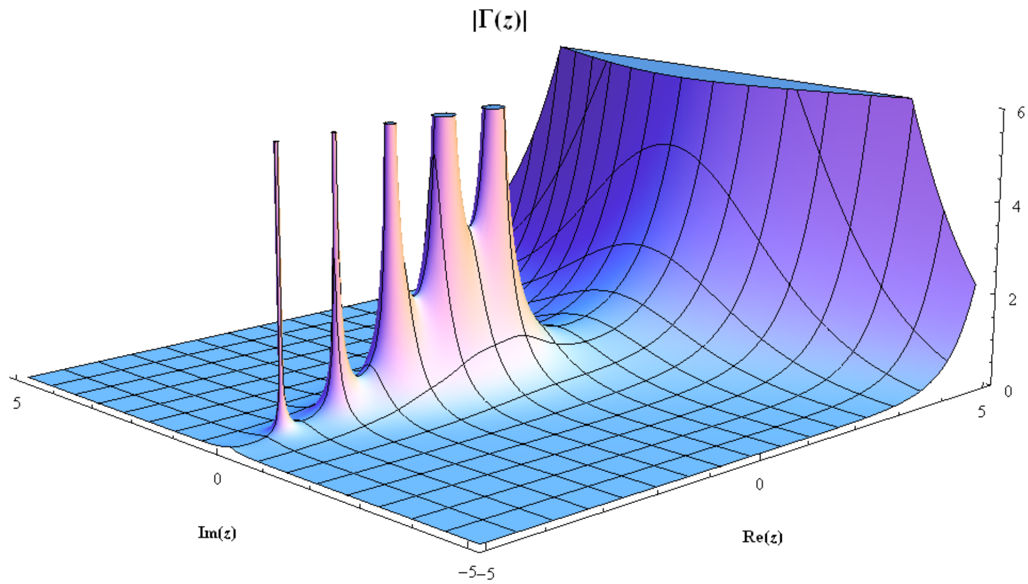


Figure 8.2: $|\Gamma(z)|$ function. The truncated spikes show clearly the singularities (simple poles) at the negative integer numbers.(taken from Wikipedia)

8.2 Logarithmically Convex

This session illustrates some important inequalities related to the $\Gamma(x)$ function. For the moment, we focus on x as a real number. We use the concept of convexity to help on the stablishment of such inequalities.

A function $g : \mathbb{R} \to \mathbb{R}$ is called **convex** on an interval $[\alpha, \beta] \subseteq \mathbb{R}$ if and only if

$$\forall a, b \in [0,1] : \exists : a + b = 1 : \forall x_1, x_2 \in [\alpha, \beta] :$$
$$g(ax_1 + bx_2) \leq ag(x_1) + bg(x_2).$$

Convextiy, means that the segment that joints two points in the graph of the function, lie under the curve. The opposite term is **concavity** . In this case the segment that joints two points of the curve lie above the curve. From calculus, convexity means that the first derivative, if it exists is increasing, and the second derivative (if it exists) is positive in the observation interval. The $\Gamma(x)$ function is convex for $x > 0$, but for $x < 0$ it switches between convex and concave at each integer interval of the form $[n, n+1]$ where $n < 0$, is integer . This switching is due to the flip on signs implied by equation 8.1.3. A function $g(x)$ is called logarithmically convex if its logarithm is convex. If g is twice differentiable then $g''(x) > 0$. In this case

$$\begin{aligned} (\log g(x))' &= \frac{g'}{g} \qquad \text{and} \\ (\log g(x))' = &= \frac{g''g - g'^2}{g^2} \qquad \text{and} \end{aligned}$$

The $\Gamma(x)$ function is logarithmically convex. We prove this.

On the other hand,

$$\begin{aligned} \Gamma(z+1) &= \lim_{n\to\infty} \frac{n!n^{z+1}}{(z+1)(z+2)\cdots(z+1+n)} \\ &= \lim_{n\to\infty} \left(z\frac{n!n^z}{(z+1)(z+2)\cdots(z+n)} \frac{n}{z+1+n} \right) \\ &= z\Gamma(z) \lim_{n\to\infty} \frac{n}{z+1+n} \\ &= z\Gamma(z). \end{aligned}$$

8.3 Analytic continuation of the Riemann ζ function.

For years, mathematicians have looked for evaluations of the zeta (ζ) function defined as

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

The function diverges for Re$s \leq 1$, and is analytic for $Re(s) > 1$. Euler gain great reputation by finding, in 1735, that $\zeta(2) = \pi^2/6$. A problem know as the Basel problem [2], and that was being searched by great mathematicians for almost a hundred years. Euler went much further and estabilshed the relation between the ζ function and the prime numbers through the formula

$$\zeta(s) = \prod_{p=\text{prime}} \frac{1}{1-p^{-s}}$$

[2] formulated by Pietro Mengli in 1644)

which relates the ζ function with a product over all prime numbers, and paved the way to the formulation of the Riemman Hypothesis; one of the greatest mathematical problems not yet solved. From the definition of the *Gamma* function 8.1.2, and multiplying this by $\zeta(z)$ we find

$$\begin{aligned} \zeta(z)\,\Gamma(z) &= \sum_{k=1}^{\infty}\left(\int_0^{\infty} \mathrm{e}^{-t} t^{z-1} dt\right) \frac{1}{k^z} \qquad (8.3.5) \\ &= \sum_{k=1}^{\infty}\int_0^{\infty} \mathrm{e}^{-kt} t^{z-1} dt \end{aligned}$$

Now

$$\sum_{k=1}^{\infty} \mathrm{e}^{-n\,t} = \frac{\mathrm{e}^{\mathrm{t}}}{1-\mathrm{e}^{-t}} = \frac{\mathrm{e}}{\mathrm{e}-1}$$

with $t > 0$. Now if $z = x + \mathrm{i}y$ then

$$\int_0^{\infty}\sum_{k=1}^{\infty} |\mathrm{e}^{-k\,t} t^{z-1}| dt = \int_0^{\infty}\sum_{k=1}^{\infty} |\mathrm{e}^{-k\,t}| t^{x-1} dt = \int_0^{\infty} \frac{t^{x-1}}{\mathrm{e}^t - 1} dt. \qquad (8.3.6)$$

This last integrals behave like e^{-t} for $t \gg 1$ and like t^{x-2} for small t ($t \approx 0$). Hence the integral converges for $x > 1$. From Fubini–Tonelli's theorem we justify the changing of order between the the sum and the integral in equations 8.3.5 and 8.3.6 and say that

$$\zeta(a)\,\Gamma(z) = \int_0^{\infty} \frac{t^{z-1}}{\mathrm{e}^t - 1} dt.$$

with $\mathrm{Re}(z) > 1$. This is the is the first result derived by Rimmann [?] [3] in his famous paper. Given that the integral diverges at $t = 1$, Riemann considered the following replacement by a contour integral. Let the contour γ be

8.4 An analytic wall

There is an interesting class of functions that can not be analytic continued because there is a hard wall that stop them. Let us assume O to be a

[3] Reprinted by Dover, New York 1953. The title in english is "On the Numbers of Primes Less Than a Given Maginute.

connected open set of f such that f is analytic on O. Let us further assume that $z_0 \in \partial O$ [4]. Let f be a an analytic function defined in O. If an analytic continuation g of f exists, such that g agrees with f in the intersection

$$O \cap B(z_0, r),$$

where $B(z_0, r)$ is a disc with center at z_0 and radius r, we say that z_0 is a **regular** point. A point is called **singular** if it is not regular. The set of regular points is open, so the set of singular points is closed. The set ∂O is a **natural boundary** if all its points are singular. This boundary is an analytic wall. The function can not be continued to the outside of O. One also says that O is a domain of holomorphy for f. The following example was set by Weierstrass in the (1840s). be defined as

$$f(z) = \sum_{n=1}^{\infty} z^{n!},$$

and the proof can be found in [**?**].

[4] Here ∂O is the boundary of O. That is the set of points such that any ball around them contains points in the interior and in the exterior of O

Chapter 9

Analytic Continuation of the Fourier Transform: The Laplace Transform

9.1 The Fourier Transform

Let us define the Fourier transform as follows

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t} dt.$$

In practice $F : \mathbb{R} \to \mathbb{C}$. What if we think of $F : \mathbb{C} \to \mathbb{C}$, where ω is a complex number $\omega = \alpha + \mathrm{i}\beta$, with α, β real numbers. Where in the complex plane is $F(\omega)$ analytic? What conditions do we have on f? Note that, for example to define $F(0)$ we need f to be integrable in the whole real line. So, at least we need that. We see that we can relax this condition under certain assumptions. Note that for ω real (that is $\beta = 0$), we just changed the phase of the exponential function but this should not impact on the convergence. We are interested in the parameter β. For this, let us expand the exponential function. That is,

$$\mathrm{e}^{-\mathrm{i}\omega t} = \mathrm{e}^{-\mathrm{i}\alpha t + \beta t} = \mathrm{e}^{-\mathrm{i}\alpha t}\mathrm{e}^{\beta t}$$

The first factor is always bounded by 1. Provided that the regular Fourier transform is well defined (bounded), we have problems. The second factor

$e^{\beta t}$ is not bounded and diverges to infinity. That is, if $\beta < 0$, the product $\beta t > 0$, for $t < 0$, while if $\beta > 0$, the product $\beta t > 0$, for $t > 0$. The only way to fix this is by having $f(t)$ converge strongly but we do not want to impose new conditions of $f(t)$ other than causality. If $f(t)$ is causal (that is if $f(t) = 0, t < 0$) then we can choose any $\beta < 0$ and there is assured convergence, since $\beta t < 0$, will only reduce the size of the integration argument. Hence we say that for causal function the function $F(\omega)$ (with $\omega \in \mathbb{C}$) is convergent in the lower complex plane. [1] We need to solve two question:

(i) Does convergence implies analyticity?

