Representations of the Dihedral Group D_3

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March 31, 2013

Abstract

This article shows a set of representations of the Dihedral group D_3.

1 Introduction

Group theory is a branch of Abstract Algebra. This theory abstracts the set of objects that have the properties of closure under a given operation (the operation on two objects of the group is a member of the group), the existence of the identity and the existence of the inverse for any member of the group. The operation should also posses the associative law.

The applications of Group Theory are numerous in the fields of solid state physics, crystallography, cryptography, theory of equations etc.

My motivation to write this pamphlet arose from reading a book in Modern Crystallography (Vainshtein, 1994). The author shows a representation of the Dihedral group D_3 as a set of six 2×2 matrices with just plus or minus ones and zeroes (page 52). He does not show how he came up with those and lists a couple of references

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(apparently from the Russian literature) which were not available to me. I though that finding this was an easy task, either trying to find them in my own or doing a Google search. I searched through thousands of website links and eye-scanned thousands of pages of books without finding anything that hinted me to solve this question until finally I found the following website link [*] Mackiw with the title "Finite Groups of 2x2 Integer Matrices" by George Mackiw. In this particular page George Mackiw motivates the diheral D_6 group with a simple geometrical selection of the base vectors for the two dimensional euclidean space. I was thinking in these terms but my algebra did not fit the results. Having something that works (this time for an hexagon instead of a triangle) makes things easier.

I think that there are several ways to solve a problem. One is to try to do it yourself without consulting external references (the re–inventing of the wheel method), another is to use any tools at hand. With the invention of the Intenet and after this, the development of the search engines, problems can be solved much quicker. So, one way to solve a problem is to find the solution somewhere else. In practice a problem is solved by a combination of the two methods above. I might be playing unfair with paper books, but a good search engine in an almost unlimited collection of resources bits any paper book or even a giant physical library. Still paper books are nice and they will not disappear (I think).

A good thing about solving a "hard" problem is that we learn many things in our way. I searched so many websites and read so many paragraphs and articles from many electronic books that I ended up learning more than I expected while trying to solve this problem; so instead of just providing the solution to my own question I will provide in addition, a summary of the different representations of the dihedral D_3 group. This summary includes as many representations of the group as I can remember now, and as far as I know, no other website link (that I came across) have those many representations. The purpose here is then to extend my memory (I would not remember much of this in 10 years) and also provide a document that could be shared with my friends, while putting my ideas in order at the same time.

[*] http://www.jstor.org /pss /2691281

2 The Dihedral Group

There are plenty of information about this group in the internet. The Wikipedia [†] website provides a good description of this group. I will describe only de D_3 group corresponding to the symmetries of an equilateral triangle. By definition a simmetry of and object is an operation that leaves the shape of the object unchanged. The triangle can be rotated by 0,120, and 240 degrees with respect to its center and it will not change. This provides three symmetries. The other symmetries are the flips (reflections) with respect to its three symmetry axes. Figure (1) shows an equilateral triangle in the top left corner, with vertices labeled as a, b, c when moving from top to botom counter–clockwise. This triangle was drawn with PowerPoint and the letters a, b, c glued to it (grouped). Then a rigid motion was applied using PowerPoint capabilities. The first operation is the identity $I = g_0$. Note that I use Vainshtein, (1994) notation for this group on the symbols g_i, when i varies between 0 and 5. So that g_0 is just a copy of the original triangle. $R_1 = g_1$ and $R_2 = g_2$ are counter–clockwise rotations of 120 and 240 degrees respectively. $F_1 = g_3$ is a flip (reflection) with respect to the vertical on the original triangle g_0, $F_2 = g_4$ is a flip with respecto to the vertical of the second group element g_1, and $F_3 = g_5$ is a flip wth respect to the vertivcal of the third group element g_2.

2.1 Representations of D_3

It is easy to verify that this set of transformation fit into an algebraic group. The Cayley table (1) is a construction of all the operations within the group. The construction of this table is as follows. The operation $g_i * g_j$ is a composition of transformations (rotations or flips) and it applies as in the composition of functions from right to left. That is g_j is performed first and then g_i is applied to the outcome of g_j. Every element g_i operated with the identity g_0 produces the same element, that is $g_i * g_0 = g_0 * g_i = g_i$. The g_1 and g_2 are rotations and therefore the composition of two rotations is a rotation with the sum of the angles as the argument. Since they are plane rotations they are commutative, that is $g_2 * g_1 = g_1 * g_2 = g_0$, $g_1 * g_1 = g_2$, $g_2 * g_2 = g_1$. The flips (reflections) have order two, so that $g_3 * g_3 = g_4 * g_4 = g_5 * g_5 = g_0$. Now, for the mixed components (rotations and flips) we have (use figure 1 to check the following operations) that $g_3 * g_1 = g_4$. That