(ii) What is the space of functions $f(t)$ that makes $F(\omega)$ analytic?

We solve these questions simultaneously. There are several ways to prove that a function is analytic. Here we use the Cauchy-Riemann conditions. This will be done in three steps:

(i) **Separate the real and imaginary parts.** That is,

$$\begin{aligned} F(\omega) &= \int_0^\infty dt\ f(t) e^{\beta t}\ e^{-i\alpha t} \\ &= \int_0^\infty dt\ f(t) e^{\beta t} \cos \alpha t - i f(t) e^{\beta t} \sin \alpha t \\ &= \int_0^\infty dt\ f(t) e^{\beta t} \cos \alpha t - i \int_0^\infty f(t) e^{\beta t} \sin \alpha t \\ &= u(\alpha, \beta) + i\, v(\alpha, \beta) \end{aligned}$$

(ii) **Show that the integral has uniform convergence,** so that the derivatives commute with the integrals, and we can move the derivaties inside the integral sign.

Appendix A reviews the concepts of uniform convergence and why differentiation (and continuity) is preserved under integration. Particularly Theorem B.7 proofs this. While we can relax the continuity conditions on f let us assume that f is continuous and ask what conditions should f have for the integrals u and v to converge uniformly.

[1] Note that if we would have chosen to define $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$, then when we said "lower" above, we should say "upper" here. The sign convention for the Fourier transform is arbitrary as along as the whole theory is consistent with this sign convention. Electrical engineers have the opposite sign convention as geophysicists.

If $f(t)$ has an exponential growth that is lower than the β power each integrand behaves a decay exponential. That is, assume $|f(t)| < C\mathrm{e}^{\gamma t}$, $C > 0$. Then

$$|f(t)\mathrm{e}^{\beta t}\cos\alpha t| \leq C\mathrm{e}^{(\beta+\gamma)t}$$

So, if $\beta+\gamma < 0$, then the integrand behaves asymptotically as a decaying exponential and the integral is bounded by the absolute value of

$$\int_0^\infty C\mathrm{e}^{(\beta+\gamma)t}dt = -\frac{C}{\beta+\gamma}$$

and so we from Theorem B.4 in Appendix A we see that the integral converges uniformly and we can interchange the integral with the derivative sign. The analysis is the same if we change $\cos\alpha t$, by $\sin\alpha t$, so both $u(\alpha,\beta)$ and $v(\alpha,\beta)$ are uniformly convergent integrals, on the complex plane of numbers $\alpha + i\beta$, with $\beta < -\gamma$, with and the space of functions such that $|f(t)| < C\mathrm{e}^{\gamma t}$ satisfies the condition for uniform convergence.

(iii) **Verify the Riemann-Cauchy conditions.** Let us find the partial derivatives $u_\alpha, u_\beta, v_\alpha$ and v_β. Since we can take the derivatives inside the integral sign due to the uniform convergence, then

$$\begin{aligned}
u_\alpha(\alpha,\beta) &= \int_0^\infty dt(-t\sin\alpha t)f(t)\mathrm{e}^{\beta t} \\
u_\beta(\alpha,\beta) &= \int_0^\infty dt\ t\cos\alpha t f(t)\mathrm{e}^{\beta t} \\
v_\alpha(\alpha,\beta) &= -\int_0^\infty dt(t\cos\alpha t)f(t)\mathrm{e}^{\beta t} \\
v_\beta(\alpha,\beta) &= -\int_0^\infty dt\ t\sin\alpha t f(t)\mathrm{e}^{\beta t}
\end{aligned}$$

Clearly $u_\alpha = v_\beta$ and $u_\beta = -v_\alpha$, which are the Cauchy-Riemann conditions that provide analyticity for the Fourier transform in the half plane unde $\beta < -\gamma$, for the space of functions $f(t)$ such that $|f(t)| < C\mathrm{e}^{\gamma}$.

This extends the Fourier transform from the real line analytically to half of the complex plane.

9.2 The Laplace Transform

The Laplace transform is defined causal functions (there is a bilateral Laplace transform but that is not of much interest; unless in this context). Let us then assume that $f(t)$ is causal (that is $f(t) = 0, t < 0$). The causal Fourier transform is defined as

$$F(\omega) = \int_0^\infty f(t)\mathrm{e}^{-\mathrm{i}\omega t} dt.$$

We would like to extend the function F away from the real axis. We want to think of $\mathrm{i}\omega$ as a general complex number and replace it but $s = \sigma + \mathrm{i}\omega$. That is we define, $F : \mathbb{C} \to \mathbb{C}$, as

$$\mathcal{L}(s) = \int_0^\infty f(t)\mathrm{e}^{-st} dt. \tag{9.2.1}$$

We are extending the domain of definition from the real line to the entire complex plane. We can borrow all the work done on the Fourier transform by identifying $\mathcal{L}(s) = F(\mathrm{i}s)$. Then while the Fourier transform is analytic in a lower half plane, the Laplace transform is analytic in a right half plane such that $\mathrm{Re}\, s > \gamma$, where we know that $|f(t)| < C\mathrm{e}^{\gamma t}$. Figure 9.1 shows the regions of analyticity of the Fourier and the Laplace transforms.

It is interesting that we can see the Laplace transform in a different way as a Fourier transform. If $s = \sigma + \mathrm{i}\omega$ then the Laplace transform is

$$\mathcal{L}(s) = \int_0^\infty f(t)\mathrm{e}^{-st} dt. = \int_0^\infty [f(t)\mathrm{e}^{-\sigma t}]\mathrm{e}^{-i\omega} dt = F(\sigma, \omega) \tag{9.2.2}$$

where $F(\sigma, \omega)$ is the Fourier transform of $f(t)\mathrm{e}^{-st}$. This provides a way to invert for the Laplace transform, since we already know how to invert for the Fourier transform. We know that if $f(t)$ is of order of an exponential in α, that is $|f(t)\mathrm{e}^{\alpha t}| < C$ for a constant real C, then for $\mathrm{Re}(s) > \alpha$ the function $g(t) = \mathrm{e}^{-\sigma t} f(t)$ is absolutely integrable. That is

$$\int_0^\infty |\mathrm{e}^{-st} f(t)| dt = \int_0^\infty |\mathrm{e}^{-s\sigma} f(t)| dt < \infty \quad , \quad \sigma > \alpha.$$

We then use the inverse of the Fourier transform in 9.2.2. That is,

$$f(t)\mathrm{e}^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^\infty F(\sigma, \omega)\, \mathrm{e}^{\mathrm{i}\omega t} d\omega \quad , \quad t > 0$$

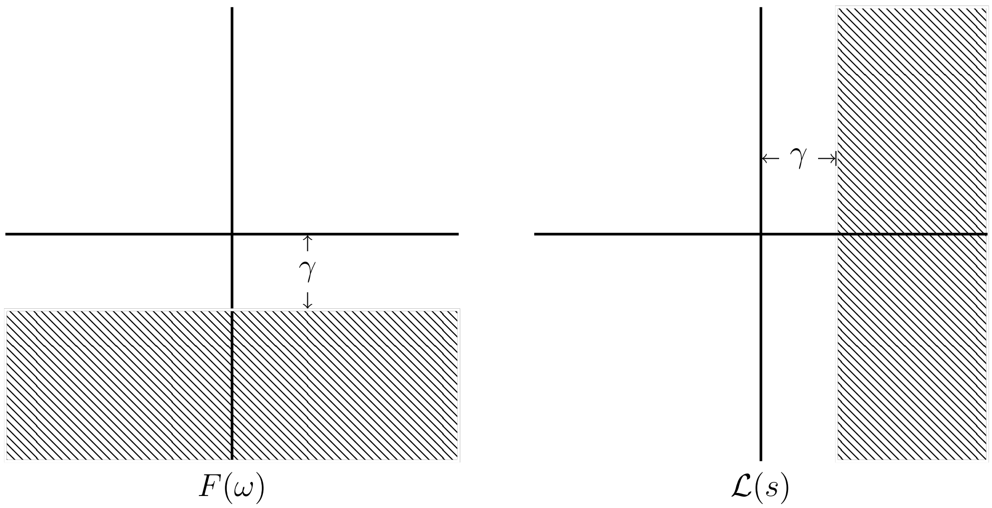


Figure 9.1: Regions of analyticity for the Fourier and Laplace transforms. Assume that the function under the integral is of exponential grow under $|f(t)| < Ce^{\gamma t}$. The hatch area represents to zone of convergence.