† http://en.wikipedia.org /wiki /Dihedral _ symmetry

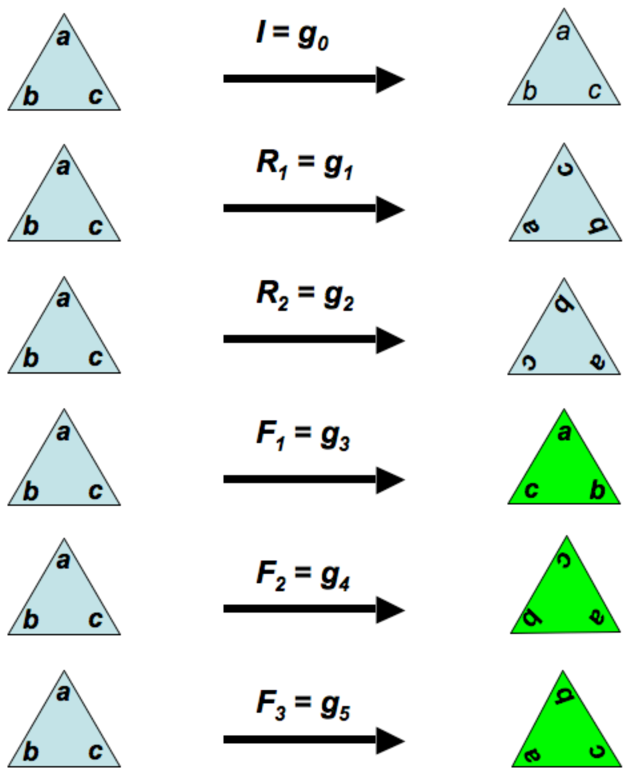

Figure 1: The symmetries of an equilateral triangle. The first three operations are counter–clockwise rotations, and the next three operations of 120 degrees

Table 1: Cayley table for the D_3 group

*	g_0	g_1	g_2	g_3	g_4	g_5
g_0	g_0	g_1	g_2	g_3	g_4	g_5
g_1	g_1	g_2	g_0	g_5	g_3	g_4
g_2	g_2	g_0	g_1	g_4	g_5	g_3
g_3	g_3	g_4	g_5	g_0	g_1	g_2
g_4	g_4	g_5	g_3	g_2	g_0	g_1
g_5	g_5	g_3	g_4	g_1	g_2	g_0

is a rotation of 120 degrees counter–clockwise followed by a reflection with respect to the vertical axis ‡. Using this result we obtain $g_4 * g_1 = g_3 * g_1 * g_1 = g_3 * g_2 = g_5$. This is counter–clockwise rotation of 240 degrees followed by a reflection with respect to the vertical axis. Using this again $g_5 * g_1 = g_3 * g_2 * g_1 = g_3$. Now $g_4 * g_2 = g_3 * g_1 * g_2 = g_3$, $g_5 * g_2 = g_3 * g_2 * g_2 = g_3 * g_1 = g_4$. From figure 1 we have $g_1 * g_3 = g_5$. This is a cascade of a reflection with respect to the vertical line and a 120 counter–clockwise rotation. Along the same lines $g_2 * g_3 = g_4$ and from here $g_4 * g_3 = g_2 * g_3 * g_3 = g_2$. Also $g_5 * g_3 = g_3 * g_2 * g_3 = g_3 * g_4 = g_3 * g_3 * g_1 = g_1$. The new operations starting at g_4 are:

$$\begin{aligned} g_1 * g_4 &= g_1 * g_2 * g_3 = g_3 \\ g_2 * g_4 &= g_2 * g_2 * g_3 = g_1 * g_3 = g_5 \\ g_3 * g_4 &= g_3 * g_2 * g_3 = g_5 * g_3 = g_1 \\ g_5 * g_4 &= g_5 * g_2 * g_3 = g_4 * g_3 = g_2 \end{aligned}$$

and finally the new operations starting with g_5 are:

$$\begin{aligned} g_1 * g_5 &= g_1 * g_2 * g_4 = g_4 \\ g_2 * g_5 &= g_2 * g_3 * g_2 = g_4 * g_2 = g_3 \\ g_3 * g_5 &= g_3 * g_2 * g_4 = g_5 * g_4 = g_2 \\ g_4 * g_5 &= g_4 * g_2 * g_4 = g_3 * g_4 = g_1. \end{aligned}$$

A good way to check for errors in the table is to observe that no repetitions of elements can appear either in any column or row. The probability of finding an error is high. The total number of right and wrong entries (combinations) in a Cayley table of order n is $n^{(n^2)}$ while the only possible right permutations is given by $(n!)!/(n! - n)!$. For the D_3 group we find that the universe of options is given by $6^{36} = 10314424798490535546171949056$ while the only possibly valid permutations is given by $720 * 719 * 718 * 717 * 716 * 715 = 136434451994755200$ so the probability (assuming, of course, that all options are equally possible, which is not true) of catching an error this way is $1 - 0.00000000001322753858 = .99999999998677246142$. By the way, it should not come to surprise Cayley theorem that states that any finite group is a

‡All reflections will be seen as with respect to the vertical axis to avoid confusion on this particular analysis

subset of permutations, since any finite group has such a Cayley table representation as the one in table (1).