That is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\sigma t} F(\sigma, \omega)\, e^{i\omega} d\omega \quad , \quad t > 0.$$

We make a change of variables. That is, we want to write this in terms of s instead of σ and ω. From $s = \sigma + i\omega$ we have $ds = d\sigma + i\, d\omega$. Now, we choose σ fixed, that is we integrate having the real part $\mathrm{Re}(s)$ fixed. Then $d\omega = ds/i$, and since $e^{\sigma + i\omega} = e^s$, we have

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s)\, e^{st}\, ds \tag{9.2.3}$$

where we rename $F(\omega, \sigma)$ by $\mathcal{L}(f(t)) = F(s)$. This is know as the **complex inversion formula** or the **Fourier-Mellin inversion formula** , and the vertical line as the **Bromwich line** . This formula is useful to do analytical work but to invert for Laplace transforms other techniques are used in practice which are outside of the domain of this document. However if we want to evaluate this integral we have to do it following contour integration. We

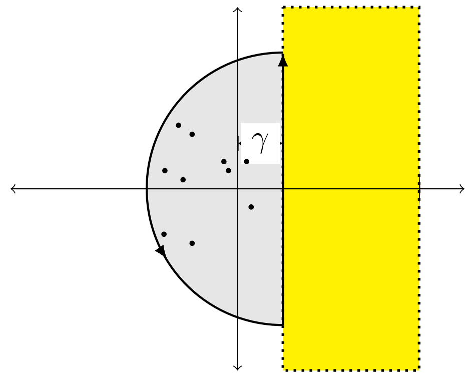


Figure 9.2: The yellow area on the right of the vertical line $\text{Re}(z) = \gamma$ is the zone of analyticity for the inverse Laplace transform. This means that all the singularities have to be to the left of this line. The contour of integration is the Brownwich line and the semicircle at its left. Black dots inside the semicircle indicate possible isolated singularies (poles).

do this now. Pick the contour following the Brownwich line from $\gamma - \text{i}r$ to $\gamma + \text{i}r$, and then through a semicircle on the left plane, centered at 0 and with radius r. Figure 9.2 sketches the contour. A contour integral on the right of the Brownwich line $\gamma = \text{Re}(z)$ will not produce anyting useful since the function is analytic there and the integral would be zero. Any useful contour should be set on the left of this Brownwich contour. A semicircle with center at the origin and radius r, is chosen. The idea is to enclose all singularities inside the semicircle. For this we can let $r \to \infty$ if necessery. We show that the for certain class of functions $F(s)$, the contribution along the semicircle arc goes to zero as $r \to \infty$, and so the contour integral will represent well the integration about the Brownwich vertical line. For example, if all singularities are poles, we can use the residue theorem and say that

$$\int_C F(s)\, \text{e}^s\, ds = 2\pi\text{i} \sum_i \text{Res}(z_i)$$

where C is the contour considered her, and $\text{Res}(z_i)$ is a residue corresponding to the pole z_i.

To prove that under certain conditions, contours around semicircles away from the origin provide no contribution to an integral, a powerful tool is used and it is call

Lemma 9.3 Jordan's Lemma: *If $\lambda > 0$ is some real constant and f an analytic function (except possibly for a finite number of poles), the integral*

$$I = \int_{C_R} f(z)\mathrm{e}^{\mathrm{i}\lambda z} dz \to 0$$

as $R \to \infty$ with C_R a semicircle with radius R on the upper half plane.

Proof: We can see how this lemma is convenient for Fourier-like integrals. We start by making natural change of variables. If we think that z is running along a circumference (center at the origin, but this is no necessary since we can always can apply translations which do not change the value of the integral) then we can say that

$$z = R\,\mathrm{e}^{\mathrm{i}\theta}$$

with $\theta \in [0, \pi]$. With this change of variable we have $dz = \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}d\theta$, and

$$I_R = \int_0^\pi d\theta\,\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta} f(R\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\lambda R\cos\theta - \lambda R\sin\theta}$$

We now use the following facts

- $$|f(R\mathrm{e}^{\mathrm{i}\theta})| \le M$$

  for some $M > 0$. That is, since $\theta \in [0, \pi]$ which is a compact set (closed and bounded), and the function f is analytic for a large enough R (so there are no poles over the curve) then the function reaches a finite maximum M.

- $$|\mathrm{i}\,\mathrm{e}^{\mathrm{i}(\lambda R\cos\theta + \theta)}| = 1,$$

  since the argument after i is real.

Then, taking the absolute value inside the integral,

$$I \le \int_0^\pi d\theta M R\mathrm{e}^{-\lambda R\sin\theta} = M\int_0^\pi d\theta R\mathrm{e}^{-\lambda R\sin\theta} = 2M\int_0^{\pi/2} d\theta R\mathrm{e}^{-\lambda R\sin\theta},$$

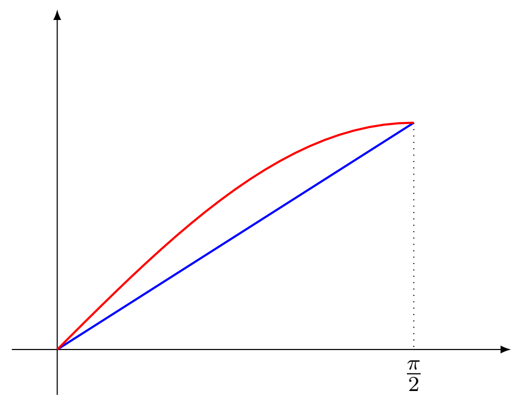


Figure 9.3: Illustration of the convexity of $\sin\theta$ in the interval $[0, \pi/2]$. The blue line is below the sine function in red.

where we use the fact that the integrand is even with respect to the axis $x = \pi/2$. Now, the function $\sin\theta$ is convex in the interval $[0, \pi/2]$, that is the line $y = (2/\pi)\theta$ is under $\sin\theta$, or in other words

$$\sin\theta > \frac{2}{\pi}\theta \quad \Rightarrow \quad -\frac{2\theta}{\pi} < -\sin\theta.$$

For an illustration of this observe Figure 9.3 Then we say that

$$I \leq 2M \int_0^{\pi/2} d\theta R \mathrm{e}^{-2R\lambda\theta/\pi} = -\cancel{2R}M \frac{\pi}{\cancel{2R}\lambda} \mathrm{e}^{-2R\theta/\pi}\Big|_0^{\pi/2} = M\pi(1 - \mathrm{e}^{-\lambda R}) \to 0$$

as $R \to \infty$, and $\lambda > 0$. If $\lambda < 0$, we need to consider the contour in the lower half plane instead. This proofs the Jordan's Lemma.

Let us return to the evaluation of the inverse Laplace transform function. We assume that, possible after some large $R = r$, all the singularities are poles and are contained in the semicircle of radius r shown in Figure 9.2. However this figure has the semicircle tilted 90 degrees with respect to the Jordan's lemma just proved.

If in the Jordan's lemma you substitute $\mathrm{i}\lambda z$ by s, we have the inverse Laplace formula in the integrand. Let us see what happens to the domain of integration.

$$s = \mathrm{i}\lambda z \quad \Rightarrow dz = \frac{ds}{\mathrm{i}\lambda}.$$

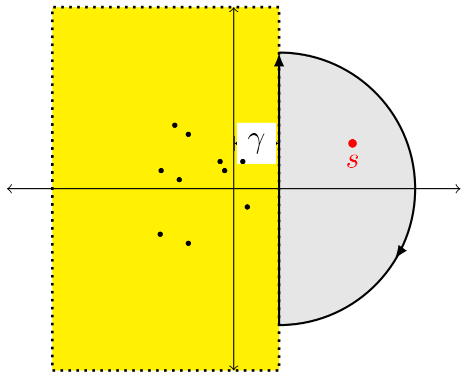


Figure 9.4: This is the mirror image of Figure 9.2, with respect to the Brownwich contour, where the integration is peformed over the analytical region of $F(z)$, which is to the right of the vertical Brownwich contour $\mathrm{Re}(z) = \sigma$. The red point is the represents single pole of the function $f(z)/(z-s)$.

Any point in the complex inverse Laplace plane is obtained by multiplying by $i = \mathrm{e}^{\mathrm{i}\pi/2}$, and scaled by λ. The scaling by λ is not an issue. Actually here $\lambda = 1$. The multiplication by i is a counter–clockwise $\pi/2$ rotation. That is, the semicircle in Jordan's theorem that is in the upper half plane, is in the inverse Laplace function in the left plane. Nothing changes for the integration over the circle, which is 0. So this proofs that the inverse Laplace transform integral is the given by as proposed above.

$$\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} F(s)\,\mathrm{e}^{s}\,ds = 2\pi\mathrm{i}\sum_i \mathrm{Res}(s_i),$$

where s_i is a pole of $F(s)$.

Knowning what we know now we can present a different derivation of the inverse Laplace transform which is not based on the inverse of the Fourier transform, but in the Cauchy Integral Theorem 5.3.4. Let $F(z)$ (note the use of z instead of s) be the Laplace transform of $f(t)$. That is $F(z) = \mathcal{L}[f(t)]$, which is analytic in the domain $D = \{z : \mathrm{Re}(z) > \gamma\}$, and such that $\lim_{|z|\to\infty} |F(z)| = 0$. Pick the integration path as defined in Figure 9.4. If we apply the Cauchy Integral Theorem along the contour C in the figure, we find that

$$-2\pi\mathrm{i}F(s) = \int_{C_-} \frac{F(z)}{z-s}dz = \int_{\gamma-\mathrm{i}R}^{\gamma+\mathrm{i}R} \frac{F(z)}{z-s}dz + \int_{C_R} \frac{F(z)}{z-s}dz$$

where the minus "-" sign at the front of the expression is because we are circulating the contour in the clockwise (negative) direction. The contour C_- is the whole contour in Figure 9.4, and the contour C_R just corresponds to the arc of circle.