2.1.1 Representations of D_3 in the space of permutations of 6 objects

We can think of the D_3 group as the group of permutations of the indices 0 through 5. For example let us consider the element corresponding to g_2 as the column under g_2. This element can then be associated with the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 0 & 1 & 5 & 3 & 4 \end{pmatrix} \tag{2.1}$$

The permutation is a bijective function of a finite dimensional set to itself. In this case, as a function, for example $g_2(3) = 5$. A short hand notation for this permutation is

$$(021)(354) \tag{2.2}$$

which indicates that the element 0 maps into the element 2, the element 2 maps into the element 1 and the element 1 circles back to the element 0; and similarly for the loop (354). With this short notation the group D_3 is represented as:

$$\begin{aligned} g_0 &= (0)(1)(2)(3)(4)(5) && (2.3) \\ g_1 &= (012)(345) \\ g_2 &= (021)(354) \\ g_3 &= (03)(15)(24) && (2.4) \\ g_4 &= (04)(13)(25) \\ g_5 &= (05)(14)(23) \end{aligned}$$

The operation between elements of the group is product of permutations (function composition). For example

$$g_3 * g_1 = (03)(15)(24) * (012)(345). \tag{2.5}$$

The index 0 goes by g_3 into index 3 and this by g_1 into index 4,
the index 1 goes by g_3 into index 5 and this by g_1 into index 3,
the index 2 goes by g_3 into index 4 and this by g_1 into index 5,
the index 3 goes by g_3 into index 0 and this by g_1 into index 1,
the index 4 goes by g_3 into index 2 and this by g_1 into index 0,
the index 5 goes by g_3 into index 1 and this by g_1 into index 2
so the outcome is $g_4 = (04)(13)(25)$ which corresponds to the result in the Cayley Table (1).

2.1.2 Representations of D_3 in the space of 6×6 matrices over the integers

We know that another way to represent permutations is as permutations of the rows or columns of the identity matrix. On the example above: The group elements $g_1 = (012)(345)$ and $g_3 = (03)(15)(24)$ can be written as matrices as follows

$$g_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad g_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and as expected

$$g_3 * g_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} = g_4.$$

2.1.3 Representations of D_3 as motions of a rigid triangle in a plane, 2×2 matrices over the real numbers

Here we acknowledge the physics of the original problem and will use the rotations and reflections in the plane as 2×2 matrices. The rotation matrix in the plain is defined as

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2.6}$$

The flip that will take a point (x, y) in the plane to its reflection with respect to the vertical axis is simply defined by the matrix

$$F = g_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.7}$$

We will follow Figure (1). The first three operations are rotations of 0, 120 and, 240 degrees. The corresponding matrices for these three groups are:

$$g_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad g_1 = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \quad g_2 = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \tag{2.8}$$

From the previous construction (and from the Cayley table (1)) we see that

$$g_4 = g_3 * g_1 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \tag{2.9}$$

and

$$g_5 = g_3 * g_2 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \tag{2.10}$$

Figure (2) shows the equilateral triangle for the identity element g_0 in Figure (1), this time with Cartesian coordinates on the vertices a, b, c as follows: $a = (0, 1), b = (-\sqrt{3}/2, -1/2,), c = (\sqrt{3}/2, -1/2)$. The reader can check that indeed g_4 and g_5 follow the mappings:

$$\begin{array}{ccccccc} a & \mapsto & c & \quad & a & \mapsto & b \\ b & \mapsto & b & & b & \mapsto & a \\ c & \mapsto & a & & c & \mapsto & c \end{array} \tag{2.11}$$

respectively. See that this is consistent with Figure (1).

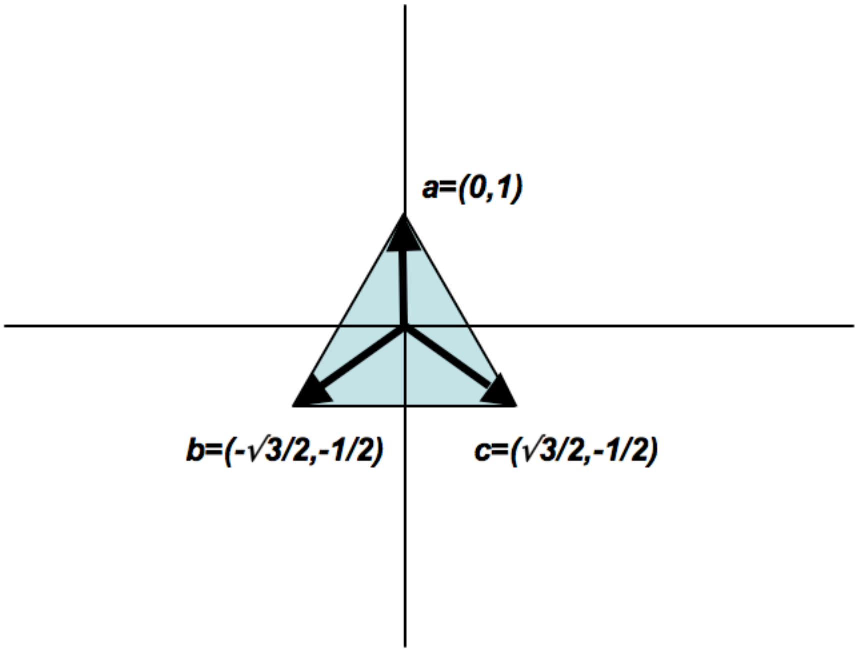

Figure 2: The symmetries of an equilateral triangle. The first three operations are counter–clockwise rotations, and the next three operations of 120 degrees

2.1.4 Representations of D_3 as motions of a rigid triangle in the space, 3×3 matrices over the real numbers

In the previous representation we saw the triangle in the 2D world. In the 2D world a flip is an improper rotation. That is, the flip is an impossible (no realizable) physical transformation (except on the virtual space of images through a mirror). The determinant of this improper rotation is -1. In 3D the rotation is a regular rotation. Figure (3) illustrates the situation. Here the flip is a regular rotation on the y axis by 180 degrees. We will construct the 3×3 matrices for this case. The classical rotations in the $x-y$ plane are given by the 3×3 matrix

$$r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.12}$$

which represents a rotation with respect to the z axis. We state that

$$g_0 = r(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.13}$$

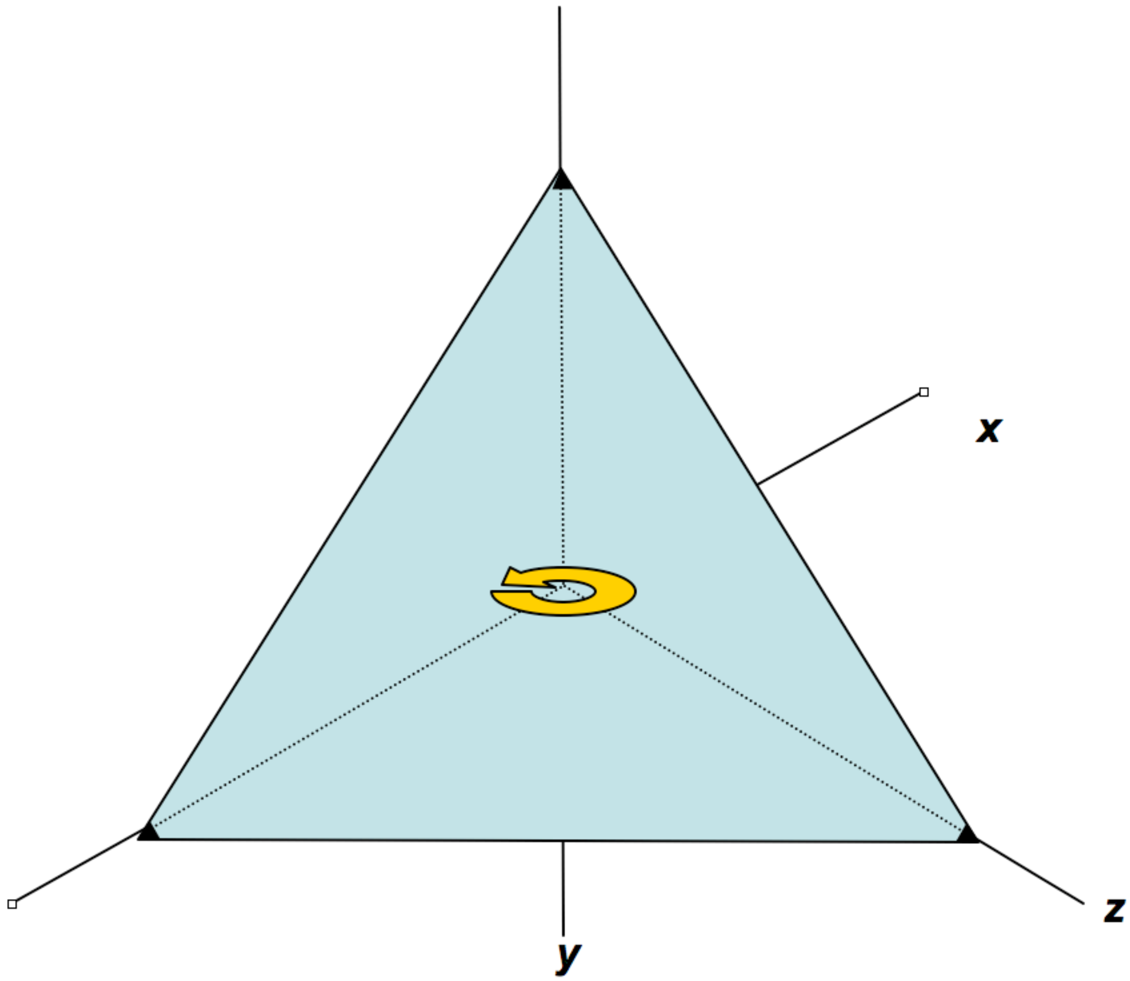

Figure 3: We can see that the flip is now a rotation of 180 degrees with respect to the y axis

$$g_1 = r(120) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.14}$$

$$g_2 = r(240) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} & 0 \\ -\sqrt{3} & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.15}$$