We show that the last integral goes to zero as the radius of the circle goes to infinity. We first parameterize the contour which is centered at $z_0 = (\gamma, 0)$, as

$$C_R = \{z : z - z_0 = R\mathrm{e}^{\mathrm{i}\theta} \ , \ \pi/2 > \theta > -\pi/2\}.$$

where $dz = \mathrm{i}\, R\, \mathrm{e}^{i}\, d\theta$. In the denominator we have

$$z - s = z - z_0 + z_0 - s = R\mathrm{e}^{\mathrm{i}\theta} + z_0 - s$$

and dividing numerator and denominator by R we find

$$\int_{C_R} \frac{F(z)}{z-s} dz = \int_{-\pi/2}^{\pi/2} \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\theta} F(z_0 + R\mathrm{e}^{\mathrm{i}\theta})}{\mathrm{e}^{\mathrm{i}\theta} + (z_0 - s)/R} d\theta.$$

We see that as $R \to \infty$ the absolute value of the numerator goes to zero. That is,

$$\lim_{R\to\infty} |\mathrm{i}\mathrm{e}^{\mathrm{i}\theta} F(z_0 + R\mathrm{e}^{\mathrm{i}\theta})| = \lim_{R\to\infty} |F(z_0 + R\mathrm{e}^{i\theta})| = 0.$$

and the denominator goes to e^{θ} which is always a number with size 1. Hence the integral over the arc of circle C_R goes to zero. We have so far that

$$F(s) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \frac{F(z)}{s - z} dz.$$

and then we have the following chain of equations

$$\begin{aligned} f(t) = \mathcal{L}^{-1} F(s) &= \mathcal{L}^{-1} \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \infty}^{\gamma + \infty} \frac{F(z)}{s - z} dz \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} F(z) \left[\mathcal{L}^{-1} \left(\frac{1}{s - z} \right) \right] dz \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} F(z) \mathrm{e}^{zt} dz. \end{aligned}$$

where used the fact that

$$\mathcal{L}^{-1}\left(\frac{1}{s-z}\right) = \mathrm{e}^{zt}.$$

We could move the Laplace operator inside the integral because the integration is uniformly continuous due to the analyticiy of the inverse Laplace transform in the Brownwich contour.

In complete analogy the Fourier transform can be seen also as a sum of the residues due to all poles inside a semicircle over the horizontal line $\mathrm{Im}(z) = \gamma$, shown in Figure 9.1.

9.4 The Z-Transform

The Z-transform is the time-discrete counter-part of the Laplace transform, written in a convenient form so it is a power series on a variable $z = \mathrm{e}^s$.

Let us consider again the Laplace transform along a continuous (time) signal as defined by equation 9.2.1. If instead of a continous signal we have a discrete signal sampled at Δt intervals. Think of $f(t)$ for example as

$$f(t) = \sum_{i=0}^{\infty} f[i]\delta(t - i\Delta t),$$

and replace this in equation 9.2.1 to get

$$F(s) = \sum_{i=0}^{\infty} f[i]\mathrm{e}^{-si\Delta t}$$

where we use square brackets [] to indicate that the signal is now discrete. It is convenient to define a mapping $z = \mathrm{e}^{s\Delta t}$, and write

$$F(z) = \sum_{i=0}^{\infty} f[i]z^{-i}$$

Geophycisists use the convention $z = \mathrm{e}^{-s}$, and here instead

$$F(z) = \sum_{i=0}^{\infty} f[i]z^{i}$$

Note that the mappings, along the line throught points $\sigma - \mathrm{i}\omega\Delta t$ and $\sigma - \mathrm{i}\omega\Delta t$ where

$$\begin{aligned} z &= \mathrm{e}^{s\Delta t} = \mathrm{e}^{\sigma\Delta t}\mathrm{e}^{\mathrm{i}\omega} = r\mathrm{e}^{\mathrm{i}\omega\Delta t} \\ z &= \mathrm{e}^{-s\Delta t} = \mathrm{e}^{-\sigma\Delta t}\mathrm{e}^{-\mathrm{i}\omega\Delta t} = r^{-1}\mathrm{e}^{-\mathrm{i}\omega\Delta t} \end{aligned}$$

with $r = \mathrm{e}^{\sigma\Delta t}$, and since $|\mathrm{e}^{-\mathrm{i}\omega\Delta t}| = 1$, the first mapping above sends the plane on the left side (the plane with singularities) to the outside of the circle, $|z| = r$ while the second maps the plane on the left to the inside of the unit circle. Figure 9.5 illustrates this.

Note that since sampling introduce periodicty. It is known that the maximum frequency after sampling is the Nyquist frequency [2]

$$f_N = \frac{1}{2\Delta t},$$

and so since at Nyquist, $\omega_N \Delta t = 2\pi f_N \Delta t$, then the circular dimensionless period is

$$\frac{\cancel{2\Delta t}\pi}{\cancel{2\Delta t}} = \pi.$$

Figure 9.5 in the center frame illustrates the stripes with distances π which indicate the periodicity regions. For simplification the zeroes are drawn only in the first region $\mathrm{Re} s \in [0, \pi]$. Not only the algebra of the power series in z makes the Z-transform easier to operate with, but the mapping into a circle instead of having an infinite number of periodic stripes make the interpretation and analysis much easier.

The **inverse Z-transform** is computed directly from the inverse Laplce transform. That is, we perform a change of variable in equation 9.2.3. $z = \mathrm{e}^{s\Delta t}$, and from here $dz = \Delta t\, \mathrm{e}^{st}\, ds = \Delta t\, z\, ds$. Hence, since $t = n\Delta t$, $\mathrm{e}^{st} = z^n$, and

$$ds = \frac{1}{\Delta t z} dz \quad \Rightarrow \quad \mathrm{e}^{st} ds = \frac{z^{n-1}}{\Delta t} dz.$$

We now look where the Brownwich line goes. As s goes from $s = \sigma - \mathrm{i}\infty$ to $s = \sigma + \mathrm{i}\infty$ we observe that

$$z = \mathrm{e}^{\sigma - \mathrm{i}\omega\Delta t} = \mathrm{e}^{\sigma}\mathrm{e}^{-\mathrm{i}\omega\Delta t} = r\mathrm{e}^{-\mathrm{i}\omega\Delta t}$$

[2] http://en.wikipedia.org/wiki/Nyquist%E2%80%93Shannon_sampling_theorem

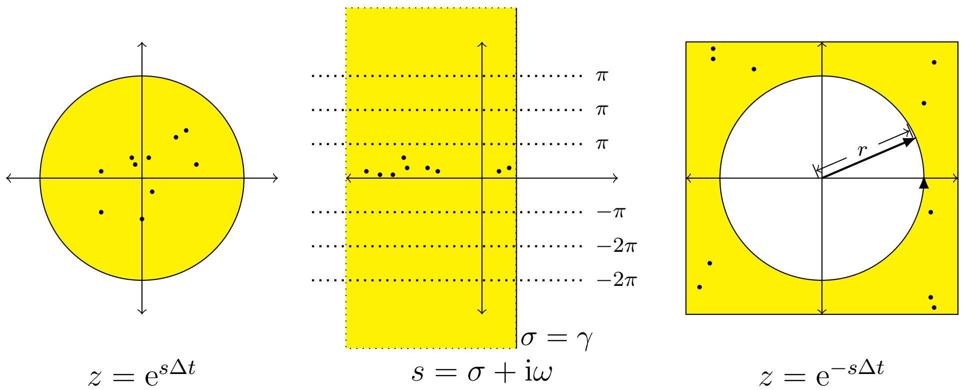


Figure 9.5: If we map the z transform using the geophysical convention (left frame) $z = \mathrm{e}^{s\Delta t}$, then the region of singularities (poles) which in the Laplace domain is at the left of the Brownwich contour which is the vertical line $\mathrm{Re}(s) = \sigma = \gamma$ (center frame) is mapped inside the unit circle. On the other hand the common convention of $z = \mathrm{e}^{-\mathrm{s}}\,\Delta t$ maps the singularities (poles) in the outside of the unit circle $|z| = r$ as shown in the right frame. Sampling introduces periodicity. That is, the stripes on the middle plot represent regions with the same values for the sampled function. Here the period is the circular dimensionless sampling frequency (Nyquist) π. The dots represent poles. They are just a sketch and do not correspond with the true map. The radius r is mapped from the location of the Brownwich contour $\mathrm{Re}s = \gamma$. If $\gamma = 0$ then $|z| = 1$ along the circle, otherwise $|z| = \mathrm{e}^{\pm\sigma\Delta t}$ with the "+" for the geophysical convention. In summary, vertical lines in the s plane map to circles in the z plane. As lines move from left to right, the circles shrink or expand, according to the convention used for the sign of s in the exponential.

where $r = e^{\sigma}$ is fixed and 1 for $\sigma = 0$. Figure 9.5 shows in the third pannel (right) the contour of integration where the circle is transversed in the contour-clockwise direction. We find then that the inverse Z-transform is given by

$$f[n] = \frac{1}{2\pi i \Delta t} \oint_{|z|=r} F(z) z^{n-1} dz.$$

Usually in the literature $\Delta t = 1$. We prefer to keep it in the formula, not only for dimensinal analysis but to for correctness.