Now the flip s is described as a rotation or 180 degrees around the y axis. That is

$$g_3 = s = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2.16}$$

$$g_4 = g_3 * g_1 = s = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2.17}$$

and

$$g_5 = g_3 * g_2 = s = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} & 0 \\ -\sqrt{3} & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2.18}$$

2.1.5 Representations of D_3 as permutations of three objects: S_3

The group of permutations of three elements S_3 is known as the *symmetric group*. The number of permutations of three objects is 6. This coincides with the D_3 group since it is a group of permutations of 3 elements of order 6. From Figure (1) it is easy to observe that

$$\begin{array}{ccc} g_0 = (a)(b)(c) & g_1 = (acb) & g_2 = (abc) \\ g_3 = (bc) & g_4 = (ac) & g_5 = (ab). \end{array} \tag{2.19}$$

The regular product of permutations produces the Cayley table (1).

2.1.6 Representations of D_3 as matrices 2×2 with ± 1 and 0 entries

This is actually the problem that motivated the writing of this pamphlet. Vainshtein, (1994) shows in his book (page 52) the following representation of the dihedral group D_3.

$$g_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad g_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad g_2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$g_3 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \quad g_4 = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \quad g_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.20}$$

Vainshtein does not explain how to obtain these matrices but he gives a hint that they were obtained by reducing some 3×3 matrices. I do not define the expression "reduce" here. He gives a couple of references that seems to be Russian articles. I do not have access to them.

One way I can get to Vainshtein set of matrices is by starting with the 2×2 system in equations (2.8, 2.9, 2.10) and transforming them to they system in equation (2.20). We start with a justification for this transformation by defining the concept of conjugate. Let us assume that an element of a group H exist, call it h, such that it produces new elements from group G by using the equation

$$\bar{g} = h^{-1} * g * h. \tag{2.21}$$

We assume that H is a group that contains G as a subset and that the operation $*$ is well defined in H. Then we say that the element $\bar{g}$ is conjugate of g. The set of elements obtained after using all elements of G is a group (an isomorphic group of G) under the same operation $*$. Let us check. The group is a subset of H, so for a subgroup we only have to check closure under product and closure under inverse. Given g_1 and g_2 in G and the corresponding $\bar{g_1}$, $\bar{g_2}$ in H we have

$$\begin{aligned} \bar{g_1} * \bar{g_2} &= (h^{-1} * g_1 * h) * (h^{-1} * g_2 * h) \\ &= h^{-1} * g_1 * (h * h^{-1}) * g_2 * h \\ &= h^{-1} * (g_1 * g_2) * h \end{aligned} \tag{2.22}$$

and since $g_1 * g_2$ is an element of the group G the conjugacy relation is well preserved under product [§] , so closure under product is maintained. Now

$$(\bar{g})^{-1} = (h^{-1} * g * h)^{-1} = h^{-1} * g^{-1} * h, \tag{2.23}$$

and, again, since g^{-1} is an element of the group G the conjugacy relation is preserved, so there is closure under the inverse operation. What makes this interesting is that the product of group elements translates (is isomorphic) into a product of matrices when we are in the space of matrices. Now in the space of matrices the relation between matrices

$$A = P^{-1}BP \tag{2.24}$$

is called an equivalence or similarity relationship. We say that the matrices A and B are

[§] I call the group operation product, some people call it addition.

equivalent or similar. But what this means is that the relation from B to A is obtained after a change of the basis of the vectorial space associated with the matrix space. So we can find some matrix P such that the matrices in system (2.8, 2.9, 2.10) is transformed into system (2.20). However the algebra here could be tedious, since this would imply the solution of a system of four quadratic equations with four unknowns. Instead I will find a "natural" base that represents the problem in the simplest form (2.20). I will find the group representation in two stages. First I find the representation for the g_1 120–degree rotation. The g_2 is obtained from $g_1 * g_1$. Then I find the representation for g_3. With all this I can compute from the Cayley table g_5 and g_6. Referring back to Figure (2) we pick as base vectors

$$
\begin{aligned}
\boldsymbol{v_1} &= \boldsymbol{c} = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix} \\
\boldsymbol{v_2} &= \boldsymbol{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}
\tag{2.25}
$$

In this new base we search for a matrix $\boldsymbol{M}$ such that $RV = MV$ where R represents a 120–degree rotation counter–clockwise and the expression in the right is a matrix matrix multiplication with the matrix V having the base vectors as its columns. That is, since we know

$$
R(v_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_2 \tag{2.26}
$$

$$
\text{and} \qquad R(v_2) = \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix} \tag{2.27}
$$

(2.28)

So we must solve the equation

$$
R(v_1) = r_{11}v_1 + r_{12}v_2 \tag{2.29}
$$

$$
R(v_2) = r_{21}v_1 + r_{22}v_2 \tag{2.30}
$$

but from

$$R(v_1) = v_2 = 0\, v_1 + 1\, v_2. \tag{2.31}$$

we see that $r_{11} = 0, r_{12} = 1$ Now the second equation is

$$\begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2\; r_{21} \\ (-1/2)\, r_{21} + r_{22} \end{pmatrix} \tag{2.32}$$

so clearly $r_{21} = -1$ and $r_{22} = -1$, hence the matrix r is given in the new basis by

$$r = g_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1. \end{pmatrix} \tag{2.33}$$

Since this matrix represents a rotation of 120 degrees counter–clockwise we will identify it with g_1. We observe that this matrix is indeed Vainshtein's, (1994) matrix g_1. Then $g_2 = g_1 * g_1 = (g_1)^2$. That is

$$g_2 = \begin{pmatrix} -1 & -1 \\ 1 & 0. \end{pmatrix} \tag{2.34}$$

We now proceed to find the reflection matrix in the new basis. We have then that the operator flip F acts as following:

$$F(v_1) \quad = \quad b = \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix} \tag{2.35}$$

$$F(v_2) \quad = \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.36}$$

Along the same lines of the operator T we should find some matrix N such that FV=NV; where the left expression represents the physical flip and the right a matrix matrix multiplication with the matrix V is having the base vectors as its columns. Hence we must solve the following system

$$F(v_1) = f_{11}v_1 + f_{12}v_2 \tag{2.37}$$

$$F(v_2) = f_{21}v_1 + f_{22}v_2 \tag{2.38}$$

for the matrix N with components f_{ij}. From the second equation in this system and equation (2.36) we have

$$F(v_2) = 0\,v_1 + 1\,v_2, \tag{2.39}$$

That is $f_{21} = 0$, $f_{22} = 1$. Now, from equations (2.35) and (2.37) we have

$$F(v_1) \;\; = \;\; \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2\, f_{11} \\ -1/2 f_{11} + f_{12} \end{pmatrix}, \tag{2.40}$$

so $f_{11} = -1$ and $f_{12} = -1$ that is g_3 is described by the matrix

$$g_3 = \begin{pmatrix} -1 & -1 \\ 0 & 1. \end{pmatrix} \tag{2.41}$$

We see that this corresponds to what Vainshtein calls g_4 but we will stick to our notation. We are ready to finish the other elements of the D_3 group. $g_4 = g_3 * g_1$. That is

$$g_4 = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}. \tag{2.42}$$

This coincides with Vainshtein's g_3. Finally $g_5 = g_3 * g_2$, this is

$$g_5 = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.43}$$

and this is Vainshtein's g_5. So we found a way to derive Vainshtein's equation (2.49).

It is interesting to observe how the "natural" basis produce the simplest matrix representation of the group. Also it is important to note that since the basis used here is not orthonormal then the matrices operators are not isometric (the matrix representation is not orthonormal), which is not the case for the two sets of matrix representations (the six 6×6 and the six $2x2$ matrices) above. That is, angles and distances are not preserved. While this can matter in geometry it does not matter under group theory.

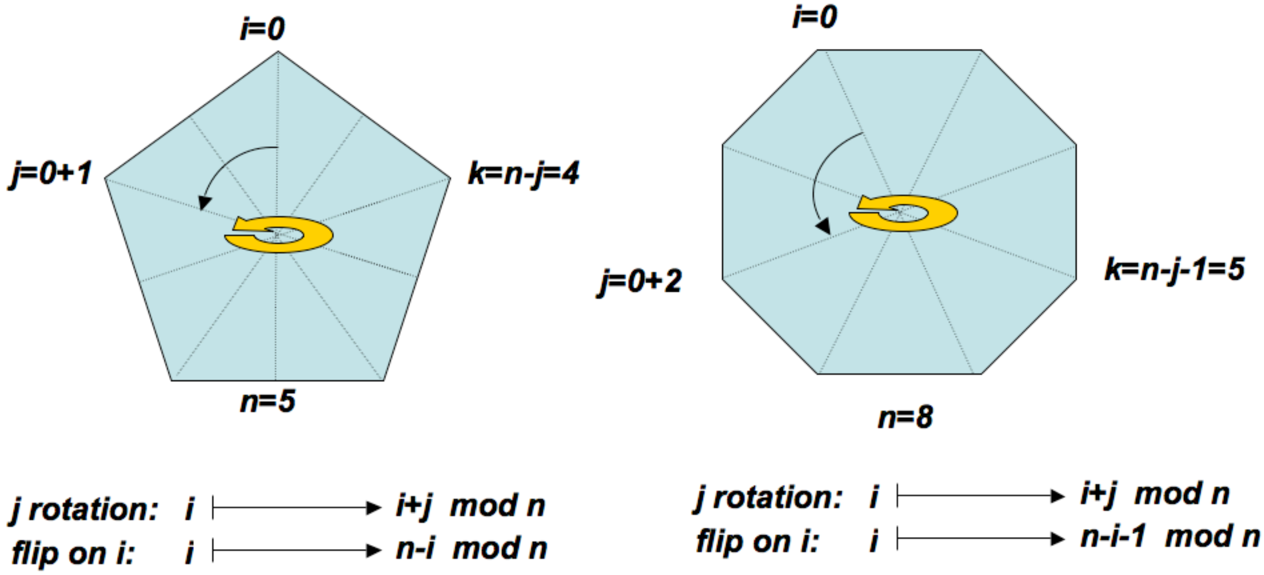

Figure 4: Rotations and reflections on a pentagon and an octagon as functions that map the vertices indices into other vertices indices