As an example of the application of the inverse Z-transform formula, let us find the inverse Z-transform of

$$X(z) = \frac{1}{2 - 3z},$$

Geophysicist consider this a function of z but elsewhere the Z-transform is considered as a function of z^{-1} (why?) We will develope this example with z^{-1} in mind. Hence we write $X(z)$ as

The result could be found by using

(i) Geometrical Series:

(a) **Left Series** That is

$$X(z) = \frac{1}{2}\frac{1}{1 - 3z/2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n (z^{-1})^{-n} = \frac{1}{2}\sum_{n=0}^{-\infty}\left(\frac{3}{2}\right)^n (z^{-1})^{n},$$

from which the coefficients are

$$x[n] = \frac{1}{2}\left(\frac{3}{2}\right)^n \quad n < 1$$

This infinite series converges for $|z| < 2/3\,,\ |1/z| > 2/3$.

(b) **Right Series**

$$\begin{aligned} X(z) &= \frac{1}{z(2\,z^{-1}-3)} \\ &= \frac{z^{-1}}{(2z^{-1}-3)} \\ &= -\frac{1}{3}\frac{z^{-1}}{(1-(2/3)z^{-1})} \\ &= -\left(\frac{1}{3}\right) z^{-1} \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{i} z^{-i} \end{aligned}$$

from which

$$X(z) = -\left(\frac{1}{3}\right) \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n} z^{-n-1} = -\left(\frac{1}{3}\right) \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} z^{-n}$$

and then

$$x[n] = -\frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = -\frac{1}{2}\left(\frac{2}{3}\right)^{n}$$

with $n \geq 1$, and converges for $|z^{-1}| < \frac{3}{2}$ or $|z| > \frac{3}{2}$.

(ii) Contour Integration: We know that

$$x[n] = \frac{1}{2\pi \mathrm{i}} \oint_C X(z) z^{n-1} dz.$$

We can use the Cauchy residue theorem. That is,

$$\frac{1}{2\pi \mathrm{i}} \oint_C X(z) z^{n-1} dz = \sum_{k=1}^{n} R_k,$$

where R_k is a residue.

We further consider several cases:

(a) $n \geq 1$
For a simple pole (and this is the case here)

$$R_k = \lim_{z \to p_k} (z - p_k) X(z) z^{n-1}.$$

since $X(z) = -1/3(z - 2/3)$, then for $p_0 = 2/3$, we consider the contour as the unit circle, for which the only pole $p_0 = 2/3$ is inside.

$$\begin{aligned} R_0[2/3, n] &= \lim_{z \to 2/3} (z - 2/3) X(z) z^{n-1} \\ &= \lim_{z \to 2/3} \left[\cancel{(z - 2/3)} \frac{-1}{3\cancel{(z - 2/3)}} z^{n-1} \right] \\ &= -\frac{1}{3} \left(\frac{2}{3} \right)^{n-1} \\ &= -\frac{1}{2} \left(\frac{2}{3} \right)^{n} \end{aligned}$$

from which the coefficients are

$$R_0[2/3, n] = -\frac{1}{2} \left(\frac{3}{2} \right)^n$$

(b) $n < 1$ Since n is integer we are talking about poles at 0 of multiplicity. If n=0, we get

$$R_0[0] = \lim_{z \to 0} \cancel{z} \frac{1}{(2 - 3z)\cancel{z}} = \lim_{z \to 0} \frac{1}{2 - 3z} = \frac{1}{2}.$$

If $n = -1$ we have a pole of multiplicity 2, from which we apply the formula:

$$R_1[0] = \lim_{z \to 0} \frac{d}{dz} \left(\frac{z^2 X(z)}{z^2} \right) = \lim_{z \to 0} \frac{d}{dz} X(z) = \frac{3}{4}.$$

In general:

$$R_n[0] = \frac{1}{(n-1)!} \frac{d^{n-1}}{d^{n-1}} X(z) \bigg|_{z=0}$$

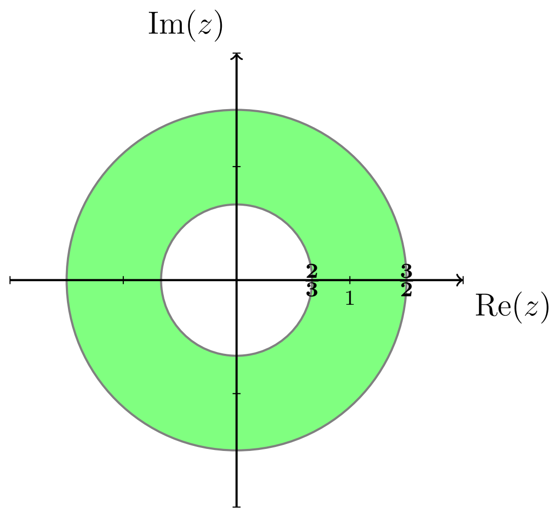


Figure 9.6: Ring of validity for Inverse Z-transform of $X(z) = 1/(2 - 3z)$. The region $|z| > 2/3$ is zone of convergence for the right (causal) series while the region $|z| < 3/2$ is zone of convergence for the left (anticausal) series.

That is, $R_n[0]$ is the $(n-1)^{th}$ coefficient of the Taylor series expansion of $f(z) = 1/(2 - 3z)$.
This is:

$$R_n[0] = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)^{-n}$$

Now we apply the residue equation, and get

$$x[n] = \begin{cases} -\frac{1}{2}\left(\frac{3}{2}\right)^{n} & n \geq 1 \quad |z^{-1}| < 3/2 \\ \\ \frac{1}{2}\left(\frac{3}{2}\right)^{-n} & n > 1 \quad |z^{-1}| > 2/3 \end{cases} \tag{9.4.4}$$

Figure 9.6 illustates the anulus of convergence for the bilateral Z-transform in equation 9.4.4

Chapter 10

Hardy Spaces

Chapter 11

Riemann Surfaces

Chapter 12

Conformal Mapping

Appendices

A Important Facts on Commuting Operators

We discuss a broad class of situations where "operators" commute. By "operators" we mean the following: limits, infinite series, infinite sequences, integrals, total derivatives, partial derivatives. That is we asked questions such as

- If a cascaded of **limit** operations could be reversed in order, without changing the result.
- If two mixed **partial derivatives** would produce the same result by reversing the order.
- If an **integral** and a **differential** operator could be applied in the reverse order,
- If **two integral** operators could reverse the order without changing the result.
- If two **infinite sums** can change the order without producing different results.
- If an **infinite sum** with an **integral** can reverse the order. That is, is it the same the sum of integrals that the integral of the sum?

To address these questions we need to review some basic facts on calculus and analysis.

Ignoring the correct commutation of operators is a common and serious source of errors both analytical and numerical with high costs in human and computing time.

A.1 Review

We review some basic concepts used in Integral and Differential Calculus and analysis.

Limit

The concept of limit [1] is at the hearth of integral and differential Calculus and analysis. Integrals, derivatives, and everything on top of them such as

[1] http://en.wikipedia.org/wiki/Limit_%28mathematics%29

integral, differential, and integro–differential equations is rooted in limits. It took hundreds of years to be formalized. Newton and Leibniz used it to formulate their concepts of integration and differentiation, in the XVII century; however, they did not formalize it. It was Cauchy (see Wikipedia site above) in 1821, followed by Weirstrass who formalized the concept of limit into the following precise definition.

Consider a real function $f(x)$ of on a set A of the real line. We say that the function f has a **limit** L at $a \in A$, if for any $\epsilon > 0$, there exists $\delta > 0$, such that

$$|f(x) - L| < \epsilon \quad , \quad \text{as long as } |x - a| < \delta.$$

We say that L is the limit of $f(x)$ as x approaches a and write in symbols

$$\lim_{x \to a} f(x) = L.$$

Proofs with limits could be tedious. This definition and the proofs with limits are hard when seen the first time. This makes, sense. Think about how long took great mathematicians to come up with a definition such as this. The proof involving limits and continuity boils down to an existence problem. We should find $\delta = \delta(\epsilon)$, provided a given ϵ such that the inequalities above are satisfied. These proofs require training, and the only way to learn about them is by doing them.

Uniform Continuity

Let us define a real function $F : A \subset \mathbb{R} \to \mathbb{R}$, where A is an arbitrary set.

1. A real function $f(x)$ is **continuous** at a point $x_0 \in A$, if for any $\epsilon > 0$, there is a $\delta > 0$, such that

$$|f(x) - f(x_0)| < \epsilon \quad , \quad \text{as long as} \quad |x - x_0| < \delta \quad , \quad \forall x \in A$$

2. A real function $f(x)$ is **continuous in** A if it is continuous in every point of A.

3. A real function $f(x)$ is **uniformly continuous** in A if for any $\epsilon > 0$, there is a $\delta > 0$, such that

$$|f(x) - f(y)| < \epsilon \quad , \quad \text{as long as} \quad |x - y| < \delta$$

with $x, y \in A$.

Note that the definition of (regular) continuity, implies locality. That is, the function is continous at a point x_0. The delta, depends on the chosen point x_0. That is, in regular continuity we can say that $\delta = \delta(\epsilon, x_0)$. For uniform continuity the δ is independent of any chosen $x_0 \in A$. In this way we say that uniform continuity is a **strong** convergence issue, while "point" continuity is a **weak** convergence issue. Uniform continuity implies (regular) continuity, but the opposite is no true. A counter-example is always good.

Think about the function $f(x) = 1/x$ in the interval $(0, 1]$. The function is continuous at each point $x \in (0, 1]$. However it is not uniformly continuous in $(0, 1]$. Let us see.