2.1.7 Representations of D_3 as a sets of functions f from the integers to the integers module 3. That is $f : \mathbb{Z} \mapsto \mathbb{Z}_3$

In general for a dihedral group of order n, D_n (sometimes also known as D_{2n}) the rotation and reflection (flip) can be described by two functions from the set of integers to the set of integers module n. For any n the rotation by j steps is given by the mapping:

$$i \mapsto i + j \mod n. \tag{2.44}$$

Here i, j are integers between 0 and $n - 1$. The reflection works different for even or odd numbers n. If n is odd, the reflection mapping is given by

$$i \mapsto n - i \mod n \tag{2.45}$$

while if n is even the mapping is given by

$$i \mapsto n - i - 1 \mod n \tag{2.46}$$

Figure (4) shows an illustration for the case of $n = 5$ and $n = 8$. We find then that the first four elements of the D_3 group as functions in the space of functions from the

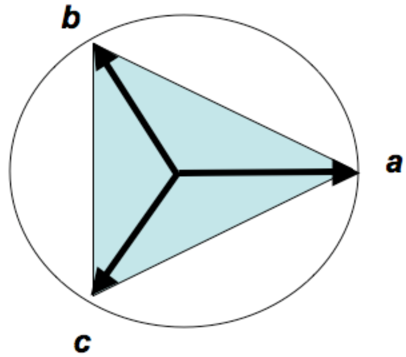

Figure 5: The equilateral triangle of Figure 1, rotated 90 degrees clock–wise

integers F to the integers module 3 $\mathbb{Z}_3$ are given by

$$\begin{aligned} g_0 : \ i \ &\mapsto i \\ g_1 : \ i \ &\mapsto i+1 \quad \bmod 3 \\ g_2 : \ i \ &\mapsto i+2 \quad \bmod 3 \\ g_3 : \ i \ &\mapsto 3-i \quad \bmod 3 \end{aligned} \tag{2.47}$$

Now, from the Cayley table (1)

$$g_4(i) = g_2(g_3(i)) = g_2(3-i) = (5-i) \bmod 3 = (2-i) \bmod 3$$

and

$$g_5(i) = g_3(g_2(i)) = g_3(i+2) = (3-i-2) \bmod 3 = (1-i) \bmod 3.$$

That is,

$$\begin{aligned} g_4 : \ i \ &\mapsto 2-i \quad \bmod 3 \\ g_5 : \ i \ &\mapsto 1-i \quad \bmod 3 \end{aligned} \tag{2.48}$$

It is left to the reader to check that these g_i satisfy the Cayley Table (1).

2.1.8 Representations of D_3 as the sets of complex numbers in the unit circle

In this case it is convenient to redraw the initial triangle in Figure (1), so that the vertix a is along the positive x axis as shown in Figure (5). In the two Cartesian plane the coordinates of a, b, c are

$$a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \quad c = \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix}. \tag{2.49}$$

In the complex plane a, b, c are the cubic roots of 1. The identity rotation g_0 is expressed as 1. That is $g_0 = a = 1$. Or as a function of the complex plane into the complex plane $g_0(z) = z$ where z is a complex number. A counter–clockwise rotation by 120 degrees is expressed as

$$g_1 = b = e^{2\pi/3\,\mathrm{i}} \tag{2.50}$$

or as a function in the complex plane $g_1(z) = b\,z$. A 240 counter–clockwise degrees rotation gives

$$g_2 = c = e^{4\pi/3\,\mathrm{i}}, \tag{2.51}$$

or as function in the complex plane $g_2(z) = c\,z$. Under the new geometry g_3 is a reflection about the x axis. In the complex plane this reflection is represented by the conjugate number obtained by reversing the sign of the imaginary component. Hence as a function in the complex plane $g_3(z) = \bar{z}$. Finally, since $g_4 = g_3 * g_1$ then in the complex plane $g_4(z) = g_3(g_1(z)) = g_3(b\,z) = \bar{b}\,\bar{z}$; and since $g_5 = g_3 * g_2$, then, in the complex plane, $g_5(z) = g_3(g_2(z)) = g_3(c\,z) = \bar{c}\,\bar{z}$. We might want to use the short hand notation

$$g_0 = a, \quad g_1 = b, \quad g_2 = c, \quad g_3 = \bar{a}, \quad g_4 = \bar{b}, \quad g_5 = \bar{c}$$

where the expression $\bar{u}$ means $\bar{u}(z) = \bar{u}\,\bar{z}$. At this point we are tempted to say that the six roots of the unit are isomorphic to D_3. That is the set of $z_j = \exp((2\pi j/6)\,\mathrm{i})$ $j = 0, 1, 2, 3, 4, 5$ is isomorphic to D_3. This is not the case. The dihedral group D_3 is generated by two elements [¶]. The group of roots of the equation $z^6 = 1$ in the complex plane is a cyclic group, that is it is generated by only one element different from the identity. This cyclic group is commutative (or Abelian) while de dihedral group D_3 is not commutative (or no Abelian). An Abelian group should have a symmetric Cayley table.