Pick a point $0 < x_0 \leq 1$. Given $\epsilon > 0$. We have to find δ such

$$|f(x) - f(x_0)| = \left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x - x_0}{x x_0}\right| < \epsilon$$

provided that $|x - x_0| < \delta$. The problem is that the denominator xx_0 can go as close as we want to zero, making the fraction as large as we want. We have to find the maximum bound of the fraction, and this is achieved by the smallest denominator, and this is the smallest product of xx_0 with x_0 is a fixed point. We can choose x away from x_0 an amount δ, as long as $|x_0 - \delta| > 0$ (we do not want to divide by zero). Choose $0 < \delta < x_0$ so that $|x_0 - x| < \delta$ and $x > x_0 - \delta > 0$, so the bound of the inequality above is

$$|f(x) - f(x_0)| < \frac{\delta}{(x_0 - \delta)x_0}$$

The trick in limits is to set

$$\epsilon = \frac{\delta}{(x_0 - \delta)x_0}$$

and find δ from this. That is

$$\delta = \frac{\epsilon x_0^2}{1 + \epsilon x_0}.$$

Then for any $\epsilon > 0$, pick δ as in the previous formula, and you can assert that $|f(x) - f(x_0)| < \epsilon$.

Clearly δ depends both on ϵ and on x_0. The dependence on ϵ can never be removed, but we can remove the dependence on x_0, by, for example, considering a closed interval $[a, 1]$, with $a > 0$. Let us see how this works. Let us consider again the quantity that we want to bound

$$\left| \frac{x - x_0}{x x_0} \right|$$

Since we know that $a > 0$ the upper bound of this expression is

$$\left| \frac{x - x_0}{x x_0} \right| < \frac{x - x_0}{a^2} = \frac{\delta}{a^2}.$$

So choosing $\delta = a^2 \epsilon$, would serve in any case. That is δ does not depend on x_0, and hence the function is uniformly continuous in the interval $[a, 1]$. This result is general. If a function is continuous in an interval $[a, b]$ it is uniformly continuous there. The proof is similar to what we showed here. In infinite dimensional spaces the key sets that correspond to our $[a, b]$ are called compacts. In $\mathbb{R}^n$ closed and bound sets are compacts. See the Heine Borel Theorem [2]. We will not expand more on this

Uniform Convergence

Parallel to the definitions of continuity and uniform continuity, there are the concepts of convergence of sequences, series, and integrals. The terms here are **pointwise** and **uniform** convergence. Let us see.

Given a sequence of real functions $\{f_n\}$ we say that

- f_n converges **pointwise** to f and we write

  $$f_n \underset{\text{pointwise}}{\longrightarrow} f$$

  if

  $$\lim_{n \to \infty} f_n(x) = f(x).$$

  Or, in symbols, if for any $\epsilon > 0$, there is an $N \in \mathbb{Z}^+$ such that

  $$|f_n(x) - f(x)| < \epsilon \quad , \quad \text{provided that } n > N.$$

[2]http://en.wikipedia.org/wiki/Heine%E2%80%93Borel_theorem

- f_n converges **uniformly** to f and we write

  $$f_n \underset{\text{uniformly}}{\rightarrow} f$$

  [3] if

  $$\lim_{n\to\infty} f_n = f,$$

  Or, in symbols, if for any $\epsilon > 0$, there is an $N \in \mathbb{Z}^+$ such that

  $$\|f_n - f\| < \epsilon \quad , \quad \text{provided that } n > N.$$

  In Calculus, it is common to choose the following norm

  $$\|f(x)\| = \|f(x)\|_\infty = \sup_{x\in A} |f(x)|.$$

  Where sup is the supremum [4] of all $f(x)$ with $x \in A$. The supremum is the least upper bound. By making sure that the supremum goes to zero, anything under it collapses to zero. That is the idea of choosing the supremum norm.

Note that in the definition of uniform convergence the particular element $x \in A$ is not important and N is independent of x. That is $N = N(\epsilon)$ alone. The sup norm will take care of all $x \in A$ simultaneously. In the pointwise definition each point x is considered individually and $N = N(\epsilon, x)$.

Uniform continuity is the correct property to have in mind when we want to assert "conservation" of limits along various sequences.

We make the following observations [5]:

- Continuity is the "conservation" of the **limit.** That is if

  $$\lim_{x\to a} f(x) = f(a),$$

  we say that f is continuous in a. This is just the combination of the definition of limit (see definition A.1) and to be continous in a. This is like saying that

  $$\lim_{x\to a} f(x) = f(\lim x \to a),$$

[3] someties the word "uniformly" is omitted, so the default notation $f_n \to f$ is for uniform convergence.

[4] http://en.wikipedia.org/wiki/Infimum_and_supremum

[5] The proofs are easy and found in calculus books

We can say that the lim "commutes" with the operator f, or goes over the operator f.

- **Continuity** is preserved in sequences of functions which converge uniformly. It is interesting that Cauchy, [6] who formally stated the definition of limit, provided a proof in 1821 which states that a convergent sum of continuous functions is always continuous. This is no true and Niels Henrik Abel in 1826 found a counter-example for it. Regardless which counter-example Abel provided, here is a counter-example of a sequence of continuous functions with pointwise convergence to a function which is non-continuous.

  Think about the functions defined by the formula $f_n(x) = x^n$, in the interval $[0, 1]$. They all are continuous. But we know that for $0 < x < 1$, $x^n \to 0$, as $n \to \infty$ and $1^n = 1$, no matter how large n is. So the pointwise (point-to-point) limit of x^n, as $n \to \infty$ is

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & x < 1 \\ 1 & x = 1. \end{cases}$$

  which is clearly non continuous in $[0, 1]$, with a jump discontinuity in the coordinate $x = 1$.

  Figure 1 illustrates this example. It is in this sense that we say that if $f_n \to f$ uniformly, for a sequence of continuous f_n functions, then f is continuous and continuity is "preserved" under the limit operation.

- **Derivatives** are "preserved" in sequences provided uniform convergence. That is, if $f_n' \to g$ uniformly, then $g = f'$, with $f = \lim_{n\to\infty} f_n$. Another way to say this is that the limit commutes with the derivative, or in symbols

$$\lim_{n\to\infty} \frac{df_n}{dx} = \frac{d \lim f_n(x)}{dx},$$

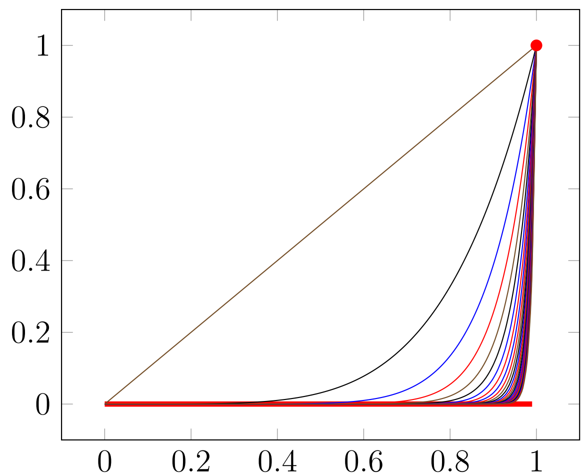
  or that the limit goes inside the derivative symbol. A shorter notation is $f'(x) = \lim_{n\to\infty} f_n'(x)$.

- **Integrals** are "preserved" in sequences provided uniform convergence. Let us assume a sequence of functions $f_n \to f$, uniformly. Then

$$\int f_n \underset{\text{pointwise}}{\to} \int f$$

[6] http://en.wikipedia.org/wiki/Uniform_convergence

Figure 1: We show the function $f_n(x) = x^n$, for $n = 1, 5, 10 \cdots 100$, $x \in [0, 1]$, and see how the limit (thick red line and dot) approches the discontinuous function $f(x) = 0$, $0 \leq x < 1$, and $f(1) = 1$.



In the Riemann sense we can consider as a domain a close interval or a finite union of close intervals. Again, this is saying that the limit commutes or goes into the integral sign, which in symbols is

$$\lim_{n\to\infty} \int_A f_n(x)dx = \int_A \lim_{n\to\infty} f_n(x)dx$$

- **Analyticiy,** is preserved under uniform convergence. That is, if a sequence $\{f_n\}$ of analytic functions in a region S converges uniformly to f, then f is analytic in S.

- **Series** can always be seen as sequences. That is a series $\sum_{n=1}^{\infty} f_n(x)$ can be seen as the sequence $S_N = \sum_{n=1}^{N} f_n$, and $S_N, N \to \infty$ is the same probelm as the infinite series above. So everything known for sequences is inherited under series. Saying that the series is **pointwise** convergent on a set A, means that the sequence $S_N(x)$ is pointwise convergent. The same applies to uniform convergence. If $S_N(x)$ converges to $S(x)$ uniformly then the series is uniformly converging. Hence the limit under uniform convergence of continuous functions is a continuous function.

Here is an interesting result. We know that

$$\int \sum f_n = \sum \int f_n$$

for any finite sum. The integral sign commutes with the sum sign. If the series of continuous functions f_n converges uniformly to f, then we can say that

$$\int_A dx \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_A dx f_n$$

The proof is simple. Consider the partial sums

$$S_N(x) = \sum_{n=1}^{N} f_n(x),$$

hence we already know that the limit goes into the integral sign for uniform convergence sequences. That is,

$$\lim_{N\to\infty} \int_A S_N(x)dx = \int_A \lim_{N\to\infty} S_N(x)dx = \int_A S(x)dx$$

with $S(x) = \sum_{n=1}^{\infty} f_n(x)$. Now,

$$\int_A S_N(x)dx = \int_A \sum_{n=1}^{N} f_n(x)dx = \sum_{n=1}^{N} \int_A f_n(x)dx =$$

Hence:

$$\int_A dx \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_A dx f_n$$

We then say that under uniform convergence the infinite series commute with the integral. When the infinite series becomes an integral and this is a form of Fubini's theorem [7], which we show later.