[¶]Generated in the same sense of Linear Algebra. That is, all operations on those two elements generate the group and each of those two elements can not be obtained from the other. We will talk more about it later.

2.1.9 Abstract representation of the dihedral D_3 group

We start from the representation under equations (2.44) through (2.46) which in general state that a j rotation (r^j) in a general dihedral group D_n is given by the mapping

$$r^j : i \mapsto i + j \qquad \mod n \tag{2.52}$$

and the reflection s is described by

$$s : i \mapsto n - i \qquad \mod n \tag{2.53}$$

if n is odd or

$$s : i \mapsto n - i - 1 \qquad \mod n \tag{2.54}$$

if n is even. The inverse of the reflection is itself since it is of order two. That is $ss = 1$ [‖] The order of any r that is not the identity 1 is n since $r^n(i) = i + n \mod n = i$. Now let us derive the conjugacy relation for n odd, on each element r^j, under s.

$$s^{-1}\, r^j\, s\,(i) = s^{-1}\, r^j\,(n-i) = s^{-1}\,(n-i+j) = s(n-i+j) = n-(n-i+j) = i-j \tag{2.55}$$

all operations mod(n). That is

$$s^{-1}\, r^j\, s = r^{-j} \tag{2.56}$$

If n is even,

$$s^{-1}\, r^j\, s\,(i) = s^{-1}\, r^j\,(n-i-1) = s(n-i+j-1) = n-(n-i+j-1)-1 = i-j \tag{2.57}$$

so in any case (even or odd) we have the conjugacy relation

$$s^{-1}\, r^j\, s = r^{-j} \tag{2.58}$$

That is, any rotation is conjugate to its inverse rotation. This shows that conjugacy in the group nomenclature coincides with conjugacy as seen in the complex plane. So, one abstract representation of the dihedral D_n group is as follows: The group has two elements that generates it (therefore the word *dihedral*). One element r is of order n and the other s is of order two. They satisfy the conjugacy relation (2.58). This is enough

[‖] It is common to find in the literature that the identity 1 is also noted as e.

to generate the group and the Cayley table (1). The group is then represented by

$$G = \left\{1, r, r^2, \ldots, r^{n-1}, r\,s, r^2\,s, \ldots\; r^{n-1}s\right\}. \tag{2.59}$$

Now, if we want to generatea Cayley table for this group we should consider it divided in two sets as follows

$$\begin{aligned} A &= \left\{1, r, r^2, \ldots, r^{n-1}\right\} \\ B &= \left\{r\,s, r^2\,s, \ldots\; r^{n-1}s\right\}. \end{aligned} \tag{2.60}$$

A is a subgroup of G so operations in A are all in A since $r^j r^k = r^{j+k \mod n}$ which belongs to A. Now operations from an element of A and an element of B produces an element of B, since $r^j(r^k s) = r^{j+k \mod n}s$. Operations between two members of B produce members of B since $(r^j s)(r^k s) = r^j(sr^k s) = r^j r^{-k}s = r^{j-k \mod n}s$ and finally operations between a member of B and a member of A produce a member of B since $(r^j s)r^k = s(s^{-1}r^j s)r^k = sr^{-j}r^k = sr^{k-j \mod n}$ and from (2.58) $sr^{k-j \mod n} = r^{j-k \mod n}s$.

I end up this section by providing a couple of internet links that complement well this document. The links are
http://www.math.uconn.edu / kconrad /math217f07 /dihedral.pdf and
http://www.math.uconn.edu / kconrad /math315f07 /dihedral2.pdf

Note: Yet another representation is using versos from the Geometric Algebra field. See the paper by David Hestens

Discussion

The dihedral D_3 group is one of the simplest groups. It has multiple representations that take us from the simple symmetries of the triangle through the set of 2x2 matrices, permutations, addition of integers module n, and multiplication of complex numbers among others. What is most interesting is that the group itself is an abstraction of so many different but related (to the group) concepts.

Being the triangle the simplest geometrical polygon, I think that the introduction of group theory through the dihedral D_3 is convenient. A great deal of group theory can be studied through the dihedral D_3 group.

Representation Theory [**] is a general theory that goes beyond this example. It has gone a long way and even there is a Journal [††] about it. We see how the group takes different forms under equivalent representations (isomorphisms). Here also there is a field that covers the sets of transformations that respect structures and this is Category Theory [‡‡]

The example of the dihedral group D_3 provides then a motivation for the study of more general subjects beyond just group theory, while providing a practical way to learn about group theory.

References

Vainshtein, B. K. 1994. *Fundamentals of Crystals: Symmetry, and Methods of Structural Crystallography*. Springer-Verlag. 1, 3, 11, 14

[**]http: //en.wikipedia.org /wiki /Group_representation

[††]http://www.ams.org/ert/

[‡‡]http://en.wikipedia.org /wiki /Category_theory