At this time we have considered functions only of one real variable $f(x)$. We can extend some of the results to several variables. In a way, what we just said above about Fubini's theorem is that we can see $f_n(x)$, as a function $f(x, n)$, where n is an index running over the natural numbers $\mathbb{N}$. We want to extend this n to consider any arbitrary number in the real line and think of $f(x, y)$ as a function in $\mathbb{R}^2$. We deal with this in the next section.

[7] http://en.wikipedia.org/wiki/Fubini%27s_theorem

A.2 Fubini's Theorem

Changing the order of integration could be a source of error. Here is an example suggested by Wikepedia [8]

Consider:

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}.$$

Compute both

$$\int_1^\infty \left(\int_1^\infty f(x,y)dy \right) dx \quad \text{and} \quad \int_1^\infty \left(\int_1^\infty f(x,y)dx \right) dy$$

This is an interesting example. The function is anti-symmetric. That is, $f(x,y) = -f(y,x)$. So reversing the order changes the sign. The integration in both cases is along the same interval $[1\ \infty[$. So we can expect that whatever we get from the integral on the left, should be the same as what we get in the integral on the right, but with opposite sign. So, unless each integral is equal to zero, we will find a contradiction of the form $a = -a$, $a \neq 0$.

Let us evaluate the inside of the left inside integral. We transform the fraction as follows

$$\frac{x^2 - y^2}{(x^2+y^2)^2} = \frac{1-(y/x)^2}{x^2(1+(y/x)^2)^2}$$

Do an initial change of variables

$$\begin{aligned} u &= \frac{y}{x} \quad \Rightarrow \quad y = xu \\ dy &= x\,du \qquad 1 \leq u \leq \infty \end{aligned}$$

We now need to integrate

$$\frac{1-u^2}{x(1+u^2)^2}.$$

Another change of variable is

$$u = \tan\theta \qquad du = \sec^2\theta d\theta \qquad \pi/4 < \theta < \pi/2$$

[8] http://en.wikipedia.org/wiki/Order_of_integration_%28calculus%29

and the expression further simplifies to

$$\frac{\sec^2\theta(1-\tan^2\theta)}{x\sec^4\theta} = \frac{1-\tan^2\theta}{x\sec^2\theta} = \frac{\cos^2\theta-\sin^2\theta}{x} = \frac{\cos 2\theta}{x}$$

This integral (with respect to θ) is easy. Its value is $\sin 2\theta/2x$. That is

$$\frac{\sin 2\theta}{2x} = \frac{\cos\theta\sin\theta}{x}$$

Since $y/x = u = \tan\theta$, then we have

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} \qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

and

$$\frac{\sin 2\theta}{2\,x} = \frac{y}{x^2+y^2}$$

and

$$\int_1^\infty f(x,y)dy = \left.\frac{y}{x^2+y^2}\right|_1^\infty = -\frac{1}{1+x^2} \qquad x \geq 1.$$

Then

$$\int_1^\infty \left(\int_1^\infty f(x,y)dy\right)dx = -\tan\theta\Big|_{\pi/4}^{\pi/2} = -\frac{\pi}{4}.$$

which is non-zero, and we know that reversing the order of integration we would get a sign change. That is

$$\int_1^\infty \left(\int_1^\infty f(x,y)dx\right)dy = \frac{\pi}{4}.$$

So, clearly changing the order of integration could be troublesome. Hence we need to be careful whenever we need to change the order of integration.

We prove Fubini's theorem first for proper integrals and then for improper integrals. Let us define a function

$$F(x) = \int_a^b f(x,y)dy, \tag{A.1}$$

where $f(x, y)$ is a real function defined in the set $[a, b] \times [c, d]$. We assume f to be continuous, but having a finite number of discontinuities or a a set of measure zero of discontinuities should not change the results. We will not deal with Lebesgue measures, but only Riemann integration.

Theorem B.5 (below) shows that if $f(x, y)$ is continuous in the rectangle $[a, b] \times [c, d]$, then the function F is uniformly continuous in $[a\ b]$. We show now that for closed domains of the type $[a, b] \times [c, d]$ the differential operator can go over the integral sign. That is,

Theorem A.3 *If in the closed rectangle $[a, b] \times [c, d]$ the function $f(x, y)$ is continuous and has a continuous derivative with respect to x, the differential operator can enter the integral. That is we can interchange the diffentiation with the integration operators. In symbols:*

$$\frac{dF(x)}{dx} = \frac{d}{dx}\int_a^b f(x, y)dy = \int_a^b \frac{\partial}{\partial x} f(x, y)dy.$$

Moreover, $F'(x)$ is continuous.

Proof: We write

$$\begin{aligned} F(x + h) - F(x) &= \int_a^b f(x + h, y)dy - \int_a^b f(x, y)dy \\ &= \int_a^b [f(x + h, y) - f(x, y)]dy. \end{aligned}$$

We use the mean theorem of calculus to write

$$f(x + h, y) - f(x, y) = hf_x(x + th, y) \quad 0 < s < 1.$$

Now since f_x is continuous in the closed rectangle, it is uniformly continuous, so

$$|f_x(x + th, y) - f_x(x, y)| < \epsilon \quad \text{whenever} \quad |h| < \delta,$$

where $\delta = \delta(\epsilon)$ that is δ does not depend on x or y. Thus,

$$\begin{aligned} \left|\frac{F(x + h) - F(x)}{h} - \int_a^b f_x(x, y)dy\right| &= \left|\int_a^b f_x(x + th, y)dy - \int_a^b f_x(x, y)dy\right| \\ &\leq \int_a^b \epsilon dy = \epsilon(b - a). \end{aligned}$$

for $|h| < \delta(\epsilon)$, provided that $h \neq 0$. Then

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \int_a^b f_x(x,y)dy = F'(x).$$

We could even have the x parameter in the limits of integration. Let us assume that we want to find $F'(x)$, for F defined as

$$\int_{g_1(x)}^{g_2(x)} f(x,y)dy,$$

where x appears not only in the integrand but in both limits of integration. We use the following expression

$$\phi(u,v,x) = \int_u^v f(x,y)dy.$$

where we split the three parts of interest into u, v, and x, with $u = g_1(x)$ and $v = g_2(x)$, where u_x and v_x exists and are continuous, and $a < g_1(x) < g_2(x) < b$, for $\alpha \leq x \leq \beta$. Let, moreover, $f(x,y)$ and $f_x(x,y)$ be continuous for $(x,y) \in [\alpha, \beta] \times [a,b]$. The function ϕ of the three independent variables u, v, x is defined for

$$\alpha \leq x \leq \beta \quad , \quad a \leq u \leq b \quad , \quad a \leq v \leq b$$

Moreover, it has continuous partial derivatives.

- $$\phi_x(u,v,x) = \frac{\partial}{\partial x} \int_u^v f(x,y)dy = \int_u^v f_x(x,y)dy$$

  due to the result just shown above.

- From the fundamental theorem of calculus

  $$\phi_v(u,v,x) = f(x,v) \quad , \quad \phi_u(u,v,x) = -f(x,u).$$

We now use the chain rule. That is

$$F(x) = \phi[g_1(x), g_2(x), x]$$

so

$$F'(x) = g_1'(x)\phi_u + g_2'(x)\phi_v + \phi_x$$

or

$$\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(x,y)dy = \int_{g_1(x)}^{g_2(x)} f_x(x,y)dy - g_1'(x)f(x,g_1(x)) + g_2'(x)f(x,g_2(x)).$$

Before proving Fubini's theorem we show a theorem which is closed related to it. This is the mixed partial derivatives, if they exist and are continuous, they "commute". Or, in other words, $f_{xy}(x,y) = f_{yx}(x,y)$. This is known as Clairet's theorem and the and details including its proof are below.

Clairet's Theorem

We first proof the following

Lemma B *Let f_{xy} and f_{yx} be continuous on rectangle $R = [a,b] \times [d,c]$. Then*

$$\int_R f_{xy}\,dA = \int_R f_{yx}\,dA = f(b,d) - f(a,d) - f(b,c) + f(a,c).$$

Proof: We show only the assertion for f_{xy}. The other assertion is shown in the same manner.

Let us see:

$$\begin{aligned}
\int_R f_{xy}dA &= \int_a^b \left(\int_c^d f_{xy}(x,y)dy\right)dx \\
&= \int_a^b \left(f_x(x,y)|_{y=c}^{y=d}\right)dx \\
&= \int_a^b [f_x(x,d) - f_x(x,c)]dx \\
&= [f(x,d) - f(x,c)]|_a^b \\
&= f(b,d) - f(a,d) - f(b,c) + f(a,c),
\end{aligned}$$

where the first equality is from the definition (in that order) and the second and third come from iterated applications of the fundamental theorem of calculus (here is where the continuity is required) to each variable y and x, in that order.

Let us now show Clairaut's theorem

Theorem B.1 *(Clairaut's) If f_{xy} and f_{yx} are continuous (in an open set), then they are equal.*

Proof: Let us assume the opposite, that is, there is point (a, b) such that

$$f_{xy}(a, b) - f_{xy}(a, b) = h \quad \text{say } h > 0$$

(if $h < 0$, then show the theorem using the reverse order of this difference). By continuity there is neighborhood centered at (a, b), in which

$$f_{xy}(x, y) - f_{yx}(x, y) \geq \frac{h}{2}.$$

Choose a rectangle $R = [a - \Delta x/2, a + \Delta x/2] \times [b - \Delta y/2] \times [b + \Delta y/2]$ inside this neighborhood. Then

$$\int_R (f_{xy} - f_{yx}) dA \geq \int_R \frac{h}{2} dA = \frac{h}{2} \Delta x \Delta y > 0.$$

This contradicts the Lemma above, since the integral of the difference between f_{xy} and f_{yx} in R should be zero. Then $f_{xy} = f_{yx}$.

We are now ready to proof the Fubini's theorem for closed intervals. This is,

Theorem B.2 *(Fubini) Let $f(x, y)$ be continuous in the rectangle $R = [a, b] \times [c, d]$. Then the integrals*

$$\int_a^b dx \int_c^d f(x, y) dy \quad \text{and} \quad \int_c^d dy \int_c^d f(x, y) dx$$

are equal.

Proof: This theorem could be proven from the very basics. That is, by using the definition of Riemann (or Lebesgue) integral as limits. However a common proof is based on the Clairaut's theorem (B.1), sometimes also known as Schwarz's theorem [9] which indicates that if the mixed (second order) partial derivatives exist in an open set or $\mathbb{R}^2$, and they are continuous, then they commute. That is for $f(x,y)$, $f_{xy}(x,y) = f_{yx}(x,y)$.

The trick is to define two auxiliary functions:

$$v(x,y) = \int_a^y f(x,s)ds \quad , \quad u(x,y) = \int_a^x v(t,y)dt$$

The function $v(x,y)$ is continuous because $f(x,s)$ is continuous. The function $u(x,y)$ is continuous because $v(t,y)$ is continuous.

We apply the fundamental theorem of calculus first to $u(x,y)$, and then to $v(x,y)$. That is

$$u_x(x,y) = v(x,y) \quad , \quad v_y(x,y) = f(x,y),$$

from which

$$u_{xy}(x,y) = f(x,y).$$

and since $f(x,y)$ is continuous, then $u_{xy}(x,y) = u_{yx}(x,y)$ and from Lemma B

$$\int_R u_{xy}\, dA = \int_R u_{yx}\, dA.$$

That is

$$\int_R f(x,y)dA = \int_R f(y,x)dA,$$

so it does not matter the order on x and y and the integral provides the same value, in this case $u(b,d) - u(a,d) - u(b,c) + u(a,c)$. Note that still $x \in [a,b]$ while $y \in [c,d]$ so care should be taken of tying the domain of definition of the particular variable.

B.3 Interchange between partial derivatives and integrals: Improper integrals

We start with a proper integral in equation B.2. That is,

$$F(x) = \int_a^b f(x,y)dy, \tag{B.2}$$

[9] http://en.wikipedia.org/wiki/Symmetry_of_second_derivatives

Let us now start the extension to improper integrals. We say that the integral

$$F(x) = \int_a^\infty f(x,y)dy$$

converges uniformly on an interval I if the family

$$F_R(x) = \int_a^R f(x,y)dy$$

converges uniformly to F on I. That is, if

$$\|F_R - F\| \to 0 \qquad \text{as} \quad R \to \infty$$

Here is a test for uniform convergence of integrals.

Theorem B.4 *Assume:*

(i) $f(x,y)$ *is continuous in* $I \times [a, \infty[$

(ii) $|f(x,y)| \leq M(y)$ *for all* $x \in I$ *and* $y \in [a, \infty[$

(iii) $\int_a^\infty M(y)dy$ *is finite (converges)*

Then

$$F(x) = \int_a^\infty f(x,y)dy$$

converges uniformly on I*.*

Proof: We know

$$F_R(x) - F(x) = \int_R^\infty f(x,y)dy$$

from which we have

$$|F_R(x) - F(x)| \leq \int_R^\infty |f(x,y)|dy \leq \int_R^\infty M(y)dy,$$

Now, since $\int_R^\infty M(y)dy$ is finite, then as $R \to \infty$, this integral should go to 0. That is regardless which x

$$|F_R(x) - F(x)| \to 0 \quad \text{as} \quad R \to \infty,$$

or

$$\|F_R - F\| \to 0 \quad \text{as} \quad R \to \infty,$$

So the integral $F(x)$ converges uniformly on I.

We now prove the continuity of F_R as inherited from the continuity of $f(x, y)$ in an interval $[a, R] \times [c, d]$. This is:

Theorem B.5 *For each finite $R > 0$, we have*

$$F_R(x) = \int_a^R f(x, y)dy$$

is uniformly continuous on $[c, d]$ if $f(x, y)$ is continuous in $[c, d] \times [a, R]$

Proof: If $f(x, y)$ is continuous in the compact set $[a, R] \times [c, d]$, then it is uniformly continuous there. That means that, for any given ϵ, there is a δ such that if

$$|x_1 - x_0| \leq \|(x_1, y_1) - (x_0, y_0)\| < \delta$$

then

$$|f(x_1, y_1) - f(x_0, y_0)| < \epsilon.$$

Hence,

$$\begin{aligned} |F_R(x_1) - F_R(x_0)| &= \left| \int_a^R f(x_1, y) - f(x_0, y)dy \right| \\ &\leq \int_a^R |f(x_1, y) - f(x_0, y)|dy \\ &\leq \epsilon \int_a^R dy = (R - a)\epsilon. \end{aligned}$$

Then for each R fixed, $|F_R(x_1) - F_R(x_0)| \to 0$, as $|x_1 - x_0| \to 0$. Hence F_R is uniformly continuous in $[c, d]$. If the convergence from F_R to F is uniform, then F is continuous. We show how this works.

We can now think about a sequence

$$f_n(x) = \int_a^n f(x, y)dy.$$

Given that f_n converges uniformly to F, and each f_n is continous then we showed above that the limit is continuous as well. That is

$$F(x) = \int_a^\infty f(x,y)dy$$

is continuous in $[c,d]$.

We now prove one form of Fubini's Theorem. Assume $f(x,y)$ is continuous in the interval $I \times [a,\infty[$, then

Theorem B.6 *(Fubini) If $f(x,y)$ is continuous in $I \times [a,\infty[$, and*

$$F(x) = \int_a^\infty f(x,y)dy$$

converges uniformly in I, then for any bounded, closed interval $[b,c] \subset I$ we have

$$\int_b^c \int_a^\infty f(x,y)dxdy = \int_a^\infty \int_b^c f(x,y)dydx$$

Proof: Let us define

$$F_n(x) = \int_a^n f(x,y)dy$$

(this is $F_R(x)$ above. We prefer n over R). We show above that each f_n is continuous, and since the convergence from F_n to F with

$$F(x) = \int_a^\infty f(x,y)dy \tag{B.3}$$

is uniform, with $F(x)$ continuous. We show this in three pieces

(i) We integrate equation B.3 between c and d to find

$$\int_c^d F(x)dx = \int_c^d \left(\int_a^\infty f(x,y)dy \right) dx.$$

This can be done because the function $F(x)$ is continuous over a closed interval. So it is integrable there.

(ii)

$$\begin{aligned}\lim_{n\to\infty}\int_c^d F_n(x)dx &= \int_c^d \lim_{n\to\infty}\left(\int_a^n f(x,y)dy\right)dx\\ &= \int_c^d F(x)dx\\ &= \int_c^d\left(\int_a^\infty f(x,y)dy\right)dx.\end{aligned}$$

(iii)

$$\begin{aligned}\lim_{n\to\infty}\int_c^d F_n(x)dx &= \lim_{n\to\infty}\int_c^d\left(\int_a^n f(x,y)dy\right)dx\\ &= \lim_{n\to\infty}\int_a^n\left(\int_c^d f(x,y)dx\right)dy.\\ &= \int_a^\infty\left(\int_c^d f(x,y)dx\right)dy.\end{aligned}$$

In the previous to the last step we applied Fubini's theorem B.2 for a rectangle.

Theorem A.3 shows the commutation of a derivative with an integral sign for a finite rectangle. We now extend this to improper integrals. That is,

Theorem B.7 *If $f(x,y)$ and $f_x(x,y)$ are continuous in $I\times[a,\infty[$,*

$$F(x)=\int_a^\infty f(x,y)dy \quad \text{converges}$$

and

$$G(x)=\int_a^\infty f_x(x,y)dy$$

converges uniformly in I then

$$F'(x)=\int_a^\infty f_x(x,y)dy.$$

Proof: Since $G(x)$ converges uniformly in I, and $f_x(x,y)$ is continuous on $I \times [a, \infty[$, then $G(x)$ is continuous on I. Then for any $b, x \in I$ we have

$$\begin{aligned}\int_b^x G(t)dt &= \int_b^x \left(\int_a^\infty f_t(t,y)dy\right) dt \\ &= \int_a^\infty \left(\int_b^x f_t(t,y)dt\right) dy \\ &= \int_a^\infty [f(x,y) - f(b,y)]dy \\ &= F(x) - F(b).\end{aligned}$$

In step two we applied the Fubini's theorem for improper integrals, then in step three and four we applied the fundamental theorem of calculus. Again, from the fundamental theorem of calculus

$$\frac{d}{dx}\int_b^x G(t)dt = G(x),$$

and from this it follows that $F(x)$ is differentiable and

$$F'(x) = G(x) = \int_a^\infty f_x(x,y)dy.$$

The theorem is proved.